Geometric Compressed Sensing and Structured Sparsity

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Zusammenfassung

Compressed Sensing ist eine neuartige Methode in der Datenverarbeitung, welche einen Vorteil aus der Tatsache zieht, dass viele Signale dünnbesetzt (engl.: sparse) sind oder zumindest eine dünnbesetzte Darstellung erlauben. Das heißt, dass nur eine kleine Anzahl an Einträgen des Signals von null verschieden sind, oder dass sich das Signal als Linearkombination weniger Elemente eines gegebenen Erzeugendensystems darstellen lässt. Die Dünnbesetztheit erlaubt dann die Rekonstruktion des Signals aus signifikant weniger Messungen im Vergleich zu traditionellen Methoden.

Im ersten Teil dieser Dissertation betrachten wir Signale mit einer gewissen geometrischen Struktur. Genauer betrachten wir Signale, welche sich als Vereinigung weniger diskreter Geraden darstellen lassen. Wir untersuchen die Frameeigenschaften des Systems aus diskreten Geraden und wenden Compressed Sensing - Methoden an, um u.a. diskrete Geraden von Punkten zu trennen.

Zudem betrachten wir Signale, welche dünnbesetzt bzgl. eines diskreten Gaborsystems sind. Dabei sind wir besonders an Gaborsystemen interessiert, welche von sogenannten Differenzmengen generiert werden. Diese können wiederum als Geraden in einem endlichdimensionalen projektiven Raum betrachtet werden. Wir werden zeigen, dass die Gaborsysteme als optimal dünnbesetzte Fusion Frames gesehen werden können, und es sich zudem um äquidistante tight Fusions Frames handelt, also um optimale Grassmannsche Packungen.

In der zweiten Hälfte der Dissertation betrachten wir die Phasenrückgewinnung (engl.: phase retrieval) von Signalen. Im Gegensatz zum Compressed Sensing, welches sich mit der Rückgewinnung von Signalen aus linearen Messungen beschäftigt, rekonstruieren wir die Signale bei der Phasenrück- gewinnung nur aus den Beträgen der entsprechenden Messungen. Im Allgemeinen müssen die Signale hier nicht notwendigerweise dünnbesetzt sein, wobei das Interesse an dieser zusätzlichen Annahme in den letzten Jahren gestiegen ist, um die nötige Anzahl der Messungen zu reduzieren. Daher werden wir uns hier genauer mit Signalen beschäftigen, welche dünnbesetzt bzgl. eines beliebigen, aber festen Erzeugendensystems sind.

Schließlich betrachten wir die Phasenrückgewinnung aus Messungen mit einem diskreten Gaborsystem, welches durch einen geeignet gewählten Vektor generiert wird, und beweisen, dass die Phasenrückgewinnung so garantiert werden kann. Für dünnbesetzte Signale können wir die Anzahl der Messungen signifikant reduzieren. Weiter diskutieren wir, welche Generatoren für das Gaborsystem geeignet sind und stellen einen Algorithmus zur Lösung des Problems auf.

Abstract

Compressed sensing is a novel methodology in data processing, which takes an advantage of the fact that most signals are sparse (have a small number of nonzeros), or admit a sparse representation, i.e., can be represented as a linear combination of few elements of a given frame (dictionary). Sparsity then allows one to recover the signal from considerably fewer linear measurements than what is required by traditional methods.

In the first part of the thesis, we exploit sparse geometric structure of the signal. We consider signals consisting of unions of a few discrete lines as the simplest case of geometric sparsity. We investigate the frame properties of the system of discrete lines and the application of compressed sensing to such signal models, for example, for separating discrete lines and points.

The second type of structured sparsity that we consider is, signals which are sparse in a dictionary of time- and frequency-shifts, i.e., a Gabor system. We are interested in Gabor systems generated by difference sets, which can be seen as lines in (finite) projective geometry. We further view this system as a fusion frame, show that it is optimally sparse, and moreover an equidistant tight fusion frame, i.e. it is an optimal Grassmannian packing.

In the second half of this thesis, we move from compressed sensing to phase retrieval: if compressed sensing studies the recovery of signals from a set of linear, non-adaptive measurements, phase retrieval tries to recover signals from only the absolute values of those measurements. In general, the signal in the phase retrieval problem is not necessarily sparse, but there has been an increased interest in the recent years in including the sparsity assumption for this problem, and by that lowering the number of measurements needed for recovery of the signal. We investigate the case when the signal itself is not sparse, but it has a sparse representation in an arbitrary dictionary.

Finally, we consider the phase retrieval problem in the case when the measurements are time- and frequency-shifts of a suitably chosen generator. We prove an injectivity condition for recovery of any signal from all time-frequency shifts, and for recovery of sparse signals, when only some of those measurements are given. We discuss which generators are suitable for sparse phase retrieval from Gabor measurements, and provide an algorithm for solving this problem.

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Chapter 1

Introduction

Compressed sensing is a novel methodology in data processing, which in the past decade has been intensively investigated and widely used by applied mathematicians, engineers, statisticians, theoretical computer scientists and others. Biology, medicine, astronomy, communication, radar and sonar signal processing, imaging science are just a few of the examples where compressed sensing has found applications. This methodology takes an advantage of the fact that most signals admit a sparse representation. In fact, sparsity allows one to recover the signal from considerably fewer measurements than what is required by traditional methods.

Mathematically, we are given a set of measurement vectors $\{a_i\}_{i=1}^m$ in $\mathbb{R}^N$, and measurements (samples) of an unknown signal $x \in \mathbb{R}^N$

$$\langle a_1, x\rangle,\ \langle a_2, x\rangle,\ \ldots,\ \langle a_m, x\rangle.$$

Often the measurements are written in a block from as $y = Ax$. The question is whether we can recover x, under the assumption that it is k-sparse, i.e., it has at most k nonzero entries. The system at the same time is underdetermined, i.e., $m << N$. This idea was conceived around 2006, with the pioneering works [26, 31, 46], and has been followed by a variety of reformulations, extensions and applications. Since then, there has been an exponential growth of articles in the field of sparse representations and compressed sensing. In the following, we give an overview of the main related questions and problems.

We sketch here the main research directions related to this question.

1. Given the measurements $y = Ax$, how can the sparse solution x be obtained? Using the notation $\|x\|_0 := |\operatorname{supp}(x)|$, we can formulate the minimization problem

$$\min \|x\|_0 \text{ subject to } Ax = y,$$

which would ideally find the desired x. However, unfortunately this problem is NP hard (see, e.g, [64, Theorem 2.17]). A relaxation to an ℓ_1 minimization was proposed in [40]:

$$\min \|x\|_1 \text{ subject to } Ax = y.$$

This problem (also called *Basis Pursuit* (BP)) can be rewritten as a linear program. Moreover, it can be successfully modified to suit the case when the measurements are contaminated by noise, by substituting $Ax = y$ with $\|Ax - y\|_2 \leq \eta$, where η is the noise level. Basis Pursuit is by far not the only possible way of recovering a sparse vector from a small number of linear measurements. Various iterative, greedy, or message passing algorithms are proposed in the literature and successfully used.

2. How many measurements are needed in order to guarantee recovery of a sparse vector x? When we speak about recovery guarantees, it is important to distinguish between *uniform* and *non-uniform* recovery. In the first case, recovery is guaranteed for all k-sparse signals simultaneously, whereas in the second case, for every fixed k-sparse vector we have a certain recovery guarantee. Furthermore, often there are guarantees which work for *almost all* k-sparse signals, in other words, for random sparse signals, and recovery is guaranteed almost surely. Moreover, in applications, the number of measurements is usually fixed, and the question is up to what level of sparsity recovery can be guaranteed.

3. What kind of properties should the measurement matrix A obey in order to guarantee recovery of sparse vectors? The most common guarantees are related to the notions of spark, mutual coherence, the restricted isometry property (RIP) and the null space property. These be discussed in details in Chapter 2. We only note here, that so far the best known guarantees can be obtained if the matrix A is random. This observation [31] brought a lot of attention to the subject, as with conventional sampling schemes it was not possible to recover an N-dimensional (k-sparse) signal from $m = O(k \ln(N/k))$ samples. Unfortunately, pure deterministic matrix constructions require the number of samples to be of the order of k^2, and the search for measurement matrices which are both suitable for applications and have good theoretical guarantees still continues.

1.1 The Future of Compressed Sensing

In 2012, Strohmer [117] and Elad [53] published two articles[1] on the current achievements and future research directions of the theories of compressed sensing and sparse and redundant representation modeling. To cover all different developments of this field is far beyond the scope of this thesis, but we state some of the questions, which are analyzed in this thesis.

1.1.1 Structured Sensing Matrices

Although the best theoretical results so far are for random matrices, the measurement matrices which are useful in practice (or are simply given by the application) are usually not Gaussian or Bernoulli. Nevertheless, they typically obey some specific structure, such as, e.g., of Fourier or Hadamard type, or time-frequency structure (Gabor systems). Gabor systems [99] are collections of time- and frequency- shifts (translations and modulations) of a chosen generator, and they have already been shown useful for a variety of applications of sparse recovery. For example, they are employed in model selection (also called sparsity pattern recovery) [9], and channel estimation and identification [100].

In the aforementioned applications, two main types of generators have proved to be particularly useful from both a theoretical and practical point of view, these are Alltop and random vectors (see, e.g., [99]). They provide theoretical guarantees which are due to the near optimal coherence properties of the Gabor systems generated by these vectors. We investigate Gabor systems in Chapters 4 and 6. In Chapter 4, we exploit how far one can go with a specific deterministic generator (using a construction from combinatorial design theory), and in Chapter 6 we explore which generators yield Gabor systems suitable for recovery from only the magnitude of the linear measurements. We will observe here also, that both random and deterministic generators are possible.

1.1.2 Beyond Sparsity (Other Simplicity Measures)

A key a priori condition for the compressed sensing idea is *sparsity*. Often in applications, the signals are not sparse themselves, but have a *sparse representation* in some dictionary. There are further extensions of this kind of simplicity measure: for example, our signal is a matrix X, which is not sparse, but has a small rank (has only few nonzero singular values). The recovery of such matrices is known as the matrix completion problem. In this case, an analogous approach to the ℓ_0 minimization would be to minimize the rank of

[1] invited contributions "offering a vision for key advances in emerging fields"

the matrix. Since this is also computationally infeasible, a relaxation to minimizing the nuclear norm was proposed [25]. Another type of sparsity is block sparsity [56], where one works under the assumption that the signal which has a block structure consists of only few nonzero blocks. Joint sparsity [39, 50], also known as multiple-measurement vectors model, is a setup when multiple signals which share the same support are measured.

Somewhere in between is yet another type of simplicity measure, which has deeper geometrical structure. It is based on the powerful concept of redundant systems called *fusion frames* [18, 38], which are frame-like collections of subspaces in a Hilbert space. In this case, one has the assumption that the signal lies in only *few subspaces*, and as we discuss in Chapter 4, it is then reasonable to minimize the mixed ℓ_1/ℓ_2 norm. Similarly to the classical frame setting, one is interested in constructing fusion frames with prescribed optimality properties in order to guarantee recovery, including, for example, analogous coherence and RIP conditions [18, 7].

1.1.3 Nonlinear Compressed Sensing

Another key aspect of compressed sensing, which is crucial for the successful recovery of sparse vectors, is *linearity,* i.e., the fact that the measurements are linear. However, driven by various applications, it is natural to ask how to proceed if the signals are measured in a non-linear way. One example is the so-called 1-bit compressed sensing [19, 104], where we are given only the sign of the measurements, ± 1, but the value is lost. Another example, and this is the one that we focus on, is the situation where we are given only the absolute values of the linear measurements. This is the so-called *phase retrieval* problem [13, 28]. Although it is traditionally a problem without sparsity prior, it is however, natural to assume sparsity of the measured signal [125, 97]. We consider the phase retrieval problem with both general and structured measurements, in Chapters 5 and 6.

Finally, we would like to mention that the properties of the dictionary which sparsely represents the signal are also very important. Having not only sparse, but also a redundant representation instead of a representation in a basis often proves helpful in order to gain stability, robustness against noise, erasures, etc. By redundant representation here we mean that the signal is sparse in a dictionary which spans the whole space, but has more elements then the dimension of the signal. Such construction are known as *finite frames*, and the rich theory of finite frames [37, 33] is often useful from a compressed sensing perspective. For example, having a small mutual coherence ($\max_{j\neq k} |\langle a_j, a_k\rangle|$) of the normalized measurement vectors guarantees recovery of sparse vectors via the ℓ_1

minimization. Likewise, collection of vectors which have minimal possible mutual coherence are interesting for frame theory and are known as *equiangular tight frames* (ETF). As we explain further, especially interesting is the question of constructing ETF with N^2 elements in dimension N, and we give more details about this problem in Chapter 4.

1.2 Main Contributions

The content of this thesis, in light of the aforementioned problems, can be divided into two logical units. In the first half (Chapters 3 and 4) we consider recovery of sparse signals from structured *linear* measurements. In the second half (Chapters 5 and 6), we investigate respective problems in the *non-linear* setting, i.e., recovery from *phaseless* measurements. In what follows, we give a detailed overview of our contributions in both areas.

1.2.1 Structured Sparse Signal Recovery from Linear Measurements

As already mentioned, many signals from various applications are sparse, in other words, consist of only few significant components. Compressed sensing takes advantage of this fact in order to recover the sparse signals from a small set of (linear) non-adaptive measurements in an efficient manner. Often, signals have additional structure, and sparsity can be seen from a different point of view, for example, as a measure of the presence of certain geometric properties in the signal. We want to exploit this geometric structure of the signals, in order to improve the existing compressed sensing results where no structure is assumed. Another goal is to broaden the range of applications of the methodology of compressed sensing, and strengthen them with theoretical guarantees.

In Chapter 3 we investigate signals consisting of unions of discrete lines as the simplest case of geometric sparsity. The first general definition of discrete lines (called *planar digital straight lines*) was given by Reveillès in 1991 [108]. There, the ideas of so called *arithmetic geometry* are developed, meaning that discrete geometric structures are defined by linear Diophantine inequalities. This construction is further developed and used in *digital* geometry [79].

Our discrete lines are a special case of the definition in [108], and we show that the standard properties of lines in Euclidean geometry are satisfied, both in two and higher dimensions. Besides the geometric properties, we are also interested in the behavior of this system from a signal processing point of view. We show that a collection of discrete lines can be modified to a unit norm tight frame. In addition, we find an expression for the spark of these frames. In the above context, we focus on two main objectives:

1. Separation of discrete lines and points

2. Recovery of unions of discrete lines from a small number of linear measurements.

A motivation for using the model of discrete lines in the separation problem is given in [47], and stems from extragalactic astronomy, where it is often required to separate lines, points and planes from $3D$ volumetric data. Some numerical experiments with such data are provided in [48]. Also, similar to our idea of separating discrete points and lines is the work on theoretical and numerical analysis of separation of curve-like and point-like structures conducted in [84], where shearlets [83] and wavelets were used to sparsely represent each of the components.

The separation problem in general appears very naturally in the context of compressed sensing, and we refer to [57, Chapter 11] for a detailed review of the achievements in this direction. The crucial idea here is the so-called Morphological Component Analysis (MCA) [116, 115], which deals with the separation of features in images which are morphologically different. If it is known that the morphologically different structures obey sparse representations in some dictionaries, which, in turn, satisfy certain incoherence conditions, then the data separation problem can be reduced to a sparse recovery problem.

In that spirit, we would like to obtain theoretical guarantees for successful separation of discrete points and lines. In this case, we need to make sure that the *measurement matrix* in this case satisfies some optimality properties, as for example small mutual coherence, or small RIP constant. The spark of a matrix (minimal number of linearly dependent columns) is also very important, because a large spark guarantees uniqueness of the sparse representation. However, since it is hard to compute in general, it is usually not a point of direct investigation [47]. It turns out that in our case it is possible to compute the spark, and obtain a better theoretical recovery guarantee, compared to, for example, the guarantees obtained via mutual coherence.

We are also interested in obtaining similar results in the case of recovery of sparse geometric structures (lines), from given linear measurements. In this case, it is important how to choose the measurements, as well as how many of them are needed for successful recovery.

In Chapter 4, we again consider discrete lines, but from a prospective of (finite) *projective* geometry. Interestingly, projective planes can be constructed using tools from combinatorial design theory [45], and one of the simplest and most popular examples is the so-called Fano plane, which is a finite projective plane of order 2, having 7 points and lines, with 3 points on every line, and 3 lines through every point. The set of points which define one line in the Fano plane is an example of block design construction called *difference set*, which is a subset $\mathcal{K}$ of $\{0, 1, \ldots, N-1\}$, $|\mathcal{K}| = K$, with parameters (N, K, λ), such that every nonzero element of $\mathbb{Z}_N$ can be represented as a difference of elements in $\mathcal{K}$ in exactly λ different ways.

The relation of the construction of difference sets to signal processing and frame theory comes from the fact that if we choose K rows of the discrete Fourier transform matrix indexed according to the elements of a difference set – as it was shown in [128] – we obtain an equiangular tight frame (ETF) of N vectors in dimension K. For other large families of ETFs inspired from design theory see [60, 78].

The problem of finding frames with optimal (in)coherence (in the sense of achieving the Welch bound, or equivalently being ETF) is of great importance not only for signal processing applications, but also for other areas of mathematics. One example is coding theory, where one seeks for maximum-Welch-bound-equality (MWBE) codebooks [128]. Another example is line packing in Grassmannian manifolds, where N lines in the K-dimensional space are sought so that the maximum chordal distance between any two lines is minimized [44, 118]. These equivalent problems are very difficult and analytic constructions are very limited, known to date only for certain parameters N and K (see [59] for a comprehensive overview of known results).

Unlike the case of Euclidean discrete lines in Chapter 3, we do not investigate the geometric properties of difference sets as lines in projective space [15, 114], but we concentrate immediately on the signal processing aspect. In the first part of Chapter 4, we regard a difference set as an element of $\mathbb{C}^N$ via its characteristic function, and investigate the following question: what type of coherence properties does the full system of modulations and translations of a difference set exhibit? As we already mentioned, the collection of all modulations of an (N, K, λ) difference set yield an ETF of N vectors in $\mathbb{C}^K$, and we are interested in properties of the system of modulations and translation in $\mathbb{C}^N$. Here the corresponding optimal packing problem is to pack N^2 lines in N-dimensional space. Although for some difference sets the mutual coherence can be asymptotically small, we show that achieving the Welch bound for full Gabor frames generated by the characteristic function of difference sets is not possible. However, in the light of compressed sensing, our numerical results show that the Gabor measurements generated by some known difference sets are suitable for recovering sparse signals, and have a recovery rate of the order of the Gabor measurements generated by random or Alltop vectors.

In the second part of Chapter 4 by grouping elements of the aforementioned Gabor system into subsystems, we generate a so-called *fusion frame* and prove that these are tight, optimally sparse, and equidistant. Finally, we solve numerically the problem of recovering sparse signals from Gabor frame and Gabor fusion frame measurements, generated by difference sets. The sparsity is understood differently in each of the cases. In the first case, the signal is a linear combination of a small number of modulations and translations of the generator. For Alltop and random generators this problem was considered in [9, 100]. The second case is fusion sparsity, when the signal is represented as a union of few subspaces (in our case translations). This problem was considered previously for general and random fusion frames [18, 7, 8].

1.2.2 Structured Sparse Signal Recovery from Non-linear Measurements

One of the most common steps to go from linear to non-linear measurements is instead of recovering a signal from its linear measurements, to recover it from the *absolute values* of those measurements. Such a setup naturally appears in a variety of applications including X-Ray crystallography, optical imaging, and electron microscopy [112, 71], and is known as the *phase retrieval problem*. We can always recover the signal x only up to a unimodular constant, because x and cx, where $|c| = 1$, always give the same measurements. To fix notation, let $F = \{f_i\}_{i=1}^m \subseteq \mathbb{K}^N$ be a set of measurement vectors, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Further, let $\mathbb{T} = \{c \in \mathbb{K} : |c| = 1\}$. The measurement process is then given by the map

$$\mathcal{M}_F : \mathbb{K}^N/\mathbb{T} \to \mathbb{R}_+^m, \quad \mathcal{M}_F(x) = \begin{bmatrix} |\langle f_1, x\rangle|^2 & |\langle f_2, x\rangle|^2 & \dots & |\langle f_m, x\rangle|^2 \end{bmatrix}^T. \tag{1.1}$$

Given $\mathcal{M}_F(x)$, the task is to recover x up to global phase. We say that F *allows phase retrieval*, if the map $\mathcal{M}_F$ is injective.

There are three main types of questions that one is interested in when studying this problem:

- Injectivity: Which properties of the measurement vectors yield necessary and/or sufficient conditions on the injectivity of the map $\mathcal{M}_F$?
- Minimal number of measurements: How many vectors in F are necessary needed to allow phase retrieval?
- Algorithms: Given the intensity measurements $\mathcal{M}_F(x)$, how can x be recovered up to global phase?

Comprehensive answers to these questions and open problems can be found in [3, 12, 13, 28].

Motivated by different applications, these general questions can also be posed only for a particular type of measurements and/or signals. One common restriction on the signal side is the sparsity prior: what kind of results can be obtained, given the assumption that the vector that we want to recover is sparse? This problem was intensively studied in [125], where the questions of injectivity, minimal number of measurements and recovery via ℓ_1 minimization were investigated, both in $\mathbb{R}$ and $\mathbb{C}$. For example, it was shown that, $2k$ is the minimal number of measurements needed for k-sparse phase retrieval ($k < N$) of real signals (without sparsity prior this number is $2N - 1$ [12]).

In Chapter 5, we answer some of the question above, for the case where the signal is not sparse itself in the standard notion, but has a *sparse representation* in a dictionary (frame). This assumption is more realistic for any type of application, and it is therefore important to know not only intuitively, but also theoretically, that similar results hold in this case as well.

The sparsity prior is interesting from an aspect of a generalization of the phase retrieval problem, where the measurements taken are not scalar products (which can be viewed as projections onto one-dimensional subspaces), but orthogonal projections onto more general subspaces, and we are given only the norms of the projections. This problem was called *phase retrieval by projections* in [20], and turned out to be a difficult topic with many open problems [32]. It was shown in [20] that any signal can be recovered from $2N-1$ "projection phaseless" measurements, and a connection to the classical phase retrieval setting was presented. A question now arises: are there similar results as in the sparse phase retrieval problem discussed above? To our knowledge, sparse phase retrieval by projections has not been investigated so far, and this thesis makes a contribution to the field by proving analogous results to [20] in the sparse setting.

In many applications like X-ray crystallography, transmission electron microscopy and coherent diffractive imaging, the measurements for the phase retrieval problem are of Fourier type, and in fact this is the initial formulation of this problem in imaging and optics: recover the signal, given only the magnitude of its Fourier transform [67, 61, 92].

Conditions on the injectivity of phase retrieval from Fourier measurements, which depend on the structure of the signal (*collision free* and some additional mild properties) were presented in [105]. It was proven that as long as the number of measurements is at least $2N - 1$, such vectors can be determined uniquely. This idea was further developed for the sparse setting in [97], and it was shown that when the dimension N is prime, $2(k^2 - k + 1)$ Fourier measurements are sufficient to guarantee unique recovery of k-sparse collision free vectors.

In Chapter 6, we focus on phase retrieval from Gabor, i.e., short-time Fourier measurements, which are time-frequency shifts of a suitably chosen generator. This type of measurements is of particular interest for many applications in speech and audio processing [95], ptychographical CDI [70], etc. We consider recovery of both arbitrary and sparse signals.

A combination of phase retrieval from Gabor measurements with sparse signals was first considered in [55], where theoretical results on the recovery of *non-vanishing* signals from a full set of N^2 Gabor measurements are obtained, and some intuition about the difficulty of recovery of sparse signals is given. Numerical results show that the latter can be effectively conducted with modification of the GESPAR algorithm [111], using less than N^2 measurements.

In the recent work [77], both theoretical and numerical investigations show, that $O(N \log^3(N))$ measurements are sufficient for recovering general signals from block circulant Fourier based measurements, and if the signal is k-sparse, only $O(k \log^5(N))$ measurements are needed. The structure of the measurements is similar to that one of Gabor systems, but at this moment it is not clear if and how their results transfer to the Gabor setting that we consider here.

Our main objective is the question of injectivity of the map (1.1), when F is a Gabor system. Using the characterization of phase retrievability via the properties of the kernel of the PhaseLift operator [13], in Chapter 6 we provide a condition on the generator which is sufficient for the corresponding Gabor system to allow phase retrieval. We show how this condition can be eased, if the signal that needs to be recovered is non-vanishing. Further, we provide two representative classes of generators, complex random signals and characteristic functions of difference sets, which satisfy the above mentioned condition. On the other hand, we show that the common Gabor generators, which are short windows or Alltop sequences, are not suitable for phase retrieval of general signals (they fail to recover sparse signals). This problem was also considered in [77].

Furthermore, we extend the injectivity condition from [13] to the sparse setting, and provide a similar, but more involved condition on the generator which can guarantee phase retrievability of sparse signals with less than N^2 measurements. When N is prime, we construct generators such that the Gabor system allows k-sparse phase retrieval from $O(k^3)$ measurements. We also generalize this result to signals which are sparse in the Fourier domain, and show that if, additionally, the nonzero elements are structured consecutively in a block, the number of measurements can be further decreased.

Both injectivity theorems naturally yield a simple algorithm for recovery of signals from phaseless Gabor measurements up to global phase, and we present a pseudo-code for it. When all N^2 measurements are given, the recovery of any signal is possible by using solely the fast Fourier transform, which makes the algorithm extremely efficient. If some of the measurements are lost, we can employ ℓ_1 minimization to recover the signal. We provide several numerical experiments to test this idea in various settings, including a recovery of discrete lines from phaseless Gabor measurements.

In the following, we briefly outline the thesis. In Chapter 2 we provide the necessary background on tools and methods from the following fields: compressed sensing and sparse recovery, frame theory (including Gabor frames and fusion frames), and the phase retrieval problem.

In Chapter 3 we start with geometric sparsity, i.e., signals which have a sparse representation in a dictionary of discrete lines. We define discrete lines in arbitrary dimension d, and show that it is possible to construct a unit norm tight frame from the collection of all lines. Furthermore, we compute the spark and the mutual coherence of the initial and the resulting systems. Finally, we consider two sparse recovery settings using discrete lines: one is the problem of separation of points and lines, and the other is recovery of lines from small number of linear measurements.

In Chapter 4 we consider lines from the point of view of projective geometry. This time, instead of translations and rotations, we consider the collection of translations and modulations (frequency shifts) of the characteristic function of a difference set, i.e., a Gabor system. We compare the performance of this system to those generated by random and Alltop vectors, both theoretically and numerically. Moreover, we consider the collection of time- and frequency-shifts as a fusion frame, and prove several optimality conditions for it: tightness, equidistance, and optimal sparsity.

In Chapter 5 we deal with non-linear measurements. We consider the phase retrieval problem for signals which are sparse in an arbitrary dictionary, and investigate the question of injectivity of those measurements both in $\mathbb{R}$ and $\mathbb{C}$. We also discuss the characterization of ℓ_1 recovery from phaseless measurements via the null space property, modified for the purpose of phase retrieval and dictionary sparse signals. At the end, we shortly discuss the question of sparse phase retrieval by projections (we are given only the norm of the projections onto a number of subspaces), and give conditions for injectivity of such measurements.

Finally, Chapter 6 is dedicated to the question of phase retrieval from Gabor measurements, both for sparse and for general signals. We prove sufficient conditions for injectivity in the cases of sparse, nonzero, and arbitrary signals, and provide examples of generators which satisfy those conditions. At the end, we propose an algorithm for recovery which we test with numerical experiments, and analyze its stability.

In the last chapter, Chapter 7, we provide a conclusion where we summarize the work conducted in this thesis, and comment on open questions for future research.

Chapter 2

Background

This chapter is divided into three parts, each representing a specific topic: sparsity and compressed sensing, frame theory, and phase retrieval. We provide the fundamentals, and to keep concise, formulate only those achievements from every area, which will be extensively used in the consecutive chapters.

In Section 2.1, we will state the basic tools and results from compressed sensing. Despite being a fairly new field, sparse representations, compressed sensing, and sparse recovery have been the subject of already a few comprehensive books [64, 57, 63, 52]. We refer the interested reader to those for detailed elaboration on the subject. In Section 2.2 we discuss the theory of finite frames, first in its general setting,followed by Gabor frames fusion frames. There is also an excellent book on finite frames, both for theory and applications [37]. Finally, in Section 2.3, we will state the classical results from phase retrieval, as well as two particular cases: sparse phase retrieval and phase retrieval from Fourier measurements.

2.1 Sparsity and Compressed Sensing

Compressed sensing is often shortly described as a methodology for the recovery of *sparse* vectors from a small number of linear, non-adaptive measurements. The reason why compressed sensing works and how it is possible to solve linear systems with much more variables than equations lies in the *sparsity constraint*. The observation that the signal we are interested in usually lies in a union of low-dimensional subspaces, gives us an intuition, why it should be possible to recover it from an under-determined system. Surely, from an application point of view, it is very rare that many of the components of the signal are zero. More often, the signal has a sparse representation. For example, if the signal is an image, it becomes sparse after some transformation (we say is sparse in some basis), like the wavelet, discrete cosine or Fourier transform. In other cases, the signal can have a sparse representation in some other, perhaps redundant system, in general called a *dictionary.* For example, a signal can consist of few time- and frequency-shifts of some vector, in which case the discrete Gabor system can sparsely represent this signal, or sometimes this dictionary is unknown, and methods like *dictionary learning* are used in order to find a dictionary which will sparsely represent a class of signals [88, 2]. However, our methods for dealing with dictionary sparsity are very closely related to those of classical sparsity. Therefore, in this chapter we focus on tha latter.

Let us start with a formal definition.

Definition 2.1. A signal $x \in \mathbb{R}^N$ is called k*-sparse*, if it has at most k nonzero components:

$$\|x\|_0 := |\{j : x(j) \neq 0\}| \leq k.$$

The set of all k-sparse vectors is denoted by Σ_k, i.e., $\Sigma_k := \{x \in \mathbb{R}^N : \|x\|_0 \leq k\}$.

Although $\|\cdot\|_0$ is not a norm by definition, this notion is useful for the formulation of the sparse recovery problems. First, it is important to note that, often signals are not exactly k-sparse, in the sense that many coefficients are not exactly zero, but they are small or negligible. In this case, an important measure is the *best* k*-term approximation* [43], defined for $x \in \mathbb{R}^N$ as

$$\sigma_k(x)_1 := \min_{\tilde{x} \in \Sigma_k} \|x - \tilde{x}\|_1. \tag{2.1}$$

If a signal x is k-sparse, then $\sigma_k(x) = 0$, and the signal x is called k*-compressible,* if $\sigma_k(x)$ is sufficiently small, or more precisely, if there exist $C, r > 0$ such that $\sigma_k(x)_1 \leq Ck^{-r}$. The question we are interested in is: given a matrix $A \in \mathbb{R}^{M \times N}$ and linear measurements of the sparse vector $y = Ax \in \mathbb{R}^m$, recover x. The sparsity k or the positions of the nonzero coefficients are not known.

A first and most intuitive approach to find a sparse vector x would be to solve

$$\min \|x\|_0 \quad \text{subject to} \quad Ax = y, \tag{P_0}$$

but as we also discussed in the Introduction, this problem is NP hard.

In (2.1), how close is x to being sparse is measured in the ℓ_1 norm, defined for $x \in \mathbb{R}^N$ as

$$\|x\|_1 = \sum_{i=1}^{N} |x|,$$

and this is not accidentally (although in come cases other norms can be considered as well). Indeed, as indicated in Figure 2.1, the ℓ_1 norm promotes sparsity the most among all other norms. This led to the idea of Chen, Donoho and Saunders [40] to substitute

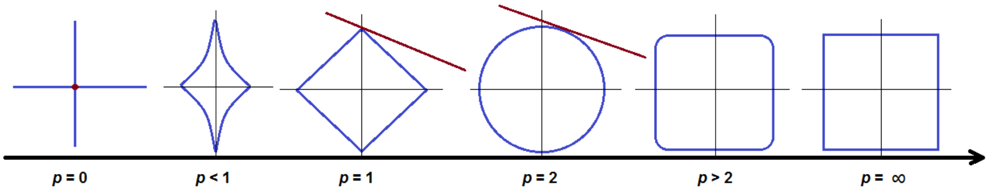


Figure 2.1: Different norms in $\mathbb{R}^2$ and the solution of $Ax = y$.

(P_0) with an alternative, which can be rewritten as easy as linear programming:

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = y. \tag{P_1}$$

This formulation is known as the ℓ_1 minimization problem, or *Basis Pursuit (BP)*. Such an approach to recovering sparse vectors from linear measurements is probably the most common and most investigated one. There are also various efficient greedy and iterative methods, as for example the orthogonal matching pursuit (OMP)[121], compressive sampling matching pursuit (CoSaMp) [96], iterative hard thresholding (IHT) [16], message passing algorithms [49] and many others. In this thesis we will be working mainly with BP. In order to successfully recover x from $y = Ax$,, it is not sufficient that x is sparse, and special properties of the matrix A need to be satisfied. When speaking about recovery guarantees for the ℓ_1 minimization problem (P_1), the most commonly used features are: the mutual coherence, the null space property, and the restricted isometry property of the matrix A. Before we discuss each of them and provide the corresponding results, we will describe a property which is related to the uniqueness of the sparse representation, or in other words, to the solution of (P_0).

Although the ℓ_0 minimization (P_0) is not suitable for recovering x practically, it is important to understand under which conditions the representation $y = Ax_0$ is unique for k-sparse x_0, or equivalently, when a k-sparse x_0 is the unique solution to (P_0). An answer to this question is given by using the *spark,* a notion introduced in [47] and defined as follows.

Definition 2.2. Let $A \in \mathbb{R}^{M \times N}$. The *spark* of A is defined as the smallest number of linearly *dependent* columns of A.

We can rewrite this definition using the ℓ_0 notation as

$$\text{spark}(A) = \min\{\|x\|_0 : x \in \mathbb{R}^N \backslash \{0\}, \text{ such that } Ax = 0\}.$$

Theorem 2.3 ([47])**.** *Let $A \in \mathbb{R}^{M \times N}$, and let $k \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *If a solution x of (P_0) satisfies $\|x\|_0 \leq k$, then this is the unique solution.*

(ii) $k < \text{spark}(A)/2$.

The idea of the theorem is intuitively clear: if the spark of A is sufficiently large, then any subchoice of less then spark(A) columns will be linearly independent, and thus the restriction of A to those columns will be injective. However, computing the spark is an NP-hard problem [120], and other easily computable properties are used in practice instead. One example is the mutual coherence, which when small, guarantees uniqueness of the solutions via both (P_0) and (P_1).

2.1.1 The Mutual Coherence

Definition 2.4. Let $A \in \mathbb{R}^{M \times N}$ be a matrix with columns $\{a_i\}_{i=1}^N \in \mathbb{R}^M$. Then its *mutual coherence* is defined as

$$\mu(A) := \max_{i \neq j} \frac{|\langle a_i, a_j \rangle|}{\|a_i\| \, \|a_j\|}. \tag{2.2}$$

We now present two results on sparse recovery including the mutual coherence.

Theorem 2.5 ([47, 54]). *Let $A \in \mathbb{R}^{M \times N}$ and let $x_0 \in R^N \backslash \{0\}$ be a common solution of (P_0) and (P_1). If*

$$\|x_0\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right), \tag{2.3}$$

then x_0 is the unique solution of both (P_0) and (P_1).

Theorem 2.6 ([47]). *Let $A \in \mathbb{R}^{M \times N}$ have a unit norm columns and a mutual coherence which satisfies*

$$(2k-1)\mu(A) < 1.$$

Then (P_1) recovers every k-sparse vectors x from $y = Ax$.

Unlike the spark, the mutual coherence is easy to calculate, and in our work we will often use it to provide theoretical guarantees for the recovery problem with particular measurement matrices. The drawback of this approach is that the mutual coherence bound on the sparsity of x (2.3) is highly suboptimal, which can explained partially by that fact that this is a worst-case bound. Attempts for improvement, which still involve a measure of the (in)coherence of A were made by introducing the so called *average coherence* [9, 10] as well as the *asymptotic incoherence* [1].

2.1.2 The Null Space Property

As we noted, the results which involve the mutual coherence give only a sufficient condition, and often provide a very rough bound on the level of sparsity that can be recovered. Also, often in compressed sensing one is able to choose the number of measurements M and the interesting question is: what is the minimal number of measurements for a given sparsity level, which can guarantee recovery? In this respect, an important role is played by the notions of *null space property* and *restricted isometry property.*

Definition 2.7. Let $A \in \mathbb{R}^{M \times N}$. Then A has the *null space property (NSP) of order k,* if, for all $h \in \mathcal{N}(A) \setminus \{0\}$ and all index sets $T \subseteq \{1, \dots, n\}$ with $|T| \leq k$,

$$\|h_T\|_1 < \frac{1}{2}\|h\|_1,$$

where $\mathcal{N}(A)$ is the null space of A and h_T is the vector h restricted to the indices in T.

Unlike the mutual coherence, the null space property is difficult to verify in general, since it requires a combinatorial search. Nevertheless, the NSP plays an important role in the achievements of theoretical results for compressed sensing. As we will see in the next theorem, it yields a characterization of the solvability of (P_1).

Theorem 2.8 ([43])**.** *Let $A \in \mathbb{R}^{M \times N}$ and $k \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *If a solution x of (P_1) satisfies $\|x\|_0 \leq k$, then it is the unique solution.*

(ii) *A satisfies the NSP of order k.*

Various modifications of the null space property appeared in the literature, as people were getting interested in classes of signals which are more general than that of the k-sparse vectors: dictionary-based NSP [41], fusion NSP [18] and others. We will also use some of these approaches in Section 5.3 of Chapter 5, to create a new NSP suitable for non-linear (phaseless) measurements of signals which are sparse in a dictionary.

2.1.3 The Restricted Isometry Property

Now we come to the last, but probably most widely known property — the restricted isometry property (RIP). One can say that compressed sensing as a theory started with the RIP and with the first recovery guarantees using it, in the series of work of Candes, Romberg and Tao [26, 30, 31], along with the work of Donoho [46]. The success of the RIP is based on the fact that many random (sub-Gaussian) or randomly chosen measurements (for example, partial Fourier) satisfy the RIP with a small RIP constant, and that guarantees success of the ℓ_1 minimization, both for exact and noisy measurements [27, 109, 90]. Also, the question of the number of measurements needed can be answered in this case: roughly $M \geq 2k \ln(N/k)$ random measurements will recover k-sparse vector x with high probability (see, e.g., [64, Chapter 9]. Having measurements only of the order of the sparsity level and not of the dimension of the signal allows to go beyond the Nyquist sampling rate [123]. This brought data acquisition to a completely new dimension, and prompted an avalanche of research in the last decade.

Definition 2.9. Let $A \in \mathbb{R}^{M \times N}$. Then A has the *restricted isometry property (RIP) of order* k, if there exists a $\delta_k \in (0,1)$ such that

$$(1-\delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_k)\|x\|_2^2 \quad \text{for all} \quad x \in \Sigma_k. \tag{2.4}$$

We see that the RIP requires that every set of k columns of A should behave like an isometry. The RIP constant can be calculated by the following expression:

$$\delta_k = \max_{T \subseteq [1,\ldots,N], |T| \leq k} \|A_T^* A_T - I_M\|_{2\to 2}. \tag{2.5}$$

Since verifying the RIP requires a search over all possible subsets of $[1,\ldots,N]$ of size at most k, the problem becomes intractable in the case of pure deterministic constructions. To overcome this problem, the *statistical RIP* was introduced [23], where we require that (2.4) holds only for all randomly chosen subsets T of size k. Another idea is to consider so-called *structured random matrices* [106], for which the RIP can be computed, and which are one step closer to deterministic constructions and therefore have certain advantages from an application point of view.

Let us state one of the most classical results for the ℓ_1 recovery of sparse vectors via RIP, which we will also use later in Chapter 3.

Theorem 2.10 ([43, 24])**.** *Let $A \in \mathbb{R}^{M \times N}$ satisfy the RIP of order $2k$ with $\delta_{2k} < \sqrt{2}-1$. Let $x \in \mathbb{R}^N$, and let $\hat{x}$ be a solution of the associated ℓ_1 problem (P_1). Then*

$$\|x - \hat{x}\|_2 \leq C \cdot \frac{\sigma_k(x)_1}{\sqrt{k}},$$

for some constant C dependending only on δ_{2k}.

The condition $\delta_{2k} \leq \sqrt{2}-1$ which guarantees recovery was constantly changing its format and improving over the years. For example, an RIP with $\delta_k < \frac{1}{3}$ is another sufficient condition for recovery of k sparse vectors via ℓ_1 [21], and this bound was then further improved to a sharp result $\delta_{2k} \leq \frac{1}{\sqrt{2}}$ by the same authors in [22].

Also, we would like to note here that there is a crucial difference between the Theorem 2.5 and Theorem 2.10. Namely, the result involving the mutual coherence is so-called *uniform* result, because it holds for all k-sparse vectors simultaneously. Whereas in the RIP result we stated in Theorem 2.10, we first fix a k-sparse vector, and thus it is referred to as a *non-uniform* result in the literature. Uniform results based on the RIP also exist, but we will not formulate them here for the sake of brevity.

2.2 Finite Frames and Fusion Frames

For more details on frame theory and its applications we refer the interested reader to the book [37] and for abstract frame theory and the history of its development to the survey paper [33].

Let N be a positive integer. We denote by $\mathcal{H}^N$ a real or complex N-dimensional Hilbert space. The Hilbert-Schmidt norm of an operator $A : \mathcal{H}^N \mapsto \mathcal{H}^N$ is defined as

$$\|A\|_{HS}^2 = \mathrm{Tr}(A^*A) = \sum_{i=1}^{N} \|Ae_i\|^2,$$

where Tr denotes the trace of an operator. We have $\|A\|_{HS} = \langle A, A\rangle_{HS}$, where

$$\langle A, B\rangle_{HS} = \mathrm{Tr}(B^*A) = \sum_{i=1}^{N} \langle Ae_i, Be_i\rangle\,,$$

with $A, B : \mathcal{H}^N \mapsto \mathcal{H}^N$, and $\{e_i\}_{i=1}^N$ any orthonormal basis of $\mathcal{H}^N$. Also, following the notation from [37], we will write ℓ_2^M for $\ell_2([1, \dots, M])$.

2.2.1 Background on Finite Frame Theory

Although frame theory is a relatively new discipline, the formal definition of a frame (given in infinite dimension) goes back to 1952 and the work of Duffin and Schaeffer [51] on non-harmonic Fourier series.

Definition 2.11. A family of vectors $\{\phi_i\}_{i=1}^M$ in a Hilbert space $\mathcal{H}^N$ is called a *finite frame* for $\mathcal{H}^N$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \phi_i\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}^N. \tag{2.6}$$

Below, we list some notions related to a frame $\{\phi_i\}_{i=1}^M$ which we will extensively use.

1. If $A = B$ in (2.6), then $\{\phi_i\}_{i=1}^M$ is called an A-*tight frame*.

2. If $A = B = 1$, then $\{\phi_i\}_{i=1}^M$ is called a *Parseval frame*.

3. If there exist a constant c such that $\|\phi_i\| = c$ for all $i = 1, 2, \ldots, M$, then $\{\phi_i\}_{i=1}^M$ is an *equal norm frame*. If $c = 1$, $\{\phi_i\}_{i=1}^M$ is a *unit norm frame*.

4. If there exists a constant c such that $|\langle \phi_i, \phi_j \rangle| = c$ for all $i \neq j$, then $\{\phi_i\}_{i=1}^M$ is called an *equiangular frame*.

Two main operators, the *analysis* and *synthesis* operator play an important role in the development of frame theory. The *analysis operator* $T : \mathcal{H}^N \mapsto \ell_2^M$ is defined as

$$Tx := (\langle x, \phi_i \rangle)_{i=1}^M, \quad x \in \mathcal{H}^N.$$

Note that $\|Tx\|^2 = \sum_{i=1}^M |\langle x, \phi_i \rangle|^2$ for all $x \in \mathcal{H}^N$. Therefore, the inequality (2.6) can be rewritten as

$$A\|x\|^2 \leq \|Tx\|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}^N.$$

Hence, $\{\phi_i\}_{i=1}^M$ is a frame if and only if T is an injective operator. The *synthesis operator* is defined as the adjoint operator T^*. It is not difficult to see that the *adjoint operator* $T^* : \ell_2^M \mapsto \mathcal{H}^N$ of T is given by

$$T^*(a_i)_{i=1}^M = \sum_{i=1}^M a_i \phi_i, \quad (a_i)_{i=1}^M \in \ell_2^M.$$

Often, we will identify the family of vectors $\{\phi_i\}_{i=1}^M$ with the matrix Φ of size $M \times N$, with columns $\phi_i, i = 1, \ldots, M$,

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_M \end{bmatrix}. \tag{2.7}$$

This is actually the matrix representation of the synthesis operator T^*, and throughout this thesis we will often write Φ instead of T^*. A concatenation of analysis and synthesis operator gives rise to the so-called *frame operator* $S : \mathcal{H}^N \mapsto \mathcal{H}^N$ which is defined as

$$Sx := T^*Tx = \sum_{i=1}^M \langle x, \phi_i \rangle \phi_i, \quad x \in \mathcal{H}^N.$$

Now, observing that $\langle Sx, x \rangle = \langle T^*Tx, x \rangle = \|Tx\|^2$ we can yet again rewrite (2.6) as

$$A\|x\|^2 \leq \langle Sx, x \rangle \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}^N.$$

Finally, we introduce the *Gram operator (Grammian)* $G : \ell_2^M \mapsto \ell_2^M$ which is defined as

$$G(a_i)_{i=1}^M := TT^*(a_i)_{i=1}^M = \sum_{i=1}^M a_i(\langle \phi_i, \phi_k \rangle)_{k=1}^M, \quad (a_i)_{i=1}^M \in \ell_2^M.$$

Certain properties like tightness and equiangularity can be expressed now in terms of the introduced operators.

Let I_M be the identity matrix of size M, and $\mathbf{1}_M$ the $M \times M$ matrix of all ones.

Proposition 2.12. *Let* $\{\phi_i\}_{i=1}^M$ *be a unit norm finite frame.*

(i) $\{\phi_i\}_{i=1}^M$ *is a tight frame if and only if* $S = T^*T = \Phi\Phi^* = AI_M$*. Moreover, if* $\{\phi_i\}_{i=1}^M$ *is a unit norm tight frame, then* $A = \frac{N}{M}$*.*

(ii) $\{\phi_i\}_{i=1}^M$ *is an equiangular frame if and only if* $G = TT^* = \Phi^*\Phi = c\mathbf{1}_M - (c-1)I_M$*.*

We also note that when $\{\phi_i\}_{i=1}^M$ is a unit norm tight frame, every element in the Hilbert space $x \in \mathcal{H}^N$ can be represented as

$$x = \frac{1}{A} \sum_{j=1}^M \langle x, \phi_j \rangle \phi_j. \tag{2.8}$$

This representation gives us some intuition, why tight frames became a very important tool in signal processing. Enormous work has been done in the last 20 years to understand better and apply frame theory in real-life problems. A comprehensive work in the direction of developing the theory of unit norm tight frames (UNTF) is given in [14]. Additional properties and their suitability for solving problems like loss of coefficients in the representation (2.8) (erasures) are presented in [35]. In Chapter 3, we will see how to construct a family of UNTF from a collection of discrete lines. As mentioned in [14], tight frames are not only important for signal processing, but are also directly connected to questions like equidistribution of points on the unit sphere, and this is moreover true when it comes to equiangular tight frames (ETF).

An important quantity that measures the relations between the frame elements is the *mutual coherence*, which we defined in (2.2). It is clear that on the one hand, when $M = N$ and Φ is an orthogonal basis, we will have $\mu(\Phi) = 0$. On the other hand, if there exist two linearly dependent vectors in the system, the mutual coherence will be $\mu(\Phi) = 1$. There is a minimal value that μ can have and it is given by the so called *Welch bound* [127],

$$\mu(\Phi) \geq \sqrt{\frac{M-N}{N(M-1)}}. \tag{2.9}$$

One interesting and very useful property of the Welch bound is that equality in (2.9) is achieved, if and only if Φ is an equiangular tight frame [118]. This can be seen from the following chain of inequalities, which hold for an arbitrary unit-norm frame $\Phi = \{\phi_i\}_{i=1}^M$ for $\mathcal{H}^N$:

$$\frac{M^2}{N} \leq \|\Phi^*\Phi\|_{HS}^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} |\langle \phi_i, \phi_j \rangle|^2 \leq M + M(M-1)\mu(\Phi)^2.$$

Note that an equality on the left-hand side is equivalent to being a tight frame.

We see again here the importance of ETFs: they are the optimally incoherent frames, but at the same time highly difficult to construct. We shall discuss this question in more detail in Section 4.2 of Chapter 4. Furthermore, as we saw in Section 2.1, one of the most common guarantees for recovery of sparse vectors from linear measurements relies on having small mutual coherence. We will use this approach to get sparse recovery guarantees in Chapter 3.

2.2.2 Gabor Frames in Finite Dimensions

One particular type of frames, a redundant system popular in applications like radar and communications [75], is the system of time- and frequency-shifts, also called a *Gabor system*. Gabor systems are named by Denis Gabor, who was considering such systems back in 1946 [65]. Here, we consider only the finite-dimensional case. For a detailed study, we refer to [99].

We work in the signal space $\mathbb{C}^N$, as a space of complex valued, N periodic functions with integer argument, $x = x(j)$, $j \in \mathbb{Z}$, which therefore always has to be assumed modulo N. We use the customary domain $[0, \dots, N-1]$ of j, but we often write $j \in \mathbb{Z}_N$ for convenience. The scalar product between two signals x and y is defined as

$$\langle x, y \rangle = \sum_{j=0}^{N-1} x(j)\overline{y(j)}.$$

The N-th root of unity will be denoted by $\omega = e^{\frac{2\pi i}{N}}$. We define the (discrete) Fourier transform $\hat{x}$ and the inverse Fourier transform $\check{x} \in \mathbb{C}^N$ as follows:

$$\hat{x}(j) = \sum_{j=0}^{N-1} x(n)\omega^{-nj}, \quad j = 0, 1, \dots, N-1,$$

$$\check{x}(n) = \frac{1}{N}\sum_{j=0}^{N-1} x(j)\omega^{nj}, \quad n = 0, 1, \dots, N-1.$$

We will also often use δ_N,the discrete periodic Kronecker delta function, defined as

$$\delta_N(j) = \begin{cases} 1, & \text{if } j \text{ is divisible by } N, \\ 0, & \text{elsewise.} \end{cases}$$

For $p \in \mathbb{Z}_N$, we define the *translation (or time-shift) operator* $T_p : \mathbb{C}^N \mapsto \mathbb{C}^N$ through

$$(T_p x)(n) = x(n-p), \quad x \in \mathbb{C}^N.$$

Further, we define for $\ell \in \mathbb{Z}_N$, the *modulation (or frequency-shift) operator* $M_\ell : \mathbb{C}^N \mapsto \mathbb{C}^N$ through

$$(M_\ell x)(n) = \omega^{\ell n} x(n), \quad x \in \mathbb{C}^N.$$

By combining translations and modulations we obtain the *time-frequency shift operators* $\pi(p,\ell) : \mathbb{C}^N \mapsto \mathbb{C}^N$,

$$\pi(p,\ell) = M_\ell T_p,$$

For a pair $\lambda = (p,\ell)$ sometimes we will use the short-hand notations $\Pi_\lambda := \pi(p,\ell)$, and $x_\lambda := \Pi_\lambda x$ for some $x \in \mathbb{C}^N$. The following result will play an important role in our investigations on Gabor systems in Chapter 6.

Proposition 2.13 ([99]). *The collection of normalized time-frequency shift operators* $\{\frac{1}{\sqrt{N}}\pi(k,l)\}_{k,l=0}^{N-1}$ *forms an orthonormal basis for the Hilbert-Schmidt space of linear operators in* $\mathbb{C}^N$. *In other words, for any* $\lambda, \mu \in \mathbb{Z}_N \times \mathbb{Z}_N$,

$$\langle \Pi_\lambda, \Pi_\mu \rangle_{HS} = N\delta_N(\mu - \lambda). \tag{2.10}$$

We also note the well-known commutation relations between translations and modulations.

Proposition 2.14 ([69]). *Let* $\lambda = (p,\ell), \mu = (q,j) \in \mathbb{Z}_N \times \mathbb{Z}_N$. *Then, we have*

$$M_\ell T_p = \omega^{\ell p} T_p M_\ell,$$
$$\Pi_\lambda \Pi_\mu = \omega^{-jp}\omega^{\ell q} \Pi_\mu \Pi_\lambda.$$

Now we have all ingredients to define a Gabor system.

Definition 2.15. A *Gabor system* generated by a vector $g \in \mathbb{C}^N \backslash \{0\}$ is the collection of all translations and modulations of g,

$$\Phi_g := \{\pi(k,l)g\}_{(k,l)\in\mathbb{Z}_N^2} = \{M_l T_k g\}_{k,l=0}^{N-1}.$$

Note that sometimes the Gabor systems are defined as a collection of some translations and modulations, $\{\pi(k,l)g\}_{(k,l)\in\Lambda}$, where $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$. Our definition would then correspond to a *full Gabor system*. The generator g is often called a *window function,* by tradition of the discrete short-time Fourier transform $V_g : \mathbb{C}^N \mapsto \mathbb{C}^{N\times N}$, which is defined as

$$V_g x(k,l) = \langle x, \pi(k,l)g\rangle = \sum_{n=0}^{N-1} x(n)\overline{g(n-k)}\omega_N^{-ln}.$$

Thus, frame coefficients of Φ_g correspond to samples of V_g.

The next proposition shows that every Gabor system forms a frame — even equal norm tight frame.

Proposition 2.16 ([87, 99])**.** *For any $g \neq 0$, the collection $\{\pi(k,l)g\}_{(k,l)\in\mathbb{Z}_N^2}$ is a equal norm tight frame for $\mathbb{C}^N$ with frame bounds $A = B = N\|g\|^2$.*

We would like to focus on two very challenging problems related to the following properties of a Gabor frame:

(i) Full spark, and

(ii) Equiangularity.

What we mean by the first condition is finding Gabor frames which have largest possible spark, i.e., $\text{spark}(\Phi_g) = N + 1$. In other words, every collection of N elements needs to be linearly independent. This condition is also known as the *Haar property* [87], or also as a property of being in a *general linear position* [89]. Finding Gabor frames which have the Haar property is also known as the discrete analogue of the HRT (Heil-Ramanathan-Topiwala) conjecture [73], which remain unsolved up to date in its general formulation: prove that time-frequency translates (not necessarily integers) of a nonzero square integrable function f on $\mathbb{R}$ are linearly independent. See the survey [72] for more details on the achievements in this direction. The question for discrete time-frequency shifts them was partially solved in 2005, when Lawrence, Pfander and Walnut proved that such construction exist for any prime dimension [87]. Fortunately, in 2013 Malikiosis solved the existence of full spark Gabor frames affirmatively for any dimension N [89].

The second property, equiangularity, will be discussed in more detail in Chapter 4, so we mention it here only briefly. The problem of finding Gabor frames which are equangular also has various names and interpretations, one of which comes from quantum mechanics and is known as the Zauner's conjecture [129]. It is still an open question if in arbitrary dimension N one can build N^2 equiangular lines (it is also conjectured that they can always be constructed as the time-frequency shifts of a given generator [107], or in our notation that there exists a generator g such that the obtained Gabor frame Φ_g

is equiangular). So far, there exist numerical experiments which confirm this conjecture for dimensions up to 67 [110], and some particular dimensions have been investigated theoretically, including the recent result [42] for dimension $N = 17$.

Unlike the tightness condition, which is satisfied by any Gabor frame, the question of small mutual coherence is strongly dependent on the generator of the Gabor frame. Because we have a fixed dimension of our frame, $N \times N^2$, we can write the Welch bound,

$$\mu(\Phi_g) \geq \sqrt{\frac{N^2 - N}{N(N^2-1)}} = \sqrt{\frac{1}{N+1}}.$$

Knowing the fact that a frame is an ETF if and only if the Welch bound is achieved [118], and since we already have tightness, we see that the difficult question related to Zauner's conjecture is a question about achieving the Welch bound with Gabor systems. We will discuss this question for a specific type of generators in Chapter 4.

We focus next on the relations and connections between the theory of sparse recovery and Gabor frames. The question we are interested in is, given a Gabor frame $\Phi_g \in \mathbb{C}^{N\times N^2}$, and measurements $y = \Phi_g x$ of an unknown sparse vector $x \in \mathbb{C}^{N^2}$, can we recover x using ℓ_1 minimization? We note here, that this question can be seen as a *sparse matrix identification* problem [100]. In such a formulation, one assumes that a matrix $\Gamma \in \mathbb{C}^{N\times N}$ has a sparse representation in the dictionary of time-frequency shift matrices, and the question is to recover Γ from the action of Γ on a test signal $g \in \mathbb{C}^N$. See [100] for more details on the subtle differences between the two problems.

Since the size of the measurement matrix Φ_g, i.e., the number of measrements is fixed, we are interested only how sparse vectors x can recover. We know from Theorem 2.5, that one way to do answer this question is via the mutual coherence. There exists generators, which have *nearly optimal* mutual coherence, or in other words for which the Welch bound is *almost attained*. In particular, it was shown in [118] that if g is an Alltop sequence [5], i.e. $g(j) = \omega_N^{j^3}$, $j = 0, \ldots, N-1$ (with N necessarily a prime), then $\mu(\Phi_g) \leq \frac{1}{\sqrt{N}}$. For general dimension N, it was shown in [100] that when g is independent and uniformly distributed on the torus complex random vector, then the mutual coherence of Φ_g comes close to $\frac{1}{\sqrt{N}}$ with high probability.

Both these results on the mutual coherence of Gabor systems generated by Alltop and random vectors lead to a conclusion about recovery of sparse signals via Theorem 2.5. The corresponding theorems are formulated in [100]. Roughly speaking, it was shown in [100] there that vectors which have sparsity up to the order of $\sqrt{N}$ or $\sqrt{N/\log N}$ can be recovered by ℓ_1 minimization. Furthermore, using a result from Tropp [122], these rates are then improved to a sparsity level of the order of N, if we assume that

x is random. Independently, similar research using the mutual coherence and Alltop sequences was conducted in [75], with a focus on the application of such a measurement process in radar (identifying target scene). Another approach was taken by the authors in [9], where a condition called *the coherence property*, involving both (worst-case) mutual coherence and average coherence is used. They obtain guarantees for recovery of the *support* of an unknown sparse signal via a one-step thresholding algorithm, from Gabor measurements generated by Alltop and random vectors. This result is then also extended to recovery not only of the support, but also of the vaues of the sparse signal.

2.2.3 Background on Fusion Frames

The needs of the fast developing high technology led to a generalization of frames: the idea is to have a collection (union) of subspaces which will span the whole space, and the subspaces are spanned by "smaller" frames. This is the complex framework of *fusion frames*, which includes the classical theory of frames as a special case (each element of the frame in this case generates a one-dimensional subspace). Since its foundation in [38, 36], fusion frames have developed into a rich theory and have found many applications, especially in parallel or distributed processing of sensor networks.

Let us state the definition of a fusion frame, in our formulation considered with all weights equal to one.

Definition 2.17. A family of subspaces $\{\mathcal{W}_i\}_{i=1}^M$ in a Hilbert space $\mathcal{H}^N$ is called a *fusion frame* for $\mathcal{H}^N$, if there exist A and B, $0 < A \le B < \infty$ such that

$$A\|x\|_2^2 \le \sum_{i=1}^{M} \|P_i(x)\|_2^2 \le B\|x\|_2^2 \quad \text{for all } x \in \mathcal{H}^N,$$

where P_i is the orthogonal projection onto $\mathcal{W}_i$.

If $A = B$ is possible, then $\{\mathcal{W}_i\}_{i=1}^M$ is called an *A-tight fusion frame*. Tightness is an important property, required for example, for minimization of the recovery error of a random vector from its noisy fusion frame measurements [85]. Among other desirable properties are equidimensionality and equidistance. They assure maximal robustness against erasures of one or more subspaces, and as we will explain further, yield optimal Grassmannian packings [85]. Equidimensionality means that all the subspaces $\{\mathcal{W}_i\}_{i=1}^M$ are of the same dimension, while to define equidistant fusion frames, we need the notion of chordal distance.

Definition 2.18. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathcal{H}^N$ with $m := \dim \mathcal{W}_1 = \dim \mathcal{W}_2$ and denote by P_j the orthogonal projection onto $\mathcal{W}_j$, $j = 1, 2$. The *chordal distance*

$d_c(\mathcal{W}_1, \mathcal{W}_2)$ between $\mathcal{W}_1$ and $\mathcal{W}_2$ is given by

$$d_c(\mathcal{W}_1, \mathcal{W}_2) = \sqrt{m - \mathrm{Tr}[P_1 P_2]}.$$

Multiple subspaces are called *equidistant* if they have pairwise equal chordal distance d_c.

It was shown in [85] that equidistant tight fusion frames are optimal Grassmannian packings, where optimality comes from the classical packing problem: For given m, M, N, find a set of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^M$ in $\mathcal{H}^N$ such that $\min_{i \neq j} d_c(i, j)$ is as large as possible. In this case we call $\{\mathcal{W}_i\}_{i=1}^M$ an *optimal packing*. An upper bound is given by the *simplex bound*

$$\frac{m(N-m)M}{N(M-1)}. \tag{2.11}$$

This is to some extent analogous to the Welch bound from classical frame theory, and we will see in Chapter 4 that fusion frames generated by so-called difference sets achieve the simplex bound (2.11).

2.2.4 Sparse Recovery with Fusion Frames

Although the main applications of fusion frames are inspired by distributed sensing, parallel processing and packet encoding, it is also possible to transfer (generalize) the classical compressed sensing methodology to the case when a signal is sparse in a fusion frame [6, 18, 58]. We will use this idea to do some numerical experiments on recovery with fusion frames in Chapter 4, and therefore we present here the model and the main achievements in this direction. Let $\{\mathcal{W}_i\}_{i=1}^M$ be a fusion frame for $\mathcal{H}^N$ and define

$$\mathcal{H}_{\mathcal{W}} := \{(x_i)_{i=1}^M : x_i \in \mathcal{W}_i \text{ for all } i = 1, \dots, M\} \subseteq \mathbb{R}^{MN}.$$

An element $x \in \mathcal{H}_{\mathcal{W}}$ is called k-sparse, if

$$\|x\|_0 := |\{i : x_i \neq 0\}| \leq k.$$

Note that here sparsity is defined as the number of nonzero components, and this is very similar to block-sparsity [56] (with an additional structure coming from the fact that $x_i \in \mathcal{W}_i$). On the other hand, if all subspaces are identical we obtain the case of joint sparsity [39, 50], also known as multiple-measurement vectors model. For more details on the hierarchy of different kind of structured sparsity models see Table 1 in [18]. Let $x^0 = (x_i^0)_{i=1}^M \in \mathcal{H}_{\mathcal{W}}$. We would like to measure x_0 with some measurement matrix $A \in \mathbb{R}^{n \times N}$ with unit-norm columns in the following way:

$$y = \{y_j\}_{j=1}^n = \{\sum_{i=1}^M a_{ij} x_i^0\}_{j=1}^n.$$

We can also rewrite it in a block form as $y = A_I x^0$, if we denote $A_I = (a_{ij} I_N)_{i,j=1}^{M,N}$. Since we denoted by $\|\cdot\|_0$ the norm which counts the number of nonzero components in Since we assume that x^0 lies in only few subspaces, and the number of those subspaces is measured by the introduced $\|\cdot\|_0$ norm, it is intuitive to pose the following minimization problem:

$$\min_{x \in \mathcal{H}} \|x\|_0 \text{ subject to } A_I x = y. \tag{2.12}$$

However, such a problem is intractable in practice, and a relaxation of the minimization function is needed. Here, since every nonzero component is itself not necessarily sparse, the idea is to take a mixed ℓ_1/ℓ_2 norm,

$$\|x\|_{2,1} := \sum_{i=1}^M \|x_i\|_2, \quad \text{where } x = (x_i)_{i=1}^M \in \mathcal{H}.$$

Thus, the minimization problem can be formulated as

$$\min_{x \in \mathcal{H}} \|x\|_{2,1} \text{ subject to } A_I x = y. \tag{2.13}$$

For computational purposes, this minimization problem 2.13 can be rewritten as a minimization process over $\mathbb{R}^{MN}$, using orthonormal bases for each subspace, $U_i \in \mathbb{R}^{N \times m_i}$, where $m_i = \dim \mathcal{W}_i$. An equivalent problem to (2.13) is

$$\min_c \|c\|_{2,1} \text{ subject to } AU(c)c = Y,$$

where $c \in \mathbb{R}^{MN}$,

$$U(c) = \begin{bmatrix} c_1^T U_1^T \\ \vdots \\ c_M^T U_M^T \end{bmatrix} \in \mathbb{R}^{M \times N}, \quad Y = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix} \in \mathbb{R}^{n \times N}.$$

We implemented this minimization problem using MATLAB and CVX and used it for the numerical experiments in Chapter 4, where we consider recovery of signals sparse in a Gabor fusion frame.

Regarding the theoretical guarantees for recovery via (2.13), both classical results involving mutual coherence and RIP could be adapted to the fusion frame setting, see [18, 6] for more details. We will formulate here only the result on the mutual coherence.

Definition 2.19. The *fusion coherence* of a fusion frame $\{\mathcal{W}_i\}_{i=1}^M$ in $\mathbb{R}^N$ and a measurement matrix $A \in \mathbb{R}^{n\times M}$ with normalized columns is defined as

$$\mu_f\left(A, \{\mathcal{W}_i\}_{i=1}^M\right) = \max_{j\neq k}\left[\langle a_j, a_k\rangle \|P_j P_k\|_2\right].$$

A crucial role in this definition is played by $\|P_j P_k\|_2$, which is the largest absolute value of the cosines of the principle angles between $\mathcal{W}_j$ and $\mathcal{W}_k$, and it is called the *incoherence parameter* in [8].

Theorem 2.20 ([18])**.** *Let $\{\mathcal{W}_i\}_{i=1}^M$ be a fusion frame for $\mathbb{R}^N$ and $A \in \mathbb{R}^{n\times M}$ with normalized columns be the measurement matrix. If there exist a solution c^0 of the system $Y = AU(c)$, with $Y \in \mathbb{R}^{n\times N}$ satisfying*

$$\|c^0\|_0 < \frac{1}{2}\left(1 + \mu_f\left(A, \{\mathcal{W}_i\}_{i=1}^M\right)^{-1}\right),$$

then this solution is the unique solution of (2.12) *and* (2.13)*.*

Surely, the following question naturally arises: are there fusion frames with small fusion coherence or fusion RIP constant [18], and how many measurements are sufficient to guarantee recovery? Using the fusion RIP, it is shown in [7] that if A is a subgaussian matrix, then fusion sparse vector can be recovered uniformly with high probability, as long as the incoherence parameter of the fusion frame is sufficiently small. The result also holds for noisy measurements, as well as approximately sparse vectors. Furthermore, using ideas from optimal packings of Grassmannian manifolds, a lower bound on the incoherence parameter was also obtained in [6].

2.3 Phase Retrieval

Phase retrieval is called the problem of recovery of a signal from the absolute values of its linear measurements. In Chapters 5 and 6 we consider phase retrieval of structured (dictionary sparse) vectors, and from structured (Gabor) measurements. Therefore we shall now briefly formulate here the main achievements and open problems in phase retrieval. First we consider the phase retrieval problem in its general setting. Then, we separately discuss the problem of sparse phase retrieval and phase retrieval from Fourier measurements.

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{T} = \{c \in \mathbb{K} : |c| = 1\}$. We denote by $\mathbb{K}^N/\mathbb{T}$ the set of equivalence classes under the equivalence relation $x \sim y \Leftrightarrow \exists c \in \mathbb{T} : x = cy$.

We say that a set of vectors $F = \{f_i\}_{i=1}^m$ in $\mathbb{K}^N$ *allows phase retrieval* or has *the phase retrieval property*, if the mapping

$$\mathcal{M}_F : \mathbb{K}^N/\mathbb{T} \to \mathbb{R}_+^m, \quad \mathcal{M}_F(x) = \left[|\langle f_1, x\rangle|^2 \quad |\langle f_2, x\rangle|^2 \quad \dots \quad |\langle f_m, x\rangle|^2 \right]^T$$

is injective. We will often refer to the measurements $\{|\langle f_i, x\rangle|^2\}_{i=1}^m$ as *phaseless, magnitude, or intensity measurements* of x.

Definition 2.21. A set of vectors $\{f_i\}_{i=1}^M$ in $\mathbb{K}^N$ satisfies the *complement property* (CP) if for every subset $S \subseteq [1, \dots, M]$ either $\{f_i\}_{i\in S}$ or $\{f_i\}_{i\in S^c}$ spans $\mathbb{K}^N$.

A simple characterization of phase retrievability is given by the complement property when $\mathbb{K} = \mathbb{R}$.

Theorem 2.22 ([12])**.** *A set of measurements $F = \{f_i\}_{i=1}^m$ in $\mathbb{R}^N$ allows phase retrieval if and only if F satisfies the complement property.*

We would like to mention here the relation between the notion of *full spark* (every set of N vectors in F is linearly independent) and the complement property. Namely, it is not difficult to see that, if a set of vectors is full spark and has $m \geq 2N - 1$ measurements, then it will necessarily satisfy the complement property. This then gives rise to deterministic constructions which allow phase retrieval, see for example the work [4] on full spark frames. We note here that, however, the constructions in [4] mostly involve complex vectors, and as we will see next, the complement property is a necessary but not a sufficient condition in the complex case. In Chapter 5 we discuss the question of full spark of real (random) measurement vectors, in the context of the complement property for sparse signals.

In the case $\mathbb{K} = \mathbb{C}$, the authors in [13] showed that the complement property is a necessary condition for phase retrievability.

Theorem 2.23 ([13])**.** *Let $F = \{f_i\}_{i=1}^M$ be a set of vectors in $\mathbb{C}^N$. If $\mathcal{M}_F$ is injective, then F satisfies the complement property.*

An example of a set of complex vectors having the complement property, but not allowing phase retrieval was also presented in [13]. However, the authors in [13] formulated another, also simple characterization of phase retrievability in the complex case, via the so-called *PhaseLift* or *super analysis* operator, which already proved effective in the question of developing algorithms for solving the phase retrieval problem [29, 11]. Let $\mathbb{H}^{N\times N}$ be the space of Hermitian $N \times N$ matrices. For a set of measurement vectors

$\{f_i\}_{i=1}^m$ in $\mathbb{C}^N$ the PhaseLift operator is defined as

$$\mathcal{A} : \mathbb{H}^{N\times N} \to \mathbb{R}_+^m, \quad H \mapsto \Big[\langle H, f_1 f_1^*\rangle_{HS} \quad \langle H, f_2 f_2^*\rangle_{HS} \quad \dots \quad \langle H, f_m f_m^*\rangle_{HS}\Big]^T.$$

Notice that the mapping $\mathcal{A}$ with $H = xx^*$ gives exactly the phaseless measurements, since

$$\begin{aligned}\mathcal{A}(xx^*)(j) &= \langle xx^*, f_j f_j^*\rangle_{HS} = \mathrm{Tr}(xx^* f_j f_j^*) = \mathrm{Tr}(f_j^* xx^* f_j)\\ &= f_j^* xx^* f_j = |\langle x, f_j\rangle|^2 = \mathcal{M}_F x(j).\end{aligned}$$

The following characterization of phase retrievability in the complex case will be essential for our research in Chapter 6.

Theorem 2.24 ([13])**.** *The mapping $\mathcal{A}$ is not injective if and only if there exists a matrix of rank 1 or 2 in the kernel of $\mathcal{A}$.*

Simply stated and with a fairly short proof, this result is still very powerful and will help us to find generators of Gabor systems in Chapter 6 which are suitable for allowing phase retrieval. At the same time, after we prove an analogue of this result in the sparse and dictionary sparse setting in Chapter 5, we will be able to elaborate on the question why certain generators like Alltop vectors and window functions are not suitable for recovery of sparse vectors from Gabor phaseless measurements.

One of the most challenging theoretical questions in phase retrieval is to find the smallest number of measurements which guarantee injectivity. For arbitrary signals in $\mathbb{R}$ the result is known and equals $2N-1$ [12] (see the remark above about the full spark), but the complex case is an open and ongoing problem. In [12], it was conjectured that $4N-2$ are necessary for phase retrival of complex signals. However, later in [17] and [103] examples of $4N-4$ vectors doing phase retrieval were presented, and therefore $4N-4$ was thought to be the correct answer [13]. Exactly Theorem 2.24 was used in [13] to support *the* $4M-4$ *conjecture*. Nevertheless, this conjecture was further disproved with a small example of 11 vectors in dimension 4 in [124], leaving space for further investigations. It is important to note that the best lower bound on the number of measurements is $4N-2\alpha(N-1)-3$ due to [74], where $\alpha(N) \le \log_2(N)$ is the number of 1's in the binary representation of $N-1$.

2.3.1 Sparse Phase Retrieval

The similarity between phase retrieval and compressed sensing becomes apparent when we add the sparsity prior in the unknown signal x. If in the compressed sensing regime we were interested in recovering a k-sparse x from measurements $y = Ax$, now we have a similar task but we are given only the absolute values of these measurements, $y = |Ax|$.

Recalling the null space property and the restricted isometry property from Section 2.1, we understand intuitively that if we want injectivity for sparse signals, we need to make sure that A has certain properties when restricted to any subset of k columns. For example, an analog of the complement property for the sparse setting was given in [97], and it was then used to formulate a necessary condition for doing k-sparse phase retrieval. As we will also discuss in Chapter 5, the complement property in the sparse case is no longer a characterization of the phase retrievability. Let us recall the sparse complement property, and the corresponding result on the k-sparse phase retrievability in $\mathbb{R}$.

Definition 2.25 ([97])**.** A given set $F = \{f_i\}_{i=1}^m$ in $\mathbb{R}^N$ has the *k-complement property*, if for all $\mathcal{S} \subseteq [1, \ldots, m]$, and all $\mathcal{K} \subseteq [1, \ldots, N]$ with $|\mathcal{K}| \leq k$, either $\{f_i^{\mathcal{K}}\}_{i \in \mathcal{S}}$ or $\{f_i^{\mathcal{K}}\}_{i \in \mathcal{S}^c}$ spans $\mathbb{R}^k$.

Here we denote by $f_i^{\mathcal{K}} \in \mathbb{R}^k$ the restriction of f_i to the coefficients in $\mathcal{K}$.

Theorem 2.26 ([97])**.** *Let $F = \{f_i\}_{i=1}^m$ in $\mathbb{R}^N$ satisfy the $2k$-complement property. If x_0 is a k-sparse vector in $\mathbb{R}^N$ and $y = \mathcal{M}_F(x_0)$, then x_0 is the unique real vector satisfying the given measurements with k or fewer nonzero elements.*

We follow an analogue approach in the case where x is sparse in a dictionary in Chapter 5, and we show that k-complement property is a necessary condition for sparse phase retrievability in $\mathbb{R}$ (in Theorem 2.26 we saw that $2kp$-complement property is sufficient). Further, we shal investigate how sharp are these results.

Regarding the question of finding the minimal number of measurements needed for sparse phase retrieval in $\mathbb{R}$, [97] showed that $4k - 1$ (random) vectors suffice to guarantee injectivity. This result was improved in [125] where it was shown that $2k$ is the minimal (necessary) number of measurements for phase retrieval of k-sparse signals, and that $2k$ generically chosen vectors allow k-sparse phase retrieval. We will extend this result in Chapter 5 to signals which are sparse in a dictionary.

The complex case for sparse phase retrieval remains unsolved regarding the minimal number of measurements. It was shown in [125] that $4k - 2$ generically chosen vectors allow sparse phase retrieval in $\mathbb{C}$, and it is conjecture there that this is the minimal number of vectors needed. In Chapter 5 we will formulate necessary and sufficient conditions for sparse phase retrieval in $\mathbb{C}$ via the PhaseLift operator $\mathcal{A}$.

2.3.2 Phase Retrieval from Fourier Measurements

As in compressed sensing, in phase retrieval we are often not in a position to choose the type of measurements, but they are given by the application. Taking this into account, one is not interested in how many measurements guarantee recovery in theory (for example, random or generic measurements), but how many computationally suitable measurements (for example, Fourier) can guarantee injectivity? The initial setting of the phase retrieval problem [92, 67] is exactly recovery from *Fourier magnitude measurements* (FMM), but because of the specificity of the problem, only a few theoretical guarantees are known. In the recent years, there has been a growing interest in this problem with exploiting additional knowledge about the structure of the signal, in particular that the signal we are interested in is sparse. It was proven in [105] that the full set of N Fourier measurements will give injectivity for k-sparse signals, as long as $k \neq 6$ and x is collision-free (to be defined below). For N prime this result was improved in [97], where it was shown that $k^2 - k + 1$ Fourier measurements guarantee uniqueness under similar additional conditions on the sparse signal x. We state this result below. Note that the signal here is still in $\mathbb{R}^N$, although the measurements are complex.

Definition 2.27. A vector $x \in \mathbb{R}^N$ is *collision free* if $x(i) - x(j) \neq x(k) - x(l)$, for all distinct $i, j, k, l \in \{n \in [0, \ldots, N-1] : x(n) \neq 0\}$.

Theorem 2.28 ([97])**.** *Let* $\{k_1, k_2, \ldots, k_m\} \subseteq [0, \ldots, 2N-1]$,

$$f_j = \begin{bmatrix} 1 & e^{-i2\pi k_j/2N} & e^{-i4\pi k_j/2N} & \cdots & e^{-i(2N-1)\pi k_j/2N} \end{bmatrix}^T,$$

and let $\mathcal{M} : \mathbb{R}^N/\mathbb{T} \to \mathbb{R}^m_+$ *be defined by*

$$\mathcal{M}x(j) = \left| \left\langle f_j, \begin{bmatrix} x^T & 0_{1\times N} \end{bmatrix}^T \right\rangle \right|^2, \quad j = 1, \ldots, m.$$

Let x_0 *be* k*-sparse and collision free vector in* $\mathbb{R}^N$*, and let* m *be a prime integer larger than* $2(k^2 - k + 1)$*. Then,* x_0 *is uniquely defined by* $y = \mathcal{A}(x_0) \in \mathbb{R}^m$ *whenever*

- $k \neq 6$*, or*
- $k = 6$ *and* $x_0(j) \neq x_0(k)$*, for some* $j, k \in \{i : i \in [0, \ldots, N-1], x_0(i) \neq 0\}$.

Later, in Chapter 6, we will show that phase retrieval from Gabor (short-time Fourier) measurements is possible with the order of k^3 measurements, without additional constraints of the signal except for the sparsity.

One could notice that so far we discussed only the injectivity of the measurements, and what type of measurements guarantee phase retrieval, but not also *how to practically* *recover* x? This question is a topic on its own, and here we will give only few references for well established methods. One of the first algorithms proposed, for recovery from Fourier magnitude measurements is the Gerchberg-Saxton algorithm [67, 61]. A novel approach is given in [111], where GESPAR, a greedy method for recovering sparse signals from Fourier magnitude measurements is proposed. With the development of compressed sensing and the idea of low-rank matrix recovery via convex optimization, a convex programming algorithm was proposed in [29] for solving the phase retrieval problem from arbitrary measurements. It was also shown in [29] that $O(N \log N)$ random phaseless measurements suffice to recover a signal with this method. In Chapter 6, we will also discuss an algorithm for recovery of both sparse and arbitrary vectors, from Gabor magnitude measurements.

Chapter 3

Discrete Lines and Sparse Recovery

3.1 Introduction

In this chapter we are interested in discrete lines as one form of geometric sparsity, and we will be investigating them from few aspects. At first, we will explore the geometric properties of discrete lines, defined in arbitrary dimension d. Their geometric structure will be crucial for the frame properties of the collection of discrete lines that we will show afterward. Our main goal is to then use these results for recovery of signals which are geometrically sparse (consist of few lines, or few points and lines) from linear measurements.

This chapter is organized as follows. We define discrete lines and explore their main geometric properties in Section 3.2. Next, in Section 3.3 we investigate the frame properties of the collection of lines, and show how to modify it to a UNTF, and we compute its mutual coherence and the spark. In Section 3.4 we consider the problem of separation of discrete points and lines, give theoretical guarantees via the mutual coherence, the spark and the RIP, and provide numerical experiments. The problem of recovery of unions of lines from linear measurements is considered in Section 3.5.

3.2 Discrete Lines in d-dimensional Space

We start with the definition of discrete lines in arbitrary d-dimensional space and their basic geometric properties. Let p be some prime number, and $d \geq 2$ some integer. By $\mathbb{Z}_p$ we denote the set of integers modulo p, and let $\mathbb{Z}_p^* := \mathbb{Z}_p \backslash \{0\}$. For any $k \in \mathbb{Z}$, $[k]_p \in \mathbb{Z}_p$ is the equivalence class of the integer k modulo p. Often we will omit the brackets $[\cdot]_p$, having in mind that all calculations are in $\mathbb{Z}_p$, and we will simply use the residual system

$\{0, \ldots, p-1\}$. We define a *discrete point* in $\mathbb{Z}_p^d$ as a d-tuple $(x_1, \ldots, x_d)$ where $x_i \in \mathbb{Z}_p$ for all $i = 1, \ldots, d$.

Definition 3.1. Let $x_0 \in \mathbb{Z}_p^d$, $m \in \mathbb{Z}_p^d \backslash \{0\}$. A *discrete line* in $\mathbb{Z}_p^d$ is a set of p discrete points given by the following system of parametric equations (given in vector form):

$$\mathcal{L}_{x_0,m} = \{(x_1, \ldots, x_d) : x = [x_0 + mt]_p, \quad t \in \mathbb{Z}_p\}. \tag{3.1}$$

Here x_0 is a point that lies on the line, m is the direction vector, and t is the parameter which takes all possible values from $\mathbb{Z}_p$. Thus, every line in $\mathbb{Z}_p^d$ consists of p points. In Figure 3.1 we plot few lines in three dimensional space viewed from two different angles. In Figure 3.2 we show examples of discrete lines in two dimensions. We will show

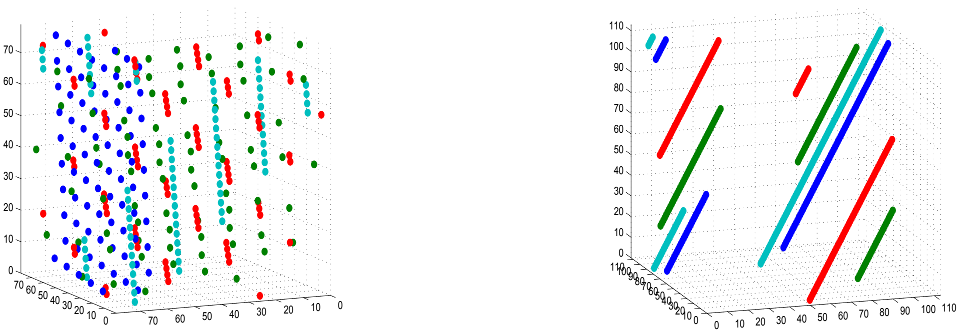


Figure 3.1: Four different lines in 3-dimensional space, $p = 113$.

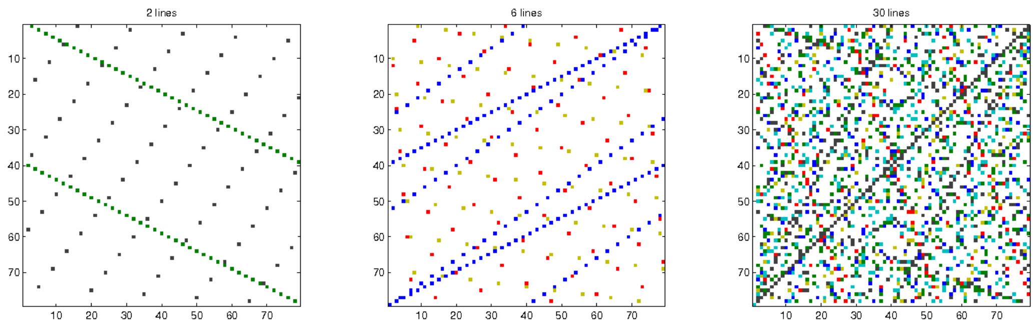


Figure 3.2: Different unions of discrete lines in two dimensions for $p = 79$.

now that the intuitive Euclidean geometric properties are preserved with such definition. Some investigation on the geometry of 3 dimensional (digital) lines is given in [62]. At first we compute the number of all different lines for fixed p. Note that there are p^d discrete points $x = (x_1, \ldots, x_d)$ living in $\mathbb{Z}_p^d$.

Proposition 3.2. *There are* $\dfrac{p^d - 1}{p - 1} p^{d-1}$ *distinct discrete lines in a* d*-dimensional space* $\mathbb{Z}_p^d$.

Proof. At first we calculate the number of lines passing through a given point (for example, passing through the origin). This line, $x = [mt]_p$, is defined by its slope (direction vector), which is a nonzero point. Some of those points have the same direction, and they will give the same line. Therefore, we divide the number of all nonzero points $p^d - 1$ by the number of points that have same direction, which is $p-1$, and obtain $\frac{p^d-1}{p-1}$. Now, for each of those lines we have all possible shifts, that is p^{d-1}. Therefore, the number of lines in $\mathbb{Z}_p^d$ is $\frac{p^d-1}{p-1}p^{d-1}$. □

We need to prove that our system of lines satisfies the standard properties of lines as geometric objects. One of the crucial properties of lines in affine geometry is that two distinct lines intersect at most at one point. The same property holds also for the discrete lines that we introduced.

Proposition 3.3. *Let $\mathcal{L}_{a,m}$ and $\mathcal{L}_{b,n}$ be two d-dimensional discrete lines. Then, one of the following holds:*

(i) $|\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}| = p$, *i.e., the lines coincide,*

(ii) $|\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}| = 1$, *i.e., the lines intersect,*

(iii) $|\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}| = 0$, *i.e., the lines are parallel or skew.*

Proof. Let us consider two arbitrary d-dimensional discrete lines,

$$\begin{aligned} \mathcal{L}_{a,m} &= \{(x_1, \dots, x_d) : x = a + mt, \quad t \in \mathbb{Z}_p\}, \\ \mathcal{L}_{b,n} &= \{(x_1, \dots, x_d) : x = b + nt, \quad t \in \mathbb{Z}_p\}. \end{aligned}$$

Finding points that will belong to $\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}$ means finding $t, \tilde{t} \in \mathbb{Z}_p$, which are solutions of the system of equations (congruences modulo p) given by

$$\begin{aligned} a_1 + m_1 t &\equiv b_1 + n_1 \tilde{t} \pmod{p}, \\ a_2 + m_2 t &\equiv b_2 + n_2 \tilde{t} \pmod{p}, \\ &\dots \\ a_d + m_d t &\equiv b_d + n_d \tilde{t} \pmod{p}. \end{aligned} \tag{3.2}$$

Since p is prime, $\mathbb{Z}_p$ is a finite field with the operations addition and multiplication modulo p. Thus we can use Gaussian elimination to solve this system. For this purpose we write it in a matrix form as $Ax = b$, where $x = \begin{bmatrix} t \\ -\tilde{t} \end{bmatrix}$, and

$$\begin{bmatrix} A \,|\, b \end{bmatrix} = \begin{bmatrix} m_1 & n_1 & b_1 - a_1 \\ m_2 & n_2 & b_2 - a_2 \\ \dots & & \\ m_d & n_d & b_d - a_d \end{bmatrix}. \tag{3.3}$$

Our system is over-determined. Hence, it will have a solution if and only if with elementary transformations we can modify it to

$$\begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \\ & \dots & \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & \tilde{c}_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ & \dots & \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.4}$$

In the first case, there will be exactly one solution modulo p, and in the second case there will be p solutions of the system. ☐

We are also interested in the conditions on the vectors m, n and $b - a$ under which the system has $1, p$, or 0 solutions, in other words, conditions for when the lines intersect or not.

Proposition 3.4. *Let p be some prime, and $\mathcal{L}_{a,m}$ and $\mathcal{L}_{b,n}$ be two d-dimensional discrete lines. Then, we have*

(i) $|\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}| = 1 \iff m \notin \operatorname{span}(n)$ *and* $b - a \in \operatorname{span}(m, n)$,

(ii) $|\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}| = 0 \iff m \in \operatorname{span}(n)$ *and* $b - a \notin \operatorname{span}(m)$ *or* $m \notin \operatorname{span}(n)$ *and* $b - a \notin \operatorname{span}(m, n)$,

(iii) $|\mathcal{L}_{a,m} \cap \mathcal{L}_{b,n}| = p \iff m \in \operatorname{span}(n)$ *and* $b - a \in \operatorname{span}(m)$.

Proof. To prove this proposition we will use the criteria that a linear system of equations $Ax = b$ has a solution if and only if $\operatorname{span}(a_1, \dots, a_N) = \operatorname{span}(a_1, \dots, a_N, b)$. In our case, in order to have a solution of (3.2), it is necessary and sufficient to have

$$\operatorname{span}(m, n) = \operatorname{span}(m, n, b - a). \tag{3.5}$$

Since the dimension of $\operatorname{span}(m, n)$ can be only 1 or 2, we have the following two cases:

$\dim(\operatorname{span}(m,n)) = \dim(\operatorname{span}(m,n,b-a)) = 2$. This case is possible if and only if the columns of the matrix A are linearly independent, i.e., $m \neq \lambda n$, for any $\lambda \neq 0$. Geometrically, that means that the two directional vectors are not collinear. Since $\dim(\operatorname{span}(m,n,b-a))$ is also 2, $b-a$ must be a linear combination of m and n. Hence, there exist α and γ such that $b-a=\alpha m+\gamma n$. This representation is actually unique, because if we assume that there is another different linear combination $b-a=\alpha_1 m+\gamma_1 n$, then we can write $(\alpha_1-\alpha)m=(\gamma-\gamma_1)n$. Since at least one of the differences is nonzero, we can divide by it and obtain that $m=\lambda n$. This is in contradiction to the fact that two vectors were linearly independent. Thus, in this case we have a unique solution of the system — the lines intersect at exactly one point.

$\dim(\operatorname{span}(m,n)) = \dim(\operatorname{span}(m,n,b-a)) = 1$. In this case the vectors are linearly dependent, i.e., $m=\lambda n$, for some nonzero λ. Geometrically, it means that the two directional vectors are collinear. In order to have (3.5), we need $b-a$ to be a linear combination of m and n, $b-a=\alpha m+\beta n=(\alpha\lambda+\beta)n=\gamma n$. The parameter γ can be any element from $\mathbb{Z}_p$ and so there are p solutions of our system. Geometrically, the vector $b-a$ has the same direction as the vector n. But this means the point b needs to lie on the same line as a, in addition both lines have the same direction — the lines coincide. For completeness of the geometrical interpretation, we also give a description of the cases where we do not have a solution:

$$\operatorname{span}(m,n) \neq \operatorname{span}(m,n,b-a).$$

This can happen in the following cases:

$\dim(\operatorname{span}(m,n)) = 1, \dim(\operatorname{span}(m,n,b)) = 2$. In this case the vectors m and n are collinear, but the vector $b-a$ is not collinear to them, $b-a\neq\alpha m$. This means the lines have the same direction, but the points b and a lie on different lines, so those lines are parallel (also coplanar).

$\dim(\operatorname{span}(m,n)) = 2, \dim(\operatorname{span}(m,n,b)) = 3$. Now m and n are not collinear, and $b-a$ is a not a linear combination of the two vectors, in this case the lines are skew - they do not intersect and there is no plane containing them both. □

Finally, we are ready to prove the last property also important for the Euclidean geometric understanding of lines.

Proposition 3.5. *For every two different points m and $m' \in \mathbb{Z}_p^d$, there exist exactly one line $\mathcal{L}_{a,m}$ containing them.*

Proof. We need to prove two things: first, that two distinct points always determine some line; and second, that this line is always unique. For the first part, let $a, b \in \mathbb{Z}_p$ be the two points that our line should contain. We thus can use a as the point x_0 in equation (3.1), and we only need a direction vector. For this purpose we construct $m := b - a$, and the line $\mathcal{L}_{a,m}$ satisfies our requirements. When the parameter $t = 0$, we will obtain the point a, and when $t = 1$, we obtain the point b.

For the second part we argue by contradiction. If we assume that the line that passes through those points is not unique, it will mean that there is a second line, containing this two points. But that means, that there are two different lines that intersect in more than one point — that is in contradiction to the result of Proposition 3.3. □

Since our final goal is to use this construction for signal processing, it will be useful to have the characteristic function of a discrete line, defined as follows.

Definition 3.6. Let p be prime, $a \in \mathbb{Z}_p^d$ and $m \in \mathbb{Z}_p^d \backslash \{0\}$. We define the *characteristic function* $\mathbf{1}_{\mathcal{L}_{a,m}}$ of a discrete line $\mathcal{L}_{a,m}$ as an element of $\mathbb{C}^p \times \mathbb{C}^p \cdots \times \mathbb{C}^p$ with values

$$\mathbf{1}_{\mathcal{L}_{a,m}}(x_1, \dots, x_d) = \begin{cases} 1, & \text{if } (x_1, \dots, x_d) \in \mathcal{L}_{a,m}, \\ 0, & \text{elsewise,} \end{cases} \tag{3.6}$$

for all $(x_1, \dots, x_d) \in \mathbb{Z}_p^d$.

Often we will consider $\mathbf{1}_{\mathcal{L}_{a,b}}$ as a re-sized vector in $\mathbb{C}^{p^d}$, using the standard lexicographic ordering. Before we investigate the frame properties of the system of characteristic functions of lines, we would like to investigate the discrete Fourier transform of a signal which represents a discrete line. Interestingly, the Fourier transform of a line geometrically is a hyperplane passing through the origin and perpendicular to the line.

Proposition 3.7. *Let $\mathcal{L}_{a,m}$ be a discrete line in $\mathbb{Z}_p^d$, where $m \in \mathbb{Z}_p^d \backslash \{0\}$ is the direction vector and $a \in \mathbb{Z}_p^d$ a point on the line. The discrete Fourier transform (DFT) of $\mathbf{1}_{\mathcal{L}_{a,m}}$ is given by*

$$\widehat{\mathbf{1}}_{\mathcal{L}_{a,m}}(n_1, \dots, n_d) = p\, e^{-\frac{2\pi i n \cdot a}{p}} \mathbf{1}_{\{(k_1, \dots, k_d) \in \mathbb{Z}_p^d \,:\, k \cdot m = 0\}}. \tag{3.7}$$

Proof. To prove this statement we just need to calculate the multidimensional discrete Fourier transform of the signal defined by (3.6) and (3.1).

$$\begin{aligned}
\widehat{\mathbf{1}}_{\mathcal{L}_{a,m}}(n_1,\ldots,n_d) &= \sum_{k_1,\ldots,k_d=0}^{p-1} \mathbf{1}_{\mathcal{L}_{a,m}}(k_1,\ldots,k_d)e^{-\frac{2\pi i n\cdot k}{p^d}} \\
&= \sum_{t=0}^{p-1} \exp\{-\frac{2\pi i(n_1(a_1+m_1t)+\ldots+n_d(a_d+m_dt))}{p^d}\} \\
&= e^{-\frac{2\pi i n\cdot a}{p^d}} \sum_{t=0}^{p-1} \exp\{-\frac{2\pi i t(n_1m_1+\ldots+n_dm_d)}{p^d}\} \\
&= p\,e^{-\frac{2\pi i n\cdot a}{p}} \delta_p(n\cdot m) = \begin{cases} p\,e^{-\frac{2\pi i n\cdot a}{p}}, & \text{if } n\cdot m = 0 \pmod{p}, \\ 0, & \text{if } n\cdot m \neq 0 \pmod{p}. \end{cases}
\end{aligned}$$

□

Remark 3.8. As we can see from the equation $n \cdot m = 0$, the DFT of a line in d dimensional space is supported on a hyperplane that passes through the origin, and lies perpendicularly to the vector m, i.e. to the line $\mathcal{L}_{a,m}$. Also, the values of the non zero entries are no longer 1, but they are complex numbers and depend on a and on the position of the point n. We can see further the geometric structure of the Fourier transform of a line if we see how many nonzero points it contains. We need the number of solutions to the following equation with d variables:

$$n_1m_1 + n_2m_2 + \ldots + n_dm_d = 0 \pmod{p}.$$

Since $m \neq 0$, at least one $m_j \neq 0$, for some $j \in \{1,\ldots,d\}$. Then, the corresponding variable n_j can be expressed in terms of the rest $d-1$ variables, and there are exactly p^{d-1} solutions to our system. We see now, then in three dimensional space, the Fourier transform of a line is a plane (with p^2 points), and when $d = 2$, a Fourier transform of a line is again a line (with p points). This observation will be important later when we will be investigating the spark of the collection of discrete lines.

3.3 Frame Properties of the Union of Discrete Lines

We are interested in signals which have a line structure, or more precisely, which are unions of few lines. Mathematically, we would like to build a dictionary of lines, a matrix Φ where each column will be one characteristic function of a line (p and d are fixed). Then, a signal x which consists of few lines can be represented as $x = \Phi z$, where z will have nonzero entries at the positions of lines which x consists of. As we know from

Chapter 2, for compressed sensing-like results which involve recovering the structured signal x, it is important to investigate the properties of the dictionary of lines Φ. First of all, we can ask whether it is a frame, whether it satisfies some optimality properties and more specific, what kind of (in)coherence properties this system has. Among the other results, we will see that we can calculate the spark of this matrix, which is in general an NP-hard problem [120]. We summarize everything shown in this section in the following theorem.

Theorem 3.9. *Let p be some prime, and $d \geq 2$ be some integer. Let further $\mathcal{L}$ be the matrix of size $M \times N$, $M = p^d, N = \frac{p^d-1}{p-1}p^{d-1}$, where each column is one characteristic function of a discrete line. Let further $\mathbf{1}_{M\times N}$ be a matrix that consist of all ones. Let $\Phi \in \mathbb{R}^{M\times N}$ be a matrix obtained from $\mathcal{L}$ in the following way:*

$$\Psi = \frac{1}{C}(\mathcal{L} - c\mathbf{1}_{M\times N}), \text{ where } C^2 = \frac{p^2(p^{d-1}-1)}{p^d-1}$$

and c is a solution of the quadratic equation

$$1 - 2c\frac{p^d-1}{p-1} + c^2\frac{p^d-1}{p-1}p^{d-1} = 0.$$

Then, Ψ is a unit norm tight frame, and, additionally,

(i) $\mu(\Psi) = \begin{cases} \dfrac{p^d - p^2 + p - 1}{p^2(p^{d-1}-1)}, & \text{if } d \geq 3, \\ 1/2, & \text{if } d = 2. \end{cases}$

(ii) $\operatorname{spark}(\mathcal{L}) = 2p, \quad \mu(\mathcal{L}) = \frac{1}{2}$.

Now we go step-by-step in proving these properties of the collection of discrete lines.

3.3.1 Constructing UNTF from a Collection of Discrete Lines

Recall that a sequence of vectors $\Phi = \{\phi_i\}_{i=1}^N$ in $\mathbb{R}^M$ is an A-*tight frame* if and only if $\Phi\Phi^T = AI_M$, where I_M is the identity matrix. Also, if $\|\phi_i\| = 1$ for all $i = 1, \ldots, N$, then $\{\phi_i\}_{i=1}^N$ is a *unit norm frame*. Of particular interest in applications are the *unit norm tight frames* (UNTF), and we will show how to construct a subclass of those from a collection of discrete lines.

For this, let $\mathcal{L}$ be a matrix of size $M \times N$, $M = p^d$ and $N = \frac{p^d-1}{p-1}p^{d-1}$, where each column ϕ_n, $n = 1, \ldots, N$ is a characteristic function of one discrete line. For the lines and the points we use the standard lexicographic ordering. We want to check if such a system of discrete lines $\{\phi_n\}_{n=1}^N$ forms a frame in $\mathbb{R}^M$. First of all, we will show how to construct a UNTF from it. The idea is to modify $\mathcal{L}$ by subtracting a carefully chosen constant c from every entry, such that the obtained matrix Ψ will satisfy $\Psi\Psi^T = AI_M$, which will prove the tightness. Moreover, we will prove that all columns have equal norm, so we can make the frame unit norm. Then, by expanding an arbitrary vector in the obtained frame, we will see that if we add the unit vector in the initial system of discrete lines, it also becomes a frame.

Let us denote by $r_m \in \mathbb{R}^N, m = 1, \dots, M$ the rows of the matrix $\mathcal{L}$. By answering the question: how many lines pass through one point, and how many lines can pass through two distinct points (see Proposition 3.5), we obtain the following result:

$$\langle r_m, r_{m'} \rangle = \begin{cases} \frac{p^d-1}{p-1}, & m = m', \\ 1, & m \neq m'. \end{cases}$$

We will also need the number of lines that contain one fixed point,

$$\langle r_m, \mathbf{1}_M \rangle = \frac{p^d - 1}{p - 1}. \tag{3.8}$$

We are now ready to calculate the elements of the matrix $\Psi\Psi^T$. We will consider the scalar product of the rows of Ψ, which we denote by $s_m = r_m - c\mathbf{1}_N$.

$$\begin{aligned} \langle s_m, s_{m'} \rangle &= \langle r_m - c\mathbf{1}_N, r_{m'} - c\mathbf{1}_N \rangle \\ &= \langle r_m, r_{m'} \rangle - 2c\langle r_m, \mathbf{1}_N \rangle + c^2 \langle \mathbf{1}_N, \mathbf{1}_N \rangle \\ &= \begin{cases} \dfrac{p^d-1}{p-1} - 2c\dfrac{p^d-1}{p-1} + c^2\dfrac{p^d-1}{p-1}p^{d-1}, & m = m', \\ 1 - 2c\dfrac{p^d-1}{p-1} + c^2\dfrac{p^d-1}{p-1}p^{d-1}, & m \neq m'. \end{cases} \end{aligned}$$

We will choose c, such that for $m \neq m'$, the scalar product is zero. Then, we will have that

$$\Psi\Psi^T = AI_N, \qquad \text{where } A = \frac{p^d-1}{p-1} - 1. \tag{3.9}$$

Such c exists, because the determinant of the quadratic equation is positive:

$$\begin{aligned} 4\left(\frac{p^d-1}{p-1}\right)^2 - 4\frac{p^d-1}{p-1}p^{d-1} &= 4\frac{p^d-1}{p-1}\left(\frac{p^d-1}{p-1} - p^{d-1}\right) \\ &= 4\frac{p^d-1}{p-1}\frac{p^{d-1}-1}{p-1} > 0. \end{aligned}$$

We have shown that the first condition $\Psi\Psi^T = AI_M$ is satisfied, i.e. our frame is *tight*.

We saw that we can modify the system of lines so that we get a frame, but what is with the initial collection of discrete lines? Is it a frame, or equivalently, can any vector in

$\mathbb{R}^N$ be represented as a linear combination of those vectors? Let us consider the columns of Φ, ϕ_n, where $n = 1, \ldots N$, and their modified version $\psi_n = \phi_n - c\mathbf{1}_M$. Every signal $f \in \mathbb{R}^N$ can be represented in terms of ψ using the frame decomposition, with coefficients $\frac{1}{p}\langle f, \psi_n\rangle$ since we have a tight frame. We are interested in this representation in terms of the initial family of vectors $\{\phi_n\}_{n=1}^N$. We will use the fact that $\sum_{n=1}^N \phi_n = \frac{p^d-1}{p-1}\mathbf{1}_M$, which follows from equation (3.8). For every signal $f \in \mathbb{R}^N$ we have:

$$
\begin{aligned}
f &= \frac{1}{A}\sum_{n=1}^{N}\langle f, \psi_n\rangle\psi_n = \frac{1}{A}\sum_{n=1}^{N}\langle f, \phi_n - c\mathbf{1}_M\rangle(\phi_n - c\mathbf{1}_M) \\
&= \frac{1}{A}\sum_{n=1}^{N}\left[\langle f, \phi_n\rangle\phi_n - c\langle f, \phi_n\rangle\mathbf{1}_M - \langle f, c\mathbf{1}_M\rangle\phi_n - \langle f, c^2\mathbf{1}_M\rangle\mathbf{1}_M\right] \\
&= \frac{1}{A}\left[\sum_{n=1}^{N}\langle f, \phi_n\rangle\phi_n - c\sum_{n=1}^{N}\langle f, \phi_n\rangle\mathbf{1}_M - c\sum_{n=1}^{N}\langle f, \mathbf{1}_M\rangle\phi_n + c^2\sum_{n=1}^{N}\langle f, \mathbf{1}_M\rangle\mathbf{1}_M\right] \\
&= \frac{1}{A}\left[\sum_{n=1}^{N}\langle f, \phi_n\rangle\phi_n - c\frac{p^d-1}{p-1}\langle f, \mathbf{1}_M\rangle\mathbf{1}_M - c\frac{p^d-1}{p-1}\langle f, \mathbf{1}_M\rangle\mathbf{1}_M + \frac{p^d-1}{p-1}p^{d-1}c^2\langle f, \mathbf{1}_M\rangle\mathbf{1}_M\right] \\
&= \frac{1}{A}\left[\sum_{n=1}^{N}\langle f, \phi_n\rangle\phi_n + (c^2\frac{p^d-1}{p-1}p^{d-1} - 2c\frac{p^d-1}{p-1})\langle f, \mathbf{1}_M\rangle\mathbf{1}_M\right] \\
&= \frac{1}{A}\left[\sum_{n=1}^{N}\langle f, \phi_n\rangle\phi_n - \langle f, \mathbf{1}_M\rangle\mathbf{1}_M\right].
\end{aligned}
$$

We see now that if we consider the family of vectors $\{\phi_n\}_{n=1}^N \cup \{\mathbf{1}_M\}$, then every signal can be represented as a linear combination of those vectors, $f = \sum_{i=1}^N \alpha_i\phi_i + \alpha_{N+1}\mathbf{1}_M$, where

$$\alpha_i = \frac{1}{p}\langle f, \phi_i\rangle,\ i = 1, \ldots, N, \quad \alpha_{N+1} = -\frac{1}{p}\langle f, \mathbf{1}_M\rangle.$$

Thus, the system of all discrete lines with the vector of all ones added, $\{\phi_n\}_{n=1}^N \cup \{\mathbf{1}_M\}$, also forms a frame.

The last step in getting a UNTF is to check if all the frame elements ψ_n, $n = 1, \ldots, N$ have equal norm, and, if yes, then normalize the frame. We will obtain this as a side result from the calculation of the mutual coherence of Ψ.

3.3.2 Mutual Coherence of d-dimensional Lines

To calculate the mutual coherence, we start with the scalar product of two columns of Ψ, ψ_n and $\psi_{n'}$. We take into account that every line contains p points, hence $\langle\phi_n, \mathbf{1}_M\rangle = p$.

We have

$$\begin{aligned}\langle \psi_n, \psi_{n'} \rangle &= \langle \phi_n - c\mathbf{1}_M, \phi_{n'} - c\mathbf{1}_M \rangle = \langle \phi_n, \phi_{n'} \rangle - 2cp + c^2p^d \\ &= \begin{cases} p - 2cp + c^2p^d, & n = n', \\ 1 - 2cp + c^2p^d, & \text{if } \phi_n \text{ and } \phi_{n'} \text{ intersect}, \\ -2cp + c^2p^d, & \text{if } \phi_n \text{ and } \phi_{n'} \text{ do not intersect}. \end{cases}\end{aligned} \tag{3.10}$$

Here, we used that $\langle \phi_n, \phi_{n'} \rangle$ is $p, 1$, or 0, depending on the geometrical position of the lines, which we proved in Proposition 3.3. Note that when $n = n'$, the norm of the vectors ψ_n are all equal, since they depend only on p and c. Therefore, we conclude that we have an *equal norm tight frame.*

Both for the mutual coherence and for getting a UNTF we need to divide by the norm of the vectors ϕ_n, and we would like to have a value that will not contain c explicitly. For this, we use the following trick:

$$\begin{aligned} \operatorname{Tr}(\Psi\Psi^T) &= \operatorname{Tr}(AI_M) = \operatorname{Tr}(pI_M), \\ \frac{p^d - 1}{p - 1} - 1 = A &= \frac{1}{M}\operatorname{Tr}(AI_M) = \frac{1}{M}\operatorname{Tr}(\Psi\Psi^T) \\ &= \frac{1}{M}\operatorname{Tr}(\Psi^T\Psi) = \frac{1}{M}\sum_{n=1}^{N}\|\psi_n\|^2 = \frac{N}{M}\|\psi_n\|^2 = \frac{1}{p}\frac{p^d - 1}{p - 1}\|\psi_n\|^2. \end{aligned}$$

Thus,

$$\|\psi_n\|^2 = \frac{p^d - 1 - p + 1}{p - 1}\frac{p - 1}{p^d - 1}p = \frac{p^2(p^{d-1} - 1)}{p^d - 1}, \quad n = 1, \dots, N.$$

Now we can rewrite (3.10) without the parameter c,

$$\langle \psi_n, \psi_{n'} \rangle = \begin{cases} \frac{p^2(p^{d-1}-1)}{p^d-1}, & n = n', \\ \frac{p^2(p^{d-1}-1)}{p^d-1} - (p-1), & \text{if } \phi_n \text{ and } \phi_{n'} \text{ intersect}, \\ \frac{p^2(p^{d-1}-1)}{p^d-1} - p, & \text{if } \phi_n \text{ and } \phi_{n'} \text{ do not intersect}, \end{cases} \tag{3.11}$$

and, dividing by the norms, yields

$$\frac{\langle \psi_n, \psi_{n'} \rangle}{\|\psi_n\|\|\psi_{n'}\|} = \begin{cases} 1, & n = n', \\ \frac{-p^2+p+p^d-1}{p^2(p^{d-1}-1)}, & \text{if } \phi_n \text{ and } \phi_{n'} \text{ intersect}, \\ \frac{-p^2+p}{p^2(p^{d-1}-1)}, & \text{if } \phi_n \text{ and } \phi_{n'} \text{ do not intersect}. \end{cases} . \tag{3.12}$$

Thus, the system $\{\frac{1}{C}\psi_n\}_{n=1}^N$ with $C = \frac{p^2(p^{d-1}-1)}{p^d-1}$ is a UNTF, The mutual coherence is the maximum of the absolute values of $\frac{-p^2+p+p^d-1}{p^2(p^{d-1}-1)}$ and $\frac{-p^2+p}{p^2(p^{d-1}-1)}$, but it is not immediately visible which of the two values is larger. As we will see, this depends on the value of d. To figure this out, we can solve the inequality

$$\left|\frac{-p^2+p}{p^2(p^{d-1}-1)}+\frac{p^d-1}{p^2(p^{d-1}-1)}\right| \le \left|\frac{-p^2+p}{p^2(p^{d-1}-1)}\right|, \tag{3.13}$$

or equivalently,

$$\left|\frac{-(p^2-p)}{p^2(p^{d-1}-1)}+\frac{p^d-1}{p^2(p^{d-1}-1)}\right| \le \frac{p^2-p}{p^2(p^{d-1}-1)}.$$

Now we can open the absolute values and have

$$-\frac{p^2-p}{p^2(p^{d-1}-1)} \le \frac{-(p^2-p)}{p^2(p^{d-1}-1)}+\frac{p^d-1}{p^2(p^{d-1}-1)} \le \frac{p^2-p}{p^2(p^{d-1}-1)},$$

and further,

$$0 \le \frac{p^d-1}{p^2(p^{d-1}-1)} \le 2\frac{p^2-p}{p^2(p^{d-1}-1)}.$$

We solve the last inequality for d, taking into account that $d \ge 2$, and $p \ge 3$,

$$p^d - 1 \le 2(p^2-p),$$

and therefore $d \le \log_p(2p^2-2p+1)$. It is not difficult to see that $\log_p(2p^2-2p+1) < 3$. Our p is prime and $p \ge 2$ so we know that the following inequality holds

$$(p-2)(p^2+1)+p+1 > 0.$$

We can open the brackets and rewrite it as $p^3 > 2p^2-2p+1$. Since $2p^2-2p+1 = 2p(p-1)+1 > 0$, we obtain the needed logarithmic form $\log_p(2p^2-2p+1) < 3$.

Therefore, for $d = 2$, the right-hand side of (3.13) is larger, and we have

$$\mu(\Psi) = \frac{-p^2+p}{p^2(p-1)} = \frac{1}{p}.$$

For d larger or equal than 3, the opposite inequality takes place, and we obtain that the mutual coherence of our system Ψ is equal to

$$\mu(\Psi) = \frac{p^d-p^2+p-1}{p^2(p^{d-1}-1)}. \tag{3.14}$$

When $d = 3$, the mutual coherence is $\dfrac{p^2+1}{(p+1)\,p^2}$.

We know that $\mu(\Psi)$ of a frame with N elements in dimension M is always great or equal than $\sqrt{\frac{N-M}{M(N-1)}}$, the Welch bound. In our case, using $M = p^d$ and $N = \frac{p^d-1}{p-1}p^{d-1}$, we obtain that the Welch bound is

$$\sqrt{\frac{N-M}{M(N-1)}} = \sqrt{\frac{\frac{p^d-1}{p-1}p^{d-1}-p^d}{p^d\left(\frac{p^d-1}{p-1}p^{d-1}-1\right)}} = \sqrt{\frac{p^d-1-p^2+p}{p^{2\,d}-p^d-p^2+p}}.$$

Comparing the result for the mutual coherence in different dimensions, we can see that the system of modified discrete lines is closest to the Welch bound in the smallest dimension, $d = 2$, when the Welch bound equals $1/\sqrt{p(p^2+p+1)}$. In other words, as the dimension d grows, the Welch bound decreases, but the mutual coherence stays of the order $1/p$. Also, because in higher dimensions two different lines still either intersect in one point, or do not intersect at all, the mutual coherence of the initial system of discrete lines, also does not improve as the dimension grows, and stays $\mu(\Phi) = \frac{1}{p}$. Still, since our mutual coherence is asymptotically going to zero as p goes to infinity, we will be able to get good recovery guarantees in Sections 3.4 and 3.5. Even better results can be obtained with the *spark*, which we can luckily compute for the dictionary of lines.

3.3.3 Spark of d-dimensional Lines

As a frame is usually a redundant system of vectors, it is important to investigate the linear independency of sub-collections of the elements of a frame. An important frame property in this context is the spark. For example, if a collection of vectors $\Phi = \{\phi_i\}_{i=1}^N$ in $\mathbb{R}^M$ has the property that every set of M vectors is linearly independent, then Φ has *full spark* [4]. Recall that the spark of a matrix Φ is defined as the minimal number of linearly dependent columns. If we denote by Σ_k the set of vectors of size N with k non zero elements, and by $\mathcal{N}(\Phi)$ the kernel of Φ, then we can write

$$\mathrm{spark}(\Phi) = \min\{k : \mathcal{N}(\Phi) \cap \Sigma_k \neq \{0\}\}. \tag{3.15}$$

To compute the spark requires in general a combinatorial search over all possible subsets of columns of Φ [120]. However, to find an upper bound can be fairly easy — we just need an example of set of vectors which are linearly dependent. We will show that this is possible for the collection of lines in d dimensional space, and then we will discuss the question of computing the spark exactly.

We note at first that it is sufficient to consider only the case $d = 2$: if we have a minimal combination of linearly dependent lines in $\mathbb{Z}_p^2$, such a combination will also exist in any arbitrary space $\mathbb{Z}_p^d$, in which those lines will be embedded.

We now provide an example of a linear combination of $2p$ lines in $\mathbb{Z}_p^2$, which sum up to zero. Let us fix two direction vectors, m and n in $\mathbb{Z}_p^2$. The union of all shifts of the lines that each of them generates fills up the whole plane $\mathbb{Z}_p^2$. Namely,

$$\bigcup_j \mathcal{L}_{x_j,m} = \mathbb{Z}_p^2, \quad \bigcup_j \mathcal{L}_{x_j,n} = \mathbb{Z}_p^2. \tag{3.16}$$

This is not difficult to see, since obviously every point $(n_1, n_2) \in \bigcup_j \mathcal{L}_{x_j,m}$ is an element of $\mathbb{Z}_p^2$, and vice versa, if we take an arbitrary point $n = (n_1, n_2) \in \mathbb{Z}_p^2$, we can find a j such that the line $\mathcal{L}_{x_j,m}$ contains the point n. For this, we can simply choose $x_j := n$, and $\mathcal{L}_{n,m}$ will be a line with direction m and containing the point n by definition. In terms of characteristic functions, (3.16) means that sum of all lines in each group will give a vector of all ones. Therefore,

$$\sum_{j\in S_1} \mathbf{1}_{\mathcal{L}_{x_j,m}} - \sum_{j\in S_2} \mathbf{1}_{\mathcal{L}_{x_j,n}} = 0.$$

The number of lines in S_1 and S_2 is the number of different shifts of a given line, and that is p. This means we found $2p$ vectors in Φ which are linearly dependent, and therefore, $\operatorname{spark}(\Phi) \leq 2p$.

The question is whether there exists a linear combination of less than $2p$ lines which gives zero. Recalling (3.15), we need to find a nonzero solution of the system $\Phi c = 0$ with fewest amount of nonzero entries. An equivalent problem would be to work in the Fourier domain, and solve $\min_c \|c\|_0$ such that $\widehat{\Phi} c = 0$, where with $\widehat{\Phi}$ we denote the matrix of all Fourier transforms of the lines in $\mathbb{Z}_p^2$.

We observe first one property of the discrete Fourier transform. Let $\omega = e^{-\frac{2\pi i}{N}}$. If we want to solve the matrix system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, \tag{3.17}$$

we can do so by inverting the Fourier transform matrix, and obtain

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} \\ 1 & \omega^{-2} & \omega^{-1} \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}, \text{ where } c = \frac{\alpha}{N}.$$

The general fact that

$$\begin{bmatrix} 1 & \omega & \dots & \omega^{p-1} \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \dots & c_p \end{bmatrix}^T = 0 \tag{3.18}$$

is possible if and only if $c_1 = \ldots = c_p = c$, can also be obtained from the following observation: Since p is prime, $1, \omega, \ldots, \omega^{p-1}$ are all the p-th roots of unity, and we know that they sum up to zero, i.e.,

$$1 + \omega + \omega^2 + \ldots + \omega^{p-1} = 0.$$

Thus, a combination with $c_j = c$, $j = 1, \ldots, p$ would give us an equality in (3.18).

On the other hand, if we assume that the entries c_j are possibly not all equal, we can make the following conclusion. If z is some primitive p-th root of unity, we can write equation (3.18) as

$$c_1 + c_2 z + \ldots + c_p z^{p-1} = 0. \tag{3.19}$$

Furthermore, we know that the cyclotomic polynomial

$$1 + z + \ldots + z^{p-2} + z^{p-1} = 0 \tag{3.20}$$

is irreducible over the integers [76], and thus: first of all, c_p must be different from zero, and second, if this is the case we can divide by it to rewrite (3.19) as

$$\tilde{c}_1 + \tilde{c}_2 z + \ldots + \tilde{c}_{p-1} z^{p-2} + z^{p-1} = 0,$$

and combining it with (3.20) we will obtain

$$(\tilde{c}_1 - 1) + \ldots + (\tilde{c}_{p-1} - 1) z^{p-2} = 0.$$

Since (3.20) is an irreducible polynomial, the last equation can hold if and only if for all $i = 1, \ldots, p-1$, $\tilde{c}_i = 1$, or equivalently, $c_i = c_p$.

We now use the property of the system (3.17) for our matrix $\widehat{\Phi}$, which contains rows of the discrete Fourier transform matrix in a particular block form (recall from Remark 3.8 that every Fourier transform of a line when $d = 2$ has also line structure, more precisely it is a line which passes through the origin and has p-th roots of unity in the nonzero values). Note also, that when $d = 2$, there are p^2 points and $p^2 + p$ different lines, so $\widehat{\Phi} \in \mathbb{C}^{p^2 \times (p^2+p)}$. Using the results from Proposition 3.7 and the above reasoning, for

$p = 3$ for example, the system $\widehat{\Phi}c = 0$ must be of the form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega \\ 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ a \\ a \\ b \\ b \\ b \\ c \\ c \\ c \\ d \\ d \\ d \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For arbitrary prime p, the vector c has the following block structure with p elements in every block:

$$c = \left[\begin{array}{cccc|cccc|c|cccc} a_1 & a_1 & \cdots & a_1 & a_2 & a_2 & \cdots & a_2 & \cdots & a_{p+1} & a_{p+1} & \cdots & a_{p+1} \end{array}\right]. \tag{3.21}$$

Thus, to find the solution of $\widehat{\Phi}c = 0$ with minimal $\|c\|_0$, we need to choose c such that it has minimal number of nonzeros, and provide $\alpha = 0$. In order to satisfy $\alpha = 0$ we need to make sure that $\sum_{i=1}^{p+1} pa_i = 0$. It is not difficult to see that the sparsest solution c satisfying $\alpha = 0$ is obtained by taking for example

$$a_1 = a,\ a_2 = -a,\ a_i = 0, \quad i = 3, \ldots, p+1,$$

where $a \in \mathbb{R}$ is some arbitrary constant. Thus, we have found the minimal number of linearly dependent vectors, $\|c\|_0 = 2p$, and that equals the spark of our initial matrix Φ.

We move now from general frame properties of the system of discrete lines to particular applications of those properties in the problem of geometrically structured signal recovery, namely separation of geometric structures and recovery of unions of lines.

3.4 Separation of Points and Lines

In this and the next section, we will restrict to the case $d = 2$, because, as we will clarify below, going to higher dimensional space does not result in improvement of the results. First of all, as we saw, the mutual coherence of the initial system of lines does not get better as we go larger d, and stays $1/p$. The is also the case with the spark: $\text{spark}(\Phi) = 2p$, independent on d. Moreover, for the signal processing problem that we would like to consider, the spark will be even lower. In particular, we will be considering a separation matrix, which except for lines as columns, also contains points (unit vectors), and possibly other structures as planes etc. The spark of the separation matrix, as we will see, also does not improve with the growth of the dimension of the space. In words this can be explain by the fact that if we fix one line (which always has p nonzeros), and take p unit vectors which have nonzeros exactly at the points of the line, we will obtain $p + 1$ linearly dependent vectors, and the spark will be at most $p + 1$, independently of the dimension d. Furthermore, if we add additional structures (like planes in three dimensional space), the spark can only get smaller. For these reasons, from now on we will fix $d = 2$.

Instead of the parametric definition of the discrete lines which we gave in Definition 3.1, it will be simpler now to use the canonical "slope-intercept" form for the equation of a line.

Definition 3.10. Let p be prime and let a and b be any integers. A *discrete line* with parameters a and b is defined as the set of points

$$\mathcal{L}_{a,b} = \{(m, [am + b]_p) : m \in \mathbb{Z}_p\}. \tag{3.22}$$

Since we are working modulo p prime, the parameters a and b from $\mathbb{Z}$ actually define the same discrete line as the parameters $[a]_p, [b]_p \in \mathbb{Z}_p$. Therefore, further in the text we will always assume that the parameters of the discrete line $\mathcal{L}_{a,b}$ belong to $\mathbb{Z}_p$, and we will use the residual system $\{0, \ldots, p - 1\}$ for their notation.

If $a = 0$, we obtain a horizontal line. Sometimes we will use the notation

$$\mathcal{H}_b = \mathcal{L}_{0,b} = \{(m, [b]_p) : m \in \mathbb{Z}_p\},$$

to specify that this is a horizontal line. Note, that with the definition that we gave, vertical lines are not included. Therefore, we give one more definition.

Definition 3.11. A vertical line is defined as

$$\mathcal{V}_a = \{([a]_p, m) : m \in \mathbb{Z}_p\}. \tag{3.23}$$

For consistence of the notation, for the vertical line we will also use a two parameter notation, $\mathcal{L}_{p,a} := \mathcal{V}_a$. For fixed prime p, we will be considering the family of all discrete lines,

$$\mathcal{L} := \{\mathcal{L}_{a,b}\}_{a=0,b=0}^{p,p-1}. \tag{3.24}$$

As we know, there are in total $p^2 + p$ lines in this family.

As we discussed in the Introduction Section, the compressed sensing methodology allows to develop algorithm for separation of two or more morphologically different structures present in one signal. This could, for instance, be line and point-like structure in images, different sounds in audio signals, separating noise etc. What one needs in each case is a sparse representation for every structure, and properties like small mutual coherence or RIP constant, such that ℓ_1 minimization succeeds.

Let p be some prime, and let $M = p^2$, $K = p^2 + p$, and $N = M + K$. We consider the task of decomposition of a signal $x \in \mathbb{R}^M$ into

$$x = x_1 + x_2,$$

where x_1 consists of points and x_2 is a union of discrete lines. In order to sparsely represent the signals x_1 and x_2 we can use the following two matrices

$$\Phi_1 := I_M, \quad \Phi_2 := \mathcal{L}. \tag{3.25}$$

I_M is the identity matrix of size M, and $\mathcal{L}$ is the $M \times K$ matrix where each column is a different normalized discrete line with parameters a, b, i.e.,

$$\mathcal{L}[\,:, k] = \frac{1}{\sqrt{p}} \mathbf{1}_{\mathcal{L}_{a_k,b_k}},$$

where

$$\begin{aligned} a_k &= \lfloor k/p \rfloor, \quad k = 0, \ldots, K-1, \\ b_k &= [k]_p, \quad k = 0, \ldots, K-1. \end{aligned}$$

We now write the sparse representations as $x_1 = \Phi_1\, c_1$, $x_2 = \Phi_2\, c_2$, where c_1 and c_2 are sparse vectors. The compressed sensing idea to separate the signals x_1 and x_2 is to solve the minimization problem

$$\min_{c_1,c_2} \|c_1\|_1 + \|c_2\|_1 \text{ such that } x = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{3.26}$$

and assign $\hat{x}_2 = \Phi_1 \hat{c}_1$ and $\hat{x}_2 = \Phi_2 \hat{c}_2$. We will often use the notation

$$\Phi_{\mathcal{L}} := \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}. \tag{3.27}$$

Equation (3.26) is a relaxed version of the ℓ_0 minimization problem, which is given by

$$\min_{c_1,c_2} \|c_1\|_0 + \|c_2\|_0 \text{ such that } x = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{3.28}$$

In order to obtain results about the success of the recovery of x via (3.26) we need to investigate the properties of the matrix $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$. We concentrate on three most important properties of this matrix: the mutual coherence, the restricted isometry property, and the spark.

3.4.1 Mutual Coherence of the Separation Matrix

In order to investigate the mutual coherence of a given measurement matrix Φ, it is useful to know the corresponding Gram matrix $\Phi^T\Phi$, which contains all possible scalar products of the columns of Φ. In the case of our separation matrix $\Phi_{\mathcal{L}}$ it is possible to compute the Gram matrix exactly, because of the geometric structure, which transform scalar products into a question of intersection of objects. We thus investigate the properties of $G_{\mathcal{L}} = \Phi_{\mathcal{L}}^T\Phi_{\mathcal{L}}$, where $\Phi_{\mathcal{L}}$ is the measurement matrix (3.27).

Lemma 3.12. *Let p be prime, and $G_{\mathcal{L}} = \Phi_{\mathcal{L}}^T\Phi_{\mathcal{L}}$ be the Gram matrix of $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$. Then, $G_{\mathcal{L}}$ is a square matrix of size $2p^2 + p$ with structure*

$$G_{\mathcal{L}} = \begin{bmatrix} I_{p^2} & \mathcal{L} \\ \mathcal{L}^T & K \end{bmatrix}, \tag{3.29}$$

where K has also specific structure, consisting of blocks of identity matrices of size p on the diagonal, and constants $\frac{1}{p}$ off the diagonal blocks. In particular, using the notation $P = \frac{1}{p}\mathbf{1}_{p\times p}$, K can be written as

$$K = \begin{bmatrix} I_p & P & \cdots & P \\ P & I_p & P & \cdots \\ \cdots & P & \ddots & P \\ P & \cdots & P & I_p \end{bmatrix}.$$

Proof. We notice at first that, since $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$, we can write

$$G_{\mathcal{L}} = \begin{bmatrix} \Phi_1^T \\ \Phi_2^T \end{bmatrix} \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} = \begin{bmatrix} \Phi_1^T\Phi_1 & \Phi_1^T\Phi_2 \\ \Phi_2^T\Phi_1 & \Phi_2^T\Phi_2 \end{bmatrix}.$$

In order to prove (3.29), we consider the four blocks separately. Using the definitions of Φ_1 and Φ_2 we can calculate:

$$\Phi_1^T \Phi_1 = I_M^T I_M = I_{p^2}, \quad \Phi_1^T \Phi_2 = (\Phi_2^T \Phi_1)^T = I_M \mathcal{L} = \mathcal{L}.$$

It is left only to consider $\Phi_2^T \Phi_2 = \mathcal{L}^T \mathcal{L} =: K$, the Gram matrix of $\mathcal{L}$. Here we use the geometric structure of discrete lines. On the diagonal we have 1 since the lines are normalized. In the diagonal blocks, off the diagonal we have zeros, since there are no intersections with lines that are in the same block — these are lines which have the same slope, i.e. lines which are parallel. The rest of the scalar products are $\frac{1}{p}$, since, as we showed in Proposition 3.3, if the lines have different slope, they intersect exactly at one point. □

Using this lemma we can now calculate the mutual and the average coherence of $\Phi_{\mathcal{L}}$. The mutual coherence of a similar separation matrix (for three dimensional objects) was shortly discussed in [47].

Theorem 3.13. *Let p be prime. The mutual coherence of the matrix $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$ is equal to*

$$\mu(\Phi_{\mathcal{L}}) = \frac{1}{\sqrt{p}}. \tag{3.30}$$

Proof. This follows directly from Lemma 3.12 — we only need to find the maximal in absolute value entry of the Gram matrix $G_{\mathcal{L}} = \Phi_{\mathcal{L}}^T \Phi_{\mathcal{L}}$. Looking at the structure of $G_{\mathcal{L}}$ described in (3.29), we see that the maximal value comes from the entries in the matrix $\mathcal{L}$, and it equals $\frac{1}{\sqrt{p}}$, □

For comparison, since the matrix $\Phi_{\mathcal{L}}$ is of size $p^2 \times (2p^2 + p)$, the Welch bound (see (2.9)) gives us

$$\mu(\Phi_{\mathcal{L}}) \geq \frac{1}{\sqrt{p(2p-1)}}.$$

This shows that the mutual coherence of our system is not optimal. Still, it goes to zero as the dimension parameter p grows, and thus we are able to conclude the following sparse recovery result.

Corollary 3.14. *Let p be some prime, $M = p^2$, $N = 2p^2 + p$. Let $\Phi_{\mathcal{L}} \in \mathbb{R}^{M \times N}$ be the block matrix $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$ and $x \in \mathbb{R}^M$. If $c_0 \in \mathbb{R}^N$ is such that $x = \Phi_{\mathcal{L}} c_0$ and*

$$\|c_0\|_0 \leq \frac{1}{2}\left(1 + \sqrt{p}\right),$$

then this c_0 is a unique solution of both (3.26) and (3.28).

Proof. This result follows directly from Theorem 3.13 and Theorem 2.5. □

We have shown in Theorem 3.13 that the mutual coherence of $\Phi_{\mathcal{L}}$ is $\frac{1}{\sqrt{p}}$, which is the maximum magnitude of the scalar product of any two columns in $\Phi_{\mathcal{L}}$. Such definition of the mutual coherence is often also called *worst-case coherence*. Another measure of the coherence of a measurement matrix is the so-called *average coherence*, first introduced in [10], and defined as

$$\nu(\Phi) = \frac{1}{N-1} \max_i \left| \sum_{j:j\neq i} \langle \phi_i, \phi_j \rangle \right| = \frac{1}{N-1} \|(\Phi^T\Phi - I_N)\mathbf{1}\|_\infty. \tag{3.31}$$

Small average coherence in conjunction with some additional constraints (strong coherence property) also guarantees recovery of sparse signals from linear measurements via one-step thresholding algorithm [10]. We calculate here the average coherence of our separation matrix $\Phi_{\mathcal{L}}$, leaving the sparse recovery guarantees using it for future investigation.

Theorem 3.15. *Let p be prime. The average coherence of the matrix $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$ is equal to*

$$\nu(\Phi_{\mathcal{L}}) = \frac{p+\sqrt{p}}{2p^2+p-1}. \tag{3.32}$$

Proof. Using the definition of the average coherence, and the structure of the Gram matrix described in Lemma 3.12, we see that we have only two different sums of the rows of $\Phi_{\mathcal{L}}^T\Phi_{\mathcal{L}} - I_N$, dependent on whether we are in the upper or lower half of $\Phi_{\mathcal{L}}^T\Phi_{\mathcal{L}}$. Thus,

$$\nu(\Phi_{\mathcal{L}}) = \frac{1}{2p^2+p-1} \max\left\{ \frac{1}{\sqrt{p}}(p+1), \frac{p}{\sqrt{p}} + \frac{1}{p}p^2 \right\} = \frac{p+\sqrt{p}}{2p^2+p-1}.$$

□

3.4.2 Restricted Isometry Property of the Separation Matrix

The restricted isometry property, in particular for deterministic measurement matrices, is very difficult to check, since it is a condition on all possible sub-collections of k columns of the matrix. One way to go around this problem is to get a bound on the restricted isometry constant from the mutual coherence. It is known that mutual coherence μ implies an RIP constant $\delta \leq (k-1)\mu$ (see Theorem 5.3 and Proposition 6.2 in [64]). Although this is not a sharp estimate, such a recovery guarantees which use RIP instead of the mutual coherence is better, in the sense that it guarantees stable and robust recovery. This is achieved with the notion of best k-term approximation, which we discussed in Chapter 2 (equation (2.1)).

Theorem 3.16. *Let p be prime. The matrix $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$ has a restricted isometry property of order k with constant $\delta = \frac{k-1}{\sqrt{p}} < 1$ whenever $k \leq \sqrt{p}$.*

Proof. Although we could view this result as a corollary from Theorem 3.13, we present the proof of it for completeness. Let K be any subset of $[1, \ldots, 2p^2 + p]$ of cardinality k. We are interested in the Gramm matrix $G_K = \Phi_{\mathcal{L}K}^T \Phi_{\mathcal{L}K}$, whose elements we denote by g_{jl},

$$g_{j,l} = \langle \Phi_{\mathcal{L}}(:, j), \Phi_c L(:, l) \rangle, \; j, l \in K.$$

If we use the notations $\phi_1, \ldots, \phi_N$ for the columns of the matrix $\Phi_{\mathcal{L}}$, we see that in Lemma 3.12 we have already calculated the possible values of the scalar products between two columns of $\Phi\mathcal{L}$, $g_{j,l} = \langle \phi_j, \phi_l \rangle$ for all $j, l \in [1, \ldots, p^2 + p]$.

When $j = l$, since all columns are normalized we have that $g_{jj} = 1$, and if $j \neq l$ the possible results are $0, \dfrac{1}{p}$ or $\dfrac{1}{\sqrt{p}}$, so we can write that $g_{jl} \leq \dfrac{1}{\sqrt{p}}$.

Now we can write $G_K = B_K + I_k$, where I_k is the identity matrix of order k, and B_K has zero on the diagonal, and at most $\dfrac{1}{\sqrt{p}}$ off the diagonal. Next we know that for a matrix A the RIP constant of order k can be computed as

$$\delta_k = \max_{K \subseteq N, |K| \leq k} \| A_K^T A_K - I_k \|_{2 \to 2}. \tag{3.33}$$

We can estimate the norm of our matrix B_K by the norm of the matrix $\tilde{B}_K$,

$$\| B_K \|_{2 \to 2} \leq \| \tilde{B}_K \|_{2 \to 2},$$

where $\tilde{B}$ is defined by

$$\tilde{b}_{jj} = 0, \quad \tilde{b}_{jl} = \frac{1}{\sqrt{p}}, \text{ for all } j \neq l.$$

For this matrix we can conclude

$$\| \tilde{B}_K \|_{2 \to 2} \leq \| \tilde{B}_K \|_{1 \to 1} = \frac{k-1}{\sqrt{p}} =: \delta.$$

This value δ is smaller then 1, whenever $k \leq \sqrt{p}$, and that concludes our proof. $\square$

In the proof we estimated each entry of B_K by the largest possible value, $\frac{1}{\sqrt{p}}$. In reality, many of the elements g_{jl} are smaller (equal to $1/p$), or equal to zero, and thus this bound is very rough. In any case, a small RIP constant guarantees successful ℓ_1 minimization, and we make the corresponding conclusion for our separation matrix in the next corollary.

Corollary 3.17. *Let p be some prime, $M = p^2$, and $N = 2p^2 + p$. Let x be some signal in $\mathbb{R}^M$ that can be represented as union of points and lines, i.e.*

$$x = x_1 + x_2 = I_M\, c_1 + \mathcal{L}\, c_2,$$

with $c = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \in \mathbb{R}^{M+N}$*. If* $k \leq \frac{\sqrt{p}}{5}$*, then,*

$$\|\hat{c} - c\|_2 \leq C \frac{\sigma_k(c)_1}{\sqrt{k}}, \tag{3.34}$$

where $\hat{c}$ is the solution of the ℓ_1 minimization problem (3.26), and $\sigma_k(c)_1$ is the error of the best k-term approximation of c.

Proof. The proof of this theorem follows immediately from the standard compressed sensing result which says that if the measurement matrix has RIP of order $2k$ with $\delta_{2k} < \sqrt{2} - 1$, then the stability condition (3.34) holds (see Theorem 2.10). We need to check for which $k > 0$

$$\delta_{2k} := \frac{2k - 1}{\sqrt{p}} < \sqrt{2} - 1.$$

We see immediately that this is true whenever $k < \frac{\sqrt{2} - 1}{2} \sqrt{p} + \frac{1}{2} \leq \frac{\sqrt{p}}{5}$, which proves the claim. Note that, if the signal c is exactly k sparse, then $\delta_k(c)_1$ equals zero, and we will have a perfect recovery. $\square$

3.4.3 Spark of the Separation Matrix

Another important characteristic in sparse recovery problems is the *spark*. As we saw in Theorem 2.3, if $\|c_0\|_0 \leq \frac{\text{spark}(\Phi)}{2}$, then c_0 it is the unique solution of the ℓ_0 minimization problem. We computed the spark of the matrix of discrete lines $\mathcal{L}$ in Theorem 3.9, and we will see now that we can find the spark of the separation matrix Φ consisting of points and lines as well.

Theorem 3.18. *Let* $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$ *be the separation matrix of points and lines. Then,*

$$\text{spark}(\Phi_{\mathcal{L}}) = p + 1.$$

Proof. Let us first observe that, since all elements in I_M are linearly independent, $\text{spark}(I_M) = p^2 + 1$. Also, as it was shown in Theorem 3.9, $\text{spark}(\mathcal{L}) = 2p$. Therefore, we see immediately that the spark can not be larger than $2p$. It is possible, however, to find less then $2p$ linearly *dependent* vectors in $\Phi_{\mathcal{L}}$. Taking into account the structure of our matrices, it is easy to find a linear combination of $p + 1$ dependent vectors. Namely, take one discrete line, for example $\phi_1^{(2)} = \mathbf{1}_{\mathcal{L}_{0,0}}$. By definition $\mathcal{L}_{0,0} = \{(m,0) : m \in \mathbb{Z}_p\}$, and therefore

$$1_{\mathcal{L}_{0,0}}(j) = \begin{cases} 1, & \text{if } j = kp+1,\ k = 0,\dots,p-1, \\ 0, & \text{otherwise.} \end{cases}$$

Now we take the p unit vectors from I_M which correspond to the support of the chosen discrete line. In this case those will be $\{e_1, e_{p+1}, \dots, e_{(p-1)p+1}\}$, where

$$e_k(j) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

It becomes clear that we can write

$$\sum_{k=0}^{p-1} e_{kp+1} - \mathbf{1}_{\mathcal{L}_{0,0}} = 0,$$

which is a linear combination of $p+1$ vectors from $\Phi_{\mathcal{L}}$ and which shows us that

$$\operatorname{spark}(\Phi_{\mathcal{L}}) \leq p+1.$$

It is left now to show that there are no less then $p+1$ linearly dependent vectors. Towards a contradiction, assume that there exists a vector $\eta \neq 0$ such that $\Phi_{\mathcal{L}}\eta = 0$, and $\|\eta\|_0 \leq p$. That means we have

$$\sum_{j\in\Lambda_1} \eta_j \phi_j^{(1)} + \sum_{j\in\Lambda_2} \eta_j \phi_j^{(2)} = 0, \tag{3.35}$$

where $|\Lambda_1| + |\Lambda_2| \leq p$. It is important that the sets Λ_1 and Λ_2 are non empty, since the spark of each of the corresponding matrices is larger than p, so there can not be p or less linearly dependent columns taken from only one of the matrices Φ_1 or Φ_2.

Let us assume at first that $|\Lambda_2| = 1$. We have one discrete line, and that vector has p non zeros, $\|\phi_j^{(2)}\|_0 = \|1_{\mathcal{L}_j}\|_0 = p$. On the other hand, the vectors from Φ_1 have each only one nonzero value, and they do not intersect. Therefore, in order to get a linear combination that will sum up to zero, Λ_1 always needs to be of the size of the support of $\sum_{j\in\Lambda_2} \eta_j \phi_j^{(2)}$, in this case $\mathbf{1}_{\mathcal{L}_j}$. That will give us exactly $p+1$ vectors, as described in the beginning of the proof.

Assume next that Λ_2 possesses two elements j_1, j_2. Since two lines intersect at most at one point, we obtain

$$\|\eta_{j_1}\phi_{j_1}^{(2)} + \eta_{j_2}\phi_{j_2}^{(2)}\|_0 \geq 2p-2.$$

In general, if we have k lines, $|\Lambda_2| = k$, they will have at most $(k-1)k/2$ intersections, and thus the linear combination of them will have support at least $kp - 2\frac{k(k-1)}{2} = k(p-k+1)$.

In order to make sure that we have 0 on the right hand side in (3.35), we need to have the same number of unit vectors from Λ_1, thus $|\Lambda_1| = k(p-k+1)$. The question now becomes, whether there exist some positive k such that

$$k(p-k+1)+k < p+1.$$

We can easily investigate the function $f(k) = k(p-k+1)+k-(p+1) = -k^2+k(p+2)-(p+1)$ and see for which k is it negative. Since $f'(k) = -2k+(p+2)$, the function has an extremal point in $\frac{p+2}{2}$, and since $f''(k) = -2 < 0$, this point is a maximum. The zeros of the function are 1 and $p+1$, therefore, the minimal positive k such that the function is strictly negative is $p+2$. This number is already larger than $p+1$, the number of vectors which we have shown that are linearly dependent. Thus, the spark can not be smaller than $p+1$. □

Corollary 3.19. *Let p be some prime, $M = p^2$ and $N = 2p^2+p$. Let further $\Phi_{\mathcal{L}} \in \mathbb{R}^{M\times N}$ be the block matrix $\Phi_{\mathcal{L}} = \begin{bmatrix} I_M & \mathcal{L} \end{bmatrix}$ and $x \in \mathbb{R}^M$. If $c_0 \in \mathbb{R}^N$ is such that $x = \Phi_{\mathcal{L}} c_0$ and*

$$\|c_0\|_0 < \frac{p+1}{2},$$

then c_0 is the unique solution of ℓ_0 minimization problem (3.28).

Proof. This result follows directly from Theorem 3.18 and Theorem 2.3. □

3.4.4 Numerical Experiments on Separation of Points and Lines

Here, we present a few numerical results on the separation problem. Recall that we are interested in solving

$$\min_{c_1,c_2} \|c_1\|_1 + \|c_2\|_1 \text{ subject to } x = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{3.36}$$

where $\Phi_1 = I_{p^2}$ is the identity matrix which sparsely represents points, and Φ_2 is a dictionary of discrete lines, $\Phi_2 = \{\mathcal{L}_{a,b}\}_{a=0,b=0}^{p,p-1}$. In Figure 3.3, we show the separation of lines and points in two dimensions with $p = 79$ visually. The sparse vector $c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ or in other words the lines and the points were chosen at random, and the CVX package [68] was used for solving the ℓ_1 minimization.

In Figure 3.4, we show the recovery rate — how many points and lines can be successfully recovered for fixed dimension of the problem. For every sparsity level k, we chose at random k points and lines, and try to recover them by solving (3.36). We count the

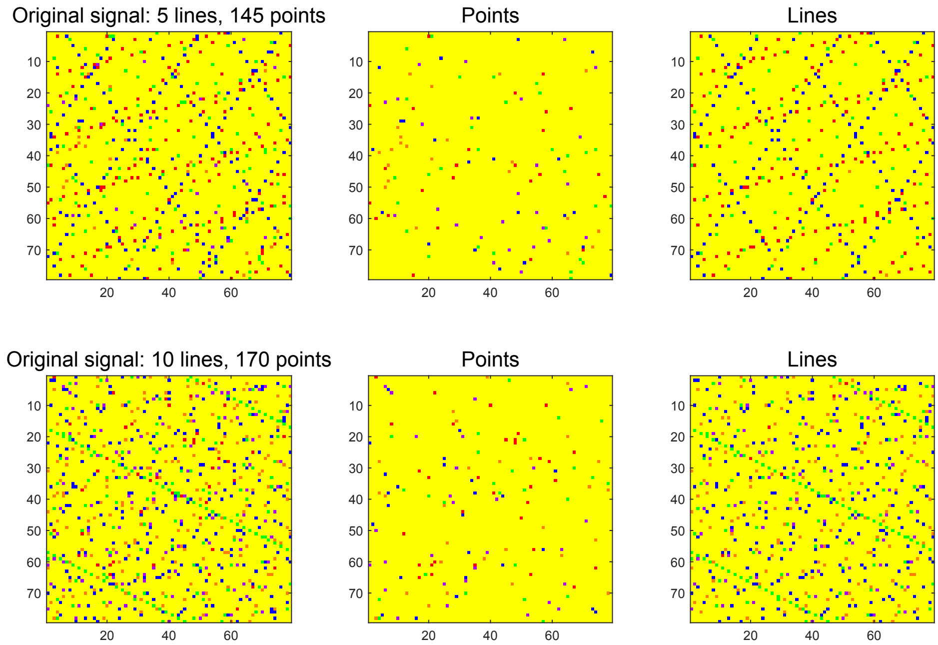


Figure 3.3: Visualized separation of points and lines via compressed sensing.

experiment as successful, if the normalized squared error of the recovered signal is smaller than 10^{-4}. We repeat this experiment $T = 100$ time for each sparsity level, and plot the successful recovery rate.

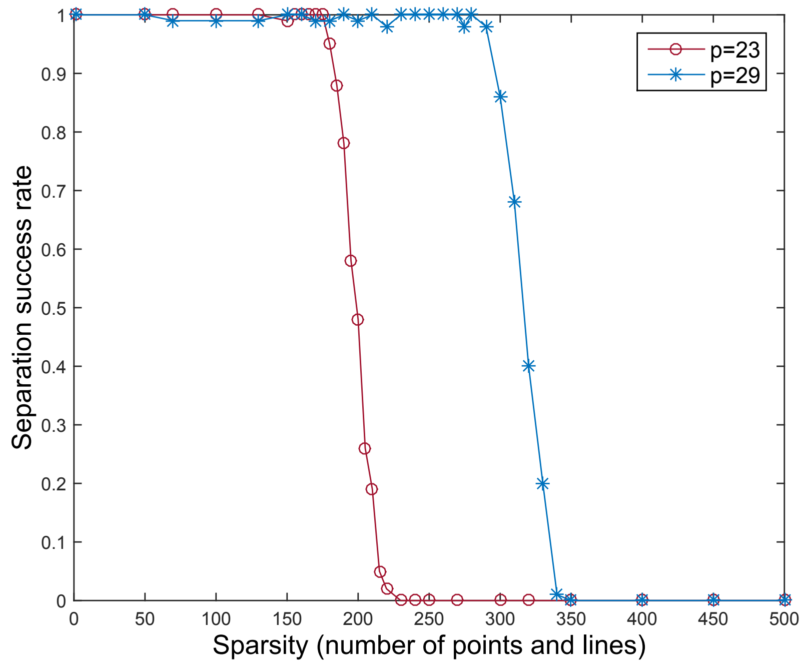


Figure 3.4: Success rate of separation for randomly chosen points and lines.

The observation that we obtain successful recovery for sparsity levels much larger then the theoretical guarantees (which were of order p and $\sqrt{p}$) can be explained by that fact that we used worst-case guarantees, which assure successful recovery uniformly, i.e. for any sparse signals, but in the numerical experiments we take the nonzero positions at random.

3.5 Compressed Sensing of Discrete Lines

We are now getting back to the investigation of signals which are unions of only discrete lines. Before we start, it will be useful to state a different formulation of the discrete Fourier transform of lines from what we proved for general d in Proposition 3.7, namely such which will use the notation of discrete lines from Definition 3.10.

Proposition 3.20. *Let p be prime. The discrete Fourier transform of the lines $\mathcal{L}_{a,b}$, where $a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p$, the horizontal lines $\mathcal{H}_b$, $b \in \mathbb{Z}_p$ and vertical lines $\mathcal{V}_a$, $a \in \mathbb{Z}_p$ are given by the formulas*

$$\widehat{\mathbf{1}}_{\mathcal{L}_{a,b}}(m,n) = e^{2\pi i \frac{ba^{-1}m}{p}} \mathbf{1}_{\mathcal{L}_{-a^{-1},0}}(m,n), \tag{3.37}$$

$$\widehat{\mathbf{1}}_{\mathcal{H}_b}(m,n) = e^{-2\pi i \frac{bn}{p}} \mathbf{1}_{\mathcal{V}_0}(m,n), \tag{3.38}$$

$$\widehat{\mathbf{1}}_{\mathcal{V}_a}(m,n) = e^{-2\pi i \frac{am}{p}} \mathbf{1}_{\mathcal{H}_0}(m,n). \tag{3.39}$$

Proof. Having in mind that $\mathcal{H}_b = \mathcal{L}_{0,b}$, we start by proving the first two equalities simultaneously. Let p be prime, and $a, b \in \mathbb{Z}_p$. By the definition of a discrete line and the discrete Fourier transform, we have

$$\begin{aligned}\widehat{\mathbf{1}}_{\mathcal{L}_{a,b}}(m,n) &= \frac{1}{p}\sum_{k,\ell=0}^{p-1} \mathbf{1}_{\mathcal{L}_{a,b}}(k,\ell) \cdot e^{-2\pi i \frac{\langle (m,n),(k,\ell)\rangle}{p}} \\ &= \frac{1}{p}\sum_{k=0}^{p-1} e^{-2\pi i \frac{mk+n[ak+b]_p}{p}} = e^{-2\pi i \frac{bn}{p}} \frac{1}{p}\sum_{k=0}^{p-1} e^{-2\pi i \frac{k(m+an)}{p}}.\end{aligned}$$

We know from the digital signal processing field (see, for example [98]) that

$$\frac{1}{p}\sum_{k=0}^{p-1} e^{-2\pi i \frac{kl}{p}} = \delta_p(l), \tag{3.40}$$

where δ_p is the periodic Kronecker delta function. Therefore, we have

$$\widehat{\mathbf{1}}_{\mathcal{L}_{a,b}}(m,n) = e^{-2\pi i \frac{bn}{p}} \cdot \mathbf{1}_{\{(m,n)\in\mathbb{Z}_p^2 : [m+an]_p = 0\}}. \tag{3.41}$$

The relation $[m+an]_p = 0$ is equivalent to $m = [-an]_p$, and hence the last set can be written as

$$\{(m,n) \in \mathbb{Z}_p^2 : [m+an]_p = 0\} = \{([-an]_p, n) : n \in \mathbb{Z}_p\}. \tag{3.42}$$

Now we see that if $a = 0$, then this set is the line $\mathcal{V}_0$. If $a \neq 0$, then a is element of $\mathbb{Z}_p^*$, which is a multiplicative group since p is prime, and thus an inverse element exists. We can write the set (3.42) as

$$\{(m, [-a^{-1}m]_p) : n \in \mathbb{Z}_p\} = \mathcal{L}_{-a^{-1},0},$$

and conclude

$$\widehat{\mathbf{1}}_{\mathcal{L}_{a,b}}(m,n) = \begin{cases} e^{2\pi i \frac{ba^{-1}m}{p}} \mathbf{1}_{\mathcal{L}_{-a^{-1},0}}(m,n), & \text{if } a \neq 0, \\ e^{-2\pi i \frac{bn}{p}} \mathbf{1}_{\mathcal{V}_0}(m,n), & \text{if } a = 0. \end{cases}$$

Let now $a \in \mathbb{Z}_p$. We are interested in the discrete Fourier transform of the vertical line $\mathcal{V}_a$. Proceeding as before,

$$\begin{aligned} \widehat{\mathbf{1}}_{\mathcal{V}_a}(m,n) &= \frac{1}{p} \sum_{k,\ell=0}^{p-1} \mathbf{1}_{\mathcal{V}_a}(k,\ell) \cdot e^{-2\pi i \frac{\langle (m,n),(k,\ell) \rangle}{p}} \\ &= \frac{1}{p} \sum_{\ell=0}^{p-1} e^{-2\pi i \frac{ma+n\ell}{p}} = e^{-2\pi i \frac{am}{p}} \frac{1}{p} \sum_{\ell=0}^{p-1} e^{-2\pi i \frac{n\ell}{p}} = e^{-2\pi i \frac{am}{p}} \delta_p(n). \end{aligned}$$

Since $n \in \mathbb{Z}_p$, $\delta_p(n)$ is not equal to zero only when $n = 0$, and thus we will have nonzero values on the set $\{(m, 0) : m \in \mathbb{Z}_p\}$. We obtain

$$\widehat{\mathbf{1}}_{\mathcal{V}_a}(m,n) = e^{-2\pi i \frac{am}{p}} \mathbf{1}_{\mathcal{H}_0}(m,n).$$

The proposition is proved. □

We are interested in the family of signals $z \in \mathbb{C}^{p^2}$ which are linear combination of characteristic functions of discrete lines. We can write each signal in our family as

$$z = \sum_{(c,d) \in \Lambda} \alpha_{c,d} \mathbf{1}_{\mathcal{L}_{c,d}}, \tag{3.43}$$

where $\Lambda \subseteq \{0, \dots, p\} \times \mathbb{Z}_p$, $|\Lambda| = k$ and $\alpha_{c,d} \in \mathbb{C}$ for all $(c,d) \in \Lambda$.

As we know, the crucial ingredient for the compressed sensing methodology is the condition of *sparsity,* and it is usually defined as the number of nonzero entries of the signal. In our case, since every line has p nonzero elements, this number would be kp, or smaller depending on the number of intersections which the lines have. However, in our setting the signals have a sparse representation in $\mathcal{L}$, and sparsity is measured as the number of geometric structures in the signal.

Definition 3.21. We say that the signal z defined in (3.43) is *k-sparse*, if $|\Lambda| \leq k$.

Thus, the sparsity is now the number of lines that the signal consists of. Instead of the signal itself, we will measure the Fourier coefficients of the signals z. For this, we set

$$x := \hat{z} = \sum_{(c,d)\in\Lambda} \alpha_{c,d} \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}}.$$

The measurements are obtained by taking M scalar products with some discrete lines, i.e.,

$$y_m = \langle \phi_m, x\rangle, \text{ where } \quad \phi_m = \mathbf{1}_{\mathcal{L}_{a_m,b_m}}, \; a_m \in \{0,\dots,p\}, b_m \in \mathbb{Z}_p, \; m = 1,\dots M.$$

If we have one signal $x \in \mathbb{C}^{p^2}$ and one measurement $\phi \in \mathbb{C}^{p^2}$, then the corresponding $y \in \mathbb{C}$ has the form

$$y = \langle \omega, x\rangle = \langle \mathbf{1}_{\mathcal{L}_{a,b}}, \sum_{(c,d)\in\Lambda} \alpha_{c,d} \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}}\rangle = \sum_{(c,d)\in\Lambda} \alpha_{c,d} \, \langle \mathbf{1}_{\mathcal{L}_{a,b}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}}\rangle.$$

We will start with investigation with the entries $\langle \mathbf{1}_{\mathcal{L}_{a,b}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}}\rangle$, i.e. the scalar product of a line with a Fourier transform of some other line.

Definition 3.22. Let p be prime. We then define a function $w_{ac} : \{0,\dots,p\}\times\{0,\dots,p\} \to \mathbb{Z}_p^*$ by

$$w_{ac} = \begin{cases} ac+1, & \text{if } a, c \neq p, \\ -a, & \text{if } a \neq p, \; c = p, \\ -c, & \text{if } a = p, \; c \neq p, \\ 1, & \text{if } a = p, \; c = p. \end{cases} \tag{3.44}$$

Theorem 3.23. *Let p be prime. Let $a, c \in \{0,\dots,p\}$, and $b, d \in \mathbb{Z}_p$. Then,*

$$\langle \mathbf{1}_{\mathcal{L}_{a,b}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}}\rangle = \begin{cases} e^{-2\pi i \frac{bdw_{ac}^{-1}}{p}}, & \text{if } w_{ac} \neq 0, \\ 0, & \text{if } w_{ac} = 0 \text{ and } b \neq 0, \\ 0, & \text{if } w_{ac} = 0, \; b = 0, \text{ and } d \neq 0, \\ p, & \text{if } w_{ac} = 0, \; b = 0, \text{ and } d = 0. \end{cases} \tag{3.45}$$

Proof. We will prove the theorem in four separate cases, according to the four possible values of w_{ac} defined in (3.44).

Case $a, c \neq p$: We are interested in the result of the scalar product of the measurement line

$$\phi = \mathbf{1}_{\mathcal{L}_{a,b}}, \quad \mathcal{L}_{a,b} = \{(m, [am+b]_p) : m \in \mathbb{Z}_p\} \tag{3.46}$$

with the signal (see Proposition 3.20, equations (3.41)-(3.42))

$$x = \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}} = e^{-2\pi i \frac{dn}{p}} \mathbf{1}_{\{([-cn]_p, n) : n \in \mathbb{Z}_p\}}. \tag{3.47}$$

The scalar product between two lines is geometrically the result of the intersections of those two lines. We are thus looking for a point that will be contained in both lines. That is equivalent to looking for a solution of the system of congruences

$$\begin{aligned} m &\equiv -cn \pmod{p}, \\ am + b &\equiv n \pmod{p}. \end{aligned}$$

We can substitute m by $-cn$ in the second equation to obtain

$$(ac+1)n \equiv b \pmod{p}. \tag{3.48}$$

Now we consider different cases.

If $ac + 1 \neq 0$, it means that its inverse element exists, and thus we can find the unique solution (intersection) $(m, n) \in \mathbb{Z}_p \times \mathbb{Z}_p$, where

$$m = -cn,\ n = (ac+1)^{-1} b.$$

The value of x at this point according to (3.47) is exactly $e^{-2\pi i \frac{d(ac+1)^{-1} b}{p}}$.

If $ac + 1 = 0$, and at the same time $b \neq 0$, the equation (3.48) can not have solution, and thus there are no intersections and the scalar product is zero. If on the other hand, $b = 0$, then this equation has p solutions - any $n \in \mathbb{Z}_p$ is a solution, which means that the two lines have the same support. Therefore the scalar product will be the sum of all values of the intersection points,

$$\langle \phi, x \rangle = \langle \mathbf{1}_{\mathcal{L}_{a,b}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}} \rangle = \sum_{n=0}^{p-1} e^{-2\pi i \frac{dn}{p}} = p\, \delta_p(d),$$

where the last equality is based on equation (3.40). According to the definition of δ_p, we have the two different cases as formulated in the statement we want to prove: $\langle \phi, x \rangle = 0$, if $d \neq 0$, and equals to p, if $d = 0$.

Case $a \neq p, c = p$: Now let ϕ be defined as in (3.46), and let the signal x be the Fourier transform of a vertical line,

$$x = \widehat{\mathbf{1}}_{\mathcal{V}_d}, \quad \widehat{\mathbf{1}}_{\mathcal{V}_d} = e^{-2\pi i \frac{dn}{p}} \mathbf{1}_{\{(n,0) : n \in \mathbb{Z}_p\}}. \tag{3.49}$$

This time we need to solve the system

$$m \equiv n \pmod{p}, \tag{3.50}$$
$$am + b \equiv 0 \pmod{p}. \tag{3.51}$$

If $a \neq 0$, an inverse element of a exist, and we can find the unique solution as $m = n = -ba^{-1}$. Substituting n in the exponent in (3.49), we obtain the first scalar product.

If $a = 0$, we see that if $b \neq 0$, there will be no solution of the system (3.50)-(3.51) – the scalar product will be zero. If $b = 0$, again, there are p solutions, and as in the previous case, the sum of the exponent can be 0 or p depending on the parameter d.

Case $a = p, c \neq p$: In this case, we need a vertical line as a measurement,

$$\phi = \mathbf{1}_{\mathcal{V}_b}, \quad \mathcal{V}_b = \{(b, m) : m \in \mathbb{Z}_p\}, \tag{3.52}$$

and signal $x = \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}}$, as defined in (3.47). Now we need to solve the system

$$-cn \equiv b \pmod{p},$$
$$n \equiv m \pmod{p}.$$

If $c \neq 0$, we can solve this system and have $n = -c^{-1}b$, which is the unique solution, and the scalar product will be $e^{2\pi i \frac{dc^{-1}b}{p}}$.

If $c = 0$, but $b \neq 0$, there will be no solution to this system, and the scalar product will be zero. If $b = 0$, again there are p solutions, and the sum is 0 or p depending on d.

Case $a = c = p$: Finally, if both the measurement and the signal are vertical lines, $\phi = \mathbf{1}_{\mathcal{V}_b}$, as in (3.52), and $x = \widehat{\mathbf{1}}_{\mathcal{V}_d}$, as in (3.49). The system of equations has the form

$$n \equiv b \pmod{p},$$
$$m \equiv 0 \pmod{p}.$$

This system is already solved uniquely, and the scalar product is thus $e^{-2\pi i \frac{bd}{p}}$. □

Example 3.1. Let $p = 3$. For the future discussion, in Table 3.1 we present the values of the scalar products $\langle \phi, x \rangle = \langle \mathbf{1}_{\mathcal{L}_{a,b}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}} \rangle$ for all possible pairs $(a, b), (c, d) \in \{0, \ldots, p\} \times \mathbb{Z}_p$. This can be easily accomplished following the results from Theorem 3.23. For convenience, we have used the notation $\epsilon_1 = e^{\frac{2\pi i}{3}}$ and $\epsilon_2 = e^{\frac{4\pi i}{3}}$.

$\langle \mathbf{1}_{\mathcal{L}_{a,b}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c,d}} \rangle=$	(cd)= 00	01	02	10	11	12	20	21	22	30	31	32
(ab)= 00	1	1	1	1	1	1	1	1	1	3	0	0
01	1	ϵ_1	ϵ_2	1	ϵ_1	ϵ_2	1	ϵ_1	ϵ_2	0	0	0
02	1	ϵ_2	ϵ_1	1	ϵ_2	ϵ_1	1	ϵ_2	ϵ_1	0	0	0
10	1	1	1	1	1	1	3	0	0	1	1	1
11	1	ϵ_1	ϵ_2	1	ϵ_2	ϵ_1	0	0	0	1	ϵ_2	ϵ_1
12	1	ϵ_2	ϵ_1	1	ϵ_1	ϵ_2	0	0	0	1	ϵ_1	ϵ_2
20	1	1	1	3	0	0	1	1	1	1	1	1
21	1	ϵ_1	ϵ_2	0	0	0	1	ϵ_2	ϵ_1	1	ϵ_1	ϵ_2
22	1	ϵ_2	ϵ_1	0	0	0	1	ϵ_1	ϵ_2	1	ϵ_2	ϵ_1
30	3	0	0	1	1	1	1	1	1	1	1	1
31	0	0	0	1	ϵ_2	ϵ_1	1	ϵ_1	ϵ_2	1	ϵ_1	ϵ_2
32	0	0	0	1	ϵ_1	ϵ_2	1	ϵ_2	ϵ_1	1	ϵ_2	ϵ_1

Table 3.1: Scalar products of discrete lines with DFT of discrete lines, for $p = 3$.

Let p be some prime. Following the notation we used in Table 3.1, we now employ the general notation:

$$\begin{aligned} \epsilon_0 &:= 0, \\ \epsilon_k &:= e^{\frac{2\pi i(k-1)}{p}}, \quad k = 1, \dots, p, \\ \epsilon_{p+1} &:= p. \end{aligned} \tag{3.53}$$

Now that we have defined the measurement process, the question left is, which of the $N = p^2 + p$ lines to choose for measuring, and how many of them are sufficient to recover a vector with given sparsity? In order to solve this problem we first investigate the full matrix denoted by $A_{\mathcal{L}} \in \mathbb{C}^{N \times N}$, where each element is defined as

$$A_{\mathcal{L}}(k, j) := \langle \mathbf{1}_{\mathcal{L}_{a_k,b_k}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c_j,d_j}} \rangle, \quad k, j = 0, \dots, N-1, \tag{3.54}$$

where $a_k = \lfloor k/p \rfloor, b_k = [k]_p$, $c_j = \lfloor j/p \rfloor$, $d_j = [j]_p$. In Table 3.1 we have seen an example of the matrix when $p = 3$.

3.5.1 Deterministically Chosen Measurements

Since we know the structure of our candidate for a measurement matrix, samples from $A_{\mathcal{L}} = \mathcal{L}\widehat{\mathcal{L}}^T$, we would like to come up with a way of choosing the lines that we measure with, and obtain theoretical guarantees for the recovery with such matrix. How many measurements (lines) are needed? Certainly one option is to choose m lines at random, but how far can we go if we want to have a purely deterministic construction? We will choose $p - 1$ measurement, namely all lines with a fixed slope, and we will investigate the properties of this measurement matrix.

Definition 3.24. Let p be prime, $M = p-1, N = p^2+p$. Let $a \in \{0,\dots,p\}$ be fixed. We define the deterministic measurement matrix Φ_0 of size $M \times N$ by

$$\Phi_0(k,j) := \langle \mathbf{1}_{\mathcal{L}_{a,b_k}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c_j,d_j}} \rangle, \quad k = 1,\dots,M,\ j = 0,\dots,N-1, \tag{3.55}$$

where $b_k = [k]_p$, $c_j = \lfloor j/p \rfloor$, $d_j = [j]_p$.

Remark 3.25. The matrix Φ_0 defined in (3.55) has a particular structure which can be described as follows:

$$\Phi_0 = \begin{bmatrix} \Psi_0 & \Psi_1 & \dots & \Psi_p \end{bmatrix},$$

where each Ψ_c, $c = 0,\dots,p$ is a block of size $p-1 \times p$. Using Theorem 3.23, we can make conclude that $w_{ac} = 0$, implies

$$\Psi_c = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $w_{ac} \neq 0$, yields

$$\Psi_c = \begin{bmatrix} e^{-2\pi i \frac{1\cdot 0 w_{ac}^{-1}}{p}} & e^{-2\pi i \frac{1\cdot 1 w_{ac}^{-1}}{p}} & \cdots & e^{-2\pi i \frac{1\cdot (p-1) w_{ac}^{-1}}{p}} \\ e^{-2\pi i \frac{2\cdot 0 w_{ac}^{-1}}{p}} & e^{-2\pi i \frac{2\cdot 1 w_{ac}^{-1}}{p}} & \cdots & e^{-2\pi i \frac{2\cdot (p-1) w_{ac}^{-1}}{p}} \\ & & \cdots & \\ e^{-2\pi i \frac{(p-1)\cdot 0 w_{ac}^{-1}}{p}} & e^{-2\pi i \frac{(p-1)\cdot 1 w_{ac}^{-1}}{p}} & \cdots & e^{-2\pi i \frac{(p-1)\cdot (p-1) w_{ac}^{-1}}{p}} \end{bmatrix}.$$

If we introduce the notation $\omega_{pac} = e^{\frac{-2\pi i w_{ac}}{p}}$, then the elements of the nonzero blocks can be simply written as

$$\Psi_c(k,j) = \omega_{pac}^{kj}, \quad k = 1,\dots,p-1, j = 0,\dots,p-1.$$

It is important to us, that in the nonzero blocks, in every row only one of the parameters in the exponent is changing, and thus every row contains all p-th roots of unity, possibly in different permutation. Furthermore, in all the columns except for the first one (which is a vector of all ones), also all p-th roots of unity except 1 appear without repeating.

In order to prove the main theorem, we need one additional property of the arrangement of the p-th roots of unity in the columns from different blocks of the matrix Φ_0.

Proposition 3.26. *Let p be some prime, and let Φ_0 be the deterministic measurement matrix* (3.55)*. Let $j, j' \in \{0,\dots,N-1\}$ define two columns in Φ_0, different from the* *zero and the unit column, and such that for some* $k \in \{1, \ldots, M\}$,

$$\Phi_0(k, j) = \Phi_0(k, j'). \tag{3.56}$$

Then, these columns are equal, i.e.

$$\Phi_0(k, j) = \Phi_0(k, j') \text{ for all } k = 1, \ldots, M.$$

Proof. Let us consider a matrix Φ_0 and two of its columns as in the condition of the proposition. Since the columns are not zero columns, according to Definition 3.24 and Theorem 3.23, we can rewrite (3.56) as

$$\exp\left(-2\pi \frac{b_k d_j w_{ac_j}^{-1}}{p}\right) = \exp\left(-2\pi \frac{b_k d_{j'} w_{ac_{j'}}^{-1}}{p}\right).$$

This is equivalent to the congruence

$$b_k d_j w_{ac_j}^{-1} \equiv b_k d_{j'} w_{ac_{j'}}^{-1} \pmod p.$$

Since $b_k \neq 0$, we can divide by it and we will have

$$d_j w_{ac_j}^{-1} \equiv d_{j'} w_{ac_{j'}}^{-1} \pmod p.$$

Now we can take any other $k' \in \{1, \ldots, p-1\}$ and multiply the last congruence by $b_{k'}$ from both sides, that will again lead to equality. Those will form all the other elements of the two columns. □

Coming to the sparse recovery results, if we want to recover x, given the measurements $y = \Phi_0 x$, we can not hope for a mutual coherence result, because $\mu(\Phi_0) = 1$ (we have some identical columns in Φ_0). However, if we focus only on signals which have not arbitrary, but random support, then we have some hope that we can guarantee recovery with a certain probability. A handy tool in this setup is the so-called *statistical RIP* [23], when we want our measurement to act like an isometry not for every sub-choice of columns, but for most of them. We can prove the following result.

Theorem 3.27. *Let p be prime, $M = p - 1, N = p^2 + p$, and let $\Phi := \frac{1}{\sqrt{M}}\Phi_0$, where $\Phi_0 \in \mathbb{C}^{M \times N}$ is the measurement matrix (3.55). Let $T \subset [0, 1, \ldots, N-1]$ be any subset of k indices, chosen uniformly and independently at random. Then, with probability $\varepsilon = r_{p,k}$, we have*

$$\delta_k = \|\Phi_T^* \Phi_T - I_k\|_{2\to 2} = \frac{k-1}{p-1}. \tag{3.57}$$

The probability $r_{p,k}$ equals to

$$r_{p,k} = \frac{\binom{p}{k} p^k}{\binom{p(p+1)}{k}}. \tag{3.58}$$

Proof. First of all, we claim that if we choose the k columns of Φ randomly, uniformly and independently, the probability of having all k columns different and nonzero is equal to $r_{p,k}$ defined in (3.58). Furthermore, for all such subsets T, the Gram matrix $\Phi_T^* \Phi_T$ will always be the same, independently of the choice of the k columns.

For simplification of the reasoning, we make the following observation. The following two sets are equal:

$$A_1 = \{T : \Phi_T(1, j) \neq \Phi_T(1, l) \text{ for all } j \neq l, j, l \in T\},$$
$$A_2 = \{T : \Phi_T(k, j) \neq \Phi_T(k, l) \text{ for all } j \neq l, j, l \in T, k = 1, \ldots, M\}.$$

This is true because of Proposition 3.26: if we assume that $A_1 \neq A_2$, there would be two columns for which $\Phi_T(1, j) \neq \Phi_T(1, l)$, but not all other elements are different. Hence, there exists $k \in \{1, \ldots, p-1\}$, such that $\Phi_T(k, j) = \Phi_T(k, l)$. But it then follows (for zero and unit columns trivially, and elsewise from Proposition 3.26), that

$$\Phi_T(k, j) = \Phi_T(k, l) \text{ for all } k = 1, \ldots, M,$$

which includes $k = 1$ as well. Therefore, we will investigate only the first row of Φ_T, and count the probability of having all coefficients of this row different and nonzero. According to the structure of Φ_0 that was described in Remark 3.25, the elements $\Phi_0(1, j), j = 1, \ldots, N$ can have the following values (possibly in different order)

$$\epsilon_0 = 0,\ \epsilon_j = e^{\frac{2\pi i (j-1)}{p}}, \quad j = 1, \ldots, p,$$

in the following quantities

$$|\{l \in [0, 1, \ldots, N-1] : \Phi(1, l) = \epsilon_j\}| = p, \qquad \text{for all } j = 0, 1, 2, \ldots, p.$$

We need to answer the following combinatorial question:

Given p elements of type ϵ_0, and p elements of each of the types $\epsilon_1, \ldots, \epsilon_p$, from the total of $p^2 + p$ elements, we select independently, uniformly at random k elements. What is the probability that all k elements are different, and ϵ_0 is not included?

Since we do not want to choose an element from type ϵ_0, we need to count the numbers of ways to choose k different elements from p types (p elements are also in each type), which is

$$\binom{p}{k} p^s.$$

We divide this by the total number of choices, $\binom{p^2+p}{k}$, and we obtain the probability $r_{p,k}$.

Let us now suppose that T is chosen such that there are no zero-columns, and all $\Phi(1,j), j \in T$ are different. We can evaluate the elements of the matrix $G_T = \Phi_T^* \Phi_T$ that we will denote by $g_{j,j'}$. Let $k = 1, \ldots, p-1$. We take $b_k = [k]_p$, $c_j = \lfloor j/p \rfloor$, $d_j = [j]_p$. As noted in Remark 3.25, since the columns are nonzero, we know that the corresponding w_{ac_j} and $w_{ac_{j'}}$ are nonzero, and we can write

$$\Phi_T^0(k,j) = \langle \mathbf{1}_{\mathcal{L}_{a,b_k}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c_j,d_j}} \rangle = \exp\left(-2\pi i \frac{b_k d_j w_{ac_j}^{-1}}{p}\right).$$

$$\Phi_T^0(k,j') = \langle \mathbf{1}_{\mathcal{L}_{a,b_k}}, \widehat{\mathbf{1}}_{\mathcal{L}_{c_{j'},d_{j'}}} \rangle = \exp\left(-2\pi i \frac{b_k d_{j'} w_{ac_{j'}}^{-1}}{p}\right).$$

We then consider the following cases.

Case $j = j'$: We have

$$g_{j,j} = \Phi_T^* \Phi_T(j,j') = \frac{1}{p-1} \sum_{k=1}^{p-1} e^{-2\pi i \frac{b_k d_j w_{ac_j}^{-1}}{p}} e^{2\pi i \frac{b_k d_j w_{ac_j}^{-1}}{p}} = 1.$$

Case $j \neq j'$:

$$\begin{aligned}
g_{j,j'} = \Phi_T^* \Phi_T(j,j) &= \frac{1}{p-1} \sum_{k=1}^{p-1} e^{-2\pi i \frac{b_k d_j w_{ac_j}^{-1}}{p}} e^{2\pi i \frac{b_k d_{j'} w_{ac_{j'}}^{-1}}{p}} \\
&= \frac{1}{p-1} \left(\sum_{k=0}^{p-1} e^{2\pi i \frac{b_k d_j w_{ac_j}^{-1}}{p}} e^{-2\pi i \frac{b_k d_{j'} w_{ac_{j'}}^{-1}}{p}} - 1 \right) \\
&= \frac{1}{p-1} \left(\sum_{k=0}^{p-1} e^{2\pi i \frac{b_k \left(d_j w_{ac_j}^{-1} - d_{j'} w_{ac_{j'}}^{-1}\right)}{p}} - 1 \right) \\
&= \frac{1}{p-1} \left(p \delta_p \left(d_j w_{ac_j}^{-1} - d_{j'} w_{ac_{j'}}^{-1} \right) - 1 \right).
\end{aligned}$$

Since we have assumed that the columns are different, the difference $d_j w_{ac_j}^{-1} - d_{j'} w_{ac_{j'}}^{-1}$ is nonzero by Proposition 3.26, and therefore $\delta_p \left(d_j w_{ac_j}^{-1} - d_{j'} w_{ac_{j'}}^{-1} \right) = 0$.

Summarizing, G_T has the following elements:

$$g_{j,j'} = \begin{cases} 1, & \text{if } j = j', \\ -\frac{1}{p-1}, & \text{elsewise.} \end{cases}$$

Therefore, we write

$$G_T = I_k + B_T,$$

where I_k is the identity matrix of size k, and B_T is a matrix with zeros on the diagonal, and $-\frac{1}{p-1}$ off the diagonal. In our case, $\Phi_T^*\Phi_T - I_k = B_T$, and under the constraint that all k columns are different and nonzero, B_T has the same structure regardless of the choice of the subset T. Since we know the elements of this matrix explicitly, we can compute

$$\|B_T\|_{2\to 2} = \frac{k-1}{p-1} := \delta,$$

and $\delta < 1$, when $k < p$. □

In Figure 3.5 we see the behavior of the $r_{p,k}$ as the sparsity changes from 1 to p, for four different values of p. As p increases, the desired value of the constant δ_k will be satisfied with high probability for larger values of k.

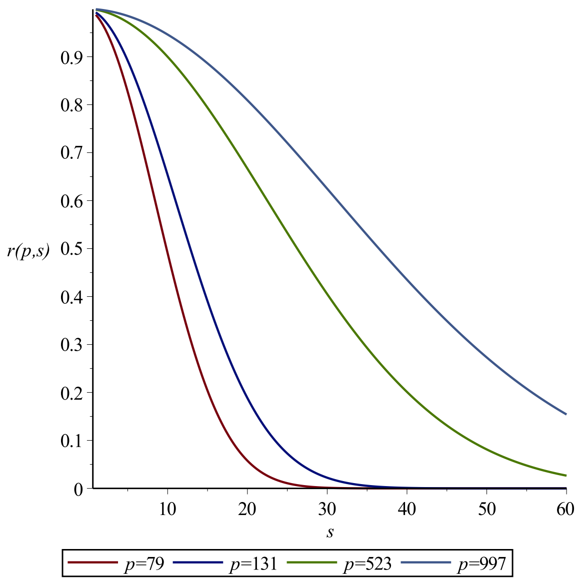


Figure 3.5: Probability of having all k randomly chosen columns of Φ different.

3.5.2 Numerical Experiments on Recovery of Lines

In this section we give some numerical results on the ℓ_1 minimization problem which we were investigating, i.e.,

$$\min_x \|x\|_1 \text{ subject to } y = M\widehat{\mathcal{L}}x. \tag{3.59}$$

We choose $p = 41$, hence the size of the signal is $N = p^2 = 1681$. We fix the number of measurements, $m = 40$ and $m = 100$ for each of the plots in Figure 3.6. Then, for fixed sparsity k, we generate sparse signals $z_0 \in \mathbb{R}^N$ with k nonzero elements, which make the sparse represenation of k discrete lines, $x_0 = \widehat{\mathcal{L}} z_0$. We measure the signal x_0 with three different matrices:

(i) M-partial Fourier. From the Fourier matrix of size $N \times N$, we pick m rows independently at random and normalize the obtained matrix by $\frac{1}{\sqrt{m}}$.

(ii) M-complex random. This matrix is of size $m \times N$ and each value is drawn randomly and uniformly from the complex standard normal distribution.

(iii) M-lines. This is the matrix consisting of m discrete lines, chosen uniformly and independently at random.

For every sparsity level, we try to recover x_0 by solving (3.59) using CVX [68], and count a recovery as successful if the normalized squared error was less than 10^{-4}. We repeat this experiment $T = 100$ times for each measurement matrix, and plot the recovery rate in Figure 3.6. We see that the complex random matrices give the best recovery rate,

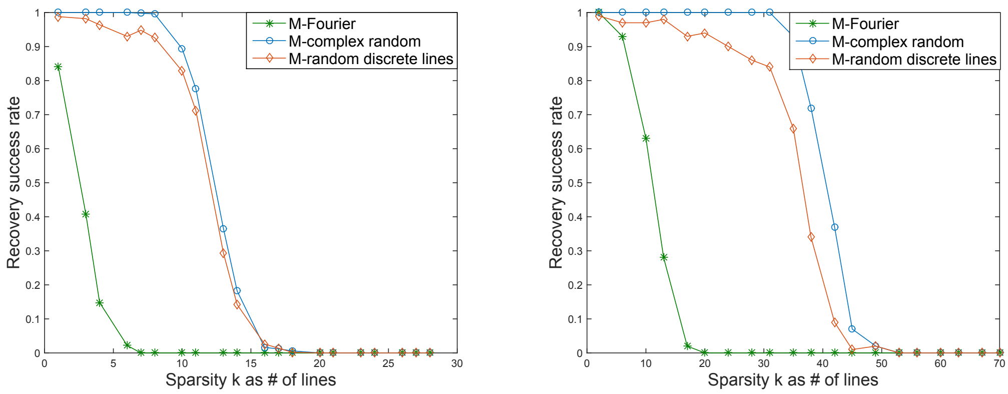


Figure 3.6: Recovery of union of discrete lines via ℓ_1 minimization.

but also that using lines for measurements is a reasonable idea which also gives decent results, especially when the number of measurements is small.

Chapter 4

Sparse Recovery with Difference Sets

4.1 Introduction

We move now from Eucledean lines to lines in a projective plane. Namely, we will consider characteristic functions of the so-called *difference sets*, and instead of collection of translations and rotations, we will have translations and modulations (frequency shifts), i.e. a Gabor system. We will investigate the frame properties of this construction, and also consider the problem of sparse recovery from linear measurements relates in this setting. This chapter is organized as follows. In Section 4.2, after introducing the main objects and their basic properties, we start with investigation of the mutual coherence of the Gabor frame generated by a characteristic function of a difference set. We provide a formula which depends on the parameters of the difference sets and study the question of achieving the Welch bound. In Section 4.3 we switch to the properties of the Gabor-like fusion frame, generated by a difference set. We prove three important properties of this construction: tightness, equidistance, and optimal sparsity. At the end of each section, we describe the mathematical models of the corresponding sparse recovery problem and provide numerical experiments to demonstrate the effectiveness of the proposed constructions in solving it.

4.2 Gabor Systems Generated by Difference Sets

We start with the definition of difference sets, which comes from combinatorial design theory [45], but have at the same time strong connections to finite projective geometry [15].

Definition 4.1. A subset $\mathcal{K} = \{u_1, \ldots, u_K\}$ of $\mathbb{Z}_N$ is called an (N, K, λ) *difference set*, if the $K(K-1)$ differences

$$(u_k - u_l) \mod N, \quad k \neq l,$$

take all possible nonzero values $1, 2, \ldots, N-1$, with each value exactly λ times.

Example 4.1. Let $N = 7$. The subset $\mathcal{K} = \{1, 2, 4\}$ is then a $(7, 3, 1)$ difference set. We can check this by considering all possible differences modulo 7, displayed in the following diagram:

–	1	2	4
1	–	6	4
2	1	–	5
4	3	2	–

This confirms that indeed every value from 1 to 6 appears exactly one time.

In order to see the geometric structure of the difference sets, let us consider the set $\mathcal{K}$ and all its translations. Taking a characteristic functions as a vector in $\mathbb{R}^7$ for every set (with numeration starting from 0), we get

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

In Figure 4.1 we have depicted all translations of the line $\{1, 2, 4\}$ and we see that we have obtained a finite projective plane with 7 lines and 7 points, with 3 points on every line and 3 lines through every point. This example is also known as the Fano plane.

For many other examples of difference sets, parameters for which they they exist or not, and open questions, see the La Jolla Difference Set Repository[1].

Recall that a *Gabor system* with window $g \in \mathbb{C}^N$ is the collection

$$\Phi_g = \{M_j T_k g\}_{j,k=0}^{N-1},$$

where $M_j g(n) = e^{\frac{2\pi i j n}{N}} g(n)$ is the modulation (or frequency-shift) operator and $T_k g(n) = g(n-k)$ for all $n = 0, \ldots, N-1$ is the translation (or time-shift) operator. We emphasize

[1] http://www.ccrwest.org/ds.html

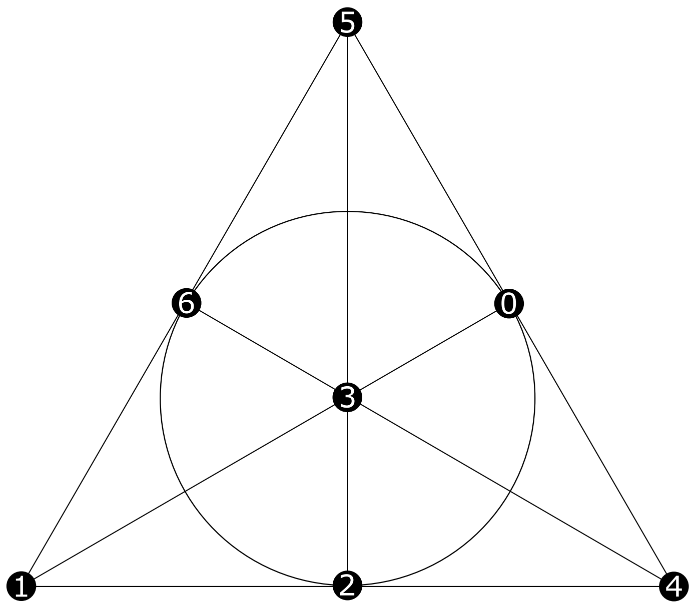


Figure 4.1: Fano plane numerated by the translations of the difference set $\mathcal{K} = \{1, 2, 4\}$.

that all the operations made in the index set $\{0, \ldots, N-1\}$ are in the sense of the group $\mathbb{Z}_N$, that is, modulo N. For every $g \neq 0$, the Gabor system is actually an $N\|g\|^2$-tight frame [87]. The coherence of this frame, however, depends strongly on the properties of the generator. It is therefore of interest for many applications to search for "optimal" generators.

We will now be investigating the Gabor frame (and later its generalization to a fusion frame), which is generated by a characteristic function of a difference set. Given a difference set $\mathcal{K}$ with parameters (N, K, λ) we denote by $\chi_{\mathcal{K}} \in \mathbb{C}^N$ its characteristic function:

$$\chi_{\mathcal{K}}(j) = \begin{cases} 1, & \text{if } j \in \mathcal{K}, \\ 0, & \text{if } j \notin \mathcal{K}. \end{cases}$$

We next note some basic, but important properties of difference sets.

Proposition 4.2. *Let $\mathcal{K}$ be a difference set with parameters (N, K, λ). Then the following is true:*

(i) $K(K-1) = \lambda(N-1)$,

(ii) $\lambda \leq K$,

(iii) $|\widehat{\chi_{\mathcal{K}}}(j)|^2 = K - \lambda, \quad$ *for all* $j \in \mathbb{Z}_N \backslash \{0\}$,

(iv) $\widehat{\chi_{\mathcal{K}}}(0) = K$.

Proof. The claim in (i) comes just from a counting argument: On the one hand, there exist $K(K-1)$ differences in total, and on the other hand, there are $N-1$ numbers that need to appear λ times.

Once we have this, for (ii) we need to check that $\frac{K(K-1)}{N-1} \leq K$. This inequality is equivalent to $K(K-1) \leq K(N-1)$, which is always true since $K \leq N$.

Finally, for the Fourier transform, we evaluate

$$|\widehat{\chi_{\mathcal{K}}}(j)|^2 = \widehat{\chi_{\mathcal{K}}}(j)\overline{\widehat{\chi_{\mathcal{K}}}(j)} = \sum_{k,k' \in \mathcal{K}} e^{\frac{-2\pi i k j}{N}} e^{\frac{2\pi i k' j}{N}} = \sum_{k,k' \in \mathcal{K}} e^{\frac{-2\pi i (k-k')j}{N}} = \sum_{k \in \mathcal{K}} 1 + \sum_{\substack{k,k' \in \mathcal{K}, \\ k \neq k'}} e^{\frac{-2\pi i (k-k')j}{N}}$$

$$= K + \lambda \sum_{\ell=1}^{N-1} e^{\frac{-2\pi i j \ell}{N}} = K + \lambda \left(\sum_{\ell=0}^{N-1} e^{\frac{-2\pi i j \ell}{N}} - 1 \right) = K - \lambda, \text{ when } j \neq 0.$$

For $j = 0$, we have $\widehat{\chi_{\mathcal{K}}}(0) = K$, proving (iv). □

Let $\mathcal{K}$ be a difference set with parameters (N, K, λ) and consider the normalized vector $v := \frac{\chi_{\mathcal{K}}}{\|\chi_{\mathcal{K}}\|} = \frac{\chi_{\mathcal{K}}}{\sqrt{K}} \in \mathbb{C}^N$. We will denote by $\Phi_{\mathcal{K}}$ the Gabor system generated by v,

$$\Phi_{\mathcal{K}} = \Phi_v = \{M_j T_k v\}_{j,k=0}^{N-1}. \tag{4.1}$$

For short, we will call $\Phi_{\mathcal{K}}$ the Gabor system generated by $\mathcal{K}$.

Consider the $N \times N^2$ matrix whose columns are the elements of the Gabor system (4.1). We also denote this matrix by $\Phi_{\mathcal{K}}$. Further, we write $\Phi_{\mathcal{K}}$ as a block matrix,

$$\Phi_{\mathcal{K}} = \begin{bmatrix} B_0 & B_1 & \dots & B_{N-1} \end{bmatrix}, \tag{4.2}$$

where each B_k is a square submatrix of size $N \times N$ with columns of fixed translation, i.e.,

$$B_k = \begin{bmatrix} M_0 T_k v & M_1 T_k v & M_2 T_k v & \dots & M_{N-1} T_k v \end{bmatrix}.$$

Example 4.2. Let $N = 7$, and let $\omega = e^{2\pi i/7}$. If we consider the difference set from Example 4.1, the corresponding matrix $\Phi_{\mathcal{K}}$, with not normalized columns for simplicity, will have the form

$$\left[\begin{array}{ccccccc|ccccccc|c|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 & \dots & 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 \\
1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

4.2.1 Coherence Properties

Going back to the general construction $\Phi_{\mathcal{K}}$ and the investigation of its coherence, we note that the Gram matrix of $\Phi_{\mathcal{K}}$, which is defined as $G = \Phi_{\mathcal{K}}^* \Phi_{\mathcal{K}}$, is closely related to the mutual coherence. Namely,

$$\mu(\Phi) = \max_{i \neq j} |G(i,j)| . \tag{4.3}$$

For our Gabor system, using the notation from (4.2), the Gram matrix can be written in the block form

$$G = \begin{bmatrix} B_0^* \\ B_1^* \\ \cdots \\ B_{N-1}^* \end{bmatrix} \begin{bmatrix} B_0 & B_1 & \cdots & B_{N-1} \end{bmatrix} = \begin{bmatrix} B_0^* B_0 & B_0^* B_1 & \cdots & B_0^* B_{N-1} \\ B_1^* B_0 & B_1^* B_1 & \cdots & B_1^* B_{N-1} \\ & & \cdots & \\ B_{N-1}^* B_0 & B_{N-1}^* B_1 & \cdots & B_{N-1}^* B_{N-1} \end{bmatrix} . \tag{4.4}$$

We will next state a property of the diagonal blocks in G, which will later turn out to be useful.

Proposition 4.3. *Under the notation given above, we have that*

$$|B_k^* B_k(j,\ell)| = \begin{cases} \sqrt{\frac{N-K}{K(N-1)}}, & \textit{if } j \neq \ell, \\ 1, & \textit{if } j = \ell, \end{cases}$$

for all $k, j, \ell = 0, \ldots, N-1$. *In particular, the diagonal blocks* $B_0^* B_0, \ldots, B_{N-1}^* B_{N-1}$ *are all equal in absolute value.*

Proof. We will first prove that any entry of the blocks $B_k^* B_k$, $k = 1, \ldots, N-1$ is equal in absolute value to the corresponding one in the first block $B_0^* B_0$. Let k be some element from $\{1, 2, \ldots, N-1\}$. Using the definition of B_k and the basic properties of translation and modulation operators, we have

$$\begin{aligned} |B_k^* B_k(j,\ell)| &= |\langle M_\ell T_k v, M_j T_k v \rangle| = |\langle e^{\frac{-2\pi i k \ell}{N}} T_k M_\ell v, e^{\frac{-2\pi i k j}{N}} T_k M_j v \rangle| \\ &= |\langle M_\ell v, M_j v \rangle| = |B_0^* B_0(j,\ell)|, \end{aligned}$$

for all $j, \ell = 0, 1, \ldots, N-1$. Now, according to the definition of B_0 and Proposition 4.2 (iii)-(iv),

$$|B_0^* B_0(j,\ell)| = \frac{1}{K} \left| \sum_{k \in \mathcal{K}} e^{\frac{2\pi i (\ell - j) k}{N}} \right| = \frac{1}{K} |\widehat{\chi_{\mathcal{K}}}(j-\ell)| = \begin{cases} \frac{1}{K}\sqrt{K-\lambda} = \sqrt{\frac{N-K}{K(N-1)}} & \text{if } j \neq \ell, \\ 1 & \text{if } j = \ell. \end{cases}$$

□

Remark 4.4. Note that, for each $k = 0, \dots, N-1$, Proposition 4.3 says that the collection of N vectors $\{M_0T_kv, M_1T_kv, \dots, M_{N-1}T_kv\}$ which spans a K-dimensional subspace of $\mathbb{C}^N$, has coherence achieving the Welch bound. Therefore, by [118, Theorem 2.3], $\{M_0T_kv, M_1T_kv, \dots, M_{N-1}T_kv\}$ is an ETF for the subspace it spans, for every $k = 0, \dots, N-1$ and it has frame bound $\frac{N}{K}$. Note, that this result was proven in [128, Theorem 1], where equiangular tight frames are called maximum-Welch-bound-equality (MWBE) codebooks. It is unclear at this point, however, what the absolute values of the entries in the off diagonal blocks are. As we will see in the next theorem, they will depend on the value of λ, and thus the mutual coherence of $\Phi_{\mathcal{K}}$ will depend on the parameters of the difference set $\mathcal{K}$.

Theorem 4.5. *Let $\Phi_{\mathcal{K}}$ be a Gabor system generated by an (N, K, λ) difference set $\mathcal{K}$. Then,*

$$\mu(\Phi_{\mathcal{K}}) = \begin{cases} \sqrt{\frac{N-K}{K(N-1)}}, & \text{if } \lambda = 1, \\ \max\{\frac{K-1}{N-1}, \sqrt{\frac{N-K}{K(N-1)}}\}, & \text{if } \lambda \neq 1. \end{cases}$$

Proof. According to the described block structure of $\Phi_{\mathcal{K}}$ and by (4.3), the mutual coherence is

$$\mu(\Phi_{\mathcal{K}}) = \max\{\max_{\substack{r\neq q\\ j,\ell}} |B_r^*B_q(j,\ell)|, \max_{j\neq\ell} |B_0^*B_0(j,\ell)|\}.$$

We have already investigated the diagonal blocks in Proposition 4.3. Next we write explicitly the elements of the Gram matrix G in the off-diagonal blocks as

$$\begin{aligned} |B_r^*B_q(j,\ell)| &= |\langle M_\ell T_q v, M_j T_r v\rangle| = |\langle M_\ell v, T_{(r-q)} M_j v\rangle| \\ &= \left|\sum_{k=0}^{N-1} v(k) e^{\frac{2\pi i k\ell}{N}} \overline{v(k-(r-q))e^{\frac{-2\pi i (k-(r-q))j}{N}}}\right| = \frac{1}{K}\left|\sum_{\substack{k\in\mathcal{K}\\ k-(r-q)\in\mathcal{K}}} e^{\frac{2\pi i(k\ell+(r-q)j-kj)}{N}}\right| \\ &= \frac{1}{K}\left|\sum_{\substack{k\in\mathcal{K}\\ k-(r-q)\in\mathcal{K}}} e^{\frac{2\pi i(\ell-j)k}{N}}\right|. \end{aligned} \tag{4.5}$$

We can simplify this expression further dependent on the properties of the difference set $\mathcal{K}$. We thus consider two separate cases.

Case $\lambda = 1$. In the final sum (4.5), in the case $q \neq r$, since $\lambda = 1$, there can be only one $k \in \mathcal{K}$, such that k and $k-(r-q)$ are both in $\mathcal{K}$. This is because there is only one way to write $r-q$ as a difference of elements in $\mathcal{K}$, and $k-(k-(r-q))$ is such a difference. Thus, we can continue (4.5) to obtain

$$|B_r^* B_q(j,\ell)| = \frac{1}{K}\left|e^{\frac{2\pi i(\ell-j)k}{N}}\right| = \frac{1}{K}, \quad \text{when } q \neq r.$$

Further, by Proposition 4.3, $|B_0^* B_0(j,\ell)| = \sqrt{\frac{N-K}{K(N-1)}}$. Therefore, when $\lambda = 1$, $\mu(\Phi) = \max\{\frac{1}{K}, \sqrt{\frac{N-K}{K(N-1)}}\} = \sqrt{\frac{N-K}{K(N-1)}}$.

Case $\lambda \neq 1$. We will estimate $\max_{r\neq q,j,\ell} |B_r^* B_q(j,\ell)|$. For fixed $r \neq q$, since $\mathcal{K}$ is a (N,K,λ) difference set we have that $\{k \in \mathcal{K} : k-(r-q) \in \mathcal{K}\}$ is a set of exactly λ elements. Then, from (4.5) it follows that for all $j, l = 0, \ldots, N-1$,

$$|B_r^* B_q(j,\ell)| \leq \frac{\lambda}{K}.$$

Note that when $j = \ell$, also by (4.5), $|B_r^* B_q(j,j)| = \frac{\lambda}{K}$. Thus $\max_{r\neq q,j,\ell} |B_r^* B_q(j,\ell)| = \frac{\lambda}{K}$. Now we just use the fact that $K(K-1) = \lambda(N-1)$ to rewrite $\frac{\lambda}{K}$ as $\frac{K-1}{N-1}$. $\square$

Remark 4.6. Although the value $\sqrt{\frac{N-K}{K(N-1)}}$ was optimal for the case of N vectors in K dimensional space, for the full Gabor frame $\Phi_{\mathcal{K}}$ the optimal Welch bound will be different. Namely, for a system of N^2 vectors in N dimensional space, the Welch bound is

$$\mu^* = \sqrt{\frac{N^2-N}{N(N^2-1)}} = \sqrt{\frac{1}{N+1}}.$$

In Table 4.1 we present several families of difference sets and the mutual coherence of the corresponding Gabor systems. For more details on the construction of these difference sets see [128]. From this table it can be seen that the mutual coherence is not as close to the optimal bound, as it was established, for example, for the Alltop vectors in [118]. It is still going asymptotically to zero as the dimension grows for the Singer family, and, as we will see in the numerical experiments, the performance of the difference sets and the Alltop vectors for the sparse recovery problem are almost identical, making our construction still interesting for applications.

Reaching the Welch bound is important not only for signal processing, but actually for many other fields, including quantum mechanics, where ETFs of N^2 elements in dimension N are known as SIC-POVMs (symmetric informationally complete positive-operator valued measure). It is in fact an open problem whether they exist for every dimension (Zauner conjecture [129]). Particular examples are also difficult to construct, and known only for certain values of N.

Table 4.1: Families of difference sets and the mutual coherence of the corresponding Gabor frames

Family	(N, K, λ)	$\mu(\Phi_{\mathcal{K}})^2$	μ^{*2}
Singer, $d = 2$	$\left(q^2 + q + 1, q + 1, 1\right)$	$\dfrac{q}{(q+1)^2}$	$\dfrac{1}{q^2 + q + 2}$
Singer, $d > 2$	$\left(\dfrac{q^{d+1} - 1}{q - 1}, \dfrac{q^d - 1}{q - 1}, \dfrac{q^{d-1} - 1}{q - 1}\right)$	$\dfrac{(q^d - q)^2}{q^2(q^d - 1)^2}$	$\dfrac{q - 1}{q^{d+1} + q - 2}$
Quadratic, $q > 7$	$\left(q, \dfrac{q-1}{2}, \dfrac{q-3}{4}\right)$	$\dfrac{(q-3)^2}{4(q-1)^2}$	$\dfrac{1}{q+1}$
Quartic, $p < 57$	$\left(p, \dfrac{p-1}{4}, \dfrac{p+3}{16}\right)$	$\dfrac{3p+1}{(p-1)^2}$	$\dfrac{1}{p+1}$
Quartic, $p > 57$	$\left(p, \dfrac{p-1}{4}, \dfrac{p+3}{16}\right)$	$\dfrac{(p-5)^2}{16(p-1)^2}$	$\dfrac{1}{p+1}$

From our reasoning above, one might conclude that it is possible to get a Gabor ETF by choosing a difference set with optimal values of the parameters K and N, such that $\mu(\Phi) = \mu^*$. Constructing a difference set with prescribed parameters is however itself a very difficult and open problem in combinatorial design theory. It is also directly connected to the optimal Grassmannian packing problem [44]. Up to now only constructions with certain pairs of parameters (N, K) are known. In any case, we will show that, unfortunately, combinations of parameters of difference sets such that corresponding Gabor system achieves the Welch bound can not exist. Such hope was probably too good to be true, since for illustration, for $N = 17$ an analytical example of a generator which gives an ETF of N^2 lines in dimension N was provided in [42], but it took over 40 pages to write its expression down.

Proposition 4.7. *Let $N > 3$. Then, there can not exist an (N, K, λ) difference set such that the corresponding Gabor system $\Phi_{\mathcal{K}}$ will form an equiangular tight frame.*

Proof. By Theorem 4.5, the mutual coherence can take only one of the two possible values:

$$\sqrt{\frac{N-K}{K(N-1)}} \text{ or } \frac{K-1}{N-1}.$$

We will now consider these two cases separately.

Let us first assume that $\mu(\Phi) = \sqrt{\frac{N-K}{K(N-1)}}$. If we want to reach the Welch bound, we need to solve $\frac{N-K}{K(N-1)} = \frac{1}{N+1}$, which implies $K = \frac{N+1}{2}$. From Proposition 4.2, we know that the corresponding λ in this case is $\frac{N+1}{4}$. However, for this set of parameters $\left(N, \frac{N+1}{2}, \frac{N+1}{4}\right)$, when $N > 3$, it is easy to check that the mutual coherence is actually $\max\left\{\sqrt{\frac{N-K}{K(N-1)}}, \frac{K-1}{N-1}\right\} = \frac{K-1}{N-1} = \frac{1}{2}$, and thus far from the Welch bound $\mu^* = \sqrt{\frac{1}{N+1}}$. It is interesting to note that, when $N = 3$, potential difference sets with parameters $(3, 2, 1)$ will achieve the Welch bound. An example for such a difference set is $\mathcal{K} = \{0, 1\}$. Its characteristic function $\tilde{g} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ (which is a member of a continuous family of generators presented in [42]) forms a Gabor frame of 9 elements which is an ETF.

Let us next see what happens if we want to reach the Welch bound with the other value, i.e. to have $\left(\frac{K-1}{N-1}\right)^2 = \frac{1}{N+1}$. For positive K this equation is solved by $K = \frac{N+1+\sqrt{N^3-N^2-N+1}}{N+1}$. But for such K and $N > 3$, the mutual coherence will actually be $\sqrt{\frac{N-K}{K(N-1)}}$ instead of $\frac{K-1}{N-1}$, and thus again we can not reach the Welch bound. Note that when $N = 3$, K in the obtained solution is again 2. Actually, $\sqrt{\frac{N-K}{K(N-1)}} = \frac{K-1}{N-1} = \frac{1}{\sqrt{N+1}} = \frac{1}{2}$, and thus we again achieve the Welch bound. $\square$

4.2.2 Compressed Sensing with Gabor Frames

We aim to recover an unknown sparse (having small number of nonzero entries) vector $x \in \mathbb{C}^{N^2}$ from its linear measurements $y = \Phi_g x$, where Φ_g is the $N \times N^2$ time-frequency shifts matrix generated by $g \in \mathbb{C}^N$. This looks like the classical compressed sensing setup [27], only with a specific deterministic measurement matrix. We can also view Φ_g as a dictionary, and x as having a sparse representation in a Gabor system.

In [100], a different prospective of the same problem is given: *sparse matrix identification*. Namely, here one is interested in matrices Γ which have a k-sparse representation in the matrix dictionary $\Phi = \{\Phi_j\}_{j=1}^M$,

$$\Gamma = \sum_{j=1}^{M} x_j \Phi_j, \text{ with } \|x\|_0 = k.$$

The measurement process can be modelled as an action of Γ on a test signal $h \in \mathbb{C}^N$,

$$\Gamma h = \Big(\sum_{j=1}^{M} x_j \Phi_j\Big) h = (\Phi_1 h | \Phi_2 h | \dots | \Phi_M h)\, x = (\Phi h) x.$$

The task is then to recover x (and by this the sparse representation of Γ) from the measurements $(\Phi h)x$. It is left to notice, that in the case of Φ_j being the time-frequency shift matrices, and h being the generator, we have exactly the first problem described in the beginning.

To recover the sparse x we will traditionally use Basis Pursuit (BP) [40], which is the convex problem given by

$$\min \|x\|_1 \text{ subject to } \Phi_g\, x = y. \tag{4.6}$$

We want to compare the results of recovery of sparse vectors using three different types of generators for Φ_g : Alltop sequences [5], complex random vectors and difference sets:

1. Alltop sequence. $g_A(j) = \frac{1}{\sqrt{N}} e^{2\pi i j^3/N}$, for prime $N \geq 5$.
2. Random vector. $g_R(j) = \frac{1}{\sqrt{N}}\epsilon_j$, where ϵ_j are independent and uniformly distributed on the torus.
3. Difference set. $g_K = \frac{1}{\sqrt{K}}\chi_{\mathcal{K}}$ for some (N, K, λ)-difference set $\mathcal{K}$.

We have chosen Alltop and random generators, since their Gabor frames have already proven to be successful for sparse recovery both theoretically and numerically in [100]. The theoretical guarantees come from the near optimality of the mutual coherence of these Gabor systems, more specifically, the following results were proven.

Theorem 4.8 ([100]). *1. Let N be prime and g_A be the Alltop window defined in 1.. If $k < \frac{\sqrt{N}+1}{2}$ then Basis Pursuit recovers all matrices $\Gamma \in \mathbb{C}^{N\times N}$ having a k-sparse representation, with respect to the time-frequency shift dictionary Φ_{g_A}.*

2. Let N be even and choose g_R to be the random unimodular window in 2.. Let $t > 0$ and suppose

$$k \leq \frac{1}{4}\sqrt{\frac{N}{2\log N + \log 4 + t}} + \frac{1}{2}. \tag{4.7}$$

Then with probability of at least $1 - e^{-t}$ Basis Pursuit recovers all matrices $\Gamma \in \mathbb{C}^{N\times N}$ having a k-sparse representation with respect to the time-frequency shift dictionary Φ_{g_R}.

As noted in [100], similar result to Theorem 4.8–2. hold also for N odd, and in both cases recovery is stable under noisy measurements. We now add also third part to those results, when the generator is a characteristic funcion of a difference set.

Theorem 4.9. *Let $\mathcal{K}$ be a difference set from the Singer family, with parameters (N, K, λ), where $N = q^2 + q + 1$ and $q = p^r$ is a power of prime. Let further g_K be its normalized characteristic function, as defined in 3.. If*

$$k < \frac{1}{2}(\sqrt{q} + \frac{1}{\sqrt{q}} + 1), \tag{4.8}$$

then Basis Pursuit recovers all $\Gamma \in \mathbb{C}^{N\times N}$ *matrices having* k*-sparse representation, with respect to the time-frequency shift dictionary* Φ_{g_K}.

Proof. We will show at first that the given difference set vector g_K has mutual coherence $\frac{\sqrt{q}}{q+1}$. From there, using Theorem 2.5 from Chapter 1, the claim will immediatelly follow. A description of the construction of Singer difference sets is given, for example, in [128]. What we need is only the parameters of this set, in our case taken with $d = 2$ and equal to

$$N = q^2 + q + 1, \quad K = q + 1, \quad \lambda = 1.$$

We can now use Theorem 4.5 to calculate the mutual coherence of Φ_{g_K},

$$\mu(\Phi_{g_K}) = \sqrt{\frac{N-K}{K(N-1)}} = \sqrt{\frac{q^2+q+1-(q+1)}{(q+1)(q^2+q)}} = \frac{\sqrt{q}}{q+1}.$$

It is left only to check what is the value of $\frac{1}{2}(\mu^{-1}+1)$ — we know from Theorem 2.5 that for sparsity less than this value, recovery is possible. We see that $\left(\frac{\sqrt{q}}{q+1}\right)^{-1} = \sqrt{q} + \frac{1}{\sqrt{q}}$, and from here the claim follows. □

From these three results, we can see that unlike the random and Alltop vectors, which guarantee recovery for all vectors which are of the order of $N^{\frac{1}{2}}$ sparse, Gabor systems based on difference sets reach only $N^{\frac{1}{4}}$ sparsity level.

We would like to see next how the difference sets compare to Alltop and random vectors numerically, despite their theoretical non-optimal coherence.

4.2.3 Numerical Experiments

To solve the Basis Pursuit problem we use CVX, a package for specifying and solving convex programs [68]. In the numerical experiment in Figure 4.2, we have chosen $N = 43$ (a prime which gives 3 modulo 4, suitable for difference sets of the Quadratic family that we will use). For fixed sparsity level k, we generate a random k-sparse vector $x \in \mathbb{C}^{N^2}$, with k nonzero values $x(j) = r_j \exp(2\pi i\theta_j)$, where r_j is drawn independently from the standard normal distribution $\mathcal{N}(0,1)$, and θ_j is drawn independently and uniformly from $[0,1)$. Then, we measure this signal with each of the three Gabor frames, and try to recover it by BP. We count the recovery as successful, if the normalized squared error was smaller than 10^{-6}. For every k we repeat this experiment $T = 500$ times, and plot the successful recovery rates in Figure 4.2. What we observe here is that all three generators have almost identical recovery rate. The complex random generator performs the best since we choose a different realization for g_R at every experiment, and the difference sets are slightly better then the Alltop in the transition level of k.

The fact that the mutual coherence can not always capture the desired properties of the Gabor frame was noted in [9], where *average coherence* was introduced. To guarantee a successful recovery via BP, certain relations between the average and the mutual coherence need to be satisfied. One can show that those particular conditions are also not satisfied by the Gabor frame generated by difference sets. Finding the correct theoretical explanation of this successful behavior in numerical experiments is an interesting question, and we leave it for future investigation.

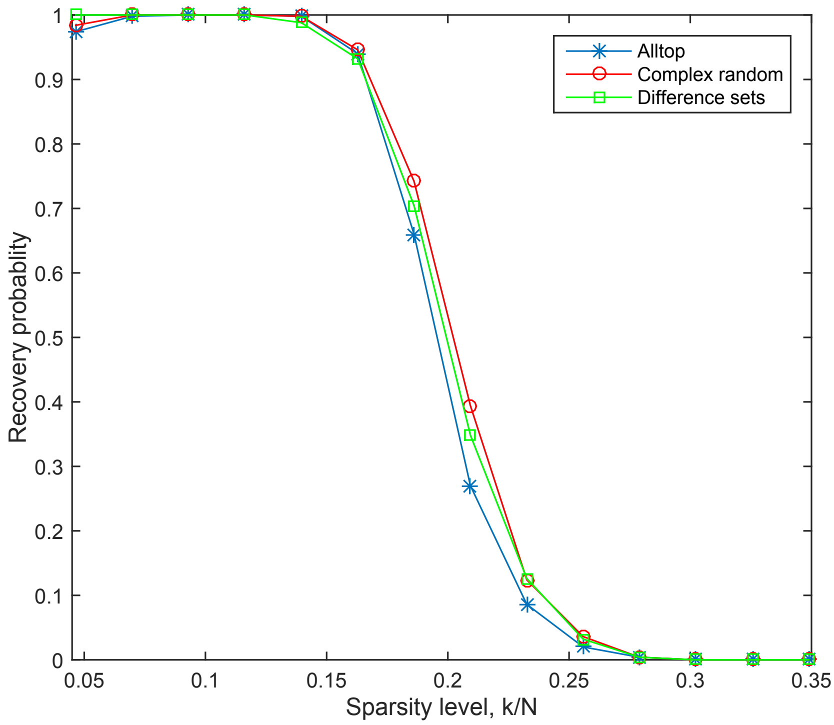


Figure 4.2: Sparse recovery from Gabor measurements.

4.3 Gabor Fusion Frames Generated by Difference Sets

We now move to the second part of this chapter, where we aim to investigate our collection of time-frequency shifts of a difference set from a perspective of fusion frames, which are collections of subspaces and generalize the notion of frames. Constructing fusion frames with prescribed "frame-like" properties is an important and challenging task. We will show how our Gabor system can be seen as a fusion frame, and that it moreover satisfies certain optimality properties which will be discussed further. We now recall the definition of fusion frames [38], in our case considered with all weights equal to one.

Definition 4.10. A family of subspaces $\{\mathcal{W}_i\}_{i=1}^M$ in $\mathbb{C}^N$ is called a *fusion frame* for $\mathbb{C}^N$, if there exist A and B, $0 < A \leq B < \infty$ such that

$$A\|x\|_2^2 \leq \sum_{i=1}^{M} \|P_i(x)\|_2^2 \leq B\|x\|_2^2 \quad \text{for all } x \in \mathbb{C}^N,$$

where for each $i = 0, \dots, N-1$, P_i denotes the orthogonal projection of $\mathbb{C}^N$ onto $\mathcal{W}_i$.

If $A = B$ is possible, then $\{\mathcal{W}_i\}_{i=1}^M$ is called an *A-tight fusion frame.* Tightness is an important property, required for example, for minimization of the recovery error of a random vector from its noisy fusion frame measurements [85]. Among other desirable properties are equidimensionality and equidistance. They provide maximal robustness against erasures of one or more subspaces, and as we will see later, yield optimal Grassmannian packings [85]. Equidimensionality means that all the subspaces $\{\mathcal{W}_i\}_{i=1}^M$ are of the same dimension, while to define equidistant fusion frames, we need the notion of chordal distance.

Definition 4.11. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathbb{C}^N$ with $m := \dim \mathcal{W}_1 = \dim \mathcal{W}_2$ and denote by P_i the orthogonal projection onto $\mathcal{W}_i$, $i = 1, 2$. The *chordal distance* $d_c(\mathcal{W}_1, \mathcal{W}_2)$ between $\mathcal{W}_1$ and $\mathcal{W}_2$ is given by

$$d_c^2(\mathcal{W}_1, \mathcal{W}_2) = m - \mathrm{Tr}[P_1 P_2],$$

where Tr denotes the trace of an operator. Multiple subspaces are called *equidistant*, if they have pairwise equal chordal distance d_c.

It was shown in [85] that equidistant tight fusion frames are optimal Grassmannian packings, where optimality comes from the classical packing problem: For given m, M, N, find a set of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^M$ in $\mathbb{C}^N$ such that $\min_{i \neq j} d_c(\mathcal{W}_i, \mathcal{W}_j)$ is as large as possible. In this case we call $\{\mathcal{W}_i\}_{i=1}^M$ an *optimal packing.* An upper bound is given by the *simplex bound*

$$\frac{m(N-m)M}{N(M-1)}.$$

This is to some extent analogous to the Welch bound from the classical frame theory, and we will see that fusion frames generated by difference sets actually achieve the simplex bound.

We will investigate the family of subspaces arising from Gabor system of difference sets, defined as follows. Let $\mathcal{K}$ be a difference set with parameters (N, K, λ) and let $v = \frac{1}{\sqrt{K}}\chi_{\mathcal{K}}$ be our generator for the Gabor system

$$\Phi_{\mathcal{K}} = \{M_j T_i v\}_{j,i=0}^{N-1}.$$

For every $i = 0, \dots, N-1$, let the subspaces $\mathcal{W}_i$ be defined as

$$\mathcal{W}_i = \operatorname{span}\{M_j T_i v\}_{j=0}^{N-1} = \{x \in \mathbb{C}^N : \operatorname{supp}(x) = \mathcal{K} + i\}. \tag{4.9}$$

We call $\mathcal{W}_{\mathcal{K}} = \{\mathcal{W}_i\}_{i=1}^N$ *a Gabor fusion frame associated to a difference set* $\mathcal{K}$. The fact that this family of subspaces is in fact a fusion frame (and more over tight) will follow from the next proposition.

Proposition 4.12 (Corollary 13.2 in [37]). *Let $\{\mathcal{W}_i\}_{i=1}^M$ be a family of subspaces in $\mathbb{C}^N$. Let $\{\phi_{ij}\}_{j=1}^{J_i}$ be an A-tight frame for $\mathcal{W}_i$ for each i. Then the following conditions are equivalent.*

(i) *$\{\mathcal{W}_i\}_{i=1}^M$ is a C-tight fusion frame for $\mathbb{C}^N$.*

(ii) *$\{\phi_{ij}\}_{i=1,j=1}^{M,J_i}$ is an AC-tight frame for $\mathbb{C}^N$.*

Theorem 4.13. *The family of subspaces $\mathcal{W}_{\mathcal{K}} = \{\mathcal{W}_k\}_{k=0}^{N-1}$ defined in (4.9) is a K-tight fusion frame.*

Proof. This property follows directly from Proposition 4.12. First of all, as noted in Remark 4.4, for every fixed i, $\{M_j T_i v\}_{j=0}^{N-1}$ is a $\frac{N}{K}$-tight (also equiangular) frame for $\mathcal{W}_i$. Also, the full system $\{M_j T_i v\}_{i,j=0}^{N-1}$ is a N-tight frame for $\mathbb{C}^N$. Thus, according to Proposition 4.12, this is equivalent to $\{\mathcal{W}_i\}_{i=0}^{N-1}$ being a K-tight fusion frame for $\mathbb{C}^N$. $\square$

4.3.1 Optimality Properties

When speaking about optimality, we speak about certain desirable properties of a frame (fusion frame), which usually arise from various application problems. For example, in the classical frame theory, as we saw in Subsection 4.2.2, small coherence of a frame applies successful sparse recovery of signals measured with this frame. The Welch bound gives us a barrier how low can one go. Achieving it is already important not only for sparse recovery, but for a variety of other problems, for example in coding theory, or quantum mechanics. Similarly, in the fusion frame theory, there exist a notion of fusion coherence, which guarantees sparse recovery [18] from fusion measurements. But apart from compressed sensing-like problems, and there the question of achieving the so called *simplex bound* arises. Yet another optimality property is *optimally sparse fusion frame*, and in the next subsections we investigate each of them for our Gabor fusion frame generated by difference sets.

4.3.1.1 Equidistant Fusion Frames

We saw that our construction produces tight fusion frames consisting of equi-dimensional subspaces. Next, we will show that they moreover have equal pairwise chordal distance.

Theorem 4.14. *The Gabor fusion frame* $\mathcal{W}_\mathcal{K} = \{\mathcal{W}_k\}_{k=0}^{N-1}$ *associated to an* (N, K, λ) *difference set* $\mathcal{K}$ *is an equidistant fusion frame with*

$$d_c^2 = \frac{K(N-K)}{N-1}.$$

Proof. Let $\mathcal{W}_{i_1}$ and $\mathcal{W}_{i_2}$ be any two different subspaces from (4.9). In order to compute $d_c^2(\mathcal{W}_{i_1}, \mathcal{W}_{i_2})$ we require $\mathrm{Tr}[P_{i_1}P_{i_2}] = \sum_{\ell=0}^{N-1}\langle P_{i_2}e_\ell, P_{i_1}e_\ell\rangle$, where $\{e_\ell\}_{\ell=0}^{N-1}$ is the canonical basis of $\mathbb{C}^N$. For this, first note that

$$P_{i_k}\, e_\ell = \frac{K}{N}\sum_{j=0}^{N-1}\langle e_\ell, T_{i_k}M_j v\rangle T_{i_k}M_j v = \frac{K}{N}\sum_{j=0}^{N-1}\overline{T_{i_k}M_j v(\ell)}\, T_{i_k}M_j v, \quad k = 1, 2.$$

This leads to

$$\begin{aligned}
\sum_{\ell=0}^{N-1}\langle P_{i_2}e_\ell, P_{i_1}e_\ell\rangle &= \frac{K^2}{N^2}\sum_{\ell=0}^{N-1}\langle\sum_{j=0}^{N-1}\overline{M_jT_{i_2}v(\ell)}\, M_jT_{i_2}v, \sum_{j'=0}^{N-1}\overline{M_{j'}T_{i_1}v(\ell)}\, M_{j'}T_{i_1}v\rangle \\
&= \frac{K^2}{N^2}\sum_{\ell=0}^{N-1}\sum_{j=0}^{N-1}\sum_{j'=0}^{N-1}\overline{M_jT_{i_2}v(\ell)}M_{j'}T_{i_1}v(\ell)\langle M_jT_{i_2}v, M_{j'}T_{i_1}v\rangle \\
&= \frac{K^2}{N^2}\sum_{j,j'=0}^{N-1}\langle M_{j'}T_{i_1}v, M_jT_{i_2}v\rangle\langle M_jT_{i_2}v, M_{j'}T_{i_1}v\rangle \\
&= \frac{K^2}{N^2}\sum_{j,j'=0}^{N-1}|\langle M_jT_{i_1}v, M_{j'}T_{i_2}v\rangle|^2 \\
&\overset{(4.5)}{=} \frac{1}{N^2}\sum_{j,j'=0}^{N-1}\left|\sum_{\substack{k\in\mathcal{K}\\ k-(i_2-i_1)\in\mathcal{K}}} e^{\frac{2\pi i(j'-j)k}{N}}\right|^2 = \frac{1}{N^2}\sum_{j,j'=0}^{N-1}|\hat{\chi}_{\mathcal{K}_{i_1-i_2}}(j-j')|^2,
\end{aligned} \tag{4.10}$$

where $\mathcal{K}_{i_1-i_2} = \{k \in \mathcal{K} : k-(i_1-i_2) \in \mathcal{K}\}$. As we have noted before, $\mathrm{card}(\mathcal{K}_{i_1-i_2}) = \lambda$. For fixed j, by Plancherel's Theorem, we have

$$\sum_{j'=0}^{N-1}|\hat{\chi}_{\mathcal{K}_{i_1-i_2}}(j-j')|^2 = \|T_j\hat{\chi}_{\mathcal{K}_{i_1-i_2}}\|^2 = \|\hat{\chi}_{\mathcal{K}_{i_1-i_2}}\|^2 = N\|\chi_{\mathcal{K}_{i_1-i_2}}\|^2 = N\lambda.$$

Now we can go back to the sum (4.10), and get the final result,

$$\sum_{\ell=0}^{N-1} \langle P_{i_2} e_\ell, P_{i_1} e_\ell \rangle = \frac{1}{N^2} \sum_{j=0}^{N-1} N\lambda = \lambda.$$

Notice that this value does not depend on the choice of the subspaces. Thus, taking into account that our subspaces have dimension K, by definition of chordal distance we obtain

$$d_c^2 = K - \mathrm{Tr}[P_1 P_2] = K - \lambda.$$

Finally, by Proposition 4.2 (ii), the claim follows. □

Corollary 4.15. *The Gabor fusion frame $\mathcal{W}_{\mathcal{K}} = \{\mathcal{W}_k\}_{k=0}^{N-1}$ associated to an (N, K, λ) difference set $\mathcal{K}$ is an optimal Grassmannian packing of N K-dimensional subspaces in $\mathbb{C}^N$.*

Proof. By Theorem 4.14, $\mathcal{W}_{\mathcal{K}}$ is a fusion frame of equidimensional subspaces with pairwise equal chordal distances d_c. It was proven in [85, Theorem 4.3], that in this case the fusion frame is tight, if and only if d_c^2 equals the simplex bound. We already know from Theorem 4.13 that our fusion frame is tight, hence the claim follows. We can also check that the simplex bound is achieved. For this set of parameters, the simplex bound equals

$$\frac{K(N-K)N}{N(N-1)} = \frac{K(N-K)}{N-1},$$

and this is exactly d_c^2. Thus, we have an optimal packing. □

4.3.1.2 Optimally Sparse Fusion Frames

The notion of optimally sparse fusion frames was introduced in [34] and means that all subspaces can be seen as spans of orthonormal basis vectors that are sparse in a uniform basis over all subspaces, and thus only few entries are present in the decomposition. This different optimality property is of great practical use when the fusion frame dimensions are large, and low-complexity fusion frame decomposition is desirable. We will show that our Gabor fusion frames defined in (4.9) are also optimally sparse.

Definition 4.16. [34] Let $\{\mathcal{W}_i\}_{i=1}^M$ be a fusion frame for $\mathbb{C}^N$ with $\dim \mathcal{W}_i = m_i$ for all $i = 1, \ldots, M$ and let $\{v_j\}_{j=1}^N$ be an orthonormal basis for $\mathbb{C}^N$. If for each $i \in \{1, \ldots, M\}$, there exists an orthonormal basis $\{\phi_{i,\ell}\}_{\ell=1}^{m_i}$ for $\mathcal{W}_i$ with the property that for each $\ell =$

$1, \ldots, m_i$ there exists a subset $J_{i,\ell} \subset \{1, \ldots, N\}$ such that

$$\phi_{i,\ell} \in \operatorname{span}\{v_j : j \in J_{i,\ell}\} \text{ and } \sum_{i=1}^{M} \sum_{\ell=1}^{m_i} |J_{i,\ell}| = k,$$

we refer to $\{\phi_{i,\ell}\}_{i=1,\ell=1}^{M,m_i}$ as an *associated k-sparse frame.* The fusion frame $\{\mathcal{W}_i\}_{i=1}^{M}$ is called *k-sparse* with respect to $\{v_j\}_{j=1}^{N}$, if it has an associated k-sparse frame and if, for any associated j-sparse frame, we have $k \leq j$.

Definition 4.17. [34] Let $\mathcal{FF}$ be a class of fusion frames for $\mathbb{C}^N$, let $\{\mathcal{W}_i\}_{i=1}^{M} \in \mathcal{FF}$, and let $\{v_j\}_{j=1}^{N}$ be an orthonormal basis for $\mathbb{C}^N$. Then $\{\mathcal{W}_i\}_{i=1}^{M}$ is called *optimally sparse in $\mathcal{FF}$ with respect to* $\{v_j\}_{j=1}^{N}$, if $\{\mathcal{W}_i\}_{i=1}^{M}$ is k_1-sparse with respect to $\{v_j\}_{j=1}^{N}$ and there does not exist a fusion frame $\{\mathcal{V}_i\}_{i=1}^{M} \in \mathcal{FF}$ which is k_2-sparse with respect to $\{v_j\}_{j=1}^{N}$ with $k_2 < k_1$.

Let $\mathcal{FF}(M, m, N)$ be the class of tight fusion frames in $\mathbb{C}^N$ which have M subspaces, each of dimension m. One example of optimally sparse fusion frames in this class is the spectral tetris construction (STFF), explained in more details in [34] and [37, Chapter 13]. For this fusion frame the following theorem is known.

Theorem 4.18. *[34] Let N, M, and m be positive integers such that $\frac{Mm}{N} \geq 2$ and $\lfloor \frac{Mm}{N} \rfloor \leq M - 3$. Then the tight fusion frame STFF (M, m, N) is optimally sparse in the class $\mathcal{FF}(M, m, N)$ with respect to the canonical basis in $\mathbb{C}^N$.*

In particular, this tight fusion frame is $mM + 2(N - \gcd(Mm, N))$-sparse with respect to the canonical basis.

We will now show that the Gabor fusion frames generated by difference sets are also optimally sparse in the corresponding class of tight fusion frames.

Theorem 4.19. *Let $\mathcal{W}_{\mathcal{K}} = \{\mathcal{W}_k\}_{k=0}^{N-1}$ be the Gabor fusion frame associated with a difference set $\mathcal{K}$ with parameters (N, K, λ). Then, $\mathcal{W}_{\mathcal{K}}$ is an optimally sparse fusion frame in the class $\mathcal{FF}(N, K, N)$ with respect to the canonical basis with sparsity KN.*

Proof. From Theorem 4.13, we know that $\mathcal{W}_{\mathcal{K}}$ is a tight fusion frame from the class $\mathcal{FF}(N, K, N)$.

Let $\{e_j\}_{j=1}^{N}$ be the canonical basis of $\mathbb{C}^N$. From the definition of $\mathcal{W}_{\mathcal{K}}$ (4.9) it follows that the elements of each subspace $\mathcal{W}_i$ are supported on the sets $\mathcal{K} + i$. Therefore, as an orthonormal basis for every $\mathcal{W}_i$ we can take

$$\{\phi_{i,\ell}\}_{\ell=1}^{K}, \text{ where } \phi_{i,\ell} = e_{k_\ell + i}, k_\ell \in \mathcal{K}.$$

Then, the corresponding sets $J_{i,\ell}$ from Definition 4.16 are each of cardinality 1, and the sparsity of $\mathcal{W}_{\mathcal{K}}$ is

$$\sum_{i=1}^{N}\sum_{\ell=1}^{K}|J_{i,\ell}| = KN.$$

Now, for any other associated j-sparse frame with sets $\{\tilde{J}_{i,\ell}\}_{i=1,\ell=1}^{N,K}$, we have that $\sum_{i=1}^{N}\sum_{\ell=1}^{K}|\tilde{J}_{i,\ell}| \geq KN$ because each $\tilde{J}_{i,\ell}$ has at least one element. Thus, $\mathcal{W}_{\mathcal{K}}$ is KN-sparse. Moreover, this also says that KN is the smallest sparsity that one can expect in $\mathcal{FF}(N,K,N)$. Therefore, $\mathcal{W}_{\mathcal{K}}$ is optimally sparse. □

Remark 4.20. For $K \geq 2$, $K \leq N-3$, we have by Theorem 4.18 that STFF (N,K,N) is optimally sparse in $\mathcal{FF}(N,K,N)$. Note that in this case the sparsity given by Theorem 4.18, $KN + 2(N - \gcd(NK,N))$ is exactly KN.

4.3.2 Compressed Sensing with Gabor Fusion Frames

Now we move to another, ideologically different notion of sparsity, which is related not to the low-complexity of the fusion frame decomposition as a whole, but to the low-complexity of the structure of the signals. Namely, in many applications like target recognition or music segmentation, the signal can be modeled as a union of components lying in only few subspaces, which gives a rise to the notion of *fusion sparse* signals. For such signals, a method of linear measurement process, and recovery via minimization problem which promotes such sparsity was developed in [18], and we explained it briefly in Section4.3 of Chapter 1.

Definition 4.21. [18] Let $\mathcal{W} = \{\mathcal{W}_j\}_{j=1}^{M}$ be a fusion frame in $\mathbb{R}^N$, and let

$$\mathcal{H}_{\mathcal{W}} := \{(x_i)_{i=1}^{M} : x_i \in \mathcal{W}_i \text{ for all } i = 1, \dots, M\}.$$

We call a vector $x \in \mathcal{H}_{\mathcal{W}}$ k-fusion sparse, if

$$\|x\|_0 := |\{j : x_j \neq 0\}| \leq k.$$

The problem we are given in short is as follows. Let $\mathcal{W} = \{\mathcal{W}_j\}_{j=1}^{M}$ be a fusion frame in $\mathbb{R}^N$. Given the condition that $x = \{x_j\}_{j=1}^{M}$, $x_j \in \mathcal{W}_j$ has only few nonzero components x_j, recover x from its measurements $y = A_P x$, where $A_P = \{a_{ij}P_j\}_{i,j=1}^{n,M}$. Here, $A = \{a_{ij}\}_{i,j=1}^{n,M}$ is the measurement matrix, and P_j are the projections to the corresponding subspaces $\mathcal{W}_j$. The fusion sparse vector x can be found by solving the minimization problem

$$\min_{x \in \mathcal{H}_{\mathcal{W}}} \|x\|_{2,1} \text{ subject to } A_P x = y. \tag{$P_{2,1}$}$$

The norm which we minimize is the mixed ℓ_1/ℓ_2 norm, which promotes "block" like sparsity, and is defined as

$$\|x\|_{2,1} = \sum_{j=1}^{M} \|x_j\|_2, \text{ where } x = \{x_j\}_{j=1}^{M},\, x_j \in \mathcal{W}_j.$$

A measure of the coherence is this time given by the so-called *fusion coherence*, defined as

$$\mu_f(A, \{\mathcal{W}_i\}_{i=1}^{M}) = \max_{j \neq k} \left[|\langle a_j, a_k \rangle| \cdot \|P_j P_k\|_2 \right].$$

Small fusion coherence guarantees recovery of fusion sparse vectors via ($P_{2,1}$), as in the classical case described in the previous section.

Theorem 4.22 ([18])**.** *Let $A \in \mathbb{R}^{n \times M}$ have normalized columns $\{a_{ij}\}_{i=1}^{M}$, let $\{\mathcal{W}_j\}_{j=1}^{M}$ be a fusion frame in $\mathbb{R}^N$, and let $Y \in \mathbb{R}^{n \times N}$. If there exists a solution c^0 of the system $Y = AU(c)$ satisfying*

$$\|c^0\|_0 < \frac{1}{2}\left(1 + \mu_f(A, \{\mathcal{W}_i\}_{i=1}^{M})^{-1}\right), \tag{4.11}$$

then this solution is the unique solution of ($P_{2,1}$).

In the case $N = 1$, we obtain the result for classical frames formulated in Theorem 2.5. A detailed theoretical description of this problem for fusion frames in general, and its importance for applications is given in [18]. Investigations on recovery from random fusion frames were conducted in [8].

We are interested in exploiting this idea for the Gabor fusion frames that we introduced in Section 4.3, and looking at the problem of recovery of vectors sparse in a Gabor fusion frame generated by a difference set. From theoretical point of view, leaving out the details, the *fusion coherence* in our case will be equal to the mutual coherence of the measurement matrix, and will not depend on the fusion frame structure. This is because one can prove that for our Gabor fusion frame, $\|P_j P_k\|_2$, the largest absolute value of the cosines of the principle angles beween $\mathcal{W}_j$ and $\mathcal{W}_k$, always equals 1, and therefore the structure of the fusion frame does not play a role in the recovery guarantees.

Nevertheless, in the next subsection we conduct numerical experiments with our Gabor fusion frame and Gaussian measurement matrices, and observe promising results.

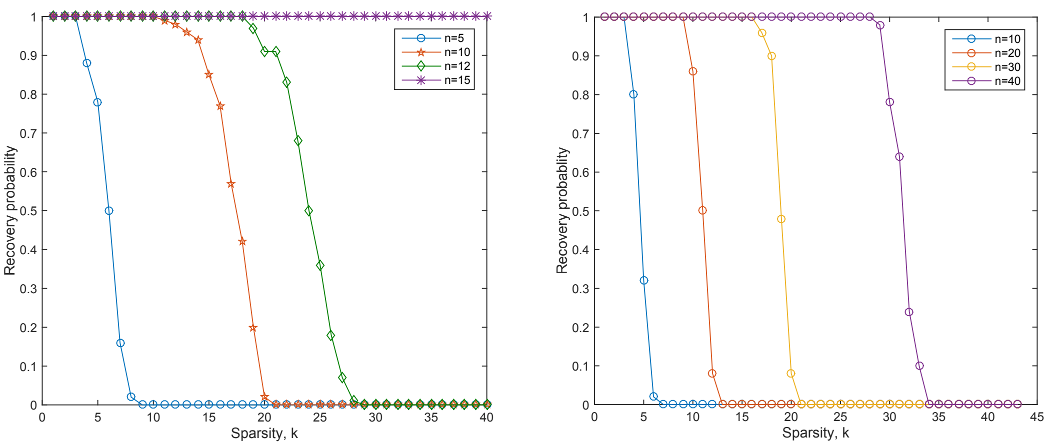
4.3.3 Numerical Experiments

The task is to use Gabor fusion frame $\mathcal{W}_{\mathcal{K}}$ generated by a difference set $\mathcal{K}$ for recovery of signals which are sparse in a fusion frame, namely, which have nonzero components lying in only few subspaces. In our case that would correspond to having only few translations.

Although it might not be clear how to numerically solve the described minimization process ($P_{2,1}$), this can be easily accomplished via the standard ℓ_1 minimization technique, by incorporating the basis vectors for each of the subspaces. The details can be found in [18]. Here the question is not only what level of sparsity are we able to recover, but also, how many measurements n in the matrix A do we need? At the same time, as we will see, the dimensions of the subspaces will play an important role.

In Figure 4.3a, we consider a (N, K, λ)-difference set with $N = 40$ and $K = 13$, which means that the dimension of the subspaces is 13, and in the experiment depicted in Figure 4.3b, we set $N = 43$ and $K = 21$. In both experiments we take the measurement matrix A to be random Gaussian. For a different number of measurements, as denoted in the legend, and for every sparsity level, we generate random k-fusion sparse vector x (with independent random Gaussian values at k subspaces chosen at random). Then, we calculate the measurements y, and try to recover back x by ($P_{2,1}$) again using CVX. We repeat each experiment $T = 100$ times, and count the recovery as successful, if the normalized mean square error was smaller than 10^{-6}. The results are presented in Figure 4.3. We observe that as expected, larger number of measurements allows for higher levels of sparsity, but also that when the dimension of the subspaces is smaller, fewer measurements are needed to recover the signal. Moreover, if the subspaces are of small dimension, and the number of measurements is sufficiently large, we can recover x independently of its sparsity level.

As we saw, the theoretical results for this problem are again not sufficient to capture this effect. Therefore, a more subtle measure of coherence is still missing in both problems presented, and these questions will be part of future research.



(a) Difference set with $N = 43, K = 13$.

(b) Difference set with $N = 43, K = 21$.

Figure 4.3: Fusion sparse recovery with Gabor fusion measurements.

Chapter 5

Phase Retrieval for Signals Having a Sparse Representation

5.1 Introduction

If compressed sensing is described as a methodology for recovery of sparse signal from *linear* measurements, sparse phase retrieval is one possible step towards a generalization of this problem to *non-linear* measurements. In this section, we would like to consider the problem of recovery of vectors which are not sparse, but have a sparse representation in some redundant system (a dictionary), from the magnitude of their linear measurements. We call this problem *dictionary sparse phase retrieval*. This chapter is organized as follows. In Section 5.2 we provide necessary and sufficient conditions for injectivity in both $\mathbb{R}$ and $\mathbb{C}$, and provide some examples of measurements which allow phase retrieval of signals which have sparse representation. In Section 5.3 we investigate the question of recovery via ℓ_1 minimization, and its characterization via the null space property, modified for this particular setting of phase retrieval and dictionary sparsity. Finally, in Section 5.4 we move to phase retrieval by projections of sparse signals.

5.2 Injectivity of the Dictionary Sparse Phase Retrieval

Let us fix notation. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. We use $\mathbb{K}_k^N$ to note the set of signals in $\mathbb{K}^N$ which have at most k nonzero elements. Furthermore, $\mathbb{K}_{kD}^N$ is the set of signals in $\mathbb{K}^N$ which have a k-sparse representation in some dictionary $D \in \mathbb{K}^{N \times n}$,

$$\mathbb{K}_{kD}^N = \{x \in \mathbb{K}^N : \exists z \in \mathbb{K}^n \text{ such that } x = Dz, \|z\|_0 \leq k\}.$$

Let $F = \{f_i\}_{i=1}^m$ be a set of measurement vectors in $\mathbb{K}^N$. Let $D = \{d_i\}_{i=1}^n$ be a fixed dictionary in $\mathbb{K}^N$. We sometimes identify those two sets with the corresponding matrices of the synthesis operators (see equation 2.7), $F \in \mathbb{K}^{N\times m}$ and $D \in \mathbb{K}^{N\times n}$. In general, we often identify a matrix $A \in \mathbb{K}^{M\times N}$ with the linear map from $\mathbb{K}^N$ to $\mathbb{K}^M$ it yields, $x \mapsto Ax$.

The phase retrieval problem consists of finding $x \in \mathbb{K}^N_{kD}$ up to a unimodular constant given the measurements $|\langle f_i, x\rangle|^2$, $i = 1,\dots,m$. Formally, we are interested in the map

$$\mathcal{M}_{FD} : \mathbb{K}^N_{kD}/\mathbb{T} \to \mathbb{R}^m_+, \quad \mathcal{M}_{FD}(x) = \begin{bmatrix} |\langle f_1, x\rangle|^2 & |\langle f_2, x\rangle|^2 & \dots & |\langle f_m, x\rangle|^2 \end{bmatrix}^T. \tag{5.1}$$

Definition 5.1. A set $F = \{f_i\}_{i=1}^m$ in $\mathbb{K}^N$ is said to have the *kD-sparse phase retrieval property*, or *allows kD-sparse phase retrieval,* if the map $\mathcal{M}_{FD}$ is injective.

We focus at first on the case $\mathbb{K} = \mathbb{R}$.

5.2.1 Dictionary Sparse Phase Retrieval in $\mathbb{R}$

We show here that the same amount of vectors — which was $2k$ — necessary for recovery of k-sparse vectors from phaseless measurements [125] is also needed when x is sparse in some arbitrary dictionary.

Theorem 5.2. *Let $F = \{f_i\}_{i=1}^m$ be a set of vectors in $\mathbb{R}^N$, and let $D = \{d_i\}_{i=1}^n$ be a dictionary which spans $\mathbb{R}^N$. Let further $k < N$. If F allows kD-sparse phase retrieval, then necessarily $m \geq 2k$.*

Proof. Towards a contradiction, let $F = \{f_i\}_{i=1}^m$ only possess $2k-1$ vectors. We will construct vectors η and $\nu \in \mathbb{R}^n$ which are k-sparse with $D\eta \neq \pm D\nu$ but $|\langle f_i, D\eta\rangle| = |\langle f_i, D\nu\rangle|$ for all $i = 1\dots m$. With this we will prove that F does not have the kD-sparse phase retrieval property.

First, since D spans the whole space, there must exist a subset T of $\{1,\dots,n\}$ with $|T| = k+1$ so that the vectors $\{d_i\}_{i\in T}$ are linearly independent. If not, the dimension of the space spanned by the dictionary will be at most $k < N$ and it hence can not be the whole of $\mathbb{R}^N$.

Now consider the subspace W of $\mathbb{R}^n$, $W = \{\alpha \in \mathbb{R}^n : \operatorname{supp}\alpha \subseteq T\}$. Because of the way T was chosen, $D\alpha \neq 0$ for all nonzero vectors $\alpha \in W$. Furthermore, we have $\dim W = k+1$.

We split the vectors in F into two groups : $\{f_i\}_{i=1}^k$ and $\{f_i\}_{i=k+1}^m$. Denote the matrices formed by these two groups as columns with F_1 and F_2, respectively. Since the space W

is of dimension $k+1$ and the image of $F_1^T D$ is at most of dimension k, there must exist a nonzero vector $\alpha \in W$ such that $F_1^T D\alpha = 0$. Similarly, the dimension of the image of $F_2^T D$ is at most $k-1$ (we assumed that $m \leq 2k-1$), and there hence exist two linearly independent vectors β and γ in W with $F_2^T D\beta = F_2^T D\gamma = 0$.

Since β and γ are linearly independent, there exists two indices $i, j \in T$ such that the subvectors (β_i, β_j) and (γ_i, γ_j) are linearly independent. There therefore exist t_0, s_0 such that

$$\begin{aligned} \alpha_i &= t_0\beta_i + s_0\gamma_i, \\ -\alpha_j &= t_0\beta_j + s_0\gamma_j. \end{aligned}$$

If $(\alpha_i, \alpha_j) \neq (0,0)$, we must also have $(s_0, t_0) \neq (0,0)$. If $\alpha_i = \alpha_j = 0$, then choose t_0 and s_0 such that

$$\begin{aligned} 1 &= t_0\beta_i + s_0\gamma_i, \\ 0 &= t_0\beta_j + s_0\gamma_j \end{aligned}$$

instead. In both cases, set $\delta := t_0\beta + s_0\gamma$ and $\eta := \alpha + \delta$, $\nu := \alpha - \delta \in \mathbb{R}^n$. Then in both cases η and ν are k-sparse. In the first case, $\operatorname{supp}\eta \subseteq T\setminus\{j\}$ and $\operatorname{supp}\nu \subseteq T\setminus\{i\}$. In the second case, both supports are contained in $T\setminus\{j\}$. In all cases, the vectors η and ν are different, both not zero, and supported on T. This implies that $D\eta \neq \pm D\nu$, since D is injective on the subspace W.

However, taking into account that by assumption $F_1^T D\alpha$ and $F_2^T D\delta$ are equal to zero, we obtain:

$$\langle f_i, D\eta\rangle = \langle f_i, D(\alpha+\delta)\rangle = \begin{cases} \langle f_i, D\delta\rangle & \text{if } i \leq k, \\ \langle f_i, D\alpha\rangle & \text{if } i > k. \end{cases}$$

$$\langle f_i, D\nu\rangle = \langle f_i, D(\alpha-\delta)\rangle = \begin{cases} -\langle f_i, D\delta\rangle & \text{if } i \leq k, \\ \langle f_i, D\alpha\rangle & \text{if } i > k. \end{cases}$$

Hence $|\langle f_i, D\eta\rangle| = |\langle f_i, D\nu\rangle|$ for all $i = 1, \ldots, m$, although $D\eta \neq \pm D\nu$. This proves that F with $2k-1$ elements can not allow kD-sparse phase retrieval, and hence $2k$ vectors are necessary for dictionary sparse phase retrieval. □

In the search of measurements which allow phase retrieval, one crucial ingredient is the characterization via the *complement property* [12, 13]. A counterpart for it in the sparse setting was introduced in [97]. Unlike for general signals, in the sparse setting the complement property is no longer a necessary and sufficient condition for injectivity, but it is still very useful since it gives a method for verifying if a system allows sparse phase retrieval. We extend this result to the dictionary sparse setting, and investigate how sharp the sparse complement property is.

Let $F = \{f_i\}_{i=1}^m$ be a set of measurement vectors and $D = \{d_i\}_{i=1}^n$ a fixed dictionary, both in $\mathbb{R}^N$. For a fixed subset $\mathcal{K} \subseteq [1, \dots, n]$, we will often use the notation

$$W_{\mathcal{K}} := \operatorname{span}\{d_i\}_{i\in\mathcal{K}}.$$

For a fixed subset $S \subseteq [1, \dots, m]$, we will denote by $F_S = \{f_i\}_{i\in S}$, and the rest of the elements in F by $F_{S^c} = \{f_i\}_{i\in S^c}$. We will often use matrix notation, and F_S^T and $F_{S^c}^T$ will be the matrices which contain as rows the elements of F_S and F_{S^c}, respectively. We will speak about the injectivity of the linear map from $\mathbb{R}^N$ to $\mathbb{R}^{|S|}$, $x \mapsto \{|\langle f_i, x\rangle\}_{i\in S}$, and we will identify it with its matrix representation $F_S^T : x \mapsto F_S^T x$.

We are now ready to formulate the k-complement property generalized to vectors which are sparse in a dictionary (see Definition 2.25 for the classical k-complement property).

Definition 5.3. A given set $F = \{f_i\}_{i=1}^m$ has the *kD-complement property* with a dictionary $D = \{d_i\}_{i=1}^n$, if for all $\mathcal{S} \subseteq [1, \dots, m]$ and all $\mathcal{K} \subseteq [1, \dots, n]$ with $|\mathcal{K}| \le k$, either F_S^T or $F_{S^c}^T$ is injective on the subspace $W_{\mathcal{K}} = \operatorname{span}\{d_i\}_{i\in\mathcal{K}}$.

Note, that if $D = I$ is the identity basis, $W_{\mathcal{K}} = \operatorname{span}\{e_i\}_{i\in\mathcal{K}} = \{x \in \mathbb{R}^n : \operatorname{supp}(x) \subseteq \mathcal{K}\}$ we obtain the k-complement property, and if additionally $k = N$, we would have the classical complement property.

We are interested when a set of vectors $F = \{f_i\}_{i=1}^m$ allows kD-phase retrieval, and we formulate this result in terms of injectivity of the map $\mathcal{M}_{FD}$ (5.1).

Theorem 5.4. *Given the notations above, the following two statements hold:*

(i) *If the map $\mathcal{M}_{FD}$ is injective, then F has the kD-complement property.*

(ii) *If F has the $2kD$-complement property, then $\mathcal{M}_{FD}$ is injective.*

Proof. (i) Assume that the mapping $\mathcal{M}_{FD}$ is injective, but the kD-complement property is not satisfied. That means, that there exist subsets $\mathcal{S} \subseteq [1, \dots, m]$ and $\mathcal{K} \subseteq [1, \dots, n]$, $|\mathcal{K}| \le k$, and nonzero vectors $x, y \in W_{\mathcal{K}}$, such that

$$\langle f_i, x\rangle = 0, \quad i \in \mathcal{S},$$
$$\langle f_i, y\rangle = 0, \quad i \in \mathcal{S}^c.$$

Then for all $i = 1, \ldots, m$, we will have $|\langle f_i, x+y\rangle| = |\langle f_i, x-y\rangle|$. By assumption, $x, y \in \mathbb{R}^N_{kD}$. Moreover, their sparse representations have support contained in $\mathcal{K}$, hence $x \pm y \in \mathbb{R}^N_{kD}$. Therefore, by injectivity of $\mathcal{M}_{FD}$ we have that $x + y = \pm(x-y)$, which is in contradiction with the assumptions that both x and y were nonzero.

(ii) Assume that F has the $2kD$-complement property, but $\mathcal{M}_{FD}$ is not injective. Then there exist $x, y \in \mathbb{R}^N_{kD}$, $x \neq \pm y$ which give the same measurements. In other words, there exist k-sparse z_x and z_y in $\mathbb{R}^n$ such that $x = Dz_x$ and $y = Dz_y$, and

$$|\langle f_i, Dz_x\rangle|^2 = |\langle f_i, Dz_y\rangle|^2, \quad i = 1, \ldots m.$$

Since we are in the real case, for every $i = 1, \ldots, m$ we can rewrite this equality as

$$\langle f_i, Dz_x + Dz_y\rangle\langle f_i, Dz_x - Dz_y\rangle = 0.$$

Define $\mathcal{S} := \{i : \langle f_i, Dz_x + Dz_y\rangle = 0\}$ and $\mathcal{K} := \operatorname{supp}(z_x) \cup \operatorname{supp}(z_y)$. Then obviously $|\mathcal{K}| \leq 2k$. We see that we have found two nonzero vectors $x+y$, $x-y \in W_{\mathcal{K}}$ such that

$$\begin{aligned}\langle f_i, x+y\rangle &= 0, \quad i \in S,\\ \langle f_i, x-y\rangle &= 0, \quad i \in S^c.\end{aligned}$$

This is in contradiction to the $2kD$-complement property. $\square$

It was shown in [97] that $2k-1$ independent Gaussian vectors satisfy the k-complement property with probability 1. From there it follows that $4k-1$ random vectors suffice to recover k-sparse vectors x uniquely from phaseless measurements. We can now show that a collection of $2k-1$ random vectors also has the kD-complement property, and thus $4k-1$ vectors allow kD-sparse phase retrieval.

Proposition 5.5. *Let $F = \{f_i\}_{i=1}^m$, $f_i \in \mathbb{R}^N$ with $m \geq 2k-1$ be randomly distributed according to a joint standard normal distribution, and let $D = \{d_i\}_{i=1}^n$ be an arbitrary dictionary in $\mathbb{R}^N$. Then F has the kD-complement property with probability* 1.

Proof. Let $\mathcal{K} \subseteq [1, \ldots, n]$, $|\mathcal{K}| \leq k$ and $\mathcal{S} \subseteq [1, \ldots, m]$ be arbitrary. We will show that either $F^T_{\mathcal{S}}$ or $F^T_{\mathcal{S}^c}$ is injective on the subspace $W_{\mathcal{K}} := \operatorname{span}\{d_i\}_{i\in\mathcal{K}}$. Note that $\dim W_{\mathcal{K}} := d_{\mathcal{K}} \leq k$. Also note that since $m \geq 2k-1 \geq 2d_{\mathcal{K}} - 1$, one of $|\mathcal{S}|$ and $|\mathcal{S}^c|$ must be at least $d_{\mathcal{K}}$. Without loss of generality, we may assume that it is $|\mathcal{S}|$.

Let $Q \in \mathbb{R}^{N\times d_{\mathcal{K}}}$ be a matrix whose columns form an orthonormal basis of $W_{\mathcal{K}}$. Define a new matrix $\widetilde{F}^T$ by choosing $d_{\mathcal{K}}$ rows of $F^T_{\mathcal{S}}$. Note that the rows of $\widetilde{F}^T Q \in \mathbb{R}^{d_{\mathcal{K}}\times d_{\mathcal{K}}}$ are still jointly normal distributed. Since the space of invertible matrices is dense in

$\mathbb{R}^{d_{\mathcal{K}}\times d_{\mathcal{K}}}$, $\widetilde{F}^T Q$ will therefore with probability 1 be injective. This however implies that $\widetilde{F}^T$ is injective on $W_{\mathcal{K}}$ with probability 1, and consequently $F_{\mathcal{S}}^T$, too. $\square$

We should note that using algebraic geometry tools, the authors in [125] showed that $2k$ generic vectors can allow k-sparse phase retrieval, which is a quite stronger result than the one obtained using the complement property in [97]. Note also, that in [97], only part (ii) of Theorem 5.4 with $D = I$ was presented (see Theorem 2.26).

It is therefore interesting to ask, how sharp the order of the complement property given in Theorem 5.4 is. Looking at part (i) of this theorem we ask: can a complement property of higher order also be guaranteed? We can observe that this is not possible already in the case $D = I$: as we just mentioned, a set of $2k$ generically chosen vectors in $\mathbb{R}^N$ allows k-sparse phase retrieval. But such sets can not have the $(k+1)$-complement property for $N > k+1$. To see this, just subdivide the vectors into two groups of cardinality k. Since there are only k in each set, none of them restricted to a set $\mathcal{K}$ of $k+1$ elements could span $\mathbb{R}^{k+1}$, and thus the $(k+1)$-complement property can not be satisfied. The $(k+1)$-complement property is hence not necessary for allowing k-sparse phase retrieval.

Now looking at part (ii), we ask if we can also observe that the $(2k-1)$-complement property is not sufficient for allowing k-sparse phase retrieval. The answer is not straight-forward, but will show that indeed one can construct a set of vectors which have the $(2k-1)$-complement property, but which do not allow k-sparse phase retrieval. To prove this, we need two lemmas. The first lemma will be the desired statement but restricted to dimension $N = 2k$, and the second lemma will help us to extended this result to larger N. Our example is for the case $D = I$, but since kD-sparsity includes in its definition the classical k-sparsity, it means that our result holds for kD-complement property as well.

Lemma 5.6. *Let $N = 2k$. Then, a set of $4k-2$ i.i.d. Gaussian vectors satisfy the $(2k-1)$-complement property with probability 1, but do not allow k-sparse phase retrieval.*

Proof. Let $A,\ B \in \mathbb{R}^{2k-1\times k}$ be matrices with i.i.d. Gaussian entries. Consider the following matrix in $\mathbb{R}^{4k-2\times 2k}$:

$$M = \begin{bmatrix} A & B \\ A & -B \end{bmatrix}. \tag{5.2}$$

We will show that the rows of this matrix satisfy the statement of the theorem. Since $2k > 2k-1$, there exists a pair $(x, y) \in \mathbb{R}^k \times \mathbb{R}^k$ not equal to $(0, 0)$ for which $Ax+By = 0$. We then have

$$M \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax \\ Ax \end{bmatrix}, \quad M \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} By \\ -By \end{bmatrix}.$$

Hence, $|M(x,0)| = |M(0,y)|$. We have found two different vectors $(x,0)$ and $(0,y)$ which are both k-sparse but give the same phaseless measurements. Therefore, M can not allow k-sparse phase retrieval.

It is left to show now that M has the $(2k-1)$-complement property with probability one. For this, it is sufficient to prove that any block of M formed by choosing $2k-1$ columns (corresponding to $\mathcal{K}$) and $2k-1$ rows (corresponding to $\mathcal{S}$ or $\mathcal{S}^c$ – at least one of those sets contains at least $2k-1$ indices since $|\mathcal{S}| + |\mathcal{S}^c| = 4k-2$) is invertible.

Let us employ the block structure of M to describe the matrix $M_S^{\mathcal{K}}$. In the choice of columns, let us say that $k_1 \leq k$ of the elements in $\mathcal{K}$ are from $[1,\ldots,k]$. Let us denote the upper block of them with P. The number of columns indexed by elements of $[k+1,\ldots,2k]$ is denoted k_2 and the corresponding upper block with Q. We arrive at a matrix of the form

$$M^{\mathcal{K}} = \begin{bmatrix} P & Q \\ P & -Q \end{bmatrix}$$

with $P \in \mathbb{R}^{2k-1\times k_1}$ and $Q \in \mathbb{R}^{2k-1\times k_2}$ with i.i.d. Gaussian entries. Note that

$$k_1 + k_2 = 2k-1, \quad \text{and } k_1, k_2 \leq k. \tag{5.3}$$

Now we have to choose rows. Because of the dual-block structure, a certain number of rows will have a corresponding pair in the other half, i.e., there are $j \in S$ such that $j + (2k-1) \in S$. Let us call the number of such rows ℓ. In particular, there exists $\tilde{S} \subseteq S$, $|\tilde{S}| = \ell$, such that $M_S^{\mathcal{K}}$ contains the block

$$\begin{bmatrix} P_{\tilde{S}} & Q_{\tilde{S}} \\ P_{\tilde{S}} & -Q_{\tilde{S}} \end{bmatrix}.$$

The rest of the rows in S do not have dual-block structure and we denote this block by

$$\begin{bmatrix} R & T \end{bmatrix}.$$

With possibly changed order of the rows, we can write the matrix $M_S^{\mathcal{K}}$ in the form

$$A = \begin{bmatrix} P_{\tilde{S}} & Q_{\tilde{S}} \\ P_{\tilde{S}} & -Q_{\tilde{S}} \\ R & T \end{bmatrix}. \tag{5.4}$$

Here $P_{\tilde{S}} \in \mathbb{R}^{\ell\times k_1}$, $Q_{\tilde{S}} \in \mathbb{R}^{\ell\times k_2}$, $R \in \mathbb{R}^{2k-1-2\ell\times k_1}$ and $T \in \mathbb{R}^{2k-1-2\ell\times k_2}$ and all have i.i.d. Gaussian entries.

If $\ell = 0$, the matrix A is simply a $\mathbb{R}^{2k-1\times 2k-1}$ Gaussian matrix, and hence almost surely invertible.

Now suppose that $\ell > 0$ and let the vector $(x, y) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ be in the kernel of A, denoted by $\mathcal{N}(A)$. Then,

$$\begin{aligned} P_{\tilde{S}}x + Q_{\tilde{S}}y &= 0 \\ P_{\tilde{S}}x - Q_{\tilde{S}}y &= 0 \\ Rx + Ty &= 0 \end{aligned} \tag{5.5}$$

and subsequently $x \in \mathcal{N}(P_{\tilde{S}})$, $y \in \mathcal{N}(Q_{\tilde{S}})$. Almost surely, the dimension of $\mathcal{N}(P_{\tilde{S}})$ will be $\max(0, k_1 - \ell)$ and the one of $\mathcal{N}(Q_{\tilde{S}}) = \max(0, k_2 - \ell)$. We now distinguish four cases.

If $k_1, k_2 \leq \ell$, then $x = 0$, $y = 0$, and hence the matrix A must be invertible.

If $k_2 \leq \ell < k_1$, then (5.3) implies that $k_1 = k$ and $k_2 = k - 1 = \ell$. Then y must be equal to 0, and therefore by (5.5), $Rx = 0$. The size of $P_{\tilde{S}}$ is in this case $k - 1 \times k$ and therefore x must lie in the 1-dimensional subspace $\mathcal{N}(P_{\tilde{S}})$. Since $2k - 2\ell - 1 = 2k - 2(k-1) - 1 = 1$, $R \in \mathbb{R}^{1\times k}$ will almost surely be injective on this subspace, and hence using the independence of R and $P_{\tilde{S}}$ we can conclude that x must be zero. The matrix A is again invertible.

If $k_1 \leq \ell < k_2$, then $k_2 = k$ and $k_1 = k - 1 = \ell$ and we can proceed as above.

Finally, if $k_1, k_2 > \ell$, then $\dim(\mathcal{N}(P_{\tilde{S}}) \times \mathcal{N}(Q_{\tilde{S}})) = k_1 - \ell + k_2 - \ell = 2k - 1 - 2\ell$ and thus (x, y) is lying on a $(2k - 1 - 2\ell)$-dimensional subspace. Almost surely, $\begin{bmatrix} R & T \end{bmatrix} \in \mathbb{R}^{2k-1-2\ell\times 2k-1}$ will be injective on this space. (Here, we used that $\begin{bmatrix} P_{\tilde{S}} & Q_{\tilde{S}} \end{bmatrix}$ and $\begin{bmatrix} R & T \end{bmatrix}$ are independent). □

We will now show that it is possible to modify a set of vectors, such that the new system has the sparse complement property of the same order, but in one dimension higher.

Lemma 5.7. *Suppose that $F \in \mathbb{R}^{m\times n}$ is a matrix with rows in $\mathbb{R}^n$ that have the k-complement property for some k. Then, if $v \in \mathbb{R}^m$ is a vector with i.i.d. Gaussian entries, then the rows of the matrix $\begin{bmatrix} F & v \end{bmatrix} \in \mathbb{R}^{m\times(n+1)}$ have the k-complement property in $\mathbb{R}^{n+1}$ with probability* 1.

Proof. We need to prove that if $v \in \mathbb{R}^m$ is a Gaussian random vector, then the rows of

$$\tilde{F} := \begin{bmatrix} F & v \end{bmatrix}$$

almost surely have the k-complement property.

In order to check if $\tilde{F}$ has the k-complement property, we choose k arbitrary columns from it indexed by $\mathcal{K} \subseteq [1, \ldots, n+1]$. If $\tilde{f}_i$ for all $i \in \mathcal{K}$ are columns in F, we know that for any $S \subseteq [1, \ldots, m]$, either $\tilde{F}_S^{\mathcal{K}}$ or $\tilde{F}_{S^c}^{\mathcal{K}}$ will span $\mathbb{R}^k$, since F itself has the k-complement property.

If one of the chosen columns is v, then $n+1 \in \mathcal{K}$, and we set $\tilde{\mathcal{K}} := \mathcal{K} \backslash \{n+1\}$. We proceed as follows. Consider a sub-choice of the rows $S \subseteq [1, \ldots, m]$. We can restrict to $|S| = k$, since necessarily $m \geq 2k-1$, given that the rows of F have the complement property. The subset $\tilde{\mathcal{K}}$ in this case has $k-1$ elements. We obtain a matrix $\begin{bmatrix} F_S^{\tilde{\mathcal{K}}} & v_S \end{bmatrix}$. Since F has the k-complement property, the columns in $F_S^{\tilde{\mathcal{K}}}$ are linearly independent – if not, we could not obtain a basis by adding one vector, which however is the case. Hence, in order for $\begin{bmatrix} F_S^{\tilde{\mathcal{K}}} & v_S \end{bmatrix}$ not to be invertible, v_S must lie in the span of $\{f_i^S\}_{i \in \mathcal{K}}$, a $(k-1)$-dimensional subspace of $\mathbb{R}^k$. Therefore, v must lie on a $(k-1)+(m-k) = (m-1)$-dimensional subspace of $\mathbb{R}^m$. This almost never happens. Since S and $\mathcal{K}$ were chosen arbitrary, the claim is proven. $\square$

Finally, we have all the ingredients to show that the $(2k-1)$-complement property can not guarantee a k-sparse phase retrieval.

Theorem 5.8. *Let $2k \leq N$. Then, there exists a set of $4k-2$ vectors which has the $(2k-1)$-complement property, but does not allow k-sparse phase retrieval.*

Proof. By Lemma 5.6, there exists a matrix $F \in \mathbb{R}^{(4k-2) \times 2k}$ which rows have the $(2k-1)$-complement property but do not allow k-sparse phase retrieval. This matrix can by Lemma 5.7 be iteratively filled up with new columns in such a way so that the resulting matrix $\widetilde{F}$ still has the $(2k-1)$-complement property, but with vectors already in dimension N. Note that the new vectors will still not allow k-sparse phase retrieval: if x, $y \in \mathbb{R}^{2k}$ are k-sparse with $x \neq \pm y$ so that $|Fx| = |Fy|$, the vectors $\tilde{x} = (x, 0)$, $\tilde{y} = (y, 0) \in \mathbb{R}^{2k} \times \mathbb{R}^{N-2k}$ are also k-sparse with $\tilde{x} \neq \pm \tilde{y}$ and $\left|\widetilde{F}\tilde{x}\right| = |Fx| = |Fy| = \left|\widetilde{F}\tilde{y}\right|$. $\square$

5.2.2 Sparse Phase Retrieval in $\mathbb{C}$

The phase retrieval problem in $\mathbb{C}$ is in general much more difficult than for real vectors, since if previously we were losing only the sign of the measurements, $\pm\langle f_i, x\rangle$, now there is a whole circle $c\,\langle f_i, x\rangle$, $|c| = 1$ which gives the same measurement in absolute value. We will write $x = y \mod \mathbb{T}$ to denote that there exists some c with norm one, such that $x = cy$.

One of the most interesting questions here is the *minimal* number of measurements needed for injectivity. For arbitrary (not necessarily sparse) signals, as we discussed in Section 2.3 of Chapter 2, this question is still open, and the last conjecture $4N-4$ [17, 103], was recently disproved in the work [124].

For sparse phase retrieval, the conjecture for the minimal number of measurements in $\mathbb{C}$ at the moment is $4k-2$ [125]. We investigate now the class of complex signals which are sparse in a given dictionary.

Let $F = \{f_i\}_{i=1}^m$ be a set of measurement vectors in $\mathbb{C}^N$. As usual, $\mathbb{C}_k^N$ is the set of d-dimensional vectors which have k or less nonzero components. We define the following two maps:

$$\mathcal{M}_F : \mathbb{C}_k^N \to \mathbb{C}_+^m : \quad (\mathcal{M}_F x)(j) = |\langle x, f_j\rangle|^2, \tag{5.6}$$

$$\mathcal{A} : \mathbb{H}^{N\times N} \to \mathbb{C}_+^m : \quad (\mathcal{A}H)(j) = \langle H, f_j f_j^*\rangle_{HS}, \tag{5.7}$$

where $\mathbb{H}^{N\times N}$ is the space of Hermitian $N\times N$ matrices. By "lifting" x and taking an H of the form xx^*, we can rewrite the phaseless measurements via the operator $\mathcal{A}$, since $|\langle x, f_i\rangle|^2 = \langle f_i, Hf_i\rangle = \langle H, f_i f_i^*\rangle_{HS}$. We refer to $\mathcal{A}$ as the *PhaseLift* operator [28], also called *super analysis operator* in [13]. It was proven in [13] that the vectors $F = \{f_i\}_{i=1}^m$ do phase retrieval of full vectors if and only if the kernel of $\mathcal{A}$ does not contain any rank 1 or 2 matrices. The corresponding result for sparse signal was so far not formulated, up to our knowledge. We generalize it directly to the dictionary sparse setting, a simplified formulation for classical sparsity shall be given in Chapter 6. Let D be a given dictionary $D = \{d_i\}_{i=1}^n$ in $\mathbb{C}^N$, and denote $W_\mathcal{K} = \operatorname{span}\{d_i\}_{i\in\mathcal{K}}$ for some $\mathcal{K} \subseteq [1,\dots,n]$.

Theorem 5.9. *Given the notations from above, the following two statements hold.*

(i) *If for every $\mathcal{K}$ with $|\mathcal{K}| = 2k$ the kernel of $\mathcal{A}$ does not contain rank 1 or 2 matrices whose range is in $W_\mathcal{K}$, then the vectors $\{f_i\}_{i=1}^m$ allow kD-phase retrieval.*

(ii) *If $\{f_i\}_{i=1}^m$ is allowing kD-phase retrieval, then for every $\mathcal{K}$ with $|\mathcal{K}| = k$, the kernel of $\mathcal{A}$ does not contain rank 1 or 2 matrices with range in $W_\mathcal{K}$.*

Proof. Let us start proving (i) by contraposition. Assume that $\{f_i\}_{i=1}^m \in \mathbb{C}^N$ is not allowing kD-sparse phase retrieval. Then there exists $x \neq y \mod \mathbb{T}$, both kD-sparse, for which

$$\langle xx^*, f_i f_i^*\rangle_{HS} = |\langle f_i, x\rangle|^2 = |\langle f_i, y\rangle|^2 = \langle yy^*, f_i f_i^*\rangle_{HS}\,,$$

i.e. $xx^* - yy^*$ is in the kernel of $\mathcal{A}$. Let $\mathcal{K} := \operatorname{supp} z_x \cup \operatorname{supp} z_y$. If the sparse representations of x and y are given by $x = Dz_x$ and $y = Dz_y$, we see that $\operatorname{ran}(xx^* - yy^*) \subseteq W_\mathcal{K}$ and further $xx^* - yy^*$ has rank less than or equal to 2. If we knew that the rank is at least one, we would have a rank 1 or 2 matrix whose range is in $W_\mathcal{K}$, with $|\mathcal{K}| = 2k$, since $|\mathcal{K}| \leq 2k$. This would be in contradiction to the assumption in (i).

To see that the condition $x \neq y \mod \mathbb{T}$ in fact implies that the rank is at least one, assume, towards a contradiction, that this is not the case, i.e that $xx^* - yy^* = 0$. Since $x \neq y \mod \mathbb{T}$, both vectors are nonzero. Hence there exists a vector $v \in \mathbb{C}^N$ such that $\langle v, x\rangle \neq 0$. Multiplying $0 = xx^* - yy^*$ with this v and rearranging terms, we arrive at

$$x = \frac{\langle v, y\rangle}{\langle v, x\rangle} y,$$

i.e $x = \lambda y$ for some $\lambda \in \mathbb{C}$. Again plugging this into $0 = xx^* - yy^*$ yields $|\lambda| = 1$. This is a contradiction to $x \neq y \mod \mathbb{T}$.

Let us now turn to (ii). Towards a contradiction, suppose that there exists a $\mathcal{K} \subseteq [1, \ldots, n]$ such that the kernel of $\mathcal{A}$ contains a Hermitian matrix H with rank 1 or 2 with range in $W_{\mathcal{K}}$. By the spectral theorem, there exists an orthonormal basis $\{\varphi_j\}_{j=1}^N$ of $\mathbb{C}^N$ consisting of eigenvectors of H, corresponding to real eigenvalues $\{\lambda_j\}_{j=1}^N$. Therefore, H can be written as $\sum_j \lambda_j \varphi_j \varphi_j^*$.

Because of the bounded rank and the fact that $H \neq 0$, either one or two of the eigenvalues are nonzero. It is clear that the eigenvectors corresponding to those eigenvalues are vectors in $W_{\mathcal{K}}$, since they form a basis of the range of H. Thus, they are kD-sparse.

Let us first consider the case where only one eigenvalue is different from zero. If we write $x = \sqrt{|\lambda_1|}\varphi_1$, then we have $H = \pm xx^*$ and hence

$$0 = \langle xx^*, f_i f_i^*\rangle = |\langle f_i, x\rangle|^2 = 0, \quad i = 1, \ldots, m.$$

This means that the two kD-sparse vectors x and 0 have the same phaseless measurements, although $x \neq 0 \mod \mathbb{T}$.

The other case is dealt with similarly. There we write $x = \sqrt{|\lambda_1|}\varphi_1$, $y = \sqrt{|\lambda_2|}\varphi_2$ and conclude that $H = \pm xx^* \pm yy^*$, where the signs depend on the signs of the eigenvalues. In any case x and y give the same measurements in absolute value. If the signs are equal, we see that $|\langle f_i, x\rangle|^2 + |\langle f_i, y\rangle|^2 = 0$ and if the signs are not equal, we obtain $|\langle f_i, x\rangle|^2 - |\langle f_i, y\rangle|^2 = 0$. Hence, we have found x and y, two kD-sparse signals which are not equal mod $\mathbb{T}$ (they are orthogonal), but give the same measurements. □

This theorem is particularly useful because it gives a method for verifying whether a system of vectors is suitable for doing kD-sparse phase retrieval in the complex case. We shall use it intensively in Chapter 6 to show which type of Gabor measurements can do (sparse) phase retrieval.

5.3 Dictionary Null Space Property and ℓ_1 Recovery

Although theoretically the most important question in the phase retrieval problem is injectivity, what one is interested in practice is a method for recovery of the vector from given phaseless measurements. In the case of classical linear measurements of sparse signals, the compressed sensing methodology is giving us a practical way of finding the sparse vector by solving an ℓ_1 minimization problem. As we recalled in Theorem 2.8 in Chapter 1, characterization of the solvability of the ℓ_1 minimization problem for spase signals is given by the null space property. In [125], this methodology was developed further in the case of non-linear measurements — when only the absolute values of the measurements are known. On the other hand, a null space property for signals which have a sparse representation in a dictionary was introduced and was investigated in [41]. We merge both ideas into one, to develop conditions for phase retrieval of signals sparse in a dictionary via ℓ_1 recovery. We focus on the case $\mathbb{K} = \mathbb{R}$.

We first recall the results from compressed sensing of dictionary sparse signals. Let $\mathbb{R}^N_{kD}$ be the set of all signals which are k-sparse in some dictionary $D \in \mathbb{R}^{N\times n}$, and let $x_0 \in \mathbb{R}^N_{kD}$. Assume we have a measurement matrix $M \in \mathbb{R}^{m\times N}$, and want to recover x_0 from the measurements $b = Mx_0 = MDz_0$. For this, we can solve the synthesis ℓ_1 minimization problem

$$\hat{z} = \arg\min_{z\in\mathbb{R}^n} \|z\|_1 \quad \text{subject to} \quad b = MDz, \tag{5.8}$$

and then find $\hat{x}$ as $D\hat{z}$. We call the ℓ_1 method *successful* when every minimizer $\hat{z}$ of (5.8) satisfies $D\hat{z} = x_0$. Note that it is not required that $\hat{z} = z_0$.

A necessary and sufficient condition for the success of the ℓ_1 recovery is the dictionary based null space property, defined as follows.

Definition 5.10. [41] Fix a dictionary $D \in \mathbb{R}^{N\times n}$. A matrix $M \in \mathbb{R}^{m\times N}$ is said to satisfy the D-NSP of order k (kD-NSP), if for any index set $\mathcal{K}$ with $|\mathcal{K}| \le k$, and any non zero $v \in D^{-1}(\mathcal{N}(M))$, there exist $u \in \mathcal{N}(D)$, such that

$$\|v_{\mathcal{K}} + u\|_1 < \|v_{\mathcal{K}^c}\|_1. \tag{5.9}$$

We use the notation $v \in D^{-1}(\mathcal{N}(M))$ to denote that $Dv \in \mathcal{N}(M)$, where $\mathcal{N}(M)$ is the null space of M.

Theorem 5.11 ([41])**.** *D-NSP is a necessary and sufficient condition for the ℓ_1 minimization* (5.8) *to successfully recover all signals in the set $\mathbb{R}^N_{kD}$.*

We are interested in the case when a signal x_0 has a sparse representation Dz_0, but additionally we are given only the magnitudes of the measurements, $b = |MDz_0|$. This is the standard dictionary sparse phase retrieval problem, only written in a matrix form — $M \in \mathbb{R}^{m\times N}$ in this case contains as rows the usual measurement vectors $\{f_i\}_{i=1}^m$ in $\mathbb{R}^N$. We want to investigate the following minimization problem

$$\hat{z} = \arg\min_{z\in\mathbb{R}^n} \|z\|_1 \quad \text{subject to} \quad b = |MDz|, \tag{5.10}$$

just having in mind that we can find x only up to a unimodular constant. We thus call this method *successful*, if every minimizer of (5.10) $\hat{z}$ satisfies $D\hat{z} = x_0 \mod \mathbb{T}$.

If we write explicitly the measurements for each $j = 1, \dots, m$, we have

$$b_j = |\langle f_j, x\rangle| = |\langle f_j, \sum_{i=1}^n z_i d_i\rangle| = |\sum_{i=1}^n z_i \langle f_j, d_i\rangle|.$$

We can now construct a new measurement matrix $G \in \mathbb{R}^{m\times n}$, where each row is

$$g_j = [\langle f_j, d_1\rangle, \dots, \langle f_j, d_n\rangle],$$

and rewrite our measurements as $b_j = |\langle z, g_j\rangle|$. Now recovery of z is the usual setting of k-sparse phase retrieval [125], only that the vectors are now in $\mathbb{R}^n$. Therefore, the question of minimal number of measurements required for phase retrieval seems straightforward, we just need to request conditions on the set $\{g_i\}_{i=1}^m$ in $\mathbb{R}^n$ instead of $\{f_i\}_{i=1}^m$ in $\mathbb{R}^N$. In matrix form, instead of M our measurement matrix is now MD, and having a null space property now for MD guarantees a recovery via ℓ_1.

This is however not entirely equivalent to the problem that we want to solve. As [41] suggests, by looking at MD we are aiming for more than what we are interested in. We do not need to successfully recover both z_0 and x_0, but only need a good estimate of x_0, and there could be many vectors $\hat{z}$ that can give us the correct x_0 (remember that D is a frame, i.e. a redundant system which spans $\mathbb{R}^n$). Following the ideas in [41] we would like to have a dictionary based null space property for phase retrieval, that will be equivalent to successful recovery of x_0 via phase retrieval with ℓ_1.

Definition 5.12. Let $D \in \mathbb{R}^{N\times n}$ be a fixed dictionary. A matrix $M \in \mathbb{R}^{m\times N}$ is said to satisfy the D-PR-NSP of order k, if for any index set $\mathcal{S} \subseteq [1, \dots, m]$ and all nonzero $u \in D^{-1}(\mathcal{N}(M_\mathcal{S}))$ and $v \in D^{-1}(\mathcal{N}(M_{\mathcal{S}^c}))$ satisfying $\|u+v\|_0 \le k$, there exists $w \in \mathcal{N}(D)$ such that

$$\|u + v + w\|_1 < \|u - v\|_1. \tag{5.11}$$

Theorem 5.13. *D-PR-NSP is a necessary and sufficient condition for ℓ_1-synthesis phase retrieval (5.10) to successfully recover all signals in the set $\mathbb{R}^N_{kD}$.*

Proof. Sufficient part. Let us assume that D-PR-NSP holds. We will show that up to a sign (5.10) is able to successfully recover all $x_0 \in \mathbb{R}^N_{kD}$. Let $b = |Mx_0|$. For a fixed $\epsilon \in \{1,-1\}^m$ set $b_\epsilon := [\epsilon_1 b_1, \dots, \epsilon_m b_m]^T$. We now consider the standard minimization problem

$$\min \|z\|_1 \text{ subject to } MDz = b_\epsilon. \tag{5.12}$$

If z_ϵ is its solution, we denote $x_\epsilon := Dz_\epsilon$. We will show that for any $\epsilon \in \{1,-1\}^m$, a solution x_ϵ satisfies

$$\|x_0\|_1 \leq \|x_\epsilon\|_1,$$

and equality holds if and only if $x_\epsilon = \pm x_0$.

Let $\epsilon^* \in \{1,-1\}^m$ be such that $b_{\epsilon^*} = MDz_0$. The corresponding solution to the minimization problem (5.12) is denoted z_{ϵ^*}.

Note that the condition D-PR-NSP implies the classical D-NSP. To show this, take any set $\mathcal{K}$, $|\mathcal{K}| \leq k$, and any nonzero $\eta \in D^{-1}(\mathcal{N}(M))$. Let $\mathcal{S} = [1, \dots, m]$, and set

$$u = \eta, \quad v = \eta_{\mathcal{K}} - \eta_{\mathcal{K}^c}.$$

Since $MDu = 0$, $M_S Du$ is zero, and thus $u \in D^{-1}(\mathcal{N}(M_{\mathcal{S}}))$. Furthermore, $\mathcal{S}^c = \emptyset$, and therefore we can write $v \in D^{-1}(\mathcal{N}(M_{\mathcal{S}^c}))$. Finally, $u + v = 2\eta_{\mathcal{K}}$, so $\|u+v\|_0 \leq k$. Thus, by D-PR-NSP, there exist an $w' = 2w \in \mathcal{N}(D)$, such that

$$\|\eta_{\mathcal{K}} + w'\|_1 < \|\eta_{\mathcal{K}^c}\|_1,$$

and this is exactly the D-NSP. Now we can use Theorem 5.11 to conclude that we can successfully recover $x_0 = Dz_{\epsilon^*}$ by solving (5.12).

Now for any $\epsilon \in \{1,-1\}^m \neq \pm\epsilon^*$ that gives us another solution z_ϵ to (5.12), we conclude the following: Set $\mathcal{S}_* = \{j : \epsilon_j = \epsilon_j^*\}$. Then,

$$\langle g_j, z_\epsilon \rangle = \begin{cases} \langle g_j, z_{\epsilon^*} \rangle, & \text{if } j \in \mathcal{S}_*, \\ -\langle g_j, z_{\epsilon^*} \rangle, & \text{if } j \in \mathcal{S}_*^c. \end{cases}$$

Set $u := z_0 - z_\epsilon$ and $v = z_0 + z_\epsilon$. We see that $u \in D^{-1}(\mathcal{N}(M_{\mathcal{S}_*}))$ and $v \in D^{-1}(\mathcal{N}(M_{\mathcal{S}_*^c}))$. Furthermore, $u + v = 2z_0$, so it is k-sparse, and we can use the D-PR-NSP to conclude that there exist a $w \in \mathcal{N}(D)$ such that

$$\|z_0 + w\|_1 < \|z_\epsilon\|_1.$$

Since $z_0 + w$ is feasible to (5.12), the strict inequality tells us that z_ϵ can not be a minimizer to (5.12). Moreover, all possible solutions $z_0 + w$ give us a successful recovery, since $D(z_0 + w) = x_0$.

Necessary part. We assume that (5.10) can successfully recover all signals in $\mathbb{R}^N_{kD}$, but D-PR-NSP is not fulfilled. Namely, there exist a set $\mathcal{S} \subseteq [1, \dots, m]$, nonzero $u \in D^{-1}(\mathcal{N}(M_\mathcal{S}))$ and $v \in D^{-1}(\mathcal{N}(M_{\mathcal{S}^c}))$ such that $\|u+v\|_0 \leq k$ and for all $w \in \mathcal{N}(D)$:

$$\|u + v + w\|_1 > \|u - v\|_1. \tag{5.13}$$

Let $z_0 = u + v$, $\|z_0\| \leq k$. By assumption, we can successfully recover the corresponding $x_0 = Dz_0$ up to a sign by (5.10). Let $\hat{z}$ be a successful minimizer, meaning that $D\hat{z} = \pm Dz_0$ and that $\|\hat{z}\|_1 \leq \|z\|_1$ for any other feasible z. In particular, $\hat{z} - z_0$ or $\hat{z} + z_0 \in \mathcal{N}(D)$, and we can write this as

$$\hat{z} = z_0 + w \mod \mathbb{T} \quad \text{for some } w \in \mathcal{N}(D). \tag{5.14}$$

We notice that

$$|\langle g_j, z_0\rangle| = |\langle g_j, u + v\rangle| = |\langle g_j, u - v\rangle|$$

for all $j = 1, \dots, m$, since $\langle g_j, u\rangle = 0$ when $j \in \mathcal{S}$ and $\langle g_j, v\rangle = 0$ when $j \in \mathcal{S}^c$. Therefore, $u - v$ is also a feasible for (5.10), and hence we must have $\|\hat{z}\|_1 \leq \|u - v\|_1$. Using (5.14), we obtain

$$\|u + v + w\|_1 = \|z_0 + w\| \leq \|u - v\|_1.$$

This is a contradiction to (5.13), and thus the theorem is proven. □

5.4 Sparse Phase Retrieval by Projections

In this section we want to consider a generalization of the measurement process that we were considering so far — instead of linear measurements of the signal, which can also be viewed as rank one projections, the measurements are projections onto subspaces, and moreover, we are then given only the norms of those projections. This is the so-called *phase retrieval by projections* problem, see [20] for recent advances and many open problems. The sparsity prior was so far not considered in this context, and we contribute to it with two results: an equivalence condition for injectivity, and the sufficient number of measurements. We should note that unlike some other problems involving measuring by projections, here sparsity is not understood as the number of subspaces on which the signal lies (as we had for example in Section 4.3 of Chapter 4), but as the number of nonzeros of the signal. We will see, however, that further generalizations are possible.

Let $\mathcal{W} = \{\mathcal{W}_j\}_{j=1}^m$ be a collection of subspaces of $\mathbb{K}^N$, where $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$. By P_j we denote the orthogonal projection onto $\mathcal{W}_j$. We want to consider the class of k-sparse signals,

$$\mathbb{K}_k^N = \{x \in \mathbb{K}^N : \|x\|_0 \leq k\},$$

and pose the question whether we can recover such signals from phaseless measurements by projections, $\{\|P_j x\|\}_{j=1}^m$. More precisely, given the following mapping:

$$\mathcal{M}_{\mathcal{W}} : \mathbb{K}_k^N/\mathbb{T} \to \mathbb{R}_+^m : \mathcal{M}_{\mathcal{W}}(x) = \begin{bmatrix} \|P_1 x\|^2 & \|P_2 x\|^2 & \dots & \|P_m x\|^2 \end{bmatrix}^T, \tag{5.15}$$

we are interested mainly in two questions: When is this mapping injective (when $\mathcal{W}$ *allows k-sparse phase retrieval by projections*), and how many measurements do we need in order to have an injective map?

5.4.1 Injectivity

Since the classical phase retrieval problem is already well investigated, one natural approach would be to transfer the problem of recovery by projections into a recovery by vectors. The authors in [20] managed to do this in the case of recovery of arbitrary signals, by incorporating the orthogonal bases of the subspaces. We will show that this idea can be used when considering a particular class of signals, for example vectors which are k-sparse.

Theorem 5.14. *The following statements are equivalent:*

(i) $\{\mathcal{W}_j\}_{j=1}^m$ *allows k-sparse phase retrieval by projections ($\mathcal{M}_{\mathcal{W}}$ is injective).*

(ii) *Any choice of $\Phi = \{\phi_{j,d}\}_{j,d=1}^{m,D_n}$, where $\{\phi_{j,d}\}_{d=1}^{D_n}$ is an orthonormal basis of $\mathcal{W}_j$, allows k-sparse phase retrieval in $\mathbb{K}_k^N$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $\mathcal{M}_{\mathcal{W}}$ is injective, i.e. for all $x, y \in \mathbb{K}_k^N$ we have

$$\|P_j x\|_2^2 = \|P_j y\|_2^2, \, j = 1, \dots, m \;\Rightarrow\; x = y \mod \mathbb{T}.$$

Consider $x,\ y \in \mathbb{K}_k^N$. We need to show that if we have a union of ONB's as above satisfying

$$|\langle x, \phi_{j,d}\rangle|^2 = |\langle y, \phi_{j,d}\rangle|^2 \;\text{ for all }\; j = 1 \dots m,\; d = 1, \dots D_n, \tag{5.16}$$

then necessarily $x = y \mod \mathbb{T}$. If however (5.16) is true, then for every j it holds in particular

$$\|P_j x\|^2 = \sum_{d=1}^{D_n} |\langle x, \phi_{j,d}\rangle|^2 = \sum_{i=1}^{D_n} |\langle y, \phi_{j,d}\rangle|^2 = \|P_j y\|^2 \,.$$

Therefore, since x, $y \in \mathbb{K}_k^N$ and $\mathcal{M}_{\mathcal{W}}$ is injective, it follows that $x = y \mod \mathbb{T}$.

(ii) $\Rightarrow$ (i). Let x, $y \in \mathbb{K}_k^N$ be vectors with $\|P_j x\|^2 = \|P_j y\|^2$ for every $j = 1, \ldots, m$. Given such vectors, independent of the sparsity assumption, for each subspace $\mathcal{W}_j$ one can construct an orthonormal basis $\{\phi_{j,d}\}_{d=1}^{D_n}$ such that $|\langle x, \phi_{j,d}\rangle| = |\langle y, \phi_{j,d}\rangle|$ for all j and d (see Lemma 3.4 of [20].) Since any collection of that kind was assumed to have the k-sparse phase retrieval property, we can conclude that $x = y \mod \mathbb{T}$. □

Note that in the proof, we never explicitly used that the signals are actually k-sparse. We could substitute the set $\mathbb{K}_k^N$ with any other subclass of signals, for example dictionary sparse $\mathbb{K}_{kD}^N$, and the result will still hold. But then for this class of signals, we need to know, when a set of measurements is doing phase retrieval, and one option is to use the corresponding complement property. As we discussed in the previous section, the k and kD-complement properties are no longer equivalent to phase retrievability, and for recovery of k-sparse signals one needs to require the $2k$-complement property. We will use this idea to construct an example of subspaces allowing k-phase retrieval by projections in the next section.

5.4.2 Sufficient Number of Measurements

The question of the number of subspaces needed for injectivity was investigated for general signals in [20]. We will prove that instead of $2N-1$, now $4k-1$ subspaces are sufficient to recover a real signal which has k nonzero entries. The idea is to transfer the problem of phase retrieval by projections into a classical phase retrieval problem, by using the orthogonality of the vectors which span the subspaces. In order to build this construction we need one lemma, but first we state the main result. We focus here on the case $\mathbb{K} = \mathbb{R}$. A similar approach can be used to obtain result for complex sparse vectors, where the result for general signals is $4N-3$ subspaces each of dimension N. The question whether this number of measurements is also necessary (both in $\mathbb{R}$ and $\mathbb{C}$) is still open even for arbitrary signals.

Theorem 5.15. *Phase retrieval by projections in $\mathbb{R}_k^N$ is possible using $4k-1$ subspaces each of dimension smaller than $k-1$.*

Proof. First, let us assume that $\{\phi_n\}_{n=1}^{4k-1}$ are vectors in $\mathbb{R}^N$ which have the $2k$-complement property, and additionally that $\{\phi_n\}_{n=1}^{2k}$ and $\{\phi_n\}_{n=2k+1}^{4k-1}$ are orthonormal sets in $\mathbb{R}^N$. We prove that such construction is possible later in Lemma 5.16. If this is true, then we can build the $4k-1$ subspaces in the following manner. Let $I_j \subseteq \{1,\dots,2k\}$, $j = 1,\dots,2k$ be at this point arbitrary subsets of indices and let P_{I_j} be the orthogonal projection onto $W_{I_j} = \operatorname{span}\{\phi_n\}_{n\in I_j}$. Analogically, let $J_j \subseteq \{2k+1,\dots,4k-1\}$, $j = 1,\dots,2k-1$, and P_{J_j} corresponds to $W_{J_j} = \operatorname{span}\{\phi_n\}_{n\in J_j}$. Set $W_{J_{2k}} = \{0\}$ for convenience. We will show that we can choose I_i, J_j such that we can recover any k-sparse vector x from the measurements $\{\|P_{I_j}x\|^2, \|P_{J_j}x\|^2\}_{j=1}^{2k}$. Because of the orthonormality condition, we can write the measurements as

$$\|P_{I_j}x\| = \sum_{n\in I_j} |\langle x,\phi_n\rangle|^2, \quad \|P_{J_j}x\| = \sum_{n\in J_j} |\langle x,\phi_n\rangle|^2, \quad j = 1,\dots,2k.$$

We can further rewrite those equations in matrix form, if we use two indicator matrices of size $2k \times 2k$:

$$\begin{aligned} A = [a_{ij}]_{i,j=1}^{2k} : &\quad a_{ij} = 1, \text{ if } j = I_j, \\ B = [b_{ij}]_{i,j=1}^{2k} : &\quad b_{ij} = 1, \text{ if } j = J_j. \end{aligned}$$

For example, for the first set of measurements, the system is

$$\begin{bmatrix} \|P_{I_1}\|^2 \\ \dots \\ \|P_{I_{2k}}\|^2 \end{bmatrix} = A \begin{bmatrix} |\langle x,\phi_1\rangle|^2 \\ \dots \\ |\langle x,\phi_{2k}\rangle|^2 \end{bmatrix}, \tag{5.17}$$

and for B we have a corresponding system which involves $\{P_{J_i}\}_{i=1}^{2k}$, and $\{\phi_i\}_{i=2k+1}^{4k-1}$. We see now, that if the matrices A and B were invertible, we could find $|\langle x,\phi_j\rangle|^2$ for all $j = 1,\dots,4k-1$. And since we know that $\{\phi_j\}_{j=1}^{4k-1}$ has the $2k$-complement property, they allow k-sparse phase retrieval by Theorem 5.4. Thus, we can recover x. Therefore, $\{\mathcal{W}_j\}_{j=1}^{4k-1}$ allow k-sparse phase retrieval by projection, and the theorem would be proven. Since we are using the proof technique from [20], the matrices A and B are in fact the same except that they are of different size. Thus, we know from Lemma 3.4 in [20] that I_j and J_j can be chosen such that A and B are invertible. □

As it was seen in the proof, the problem boils down to finding an orthonormal system of vectors in $\mathbb{R}^N$ which has the $2k$-complement property. We show now that it is possible to construct such. We note at this point the connection between the *spark* and phase retrievability which was noticed in [13]: a set of $m \geq 2N-1$ vectors in $\mathbb{R}^N$ which is *full spark* (every collection of N vectors is linearly independent, or equivalently spans $\mathbb{R}^N$)

necessarily satisfies the complement property. If one is interested in the k-complement property, then any collection of vectors restricted to a subset of k elements has to be full spark.

Lemma 5.16. *Let $2k \leq N$. Then there exists a set of $2k-1$ orthonormal vectors in $\mathbb{R}^N$ which have the k-complement property.*

Proof. Let $U \in O(N)$ be an orthogonal matrix chosen at random from the orthogonal group according to the Haar measure. Consider its first $2k-1$ columns $(u_i)_{i=1}^{2k-1}$. We claim that these vectors have the k-complement property with probability 1.

To see why, let $\mathcal{S} \subseteq [1,\ldots,2k-1]$ and $\mathcal{K} \subseteq [1,\ldots,N]$, $|\mathcal{K}| = k$ be arbitrary. We need to prove that either $\{u_i^{\mathcal{K}}\}_{i\in\mathcal{S}}$ or $\{u_i^{\mathcal{K}}\}_{i\in\mathcal{S}^c}$ spans $\mathbb{R}^k$. One of $\mathcal{S}$ and $\mathcal{S}^c$ has cardinality at least k, and we can assume without loss of generality that it is $\mathcal{S}$. In order for $\{\psi_i^{\mathcal{K}}\}_{i\in\mathcal{S}}$ to span $\mathbb{R}^k$, there must exist a sub-choice of them which is linearly independent. This is then a $k \times k$ block of the original matrix U, and we will show that it has full rank with probability 1.

It is clear that it is sufficient to consider only the upper left $k \times k$ block of U, since by multiplying U with permutation matrices, which are orthogonal, we can move any block to the upper left position. Let $A \subseteq O(N)$ be the set of all orthogonal matrices with invertible upper left $k \times k$ block. We want to show that $\mu(A) = 1$, where μ denotes the Haar measure.

There is a useful characterization of the Haar measure in terms of the standard normalized measure on the sphere $\mathbb{S}^{N-1}$, denoted by σ_{N-1}. For every $A \subseteq O(N)$, and for every $\eta \in \mathbb{S}^{N-1}$,

$$\mu(A) = \sigma_{N-1}(\{V\eta : V \in A\}).$$

See, for example [82] for more details. Therefore, it is sufficient to show that $\mathcal{O} = \{Ve_1 : V \in A\}$ has full measure in $\mathbb{S}^{N-1}$. We will prove that $\mathcal{O}$ contains the set

$$\mathcal{P} = \left\{\eta = (\eta_1,\ldots\eta_N) \in \mathbb{S}^{N-1} : (\eta_1,\ldots,\eta_k) \neq 0\right\}.$$

Since this set has full measure in $\mathbb{S}^{N-1}$, that will conclude our proof.

Let $\mathcal{K} = [1,\ldots,k]$ and let further $\eta \in \mathcal{P}$. We will construct a matrix $U \in O(N)$ with invertible upper left $k \times k$ block and with $Ue_1 = \eta$ as follows:

1. First, let η be the first column of U. This secures $Ue_1 = \eta$.

2. Now choose $k-1$ vectors $\tilde{u}_2,\ldots,\tilde{u}_k$ in $\mathbb{R}^k$ which together with the normalized version of $\eta_{\mathcal{K}}$ form an ONB of $\mathbb{R}^k$. Take these vectors and fill up the upper left block of U. This block is then invertible.

3. Finally, fill the empty entries of the columns 2 to k with zeroes. Then the full first k columns are all normalized and mutually orthogonal. These can hence be completed with $N-k$ vectors $\tilde{v}_{k+1}, \ldots \tilde{v}_n$ to an ONB of $\mathbb{R}^N$. Choose these vectors as the last $N-k$ columns of U, which then becomes an element of $O(N)$.

In matrix form, we have obtained

$$U = \begin{bmatrix} \eta_{\mathcal{K}} & \tilde{U} & \tilde{V}_{\mathcal{K}} \\ \eta_{\mathcal{K}^c} & 0 & \tilde{V}_{\mathcal{K}^c} \end{bmatrix}.$$

This proves that $\mathcal{P} \subseteq \mathcal{O}$, and therefore the lemma is proven. □

Notice that the result for non-sparse (arbitrary) signals in the real case is that $2N-1$ subspaces each of any dimension less than N are sufficient for phase retrieval by projections [20]. There, the initial result is about subspaces of dimension less than $N-1$, and it is then shown how this number can be relaxed to N. Similar approach could be used to go from $k-1$ to k in the case of sparse signals. Also, the number of subspaces $4k-1$ could be decreased to $2k-1$. Looking at the proof of Theorem 5.15, we see that one only needs to find $2k-1$ vectors which allow k-sparse phase retrieval (previously we had $4k-1$ which had the $2k$-complement property), and additionally that they can be divided into two orthonormal sets in $\mathbb{R}^N$. The authors in [20] also mention that similar results hold in the complex case. This should be true also in the case of sparse signals, but we leave the details for further investigation.

Chapter 6

Phase Retrieval from Gabor Measurements

6.1 Introduction

In this Chapter we bring together the ideas from the last two chapters: Gabor measurement vectors, i.e., time-frequency shifts of a suitably chosen generator from Chapter 4, and the phase retrieval problem, i.e., recovery from the magnitude of the linear measurements from Chapter 5. Namely, we consider the problem of phase retrieval from Gabor measurements. We investigate condtions which guarantee injectivity of the phaseless Gabor measurements, and which type of generators satsify those conditions.

At first, we will provide injectivity condition for recovery of arbitrary signals from the full set of Gabor magnitude measurements, with a special case of signals which are non-vanishing. We will see that Gabor system generated by difference sets satisfy this condition and thus allow phase retrieval, while Alltop and random generators are not suitable for phase retreival of general signals. Next, we focus on injectivity condition for sparse signals. Here we will see that generators which are window functions in the Fourier domain are suitable for sparse phase retrieval. Finally, we will provide an algorithm for recovery of signals from Gabor magnitude measurements, and a modification of it in the case the signal is sparse. We will provide numerical experiments to compare this algorithm with known phase retrieval algorithms, and we will also discuss its stability. At the end, we will make a connection to the work conducted in Chapter 3. We will test the proposed algorithm on recovery of unions of discrete lines from the full set of Gabor magnitude measurements.

The remainder of this chapter is organized as follows. Section 6.2 is dedicated to the phase retrieval question for general signals, from all N^2 Gabor measurements. In Section 6.3, we focus on the sparse setting, and show that k-sparse phase retrieval is possible with order of k^3 Gabor measurements. A detailed description of the algorithm that we propose, and its empirical evaluation is presented in Section 6.4.

6.2 An Injectivity Condition for Arbitrary Signals

We want to pose the question under what conditions a signal x from some class $\mathcal{C} \subseteq \mathbb{C}^N$ can be recovered from a set of its *Gabor phaseless (magnitude) measurements* $\{|\langle x, g_\lambda\rangle|^2\}_{\lambda\in\Lambda}$, $\Lambda \subseteq \mathbb{Z}_N^2$. Recall that for $\lambda = (p, l) \in \mathbb{Z}_p \times \mathbb{Z}_p$ we use the notation $g_\lambda := \Pi_\lambda g$, where $\Pi_\lambda := M_\ell T_p$, and the translation and the modulation operators are defined for every $n \in \mathbb{Z}_N$ as

$$(T_p x)(n) = x(n-p), \quad (M_\ell x)(n) = \omega^{\ell n} x(n).$$

Since these measurements are invariant under multiplication with $c \in \mathbb{T} = \{c \in \mathbb{C}, |c| = 1\}$, the best we can hope for is to recover x up to a global phase. If we denote by $\mathcal{C}/\mathbb{T}$ the set of equivalence classes under the equivalence relation $x \sim y \Leftrightarrow \exists c \in \mathbb{T} : x = cy$, we can formally pose the problem as follows: Under what conditions on g is the map

$$\mathcal{M}_G : \mathcal{C}/\mathbb{T} \to \mathbb{R}_+^{|\Lambda|}, \quad x \mapsto \{|\langle x, g_\lambda\rangle|^2\}_{\lambda\in\Lambda}$$

injective?

Definition 6.1. We say that the Gabor system $G = \{g_\lambda\}_{\lambda\in\Lambda}$ associated to a generator $g \in \mathbb{C}^N$ is *allowing phase retrieval for* $\mathcal{C}$ (or has the *phase retrieval property*), if the map $\mathcal{M}_G$ is injective.

6.2.1 Injectivity for Full Gabor Measurements

We start by considering the problem of recovering arbitrary signals from all measurements, i.e. $\mathcal{C} = \mathbb{C}^N$ and $\Lambda = \mathbb{Z}_N^2$. In order to investigate which Gabor frames are allowing phase retrieval for this class, we will use a well known characterization of the phase retrieval property in the complex case, given via the properties of the kernel of the *PhaseLift* operator, also called *super analysis operator* in [13]. For a set of measurement vectors $\{f_i\}_{i=1}^m$ in $\mathbb{C}^N$ this operator is defined as

$$\mathcal{A} : \mathbb{C}^{N\times N} \to \mathbb{C}^m, \quad H \mapsto \{\langle H, f_i f_i^*\rangle_{HS}\}_{i=1}^m, \tag{6.1}$$

Notice that the mapping $\mathcal{A}$ with $H = xx^*$ gives exactly the phaseless measurements, $\langle H, f_i f_i^* \rangle_{HS} = |\langle x, f_i \rangle|^2$. Also note that in the previous chapter we chose the set of Hermitian matrices $\mathbb{H}^{N\times N}$ as the domain of $\mathcal{A}$. We define $\mathcal{A}$ this way to avoid some technicalities. However, the space of Hermitian matrices is a very natural domain in the context of phase retrieval, as we know from the following result.

Theorem 6.2 ([13]). *A set of measurement vectors $\{f_i\}_{i=1}^m$ in $\mathbb{C}^N$ allows phase retrieval if and only if the kernel of the associated map $\mathcal{A}$ does not contain any Hermitian matrices of rank 1 or 2.*

With this theorem, we can prove that the full set of N^2 Gabor phaseless measurements allows phase retrieval, as long as a simple condition is satisfied.

Theorem 6.3. *Let $g \in \mathbb{C}^N$ be a generator for which*

$$\langle g, g_\lambda \rangle \neq 0 \tag{6.2}$$

for every $\lambda \in \mathbb{Z}_N^2$. Then the corresponding Gabor frame $G = \{g_\lambda\}_{\lambda\in\mathbb{Z}_N^2}$ allows phase retrieval.

Proof. Theorem 6.2 suggests that we should investigate $\langle H, g_\lambda g_\lambda^* \rangle_{HS}$ for $H \in \mathbb{H}^{N\times N}$. Using the fact that the matrices $\Pi_\lambda = M_l T_p$ are unitary, and that the collection of them forms a basis in $\mathbb{C}^{N\times N}$ [99], $\langle \Pi_\lambda, \Pi_\mu \rangle_{HS} = N\delta_{\mu,\lambda}$, we can write H in terms of the elements of this basis:

$$H = \frac{1}{N} \sum_{\mu\in\mathbb{Z}_N^2} \langle \Pi_\mu, H \rangle_{HS} \Pi_\mu.$$

If $\mu = (p, \ell)$, we have

$$\langle \Pi_\mu, H \rangle_{HS} = \sum_{i\in\mathbb{Z}_N} \langle \Pi_\mu e_i, He_i \rangle = \sum_{i\in\mathbb{Z}_N} \left\langle \omega^{\ell(i+p)} e_{i+p}, He_i \right\rangle = \sum_{i\in\mathbb{Z}_N} \omega^{-\ell i} \langle e_i, He_{i-p} \rangle = \widehat{\mathcal{H}}_p(\ell),$$

where $\widehat{\mathcal{H}}_p$ denotes the (discrete) Fourier transform of the vector $\mathcal{H}_p$, defined by

$$\mathcal{H}_p(i) := H_{i,i-p}. \tag{6.3}$$

Note that $\mathcal{H}_p$ is in some sense the p-th 'band' of the matrix H. It hence holds

$$N \langle H, g_\lambda g_\lambda^* \rangle_{HS} = N \langle g_\lambda, Hg_\lambda \rangle = \sum_{p,\ell} \left\langle g_\lambda, \widehat{\mathcal{H}}_p(\ell)\Pi_{(p,\ell)} g_\lambda \right\rangle = \sum_{p,\ell} \widehat{\mathcal{H}}_p(\ell) \langle g_\lambda, \Pi_{(p,\ell)} g_\lambda \rangle.$$

If we write $\lambda = (q, j)$, we know by the commutation relations between translations and modulations (see Proposition 2.14) that $\Pi_{(p,\ell)} g_\lambda = \Pi_{(p,\ell)} \Pi_{(q,j)} g = \omega^{-jp} \omega^{\ell q} \Pi_{(q,j)} \Pi_{(p,\ell)} g$.

Using this, and the fact that Π_λ is unitary, we arrive at

$$N \langle g_\lambda, H g_\lambda \rangle = \sum_{p,\ell} \omega^{-jp} \omega^{\ell q} \widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle . \tag{6.4}$$

Now, assume that (6.4) vanishes for all $\lambda = (q, j) \in \mathbb{Z}_N^2$. Fixing j, we see that the above expression is just the value of the Fourier transform of the vector $V^q \in \mathbb{C}^N$ with pth entry

$$V^q(p) = \sum_\ell \omega^{\ell q} \widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle$$

evaluated at j. Since (6.4) equals zero for all j, the vector V^q vanishes for every q. Further, we observe that $V^q(p)$ is N times the value at q of the inverse Fourier transform of the vector $w^p \in \mathbb{C}^N$, where

$$w^p(l) = \widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle . \tag{6.5}$$

This expression must therefore be equal to zero for all p and ℓ. With the assumption on the generator, we conclude that all the vectors $\widehat{\mathcal{H}}_p$ must vanish, and therefore also H. H can hence not have rank 1 or 2, and the proof is finished. □

Carefully going through the argument of the last proof, we see that it did not assume that the rank of H is 1 or 2. Therefore, the proof actually shows that $\mathcal{A}$ is an injective linear map. We use this idea to prove the following theorem.

Theorem 6.4. *Let $g \in \mathbb{C}^N$ be such that $\langle g, g_\lambda \rangle \neq 0$ for all $\lambda \in \mathbb{Z}_N^2$. Then, the N^2 rank-1 operators $\{g_\lambda g_\lambda^*\}_{\lambda \in \mathbb{Z}_N^2}$ form a frame for $\mathbb{C}^{N \times N}$ (equipped with the Hilbert-Schmidt norm) and hence a basis. The frame bounds are given by*

$$A = N \cdot \min_{\lambda \in \mathbb{Z}^2} |\langle g, g_\lambda \rangle|^2 , \quad B = N \cdot \max_{\lambda \in \mathbb{Z}^2} |\langle g, g_\lambda \rangle|^2 .$$

Proof. What we need to prove is that for every $H \in \mathbb{C}^{N \times N}$, there exist $0 \leq A \leq B$ such that

$$A \|H\|_{HS}^2 \leq \sum_{\lambda \in \mathbb{Z}^2} |\langle H, g_\lambda g_\lambda^* \rangle_{HS}|^2 \leq B \|H\|_{HS}^2 .$$

In other words, we need to prove that $A \|H\|_{HS}^2 \leq \|\mathcal{A}(H)\|^2 \leq B \|H\|_{HS}^2$, where $\mathcal{A}$ is the PhaseLift operator (6.1). Using the notation of the proof of Theorem 6.3, the formula (6.4) states that the N-tuple $\{V^q\}_{q=1}^N \in (\mathbb{C}^N)^N$ is obtained by performing inverse Fourier transforms of the columns of the matrix $\{N \langle H, g_\lambda g_\lambda^* \rangle_{HS}\}_{\lambda \in \mathbb{Z}_N^2} = N\mathcal{A}(H)$. Hence, their norms are related as follows:

$$\|N\mathcal{A}(H)\|^2 = \left\|\{\hat{V}^q\}_{q=1}^N\right\|^2 = N\left\|\{V^q\}_{q=1}^N\right\|^2.$$

Using the same argument, we obtain

$$\left\|\{V^q\}_{q=1}^N\right\|^2 = \left\|\{N\check{w}^p\}_{p=1}^N\right\|^2 = \frac{N^2}{N}\left\|\{w^p\}_{p=1}^N\right\|^2 \text{ and } \left\|\{\widehat{\mathcal{H}}^p\}_{p=1}^N\right\| = N\left\|\{\mathcal{H}^p\}_{p=1}^N\right\|^2.$$

The N-tuples $\{\widehat{\mathcal{H}}^p\}_{p=1}^N$ and $\{w^p\}_{p=1}^N$ are related through (6.5). Therefore, if we define $\alpha = \min_{\lambda\in\mathbb{Z}^2} |\langle g, g_\lambda\rangle|$, $\beta = \max_{\lambda\in\mathbb{Z}^2} |\langle g, g_\lambda\rangle|$, we have

$$\left\|\{w^p\}_{p=1}^N\right\|^2 = \sum_{(p,\ell)\in\mathbb{Z}_N^2} |w^p(\ell)|^2 = \sum_{(p,\ell)\in\mathbb{Z}_N^2} \left|\widehat{\mathcal{H}}_p(\ell)\langle g, g_{p,\ell}\rangle\right|^2 \begin{array}{l} \le \beta^2 \left\|\{\widehat{\mathcal{H}}^p\}_{p=1}^N\right\|^2 \\ \ge \alpha^2 \left\|\{\widehat{\mathcal{H}}^p\}_{p=1}^N\right\|^2 \end{array}.$$

Finally, the matrix H is obtained by merely permuting the elements of the array $\{\mathcal{H}^p\}_{p=1}^N$. Hence $\|H\|_{HS}^2 = \left\|\{\mathcal{H}^p\}_{p=1}^N\right\|^2$. Combining everything, we obtain

$$\begin{aligned}\|\mathcal{A}(H)\|^2 &= \frac{N}{N^2}\left\|\{V^q\}_{q=1}^N\right\|^2 = \left\|\{w^p\}_{p=1}^N\right\|^2 \\ &\begin{cases} \le \beta^2 \left\|\{\widehat{\mathcal{H}}^p\}_{p=1}^N\right\|^2 & = N\beta^2\left\|\{\mathcal{H}^p\}_{p=1}^N\right\|^2 = N\beta^2\|H\|_{HS}^2, \\ \ge \alpha^2 \left\|\{\widehat{\mathcal{H}}^p\}_{p=1}^N\right\|^2 & = N\alpha^2\left\|\{\mathcal{H}^p\}_{p=1}^N\right\|^2 = N\alpha^2\|H\|_{HS}^2, \end{cases}\end{aligned}$$

which is exactly what we wanted to prove. □

We should note here that the result of Theorem 6.4 is known from the context of sampling of operators [102, Theorem 15]. Also, more general result in terms of the singular values of the operators $\{g_\lambda g_\lambda^*\}_{\lambda\in\mathbb{Z}_N^2}$ instead of the values of the frame bounds is given by the same authors in [101].

6.2.2 Recovery of Non-Vanishing Vectors

We will now show that if we are interested in recovery of only non-vanishing vectors, weaker condition on the generator can be assumed.

Definition 6.5. A vector $x \in \mathbb{C}^N$ is called *non-vanishing* (or *full*), if all its entries are nonzero, i.e.

$$x(n) \neq 0, \qquad \text{for all } n = 0, \dots, N-1.$$

By $\mathcal{C}_f$ we denote the class of all non-vanishing signals in $\mathbb{C}^N$.

This situation is much easier to handle, because, intuitively, the non-presence of "holes" in the signals keeps the phases of the entries coupled. We will use the same technique as in Theorem 6.3 to prove that the injectivity condition can be weakened in this setting. Note that we are still assuming that all measurements are known.

Theorem 6.6. *Assume that*

$$\langle g, g_{p,\ell} \rangle \neq 0 \textit{ for } p = 0, 1 \textit{ and } \ell \in \mathbb{Z}_N. \tag{6.6}$$

Then the Gabor frame $G = \{g_\lambda\}_{\lambda \in \mathbb{Z}_N^2}$ is allowing phase retrieval for $\mathcal{C}_f$.

Proof. Assume that (6.6) is satisfied, and that x and y are full vectors which give the same Gabor phaseless measurements. Then $H := xx^* - yy^*$ is in the kernel of $\mathcal{A}$, since for every $\lambda \in \mathbb{Z}_N^2$ we have

$$\mathcal{A}(xx^* - yy^*)(\lambda) = \langle g_\lambda, (xx^* - yy^*) g_\lambda \rangle = |\langle g_\lambda, x \rangle|^2 - |\langle g_\lambda, y \rangle|^2 = 0.$$

The proof of Theorem 6.3 then implies that $\widehat{\mathcal{H}}_p = 0$ for $p = 0$, 1, i.e. that $\mathcal{H}_0 = \mathcal{H}_1 = 0$. Remembering that $\mathcal{H}_p(i) = H_{i,i-p}$, we arrive at

$$0 = x(i)\bar{x}(i) - y(i)\bar{y}(i) = x(i)\bar{x}(i-1) - y(i)\bar{y}(i-1), \quad i = 0, \ldots, N-1.$$

The first equality simply says that $|x(i)| = |y(i)|$, i.e. that there exists numbers $\epsilon_i \in \mathbb{T}$ so that $x(i) = \epsilon_i y(i)$ for all i. Inserting this into the second equation yields

$$0 = y(i)\bar{y}(i-1)(\epsilon_i \bar{\epsilon}_{i-1} - 1).$$

Since all entries of y are assumed to be nonzero, it follows that $\epsilon_i = \epsilon_{i-1}$, i.e. $\epsilon_i = \epsilon_0 =: c \in \mathbb{T}$ for all i. Hence $x = cy$ for a $c \in \mathbb{T}$, and x and y are equal mod $\mathbb{T}$. □

Remark 6.7. A similar result was proven in [55]. There, it was only assumed that $\langle g, g_{p,\ell} \rangle \neq 0$ for $p = 0$ and all $\ell \in \mathbb{Z}_N$. However, in this case further constraints on the generators need to be made: g must be a window of length $W \geq 2$, where $N \geq 2W - 1$ and N and $W - 1$ are coprime. Our result, on the other hand, works for more general generators and any N.

6.2.3 Generators Which Allow Phase Retrieval

We will present two types of signals, one random and one deterministic, which satisfy condition (6.2), and thus can be used for phase retrieval of signals from all N^2 Gabor measurements.

6.2.3.1 Complex Random Vectors as Generators

We start by considering a probabilistic approach, a common strategy in signal recovery in general.

Proposition 6.8. *Let g be a vector in $\mathbb{C}^N$, randomly distributed according to the complex standard normal distribution. Then, the condition $\langle g, g_\lambda \rangle \neq 0$ for all $\lambda \in \mathbb{Z}_N^2$ is satisfied with probability* 1*.*

Proof. Since there are only finitely many λ's, it suffices to prove that $\langle g, \Pi_\lambda g \rangle \neq 0$ with probability 1 for one arbitrary λ. Since Π_λ is a unitary operator, there exists an orthonormal basis $\{q_i\}_{i=1}^N$ of $\mathbb{C}^N$ and $c_i \in \mathbb{T}$ with

$$\Pi_\lambda = \sum_{i=1}^N c_i q_i q_i^*.$$

If we expand g in this basis, i.e. $g = \sum_i h_i q_i$, then the vector $h \in \mathbb{C}^N$ will also be distributed according to the complex standard normal distribution [66]. We have $\Pi_\lambda g = \sum_i c_i h_i q_i$, and hence

$$\langle g, \Pi_\lambda g \rangle = \sum_{i=1}^N c_i \left| h_i \right|^2.$$

In order for g to not satisfy (6.2), the random variable $\mathfrak{h} = \{|h_i|^2\}_{i=1}^N$ on $\mathbb{R}_+^N$ must hence lie in the subspace of $\mathbb{R}^N$ defined by

$$\left\{ v : \sum_{i=1}^n c_i v_i = 0 \right\}.$$

Since this space has dimension $N-1$, the set has Lebesgue measure zero. If we prove that $\mathfrak{h}$ has a distribution which has a density with respect to the Lebesgue measure on $\mathbb{R}_+^N$ which is almost never zero, we are done. This is however not hard to see, since the variables $|h_i|^2 = |a_i|^2 + |b_i|^2$, $i = 1, \dots, N$ are independently distributed according to the χ_2^2-distribution, which has density $\rho(x) = \frac{1}{2}\exp(-x/2)$ on $\mathbb{R}_+$. □

6.2.3.2 Difference Sets as Generators

The second example are difference sets, a construction coming from combinatorial design theory [45], which we introduced in Chapter 4, Section 4.2. Interestingly, the set of all modulations and translations of a difference set has exactly the property desired for phase retrieval. Moreover, we will show that the Alltop sequences and random vectors, which were almost optimal for the applications in Chapter 4, are now not suitable for recovery of arbitrary vectors from phaseless measurements. Let us recall the definition of a difference set for completeness.

Definition 6.9. A subset $\mathcal{K} = \{u_1, \dots, u_K\}$ of $\mathbb{Z}_N$ is called an (N, K, ν) *difference set* if the $K(K-1)$ differences

$$(u_k - u_l) \mod N, \quad k \neq l$$

take all possible nonzero values $1, 2, \dots, N-1$, with each value appearing exactly ν times.

Given a difference set $\mathcal{K}$ with parameters (N, K, ν) we denote by $\chi_{\mathcal{K}} \in \{0,1\}^N$ its characteristic function. We now prove that if such characteristic functions are used as generators, the corresponding Gabor frames will satisfy (6.2), and hence allow phase retrieval for arbitrary signals.

Proposition 6.10. *Let N be an integer with a prime factorization $N = p_1^{a_1} \dots p_r^{a_r}$. Let $\mathcal{K}$ be a difference set with parameters (N, K, ν), such that*

$$\nu < \min\{p_1, \dots, p_r\}. \tag{6.7}$$

Then, for $g = \chi_{\mathcal{K}}$,

$$\langle g, g_\mu \rangle \neq 0 \text{ for every } \mu \in \mathbb{Z}_N^2. \tag{6.8}$$

Proof. Let $\mu = (q, j)$, with both $q, j \neq 0$. By just using the definition of g_μ and $\mathcal{K}$ we obtain

$$\langle g, g_\mu \rangle = \sum_{n \in \mathbb{Z}_N} g(n)(M_j T_q g)(n) = \sum_{n \in \mathbb{Z}_N} g(n) g(n-q) \omega^{jn} = \sum_{\substack{n \in \mathcal{K} \text{ and} \\ n-q \in \mathcal{K}}} \omega^{jn}. \tag{6.9}$$

Now, taking into account the nature of a difference set, we can conclude that in the set

$$\{n : n \in \mathcal{K}, n - q \in \mathcal{K}\}$$

there will be always exactly ν elements (because for $q \in \mathbb{Z}_N$ there are exactly ν ways to be written as a difference of elements in $\mathcal{K}$, and $n - (n-q)$ are such differences).

If $\nu = 1$, we are left with a single ω^{jn_0} and then certainly the sum is different from zero.

If $\nu \neq 1$, we have a sum of ν different N-th roots of unity, and we will show that with the given assumptions on the difference set, (6.8) holds. We use the following result from [86] about the vanishing sums of roots of unity. The main theorem in the work [86] states that for any $N = p_1^{a_1} \dots p_r^{a_r}$, the only possible amounts of N-th roots of unity that can sum up to zero is given by $M_1 p_1 + \ldots + M_r p_r$. Here the M_i are any non-negative integers (0 is included). Now it is clear that the condition $\nu < \min\{p_1, \ldots p_r\}$ will ensure that we will never have a vanishing sum.

If now $\mu = (0, j)$ the sum will go over the full set $\mathcal{K}$, and since $\mathcal{K}$ is a difference set, again this sum is non vanishing (see, for example, the proof of Proposition 4.3).

Finally, in the last case $\mu = (q, 0)$, we have a sum of ν ones, and therefore we have proven (6.8) for all cases $\mu \in \mathbb{Z}_N^2$. □

Example 6.1. We now provide some examples of families of difference sets, which satisfy the condition from Proposition 6.10.

Family 1: Quadratic Difference Sets. Let $q = p^r = 3 \pmod 4$ be a power of a prime and

$$N = q,\ K = \frac{q-1}{2},\ \nu = \frac{q-3}{4}.$$

Then $u = \{t^2 : t \in \mathbb{Z}_N \backslash \{0\}\}$ is a (N, K, ν) difference set. If $r = 1$, condition (6.7) is satisfied.

Family 2: Quartic Difference Sets. Let $p = 4a^2 + 1$ be a prime with a odd, and

$$N = p,\ K = \frac{p-1}{4},\ \nu = \frac{p-5}{16}.$$

Then $u = \{t^4 : t \in \mathbb{Z}_N \backslash \{0\}\}$ is a (N, K, ν) difference set and additionally $\nu < N$.

6.2.4 Generators Which Do Not Allow Phase Retrieval

We now consider two cases for which condition (6.2) is *not* satisfied, and show that this in fact implies that the Gabor frames do not allow phase retrieval in these cases.

Proposition 6.11. *Let $g \in \mathbb{C}^N$ be a generator such that one of the following two conditions is satisfied*

$$\langle g, g_{\hat{p},\ell} \rangle = 0, \quad \textit{for fixed } \hat{p} \in \mathbb{Z}_N \backslash \{0\} \textit{ and all } \ell \in \mathbb{Z}_N. \tag{6.10}$$

$$\langle g, g_{\hat{p},\ell} \rangle = 0, \quad \textit{for } \hat{p} = 0 \textit{ and all } \ell \in \mathbb{Z}_N \backslash \{0\}. \tag{6.11}$$

Then, the corresponding Gabor frame $G = \{g_\lambda\}_{\lambda \in \mathbb{Z}_N^2}$ does not allow phase retrieval for $\mathbb{C}_N$.

Proof. Let us first assume that condition (6.10) is satisfied. We consider the matrix $H_1 \in \mathbb{H}^{N\times N}$, defined by

$$H_1 = e_0 e_{-\hat{p}}^* + e_{-\hat{p}} e_0^*$$

(e_0 is the 'first unit vector' - remember that we are always considering indices from $\mathbb{Z}_N$). This matrix has rank 2, and it lies in the kernel of the PhaseLift operator associated to the Gabor frame defined in (6.1). To see this, note that using the notation of the proof of Theorem 6.3 we have

$$\mathcal{H}_p(i) = H_{i,i-p} = \begin{cases} 1 & \text{if } i = 0, p = \hat{p}, \\ 1 & \text{if } i = -\hat{p}, p = -\hat{p}, \\ 0 & \text{else.} \end{cases}$$

In other words, $\mathcal{H}_p = 0$ for all $p \neq \pm\hat{p}$. Since $\langle g, g_{p,\ell}\rangle = \omega^{-\ell p}\overline{\langle g, g_{-p,-\ell}\rangle}$, equation (6.10) also implies $\langle g, g_{-\hat{p},\ell}\rangle = 0$ for all $\ell \in \mathbb{Z}_N$. These two facts prove that

$$\widehat{\mathcal{H}}_p(\ell)\,\langle g, g_{p,\ell}\rangle = 0$$

for all ℓ and p. Using the technique of the proof of Theorem 6.3 backwards, it follows $\mathcal{A}(H_1) = 0$. The matrix H_1 that we have found has rank 2 and it is in the kernel of $\mathcal{A}$. Therefore, by Theorem 6.2, the Gabor frame can not allow phase retrieval.

Now we assume that (6.11) is satisfied. In this case we define a rank 2 matrix in $\mathbb{H}^{N\times N}$ by

$$H_2 = e_0 e_0^* - e_1 e_1^*.$$

For this matrix, $\mathcal{H}_p = 0$ for $p \neq 0$. Also $\widehat{\mathcal{H}}_0(0) = \sum_i H_{i,i} = 0$. Because of these two facts and the assumption on g, we again have

$$\widehat{\mathcal{H}}_p(\ell)\,\langle g, g_{p,\ell}\rangle = 0 \quad \text{for all } (p,\ell) \in \mathbb{Z}_N^2,$$

and H_2 will by the same argument as before be in the kernel of $\mathcal{A}$. Phase retrieval is again not possible. $\square$

Example 6.2. We now give two examples, for which the conditions of the previous proposition are satisfied.

Short windows: The condition (6.10) is satisfied if the generator g is a "short window". More precisely, if $\operatorname{supp} g \subseteq [K_1, K_2]$ for $|K_1 - K_2| < \frac{N}{2}$, then g and $M_\ell T_p g$ will have

$$H_1 = \begin{bmatrix} 0 & & 1 & & \\ & \ddots & & & \\ 1 & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \quad H_2 = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

Figure 6.1: The matrices H_1 and H_2 used in the proof of Proposition 6.11.

disjoint supports for some p's and hence have a vanishing scalar product. Using a window as a generator is a core idea in short-time Fourier analysis [99].

Alltop sequence: It can be easily shown that the Alltop sequence [5], which is defined as $\{\frac{1}{\sqrt{N}}\omega^{n^3}\}_{n=0}^{N-1}$, has the property (6.11). As we saw in Chapter 4, this generator is often and successfully used in sparse signal recovery from linear Gabor measurements [9, 99].

However, both these families of signals can not be used for phase retrieval, when we are interested in recovery of all signals in $\mathbb{C}^N$.

6.3 An Injectivity Condition for Sparse Signals

Here we would like to focus on recovery of sparse signals, or signals sparse in a dictionary. Since sparse vectors in some sense are k– and not N-dimensional, one would hope that the number of measurements required to recover them can be decreased (in our case, $\Lambda < N^2$). This, and other, questions for sparse signals were considered and answered for general measurement vectors in [97] and [125]. We generalized some of them to the dictionary sparse setup in Chapter 5. We proved Theorem 5.9, a counterpart of Theorem 6.2 in the case of dictionary sparse signals. We formulate it here in the case of classical sparsity, i.e. $D = I$. Then with the help of it we will investigate the case of Gabor measurements as in the previous section.

Let $\{f_j\}_{j=1}^m$ be a set of measurement vectors in $\mathbb{C}^N$ and let $\mathcal{A}_\mathcal{K}$ be the PhaseLift operator restricted onto a set $\mathcal{K} \subseteq [1, \ldots, N], |\mathcal{K}| = k$, and defined as

$$\mathcal{A}_\mathcal{K} : \mathbb{C}^{k\times k} \to \mathbb{C}_+^m : \quad (A_\mathcal{K} H)(j) = \langle H, f_j^\mathcal{K} f_j^{\mathcal{K}*}\rangle_{HS}.$$

Theorem 6.12. *Given the notations from above, the following two statements hold.*

(i) *If $\mathcal{M}_F$ is not injective, then there exist $\mathcal{K} \subseteq [1, \ldots, N]$, $|\mathcal{K}| \leq 2k$, such that $\mathcal{A}_\mathcal{K}$ has a Hermitian matrix with rank 1 or 2 in its null space.*

(ii) *If there exist $\mathcal{K} \subseteq [1, \ldots, N]$, $|\mathcal{K}| \leq k$, such that $\mathcal{A}_\mathcal{K}$ has a rank 1 or 2 Hermitian matrix in its null space, then $\mathcal{M}_F$ is not injective.*

We will prove a condition under which a subset of our Gabor frame $\{g_\lambda\}_{\lambda \in \mathbb{Z}_N^2}$ with $\sim k^3$ elements allows k-sparse phase retrieval, when N is prime. For general N, it would still be possible to go below the full set of measurements, N^2. We will need a special form of the discrete uncertainty principle, which involves the sum of the "spread" of the signal and its Fourier transform. Let us start with a general observation.

Lemma 6.13. *Assume that for all nonzero vectors $f \in \mathbb{C}^N$*

$$\|f\|_0 + \|\hat{f}\|_0 \geq N - \theta_N \tag{6.12}$$

holds for some number θ_N. Then, if f is k-sparse ($\|f\|_0 = k$), and $\hat{f}$ is known to have $\theta_N + k + 1$ zero-entries, then f necessarily has to vanish.

This statement follows immediately by contradiction. The question is whether (6.12) is a reasonable assumption. In [119] it is proved that when N is prime, (6.12) holds with θ_N equal to -1. For general N, by the standard multiplicative uncertainty principle and the geometric mean-arithmetic mean inequality, one can derive (6.12) with $\theta_N = N - 2\sqrt{N}$. A more involved inequality for general N was obtained in [91] and will be discussed later on.

Before we proceed with a condition on the generator g for sparse phase retrieval, we will first prove a more general statement about recovery of sparse matrices from *linear* measurements, which is interesting on its own. We will be interested in the following class of signals,

$$\mathfrak{H}_K = \left\{H \in \mathbb{C}^{N\times N} : \exists \mathcal{K} \subseteq [1, \ldots N], |\mathcal{K}| = k : H_{ij} = 0 \text{ if } (i,j) \notin \mathcal{K} \times \mathcal{K}\right\}.$$

Theorem 6.14. *Let N be such that the uncertainty principle (6.12) holds, and let $\lambda = (p, l) \in \mathbb{Z}_N^2$. Let g have the following property: for each ℓ, the sequence $c_p = (\langle g, g_{p,\cdot}\rangle)$ formed by letting ℓ run obeys*

$$\theta_N + k + 1 \leq \|c_p\|_0 \leq \hat{k} \tag{6.13}$$

for some K and $\hat{k}$. Then, for any subsets $A \subseteq \mathbb{Z}_N$, $B \subseteq \mathbb{Z}_N$ with

$$|A| \geq \theta_N + \hat{k} + 1, \quad |B| \geq \theta_N + k^2 - k + 2,$$

the following holds. If a matrix $H \in \mathfrak{H}_K$ satisfies $\{\langle g_\lambda g_\lambda^*, H\rangle_{HS}\}_{\lambda \in A\times B} = 0$, then $H = 0$.

Proof. Let $H \in \mathfrak{H}_K$ satisfy $\langle g_\lambda g_\lambda^*, H \rangle_{HS} = 0$ for $\lambda \in A \times B$ and let $\mathcal{K}$ be such that $H_{ij} = 0$ if $(i,j) \notin \mathcal{K} \times \mathcal{K}$. We will prove that H then must be 0. Recall the notation from the proof in Theorem 6.3, $\mathcal{H}_p(i) = H_{i,i-p}$. Since $H_{i,i-p}$ is zero, if $(i, i-p)$ is not in $\mathcal{K} \times \mathcal{K}$, we can conclude that

$$\mathcal{H}_p(i) = H_{i,i-p} = 0 \quad \text{ if } i \notin \mathcal{K} \cap (\mathcal{K} + p).$$

This proves the following properties:

1. The vectors $\mathcal{H}_p$ are k-sparse.

2. $\mathcal{H}_p$ is zero for all but at most $k^2 - k + 1$ different values for p. To see this, notice first that $\mathcal{H}_p = 0$ if $p \notin \mathcal{K} - \mathcal{K}$. This is because if $\mathcal{H}_p(i) \neq 0$, then $i \in \mathcal{K}$ and there additionally exists a $j \in \mathcal{K}$ with $i = j + p$. It follows $p = i - j \in \mathcal{K} - \mathcal{K}$. And we know that the set $\mathcal{K} - \mathcal{K}$ has at most $|\mathcal{K}|\,(|\mathcal{K}| - 1) + 1 = k^2 - k + 1$ elements.

Now using the same argument as in the proof of Theorem 6.3, we arrive at

$$0 = N \langle g_\lambda g_\lambda^*, H \rangle = \sum_{p,\ell} \omega^{-jp} \omega^{\ell q} \widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle \quad \text{ for all } \lambda = (q,j) \in A \times B. \tag{6.14}$$

Fixing j, the sum in (6.14) is the value at j of the discrete Fourier transform of the vector V^q defined as

$$V^q(p) = \sum_\ell \omega^{\ell q} \widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle . \tag{6.15}$$

Because of 2., these vectors are all $(k^2 - k + 1)$-sparse. Further, (6.14) proves that their Fourier transforms vanish at all $j \in B$, i.e. at $\theta_N + (k^2 - k + 2)$ points. The discrete uncertainty principle (6.12) implies that V^q must equal zero.

Considering (6.15), the fact that $V^q(p) = 0$ proves that the inverse Fourier transform of the vector, which we denote by

$$w^p(\ell) = \widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle$$

vanishes at the values $q \in A$, i.e. at $\theta_N + \hat{k} + 1$ values. Because of our assumption on g, w^p is however $\hat{k}$-sparse. We can therefore again conclude that

$$\widehat{\mathcal{H}}_p(\ell) \langle g, g_{p,\ell} \rangle = 0 \quad \text{ for all } (p,\ell) \in \mathbb{Z}_N^2.$$

Hence, if $\langle g, g_{p,\ell} \rangle \neq 0$, $\widehat{\mathcal{H}}_p(\ell)$ must be 0. Due to our assumption on g, this happens for at least $\theta_N + 2k + 1$ ℓ's for every p. Because of 1., this is sufficient to prove that $\mathcal{H}_p = 0$ for all p, and H therefore must be 0. $\square$

We now use the theorem we have just proved, to provide a condition, when a Gabor frame can do k-sparse phase retrieval.

Theorem 6.15. *Let N be such that the uncertainty principle (6.12) holds, and let $\lambda = (p, l) \in \mathbb{Z}_N^2$. Let $g \in \mathbb{C}^N$ be a generator which satisfies the following condition: for each ℓ, the sequence $c_p = (\langle g, g_{p,\cdot} \rangle)$ formed by letting ℓ run obeys*

$$\theta_N + 2k + 1 \leq \|c_p\|_0 \leq \hat{k} \tag{6.16}$$

for some k and $\hat{k}$. Then, for any subsets $A \subseteq \mathbb{Z}_N$, $B \subseteq \mathbb{Z}_N$ with

$$|A| \geq \theta_N + \hat{k} + 1, \quad |B| \geq \theta_N + (2k)^2 - 2k + 2,$$

the set

$$\{g_\lambda,\, \lambda \in A \times B\} \tag{6.17}$$

allows k-sparse phase retrieval.

Proof. We will use part (i) of Theorem 6.12 to show that k-sparse phase retrieval is possible for the system (6.17). Let H be an Hermitian operator with values in $\mathbb{C}^N_{\mathcal{K}} = \{x \in \mathbb{C}^N, \operatorname{supp}(x) \subseteq \mathcal{K}\}$ for some $\mathcal{K}$ with $|\mathcal{K}| = 2k$ for which $\mathcal{A}(H) = 0$ (where $\mathcal{A}$ is the PhaseLift operator associated with (6.17)). We will prove that H must be zero, from which the claim follows. Since the range of H is contained in $\mathbb{C}^N_{\mathcal{K}}$, we know that $H_{i,j} = 0$ if $i \notin \mathcal{K}$. Since $H_{i,j} = \overline{H_{j,i}}$, we also have $H_{i,j} = 0$ if $j \notin \mathcal{K}$. We can conclude that $H \in \mathfrak{H}_{2K}$, and by Theorem 6.14 it immediately follows that H is zero, thus the theorem is proved. □

Let us make few remarks related to Theorems 6.15 and 6.14.

1. If $\theta_N + 4k^2 - 2k + 2 \geq N$, then the same theorem holds for $B = \mathbb{Z}_N$ and any A with $|A| \geq \theta_N + \hat{k} + 1$. We can therefore also in this case reduce the number of measurements from N^2 to $(\theta_N + \hat{k} + 1)N$. Also note that since $\hat{k}$ must not be smaller than $2k$, the theorem does not yield any enhanced results for non-sparse vectors (we need the sequences c_p to be $\hat{k}$-sparse to reduce the number of measurements, but to have at least $2k$ nonzero elements to ensure injectivity for k-sparse signals).

2. We note, that when N is prime, the conditions of Theorem 6.15 become much simpler. Namely,

$$2k \leq \|c_p\|_0 \leq \hat{k},$$

and the sets A and B should fulfill

$$|A| \geq \hat{k}, \quad |B| \geq (2k)^2 - 2k + 1.$$

Thus, for example, if we can find a Gabor system for which the inequality (6.16) is fulfilled as an equality, we will be able to do k-sparse phase retrieval with order of only $O(k \min(N, k^2))$ measurements.

3. When N is not prime, we have $\theta_N = N - 2\sqrt{N}$, and the number of needed measurements is not as good as in the prime case, since we obtain

$$|A| \cdot |B| \geq (N - 2\sqrt{N} + \hat{k} + 1)(N - 2\sqrt{N} + (2k)^2 - 2k + 2),$$

but some improvement over N^2 could still be obtained in some cases. Furthermore, an extension of [119] from N prime to general N was published in [91], in the form of the following property:

Let $d_1 < d_2$ be two consecutive divisors of N. If $d_1 \leq k = \|f\|_0 \leq d_2$, then

$$\|\hat{f}\|_0 \geq \frac{N}{d_1 d_2}(d_1 + d_2 - k).$$

Our function θ will in this case explicitly depend on k and be equal to $N - k + \frac{N}{d_1 d_2}(d_1 + d_2 - k)$. The smaller this value is, the less measurements will be needed for k-sparse injectivity.

4. Theorem 6.14 is interesting from a different perspective, since $\mathfrak{H}_K$ can be viewed as a set of k^2-sparse vectors in $\mathbb{C}^{N^2}$ whose sparsity has a special structure. Thus, we have provided a deterministic construction which can theoretically recover those vectors from $O(k^3)$ linear measurements. This is interesting since we know from conventional compressed sensing results [64], that deterministic constructions for stable recovery of k^2-sparse vectors require $O(k^4)$ linear measurements, whereas random constructions only need $O(k^2)$ measurements. Finding deterministic constructions which can accept sparsity levels on the order higher than the square root of the number of measurements is known as breaking the "square-root bottleneck" [93]. Although in our case $O(m)$ measurements are needed for sparsity level $m^{2/3}$, one has to bear in mind that the sparsity of the vectors is structured, and that our result is only about the injectivity of the measurements. In particular, we do not prove any recovery guarantees for a specific algorithm. Hence, we have not broken the square-root bottleneck, but the theorem can be seen as a step towards providing new results in this direction. Furthermore, we can introduce even more structure to the signal, and decrease the number of measurements needed even further. Namely, if we assume that the set $\mathcal{K}$ contains K *consecutive* indices, or if we

in other words have “block sparse” matrix structure, we can decrease the number of measurements from $O(k^3)$ to $O(k^2)$, when N is prime. We will prove this result below.

We will show below that if the signal is structured further, we can decrease the number of measurements needed from k^3 to k^2. We are interested in the following class of signals:

$$\mathfrak{B}_K = \{H \in \mathbb{C}^{N\times N} : \exists k \in [1,\dots,N] \text{ such that } \mathcal{K} = \{k, k+1, \dots, k+K-1\} \\ \text{and } H_{ij} = 0 \text{ if } (i,j) \notin \mathcal{K}\times\mathcal{K}\}.$$

Corollary 6.16. *Let N be such that the uncertainty principle (6.12) holds for some θ_N. Let $(p,\ell) \in \mathbb{Z}_N^2$. Let $g \in \mathbb{C}^N$ be a generator which satisfies (6.16). Then, for any subsets $A \subseteq Z_N$, $B \subseteq Z_N$ with*

$$|A| \geq \theta_N + \hat{k} + 1, \quad |B| \geq \theta_N + 2k,$$

the following holds: if a matrix $H \in \mathfrak{B}_K$ satisfies $\{\langle g_\lambda g_\lambda^*, H\rangle_{HS}\}_{\lambda\in A\times B} = 0$, then $H = 0$.

Proof. This corollary will follow straight forward from a simple observation about the set $\mathcal{K}$, when it contains consecutive indices. Namely, the number of elements of $\mathcal{K} - \mathcal{K}$, which we counted in point 2. in the proof of Theorem 6.14, will now be $2k-1$ instead of $k^2 - k + 1$. That allows us to take only $\theta_N + 2k$ elements in the set B. The rest of the proof and the subsequent conclusions remain unchanged. □

It is now easy to see that when $\hat{k} = k$ and N is prime, since $\theta_N = -1$, the number of measurements needed for injectivity is $|A| \cdot |B| \geq k(2k-1)$. This is of the order of the sparsity of H (counted as the number of nonzeros in the matrix), k^2.

6.3.1 Functions Window in the Fourier Domain as Generators

As in the previous section, we now provide an example of a generator g which fulfills the condition for sparse phase retrieval from Gabor measurements.

Proposition 6.17. *Let N be prime and $2k+1 < N$. Further, let $v \in \mathbb{C}^N$ be a window of length $k+1$, $v = \chi_{[0,k]}$, where χ_A denotes the characteristic function on the set $A \subseteq \mathbb{Z}_N$. Moreover, let g be defined by $\hat{g} = v$, Then, g satisfies (6.16) with $\hat{k} = 2k+1$ and therefore, $(2k+1)\min(4k^2 - 2k + 1, N)$ measurements from the Gabor frame with g as generator will do k-sparse phase retrieval.*

Proof. The Plancherel formula implies that

$$c_p(\ell) = \langle \hat{g}, T_\ell M_p \hat{g} \rangle \quad \text{for all } \ (p, \ell) \in \mathbb{Z}_N^2$$

Therefore,

$$\langle \hat{g}, T_\ell M_p \hat{g} \rangle = \sum_{m \in \mathbb{Z}_N} \overline{v(m)} v(m-\ell) \omega^{p(m-\ell)} = \sum_{\substack{m \in [0,k] \text{ and} \\ m-\ell \in [0,k]}} \omega^{p(m-\ell)}, \tag{6.18}$$

because of the way we defined v. Note that since $2k < N$, this sum is empty for $|\ell| > k$. Therefore, the sequences c_p are $2k+1$-sparse for every p, i.e. $\hat{k}$-sparse.

It remains to prove that for $|\ell| \leq k$, the expression above is not zero, and hence $\|c_p\|_0 = 2k+1$. It suffices to consider $\ell \geq 0$, since the other case can be obtained from this one by the substitution $\ell \to -\ell$. Using the formula for geometric sums, we obtain

$$\sum_{\ell \leq m \leq k} \omega^{p(m-\ell)} = \sum_{n=0}^{k-\ell} \omega^{pn} = \begin{cases} \dfrac{1-\omega^{p(k-\ell+1)}}{1-\omega^p}, & \text{if } p \neq 0, \\ k-\ell+1, & \text{if } p = 0. \end{cases}$$

The only way this could be zero when $\ell \leq k$ is that $p \neq 0$ and $1 - \omega^{p(k-\ell+1)} = 0$. This would however mean that N is a divisor of $p(k-\ell+1) \neq 0$. Since N is prime and both p and $(k-\ell+1)$ are smaller than N, this cannot be the case. Therefore, from Theorem 6.15 we conclude that any subsets $A \subseteq \mathbb{Z}_N$, $B \subseteq \mathbb{Z}_N$ with

$$|A| \geq 2k+1, \quad |B| \geq (2k)^2 - 2k + 2$$

will do k-sparse phase retrieval. □

The choice of $\hat{g}$ as a characteristic function of $[0, k]$ is not necessary – any generically chosen function with support on $[0, k]$ will also lead to a Gabor system with the same properties. To see this, note that if $|\ell| < k$, the expression (6.18) is a non-trivial polynomial in the variables $\mathfrak{re}(v), \mathfrak{im}(v)$. Since we have proved that there is a particular choice of v so that all polynomials do not vanish on v, (6.16) will be satisfied for generic v.

The matrices provided to prove that the frames considered in Section 6.2.4 do not allow phase retrieval were all matrices with range in $\mathbb{C}^N_{\mathcal{K}}$ for a $\mathcal{K}$ with $|\mathcal{K}| = 2$. Hence, the considerations made there in fact proved that the frames are not allowing phase retrieval for $\mathcal{C}_k$ for any $k \geq 2$ (although they might still allow phase retrieval for some other class of signals).

6.3.2 Signals Sparse in Fourier domain

After spending some time discussing the standard sparsity case, it is worth noting that similar results hold for signals which are sparse in the Fourier basis (dictionary) F. Recall the famous commutation relation of F with translations and modulations:

$$\Pi_{(p,\ell)}F = M_\ell T_p F = F T_\ell M_{-p} = \omega^{\ell p} F \Pi_{(-p,\ell)} \quad \text{for all } (p,\ell) \in \mathbb{Z}_N^2. \tag{6.19}$$

This formula allows us to translate the results provided in the previous section to this new setting.

Theorem 6.18. *Let N be such that the uncertainty principle (6.12) holds and F denote the Fourier basis. Let g have the following property: for each ℓ, the sequence $\tilde{c}_\ell = (\langle g, g_{\cdot,\ell}\rangle)$ formed by letting p run obeys*

$$\theta_N + 2k + 1 \leq \|\tilde{c}_\ell\|_0 \leq \hat{k} \tag{6.20}$$

for some k and $\hat{k}$. Then, for any subsets $A, B \subseteq \mathbb{Z}_N$ with

$$|A| \geq \theta_N + (2k)^2 - (2k) + 2, \quad |B| \geq \theta_N + \hat{k} + 1,$$

the set $\{g_\lambda, \lambda \in A \times B\}$ allows Fk-sparse phase retrieval.

Proof. We would like to apply now the general Theorem 5.9, with $D = F$. Using the notation of that theorem, let H be an arbitrary Hermitian matrix with range contained in $W_{\mathcal{K}}$ for some $\mathcal{K}$ with $|\mathcal{K}| = 2k$. We may write $H = FH^F F^*$ for some other Hermitian H^F, which then has a range which is contained in $\mathbb{C}^N_{\mathcal{K}}$. Let us now proceed as in the proof of Theorem 6.3 and calculate

$$\begin{aligned}\left\langle \Pi_{(p,\ell)}, H\right\rangle_{HS} &= \operatorname{tr}(\Pi^*_{(p,\ell)} F H^F F^*) = \operatorname{tr}(F^*\Pi^*_{(p,\ell)} F H^F) = \left\langle F^*\Pi_{(p,\ell)}F, H^F\right\rangle_{HS} \\ &= \left\langle \omega^{\ell p}\Pi_{-p,\ell}, H^F\right\rangle_{HS} = \omega^{-\ell p}\widehat{\mathcal{H}}^F_{-\ell}(p).\end{aligned}$$

We used the commutation relation (6.19), and the fact that F is unitary. We arrive at

$$N\left\langle g_\lambda, Hg_\lambda\right\rangle = \sum_{p,\ell} \omega^{-\ell p}\widehat{\mathcal{H}}^F_{-\ell}(p)\left\langle g_\lambda, \Pi_{(p,\ell)}g_\lambda\right\rangle.$$

This formula is very similar to (6.14), essentially, the only difference is that the roles of p and ℓ have interchanged. Further, the vectors $\mathcal{H}^F$ have the same properties as the vectors $\mathcal{H}$ in the proof of Theorem 6.14 (since H^F has the same properties as H). These two facts makes it clear that we can use the exact same technique as in that proof to prove this theorem. We leave the details to the reader. □

It is not hard to construct a concrete example of a generator g which fulfills the condition (6.20). We only have to note that the roles of translations and modulations have been interchanged. Hence, we should no longer use a g which has short support in Fourier domain, but instead one with short support in spatial domain. With this insight, we may use the exact same steps as in the proof of Proposition 6.17 to deduce the following.

Proposition 6.19. *Let N be prime and $2k+1 < N$. Then $(2k+1)\min(4k^2-2k+1, N)$ measurements from a Gabor frame generated by generic windows g of length $k+1$ will do Fk-sparse phase retrieval.*

6.4 An Algorithm for Phase Retrieval Using Gabor Measurements

The idea of the proof of Theorem 6.15 can be used to design an algorithm to reconstruct signals from their Gabor phaseless measurements. We start by recovering H, as in the proof, and then we compute the closest rank 1 operator xx^* by spectral decomposition of H. A detailed description is given in Algorithm 1.

Algorithm 1: Simple Gabor Phase Retrieval (SGPR)

Data: A generator $g \in \mathbb{C}^N$, sets $A, B \subseteq \mathbb{Z}_N$, the measurements
$b(q,j) = N\left|\langle x, g_{q,j}\rangle\right|^2, (q,j) \in A \times B$.
Result: An estimate $x_0 \in \mathbb{C}^N$ of x.

1 **for** $q = 0 \ldots N-1$ **do**
  Solve $\hat{V}^q(j) = b(q,j)$, $j \in B$ for V^q.
2 **for** $p = 0 \ldots N-1$ **do**
  Solve $N \cdot \check{w}^p(q) = V^q(p)$, $q \in A$ for w^p.
3 **for** $p = 0 \ldots N-1$ **do**
  for $\ell = 0 \ldots N-1$ **do**
    if $\langle g, g_{p,l}\rangle \neq 0$ **then**
      Set $\hat{\mathcal{H}}_p(\ell) = w^p(\ell)/\langle g, g_{p,l}\rangle$.
      Add ℓ to the set Λ_p.
  Solve $\widehat{\mathcal{H}}_p(\ell) = w^p(\ell)/\langle g, g_{p,l}\rangle$, $\ell \in \Lambda_p$ for $\mathcal{H}_p$.
Reconstruct H from $H(i, i-p) = \mathcal{H}_p(i)$
Calculate the eigenpair (λ, v) of H corresponding to the largest eigenvalue.
Set $x_0 = \sqrt{\lambda}x$.

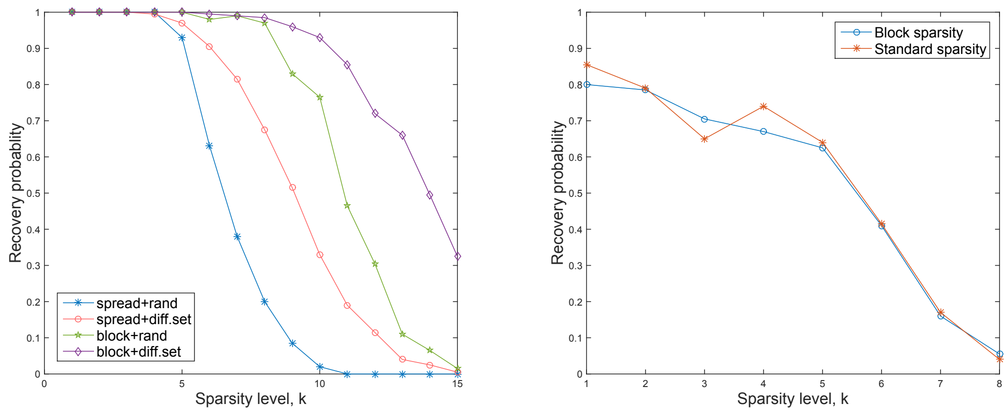
In steps (1), (2) and (3) one has to invert a Fourier transform. If all values of the transformed vector are known, one can simply use the standard fast inverse Fourier transform to compute this, and the signal will be perfectly recovered. If one on the other hand does not know all the values (not all Gabor measurements are given), some other method has to be used, where sparsity can be employed. We have chosen Basis

Pursuit [40]. This is a standard approach in compressed sensing when looking for a sparse solution x of the equation $Ax = b$. The algorithm consists of solving the following optimization problem:

$$\min \|x\|_1 \text{ s.t. } Ax = b.$$

We solve this problem with CVX, a package for specifying and solving convex programs [68].

We now present the results of the numerical experiments for testing Algorithm 1. In Figure 6.2, we plot the success rate of recovery of sparse signals via Algorithm 1. We have fixed the length of the signal $N = 67$ (a prime which gives $3 \pmod 4$), as needed for difference sets of *Family 1*). We want to recover two types of signals: k-sparse signals, where k nonzero random values are distributed on a random support, and k-sparse block signals, where one random block of k subsequent entries is assigned k random values.



(A) Partial modulations only. (B) Partial modulations and translations.

Figure 6.2: Success rate of Gabor phase retrieval via SGPR.

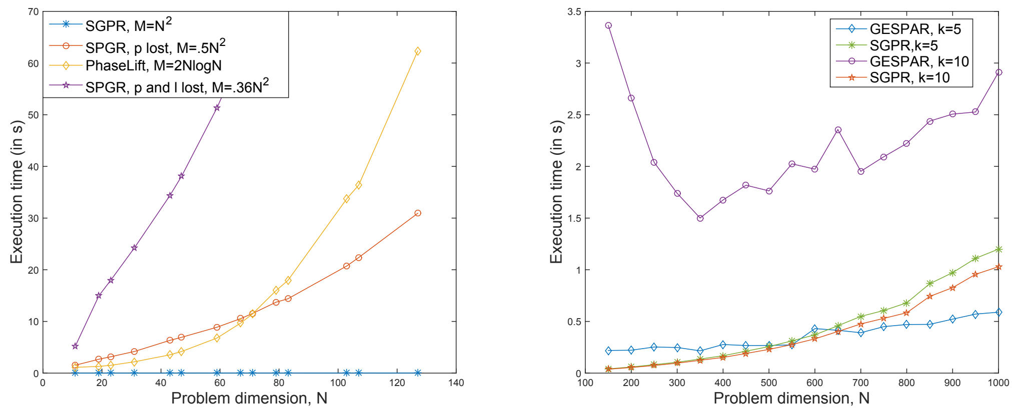
In Figure 6.2A, we also have chosen two different generators for the Gabor system: a complex random signal, and a characteristic function of a difference set, described in Section 6.2. Here, we use $0.5N^2$ measurements, namely all N translations, and random $0.5N$ from the modulations. With this setup, we use ℓ_1 minimization only in the Step (1), and in (2) and (3) we use the fast Fourier transform. For a fixed sparsity from 1 to 15, we repeat the experiment $T = 200$ times, and count a trial as successful, if the normalized squared error is smaller than 10^{-2}.

In Figure 6.2B, we do the same experiment, but we take partial measurements in both directions: translation and modulation. Namely, for $N = 67$, we take $0.52N$ translations, and $0.7N$ modulations at random. The generator here, as described in Proposition 6.17, is a short Fourier window, with length 8. Now, we need to use Basis Pursuit in all steps

(1), (2) and (3), which in turn leads to a lower recovery rate. We made $T = 100$ trials for every sparsity level.

In Figure 6.3A, we test the speed of our algorithm in comparison to the PhaseLift algorithm [28], implemented using the CVX package. We also use Gabor measurements for it, but only $2\log(N)N$, taken at random. We plot the average execution time over $T = 50$ trials, and see that as the dimension grows, our method becomes faster, although the number of measurements is much larger. Also, if we are using the full set of measurements, the time needed is incomparably smaller to both of the other methods – since then there is no minimization problem included. In this case, also, we will always recover the signal with probability 1, independently of the sparsity level.

In Figure 6.3B, we compare the execution time of Algorithm 1 from all N^2 measurements to the GESPAR algorithm [111], a greedy algorithm for recovery of sparse signals from Fourier magnitude measurements (in our experiment we use $2N$ measurements). This algorithm is very fast for high dimensions, but since it is iterative, it becomes slower as the sparsity increases for a fixed dimension of the signal. We illustrate this behavior in Figure 6.3B, where for every dimension, we measure the average time of recovery of signals with sparsity $k = 5$ and $k = 10$. We see that the GESPAR algorithm is faster, when we want to recover a signal which has only 5 nonzeros, but if this number is larger, our algorithm becomes faster than the GESPAR, since it does not strongly depend on the sparsity level.



(A) SGPR vs. PhaseLift. (B) SGPR vs. GESPAR.

Figure 6.3: Recovery time of Gabor phase retrieval.

We would like to mention that our algorithm for all N^2 measurements is also stable to additive noise in the measurements. This follows from Theorem 6.4 and can be intuitively explained by the fact that the only troublesome part is the division in Step (3). If the generator g is such, that the values $\langle g, g_{p,l}\rangle$ are bounded away from zero, one can guarantee robustness to noise. In the next section we provide more details on how small can the value $\langle g, g_{p,l}\rangle$ be in the case g is a difference set. For the recovery from less than N^2 measurements, we leave the detailed investigation on this matter for future work.

6.4.1 Stability of the Gabor Phaseless Measurements Generated by Difference Sets

The question of how small can the values $\langle g, g_\mu\rangle$, for a generator $g \in \mathbb{C}^N$ and $\mu = (q, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$ is a question about structured partial sums of roots of unity, as we can see from the following chain of equalities:

$$\langle g, g_\mu\rangle = \sum_{n\in\mathbb{Z}_N} g(n)(M_jT_qg)(n) = \sum_{n\in\mathbb{Z}_N} g(n)g(n-q)\omega^{jn} = \sum_{\substack{n\in\mathcal{K} \text{ and}\\ n-q\in\mathcal{K}}} \omega^{jn}. \tag{6.21}$$

Let us denote the set over which we sum the roots of unity:

$$\mathcal{K}_q := \{n : n \in \mathcal{K}, n-q \in \mathcal{K}\}. \tag{6.22}$$

The question how small can a sum of roots of unity be is of interest on itself in number theory, and was posed as an open problem for arbitrary sets in [94]. Many years later, a lower bound for N prime was given in [81] and was included in the survey paper [113]. We can make use of this result in the case g is a characteristic function of a difference set, and therefore we present it here.

Theorem 6.20 ([81]). *Let N be some prime. Then, for any set $A \subset \mathbb{Z}_N$, with $|A| = n$, $3 \le n \le N-1$, it holds:*

$$\min_{j\in\mathbb{Z}_N} \left|\sum_{k\in A} \omega^{jk}\right| > n^{-(N-3)/4}. \tag{6.23}$$

Let us assume now that $\mathcal{K}$ is an (N, K, λ)-difference set. We need to investigate $\mathcal{K}_q$ (6.22) using the structure of difference sets. As we know, the cardinality of $\mathcal{K}_q$ is λ. Thus, we immediately obtain a bound $\lambda^{-(N-3)/4}$.

First of all, we can find constructions for which $\lambda = 1$, and thefore the sum (6.21) will be equal to 1. We use the La Jolla Repository[1] to find such difference sets. For dimensions up to 1000, there are 17 possible values for N which give a difference set with $\lambda = 1$. We can also find some examples analytically. The Singer difference sets [114] have parameters

$$N = \frac{q^{d+1}-1}{q-1},\ K = \frac{q^d-1}{q-1},\ \lambda = \frac{q^{d-1}-1}{q-1},$$

[1] http://www.ccrwest.org/ds.html

where $q = p^r$ is a power of a prime, and $d \geq 2$ is a positive integer. Fixing $d = 2$ will gives us $\lambda = 1$ and thus a Singer difference sets with parameters

$$(q^2 + q + 1, q + 1, 1)$$

will guarantee stability of the Algorithm 1.

Now for general $\lambda \geq 3$, the result of Theorem 6.20 holds for any difference set, and we used only the fact that the set $\mathcal{K}_q$ has λ elements. Is it possible that the lower bound $\lambda^{-(N-3)/4}$ can be improved, by using the inner structure of a particular family of difference sets?

Let us consider the quadratic family of difference sets. It has parameters $(p, \frac{p-1}{2}, \frac{p-3}{4})$, where p is some odd prime with $p \equiv 3 \pmod 4$. The elements of the difference set are then given by the formula

$$\mathcal{K} = \{t^2 : t \in \mathbb{Z}_p^*\}.$$

We are interested in estimating $\sum_{n \in \mathcal{K}_q} \omega_p^{jn}$ for every $(q, j) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*$. The elements of the set $\mathcal{K}_q = \{t \in \mathcal{K} : t - q\}$ can be described as solutions of a system of equations, namely each condition is equivalent to:

$$\begin{aligned} t \in& \mathcal{K} \iff \exists r \in \mathbb{Z}_p^* : t = r^2, \\ t - q \in& \mathcal{K} \iff \exists s \in \mathbb{Z}_p^* : t = s^2 + q. \end{aligned}$$

In number theory [76], those integers t which are congruent to a perfect square modulo p are called *quadratic residues* $(t \mathrel{\mathrm{R}} p)$. Otherwise, t is called a *quadratic nonresidue* $(t \mathrel{\mathrm{R}} p)$. A useful notation related to quadratic residues is the *Legendre symbol*. It is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \text{ divides } a, \\ 1, & \text{if } a \mathrel{\mathrm{R}} p, \\ -1, & \text{if } a \mathrel{\mathrm{N}} p. \end{cases} \tag{6.24}$$

In our case, we need t and $t - q$ to both be quadratic residues, i.e. t such that $\left(\frac{t}{p}\right) = 1$ and $\left(\frac{t+q}{p}\right) = 1$. Let us for simplicity consider the case $q = 1$. That means, that t and $t - 1$ need to be quadratic residues, or in other words, we need all the quadratic residues of $\mathbb{Z}_p$ which are followed by quadratic residue. Interestingly, we can rewrite our sum of roots of unity (6.21) using the Legendre symbol:

$$\frac{1}{2} \sum_{t \in \mathbb{Z}_p^*} \left(\left(\frac{t}{p}\right) + 1 \right) \left(\left(\frac{t+q}{p}\right) + 1 \right) \omega_p^{jt} \tag{6.25}$$

It is easy to see that, when both t and $t+q$ are quadratic residue, the coefficient before the corresponding root of unity will be 1. If at least one of them is nonresidue, the coefficient will be 0 and that term will vanish. The case $t=0$ is already excluded. By opening the brackets and using the multiplicative property of the Legendre symbol we can split the sum into four parts. We will omit the normalization factor for simplicity.

$$\sum_{t\in\mathbb{Z}_p^*}\left(\frac{t}{p}\right)\left(\frac{t+q}{p}\right)\omega_p^{jt}+\sum_{t\in\mathbb{Z}_p^*}\left(\frac{t}{p}\right)\omega_p^{jt}+\sum_{t\in\mathbb{Z}_p^*}\left(\frac{t+q}{p}\right)\omega_p^{jt}+\sum_{t\in\mathbb{Z}_p^*}\omega_p^{jt}=:A+B+C+D. \tag{6.26}$$

We would like to try to obtain a lower bound on this expression. Let us start from backwards. The last sum D is the easiest to calculate: we can add and subtract 1 and rewrite it as

$$D=\sum_{t\in\mathbb{Z}_p}\omega_p^{jt}-1.$$

It is clear that $D=-1$, since $j\neq 0$ and the sum therefore equals zero. Next, with few change of variables, we can transform C into the form of B, and further eliminate the parameter j from the sum.

$$\begin{aligned}\sum_{t\in\mathbb{Z}_p^*}\left(\frac{t+q}{p}\right)\omega_p^{jt}&=\sum_{t\in\mathbb{Z}_p}\left(\frac{t}{p}\right)\omega_p^{(t-q)j}-1\\&=\omega_p^{-qj}\sum_{t\in\mathbb{Z}_p}\left(\frac{t}{p}\right)\omega_p^{tj}-1=\omega_p^{-qj}\left(\frac{j}{p}\right)\sum_{t\in\mathbb{Z}_p^*}\left(\frac{t}{p}\right)\omega_p^{t}-1.\end{aligned}$$

The expression $\sum_{t\in\mathbb{Z}_p^*}\left(\frac{t}{p}\right)\omega_p^t$ is a so called quadratic Gauss sum, and it is well investigated in number theory, see for example [76, Chapter 6]. When p is odd prime, congruent to 3 modulo 4, as in our case, it is equal to $i\sqrt{p}$. Therefore, for the sums B and C we can conclude:

$$\begin{aligned}B&=\left(\frac{j}{p}\right)i\sqrt{p},\\C&=\omega_p^{-qj}\left(\frac{j}{p}\right)i\sqrt{p}-1.\end{aligned}$$

We note here that expressions of the type ω_p^t+1 can be estimated by $\frac{\pi}{p}$ [94].

We are left to investigate the first sum, A. This is actually a version of the so called Kloosterman sums [80], or more general finite field hypergeometric sums [126], and in the case of prime p, an upper bound on the absolute value of the sum is known, $2\sqrt{p}$. Regarding a lower bound, it seems like the best one can do is a bound like $(4p)^{(2-p)/2}$[126], which at the end is not much better then the bound that we got from Theorem 6.20, where we did not use the inner structure of the quadratic difference sets. Taking into account everything that we discussed, it seem that the best lower bound on $|\langle g, g_\mu \rangle|$ for g generated by an (N, K, λ) difference set with parameter $\lambda \neq 1$ is

$$|\langle g, g_\mu \rangle| \geq \lambda^{-(N-3)/4}.$$

One approach to improve this result would be to show that for *almost all* $\mu = (q, j)$ a lower bound exists — we leave this question for further investigations.

In conclusion, we investigate the effect of noise numerically. In Figure 6.4, we fix the dimension $N = 127$, and try to recover an unknown vector $x \in \mathbb{R}^N$ from all N^2 noisy Gabor measurements via Algorithm 1. For every level of the additive Gaussian noise, we recover x from Gabor measurements generated by a difference set and by a complex random signal. We repeat every experiment $T = 500$ times. As we can see in Figure 6.4, the measurements generated by a difference set give smaller recovery error than the complex random vectors.

We did not investigate theoretically the stability of the Gabor measurements generated by random complex vectors. Also, the question of stability of Algorithm 1 when only part of the measurements are given is left open. In this case, the result will be influenced by the stability of the ℓ_1 minimization problem for sparse signals from partial Fourier measurements. We leave those questions for further investigations.

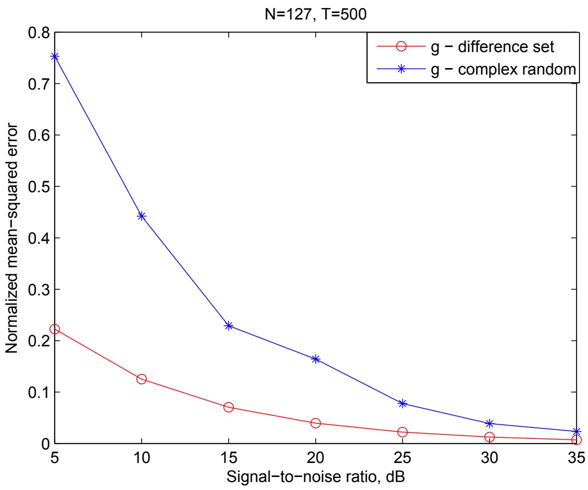


Figure 6.4: The effect of noise on the reconstruction error.

6.4.2 Phase Retrieval of Discrete Lines

In this subsection we would like to combine the ideas of Gabor phase retrieval with the problem of recovery of discrete lines from Chapter 3. The motivation comes from the numerical experiments of the GESPAR algorithm provided in [111], which includes recovery of 2D sparse signals (images) from the magnitude of its Fourier measurements.

In Figure 6.5, we present an example of the result obtained for recovery of 30 points from an image of size 32×32, $N = 32^2 = 1024$. The time it takes GESPAR to recover x from its N Fourier magnitude measurements is approximately the same as the time it takes SGPR to recover x from its N^2 Gabor magnitude measurements. We would like to consider $n \times n$ signals (where n is some prime number) which consists of union of discrete lines (see Figure 3.2), and recover them from the full set of N^2, $N = n^2$ Gabor measurements via SGPR. We use here a complex random vector as a generator for the Gabor system, although there are difference sets of dimension n^2 which satisfy the condition for successful phase retrieval (6.2), and therefore can be also used.

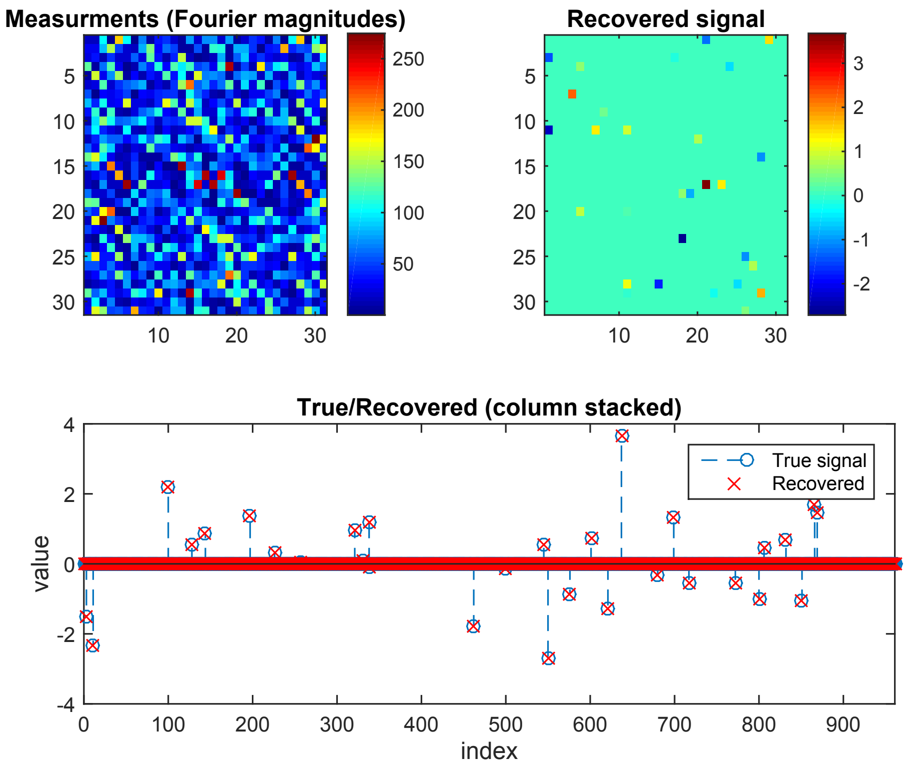


Figure 6.5: Recovery of points from Fourier magnitude measurements.

Note that the GESPAR algorithm is not able to recover more then one discrete line since that is already N nonzero elements in the signal (maximal sparsity possible). Our algorithm, on the other hand, does not depend on the sparsity level, and thus can recover any amount of lines in the signal. However, it requires n^4 measurements, since the dimension of the signal is n^2, and if we want efficient and fast recovery, we need to use all Gabor measurements.

In Figure 6.6, using the template of the code for the GESPAR algorithm, but with recovery via Algorithm 1, we present a successful recovery of 3 lines in dimension $n = 41$. It is also worth to note that since we only do Fourier and inverse Fourier transform in our algorithm, it is extremely fast even for recovery from $n^4 = 2825761$ measurements. Furthermore, algorithms which involve solving an optimization problem (like PhaseLift [29]) would take much longer regardless of the number of measurements, because the dimension of the signal $n^2 = 1681$ is already very large.

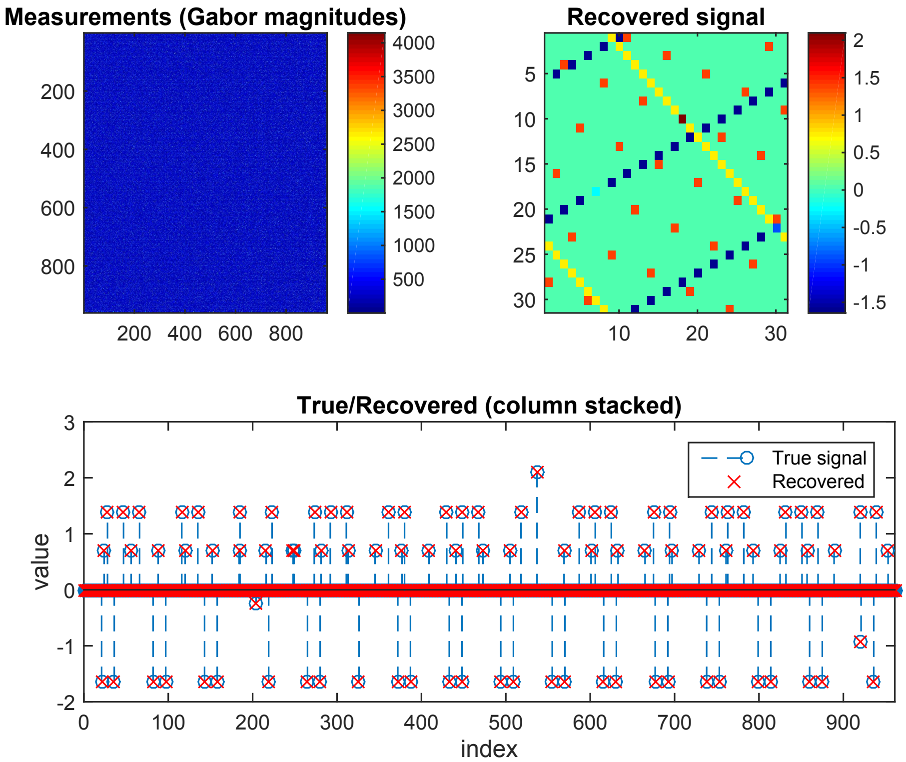


Figure 6.6: Recovery of discrete lines from Gabor magnitude measurements.

Chapter 7

Conclusion

In this thesis, we showed that the efficiency of compressed sensing for large data can be optimized by using deterministic constructions and dictionaries adapted to the class of signals one is interested in.

In Chapter 3, we employed the geometric structure of the signals to improve some known results of compressed sensing for the class of discrete lines. We showed how to construct a unit norm tight frame from the collection of discrete lines, and we computed the mutual coherence as well as the spark of such a system. We used the methodology of compressed sensing to recover discrete lines from a small number of linear measurements, and to separate lines from points in a given signal.

Next, in Chapter 4 we used another type of structure, that is the collection of time- and frequency-shifts generated by difference sets. There, we investigated the frame and fusion frame properties of this Gabor system, and showed that from the aspect of a fusion frame, this construction yields a family of optimally sparse, equidistant tight fusion frames. In relation to compressed sensing, we tested numerically the recoverability of sparse and fusion sparse signals from Gabor measurements generated by difference sets. We saw that the results of the numerical experiments surpassed the theoretical guarantees based on the mutual coherence. It is interesting, therefore, to search for recovery conditions which will go beyond the worst-case coherence, not only for Gabor systems, but also for other structured deterministic measurement matrices.

Further, we analyzed the question of recovery of signals that are sparse in a dictionary, given only the magnitudes of the linear measurements. This is called the dictionary sparse phase retrieval problem. In Chapter 5 we obtained results about the injectivity of such a measurement process, and we investigated the relation to ℓ_1 minimization via the null space property, suited for dictionary sparse signals and phaseless measurements.

We also addressed the problem of sparse phase retrieval by projections, which was so far not covered in the literature, and proved several results in this direction. Phase retrieval by projections is a difficult topic with many unsolved problems, and it is appealing to address those also in the case when this signal is sparse, or admits a sparse representation.

Finally, we investigated the problem of sparse phase retrieval from Gabor measurements in Chapter 6. Firstly, we obtained a condition on the generator, which is easy to check and which guarantees a recovery of arbitrary signals from the full set of Gabor magnitude measurements. Then, we modified this condition in the case of k-sparse signals, and found that of the order of k^3 measurements are needed for recovery. We showed that the number of measurements can be further decreased to k^2, if an additional structure of the signal is assumed, namely, that it consists of a single block of nonzeros. We presented examples of families of generators which allow phase retrieval, both in the general and in the sparse case. Finally, we proposed an algorithm for recovery from Gabor magnitude measurements. This algorithm uses only the fast Fourier and inverse fast Fourier transforms in order to recover a signal from the full set of measurements. We discussed its stability as well, and provided numerical experiments. The algorithm is adopted to the recovery of sparse signals from fewer measurements, and it then uses ℓ_1 minimization with partial Fourier measurement matrices. The investigation of the stability of the condition which guarantees phase retrieval led to an interesting connection to additive number theory. It is interesting to explore this connection further, and to find connections to other aspects of signal processing.

We conclude that driven by various applications, it is advantageous to exploit additional structure of the signal that needs to be recovered and of the measurements which are used. This approach leads to improved recovery guarantees from a compressed sensing point of view, as well as to a rich mathematical theory also interesting beyond signal processing.

Bibliography

[1] B. Adcock, A. C. Hansen, C. Poon, and B. Roman. Breaking the coherence barrier: A new theory for compressed sensing. *arXiv preprint arXiv:1302.0561*, 2013.

[2] M. Aharon, M. Elad, and A. Bruckstein. K-SVD: An algorithm for designing overcomplete dictionaries for sparse representation. *IEEE Trans. Signal Process.*, 54(11):4311–4322, 2006.

[3] B. Alexeev, A. S. Bandeira, M. Fickus, and D. G. Mixon. Phase retrieval with polarization. *SIAM J. Imag. Sci.*, 7(1):35–66, 2014.

[4] B. Alexeev, J. Cahill, and D. G. Mixon. Full spark frames. *J. Fourier Anal. Appl.*, 18(6):1167–1194, 2012.

[5] W. O. Alltop. Complex sequences with low periodic correlations. *IEEE Trans. Inf. Theory*, 26(3):350–354, 1980.

[6] U. Ayaz. *Sparse Recovery with Fusion Frames.* PhD thesis, 2014.

[7] U. Ayaz, S. Dirksen, and H. Rauhut. Uniform recovery of fusion frame structured sparse signals. *arXiv preprint arXiv:1407.7680*, 2014.

[8] U. Ayaz and H. Rauhut. Sparse recovery with fusion frames via rip. *Proc. SampTA*, 2013.

[9] W. U. Bajwa, R. Calderbank, and S. Jafarpour. Why Gabor frames? Two fundamental measures of coherence and their role in model selection. *J. Commun. Networks*, 12(4):289–307, 2010.

[10] W. U. Bajwa, R. Calderbank, and D. G. Mixon. Two are better than one: fundamental parameters of frame coherence. *Appl. Comput. Harmon. Anal.*, 33(1):58–78, 2012.

[11] R. Balan, B. G. Bodmann, P. G. Casazza, and D. Edidin. Painless reconstruction from magnitudes of frame coefficients. *J. Fourier Anal. Appl.*, 15(4):488–501, 2009.

[12] R. Balan, P. Casazza, and D. Edidin. On signal reconstruction without phase. *Appl. Comput. Harmon. Anal.*, 20(3):345–356, 2006.

[13] A. S. Bandeira, J. Cahill, D. G. Mixon, and A. A. Nelson. Saving phase: Injectivity and stability for phase retrieval. *Appl. Comput. Harmon. Anal.*, 37(1):106–125, 2014.

[14] J. J. Benedetto and M. Fickus. Finite normalized tight frames. *Adv. Comput. Math.*, 18(2-4):357–385, 2003.

[15] G. Berman. Finite projective plane geometries and difference sets. *Trans. Amer. Math. Soc.*, 74:492–499, 1953.

[16] T. Blumensath and M. E. Davies. Iterative hard thresholding for compressed sensing. *Appl. Comput. Harmon. Anal.*, 27(3):265–274, 2009.

[17] B. G. Bodmann and N. Hammen. Stable phase retrieval with low-redundancy frames. *Adv. Comput. Math.*, 41(2):317–331, 2015.

[18] P. Boufounos, G. Kutyniok, and H. Rauhut. Sparse recovery from combined fusion frame measurements. *IEEE Trans. Inf. Theory*, 57(6):3864–3876, 2011.

[19] P. T. Boufounos and R. G. Baraniuk. 1-bit compressive sensing. In *42nd Annual Conf. Inf. Sci. Syst.*, pages 16–21. IEEE, 2008.

[20] J. Cahill, P. G. Casazza, J. Peterson, and L. Woodland. Phase retrieval by projections. *arXiv preprint arXiv:1305.6226*, 2013.

[21] T. T. Cai and A. Zhang. Sharp rip bound for sparse signal and low-rank matrix recovery. *Appl. Comput. Harmon. Anal.*, 35(1):74–93, 2013.

[22] T. T. Cai and A. Zhang. Sparse representation of a polytope and recovery in sparse signals and low-rank matrices. *IEEE Trans. Inform. Theory*, 60(1):122–132, 2014.

[23] R. Calderbank, S. Howard, and S. Jafarpour. Construction of a large class of deterministic sensing matrices that satisfy a statistical isometry property. *IEEE J. Sel. Top. Signal Process.*, 4(2):358–374, 2010.

[24] E. J. Candès. The restricted isometry property and its implications for compressed sensing. *C.R. Math.*, 346(9):589–592, 2008.

[25] E. J. Candès and B. Recht. Exact matrix completion via convex optimization. *Found. Comput. Math.*, 9(6):717–772, 2009.

[26] E. J. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theory*, 52(2):489–509, 2006.

[27] E. J. Candès, J. K. Romberg, and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math.*, 59(8):1207–1223, 2006.

[28] E. J. Candès, T. Strohmer, and V. Voroninski. PhaseLift: exact and stable signal recovery from magnitude measurements via convex programming. *Comm. Pure Appl. Math.*, 66(8):1241–1274, 2013.

[29] E. J. Candès, T. Strohmer, and V. Voroninski. PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming. *Comm. Pure Appl. Math.*, 66(8):1241–1274, 2013.

[30] E. J. Candès and T. Tao. Decoding by linear programming. *IEEE Trans. Inf. Theory*, 51(12):4203–4215, 2005.

[31] E. J. Candès and T. Tao. Near-optimal signal recovery from random projections: universal encoding strategies? *IEEE Trans. Inf. Theory*, 52(12):5406–5425, 2006.

[32] P. G. Casazza, , and L. M. Woodland. Phase retrieval by vectors and projections. *Contemporary Math.*, to appear.

[33] P. G. Casazza. The art of frame theory. *Taiwanese J. Math.*, 4(2):129–201, 2000.

[34] P. G. Casazza, A. Heinecke, and G. Kutyniok. Optimally sparse fusion frames: existence and construction. In *Proc. SampTA'11*, Singapore, 2011.

[35] P. G. Casazza and J. Kovačević. Equal-norm tight frames with erasures. *Adv. Comput. Math.*, 18(2-4):387–430, 2003.

[36] P. G. Casazza and G. Kutyniok. Frames of subspaces. In *Wavelets, frames and operator theory*, volume 345 of *Contemp. Math.*, pages 87–113. Amer. Math. Soc., Providence, RI, 2004.

[37] P. G. Casazza and G. Kutyniok, editors. *Finite frames: Theory and applications.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.

[38] P. G. Casazza, G. Kutyniok, and S. Li. Fusion frames and distributed processing. *Appl. Comput. Harmon. Anal.*, 25(1):114–132, 2008.

[39] J. Chen and X. Huo. Theoretical results on sparse representations of multiple-measurement vectors. *IEEE Trans. Signal Process.*, 54(12):4634–4643, 2006.

[40] S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursuit. *SIAM J. Sci. Comput.*, 20(1):33–61, 1998.

[41] X. Chen, H. Wang, and R. Wang. A null space property approach to compressed sensing with frames. *arXiv preprint arXiv:1302.7074*, 2013.

[42] T.-Y. Chien. *Equiangular lines, projective symmetries and nice error frames*. PhD thesis, University of Auckland, 2015.

[43] A. Cohen, W. Dahmen, and R. DeVore. Compressed sensing and best k-term approximation. *J. Amer. Math. Soc.*, 22(1):211–231, 2009.

[44] J. H. Conway, R. H. Hardin, and N. J. A. Sloane. Packing lines, planes, etc.: packings in Grassmannian spaces. *Experiment. Math.*, 5(2):139–159, 1996.

[45] J. H. Dinitz and D. R. Stinson. *Contemporary design theory: A collection of Surveys*, volume 26. John Wiley & Sons, 1992.

[46] D. L. Donoho. Compressed sensing. *IEEE Trans. Inf. Theory*, 52(4):1289–1306, 2006.

[47] D. L. Donoho and M. Elad. Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ_1 minimization. *Proc. Natl. Acad. Sci.*, 100(5):2197–2202, 2003.

[48] D. L. Donoho, O. Levi, J. L. Starck, and V. Martinez. Multiscale geometric analysis for 3d catalogs. In *Astronomical Telescopes and Instrumentation*, pages 101–111. International Society for Optics and Photonics, 2002.

[49] D. L. Donoho, A. Maleki, and A. Montanari. Message-passing algorithms for compressed sensing. *Proc. Natl. Acad. Sci. U.S.A.*, 106(45):18914–18919, 2009.

[50] M. F. Duarte, S. Sarvotham, D. Baron, M. B. Wakin, and R. G. Baraniuk. Distributed compressed sensing of jointly sparse signals. In *Asilomar Conf. Signals, Sys., Comput*, pages 1537–1541, 2005.

[51] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[52] M. Elad. *Sparse and redundant representations*. Springer, New York, 2010. From theory to applications in signal and image processing, With a foreword by Alfred M. Bruckstein.

[53] M. Elad. Sparse and redundant representation modeling—what next? *IEEE Signal Process Lett.*, 19(12):922–928, 2012.

[54] M. Elad and A. M. Bruckstein. A generalized uncertainty principle and sparse representation in pairs of bases. *IEEE Trans. Inf. Theory*, 48(9):2558–2567, 2002.

[55] Y. Eldar, P. Sidorenko, D. Mixon, S. Barel, and O. Cohen. Sparse phase retrieval from short-time Fourier measurements. *IEEE Signal Process Lett.*, 22(5), 2015.

[56] Y. C. Eldar, P. Kuppinger, and H. Bölcskei. Block-sparse signals: uncertainty relations and efficient recovery. *IEEE Trans. Signal Process.*, 58(6):3042–3054, 2010.

[57] Y. C. Eldar and G. Kutyniok, editors. *Compressed sensing*. Cambridge University Press, Cambridge, 2012. Theory and applications.

[58] Y. C. Eldar and M. Mishali. Robust recovery of signals from a structured union of subspaces. *IEEE Trans. Inf. Theory*, 55(11):5302–5316, 2009.

[59] M. Fickus and D. G. Mixon. Tables of the existence of equiangular tight frames. *arXiv preprint arXiv:1504.00253*, 2015.

[60] M. Fickus, D. G. Mixon, and J. C. Tremain. Steiner equiangular tight frames. *Linear Algebra Appl.*, 436(5):1014–1027, 2012.

[61] J. R. Fienup. Phase retrieval algorithms: a comparison. *App. Opt.*, 21(15):2758–2769, 1982.

[62] O. Figueiredo and J.-P. Reveillès. A contribution to 3d digital lines. In *5th Colloquium Discrete Geometry and Computer Imagery*, number LSP-CONF-1995-002, pages 187–198, 1995.

[63] M. Fornasier. Numerical methods for sparse recovery. In *Theoretical foundations and numerical methods for sparse recovery*, volume 9 of *Radon Ser. Comput. Appl. Math.*, pages 93–200. Walter de Gruyter, Berlin, 2010.

[64] S. Foucart and H. Rauhut. *A mathematical introduction to compressive sensing*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.

[65] D. Gabor. Theory of communication. *Jour. Inst. Elec. Eng. (London)*, 93(26):429–457, 1946.

[66] R. G. Gallager. Circularly-symmetric Gaussian random vectors, 2008.

[67] R. W. Gerchberg and S. W. O. A practical algorithm for the determination of phase from image and diffraction plane pictures. *Optik*, 35:237–246, 1972.

[68] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. http://cvxr.com/cvx, Mar. 2014.

[69] K. Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[70] M. Guizar-Sicairos and J. R. Fienup. Phase retrieval with transverse translation diversity: a nonlinear optimization approach. *Opt. Express*, 16(10):7264–7278, 2008.

[71] R. W. Harrison. Phase problem in crystallography. *JOSA A*, 10(5):1046–1055, 1993.

[72] C. Heil. Linear independence of finite Gabor systems. In *Harmonic analysis and applications*, Appl. Numer. Harmon. Anal., pages 171–206. Birkhäuser Boston, Boston, MA, 2006.

[73] C. Heil, J. Ramanathan, and P. Topiwala. Linear independence of time-frequency translates. *Proc. Amer. Math. Soc.*, 124(9):2787–2795, 1996.

[74] T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. *Comm. Math. Phys.*, 318(2):355–374, 2013.

[75] M. A. Herman and T. Strohmer. High-resolution radar via compressed sensing. *IEEE Trans. Signal Process.*, 57(6):2275–2284, 2009.

[76] K. Ireland and M. Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[77] M. Iwen, A. Viswanathan, and Y. Wang. Fast phase retrieval for high-dimensions. *arXiv preprint arXiv:1501.02377*, 2015.

[78] J. Jasper, D. G. Mixon, and M. Fickus. Kirkman equiangular tight frames and codes. *IEEE Trans. Inf. Theory*, 60(1):170–181, 2014.

[79] R. Klette and A. Rosenfeld. *Digital geometry: Geometric methods for digital picture analysis*. Elsevier, 2004.

[80] H. D. Kloosterman. On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$. *Acta Math.*, 49(3-4):407–464, 1927.

[81] S. V. Konyagin and V. F. Lev. On the distribution of exponential sums. *Integers*, pages A1, 11, 2000.

[82] S. G. Krantz and H. R. Parks. *Geometric integration theory*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2008.

[83] G. Kutyniok and W.-Q. Lim. Compactly supported shearlets are optimally sparse. *Journal of Approximation Theory*, 163(11):1564–1589, 2011.

[84] G. Kutyniok and W.-Q. Lim. Image separation using wavelets and shearlets. In *Curves and surfaces*, pages 416–430. Springer, 2012.

[85] G. Kutyniok, A. Pezeshki, R. Calderbank, and T. Liu. Robust dimension reduction, fusion frames, and grassmannian packings. *Appl. Comput. Harmon. Anal.*, 26(1):64–76, 2009.

[86] T. Y. Lam and K. H. Leung. On vanishing sums of roots of unity. *J. Algebra*, 224(1):91–109, 2000.

[87] J. Lawrence, G. E. Pfander, and D. Walnut. Linear independence of Gabor systems in finite dimensional vector spaces. *J. Fourier Anal. Appl.*, 11(6):715–726, 2005.

[88] J. Mairal, F. Bach, J. Ponce, and G. Sapiro. Online learning for matrix factorization and sparse coding. *J. Mach. Learn. Research*, 11:19–60, 2010.

[89] R.-D. Malikiosis. A note on Gabor frames in finite dimensions. *Appl. Comput. Harmon. Anal.*, 38(2):318–330, 2015.

[90] S. Mendelson, A. Pajor, and N. Tomczak-Jaegermann. Uniform uncertainty principle for Bernoulli and subgaussian ensembles. *Constr. Approx.*, 28(3):277–289, 2008.

[91] R. Meshulam. An uncertainty inequality for finite abelian groups. *European J. Combin.*, 27(1):63–67, 2006.

[92] R. P. Millane. Phase retrieval in crystallography and optics. *JOSA A*, 7(3):394–411, 1990.

[93] D. G. Mixon. Explicit matrices with the restricted isometry property: Breaking the square-root bottleneck. *arXiv preprint arXiv:1403.3427*, 2014.

[94] G. Myerson. Unsolved problems: How small can a sum of roots of unity be? *Amer. Math. Monthly*, 93(6):457–459, 1986.

[95] S. H. Nawab, T. F. Quatieri, and J. S. Lim. Signal reconstruction from short-time Fourier transform magnitude. *IEEE Trans. Acoust. Speech Signal Process.*, 31(4):986–998, 1983.

[96] D. Needell and J. A. Tropp. Cosamp: Iterative signal recovery from incomplete and inaccurate samples. *Appl. Comput. Harmon. Anal.*, 26(3):301–321, 2009.

[97] H. Ohlsson and Y. C. Eldar. On conditions for uniqueness in sparse phase retrieval. In *IEEE Int. Conf. Acoust. Speech Signal Process. 2014 (ICASSP)*, pages 1841–1845. IEEE, 2014.

[98] A. Oppenheim, A. Willsky, and S. Nawab. *Signals and Systems*. Prentice-Hall signal processing series. Prentice Hall, 1997.

[99] G. E. Pfander. Gabor frames in finite dimensions. In *Finite frames*, pages 193–239. Springer, 2013.

[100] G. E. Pfander, H. Rauhut, and J. Tanner. Identification of matrices having a sparse representation. *IEEE Trans. Signal Process.*, 56(11):5376–5388, 2008.

[101] G. E. Pfander and P. Zheltov. Estimation of overspread scattering functions. *IEEE Trans. Signal Process.*, 63(10):2451–2463, 2014.

[102] G. E. Pfander and P. Zheltov. Sampling of stochastic operators. *IEEE Trans. Inf. Theory*, 60(4):2359–2372, 2014.

[103] F. Philipp. Phase retrieval from 4n–4 measurements: A proof for injectivity. In *Proc. Appl. Math. Mech.*, 2014.

[104] Y. Plan and R. Vershynin. Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach. *IEEE Trans. Inf. Theory*, 59(1):482–494, 2013.

[105] J. Ranieri, A. Chebira, Y. M. Lu, and M. Vetterli. Phase retrieval for sparse signals: Uniqueness conditions. *arXiv preprint arXiv:1308.3058*, 2013.

[106] H. Rauhut, J. Romberg, and J. A. Tropp. Restricted isometries for partial random circulant matrices. *Appl. Comput. Harmon. Anal.*, 32(2):242–254, 2012.

[107] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves. Symmetric informationally complete quantum measurements. *J. Math. Phys.*, 45(6):2171–2180, 2004.

[108] J.-P. Reveillès. *Géométrie discrete, calcul en nombres entiers et algorithmique.* PhD thesis, 1991.

[109] M. Rudelson and R. Vershynin. On sparse reconstruction from Fourier and Gaussian measurements. *Comm. Pure Appl. Math.*, 61(8):1025–1045, 2008.

[110] A. Scott and M. Grassl. Symmetric informationally complete positive-operator-valued measures: A new computer study. *Journal of Mathematical Physics*, 51(4):042203, 2010.

[111] Y. Shechtman, A. Beck, and Y. C. Eldar. GESPAR: efficient phase retrieval of sparse signals. *IEEE Trans. Signal Process.*, 62(4):928–938, 2014.

[112] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev. Phase retrieval with application to optical imaging: a contemporary overview. *IEEE Signal Process Mag.*, 32(3):87–109, 2015.

[113] I. D. Shkredov. Fourier analysis in combinatorial number theory. *Uspekhi Mat. Nauk*, 65(3(393)):127–184, 2010.

[114] J. Singer. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43(3):377–385, 1938.

[115] J.-L. Starck, D. L. Donoho, and E. J. Candès. Astronomical image representation by the curvelet transform. *Astron. Astrophys.*, 398(2):785–800, 2003.

[116] J. L. Starck, M. Elad, and D. L. Donoho. Image decomposition via the combination of sparse representations and a variational approach. *IEEE Trans. Image Process.*, 14(10):1570–1582, 2005.

[117] T. Strohmer. Measure what should be measured: progress and challenges in compressive sensing. *IEEE Signal Process Lett.*, 19(12):887–893, 2012.

[118] T. Strohmer and R. W. Heath. Grassmannian frames with applications to coding and communication. *Appl. Comput. Harmon. Anal.*, 14(3):257–275, 2003.

[119] T. Tao. An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.*, 12(1):121–128, 2005.

[120] A. M. Tillmann and M. E. Pfetsch. The computational complexity of the restricted isometry property, the nullspace property, and related concepts in compressed sensing. *IEEE Trans. Inf. Theory*, 60(2):1248–1259, 2014.

[121] J. Tropp, A. C. Gilbert, et al. Signal recovery from random measurements via orthogonal matching pursuit. *IEEE Trans. Inf. Theory*, 53(12):4655–4666, 2007.

[122] J. A. Tropp. On the conditioning of random subdictionaries. *Appl. Comput. Harmon. Anal.*, 25(1):1–24, 2008.

[123] J. A. Tropp, J. N. Laska, M. F. Duarte, J. K. Romberg, and R. G. Baraniuk. Beyond Nyquist: efficient sampling of sparse bandlimited signals. *IEEE Trans. Inf. Theory*, 56(1):520–544, 2010.

[124] C. Vinzant. A small frame and a certificate of its injectivity. *arXiv preprint arXiv:1502.04656*, 2015.

[125] Y. Wang and Z. Xu. Phase retrieval for sparse signals. *Appl. Comput. Harmon. Anal.*, 37(3):531–544, 2014.

[126] A. Weil. On some exponential sums. *Proc. Nat. Acad. Sci. U. S. A.*, 34:204–207, 1948.

[127] L. Welch. Lower bounds on the maximum cross correlation of signals (corresp.). *IEEE Trans. Inf. Theory*, 20(3):397–399, 1974.

[128] P. Xia, S. Zhou, and G. B. Giannakis. Achieving the Welch bound with difference sets. *IEEE Trans. Inf. Theory*, 51(5):1900–1907, 2005.

[129] G. Zauner. *Quantendesigns: Grundzüge einer nichtkommutativen Designtheorie*. PhD thesis, University of Vienna, 1999.