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Stability analysis in the inverse Robin transmission problem

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In this paper, we consider the conductivity problem with piecewise-constant conductivity and Robin-type boundary condition on the interface of discontinuity. When the quantity of interest is the jump of the conductivity, we perform a local stability estimate for a parameterized non-monotone family of domains. We give also a quantitative stability result of local optimal solution with respect to a perturbation of the Robin parameter. In order to find an optimal solution, we propose a Kohn–Vogelius-type cost functional over a class of admissible domains subject to two boundary values problems. The analysis of the stability involves the computation of first-order and second-order shape derivative of the proposed cost functional, which is performed rigorously by means of shape-Lagrangian formulation without using the shape sensitivity of the states variables. © 2016 The Author. *Mathematical Methods in the Applied Sciences* Published by John Wiley & Sons Ltd.

Keywords: stability analysis; second-order shape derivative; Lagrange formulation

1. Introduction

The problem of reconstructing the jump of conductivity is a classical inverse problem. Such problem arises in many physical situations such as electrical impedance tomography (for instance, [1–5]). In [6], a partial differential equation with Robin-type transmission conditions, which models the situation where the corrosion takes place between two layers of a non-homogenous medium, is considered. The authors provide an algorithm for the recovery of the Robin parameter and the jump set of the conductivity, either independently or simultaneously. A uniqueness result was proved for the same problem in [7].

In this paper, we consider the same model problem as in [7]. We give a local stability estimate for a non-monotone parameterized family of domains. Let us recall that a similar result was proved in [8] for the classical conductivity problem. We give also a quantitative stability result of the interface of the conductivity when the Robin parameter is uncertainly known.

For the computation of local optimal solution of the shape problem, we propose a Kohn–Vogelius-type functional. The stability analysis require the computation of the first-order and second-order shape derivative of the proposed shape functional. For shape analysis, we use the velocity method [9, 10].

The material/shape derivative method is known to be very hard for the computation of the first-order and second-order shape derivatives, and it require more regularity for the states variables. In this work, we follow the Lagrange method in the spirit of [9] to compute the first-order and second-order shape derivatives of the proposed shape functional. For the computation of the shape gradient, we use Lagrangian method combined with the use of theorem on the derivative of a MinMax with respect to a parameter. Such method is well known and extensively used in mechanical sciences, mathematical programming, and optimal control theory. Their application to shape sensitivity analysis is not completely straightforward because it leads to the time dependence of the underlying function spaces appearing in the MinMax formulation. There are two techniques to overcome this difficulty: the *function space parameterization* and the *function space embedding* methods. The first one will be used here.

For the computation of the shape Hessian, we follow the method given in [11] to differentiable semiconvex cost functionals.

This methods have the advantage of providing the first-order and second-order shape derivative without the need to compute the material derivative of the partial differential equations.

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The rest of the paper is outlined as follows: In Section 2, we present the model problem. In Section 3, we give a local stability estimate. In Section 4, we formulate the shape optimization problem, and we show the existence of a solution. The stability analysis of the optimization problem is performed in Section 5.

2. Problem statement

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary $\partial\Omega$ and $\omega \in \mathcal{O}$, where

$$\mathcal{O} := \{\omega \text{ open with Lipschitz boundary } \partial\omega, \omega \subset \Omega,\ \mathrm{dist}(\partial\omega, \partial\Omega) > \delta\}$$

for some positive constant δ, see Figure 1. For a given current density $g \in H^{-1/2}(\partial\Omega)$, a Robin coefficient $p \in \mathbb{R}_+^*$ and a piecewise constant conductivity $\sigma := \sigma_1 + (\sigma_2 - \sigma_1)\chi_\omega, \sigma_1, \sigma_2 \in \mathbb{R}_+^*$, where χ_ω is the characteristic function of ω, the potential u satisfies the following Neumann problem with Robin-type transmission conditions:

$$\begin{cases} -\mathrm{div}(\sigma\nabla u) = 0 & \text{in } \Omega \setminus \partial\omega, \\ [\![u]\!] = 0 & \text{on } \partial\omega, \\ [\![\sigma\partial_\nu u]\!] = pu & \text{on } \partial\omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν is the unit normal vector to the interface $\partial\omega$ or $\partial\Omega$ pointing outward of ω or Ω, respectively, and $[\![.]\!]$ means the jump across the interface $\partial\omega$. From the physical point of view, problem (1) can be viewed as a model for corrosion detection [12–14]. The weak solution to problem (1) is defined by

$$\begin{cases} \text{Find } u \in H^1(\Omega) \quad \text{such that} \\ \int_\Omega \sigma\nabla u \cdot \nabla v\, dx + \int_{\partial\omega} puv\, ds = \int_{\partial\Omega} gv\, ds \quad \text{for all } v \in H^1(\Omega). \end{cases} \tag{2}$$

The existence and uniqueness of the weak solution follows from the Riesz representation theorem.

The inverse problem under consideration is the following:

$$\textit{Find } \omega \in \mathcal{O} \textit{ knowing the pair } (u_{|\partial\Omega} := f, g). \tag{3}$$

Recently, a uniqueness result for the simultaneous identification of the conductivity σ, the Robin parameter p, and the interface $\partial\omega$ is established in [7]. Other references studying various inverse corrosion problems can be found in [12, 14–16].

3. Local Lipschitz stability

In this section, we shall establish a local Lipschitz stability for the inverse Robin transmission problem formulated in (3), where the potential u is measured only on some part γ of the boundary $\partial\Omega$ with positive measure. The stability analysis is performed in the framework of shape optimization techniques (e.g., shape derivatives with respect to a parameterized non-monotone family of domains ω_t).

Before giving the statement of our main result, we introduce some notations and definitions in the following subsection.

3.1. *Elements of shape calculus*

In this subsection, we recall some basic facts about the velocity method from shape optimization used to calculate the shape derivative of the functional J (for instance, [9, 10]). In the velocity (or speed) method, a domain Ω is deformed by the action of a velocity field V. The evolution of the domain is described by the following dynamical system:

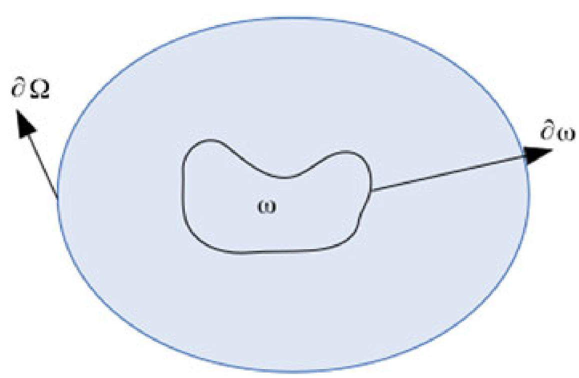

Figure 1. Representation of the domain Ω. [Colour figure can be viewed at wileyonlinelibrary.com]

$$\begin{cases} \dfrac{d}{dt}x(t) = V(x(t)), t \in [0,\varepsilon), \quad \varepsilon \in \mathbb{R}_+^*. \\ x(0) = X \in \mathbb{R}^2 \end{cases} \tag{4}$$

Suppose that V is continuously differentiable and has compact support in Ω, that is, $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$. Then, the ordinary differential Eq. 4 has a unique solution on $[0, \varepsilon)$. This allows us to define the diffeomorphism

$$T_t : \mathbb{R}^2 \to \mathbb{R}^2 : X \mapsto T_t(X) := x(t). \tag{5}$$

For $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$, the domain Ω is globally invariant by the transformation T_t, that is, $T_t(\Omega) = \Omega$ and $T_t(\partial\Omega) = \partial\Omega$. For $t \in [0, \varepsilon)$, T_t is invertible. Furthermore, for sufficiently $\varepsilon > 0$, the Jacobian determinant $\xi_V(t)$ is strictly positive

$$\forall\ t \in [0,\varepsilon), \quad \xi_V(t) = \det DT_t(X) > 0, \tag{6}$$

where $DT_t(X)$ is the Jacobian matrix of the transformation T_t associated with the velocity field V. In the sequel, we use the following notation: M^{-1} for the inverse of M and M^{-*} for its transpose. We also denote by

$$w_V(t) := \xi_V(t)|(DT_t)^{-*}\nu|, \tag{7}$$

the tangential Jacobian of T_t on $\partial\omega$ and

$$A_V(t) := \xi_V(t) DT_t^{-1} DT_t^{-*}. \tag{8}$$

We will also need the following assumption:
Assumption (H_0)*:* Given (α, β) and (α', β') satisfying $0 < \alpha < \beta$ and $0 < \alpha' < \beta'$, we can find $\varepsilon > 0$ such that

$$\forall \eta \in \mathbb{R}^2, \quad \alpha|\eta|^2 \le \sigma A_V(t)\eta \cdot \eta \le \beta|\eta|^2, \text{ and } \alpha' \le p|w_V(t)| \le \beta' \quad \text{for } t \in [0,\varepsilon). \tag{9}$$

We denote u_t the solution of (1) with $\omega_t := T_t(\omega)$ in place of ω, and $u^t = u_t \circ T_t$. The function u^t is defined on the fixed domain Ω, and its material derivative (or Lagrangian derivative) is defined by

$$\dot{u}(x) := \lim_{t \to 0} \frac{u^t(x) - u(x)}{t}, \quad \forall\ x \in \Omega.$$

The shape derivative (or Eulerian derivative) is defined by

$$u' = \dot{u} - \nabla u \cdot V.$$

Now, we are ready to state our main result.

Theorem 1
Let γ be a subset of $\partial\Omega$ with positive measure . We assume that

(i) g is not identically equal to zero;
(ii) there exists a vector field $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$ such that the following sets:

$$\Sigma_+ := \{x \in \partial\omega; V \cdot \nu > 0\}, \quad \Sigma_- := \{x \in \partial\omega; V \cdot \nu < 0\},$$

are both non-empty;
(iii) there exists an open subset $\Gamma \subset \Sigma_+$ such that

$$\operatorname{dist}(\bar{\Gamma}, \overline{\Sigma_+ \setminus \Gamma} \cup \overline{\Sigma_-}) = \inf_{x \in \bar{\Gamma}, y \in \overline{\Sigma_+ \setminus \Gamma} \cup \overline{\Sigma_-}} |x - y| > 0,$$

(iv) $p > 0$ and σ_1, σ_2 are chosen such that $p + \kappa [\![\sigma]\!] < 0$, where κ is the mean curvature of $\partial\omega$.

Then, we have the stability result:

$$\lim_{t \to 0} \frac{\left\| u^t_{|_\gamma} - u_{|_\gamma} \right\|_{L^2(\gamma)}}{|t|} > 0. \tag{10}$$

Remark 1
We can construct a vector field V satisfying condition *(ii)* and *(iii)* as follows. Let z_1, z_2, z_3 be distinct points on the boundary $\partial\omega$. Let $r > 0$ sufficiently small such that

$$\overline{B(z_i, 2r)} \cap \overline{B(z_j, 2r)} = \emptyset \quad i \neq j.$$

Here, $B(z_i, r)$ denotes the ball of center z_i and radius r. For $i = 1 \ldots 3$, let $\phi_i \in C_c^\infty(\mathbb{R}^2)$ be such that

$$\phi_i > 0, \quad \phi = 1 \text{ in } \overline{B(z_i, r)} \quad \text{and supp } \phi_i \subset B(z_i, 2r).$$

We set $V = (\phi_1 + \phi_2 - \phi_3)\psi\tilde{\nu}$, where $\tilde{\nu}$ is a C^1-extension of the normal vector ν, $\psi \in C_c^\infty(\mathbb{R}^2)$, $\psi = 1$ in a neighborhood of ω and supp $\psi \subset \Omega$. Because $V \cdot \nu = \phi_1 + \phi_2 - \phi_3$ on $\partial\omega$, we obtain $\Sigma_+ = \Gamma \cup \Gamma'$ where $\bar{\Gamma} \subset B(z_1, 2r)$ and $\overline{\Gamma'} \subset B(z_2, 2r)$. We have also $\overline{\Sigma_-} \subset B(z_3, 2r)$. Therefore,

$$\left(\overline{\Sigma_+ \setminus \Gamma} \cup \overline{\Sigma_-}\right) \cap \bar{\Gamma} \subset (B(z_2, 2r) \cup B(z_3, 2r)) \cap B(z_1, 2r) = \emptyset.$$

This imply that

$$dist\left(\left(\overline{\Sigma_+ \setminus \Gamma} \cup \overline{\Sigma_-}\right), \bar{\Gamma}\right) > 0.$$

Let us recall that this example was considered in [8] to prove local Lipshitz stability for the conductivity problem by measurements of the Neumann data.

The proof of Theorem 1 will follow from the fact that the map $t \to u^t$ is differentiable at $t = 0$, and its derivative is not identically equal to zero on γ. Before giving the proof, we need the following result.

Theorem 2
The state u has a material derivative $\dot{u} \in H^1(\Omega)$ that solves

$$\int_\Omega \sigma \nabla \dot{u} \cdot \nabla v \, dx + \int_{\partial\omega} p\dot{u}v \, ds = -\int_\Omega \sigma A_V'(0) \nabla u \cdot \nabla v \, dx - \int_{\partial\omega} p w_V'(0) uv \, ds \quad \forall v \in H^1(\Omega), \tag{11}$$

where

$$A_V'(0) = \operatorname{div}(V) - DV - DV^* \textit{ and } w_V'(0) = \operatorname{div}_\tau(V).$$

The state u is shape differentiable, and its shape derivative u' solves the following system:

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega \setminus \bar{\omega} \textit{ and } \omega, \\ [\![u']\!] = \left(\dfrac{[\![\sigma]\!]}{\sigma_1} \partial_\nu u^- - \dfrac{p}{\sigma_1} u\right) V \cdot \nu & \text{on } \partial\omega, \\ [\![\sigma \partial_\nu u']\!] - pu' = [\![\sigma]\!] \operatorname{div}_\tau (\nabla_\tau u V \cdot \nu) & \\ \qquad\qquad + p\left(\partial_\nu u^- + \kappa u\right) V \cdot \nu & \text{on } \partial\omega, \\ \partial_\nu u' = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Before proving Theorem 2, we make some comments. The derivative u' is not continuous across the interface $\partial\omega$. As a consequence u' cannot be in $H^1(\Omega)$, it belongs only to $H^1(\Omega \setminus \bar{\omega}) \cup H^1(\omega)$. We will explain how both the derivative and (12) can be obtained by the classical methods of shape optimization.

Proof
We compose the proof in four parts: first, transport the problem on a fixed domain, then prove weak convergence of the material derivative, then strong convergence, and return to the shape derivative.

First step: The transported solution u^t solves the variational equation:

$$\forall v \in H^1(\Omega), \quad \int_\Omega \sigma A_V(t) \nabla u^t \cdot \nabla v \, dx + \int_{\partial\omega} p w_V(t) u^t v \, ds = \int_{\partial\Omega} W_V(t) g \circ T_t v \, ds.$$

Second step: Subtracting the variational equation solved by u, and using the fact that $T_t(x) = x$ on $\partial\Omega$, we obtain

$$\begin{aligned} \int_\Omega \sigma A_V(t) \left(\frac{\nabla u^t - \nabla u}{t}\right) \cdot \nabla v \, dx + \int_{\partial\omega} p w_V(t) \left(\frac{u^t - u}{t}\right) v \, ds &= \int_\Omega \sigma \left(\frac{I - A_V(t)}{t}\right) \nabla u \cdot \nabla v \, dx + \int_{\partial\omega} p \left(\frac{I - w_V(t)}{t}\right) uv \, ds \\ &\quad + \int_{\partial\Omega} (w_V(t) g \circ T_t - g) \, v \, ds \\ &= \int_\Omega \sigma \left(\frac{I - A_V(t)}{t}\right) \nabla u \cdot \nabla v \, dx + \int_{\partial\omega} p \left(\frac{I - w_V(t)}{t}\right) uv \, ds. \end{aligned} \tag{13}$$

Plugging $(u^t - u)/t$ as a test function, we obtain from assumption H_0 (9).

$$\alpha \left\| \frac{\nabla(u^t-u)}{t} \right\|^2_{L^2(\Omega)} + \alpha' \left\| \frac{u^t-u}{t} \right\|^2_{L^2(\partial\omega)} \leq \left\| \sigma \frac{I-A_V(t)}{t} \right\|_\infty \|\nabla u\|_{L^2(\Omega)} \left\| \frac{\nabla(u^t-u)}{t} \right\|_{L^2(\Omega)} + \left\| p\frac{1-w_V(t)}{t} \right\|_\infty \|u\|_{L^2(\partial\omega)} \left\| \frac{u^t-u}{t} \right\|_{L^2(\partial\omega)}.$$

From Young's inequality and the fact that $\|u\|^2 := \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\omega)}$ is a norm on $H^1(\Omega)$ equivalent to the natural norm (Proposition 2), we deduce that

$$\left\| \frac{u^t-u}{t} \right\|^2_{H^1(\Omega)} \leq C\left(\left\| \sigma \frac{I-A_V(t)}{t} \right\|_\infty \|\nabla u\|^2_{L^2(\Omega)} + \left\| p\frac{1-w_V(t)}{t} \right\|_\infty \|u\|^2_{L^2(\partial\omega)} \right),$$

where C is a positive constant. Therefore, $(u^t-u)/t$ is bounded in $H^1(\Omega)$. Hence, the sequence is weakly convergent in $H^1(\Omega)$, and its weak limit is the material derivative $\dot{u}$ of u.

Third step: We show the strong convergence of $(u^t-u)/t$ in $H^1(\Omega)$. Passing to the limit $t \to 0$ in (13) yields

$$\int_\Omega \sigma \nabla \dot{u} \cdot \nabla v\, dx + \int_{\partial\omega} p\dot{u}v\, ds = -\int_\Omega \sigma A'_V(0)\nabla u \cdot \nabla v\, dx - \int_{\partial\omega} pw'_V(0)uv\, ds.$$

Therefore, $\dot{u}$ satisfy (11) in Theorem 2. This enable us to show the strong convergence in $H^1(\Omega)$; indeed, setting $v = (u^t-u)/t$ in (13), we obtain

$$\begin{aligned} \int_\Omega \sigma A_V(t)\nabla v \cdot \nabla v\, dx + \int_{\partial\omega} pw_V(t)v^2\, ds &= \int_\Omega \sigma \frac{I-A_V(t)}{t}\nabla u \cdot \nabla v\, dx + \int_{\partial\omega} p\frac{I-w_V(t)}{t} uv\, ds \\ &= \int_\Omega \sigma(A_V(t)-I)\nabla v \cdot \nabla v\, dx + \int_{\partial\omega} p(w_V(t)-1)v^2\, ds \\ &\quad + \int_\Omega \sigma \frac{I-A_V(t)}{t}\nabla u^t \cdot \nabla v\, dx + \int_{\partial\omega} p\frac{1-w_V(t)}{t}u^t v\, ds \\ &= \mathfrak{E}_1(t) + \mathfrak{E}_2(t), \end{aligned} \tag{14}$$

where

$$\mathfrak{E}_1(t) = \int_\Omega \sigma(A_V(t)-I)\nabla v \cdot \nabla v\, dx + \int_{\partial\omega} p(w_V(t)-1)v^2\, ds,$$

and

$$\mathfrak{E}_2(t) = \int_\Omega \sigma \frac{I-A_V(t)}{t}\nabla u^t \cdot \nabla v\, dx + \int_{\partial\omega} p\frac{1-w_V(t)}{t}u^t v\, ds.$$

We have

$$\lim_{t\to 0} A_V(t) = I, \qquad \lim_{t\to 0} w_V(t) = 1, \qquad \lim_{t\to 0} \frac{I-A_V(t)}{t} = -A'_V(0), \qquad \lim_{t\to 0} \frac{1-w_V(t)}{t} = -w'_V(0).$$

Therefore,

$$\mathfrak{E}_1(t) \longrightarrow 0 \text{ and } \mathfrak{E}_2(t) \longrightarrow -\int_\Omega \sigma A'_V(0)\nabla u \cdot \nabla \dot{u}\, dx - \int_{\partial\omega} pw'_V(0)u\dot{u}\, ds \text{ as } t \to 0.$$

Using (11), we conclude that

$$\mathfrak{E}_2(t) \longrightarrow \int_\Omega \sigma |\nabla \dot{u}|^2\, dx + \int_{\partial\omega} p\dot{u}^2\, ds.$$

From the weak convergence of v to $\dot{u}$, and the fact that $\left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\omega)} \right)^{1/2}$ is a norm equivalent to the norm of $H^1(\Omega)$ (Proposition 2), we deduce the strong convergence of v to $\dot{u}$ in $H^1(\Omega)$.

Forth step: We deduce that the equation satisfied the shape derivative $u' = \dot{u} - \nabla u \cdot V$. We have the classical identity (for instance, [17]).

$$-\nabla u \cdot A'(0)\nabla v = \operatorname{div}(b) - (\nabla u \cdot V)\Delta v - (\nabla v \cdot V)\Delta u,$$

where

$$b = (\nabla u \cdot V)\nabla v + (\nabla v \cdot V)\nabla u - (\nabla u \cdot \nabla v)V.$$

From Eq. 11, we have

$$\int_\Omega \sigma \nabla \dot{u} \cdot \nabla v \, dx + \int_{\partial\omega} p\dot{u}v \, ds = \int_\Omega \sigma \operatorname{div}(b) \, dx - \int_\Omega \sigma(\nabla u \cdot V)\Delta v \, dx$$
$$- \int_\Omega \sigma(\nabla v \cdot V)\Delta u \, dx - \int_{\partial\omega} puw'_V(0)v \, ds.$$

From the divergence theorem, and integration by parts, yields

$$\int_\Omega \sigma \nabla \dot{u} \cdot \nabla v \, dx + \int_{\partial\omega} p\dot{u}v \, ds = -\int_{\partial\omega} [\![\sigma(\nabla v \cdot V)\partial_\nu u]\!] \, ds + \int_{\partial\omega} [\![\sigma(\nabla u \cdot \nabla v)V \cdot \nu]\!] \, ds$$
$$- \int_{\partial\omega} puw'_V(0)v \, ds + \int_\Omega \sigma \nabla(\nabla u \cdot V)\nabla v \, dx.$$

Finally, we obtain

$$\begin{aligned}
&\int_\Omega \sigma \nabla(\dot{u} - \nabla u \cdot V) \cdot \nabla v \, dx + \int_{\partial\omega} p(\dot{u} - \nabla u \cdot V)v \, ds \\
&\quad = -\int_{\partial\omega} [\![\sigma(\nabla v \cdot V)\partial_\nu u]\!] \, ds + \int_{\partial\omega} [\![\sigma(\nabla u \cdot \nabla v)V \cdot \nu]\!] \, ds \\
&\qquad - \int_{\partial\omega} p(\nabla u \cdot V)v \, ds - \int_{\partial\omega} puw'_V(0)v \, ds \\
&\quad = [\![\sigma]\!] \int_{\partial\omega} \nabla_\tau u \cdot \nabla_\tau v V \cdot \nu \, ds - \int_{\partial\omega} p(\nabla u \cdot V)v \, ds - \int_{\partial\omega} \operatorname{div}_\tau(V)puv \, ds \\
&\quad = -[\![\sigma]\!] \int_{\partial\omega} \operatorname{div}_\tau(\nabla_\tau uV \cdot \nu)v \, ds - \int_{\partial\omega} p(\partial_\nu u^- + \kappa u)vV \cdot \nu \, ds.
\end{aligned}$$

Therefore, the shape derivative $u' = \dot{u} - \nabla u \cdot V$ solves the following equation:

$$\int_\Omega \sigma \nabla u' \cdot \nabla v \, dx + \int_{\partial\omega} pu'v \, ds = -[\![\sigma]\!] \int_{\partial\omega} \operatorname{div}_\tau(\nabla_\tau uV \cdot \nu)v \, ds - \int_{\partial\omega} p(\partial_\nu u^- + \kappa u)vV \cdot \nu \, ds.$$

Because $u' \in H^1(\Omega \setminus \bar{\omega}) \cup H^1(\omega)$, we obtain

$$-\int_{\Omega\setminus\bar{\omega}} \sigma_1 \Delta u' v \, dx - \int_\omega \sigma_2 \Delta u' v \, dx + \int_{\partial\omega} \left(pu' - [\![\sigma \partial_\nu u']\!]\right) v \, ds = -[\![\sigma]\!] \int_{\partial\omega} \operatorname{div}_\tau(\nabla_\tau uV \cdot \nu)v \, ds$$
$$- \int_{\partial\omega} p(\partial_\nu u^- + \kappa u)vV \cdot \nu \, ds.$$

This imply that $\Delta u' = 0$ in $\Omega \setminus \bar{\omega}$ and ω with the transmission condition

$$[\![\sigma \partial_\nu u']\!] - pu' = [\![\sigma]\!] \operatorname{div}_\tau(\nabla_\tau uV \cdot \nu) + p(\partial_\nu u^- + \kappa u)V \cdot \nu. \tag{15}$$

It remains to compute the jump $[\![u']\!]$. Because $\dot{u} \in H^1(\Omega)$, we have

$$[\![u']\!] = -[\![\partial_\nu u]\!] V \cdot \nu, \tag{16}$$

and from the transmission condition $[\![\sigma \partial_\nu u]\!] = pu$, we obtain

$$[\![u']\!] = \left(\frac{[\![\sigma]\!]}{\sigma_1} \partial_\nu u^- - \frac{p}{\sigma_1} u \right) V \cdot \nu, \tag{17}$$

which concludes the proof. □

Proof of Theorem 1

By definition of the material derivative, the limits in (10) is given by

$$\lim_{t \to 0} \frac{\left\| u^t_{|\gamma} - u_{|\gamma} \right\|_{L^2(\gamma)}}{|t|} = \left\| \dot{u}_{|\gamma} \right\|_{L^2(\gamma)} = \left\| (u' - \nabla u \cdot V)_{|\gamma} \right\|_{L^2(\gamma)}.$$

Because $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$, we have $V_{|\gamma} = 0$. Therefore, it is enough to prove that $\left\| u'_{|\gamma} \right\|_{L^2(\gamma)} > 0$. Assume that $\left\| u'_{|\gamma} \right\|_{L^2(\gamma)} = 0$. Then, from (12), we have

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega \setminus \bar{\omega}, \\ u' = 0 & \text{on } \gamma, \\ \partial_\nu u' = 0 & \text{on } \gamma. \end{cases}$$

By the unique continuation property for the Laplace operator, we deduce that $u' = 0$ in $\Omega \setminus \bar{\omega}$. From assumption (*iii*), we deduce that the subset

$$\Sigma_0 := \{x \in \partial\omega; V \cdot \nu = 0\}$$

is non-empty. Using again (12), u' solves the overdetermined problem:

$$\begin{cases} \Delta u' = 0 & \text{in } \omega, \\ u' = 0 & \text{on } \Sigma_0, \\ \partial_\nu u' = 0 & \text{on } \Sigma_0. \end{cases}$$

Therefore, we obtain $u' = 0$ in ω by the unique continuation property. Consequently, we conclude from (15) and (17) that

$$[\![\sigma]\!]\, \partial_\nu u^- = pu \text{ on } \partial\omega \setminus \Sigma_0, \tag{18}$$

and

$$[\![\sigma]\!]^2 \int_{\partial\omega \setminus \Sigma_0} \nabla_\tau u \cdot \nabla_\tau v V \cdot \nu \, ds = \int_{\partial\omega \setminus \Sigma_0} \left(p^2 + \kappa p\, [\![\sigma]\!]\right) uvV \cdot \nu \, ds. \tag{19}$$

Using again assumption (*iii*), we can find an open subset $\Theta \subset \mathbb{R}^2$ such that

$$\bar{\Gamma} \subset \Theta \quad \text{and} \quad \Theta \cap (\overline{\Sigma_+ \setminus \Gamma} \cup \Sigma_-) = \emptyset.$$

Let $\psi \in C^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}(\psi) \subset \Theta$ and $\psi = 1$ in a neighborhood of $\bar{\Gamma}$. Let $z \in C^2$ be the solution of the following problem:

$$\begin{cases} \Delta z = 0 & \text{in } \omega, \\ z = \psi u & \text{on } \partial\omega. \end{cases}$$

From Equation 19 and using the fact that $z = 0$ in $(\Sigma_+ \cup \Sigma_-) \setminus \Gamma$, we obtain

$$[\![\sigma]\!]^2 \int_\Gamma |\nabla_\tau u|^2 V \cdot \nu \, ds = \int_\Gamma \left(p^2 + \kappa p\, [\![\sigma]\!]\right) u^2 V \cdot \nu \, ds.$$

From assumption (*iv*), we obtain $\nabla_\tau u = 0$ and $u = 0$ in Γ. From (16) and (18), we have $\partial_\nu u^+ = 0$ on Γ. According to the unique continuation property, $u = 0$ in $\Omega \setminus \omega$, which contradicts the assumption that g is not identically equal to zero, and the proof is completed. □

Remark 2
Theorem 1 prove the 'local continuity' of the map

$$\begin{aligned} \eta : L^2(\gamma) &\longrightarrow \mathcal{O} \\ u_t &\longmapsto \omega_t \end{aligned}$$

for t sufficiently small t, and $\mathcal{O}$ is the set of admissible domains equipped with an appropriate topology.

Let us consider the topology induced by the Hausdorff metric (see [9] for more details). Then, we have

$$d_H(\omega_t, \omega) \leq \|V\|_{L^\infty(\Omega)} |t|. \tag{20}$$

Thus, from Theorem 1 and inequality (20), we deduce a 'local directional continuity' of η, because for $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$, there exists a constant $C(V) > 0$ such that

$$d_H(\omega_t, \omega) \leq \frac{\|V\|_{L^\infty(\Omega)}}{C(V)} \|u_t - u\|_{L^2(\gamma)},$$

for sufficiently small t.

In the following section, we introduce the minimization problem to solve numerically the inverse problem (3), and we prove the existence of optimal solution.

4. The minimization problem

A typical approach to solve the inverse problem (3) numerically is to consider the so-called *Kohn–Vogelius* functional

$$J(\omega, u_N, u_D) = \alpha_1 \int_\Omega |\nabla(u_N - u_D)|^2 \, dx + \alpha_2 \int_\Omega |u_N - u_D|^2 \, dx. \tag{21}$$

Here, $\alpha_1, \alpha_2 \in \mathbb{R}_+^*$ are parameters, u_N is the solution of the Neumann problem

$$\begin{cases} -\mathrm{div}(\sigma \nabla u_N) = 0 & \text{in } \Omega \setminus \partial\omega, \\ [\![u_N]\!] = 0 & \text{on } \partial\omega, \\ [\![\sigma \partial_\nu u_N]\!] = p u_N & \text{on } \partial\omega, \\ \partial_\nu u_N = g & \text{on } \partial\Omega, \end{cases} \tag{22}$$

and u_D is the solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}(\sigma \nabla u_D) = 0 & \text{in } \Omega \setminus \partial\omega, \\ [\![u_D]\!] = 0 & \text{on } \partial\omega, \\ [\![\sigma \partial_\nu u_D]\!] = p u_D & \text{on } \partial\omega, \\ u_D = f & \text{on } \partial\Omega, \end{cases} \tag{23}$$

where $f \in H^{1/2}(\partial\Omega)$ is a measurement of the potential corresponding to the input flux g.

The variational formulation corresponding to (22) is given by

$$\begin{cases} \text{Find } u \in H^1(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_{\partial\omega} p u v \, ds = \int_{\partial\Omega} g v \, ds \quad \text{for all } v \in H^1(\Omega). \end{cases} \tag{24}$$

For the Dirichlet problem (23), the constraint $u_D = f$ makes the Sobolev space dependent on f. To get around this difficulty, we introduce a Lagrange multiplier, and we obtain the variational formulation [9, Sec 6.2, p 433]:

$$\int_\Omega \sigma \nabla u_D \cdot \nabla v \, dx + \int_{\partial\omega} p u_D v \, ds + \int_{\partial\Omega} (f - u_D)\mu \, ds = 0 \quad \forall v \in H^1_{0,\sigma}(\Omega), \mu \in H^{-1/2}(\partial\Omega),$$

where

$$H^1_{0,\sigma}(\Omega) := \left\{ v \in H^1_0(\Omega) : \mathrm{div}(\sigma \nabla v) \in L^2(\Omega) \right\}.$$

Writing the saddle point equation for the Lagrangian, one obtains $\mu = \sigma_1 \partial_\nu v$, and the weak solution of the Dirichlet problem is then defined by

$$\begin{cases} \text{Find } u \in H^1(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_{\partial\omega} p u v \, ds + \int_{\partial\Omega} (f - u)\sigma_1 \partial_\nu v \, ds \quad \text{for all } v \in H^1_{0,\sigma}(\Omega). \end{cases} \tag{25}$$

In this paper, we consider the situation when the Robin parameter p is known, and we aim to reconstruct the domain ω. The corresponding minimization problem is the following:

$$\begin{cases} \text{minimize } J(\omega, u_N, u_D) \\ \text{subject to } \omega \subset \mathcal{O}, u_N \text{ and } u_D \text{ solutions of (22) and (23)} \end{cases} \tag{26}$$

An optimal solution $\omega^* = \omega^*(p)$ of (26) if exists depend on p trough the state equations.

Let us define the reduced functional

$$\mathcal{J}(\omega, p) := J(\omega, u_N(\omega, p), u_D(\omega), p).$$

We have the following theorem.

Theorem 3

The minimization problem (26) has at least one solution.

Before proving Theorem 3, we need some auxiliary results.

Definition 1 ([18])
Let A, B be two subsets of $\mathbb{R}^d$, $d \geq 2$, and define

$$d(x,B) = \inf_{y \in B} |x-y|, \quad \rho(A,B) = \sup_{x \in A} d(x,B), \quad d_H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

d_H is the Hausdorff distance between A and B; it defines a topology on the closed bounded sets of $\mathbb{R}^d$.

Let (Ω_n) be a sequence of open subsets of a fixed closed domain D and Ω be an open subset of D. We say that the sequence (Ω_n) converges on Ω in the Hausdorff sense, and we denote $\Omega_n \xrightarrow{H} \Omega$ if $\lim_{n\to\infty} d_H(D \setminus \Omega_n, D \setminus \Omega) = 0$.

Definition 2 ([18])
Let ξ be a unitary vector of $\mathbb{R}^d$, $d \geq 2$, ε be a real number strictly positive, and $y \in \mathbb{R}^d$. We call a cone with vertex y, direction ξ, and dimension ε, the set defined by

$$C(y,\xi,\varepsilon) := \left\{x \in \mathbb{R}^d :< x-y, \xi >_{\mathbb{R}^d} \geq \cos(\varepsilon)\|x-y\|_{\mathbb{R}^d} \text{ and } 0 < \|x-y\|_{\mathbb{R}^d} < \varepsilon\right\},$$

where $< .,. >_{\mathbb{R}^d}$ is the euclidean scalar product of $\mathbb{R}^d$ and $\|.\|_{\mathbb{R}^d}$ is the associated euclidean norm.

An open set Ω of $\mathbb{R}^d$ verifies the ε-cone property, if for $x \in \partial\Omega$, there exists a unitary vector ξ_x of $\mathbb{R}^d$ such that

$$\text{for all } y \in \bar{\Omega} \cap B(x,\varepsilon), \quad C(y,\xi_x,\varepsilon) \subset \Omega,$$

where $B(x,\varepsilon)$ denotes the open ball with center x and radius ε.

Proposition 1 ([18])
An open bounded set Ω of $\mathbb{R}^d$ has the ε-cone property if and only if Ω has a Lipschitz continuous boundary with constant $k(\varepsilon) > 0$.

Proof
(of Theorem 3) It is clear that $\inf \mathcal{J}(\omega)$ is finite. Therefore, there exists a minimizing sequence $\omega_n \in \mathcal{O}$ such that

$$\lim_{n\to\infty} \mathcal{J}(\omega_n) = \inf_{\omega \in \mathcal{O}} \mathcal{J}(\omega).$$

The sequence $\omega_n \in \mathcal{O}$ is bounded. According to [19, Theorem 2.4.10], there exists $\omega^* \in \mathcal{O}$, and a subsequence still denoted ω_n such that ω_n converges to ω^* in the sense of Hausdorff and in the sense of characteristic functions.

By definition, $u_N(\omega_n)$ and $u_D(\omega_n)$ solve

$$\int_\Omega \chi_{\Omega\setminus\overline{\omega_n}}\sigma_1\nabla u_N(\omega_n)\cdot\nabla v\,dx + \int_\Omega \chi_{\omega_n}\sigma_2\nabla u_N(\omega_n)\cdot\nabla v\,dx + \int_\Omega \chi_{\omega_n}\operatorname{div}(pu_N(\omega_n)v\nu)\,dx = \int_{\partial\Omega} gv\,ds \quad \forall\, v \in H^1(\Omega). \tag{27}$$

$$\int_\Omega \chi_{\Omega\setminus\overline{\omega_n}}\sigma_1\nabla u_D(\omega_n)\cdot\nabla v\,dx + \int_\Omega \chi_{\omega_n}\sigma_2\nabla u_D(\omega_n)\cdot\nabla v\,dx + \int_\Omega \chi_{\omega_n}\operatorname{div}(pu_D(\omega_n)v\nu)\,dx + \int_{\partial\Omega}(f-u_D(\omega_n))\sigma_1\partial_\nu v\,ds, \tag{28}$$

for all $v \in H^1_{0,\sigma}(\Omega)$. Taking $v = u_N(\omega_n)$ in (27), we obtain the estimate

$$\|u_N(\omega_n)\|_{H^1(\Omega)} \leq c\|g\|_{L^2(\Omega)}, \tag{29}$$

where c depends only on Ω. Thus, we may extract a subsequence still denoted $u_N(\omega_n)$ such that

$$u_N(\omega_n) \to u_N^* \quad \text{strongly in } L^2(\Omega) \text{ and } \nabla u_N(\omega_n) \rightharpoonup \nabla u_N^* \quad \text{weakly in } L^2(\Omega). \tag{30}$$

By the same way, we can prove that

$$u_D(\omega_n) \to u_D^* \quad \text{strongly in } L^2(\Omega) \text{ and } \nabla u_D(\omega_n) \rightharpoonup \nabla u_D^* \quad \text{weakly in } L^2(\Omega). \tag{31}$$

The pointwise a.e. convergence of the characteristic functions χ_{ω_n} to χ_{ω^*} and $\chi_{\Omega\setminus\overline{\omega_n}}$ to $\chi_{\Omega\setminus\overline{\omega^*}}$ with (30) and (31) yields at the limit, in (27) and (28)

$$\int_\Omega \sigma\nabla u_N^*\cdot\nabla v\,dx + \int_{\partial\omega^*} pu_N^* v\,ds = \int_{\partial\Omega} gv\,ds \quad \forall\, v \in H^1(\Omega),$$

$$\int_\Omega \sigma\nabla u_D^*\cdot\nabla v\,dx + \int_{\partial\omega^*} pu_D^* v\,ds + \int_{\partial\Omega}(f-u_D^*)\sigma_1\partial_\nu v\,ds = 0 \quad \forall\, v \in H^1_{0,\sigma}(\Omega).$$

Because of the uniqueness of the limit, we conclude that $u_N^* = u_N(\omega^*)$ and $u_D^* = u_D(\omega^*)$. The lower semi-continuity of the L^2-norm imply

$$\mathcal{J}(\omega^*) \leq \lim_{n\to\infty} \inf \mathcal{J}(\omega_n) \leq \mathcal{J}(\omega),$$

which concludes the proof. □

5. Shape stability wit respect to the Robin parameter

In this section, we study the situation where the Robin parameter p is known with some uncertainty, and we are looking for the geometry ω, that is, this corresponds to problem (26). The optimality conditions for problem (26) reads

$$D_\omega \mathcal{J}(\omega, p)(\theta) = 0 \text{ for all } \theta \in \mathcal{D}^1(\Omega, \mathbb{R}^2).$$

It can be rewritten as

$$\text{Find } \omega^*(p) \text{ such that } D_\omega \mathcal{J}(\omega^*(p), p) = 0 \tag{32}$$

where $D_\omega \mathcal{J}(\omega^*(p), p) : \mathcal{D}^1(\Omega, \mathbb{R}^2) \to \mathbb{R}$.

In this paper, we are interested in the stability of the optimal solution $\omega^*(p)$ of the minimization problem (26) with respect to p. More precisely, we study the variation $\omega^*(\tilde{p})$ of the optimal solution $\omega^*(p)$ when the exact Robin parameter p takes an uncertain value $\tilde{p}$.

Because the parameter-to-solution map $p \mapsto \omega^*(p)$ is not properly speaking a function of p, instead we consider a parameterization

$$\omega^*(\tilde{p}) = (I + \theta^*(\tilde{p}))(\omega^*(p)),$$

and we differentiate the map $p \mapsto \theta^*(p)$. This is performed in the following by applying the implicit function theorem to the optimality conditions $D_\omega \mathcal{J}(\omega^*(p), p)(\theta) = 0$. Essentially, the idea is to linearize $D_\omega \mathcal{J}(\omega^*(p), p) = 0$ to obtain

$$D^2_\omega \mathcal{J}(\omega^*(p), p)(\partial_p \theta^*(p)) + \partial_p D_\omega \mathcal{J}(\omega^*(p), p) = 0.$$

This yields the following result.

Theorem 4
If $D^2_\omega \mathcal{J}(\omega^*(p), p)$ is invertible, then the first-order approximation of $\theta^*(\tilde{p})$ is given by

$$\theta^*(\tilde{p}) \approx \partial_p \theta^*(p) = -(D^2_\omega \mathcal{J}(\omega^*(p)))^{-1} \partial_p D_\omega \mathcal{J}(\omega^*(p)). \tag{33}$$

In (33), the computation of second-order shape derivatives of $\mathcal{J}$ is required. Therefore, in what follows, we compute first-order and second-order shape derivatives and the second-order mixed derivative of $\mathcal{J}(\omega, p)$. To obtain the expression of the shape derivative, it is convenient to simply compute the directional derivative given by Definition 3.

5.1. Shape derivative of the functional J

In this subsection, we perform the analysis of the shape derivative of the functional J.

5.1.1. Lagrange formulation and adjoints states. Denote by $\boldsymbol{u} = (\boldsymbol{u}_N, \boldsymbol{u}_D)$, $\boldsymbol{u}^a = (\boldsymbol{u}_N^a, \boldsymbol{u}_D^a)$, $u = (u_N, u_D)$, $u^a = (u_N^a, u_D^a)$ and introducing the Lagrangian functional

$$\begin{aligned}\mathcal{L}(\omega, \boldsymbol{u}, \boldsymbol{u}^a) := &\int_\Omega \alpha_1 |\nabla(\boldsymbol{u}_N - \boldsymbol{u}_D)|^2 + \alpha_2 |\boldsymbol{u}_N - \boldsymbol{u}_D|^2 \, dx \\ &+ \int_\Omega \sigma \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^a \, dx + \int_{\partial\omega} p\boldsymbol{u} \cdot \boldsymbol{u}^a \, ds - \int_{\partial\Omega} g \boldsymbol{u}_N^a \, ds \\ &+ \int_{\partial\Omega} (f - \boldsymbol{u}_D^a) \sigma_1 \partial_\nu \boldsymbol{u}_D^a \, ds.\end{aligned}$$

Then, it is easy to check that

$$J(\omega, u(\omega)) = \min_{\boldsymbol{u} \in H^1(\Omega)^2} \sup_{\boldsymbol{u}^a \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega)} \mathcal{L}(\omega, \boldsymbol{u}, \boldsymbol{u}^a),$$

because

$$\sup_{\boldsymbol{u}^a \in H^1(\Omega)\times H^1_{0,\sigma}(\Omega)} \mathcal{L}(\omega, \boldsymbol{u}, \boldsymbol{u}^a) = \begin{cases} J(\omega, u(\omega)) \text{ if } \boldsymbol{u} = u \\ \quad +\infty \text{ otherwise.} \end{cases}$$

We can easily shown that the functional $\mathcal{L}$ is convex continuous with respect to $\boldsymbol{u}$ and concave continuous with respect to $\boldsymbol{u}^a$. Therefore, according to Ekeland and Temam [20], the functional $\mathcal{L}$ has a saddle point (u, u^a) if and only if (u, u^a) solve the following system:

$$\partial_{\boldsymbol{u}} \mathcal{L}(\omega, u, u^a)(\hat{\boldsymbol{u}}) = 0 \qquad \forall \hat{\boldsymbol{u}} \in H^1(\Omega)^2,$$

$$\partial_{\boldsymbol{u}^a} \mathcal{L}(\omega, u, u^a)(\hat{\boldsymbol{u}^a}) = 0 \qquad \forall \hat{\boldsymbol{u}^a} \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega).$$

This yields that $\mathcal{L}$ has a saddle point (u, u^a), where the states u_N and u_D are the unique solutions of (22) and (23), respectively, and the adjoint states u_N^a and u_D^a solve the following problems:

$$\begin{gathered}\int_\Omega \sigma \nabla u_N^a \cdot \nabla \hat{\varphi}_N \, dx + \int_{\partial\omega} p u_N^a \hat{\varphi}_N \, ds = -2 \int_\Omega \alpha_1 \nabla(u_N - u_D) \cdot \nabla \hat{\varphi}_N + \alpha_2 (u_N - u_D) \hat{\varphi}_N \, dx \\ \forall \hat{\varphi}_N \in H^1(\Omega),\end{gathered} \tag{34}$$

$$\begin{gathered}\int_\Omega \sigma \nabla u_D^a \cdot \nabla \hat{\varphi}_D \, dx + \int_{\partial\omega} p u_D^a \hat{\varphi}_D \, ds = 2 \int_\Omega \alpha_1 \nabla(u_N - u_D) \cdot \nabla \hat{\varphi}_D + \alpha_2 (u_N - u_D) \hat{\varphi}_D \, dx \\ \forall \hat{\varphi}_D \in H^1_{0,\sigma}(\Omega).\end{gathered} \tag{35}$$

The previous analysis holds also for the functional depending on the transformed domain $\omega_t = T_t(\omega)$. Thus, we obtain

$$J(\omega_t, u(\omega_t)) = \min_{\boldsymbol{u} \in H^1(\Omega)^2} \sup_{\boldsymbol{u}^a \in H^1(\Omega)\times H^1_{0,\sigma}(\Omega)} \mathcal{L}(\omega_t, \boldsymbol{u}, \boldsymbol{u}^a).$$

The corresponding saddle point $(u(\omega_t), u^a(\omega_t))$ is characterized by

$$\partial_{\boldsymbol{u}} \mathcal{L}(\omega_t, u(\omega_t), u^a(\omega_t))(\hat{\boldsymbol{u}}) = 0 \qquad \forall \hat{\boldsymbol{u}} \in H^1(\Omega)^2,$$

$$\partial_{\boldsymbol{u}^a} \mathcal{L}(\omega_t, u(\omega_t), u^a(\omega_t))(\hat{\boldsymbol{u}^a}) = 0 \qquad \forall \hat{\boldsymbol{u}^a} \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega),$$

Theorem 5 (First-order shape derivative)
The shape derivative of the functional $\mathcal{J}$ in the direction $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$ is given by

$$\begin{aligned} D_\omega \mathcal{J}(\omega; V) = & \int_\Omega \alpha_1 A_V'(0) \nabla(u_N - u_D) \cdot \nabla(u_N - u_D) + \alpha_2 |u_N - u_D|^2 \xi_V'(0) \, dx \\ & + \int_\Omega \sigma A_V'(0) \nabla u \cdot \nabla u^a \, dx + \int_{\partial\omega} p u \cdot u^a w_V'(0) \, ds. \end{aligned} \tag{36}$$

Alternatively, the shape derivative can be rewritten in a more structured way using the tensor representation:

$$D_\omega \mathcal{J}(\omega, p)(V) = \int_\Omega S(u, u^a) \cdot DV \, dx + \int_{\partial\omega} S_0(u, u^a) \cdot D_\tau V \, ds, \tag{37}$$

where

$$\begin{aligned} S(u, u^a) := & -2\alpha_1 \left(\nabla(u_N - u_D) \otimes \nabla(u_N - u_D)\right) - \sigma \left(\nabla u \otimes \nabla u^a + \nabla u^a \otimes \nabla u\right) \\ & + \left[\sigma \nabla u \cdot \nabla u^a + \alpha_2 |u_N - u_D|^2 + \alpha_1 \nabla(u_N - u_D) \cdot \nabla(u_N - u_D)\right] I, \end{aligned}$$

and

$$S_0(u, u^a) := p(u \cdot u^a) I.$$

Proof
Let us consider transformations T_t defined in (5). Our aim is to compute the derivative of the functional J using Theorem 8. In order to differentiate $\mathcal{L}(\omega_t, \boldsymbol{u}, \boldsymbol{u}^a)$ with respect to t, the integrals in $\mathcal{L}(\omega_t, \boldsymbol{u}, \boldsymbol{u}^a)$ on the domain ω_t needs to be transported back on the reference domain ω using the transformation T_t. However, composing by T_t inside the integrals creates terms $\boldsymbol{u} \circ T_t$, and $\boldsymbol{u}^a \circ T_t$, which might be

non-differentiable. To avoid this problem, we need to parameterize the space $H^1(\Omega)$ by composing the elements of $H^1(\Omega)$ with T_t^{-1}. Following this argument, we rewrite

$$G(t, \boldsymbol{u}, \boldsymbol{u}^a) := \mathcal{L}\left(\omega_t, \boldsymbol{u} \circ T^{-1}, \boldsymbol{u}^a \circ T_t^{-1}\right).$$

After change of variable, we obtain

$$\begin{aligned} G(t, \boldsymbol{u}, \boldsymbol{u}^a) = & \int_\Omega \alpha_1 A_V(t) \nabla(\boldsymbol{u}_N - \boldsymbol{u}_D) \cdot \nabla(\boldsymbol{u}_N - \boldsymbol{u}_D) + \alpha_2 |\boldsymbol{u}_N - \boldsymbol{u}_D|^2 \xi_V(t)\, dx \\ & + \int_\Omega \sigma A_V(t) \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^a \, dx + \int_{\partial\omega} p\boldsymbol{u} \cdot \boldsymbol{u}^a w_V(t)\, ds \\ & - \int_{\partial\Omega} g \circ T_t \boldsymbol{u}_N^a w_V(t)\, ds + \int_{\partial\Omega} (f \circ T_t - \boldsymbol{u}_D^a) \sigma_1 \partial_\nu \boldsymbol{u}_D^a w(t)\, ds, \end{aligned} \tag{38}$$

where $\xi_V(t)$, $w_V(t)$, and $A_V(t)$ are defined in (6), (7), and (8).

Note that in (38), the integrals on $\partial\Omega$ are unchanged because $T_t = I$ on $\partial\Omega$. The functional G has a saddle point $(u^t, u^{a,t}) := ((u_N^t, u_D^t), (u_N^{a,t}, u_D^{a,t}))$ that solve

$$\int_\Omega \sigma A_V(t) \nabla u_N^t \cdot \nabla \psi_N \, dx + \int_{\partial\omega} p u_N^t \psi_N w_V(t)\, ds = \int_{\partial\Omega} g \circ T_t \psi_N w_V(t)\, ds$$

$$\int_\Omega \sigma A_V(t) \nabla u_D^t \cdot \nabla \psi_D \, dx + \int_{\partial\omega} p \varphi_D \psi_D w_V(t)\, ds = -\int_{\partial\Omega} (f \circ T_t - \varphi_D) \sigma_1 \partial_\nu \psi_D w_V(t)\, ds,$$

$$\begin{aligned} & \int_\Omega \sigma A_V(t) \nabla_1 u_N^{a,t} \cdot \nabla \varphi_N \, dx + \int_{\partial\omega} p u_N^{a,t} w_V(t) \varphi_N \, ds = \\ & \qquad -2 \int_\Omega \alpha_1 A_V(t) \nabla(u_N^t - u_D^t) \cdot \nabla \varphi_N + \alpha_2 (u_N^t - u_D^t) \xi_V(t) \varphi_N \, dx \end{aligned}$$

$$\begin{aligned} & \int_\Omega \sigma A_V(t) \nabla u_D^{a,t} \cdot \nabla \varphi_D \, dx + \int_{\partial\omega} p u_D^{a,t} w_V(t) \varphi_D \, ds = \\ & \qquad 2 \int_\Omega \alpha_1 A_V(t) \nabla(u_N - u_D) \cdot \nabla \varphi_D + \alpha_2 (u_N^t - u_D^t) \xi_V(t) \varphi_D \, dx, \end{aligned}$$

for all $(\psi_N, \psi_D, \varphi_N, \varphi_D) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega)$.

Under hypothesis of Theorem 8, we obtain

$$\begin{aligned} D\mathcal{J}(\omega; V) &= \partial_t G(t, u, u^a)\big|_{t=0} \\ &= \int_\Omega \alpha_1 A_V'(0) \nabla(u_N - u_D) \cdot \nabla(u_N - u_D) + \alpha_2 |u_N - u_D|^2 \xi_V'(0)\, dx \\ &\quad + \int_\Omega \sigma A_V'(0) \nabla u \cdot \nabla u^a \, dx + \int_{\partial\omega} p u \cdot u^a w_V'(0)\, ds. \end{aligned}$$

Using the algebraic relations

$$A_V'(0) \nabla u \cdot \nabla u^a = (\nabla u \cdot \nabla u^a) I \cdot DV - \left(\sum_{j=1}^2 \nabla u_j \otimes \nabla u_j^a + \nabla u_j^a \otimes \nabla u_i \right) \cdot DV$$

and

$$\operatorname{div}_\tau V := I \cdot D_\tau V,$$

we obtain the expression (37).

Verification of hypothesis of Theorem 8: Introduce the sets

$$X(t) := \left\{ x^t \in H^1(\Omega)^2 : \sup_{y \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega)} G(t, x^t, y) = \inf_{x \in H^1(\Omega)^2} \sup_{y \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega)} G(t, x, y) \right\},$$

and

$$Y(t) := \left\{ y^t \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega) : \inf_{x \in H^1(\Omega)^2} G(t, x, y^t) = \sup_{y \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega)} \inf_{x \in H^1(\Omega)^2} G(t, x, y) \right\}.$$

We obtain

$$\forall\ t \in [0,\varepsilon] \quad S(t) = X(t) \times Y(t) = \{u^t, u^{a,t}\} \neq \emptyset,$$

and assumption (H_1) is satisfied.
Assumption (H_2)*:* The partial derivatives $t \to \partial_t G(t,\varphi,\psi)$ exist everywhere in $[0,\epsilon]$, and the condition (H_2) is satisfied.
Assumption (H_3) *and* (H_4) *:* Due to the strong continuity of $A_V(t)$, $w_V(t)$ as a functions of t, the assumption H_0, and the boundness of $(u_N^t, u_D^t, v_N^t, v_D^t)$, one can deduce the strong convergence $u_N^t \to u_N$, $u_D^t \to u_D$, $u_N^{a,t} \to u_N^a$ in $H^1(\Omega)$, and $u_D^{a,t} \to u_D^a$ in $H_0^1(\Omega)$. Finally, in view of the strong continuity of $(t,\boldsymbol{u}) \to \partial_t G(t,\boldsymbol{u},\boldsymbol{u}^a)$ and $(t,\boldsymbol{u}^a) \to \partial_t G(t,\boldsymbol{u},\boldsymbol{u}^a)$, assumptions H_3 and H_4 are verified. □

5.2. Second-order shape derivative

In this subsection, we compute the second-order shape derivative of the Kohn–Vogelius cost functional J using a general method that applies to differentiable semiconvex cost functionals.

5.2.1. Lagrange formulation and adjoints states.

For the second shape derivative, we need two vector fields V and $\hat{V}$ in $\mathcal{D}^1(\Omega;\mathbb{R}^N))$ and the expression of the first shape derivative $D\mathcal{J}(T_t(\omega);V)$, where $T_t(\omega)$ is the perturbation of the domain ω by T_t defined in (5) correspond to a vector filed $\hat{V}$. Then, we can express $D\mathcal{J}(T_t(\omega);V)$ as the min-sup of new Lagrangian

$$D\mathcal{J}(T_t(\omega);V) = \min_{\boldsymbol{U}\in H^1(\Omega)^3\times H^1_{0,\sigma}(\Omega)} \sup_{\boldsymbol{W}\in H^1(\Omega)^3\times H^1_{0,\sigma}(\Omega)} \mathcal{L}_2(\omega_t,\boldsymbol{U},\boldsymbol{W}),$$

where $\boldsymbol{U} := (\boldsymbol{u}_N, \boldsymbol{u}_D, \boldsymbol{u}_N^a, \boldsymbol{u}_D^a)$, $\boldsymbol{W} := (\boldsymbol{w}_N, \boldsymbol{w}_D, \boldsymbol{w}_N^a, \boldsymbol{w}_D^a)$ and $\mathcal{L}_2$ is given by

$$\begin{aligned}\mathcal{L}_2(\omega_t,\boldsymbol{U},\boldsymbol{W}) = &\int_\Omega \alpha_1 A_V'(0)\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)\cdot\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D) + \alpha_2|\boldsymbol{u}_N-\boldsymbol{u}_D|^2\xi_V'(0)\,dx\\
&+\int_\Omega \sigma A_V'(0)\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^a\,dx + \int_{\partial\omega} p\boldsymbol{u}\cdot\boldsymbol{u}^a w_V'(0)\,ds.\\
&+\int_\Omega \sigma\nabla\boldsymbol{U}\cdot\nabla\boldsymbol{W}\,dx + \int_{\partial\omega} p\boldsymbol{U}\cdot\boldsymbol{W}\,ds - \int_{\partial\Omega} g\boldsymbol{w}_N\,ds + \int_{\partial\Omega}(f-\boldsymbol{u}_D)\sigma_1\partial_\nu\boldsymbol{w}_D\,ds\\
&+2\int_\Omega \alpha_1\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)\cdot\nabla\boldsymbol{w}_N^a + \alpha_2(\boldsymbol{u}_N-\boldsymbol{u}_D)\boldsymbol{w}_N^a\,dx\\
&-2\int_\Omega \alpha_1\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)\cdot\nabla\boldsymbol{w}_D^a + \alpha_2(\boldsymbol{u}_N-\boldsymbol{u}_D)\boldsymbol{w}_D^a\,dx.\end{aligned}$$

The adjoint state $W := (w_N, w_D, w_N^a, w_D^a)$ solves the following equations:

$$\partial_{\boldsymbol{U}}\mathcal{L}_2(\omega,U,W)(\hat{U}) = 0 \qquad \forall\,\hat{U}\in H^1(\Omega)^2\times H^1_{0,\sigma}(\Omega)^2,$$

or equivalently

$$\int_\Omega \sigma\nabla w_N^a\cdot\nabla\psi_N\,dx + \int_{\partial\omega} pw_N^a\psi_N\,ds + \int_\Omega \sigma A_V'(0)\nabla u_N^a\cdot\nabla\psi_N\,dx + \int_{\partial\omega} pu_N^a w_V'(0)\psi_N\,ds = 0 \quad \forall\psi_N\in H^1(\Omega), \tag{39}$$

$$\int_\Omega \sigma\nabla w_D^a\cdot\nabla\psi_D\,dx + \int_{\partial\omega} pw_D^a\psi_N\,ds + \int_\Omega \sigma A_V'(0)\nabla u_D^a\cdot\nabla\psi_D\,dx + \int_{\partial\omega} pu_D^a w_V'(0)\psi_D\,ds = 0 \quad \forall\psi_D\in H^1_{0,\sigma}(\Omega), \tag{40}$$

$$\begin{aligned}&\int_\Omega \sigma\nabla w_N\cdot\nabla\varphi_N\,dx + \int_{\partial\omega} pw_N\varphi_N\,ds + 2\int_\Omega \alpha_1\nabla(w_N^a-w_D^a)\cdot\nabla\varphi_N + \alpha_2(w_N^a-w_N^a)\varphi_N\,dx\\
&+2\int_\Omega \alpha_1 A_V'(0)\nabla(u_N-u_D)\cdot\nabla\varphi_N + \alpha_2(u_N-u_D)\xi_V'(0)\varphi_N\,dx + \int_\Omega \sigma A_V'(0)\nabla u_N^a\cdot\nabla\varphi_N\,dx\\
&+\int_{\partial\omega} pu_N^a w_V'(0)\varphi_N\,ds = 0 \quad \forall\varphi_N\in H^1(\Omega),\end{aligned} \tag{41}$$

$$\begin{aligned}&\int_\Omega \sigma\nabla w_D\cdot\nabla\varphi_D\,dx + \int_{\partial\omega} pw_D\varphi_D\,ds - 2\int_\Omega \alpha_1\nabla\left(w_N^a-w_D^a\right)\cdot\nabla\varphi_D + \alpha_2\left(w_N^a-w_D^a\right)\varphi_D\,dx\\
&-2\int_\Omega \alpha_1 A_V'(0)\nabla(u_N-u_D)\cdot\nabla\varphi_D + \alpha_2(u_N-u_D)\xi_V'(0)\varphi_D\,dx + \int_\Omega \sigma A_V'(0)\nabla u_D^a\cdot\nabla\varphi_D\,dx\\
&+\int_{\partial\omega} pu_D^a w_V'(0)\varphi_D\,ds = 0 \quad \forall\,\varphi_D\in H^1_{0,\sigma}(\Omega).\end{aligned} \tag{42}$$

Theorem 6 (Second-order shape derivative)
The second-order shape derivative of the functional $\mathcal{J}$ in the directions V and $\hat{V}$ is given by

$$\begin{aligned}
D^2_\omega \mathcal{J}(\omega,p)(V,\hat{V}) = &\int_\Omega \alpha_1 \mathbb{P}'(0)\nabla(u_N-u_D)\cdot\nabla(u_N-u_D)+\alpha_2\mathbb{Q}'(0)|u_N-u_D|^2\,dx\\
&+\int_\Omega \sigma\mathbb{P}'(0)\nabla u\cdot\nabla u^a\,dx+\int_{\partial\omega} pu\cdot u^a\mathbb{Q}'_\tau(0)\,ds\\
&+\int_\Omega \sigma A'_{\hat{V}}(0)\nabla U\cdot\nabla W\,dx+\int_{\partial\omega} pU\cdot Ww'_{\hat{V}}(0)\,ds\\
&+2\int_\Omega \alpha_1 A'_{\hat{V}}(0)\nabla(u_N-u_D)\cdot\nabla(w^a_N-w^a_D)+\alpha_2(u_N-u_D)(w^a_N-w^a_D)\xi'_{\hat{V}}(0)\,dx,
\end{aligned}$$

where

$$\mathbb{P}'(0)=\nabla(\mathrm{div}(V))\hat{V}I-D^2V\hat{V}-D^2V^*\hat{V},\quad \mathbb{Q}'(0)=\nabla(\mathrm{div}(V))\hat{V}+\mathrm{div}(V)\,\mathrm{div}(\hat{V}),\text{ and }\mathbb{Q}'_\tau(0)=\nabla(\mathrm{div}_\tau(V))\hat{V}+\mathrm{div}_\tau(V)\,\mathrm{div}_\tau(\hat{V}).$$

Proof
Introduce the Lagrangian

$$G_2(t,\boldsymbol{U},\boldsymbol{W}):=\mathcal{L}_2\left(\omega_t,\boldsymbol{U}\circ T_t^{-1},\boldsymbol{W}\circ T_t^{-1}\right).$$

After change of variable, we obtain

$$\begin{aligned}
G_2(t,\boldsymbol{U},\boldsymbol{W}) = &\int_\Omega \alpha_1 A_{\hat{V}}(t)A'_V(0)\circ T_t\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)\cdot\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)+\alpha_2|\boldsymbol{u}_N-\boldsymbol{u}_D|^2\xi'_V(0)\circ T_t\xi_{\hat{V}}(t)\,dx\\
&+\int_\Omega \sigma A_{\hat{V}}(t)A'_V(0)\circ T_t\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^a\,dx+\int_{\partial\omega} p\boldsymbol{u}\cdot\boldsymbol{u}^a w'_V(0)\circ T_t w_{\hat{V}}(t)\,ds\\
&+\int_\Omega \sigma A_{\hat{V}}(t)\nabla\boldsymbol{U}\cdot\nabla\boldsymbol{W}\,dx+\int_{\partial\omega} p\boldsymbol{U}\cdot\boldsymbol{W}\,w_{\hat{V}}(t)\,ds-\int_{\partial\Omega} g\boldsymbol{w}_N\,ds+\int_{\partial\Omega}(f-\boldsymbol{u}_D)\sigma_1\partial_\nu\boldsymbol{w}_D\,ds\\
&+2\int_\Omega \alpha_1 A_{\hat{V}}(t)\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)\cdot\nabla\boldsymbol{w}^a_N+\alpha_2(\boldsymbol{u}_N-\boldsymbol{u}_D)\boldsymbol{w}^a_N\xi_{\hat{V}}(t)\,dx\\
&-2\int_\Omega \alpha_1 A_{\hat{V}}(t)\nabla(\boldsymbol{u}_N-\boldsymbol{u}_D)\cdot\nabla\boldsymbol{w}^a_D+\alpha_2(\boldsymbol{u}_N-\boldsymbol{u}_D)\boldsymbol{w}^a_D\xi_{\hat{V}}(t)\,dx.
\end{aligned}$$

Under hypotheses of Theorem 8, we have

$$\begin{aligned}
\partial_t G_2(t,U,W)\big|_{t=0} = &\int_\Omega \alpha_1 \mathbb{P}'(0)\nabla(u_N-u_D)\cdot\nabla(u_N-u_D)+\alpha_2\mathbb{Q}'(0)|u_N-u_D|^2\,dx\\
&+\int_\Omega \sigma\mathbb{P}'(0)\nabla u\cdot\nabla u^a\,dx+\int_{\partial\omega} pu\cdot u^a\mathbb{Q}'_\tau(0)\,ds\\
&+\int_\Omega \sigma A'_{\hat{V}}(0)\nabla U\cdot\nabla W\,dx+\int_{\partial\omega} pU\cdot Ww'_{\hat{V}}(0)\,ds\\
&+2\int_\Omega \alpha_1 A'_{\hat{V}}(0)\nabla(u_N-u_D)\cdot\nabla(w^a_N-w^a_D)+\alpha_2(u_N-u_D)(w^a_N-w^a_D)\xi'_{\hat{V}}(0)\,dx.
\end{aligned}$$

Now, we need to check hypotheses of Theorem 8. The Lagrangian G_2 is affine in $\boldsymbol{W}=(\boldsymbol{w}_N,\boldsymbol{w}_D,\boldsymbol{w}^a_N,\boldsymbol{w}^a_D)$ but not necessarily convex in $\boldsymbol{U}=(\boldsymbol{u}_N,\boldsymbol{u}_D,\boldsymbol{u}^a_N,\boldsymbol{u}^a_D)$. However, it is semi-convex in $\boldsymbol{U}=(\boldsymbol{u}_N,\boldsymbol{u}_D,\boldsymbol{u}^a_N,\boldsymbol{u}^a_D)$, and we can apply Theorem 8 of Correa and Seeger for the Lagrangian G_2. The reader is referred to [11] for more details about the differentiability of semi-convex cost functionals. For the verifications of hypotheses, the technique is the same as in the proof of Theorem 5. Therefore, we shall not repeat it here. □

Theorem 7 (Second-order mixed derivative)
The functional $D_\omega\mathcal{J}(\omega,V)$ is Gateaux differentiable, and its Gateaux derivative at $p\in L^\infty(\partial\omega)$ in the direction $\hat{p}$ is given by

$$\partial_p D_\omega\mathcal{J}(\omega,V)\hat{p}=\int_{\partial\omega}\hat{p}(U\cdot W+u\cdot u^a w'_V(0))\,ds. \tag{43}$$

Proof
Let $p_t=p+t\hat{p}$, where $\hat{p}\in L^\infty(\partial\omega)$ and $t\in\mathbb{R}$ is sufficiently small parameter. As in the previous sections, and under the hypothesis of Theorem 8, we have

$$D_pD_\omega\mathcal{J}(\omega,V)\hat{p})=\partial_t\mathcal{L}_2(p_t,U,W))\Big|_{t=0},$$

where

$$\begin{aligned}\mathcal{L}_2(p_t, \boldsymbol{U}, \boldsymbol{W}) = &\int_\Omega \alpha_1 A_V'(0)\nabla(\boldsymbol{u}_N - \boldsymbol{u}_D)\cdot\nabla(\boldsymbol{u}_N - \boldsymbol{u}_D) + \alpha_2|\boldsymbol{u}_N - \boldsymbol{u}_D|^2 \xi_V'(0)\, dx\\ &+ \int_\Omega \sigma A_V'(0)\nabla \boldsymbol{u}\cdot\nabla \boldsymbol{u}^a\, dx + \int_{\partial\omega} p_t \boldsymbol{u}\cdot \boldsymbol{u}^a w_V'(0)\, ds.\\ &+ \int_\Omega \sigma \nabla \boldsymbol{U}\cdot\nabla \boldsymbol{W}\, dx + \int_{\partial\omega} p_t \boldsymbol{U}\cdot \boldsymbol{W}\, ds - \int_{\partial\Omega} g \boldsymbol{w}_N\, ds + \int_{\partial\Omega}(f - \boldsymbol{u}_D)\sigma_1 \partial_\nu \boldsymbol{w}_D\, ds\\ &+ 2\int_\Omega \alpha_1 \nabla(\boldsymbol{u}_N - \boldsymbol{u}_D)\cdot\nabla \boldsymbol{w}_N^a + \alpha_2(\boldsymbol{u}_N - \boldsymbol{u}_D)\boldsymbol{w}_N^a\, dx\\ &- 2\int_\Omega \alpha_1 \nabla(\boldsymbol{u}_N - \boldsymbol{u}_D)\cdot\nabla \boldsymbol{w}_D^a + \alpha_2(\boldsymbol{u}_N - \boldsymbol{u}_D)\boldsymbol{w}_D^a\, dx.\end{aligned}$$

and

$$\partial_t \mathcal{L}_2(p_t, U, W)\Big|_{t=0} = \int_{\partial\omega} \hat{p}(U\cdot W + u\cdot u^a w_V'(0))\, ds.$$

From the aforementioned equation yields (43). □

Appendix A

Proposition 2

Let Ω be a bounded connexe domain in $\mathbb{R}^2$ with C^1 boundary $\partial\Omega$. The mapping

$$u \to \|u\| := \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\omega)}\right)^{1/2}$$

is a norm on $H^1(\Omega)$ equivalent to the natural norm.

Proof

It is clear that if $\|u\| = 0$, then $\nabla u = 0$ in $L^2(\Omega)$ and $u = 0$ in $L^2(\partial\omega)$. We can deduce that $\nabla u = 0$ in the sense of distribution and thus u is constant in Ω. Because $u = 0$ in $L^2(\partial\omega)$, we conclude that $u = 0$ in Ω. In the next step, we prove the equivalence with the H^1-norm.

From the trace theorem, we have $\|u\|_{L^2(\partial\omega)} \le C\|u\|_{H^1(\Omega)}$. Therefore, $\|u\| \le C_1\|u\|_{H^1(\Omega)}$, where C_1 is a positive constant. Still to prove that $\|u\| \ge C_2\|u\|_{H^1(\Omega)}$. By contradiction, suppose that there exists a sequence (u_n) in $H^1(\Omega)$ with $\|u_n\|_{L^2(\Omega)} = 1$ such that $\|u_n\|_{L^2(\Omega)} \ge n\|u\|$. Then, $\nabla u_n \to 0$ in $L^2(\Omega)$ and $u_n \to 0$ in $L^2(\partial\omega)$ as $n \to 0$. The sequence u_n is bounded in $H^1(\Omega)$. From the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, we can extract a subsequence still denoted (u_n) such that $u_n \to \bar{u}$ in $L^2(\Omega)$. Thus, we have $u_n \to \bar{u}$ and $\nabla u_n \to \nabla\bar{u}$ in the sense of distribution. By the uniqueness of the limit, we conclude that $\bar{u} = 0$ in Ω, which contradict the fact that $\lim_{n\to\infty}\|u_n\|_{L^2(\Omega)} = \|\bar{u}\|_{L^2(\Omega)} = 1$. Therefore, there exists a positive constant c such that $\|u\|^2_{L^2(\Omega)} \le c\left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\omega)}\right)$, which ends the proof. □

Definition 3

- Given a velocity field $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$ and denotes T_t^V the corresponding deformations. A functional $J : \Omega \to \mathbb{R}$ is said to have an Eulerian semiderivative at Ω in the direction V if the following limits exists and is finite:

$$\lim_{t\to 0}\frac{J(T_t^V(\Omega)) - J(\Omega)}{t}.$$

  Whenever it exists, it is denoted by $dJ(\Omega; V)$. The shape functional J is said to be shape differentiable if $dJ(\Omega; V)$ exists for all $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$ and the map $V \to dJ(\Omega; V)$ is linear and continuous.
- Let $V, \hat{V} \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$ and denotes $T_t^V, T_t^{\hat{V}}$ the corresponding deformations of V and $\hat{V}$, respectively. Assume that for all $t \in [0, \epsilon)$, $dJ(T_t^{\hat{V}}(\Omega); V)$ exists at $T_t^{\hat{V}}(\Omega)$ in the direction V. The functional J is said to have a second-order Eulerian semi-derivative at Ω in the directions $(V, \hat{V})$ if the following limit exists:

$$\lim_{t\to 0}\frac{dJ(T_t^{\hat{V}}(\Omega); V) - dJ(\Omega; V)}{t}.$$

  Whenever it exists, it is denoted by $d^2J(\Omega; V; \hat{V})$. The shape functional J is said to be twice shape differentiable if $d^2J(\Omega; V; \hat{V})$ exists for all $V, \hat{V} \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$ and the map $(V, \hat{V}) \to d^2J(\Omega; V; \hat{V})$ is bilinear and continuous.

Definition 4
We say that the functional $(t,x) : [0,\epsilon]\times X \to F(t,x)$ is semiconvex in x if there exists a family of continuous convex functionals $C(t,x)$ on the Banach space X such that $F(t,x) + C(t,x)$ is convex in x. This means that $F(t,.) + C(t,.)$ and $C(t,.)$ both have directional derivatives, and hence, $F(t,.)$ also has a directional derivative: the following limit exists:

$$dF(t,x;\hat{x}) = \lim_{\theta\to 0^+} \frac{F(t,x+\theta\hat{x}) - F(t,x)}{\theta}.$$

A.1. An abstract differentiability result

We first introduce some notations. Consider the functional

$$G : [0,\varepsilon] \times X \times Y \to \mathbb{R} \tag{A.1}$$

for some $\varepsilon > 0$ and the Banach spaces X and Y. For each $t \in [0,\varepsilon]$, define

$$g(t) = \inf_{x\in X}\sup_{y\in Y} G(t,x,y), \qquad h(t) = \sup_{y\in Y}\inf_{x\in X} G(t,x,y), \tag{A.2}$$

and the associated sets

$$X(t) = \left\{x^t \in X : \sup_{y\in Y} G(t,x^t,y) = g(t)\right\}, \tag{A.3}$$

$$Y(t) = \left\{y^t \in Y : \inf_{x\in X} G(t,x,y^t) = h(t)\right\}. \tag{A.4}$$

Note that inequality $h(t) \leq g(t)$ holds. If $h(t) = g(t)$, the set of saddle points is given by

$$S(t) := X(t) \times Y(t). \tag{A.5}$$

We state now a simplified version of a result from [9], which gives realistic conditions that allows to differentiate $g(t)$ at $t = 0$. The main difficulty is to obtain conditions that allow to exchange the derivative with respect to t and the inf-sup in (A.2).

Theorem 8 (Correa and Seeger[9])
Let X, Y, G, and ε be given as previously. Assume that the following assumptions hold:

(H1) $S(t) \neq \emptyset$ for $0 \leq t \leq \varepsilon$.
(H2) The partial derivative $\partial_t G(t,x,y)$ exists for all $(t,x,y) \in [0,\varepsilon] \times X \times Y$.
(H3) For any sequence $\{t_n\}_{n\in\mathbb{N}}$, with $t_n \to 0$, there exist a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and $x^0 \in X(0), x_{n_k} \in X(t_{n_k})$ such that for all $y \in Y(0)$,

$$\lim_{t\searrow 0, k\to\infty} \partial_t G(t,x_{n_k},y) = \partial_t G(0,x^0,y),$$

(H4) For any sequence $\{t_n\}_{n\in\mathbb{N}}$, with $t_n \to 0$, there exist a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and $y^0 \in Y(0), y_{n_k} \in Y(t_{n_k})$ such that for all $x \in X(0)$,

$$\lim_{t\searrow 0, k\to\infty} \partial_t G(t,x,y_{n_k}) = \partial_t G(0,x,y^0),$$

Then, there exists $(x^0,y^0) \in X(0) \times Y(0)$ such that

$$\frac{dg}{dt}(0) = \partial_t G(0,x^0,y^0).$$

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