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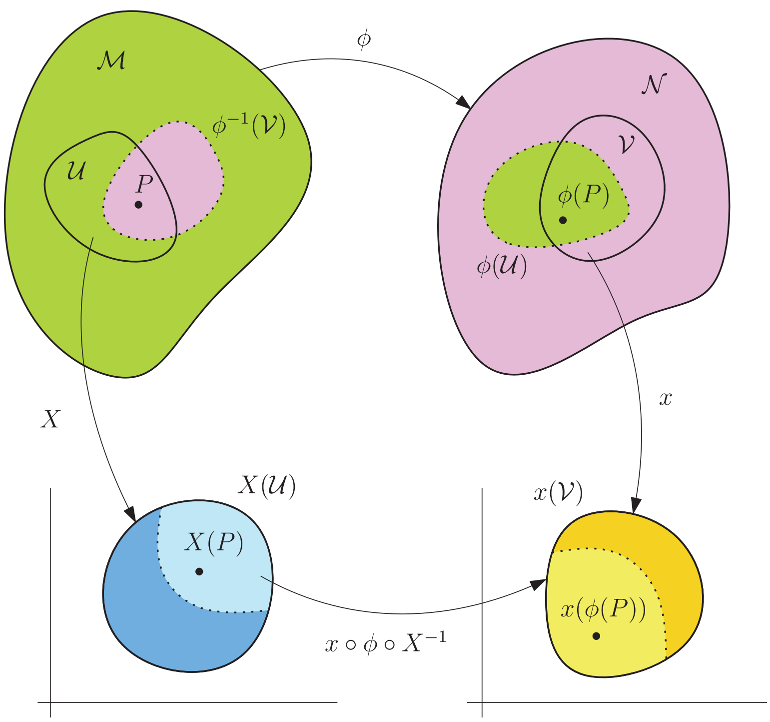
Differential Geometry Applied to Continuum Mechanics



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Preface by the Editor

The astonishing pace in the development of the finite element method (FEM) in engineering sciences and its broad application in the industry and in engineering practice have entered the areas of soil mechanics and geotechnical engineering long ago. Although the perception of a soil continuum is a controversial issue, it is generally accepted that numerical simulations based on FEM can considerably improve the understanding of the physical processes involved in situ and the interpretation of measuring data of experimental tests.

Systematically, the notion of a continuum is part of the mathematic branch of differential geometry. Therefore, it is advantageous to analyze and to discuss the topics of continuum mechanics, in particular soil mechanics, by applying the geometric terminology. Accordingly, the soil continuum shall be understood as a differentiable manifold that does not need to have a Euclidian structure, and its stress and density states are be described by coordinate-independent tensor fields. Engineers not acquainted with tensor calculus and the geometric method are at a disadvantage, because they do not have full command of the scientific fundamentals of their discipline and thus can hardly benefit from new developments. They will moreover run the risk of blindly trusting the results of numerical simulations instead of questioning them.

The presented work is a fundamental introduction into differential geometry and its application to continuum mechanics. It is addressed at scientific engineers, but also at engineers in practice and graduate students interested in the field. Another objective of the work is to revise the theoretical fundamentals of the Arbitrary Lagrangian-Eulerian (ALE) formulation of continuum mechanics by placing emphasis on the geometric background. The ALE formulation can be seen as a unification of the Lagrangian and Eulerian formulations in order to combine the advantages of both viewpoints. It is currently a topic of research at the Soil Mechanics and Geotechnical Engineering Division, in which the penetration of piles into sand is being simulated numerically by using a finite element model. A publication in the institute series is being prepared.

The author currently is a research associate at the Technical University of Berlin. He was able to investigate the topic during his research activity, which is gratefully acknowledged here. Parts of the work on the ALE formulation have been carried out with the financial support of the DFG (German Research Foundation), which is also gratefully acknowledged.

Stavros A. Savidis
Berlin, February 2009

Preface by the Author

In the year 2004/2005, after my studies of civil engineering and becoming a research associate at the Soil Mechanics and Geotechnical Engineering Division, I attended a lecture by Prof. Dr.-Ing. Gerd Brunk at the Technical University of Berlin. The lecture was about tensor analysis and continuum physics, but it made me wonder since geometry was predominant, and "index gymnastics" and mechanics were solely treated in applications. Inspired by this lecture and the famous book by Marsden and Hughes, I began my research work on an Arbitrary Lagrangian-Eulerian (ALE) approach to the finite element simulation of penetration processes in sand. Because the continuum mechanical background is massive and essentially based on the geometry of point spaces, I have written down this paper with the initial objective to compile important formulae and basic results. However, the reader will notice that the final version goes beyond to some extend.

I would like to thank Prof. Dr.-Ing. Stavros A. Savidis who gave me the opportunity to investigate as a geotechnical engineer such a theoretic topic, and my colleague Dr.-Ing. Frank Rackwitz for discussion and helpful suggestions. Last but not least I would like to thank the developers of the LaTeX program for enabling everyone to do beautiful typeset of complex mathematics.

Disclaimer

This paper is not intended to serve as a monograph for specialists about differential geometry and continuum mechanics. Many interesting topics have been omitted and many of the presented key facts and basic results are stated without proofs; they may be found in the standard textbooks, e.g. [1, 2, 3, 4, 5, 6, 7, 8]. Comprehensive treatises on linear geometry and linear algebra are, for example, [9] and [10].

Daniel Aubram
Berlin, November 2008

Abstract

Differential geometry provides the suitable background to present and discuss continuum mechanics with an integrative and mathematically precise terminology. By starting with a review of linear geometry in affine point spaces, the paper introduces modern differential geometry on manifolds including the following topics: topology, tensor algebra, bundles and tensor fields, exterior algebra, differential and integral calculi. The tools worked out are applied subsequently to basic topics of continuum mechanics. In particular, kinematics of a material body and balance of mass are formulated by applying the geometric terminology, the principles of objectivity and material frame indifference of constitutive equations are examined, and a clear distinction of the Lagrangian formulation from the Eulerian formulation is drawn. Moreover, the paper outlines a generalized Arbitrary Lagrangian-Eulerian (ALE) formulation of continuum mechanics on differentiable manifolds. As an essential part, the grid manifold introduced therein facilitates a consistent description of the relations between the material body, the ambient space and the arbitrary reference domain of the ALE formulation. Not least, the objective of the paper is to provide a compilation of important formulae and basic results —some of them with a full proof— frequently used by the community. If practical, point arguments and changes in points within equations will be clearly indicated, and component and direct (or absolute) tensor notation will be applied as needed, avoiding a single-track approach to the subject.

Keywords: differential geometry; continuum mechanics; large deformations; Arbitrary Lagrangian-Eulerian; manifold; tensor analysis

Zusammenfassung

Die Differentialgeometrie bietet den geeigneten Hintergrund, um die Kontinuumsmechanik mit einer einheitlichen und mathematisch präzisen Terminologie darzulegen und zu diskutieren. Ausgehend von einem Rückblick auf die lineare Geometrie in affinen Punkträumen führt die Arbeit in die moderne Differentialgeometrie auf Mannigfaltigkeiten unter Berücksichtigung der folgenden Themen ein: Topologie, Tensoralgebra, Bündel und Tensorfelder, Äußere Algebra sowie Differential- und Integralkalküle. Die erarbeiteten Werkzeuge werden anschließend auf grundlegende Themen der Kontinuumsmechanik angewendet. Insbesondere wird die Kinematik eines materiellen Körpers und die Massenbilanz vom geometrischen Standpunkt heraus formuliert, das Prinzip der Objektivität von Tensoren und von Materialgleichungen wird untersucht, und es wird der Unterschied zwischen der Lagrange'schen und der Euler'schen Formulierung auf klärende Weise dargestellt. Desweiteren skizziert die Arbeit eine verallgemeinerte Arbitrary Lagrangian-Eulerian (ALE) Formulierung der Kontinuumsmechanik auf differenzierbaren Mannigfaltigkeiten. Als wesentlicher Bestandteil ermöglicht dabei die eingeführte Gittermannigfaltigkeit eine konsistente Beschreibung der Beziehungen zwischen dem materiellen Körper, dem umgebenden Raum und dem beliebigen Referenzgebiet der ALE Formulierung. Nicht zuletzt besteht die Zielsetzung der Arbeit darin, wichtige Formeln und grundlegende Ergebnisse auf den behandelten Gebieten teilweise auch mit vollständigem Beweis zusammenzustellen. Sofern es zweckmäßig erscheint, werden Punktargumente und der Wechsel der Bezugspunkte in den Gleichungen hervorgehoben. Außerdem wird je nach Bedarf sowohl die Komponentenschreibweise, als auch die direkte oder absolute Schreibweise von Tensoren angewendet und dadurch ein eingleisiges Vorgehen vermieden.

Schlagworte: Differentialgeometrie; Kontinuumsmechanik; große Verformungen; Arbitrary Lagrangian-Eulerian; Mannigfaltigkeit; Tensoranalysis

God is a geometer.

—Plato

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Notation

General Conventions

Calligraphic $(\mathcal{M}, \mathcal{N}, \ldots)$	sets, vector spaces and manifolds
Italic lightface $(a, b, A, B, \ldots)$	points, chart mappings, scalars, scalar functions, coordinate indices, and labels
Greek lightface $(\alpha, \beta, \varPhi, \varPsi \ldots)$	point mappings, and coordinate indices
Boldface $(\boldsymbol{v}, \boldsymbol{\alpha}, \boldsymbol{A}, \boldsymbol{\mathfrak{T}}, \ldots)$	linear maps, vectors, tensors, linear forms, differential forms, and constitutive functions

Italics is also used for theorems, propositions, corollaries, and terminology just defined. The Halmos symbol $\blacksquare$ indicates the end of a proof, and $\diamondsuit$ indicates the end of a definition.

Symbols and Notations

$\mathbb{R}, \mathbb{N}$	real numbers, natural numbers
$\emptyset$	empty set
$\times$	cartesian product; direct product of sets
$\oplus$	direct sum
$\otimes$	tensor product; dyadic product
$\wedge$	exterior product; wedge product
$\langle \cdot, \cdot \rangle$	inner product
$f \circ g = f(g)$	composition of maps f, g
$\mathcal{U} \subset \mathcal{M}$	$\mathcal{U}$ is a subset (or subspace) of $\mathcal{M}$
$\dim(\mathcal{M}_{n_{\mathrm{dim}}}) \equiv n_{\mathrm{dim}}$	dimension of $\mathcal{M}_{n_{\mathrm{dim}}}$
$\boldsymbol{A}(\boldsymbol{v}) \equiv \boldsymbol{A}\boldsymbol{v}$	linear map applied to a vector $\boldsymbol{v}$
$\boldsymbol{v} = \overrightarrow{PQ}$	vector pointing from P to Q
$d(P, Q)$	distance of the points P, Q
$\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{n_{\mathrm{dim}}}\} \equiv \{\boldsymbol{g}_i\}$	vector basis
$(O, \boldsymbol{g}_i)$	frame of reference with origin O

$\xi^\star, \xi_\star$	pullback, pushforward by a map ξ
$\mathrm{SE}(\mathcal{M})$	special or proper Euclidian group of a point space $\mathcal{M}$
$\mathrm{SO}(\mathcal{V})$	special or proper orthogonal group of a vector space $\mathcal{V}$
$(\mathcal{S}, \mathscr{T})$	topological space; set $\mathcal{S}$ with topology $\mathscr{T}$
$\mathrm{int}(\mathcal{S}), \mathrm{cl}(\mathcal{S}), \partial\mathcal{S}$	interior, closure, and boundary of $\mathcal{S}$
$(\mathcal{U}, x)$, $x(\mathcal{U}) \subset \mathbb{R}^{n_{\dim}}$	coordinate system on $\mathcal{U}$; chart of $\mathcal{U}$
$\{x^i\}_P = x(P)$	coordinates of the point $P \in \mathcal{U}$ in a chart $(\mathcal{U}, x)$
$\mathfrak{A}(\mathcal{M}) = \{(\mathcal{U}_i, x_i)\}_{i\in\mathcal{I}}$	atlas of the manifold $\mathcal{M} = \bigcup_{i\in\mathcal{I}} \mathcal{U}_i$
$c : \mathbb{R} \to \mathcal{M}$	curve on $\mathcal{M}$
$f : \mathcal{M} \to \mathbb{R}$	real function or scalar field on $\mathcal{M}$
$T_P\mathcal{M} = \{P\} \times \mathcal{V}_n$	n-dimensional tangent space at $P \in \mathcal{M}$
$T_P^*\mathcal{M}$	cotangent space; dual space
$T\mathcal{M} = \bigcup_{P\in\mathcal{M}} T_P\mathcal{M}$	tangent bundle
$T\phi : T\mathcal{A} \to T\mathcal{B}$	tangent map; differential of a map $\phi : \mathcal{A} \to \mathcal{B}$
$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$, $\mathbf{d}x^i$	basis vectors, dual basis vectors in a tangent space
$\boldsymbol{T} \cdot \boldsymbol{S}$, $\boldsymbol{T} : \boldsymbol{S}$	single contraction, double contraction of tensors
$\boldsymbol{a}^*(\boldsymbol{v}) \equiv \boldsymbol{a}^* \cdot \boldsymbol{v}$	linear form applied to a vector $\boldsymbol{v}$
$\boldsymbol{w}[f] = \mathbf{d}f \cdot \boldsymbol{w}$	directional derivative of a function f along $\boldsymbol{w}$
$\boldsymbol{T}^*, \boldsymbol{T}^{\mathrm{T}}$	adjoint, transpose of a tensor
$^\flat$, $^\sharp$	index lowering operator, index raising operator
$\pi : \mathcal{E} \to \mathcal{M}$, $(\mathcal{E}, \pi, \mathcal{M})$	fibre bundle over $\mathcal{M}$
$\boldsymbol{\sigma} : \mathcal{M} \to T\mathcal{M}$	tangent bundle section; vector field on the manifold $\mathcal{M}$
$\Gamma(T\mathcal{M})$	set of all vector fields (bundle sections) on $\mathcal{M}$
$\Gamma(T^p_q(\mathcal{M})) = \mathfrak{T}^p_q(\mathcal{M})$	set of all $\binom{p}{q}$-tensor fields on $\mathcal{M}$
$\Gamma(\bigwedge^k T^*\mathcal{M}) = \Omega^k(\mathcal{M})$	set of all k-forms on $\mathcal{M}$
$\mathsf{Alt} : T^0_k(\mathcal{M}) \to \bigwedge^k T^*\mathcal{M}$	alternation mapping
$\boldsymbol{i_u} : \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$	interior product
$\boldsymbol{\nabla_v}$	covariant derivative along $\boldsymbol{v}$
$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial t} + \pounds_{\boldsymbol{u}}\boldsymbol{T}$	Lie derivative of a tensor field $\boldsymbol{T}$ along $\boldsymbol{u}$
$\mathbf{d}\boldsymbol{\omega}$	exterior derivative of a k-form $\boldsymbol{\omega}$
$\epsilon_{\nu_1,\ldots,\nu_n}$	Levi-Civita symbol; permutation symbol
$\mathbf{d}\boldsymbol{v} \in \Omega^n(\mathcal{M})$	Riemannian volume form on $\mathcal{M}$; $\boldsymbol{g}$-volume
J_ϕ	Jacobian of a map ϕ
$I, J, K, \ldots$	coordinate indices (Lagrangian formulation)
$i, j, k, \ldots$	coordinate indices (Eulerian formulation)
$\alpha, \beta, \gamma, \ldots$	coordinate indices (ALE formulation)

Chapter 1

Introduction

It is unquestioned for a long time that natural sciences can benefit from differential geometry, since it makes a comprehensive theory of gravitation possible in general relativity. The so-called geometric mechanics [1, 2, 11, 12, 13, 14, 15, 16], however, has been implemented more recently and thus it is applied to a lesser extend, especially in engineering sciences. Important examples are found in the theories of rods and shells, in the Lagrange-Hamilton formalism, and relativistic elasticity. In the theory of materials, the Lie derivative serves to obtain objective rates of stress measures, but it is often used detached from the overall geometric context.

Differential geometry [1, 3, 4, 5, 6, 7] provides the suitable background to present and discuss the subjects of mechanics, and especially continuum mechanics, with an integrative and mathematically precise terminology. It clarifies basic concepts and opens up deep examination even of complex issues. For example, geometry reveals that the determinant of the Jacobian matrix is not an invariant scalar, and that the question, whether the first Piola-Kirchhoff stress tensor is symmetric or not, does not make sense. However, differential geometry demands a large investment of effort and persistency from its students.

Continuum mechanics [2, 8] is typically prepared for the Euclidian point space. This is motivated by applications in engineering sciences, which prefer as simple spaces as possible. In some cases, e.g. for shells, it is reasonable to use local curvilinear coordinates instead of the global cartesian coordinates. The former are not affine, that is, they do not transform linearly as cartesian coordinates, and additional terminology —metric, covariant and contravariant basis vectors etc.— enters. If the shell is understood as embedded in the Euclidian space, the terms mentioned can refer to the ambient structure, namely if the position vectors of points are identified with those points. As a result, the covariant vector basis, for example, arises from the partial derivatives of the position vector with respect to the curvilinear coordinates of the related point on the shell. The existence of a global position vector, however, already requires a global linear structure of space. Without embedding the shell in the Euclidian space, a terminology built on position vectors, and hence the description of the shell's geometry and kinematics will collapse.

The introduction of manifolds [17] facilitates a consistently local description of the geometry of structures without the need to embed them in a linear point space. The definition of position vectors is not possible, because no global origin exists. The creation of a vector basis at every point of the manifold rather succeeds by a one-to-one mapping between a neighborhood of the point and a local coordinate system, which necessarily is curvilinear. Instead of defining additional terms (metric, covariant and contravariant basis vectors etc.) as for curvilinear coordinates in Euclidian spaces, elementary operations are established of which the structure of space results in a natural way.

Tensor analysis on manifolds delivers the standard tools to develop observer-invariant or covariant theories. Covariance is an essential requirement for physical equations [2, 6, 7, 13, 18, 19]. In its passive interpretation, covariance asks for the form-invariance of the equation structure; clearly, under coordinate transformations, terms must not be dropped or added. However, a difficulty of tensor analysis on manifolds rests on the fact that vectors and tensors are defined locally, and equations have to be formulated such that point mappings are included. This means that equations of the kind $\boldsymbol{v}(P) = \boldsymbol{v}'(Q)$, as known from affine point spaces, are not permitted, because the vectors $\boldsymbol{v}, \boldsymbol{v}'$ are given at different points P and Q and, therefore, they cannot be compared. In the literature one often finds the analogous statement: "In a manifold there are no vectors". Thus, although formulae become more complicated, point arguments and changes in points will be clearly indicated in the paper.

Some remarks on tensor notation should be made here. In the literature, vectors and tensors are often reduced to their components by adopting the component notation, because the split-up of tensors into single components will be necessary anyway if concrete calculations and implementations into computer programmes are intended. A tensor, originally proposed as a coordinate-invariant object, is then understood as a quantity with indices that transforms by a certain rule. The invariance of the tensor, however, does not result from the transformation of its components alone, since one needs the background knowledge that its basis transforms inversely. On the other hand, by applying the so-called direct or absolute notation, tensor equations have the same form in any coordinate system, as desired, but the computation of such an abstract tensor equation can be cumbersome in non-cartesian coordinate systems. The third choice is the local notation of tensors, where the associated tensor basis accompanies the components. The relation between the absolute tensor and its components is always available, and the referential coordinate frame is set by the basis. Local notation is also useful to proof several expressions which are given in absolute resp. direct notation. However, the notational preference should depend on the problem under consideration, and the paper thus avoids a single-track approach to the subject.

Continuum mechanics on manifolds from the classic or traditional Lagrangian and Eulerian viewpoints is presented, for example, in [2]. For the solution of initial boundary value problems involving large material deformations or complex fluid structure interaction, the Arbitrary Lagrangian-Eulerian (ALE) formulation [20, 21, 22] has been approved as an efficient framework. ALE frameworks implemented in the finite element method [23] have become an important numerical simulation tool, e.g. for metal forming, free surface flow and impact processes. The Lagrangian and Eulerian formulations are generalized within the ALE formulation, as a time-dependent reference domain uncoupled from the material body and its configurations in the ambient space is used to describe the physical quantities under consideration. Due to the generalization, however, the governing equations become more complicated, especially when compared to the Lagrangian formulation in solid mechanics.

In the references cited, the ALE formulation for Euclidian spaces is discussed in detail, but to the knowledge of the author, no reference is available in which the framework is extended to manifolds or Riemannian spaces, or which respond closer to the geometry of the ALE formulation. Therefore, another objective of the paper is to extend the geometric methods of continuum mechanics by the Arbitrary Lagrangian-Eulerian formulation on manifolds. It will be restricted to the geometry and kinematics of a body and to the conservation of mass. Other balance equations as well as dynamical and material theoretical aspects will not be considered.

The structure of the paper is as follows. Chapter 2 reviews some basic results of linear geometry. In chapter 3, an introduction into the terminology of modern differential geometry will be given: topology, tensor algebra, bundles and tensor fields, exterior algebra, differential and integral calculi. After applying the terminology to continuum mechanics in the classic Lagrangian and Eulerian formulations in chapter 4, the ALE formulation will be geometrically introduced and implemented into the overall continuum mechanical context. The paper closes with some concluding remarks in chapter 5.

Chapter 2

Review of Linear Geometry

The following chapter should motivate the construction of vector spaces and mappings on general manifolds. It is devoted to the important topics of vector algebra and affine point spaces used in continuum mechanics and the theory of materials, including the concept of objectivity.

2.1 Vectors and Linear Maps

Definition 2.1.1. A set $\mathcal{V}$ together with an addition

$$\begin{array}{rcl} \mathcal{V} \times \mathcal{V} & \to & \mathcal{V} \\ (\boldsymbol{a}, \boldsymbol{b}) & \mapsto & \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \end{array}$$

and a scalar multiplication

$$\begin{array}{rcl} \mathbb{R} \times \mathcal{V} & \to & \mathcal{V} \\ (\lambda, \boldsymbol{a}) & \mapsto & \lambda \boldsymbol{a} = \boldsymbol{a} \lambda \ , \forall \lambda \in \mathbb{R} \, , \end{array}$$

with existent unique neutral and inverse elements for both addition and scalar multiplication, is called *vector space* or *linear space over the body* $\mathbb{R}$. The elements of $\mathcal{V}$ are called *vectors*. ◊

Definition 2.1.2. Let $\mathcal{V}$ be a vector space. If there exists a *positive bilinear mapping* respectively an *inner product*

$$\begin{array}{rrcl} \langle \cdot , \cdot \rangle : & \mathcal{V} \times \mathcal{V} & \to & \mathbb{R} \\ & (\boldsymbol{a}, \boldsymbol{b}) & \mapsto & \langle \boldsymbol{a}, \boldsymbol{b} \rangle \ = \ \langle \boldsymbol{b}, \boldsymbol{a} \rangle \ , \end{array}$$

then $\mathcal{V}$ is said to *have a metric* or to *be metrizable*, and it is called *Euclidian vector space.* ◊

Definition 2.1.3. If the equivalence

$$a^1 \boldsymbol{g}_1 + a^2 \boldsymbol{g}_2 + \ldots + a^n \boldsymbol{g}_n = \boldsymbol{0} \qquad \Leftrightarrow \qquad a^i = 0, \, \forall \, i = 1, 2, \ldots, n$$

holds, where $\mathbf{0}$ is the zero vector, then the vectors $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n$ are called *linearly independent*, otherwise *linearly dependent*. If n vectors $\boldsymbol{g}_i$ are linearly independent, but $(n+1)$ vectors $\boldsymbol{g}_i$ are always linearly dependent, then $n \equiv n_{\text{dim}}$ is the *dimension* of $\mathcal{V}_{n_{\text{dim}}}$ and $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{n_{\text{dim}}}\} \equiv \{\boldsymbol{g}_i\} \in \mathcal{V}_{n_{\text{dim}}}$ is called *basis of* $\mathcal{V}_{n_{\text{dim}}}$. The latter can then be expressed by

$$\mathcal{V}_{n_{\text{dim}}} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \ldots \oplus \mathcal{U}_{n_{\text{dim}}} ,$$

where $\oplus$ denotes the *direct sum* (see definition below) of the subspaces $\mathcal{U}_i \subset \mathcal{V}_{n_{\text{dim}}}$, and $\boldsymbol{g}_i \in \mathcal{U}_i$ for all $i \in \{1, \ldots, n_{\text{dim}}\}$. $\diamondsuit$

Definition 2.1.4. Let $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{V}$ be subspaces of a vector space $\mathcal{V}$. The *sum* $\mathcal{U}_1 + \mathcal{U}_2$ is the subspace of $\mathcal{V}$ spanned by the elements of $\mathcal{U}_1 \cup \mathcal{U}_2$. The *direct sum* $\mathcal{U}_1 \oplus \mathcal{U}_2$ is the sum $\mathcal{U}_1 + \mathcal{U}_2$ together with the property $\mathcal{U}_1 \cap \mathcal{U}_2 = \{\mathbf{0}\}$. $\diamondsuit$

Proposition 2.1.5. *In a vector space $\mathcal{V}_n$, every vector $\boldsymbol{v}$ can be represented by*

$$\boldsymbol{v} = v^1 \boldsymbol{g}_1 + v^2 \boldsymbol{g}_2 + \ldots + v^n \boldsymbol{g}_n = \sum_{i=1}^{n} v^i \boldsymbol{g}_i \equiv v^i \boldsymbol{g}_i \, .$$

where $v^1, v^2, \ldots, v^n \in \mathbb{R}$ and $\{\boldsymbol{g}_i\} \in \mathcal{V}_n$ is a basis.

PROOF. Since $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ is linearly independent and $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n, \boldsymbol{v}\}$ is linearly dependent, $\boldsymbol{v}$ can be expressed as a linear combination of the $\boldsymbol{g}_i$ according to 2.1.3. The identity on the right hand side of the proposition is due to the Einstein summation convention. $\blacksquare$

Definition 2.1.6. One refers to 2.1.5 as the *local notation* and to $\boldsymbol{v} = v^i \boldsymbol{g}_i$ as the *local representative of* $\boldsymbol{v}$. The v^i are the *coordinates* or *components of* $\boldsymbol{v}$ *with respect to the basis* $\{\boldsymbol{g}_i\}$. $\diamondsuit$

By 2.1.3 and 2.1.5, the basis vectors can be arbitrarily chosen. They are, in general, neither orthogonal nor normalized. Moreover, no origin has been used to define the v^i or $\boldsymbol{g}_i$, that is, points do not exist in vector spaces.

Example 2.1.7. Every row resp. n-tuple $\{v^1, v^2, \ldots, v^n\} \in \mathbb{R}^n$, as known from linear algebra, is a vector, and if $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is the canonical basis in $\mathbb{R}^n$, then $\{v^1, v^2, \ldots, v^n\} = v^i \boldsymbol{e}_i$ is the local representative.

Definition 2.1.8. Let $\mathcal{V}, \mathcal{W}$ be vector spaces. A map $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$, with

1. $\boldsymbol{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{A}\boldsymbol{v}_1 + \boldsymbol{A}\boldsymbol{v}_2 \in \mathcal{W}, \ \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{V}$ and
2. $\boldsymbol{A}(\lambda \boldsymbol{v}) = \lambda(\boldsymbol{A}\boldsymbol{v}) \in \mathcal{W}, \ \forall\, \lambda \in \mathbb{R}$,

where $\boldsymbol{A}\boldsymbol{v} \equiv \boldsymbol{A}(\boldsymbol{v})$, is called *linear transformation*, *linear map* or *homomorphism*. If $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$ is bijective such that $\boldsymbol{A}^{-1}$ is its *inverse* and both $\mathcal{V}$ and $\mathcal{W}$ have the same dimension, then $\boldsymbol{A}$ is called *isomorphism*. Linear transformations $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$ are called *endomorphisms*, and isomorphisms with $\mathcal{V} = \mathcal{W}$ are called *automorphisms*. $\diamondsuit$

Definition 2.1.9. The linear transformation $\boldsymbol{I} : \mathcal{V} \to \mathcal{V}$ defined through $\boldsymbol{I}\boldsymbol{v} = \boldsymbol{v}$, $\forall \boldsymbol{v} \in \mathcal{V}$, is called the *identity map on* $\mathcal{V}$. If $\boldsymbol{A}$ is an automorphism on $\mathcal{V}$, then $\boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1}$. $\diamondsuit$

Definition 2.1.10. Let $\boldsymbol{A} : \mathcal{V}_{n_{\mathrm{dim}}} \to \mathcal{W}_{m_{\mathrm{dim}}}$ be a linear transformation and let $\{\boldsymbol{g}_i\}$ and $\{\boldsymbol{h}_\alpha\}$ be bases of $\mathcal{V}_{n_{\mathrm{dim}}}$ and $\mathcal{W}_{m_{\mathrm{dim}}}$, respectively. The images $\boldsymbol{A}\boldsymbol{g}_i$ can be expanded in the basis $\boldsymbol{h}_\alpha$ by

$$\boldsymbol{A}\boldsymbol{g}_i = A^\alpha{}_i \boldsymbol{h}_\alpha$$

through $(m_{\mathrm{dim}} \times n_{\mathrm{dim}})$ numbers $A^\alpha{}_i$. The matrix arrangement $(A^\alpha{}_i)$ of these numbers is referred to as the *matrix of the linear transformation* $\boldsymbol{A}$ *with respect to* $\{\boldsymbol{h}_\alpha\}$, in which α (the left index) denotes the row index and i is the column index, respectively —this is because a matrix of a linear transformation $\boldsymbol{A} : \mathcal{V}_{n_{\mathrm{dim}}} \to \mathcal{W}_{m_{\mathrm{dim}}}$ is understood as a map $(A^\alpha{}_i) : \mathbb{R}^{n_{\mathrm{dim}}} \to \mathbb{R}^{m_{\mathrm{dim}}}$, where $\mathbb{R}^{n_{\mathrm{dim}}}$ denotes the columns having n_{dim} elements. $\diamondsuit$

Corollary 2.1.11. *By 2.1.5, the basis vectors* $\boldsymbol{h}_\alpha$ *in 2.1.10 have an expression in the basis* $\{\boldsymbol{g}_i\}$*, say* $\boldsymbol{h}_\alpha = C^j{}_\alpha \boldsymbol{g}_j$*, yielding* $\boldsymbol{A}\boldsymbol{g}_i = D^j{}_i\, \boldsymbol{g}_j$*, where* $D^j{}_i = C^j{}_\alpha A^\alpha{}_i$ *are the components of* $\boldsymbol{A}$ *with respect to* $\{\boldsymbol{g}_i\}$*.*

It can be shown that every linear transformation $\boldsymbol{A} : \mathcal{V}_{n_{\mathrm{dim}}} \to \mathcal{W}_{m_{\mathrm{dim}}}$ has a component matrix $(A^\alpha{}_i)$, and that this matrix is unique.

Definition 2.1.12. Consider the special situation $\boldsymbol{A} = \boldsymbol{I}$ in 2.1.10, where the base vectors $\boldsymbol{g}_i \in \mathcal{V}$ are expanded in another basis $\{\boldsymbol{g}_{i'}\}$ of the same vector space:

$$\boldsymbol{g}_i = B^{i'}{}_i\, \boldsymbol{g}_{i'}\,.$$

In the previous equation, the sum is over primed indices only. The matrix of $\boldsymbol{I}$, written $(B^{i'}{}_i)$, is called the *inverse matrix of the change of basis*, that is, $(B^{i'}{}_i)^{-1} = (B^i{}_{i'})$ arranges the components of the direct change of basis, $\boldsymbol{g}_{i'} = B^i{}_{i'} \boldsymbol{g}_i$. Thus, $\boldsymbol{g}_j = B^{i'}{}_j\, B^i{}_{i'}\, \boldsymbol{g}_i = B^i{}_{i'}\, B^{i'}{}_j\, \boldsymbol{g}_i = \delta^i{}_j\, \boldsymbol{g}_i$ and $\boldsymbol{I}\boldsymbol{g}_j = \delta^i{}_j\, \boldsymbol{g}_i$, where $\delta^i{}_j$ is called the *Kronecker delta.* $\diamondsuit$

Corollary 2.1.13. *As a (fixed) vector* $\boldsymbol{v} = v^i \boldsymbol{g}_i = v^{i'} \boldsymbol{g}_{i'} = \boldsymbol{I}\boldsymbol{v}$ *is coordinate-invariant, under a change of basis* $\boldsymbol{g}_{i'} = B^i{}_{i'} \boldsymbol{g}_i$*, the components of* $\boldsymbol{v}$ *transform with the inverse matrix of the change of basis, that is,* $v^{i'} = v^i\, B^{i'}{}_i$*.*

Definition 2.1.14. Given a matrix $(B^{i'}{}_i)$ and a vector $\boldsymbol{v} = v^i \boldsymbol{g}_i$, then a transformation rule

$$\boldsymbol{v} \mapsto v^i B^{i'}{}_i\, \boldsymbol{g}_{i'}$$

has two interpretations. The *active transformation* changes the vector by a linear map $\boldsymbol{B}$ defined through $\boldsymbol{B}\boldsymbol{g}_i = B^{i'}{}_i\, \boldsymbol{g}_{i'}$, so $(B^{i'}{}_i)$ is the matrix of that transformation. From the second viewpoint, the prescribed rule defines a *passive transformation* $\boldsymbol{I}\boldsymbol{g}_i = B^{i'}{}_i \boldsymbol{g}_{i'}$ that keeps the vector fixed. In this case, $(B^{i'}{}_i)$ is the inverse matrix of the change of basis and $v^{i'} = v^i B^{i'}{}_i$ are the components of the same vector with respect to the changed basis. $\diamondsuit$

As studied later in the text, the passive transformation rule identifies the vector with a rank-one tensor. However, both interpretations are of fundamental importance in physics, especially in continuum mechanics.

Definition 2.1.15. A map $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$ of Euclidian vector spaces $\mathcal{V}$ and $\mathcal{W}$ is called *isometry*, if

$$\langle \boldsymbol{Aa}, \boldsymbol{Ab} \rangle_{\mathcal{W}} = \langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathcal{V}}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}$. An isometry $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$ is called an *orthogonal map*, having the properties $\det \boldsymbol{A} = \pm 1$ and $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$, where $\boldsymbol{A}^{\mathrm{T}}$ is the *transpose* or *adjoint* of $\boldsymbol{A}$. A more general definition of an orthogonal map includes the isometries $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$ and then stipulates $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}_{\mathcal{V}}$ and $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}_{\mathcal{W}}$. $\diamondsuit$

Proposition 2.1.16. *(Without proof.) A linear map $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$ is orthogonal, if and only if its matrix with respect to an orthonormal basis is orthogonal.*

2.2 Affine Point Spaces

Definition 2.2.1. Let $\mathcal{S} = \{A, B, C, \ldots\}$ be a set of points and $\mathcal{V}$ a vector space. The pair $(\mathcal{S}, \mathcal{V})$ —or simply $\mathcal{S}$ if the meaning is clear from the context— together with the map

$$\begin{array}{ccc} \mathcal{S} \times \mathcal{S} & \to & \mathcal{V} \\ (A, B) & \mapsto & \overrightarrow{AB} \end{array}$$

is called an *affine point space*, if the following axioms are satisfied:

1. For every $A \in \mathcal{S}$ and every $\boldsymbol{v} \in \mathcal{V}$ there is a unique $A + \boldsymbol{v} = B \in \mathcal{S}$, so that $\boldsymbol{v} = \overrightarrow{AB}$.
2. If $\overrightarrow{AB} = \overrightarrow{CD}$, then $\overrightarrow{AC} = \overrightarrow{BD}$ also holds (parallelogram axiom).

If $\mathcal{V}$ moreover has a metric according to 2.1.2, then $(\mathcal{S}, \mathcal{V})$ is called a *Euclidian point space*. $\diamondsuit$

Definition 2.2.2. Let $\mathcal{S}$ be an affine point space, and A, B, C points, then $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, $\overrightarrow{BA} = -\overrightarrow{AB}$ and $\overrightarrow{AA} = \boldsymbol{0}$ defines the *vector sum*, the *inverse element* and the *neutral element*, respectively. $\diamondsuit$

Example 2.2.3. $(\mathbb{R}^n, \mathbb{R}^n)$ is the simplest affine point space. A point $A \in \mathbb{R}^n$ is identified with its coordinates, and, because of $A + \boldsymbol{v} = B \in \mathbb{R}^n$ by definition, $\boldsymbol{v}$ is also an element of $\mathbb{R}^n$.

The affine structure imposes a global parallelism on the standard Euclidian point space. It is notable that parallelism —as well as terminology like distance and angle— does not make sense in abstract vector spaces, but only in affine spaces.

Corollary 2.2.4. *If $\boldsymbol{v}(A) = \overrightarrow{AB}$ is a vector with base point A, and $\overrightarrow{AB} = \overrightarrow{CD}$, then*

$$\boldsymbol{v}(A) = \boldsymbol{v}(C)\,.$$

Conclude that the parallelogram axiom of definition 2.2.1 renders affine point spaces flat —this is the fundamental difference between flat spaces and manifolds.

Definition 2.2.5. Let $\mathcal{S}$ be an Euclidian point space and $\boldsymbol{v} = \overrightarrow{PQ} \in \mathcal{V}$, then

$$d(P,Q) = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} = |\boldsymbol{v}| \ ,$$

is the *distance* of the points $P, Q \in \mathcal{S}$. ♢

Definition 2.2.6. Let $\mathcal{S}$ be an affine point space, $O, P \in \mathcal{S}$, and $\{\boldsymbol{g}_i\} \in \mathcal{V}_{n_{\text{dim}}}$ a basis —a basis can be obtained from a set of $(n_{\text{dim}} + 1)$ points. The $(n_{\text{dim}} + 1)$-tuple $(O, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{n_{\text{dim}}}) \equiv (O, \boldsymbol{g}_i)$ is called *frame of reference*, shortly: *frame*, in which the $\boldsymbol{g}_i$ are descriptively attached to the *origin* O (figure 2.1). $\mathcal{S}$ then becomes the *frame space*, and $\overrightarrow{OP} = \boldsymbol{x} \in \mathcal{V}_{n_{\text{dim}}}$ is called *position vector of P with respect to O*. The local representative $\boldsymbol{x} = x^i \boldsymbol{g}_i$ includes the *affine coordinates* $\{x^1, x^2, \ldots, x^{n_{\text{dim}}}\} \equiv \{x^i\}$ *of P with respect to the frame* $(O, \boldsymbol{g}_i)$. ♢

Note that in Euclidian point spaces, "frame" is used as a synonym for "Euclidian observer".

Definition 2.2.7. Given a frame $(O, \boldsymbol{g}_i)$ in the space $\mathcal{S}$, one may construct *coordinate lines* by varying one affine coordinate and keeping the other coordinate values fixed, i.e. by changing one coordinate (component) x^i of the position vector. The family of lines obtained is called *affine coordinate system on $\mathcal{S}$* and is denoted by the pair $(\mathcal{S}, x)$. If every coordinate line is orthogonal to each of the other coordinate lines, the affine coordinate system is called *cartesian.* ♢

Although affine coordinates are uniquely determined by the chosen frame, one should be careful with the difference between the frame and the coordinate system. Without a frame it does not make sense to talk about coordinate systems!

One should also be careful with the terminology "position vector", because it depends on the frame, i.e. $\overrightarrow{OP} \neq \overrightarrow{O'P}$. It becomes clear that $\overrightarrow{OP}$ is an honest vector when viewed from a different frame. The following two important results may help:

Theorem 2.2.8 (Transformation of Affine Coordinates). *The coordinate functions of every two affine coordinate systems transform linearly with* $x^{i'} = \chi^{i'}(x^1, \ldots, x^{n_{\text{dim}}}) = B^{i'}_{\ i} x^i + c^{i'}$, *where* $B^{i'}_{\ i}$ *are the components of the inverse matrix of the change of basis and* $c^{i'}$ *are constants.*

Proof. Clearly, the coordinates of the same $P \in \mathcal{S}$ under a *change of framing* $(O, \boldsymbol{g}_i) \mapsto (O', \boldsymbol{g}_{i'})$ have to be calculated. To this end, let $\overrightarrow{OP} = \boldsymbol{x} = x^i \boldsymbol{g}_i$ and $\overrightarrow{O'P} = \hat{\boldsymbol{x}} = x^{i'} \boldsymbol{g}_{i'}$ denote the position vectors of P in the two different frames, that is, $P = O + \boldsymbol{x} = O' + \hat{\boldsymbol{x}}$ (figure 2.1).

Expanded in the primed basis, $\boldsymbol{x}$ becomes $\overrightarrow{OP}_{\{\boldsymbol{g}_{i'}\}} = \boldsymbol{I}\boldsymbol{x} = B^{i'}_{\ i} x^i \boldsymbol{g}_{i'}$, where 2.1.12 has been applied. Setting $\boldsymbol{c} = c^{i'} \boldsymbol{g}_{i'} = \overrightarrow{O'O}$, then in the primed frame the coordinates of $\overrightarrow{O'P} = \overrightarrow{O'O} + \overrightarrow{OP}$ resp. $\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{c}$ are

$$x^{i'} = B^{i'}_{\ i} x^i + c^{i'} = \chi^{i'}(x^1, \ldots, x^{n_{\text{dim}}})$$

as desired. ■

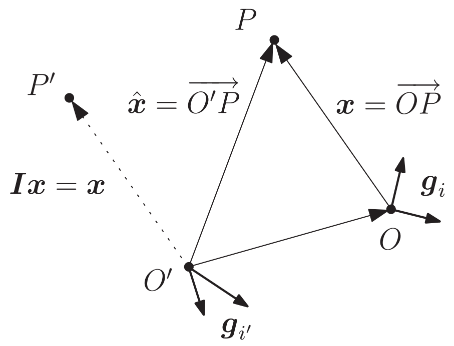

Figure 2.1: Position vectors of a point P with respect to different frames $(O, \boldsymbol{g}_i)$ and $(O', \boldsymbol{g}_{i'})$.

Proposition 2.2.9. *A vector of an affine point space ($\neq$ position vector!) is coordinate-invariant (or frame-indifferent) and it fits 2.1.13, whereas a position vector does not.*

Proof. Additional to the previously described situation, let $Q \in \mathcal{S}$ be another point and let $\overrightarrow{OQ} = y^i \boldsymbol{g}_i$ and $\overrightarrow{O'Q} = y^{i'} \boldsymbol{g}_{i'}$ denote its position vectors in the two frames. Then by 2.2.1 and 2.2.2, $\boldsymbol{v} = Q - P = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{O'Q} - \overrightarrow{O'P}$. Apply 2.2.8 to get

$$\begin{aligned} \boldsymbol{v} = (y^{i'} - x^{i'})\boldsymbol{g}_{i'} &= \Big((B^{i'}_{\ i} y^i + c^{i'}) - (B^{i'}_{\ i} x^i + c^{i'}) \Big) \boldsymbol{g}_{i'} \\ &= (y^i - x^i) B^{i'}_{\ i} \boldsymbol{g}_{i'} = (y^i - x^i)\boldsymbol{g}_i \end{aligned}$$

in local notation. This is already 2.1.13. On the other hand, $B^{i'}_{\ i} x^i \boldsymbol{g}_{i'} = x^i B^{i'}_{\ i} B^{j}_{\ i'} \boldsymbol{g}_j = x^i \boldsymbol{g}_i = \overrightarrow{OP}$ is also a vector, but $B^{i'}_{\ i} x^i \neq x^{i'}$ if $O \neq O'$ by 2.2.8, i.e. the position of P with respect to $(O, \boldsymbol{g}_i)$ and $(O', \boldsymbol{g}_{i'})$ does not transform in terms of 2.1.13. ■

Corollary 2.2.10. *Let $\overrightarrow{OO'} = a^i \boldsymbol{g}_i$, $\overrightarrow{O'O} = c^{i'} \boldsymbol{g}_{i'}$ and $\boldsymbol{g}_{i'} = B^{i}_{\ i'} \boldsymbol{g}_i$ be the translation vectors and the change of basis of two frames $(O, \boldsymbol{g}_i)$ and $(O', \boldsymbol{g}_{i'})$, respectively. Since $\overrightarrow{OO'} = -\overrightarrow{O'O}$, one has*

$$a^i = -c^{i'} B^{i}_{\ i'} \qquad \text{and} \qquad c^{i'} = -a^i B^{i'}_{\ i} \, .$$

Corollary 2.2.11. *The Jacobian matrix of the transformation of affine coordinates 2.2.8 is given by the inverse matrix of the change of basis, i.e.*

$$\frac{\partial \chi^{i'}}{\partial x^j} = B^{i'}_{\ i} \, .$$

Definition 2.2.12. Let $(\mathcal{S}, \mathcal{V})$ and $(\mathcal{T}, \mathcal{W})$ be two affine point spaces, $O, P, Q \in \mathcal{S}$, and $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$ a linear map. A map $\xi : \mathcal{S} \to \mathcal{T}$ is referred to as an *affine transformation*, if for every $P = O + \overrightarrow{OP}$,

$$\xi(P) = \xi(O) + \boldsymbol{A}(\overrightarrow{OP}) \quad \in \mathcal{T} \, .$$

$\overrightarrow{\xi(O)\xi(P)} = \boldsymbol{A}(\overrightarrow{OP}) = \xi_\star(\overrightarrow{OP})$ is called the *pushforward of* $\overrightarrow{OP}$ *by* ξ. For Euclidian $\mathcal{V}$ and $\mathcal{W}$, the affine transformation ξ is called *affine isometry*, if

$$d_{\mathcal{T}}(\xi(P), \xi(Q)) = d_{\mathcal{S}}(P, Q)\,,$$

where $d_{\mathcal{S}}, d_{\mathcal{T}}$ are the distance functions on $\mathcal{S}$ and $\mathcal{T}$, respectively. ◇

Definition 2.2.13. Let $\xi : \mathcal{S} \to \mathcal{S}$, $P \mapsto \xi(P) = \xi(O) + \boldsymbol{Q}(\overrightarrow{OP})$, where $\boldsymbol{Q} : \mathcal{V} \to \mathcal{V}$ is an automorphism, be bijective, then a *group of affine transformations* can be established, which should not be presented here. If $\xi : \mathcal{S} \to \mathcal{S}$ is an affine isometry, then $\boldsymbol{Q}$ is an isometry and orthogonal. Moreover, if ξ preserves orientation, i.e. the determinant is $\det \boldsymbol{Q} = +1$, then it is called a *(superposed) rigid motion* and $\boldsymbol{Q}$ —now *proper* orthogonal— is called *rotation*. Rigid motions belong to the so-called *special Euclidian group*, denoted as $\mathrm{SE}(\mathcal{S})$, and rotations belong to the *special orthogonal group* $\mathrm{SO}(\mathcal{V})$. ◇

Proposition 2.2.14. *Under affine isometries $\xi : \mathcal{S} \to \mathcal{S}$, every vector $\boldsymbol{v}$ transforms according to*

$$\boldsymbol{v}' = \boldsymbol{Q}\boldsymbol{v}\,,$$

where $\boldsymbol{Q}$ is the orthogonal map of ξ.

Proof. Let $O, P, Q \in \mathcal{S}$ be points, then $\boldsymbol{v} = Q - P$ is a vector. Applying 2.2.12, $\boldsymbol{v}' = \xi(Q) - \xi(P) = \Big(\xi(O) + \boldsymbol{Q}(\overrightarrow{OQ})\Big) - \Big(\xi(O) + \boldsymbol{Q}(\overrightarrow{OP})\Big) = \boldsymbol{Q}(\overrightarrow{OQ}) - \boldsymbol{Q}(\overrightarrow{OP})$. Since $\boldsymbol{Q}$ is linear, $\boldsymbol{Q}(\overrightarrow{OQ} - \overrightarrow{OP}) = \boldsymbol{Q}(Q - P)$ and the assertion follows. ■

Proposition 2.2.15. *Let $\xi : \mathcal{S} \to \mathcal{S}$, $P \mapsto \xi(P) = \xi(O) + \boldsymbol{Q}(\overrightarrow{OP})$ be a superposed rigid motion, $(O, \boldsymbol{g}_i)$ a frame and $\{\boldsymbol{g}_{i'}\}$ a basis defined through $\boldsymbol{Q}\boldsymbol{g}_i = Q^{i'}{}_i \boldsymbol{g}_{i'}$. The components of the position vector $\overrightarrow{O\xi(P)} = \boldsymbol{x}'$ with respect to the $\boldsymbol{g}_{i'}$ are then given by*

$$x^{i'} = Q^{i'}{}_i\, x^i + c^{i'}\,,$$

where $c^{i'}$ are the components of the translation vector $\overrightarrow{O\xi(O)} = \boldsymbol{c}$, and x^i are the components of $\overrightarrow{OP} = \boldsymbol{x}$ with respect to the basis $\{\boldsymbol{g}_i\}$.

Proof. Since $\boldsymbol{Q}\boldsymbol{x} = \xi_\star \boldsymbol{x}$ by 2.2.12, $\overrightarrow{O\xi(P)} = \overrightarrow{O\xi(O)} + \overrightarrow{\xi(O)\xi(P)}$ becomes $\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}$. The result then is obtained by substituting the local representatives $\boldsymbol{x} = x^i \boldsymbol{g}_i$, $\boldsymbol{x}' = x^{i'} \boldsymbol{g}_{i'}$ and $\boldsymbol{c} = c^{i'} \boldsymbol{g}_{i'}$, and using the definition of the primed basis $\{\boldsymbol{g}_{i'}\}$. ■

Instead of understanding the affine isometry $\xi : \mathcal{S} \to \mathcal{S}$ as a superposed rigid motion relative to a fixed Euclidian frame $(O, \boldsymbol{g}_{i'})$, it may also be interpreted as a change of Euclidian framing resp. a change of Euclidian observers $(O, \boldsymbol{g}_i) \mapsto \xi(O, \boldsymbol{g}_i) = (O', \boldsymbol{Q}\boldsymbol{g}_i)$, i.e. a relative motion between Euclidian frames (also called a *quasi-motion* [see 7]). This can be justified as follows: if $(O, \boldsymbol{g}_i)$ is a "rigid" frame that moves with a rigid motion ξ, then $(O', \boldsymbol{Q}\boldsymbol{g}_i)$ is the dragged-along frame at $O' = \xi(O)$ (figure 3.5).

However, an affine isometry only requires both frames to measure the same distance between points, so $\boldsymbol{Q}$ from the change of a Euclidian framing needs to be orthogonal,

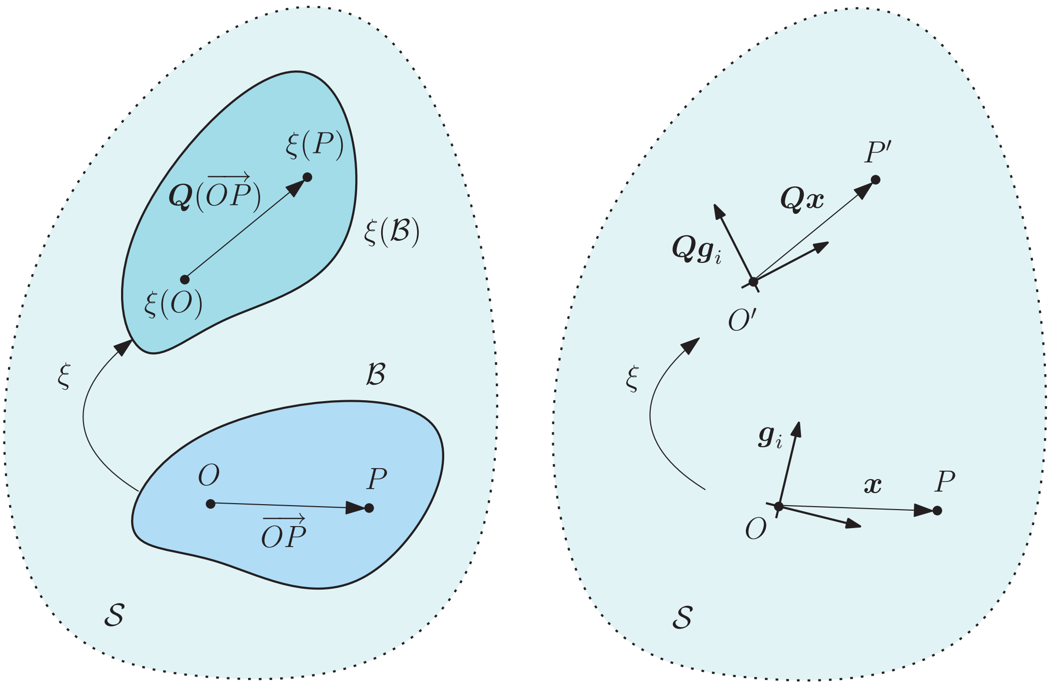

Figure 2.2: Active objectivity of the vector $\boldsymbol{x} = \overrightarrow{OP}$ in the Euclidian point space $\mathcal{S}$ with respect to a superposed rigid motion of a subset $\mathcal{B} \subset \mathcal{S}$ (*left*), and with respect to a change of framing (*right*).

but not *proper* orthogonal as for superposed rigid motions. By keeping this difference in mind, proposition 2.2.15 is equivalent to the following:

Proposition 2.2.16. *Let $\boldsymbol{Q}$ be orthogonal resp. an isometry, $(O, \boldsymbol{g}_i) \mapsto \xi(O, \boldsymbol{g}_i) = (O', \boldsymbol{Qg}_i)$ a change of framing in the Euclidian point space $\mathcal{S}$ and $\{\boldsymbol{g}_{i'}\}$ a basis defined through $\boldsymbol{Qg}_i = Q^{i'}{}_i \boldsymbol{g}_{i'}$. The components of the position vector $\overrightarrow{OP'} = \boldsymbol{x}'$ with respect to the $\boldsymbol{g}_{i'}$ are then given by*

$$x^{i'} = Q^{i'}{}_i x^i + c^{i'} ,$$

where $c^{i'}$ are the components of the translation vector $\overrightarrow{OO'} = \boldsymbol{c}$, and x^i are the components of $\overrightarrow{OP} = \boldsymbol{x}$ with respect to the basis $\{\boldsymbol{g}_i\}$.

PROOF. Isometric $\boldsymbol{Q}$'s imply that P' has the same coordinates with respect to the frame $(O', \boldsymbol{Qg}_i)$ as P has with respect to $(O, \boldsymbol{g}_i)$ —except for reordering of indices if $\det \boldsymbol{Q} = -1$. Therefore,

$$\overrightarrow{O'P'} = \boldsymbol{Q}(\overrightarrow{OP}) = \boldsymbol{Qx} = x^i \boldsymbol{Qg}_i = Q^{i'}{}_i x^i \boldsymbol{g}_{i'} ,$$

and $\overrightarrow{OP'} = \overrightarrow{OO'} + \overrightarrow{O'P'}$ resp. $\boldsymbol{x}' = \boldsymbol{Qx} + \boldsymbol{c}$ as before. Substitution of $\boldsymbol{x}' = x^{i'} \boldsymbol{g}_{i'}$ and $\boldsymbol{c} = c^{i'} \boldsymbol{g}_{i'}$ then gives the result. ■

The connection between the formula 2.2.15 resp. 2.2.16 and the formula 2.2.8 is similar to that of a map $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$ and the identity map $\boldsymbol{I}$ in a vector space. Hence, it seems likely to "rewrite" definition 2.1.14 for Euclidian point spaces.

Definition 2.2.17. Let $(\mathcal{S}, \mathcal{V})$ be an Euclidian point space. A vector $\boldsymbol{v} \in \mathcal{V}$ which transforms according to the rule

$$\boldsymbol{v}' = \xi_\star \boldsymbol{v} \,,$$

is called *objective under the transformation* $\xi_\star$, shortly: *objective*. The transformation rule, which depends on the map ξ that belongs to the transformation, has two interpretations.

By applying the *active interpretation of objectivity*, the map $\xi : \mathcal{S} \to \mathcal{S}$ is understood as a superposed rigid motion, transforming every point $P = (O + \overrightarrow{OP})$ into $\xi(P) = \xi(O) + \boldsymbol{Q}(\overrightarrow{OP})$ and keeping the Euclidian frame fixed. In this case, $\xi_\star \boldsymbol{v} = \boldsymbol{Q}\boldsymbol{v}$ as in 2.2.14, where $\boldsymbol{Q} \in \mathrm{SO}(\mathcal{V})$, and $x^{i'} = Q^{i'}_{\ i} x^i + c^{i'}$ are the affine coordinates of $\xi(P)$ in terms of the coordinates of P and $\xi(O)$ with respect to O. $(Q^{i'}_{\ i})$ is the matrix of $\boldsymbol{Q}$ with respect to a suitable basis, and $c^{i'}$ are the components of $\overrightarrow{O'O}$. Another way to define active objectivity is to interpret ξ as a change of Euclidian framing $(O, \boldsymbol{g}_i) \to (O', \boldsymbol{Q}\boldsymbol{g}_i)$ (cf. proposition 2.2.16 and figure 3.5).

On the other hand, if the *passive interpretation of objectivity* is applied, $\xi : (O, \boldsymbol{g}_i) \to (O', \boldsymbol{g}_{i'})$ is a change of framing on the fixed Euclidian point space and which keeps the vector fixed, that is, $\xi_\star \boldsymbol{v} = \boldsymbol{I}\boldsymbol{v}$ as in 2.2.9. In this case, $x^{i'} = B^{i'}_{\ i} x^i + c^{i'}$ are the affine coordinates of the same (fixed) point in different frames, where $(B^{i'}_{\ i})$ is the inverse matrix of the change of basis. Therefore, passive objectivity phrases the transformation properties of $\boldsymbol{v}$ under a change of affine coordinates. ◇

In general, it is the active interpretation of objectivity that is applied to continuum mechanics and the theory of materials in Euclidian point spaces; this will be investigated in section 4.5. The passive interpretation of objectivity, however, relates to the standard transformation rule for vectors and tensors under a change of basis (or coordinates).

It is again emphasized that the active and passive interpretations of objectivity in Euclidian point spaces correspond to the active and passive transformations of vectors defined in 2.1.14, respectively. The reader should keep track of the active-passive concepts in the remainder of the paper.

Chapter 3

Differential Geometry

3.1 Topology and Manifolds

Manifolds are more general affine point spaces. Engineers nowadays accept the term "manifold" as the mathematical expression for a continuum that has a differentiable structure, but the rich theory behind is often ignored. However, differentiability as it stands means that there is some continuity. Topology is the basic field for the study of continuity and for the creation of a general terminology of space —a manifold is also a topological space. Descriptively, a topology carries the relations or interconnections between elements of a point set.

To non-mathematicians, managing books on topology and tensor analysis on manifolds may become a difficult challenge. The following chapter should assemble the topics needed by applying a notation consistent with the rest of the text.

Definition 3.1.1. Let $\mathcal{S}$ be a set. A *topology* $\mathscr{T}$ is a collection of subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$, called *open sets*, which satisfy the following axioms:

1. $\emptyset \in \mathscr{T}$ and $\mathcal{S} \in \mathscr{T}$. ($\emptyset$ is the empty set.)
2. If $\mathcal{A}, \mathcal{B} \in \mathscr{T}$, then $\mathcal{A} \cap \mathcal{B} \in \mathscr{T}$ also holds.
3. Let $\mathcal{I} \subset \mathbb{N}$ be a set of indices. If $\mathcal{A}_i \in \mathscr{T}$, $\forall\, i \in \mathcal{I}$, then $\bigcup_{i \in \mathcal{I}} \mathcal{A}_i \in \mathscr{T}$ also holds.

The pair $(\mathcal{S}, \mathscr{T})$ is referred to as the *topological space*, but write $\mathcal{S}$ instead of $(\mathcal{S}, \mathscr{T})$ if the meaning is clear. A subset $\mathcal{A} \subset \mathcal{S}$ is called *closed*, if $\mathcal{S} \backslash \mathcal{A} \in \mathscr{T}$ is open, i.e. closed sets are the complements of open sets. From the definition, the *trivial topology* $\mathscr{T} = \{\emptyset, \mathcal{S}\}$ of $\mathcal{S}$ is obvious. The elements of a topological space are called *points*; these can be geometric points, material particles, thermodynamic states etc. $\diamond$

Corollary 3.1.2. *Since $\mathcal{S} \backslash \mathcal{S} = \emptyset$ and $\mathcal{S} \backslash \emptyset = \mathcal{S}$, the sets $\mathcal{S}$ and $\emptyset$ are both open and closed.*

Definition 3.1.3. The *interior* or *open cover* $\mathrm{int}(\mathcal{M})$ of a topological space $\mathcal{M} \subset \mathcal{S}$ is the union of all open sets $\mathcal{U}$ which completely lie in $\mathcal{M}$, i.e. $\mathrm{int}(\mathcal{M}) = \bigcup_{\mathcal{U}} \{\mathcal{U} \,|\, \mathcal{U} \in \mathscr{T}$ and $\mathcal{U} \subset \mathcal{M}\}$. The *closure* $\mathrm{cl}(\mathcal{M})$ is the smallest closed set which completely includes

$\mathcal{M}$, i.e. $\mathrm{cl}(\mathcal{M}) = \mathcal{S} - \bigcup_{\mathcal{U}} \{\mathcal{U}\,|\,\mathcal{U} \in \mathscr{T}$ and $\mathcal{U} \subset (\mathcal{S}\backslash\mathcal{M})\}$. The difference $\mathrm{cl}(\mathcal{M}) - \mathrm{int}(\mathcal{M}) = \partial\mathcal{M}$ is called *boundary* of $\mathcal{M}$. ♢

Definition 3.1.4. A topological space $\mathcal{S}$ is called *discrete*, if it has a *discrete topology* $\mathscr{T} = \{\mathcal{A}\,|\,\mathcal{A} \subset \mathcal{S}\}$, that is, if all subsets are open. However, since $\mathcal{S}\backslash\mathcal{A}$ is also a subset, all subsets in discrete topological spaces are both open and closed. ♢

Definition 3.1.5. Let $\mathcal{S}$ be a topological space with topology $\mathscr{T}$. An *(open) neighborhood* of a point $P \in \mathcal{S}$ is an open set $\mathcal{U} \in \mathscr{T}$ such that $P \in \mathcal{U}$. A point P is called *isolated*, if $\{P\}$ is open. A *basis* for the topology of $\mathcal{S}$ is a sequence or collection $\mathscr{B}$ of open sets such that every open set of $\mathcal{S}$ is a union of elements of $\mathscr{B}$. ♢

Corollary 3.1.6. *In a discrete topology every point is isolated.*

Example 3.1.7. The set of integers as well as the set of nodes in a finite element mesh are discrete topological spaces.

Definition 3.1.8. Let $\mathcal{I} \subset \mathbb{N}$ be a set of indices, then a topological space $\mathcal{S}$ is called *first countable*, if for each $P \in \mathcal{S}$ there is a countable collection $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ of neighborhoods $\mathcal{U}(P) \subset \mathcal{S}$ such that for any $\mathcal{U}(P)$, there is a $k \in \mathbb{N}$ so $\mathcal{U}_k(P) \subset \mathcal{U}(P)$. $\mathcal{S}$ is called *second countable*, if it has a countable basis, i.e. the topology of $\mathcal{S}$ has a finite number of open sets. ♢

Intuitively, second countable means that there is at least one way of covering the space with a finite number of sets. Note that every second countable space is also first countable, but not conversely.

Definition 3.1.9. A topological space $\mathcal{S}$ is referred to as a *Hausdorff space*, if every two points $P, Q \in \mathcal{S}$, $P \neq Q$, can be separated by neighborhoods $\mathcal{U}(P) \subset \mathcal{S}$ and $\mathcal{V}(Q) \subset \mathcal{S}$ such that

$$\mathcal{U} \cap \mathcal{V} = \emptyset\,.$$

♢

Proposition 3.1.10. *In a Hausdorff space the singleton sets are closed.*

Proof. Let $(\mathcal{S}, \mathscr{T})$ be a Hausdorff space, $P \in \mathcal{S}$, then for each $Q \in \mathcal{S}\backslash\{P\}$ there is an (open) neighborhood $\mathcal{U}(Q) \subset \mathcal{S}$ of Q such that $P \notin \mathcal{U}$, so $\mathcal{U} \subset \mathcal{S}\backslash\{P\}$. Hence, by the first countability condition $\mathcal{S}\backslash\{P\}$ is open, and so $\{P\} = \mathcal{S}\backslash(\mathcal{S}\backslash\{P\})$ is closed. ■

Definition 3.1.11. A *homeomorphism* is a bijective map $h : \mathcal{S} \to \mathcal{T}$, where both h and h^{-1} are continuous. A homeomorphism preserves the topology of a topological space. If two topological spaces $\mathcal{S}$, $\mathcal{T}$ are *homeomorphic*, then $\dim(\mathcal{S}) = \dim(\mathcal{T})$. ♢

Example 3.1.12. Every r-adaption or regularization of a finite element mesh at fixed mesh topology represents a homeomorphism. By contrast, the map $\mathbb{R} \to \mathbb{Z}$ is no homeomorphism.

Definition 3.1.13. An *n-dimensional topological manifold* $\mathcal{M}$ is a second countable Hausdorff space together with a homeomorphism

$$\begin{aligned} x :\ \mathcal{U} \subset \mathcal{M} \quad &\to \quad \mathcal{X} \subset \mathbb{R}^n \\ P \quad &\mapsto \quad x(P) = \{x^1, x^2, \ldots, x^n\}_P \equiv \{x^i\}_P,\ \exists\, x^{-1}\,, \end{aligned}$$

for any neighborhood $\mathcal{U} \subset \mathcal{M}$ of P. The functions $\{x^i\}_P$ are called *coordinates of* P. The pair $(\mathcal{U}, x)$ including the *chart map* $x(\mathcal{U}) = \mathcal{X} \subset \mathbb{R}^n$ of the neighborhood $\mathcal{U}(P)$ is called *chart* or *local coordinate system.* $\diamondsuit$

Definition 3.1.14. If for a certain system of local coordinates $\{x^i\}_\mathcal{U} = x(\mathcal{U})$ on $\mathcal{U} \subset \mathcal{M}$ the inverse of the chart function x^{-1} does not exist, the coordinate system is called *singular*, otherwise it is called *regular*. The inverse x^{-1} leads to the point $P \in \mathcal{U}$ expressed by coordinates:

$$P = x^{-1}(x^1, x^2, \dots, x^n)\,. \qquad \diamondsuit$$

The quintessence of manifolds is found in the homeomorphism 3.1.13. Charts enable measurement on manifolds, because every point is assigned to a tuple of real numbers.

Example 3.1.15. The simplest example of a manifold is $\mathbb{R}^n$ itself, that has the global chart $(\mathbb{R}^n, \mathrm{Id})$. Any n-dimensional vector space is also a manifold with a global chart: choose a basis $\{\boldsymbol{g}_i\}$, then $\mathcal{V}_n \to \mathbb{R}^n$, $\boldsymbol{v} \mapsto \{v^1, v^2, \dots, v^n\}$ is the corresponding chart map, in which $\boldsymbol{v} = v^i \boldsymbol{g}_i$.

Example 3.1.16. The thermodynamic state space is a two-dimensional manifold, because in the chart the state is labelled by two independent quantities, namely pressure and temperature.

Definition 3.1.17. Let $\mathcal{U}, \mathcal{U}' \subset \mathcal{M}$, $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$ an overlap and $(\mathcal{U}, x)$, $(\mathcal{U}', x')$ regular charts. Then the continuous map

$$x' \circ x^{-1}\big|_{x(\mathcal{U}\cap\mathcal{U}')} : x\left(\mathcal{U} \cap \mathcal{U}'\right) \quad \to \quad x'\left(\mathcal{U} \cap \mathcal{U}'\right)$$

is called *chart transition* or *change of coordinates.* $\circ$ is the composition operator. A chart transition $x' \circ x^{-1}|_{x(\mathcal{U}\cap\mathcal{U}')}$ is also called the *relabelling of the subset* $\mathcal{U} \cap \mathcal{U}'$. $\diamondsuit$

Definition 3.1.18. A collection $\mathfrak{A}(\mathcal{M}) = \{(\mathcal{U}_i, x_i)\}_{i \in \mathcal{I}}$ of charts of the manifold $\mathcal{M} = \bigcup_{i\in\mathcal{I}} \mathcal{U}_i$, where $\mathcal{I} \subset \mathbb{N}$, is called *atlas of* $\mathcal{M}$. $\diamondsuit$

Definition 3.1.19. A manifold $\mathcal{M}$ is called a *differentiable manifold*, if for every two charts $(\mathcal{U}, x)$, $(\mathcal{U}', x') \in \mathfrak{A}(\mathcal{M})$ the chart transition $x' \circ x^{-1}|_{x(\mathcal{U}\cap\mathcal{U}')}$ is continuous differentiable. If the chart transition is k-fold continuous differentiable, where $k \in \mathbb{N}$, then the manifold is called a *C^k-manifold.* For $k = 0$, $\mathcal{M}$ is a topological manifold and for $k \to \infty$ it is called *smooth* or *C^∞-manifold.* $\diamondsuit$

Definition 3.1.20. Let $\mathcal{M}$ be a C^1-manifold, $\mathcal{U}, \mathcal{U}' \subset \mathcal{M}$, and $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$ an overlap. For a point $P \in \mathcal{U} \cap \mathcal{U}'$, having coordinates $\{x^i\}_P \in x(\mathcal{U})$ and $\{x^{i'}\}_P \in x'(\mathcal{U}')$, respectively, the transformation of the coordinate differentials involves the *Jacobian matrix* of the coordinate functions at P,

$$\mathrm{d}x^{i'} = \left(\frac{\partial(x^{i'} \circ x^{-1})}{\partial x^j}\right)(P)\, \mathrm{d}x^j\,,$$

in which $x^{i'} \circ x^{-1}$ is in general not linear in x^i. $\diamondsuit$

Definition 3.1.21. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a continuous map, where $\mathcal{M}$ and $\mathcal{N}$ may have different dimensions. Furthermore, let $\mathcal{U}(P) \subset \mathcal{M}$ be the neighborhood of a point $P \in \mathcal{M}$, $\mathcal{V}(\phi(P)) \subset \mathcal{N}$ a neighborhood of $\phi(P) \in \mathcal{N}$, and $\phi^{-1}(\mathcal{V}) \cap \mathcal{U} \neq \emptyset$. Let $(\mathcal{U}, X)$, $(\mathcal{V}, x)$ be appropriate charts, then $x \circ \phi \circ X^{-1}$ defines the *chart transition concerning ϕ with respect to X* (figure 3.1):

$$x \circ \phi \circ X^{-1}\big|_{X(\phi^{-1}(\mathcal{V})\cap\mathcal{U})} : \quad X\left(\phi^{-1}(\mathcal{V}) \cap \mathcal{U}\right) \quad \to \quad x\left(\phi^{-1}(\mathcal{V}) \cap \mathcal{U}\right) .$$

The chart transition is also called the *local representative* or the *localization of ϕ*. If x^i denote the coordinate functions of $(\mathcal{V}, x)$, one abbreviates $\phi^i = x^i \circ \phi \circ X^{-1}$, so $x^i \circ \phi = \phi^i(X)$. The map ϕ is called *differentiable at $P \in \phi^{-1}(\mathcal{V}) \cap \mathcal{U}$*, if $x \circ \phi \circ X^{-1}$ is differentiable at P. ◊

Note that chart transitions concerning maps $\phi : \mathcal{M} \to \mathcal{N}$ reduce to 3.1.17 if $\phi = \mathrm{Id}$ is the identity map on $\mathcal{M}$.

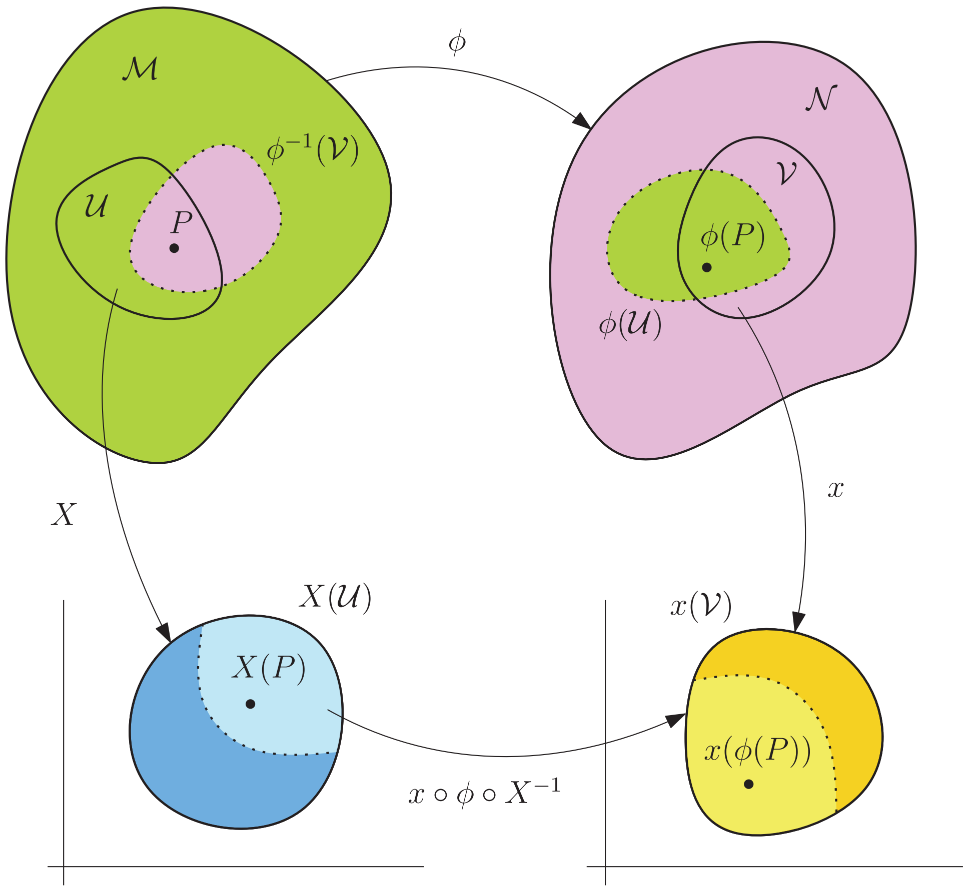

Figure 3.1: Localization of a continuous map ϕ.

Definition 3.1.22. Consider the aforementioned situation. A map $\phi : \mathcal{M} \to \mathcal{N}$ is called *regular*, if the Jacobian matrix $\frac{\partial \phi^i}{\partial X^I}$, with $\{X^I\}_P \in X(\mathcal{U})$, is invertible at $P \in \mathcal{M}$ (cf. theory of parametric surfaces). ◊

Definition 3.1.23. Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds. A bijective differentiable map

$$\phi : \mathcal{M} \rightarrow \mathcal{N}, \exists\, \phi^{-1}$$

is called *diffeomorphism*, if both ϕ and ϕ^{-1} are continuous differentiable. ♢

Corollary 3.1.24. *(i) If $\phi : \mathcal{M} \rightarrow \mathcal{N}$ is a diffeomorphism and $\mathcal{M}$ is a C^k-manifold, then $\mathcal{N} = \phi(\mathcal{M})$ is also C^k, that is, diffeomorphisms preserve or hand down the (differentiable) structure of $\mathcal{M}$. (ii) Every diffeomorphism is regular.*

Definition 3.1.25. Let $\phi : \mathcal{M} \rightarrow \mathcal{N}$ be a diffeomorphism, $\mathcal{U} \subset \mathcal{M}$, $\mathcal{V} \subset \mathcal{N}$ subsets, and let $(\mathcal{U}, X)$, $(\mathcal{V}, x)$ be appropriate charts, then one refers to ϕ as the *active diffeomorphism* and to $\tilde{\phi} = x \circ \phi \circ X^{-1}$ as the *passive diffeomorphism*, respectively (figure 3.2). ♢

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "\phi"] \arrow[d, "X"'] & \mathcal{V} \arrow[d, "x"] \\
\mathcal{X} \subset \mathbb{R}^{n_{\dim}} \arrow[r, "\tilde{\phi}"'] & \mathcal{Y} \subset \mathbb{R}^{n_{\dim}}
\end{tikzcd}$$

Figure 3.2: Active diffeomorphism ϕ and passive diffeomorphism $\tilde{\phi}$.

The difference between those maps is often ignored in applications of differential geometry; ϕ will be identified with $\tilde{\phi}$ also later in this text in order to simplify notation. However, the difference between ϕ and $\tilde{\phi}$ plays a fundamental role in general relativity, gauge theory, and even in the theory of materials, since it had been noticed by Einstein in his famous "Hole Argument" ([24], pp. 1066-1067, also [18, 19]). A consequence of this argument is that spacetime in Einstein's theory of gravitation does not exist independently of the matter within it. The Hole Argument is embedded in the *principle of general covariance*, which will be investigated in section 4.5.

Definition 3.1.26. Let $\mathcal{S}$, $\mathcal{N}$ be manifolds, $m = \dim(\mathcal{S})$, $n = \dim(\mathcal{N})$ and $m \leq n$. $\mathcal{S} \subset \mathcal{N}$ is called *submanifold of $\mathcal{N}$*, if for every $P \in \mathcal{S}$ there exists a neighborhood $\mathcal{U} \subset \mathcal{N}$ and a chart $(\mathcal{U}, x)$, such that for $\mathcal{S} \cap \mathcal{U} \neq \emptyset$,

$$x(\mathcal{S} \cap \mathcal{U}) = \big(\mathbb{R}^m \times \{x^{m+1} = \ldots = x^n = 0\}\big) \cap x(\mathcal{U}) \qquad \text{and} \qquad x(P) = 0\,.$$

The definition phrases that the charts of $\mathcal{S}$ and $\mathcal{N}$ have to be compatible, and that the chart of $\mathcal{U}$ in $x(\mathcal{S} \cap \mathcal{U})$ is centered at P. ♢

Example 3.1.27. A shell is an (m=2)-dimensional submanifold in a three-dimensional space.

3.2 The Tangent Space

In affine point spaces, a vector is identified with parallel translation (see section 2.2): a specific vector can be attached to every point of the affine point space or, equivalently, affine point spaces are flat and any point could be an origin of that vector. For a manifold, it is not possible to define a parallel translation, because at this stage reached, there is not even a connecting path between the points. Consequently, there are no vectors in the traditional sense.

In order to geometrize manifolds further, this section gives a blue print of the tangent space, that is, a vector space attached to each point of the manifold, by beginning with curvilinear coordinates in affine point spaces.

Definition 3.2.1. Let $\mathcal{S}$ be an m-dimensional affine point space and $y^1, \ldots, y^m$ the coordinate functions of an affine coordinate system $(\mathcal{S}, y)$. Let $(\mathcal{A}, x)$, with coordinate functions $x^1, \ldots, x^n$, $n \leq m$, be a coordinate system on an n-dimensional manifold $\mathcal{A}$. A *curvilinear coordinate system* on $\mathcal{A}$ embedded in $\mathcal{S}$ —such that $\mathcal{A} \subset \mathcal{S}$ is a submanifold— is determined by m nonlinear functions f^i involved in the map

$$\begin{array}{rcl} \mathbb{R}^n & \rightarrow & \mathbb{R}^m \\ (x^1, \ldots, x^n) & \mapsto & y^i = f^i(x^1, \ldots, x^n)\,, \quad i = 1, \ldots, m\,, \end{array}$$

so that $y^i = f^i$ on $\mathcal{A}$ in $\mathcal{S}$. ♢

Therefore, the simple rule for the change of affine coordinates 2.2.8, $y^{i'} = \chi^{i'}(y^1, \ldots, y^m) = B^{i'}_{\ i}\, y^i + c^{i'}$, where the $B^{i'}_{\ i}$ and $c^{i'}$ are constants and do not depend on the point, changes to $y^{i'} = (\chi^{i'} \circ f^i)(x^1, \ldots, x^n)$ if curvilinear coordinates are involved.

With curvilinear coordinates in Euclidian spaces, a vector basis of the tangent space at point $P \in \mathcal{A} \subset \mathcal{S}$ can be constructed by taking the partial derivatives of the position vector $\overrightarrow{OP} = \boldsymbol{x}$, where $O \in \mathcal{S}$ is another point, with respect to the curvilinear coordinates: $\boldsymbol{g}_i = \frac{\partial \boldsymbol{x}}{\partial x^i}$. Indeed, since $\mathrm{d}\boldsymbol{x} = \mathrm{d}x^i \boldsymbol{g}_i$, the $\boldsymbol{g}_i$ are linearly independent. Hence, consulting 2.1.12 and 2.2.11 proofs the following result.

Proposition 3.2.2. *Under a transformation $x^{i'} = \zeta^{i'}(x^1, \ldots, x^n)$ of (curvilinear) coordinates, which is assumed to be invertible, the tangent basis vectors transform according to*

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{x}}{\partial x^i} = \frac{\partial \boldsymbol{x}}{\partial x^{i'}} \frac{\partial \zeta^{i'}}{\partial x^i} = \frac{\partial \zeta^{i'}}{\partial x^i} \boldsymbol{g}_{i'}\,,$$

that is, the inverse matrix of the change of basis coincides with the Jacobian matrix of the change of coordinates.

To simplify notation in this section, $\frac{\partial x^{i'}}{\partial x^i}$ instead of $\frac{\partial \zeta^{i'}}{\partial x^i}$ shall denote the Jacobian matrix of the change of coordinates $x^i \mapsto x^{i'}$, and $\frac{\partial x^i}{\partial x^{i'}}$ is its inverse.

A linear transformation changes a vector, but the vector itself is a coordinate-invariant geometric object. For example, parallel translation is invariant because $\boldsymbol{v} = \overrightarrow{AA'} = \overrightarrow{BB'}$ is independent of the origin and no coordinate system is involved. Therefore, $\boldsymbol{v} = v^i \boldsymbol{g}_i = v^{i'} \boldsymbol{g}_{i'}$ must hold for arbitrary bases $\{\boldsymbol{g}_i\}$, $\{\boldsymbol{g}_{i'}\}$. By 3.2.2, the basis vectors transform with the inverse Jacobian matrix of the change of coordinates, i.e. $\boldsymbol{g}_{i'} = \frac{\partial x^i}{\partial x^{i'}} \boldsymbol{g}_i$, so

$$v^{i'} \boldsymbol{g}_{i'} = \underbrace{v^{i'} \frac{\partial x^i}{\partial x^{i'}}}_{=v^i} \boldsymbol{g}_i \qquad \Leftrightarrow \qquad v^{i'} = \frac{\partial x^{i'}}{\partial x^i} v^i\,.$$

This transformation rule identifies the vector components with components of a 1-fold contravariant tensor:

Definition 3.2.3. A set of coordinate-dependent functions t^i which transform under chart transitions (or changes of coordinates) $x^i \mapsto x^{i'}$ according to the rule

$$t^{i'} = \frac{\partial x^{i'}}{\partial x^i} t^i$$

are called *components of a 1-fold contravariant tensor.* A set of coordinate-dependent functions t_i which transform under the same chart transitions according to the rule

$$t_{i'} = \frac{\partial x^i}{\partial x^{i'}} t_i$$

are called *components of a 1-fold covariant tensor.* ◊

The terminology "covariant" and "contravariant" is introduced in order to distinct the behavior under chart transitions, so covariant means "having the same transformation rule as the basis vectors". However, "subscript" and "superscript" would be also an appropriate distinction.

Proposition 3.2.4. *On differentiable manifolds there are 1-fold covariant and 1-fold contravariant tensors.*

Proof. Let $\mathcal{M}$ be an n-dimensional differentiable manifold, $\mathcal{I} \subset \mathbb{R}$ an open interval of the real line and $s : \mathcal{I} \to \mathcal{M}$ a curve on $\mathcal{M}$, so that $s(t) \in \mathcal{U} \subset \mathcal{M}$ for some open neighborhood $\mathcal{U}$. In a chart $(\mathcal{U}, x)$, s induces the map $s(t) \mapsto x^i \circ s(t) = s^i(t) \subset \mathbb{R}^n$ for every coordinate line x^i. By using the Jacobian matrix defined in 3.1.20, a change of coordinates $x^i \mapsto x^{i'}$ transforms the derivations of the parameterized coordinates s^i at fixed t according to the scheme

$$\frac{\mathrm{d}s^{i'}}{\mathrm{d}t} = \frac{\partial x^{i'}}{\partial x^i}\frac{\mathrm{d}s^i}{\mathrm{d}t}\,.$$

So, by 3.2.3, 1-fold contravariant tensors on differentiable manifolds do exist.

Next, consider a real C^1 function resp. a C^1 scalar field $\tilde{f} : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U} \subset \mathcal{M}$. In a chart $(\mathcal{U}, x)$, $\tilde{f}$ induces the field $f(x) = \tilde{f} \circ x^{-1} \subset \mathbb{R}^n$ as a function of the coordinates x^i. A change of coordinates $x^i \mapsto x^{i'}$ transforms the derivations of f with respect to the coordinates according to the scheme

$$\frac{\partial f'}{\partial x^{i'}} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial f}{\partial x^i}\,.$$

Conclude that on differentiable manifolds there are also 1-fold covariant tensors. ■

Definition 3.2.5. Let $\tilde{f} : \mathcal{M} \to \mathbb{R}$ be C^1, and let $f(x) = \tilde{f} \circ x^{-1}$ be its picture in a chart. One defines the *directional derivative of $\tilde{f}$ along a curve $s(t) \subset \mathcal{M}$ at point* $s(0)$ through

$$\left.\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\partial f}{\partial x^i}\frac{\partial s^i}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\partial f}{\partial x^i}\frac{\mathrm{d}s^i}{\mathrm{d}t}\right|_{t=0} = \frac{\partial f}{\partial x^i} w^i = D_{\boldsymbol{w}} f = \boldsymbol{w}[f]\,.$$

The $w^i = \frac{\mathrm{d}s^i}{\mathrm{d}t}\Big|_{t=0} = \boldsymbol{w}[x^i]$ are tangential to the curve $s(t)$ through point $s(0)$, and they are called *components of the tangent vector of the curve at* $s(0)$. The w^i are well-defined up to re-parametrizations

$$\bar{w}^i = \frac{\mathrm{d}s^i}{\mathrm{d}\bar{t}} = w^i \frac{\mathrm{d}t}{\mathrm{d}\bar{t}} \,. \qquad \diamondsuit$$

Proposition 3.2.6. *The directional derivative of a real function is coordinate-invariant.*

Proof. By using 3.2.4,

$$\frac{\partial f}{\partial x^{i'}} w^{i'} = \left(\frac{\partial x^i}{\partial x^{i'}} \frac{\partial f}{\partial x^i}\right)\left(\frac{\partial x^{i'}}{\partial x^i} w^i\right) = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial f}{\partial x^i} w^i = \frac{\partial f}{\partial x^i} w^i \,. \qquad \blacksquare$$

Corollary 3.2.7. *If 3.2.5 is written independently of f, then $\boldsymbol{w}[\cdot] = w^i \frac{\partial}{\partial x^i}$ is also coordinate-invariant. Application to the coordinate functions $x^i : \mathcal{U} \to \mathbb{R}$ shows that the $\frac{\partial}{\partial x^i} \in \mathbb{R}^{n_{\dim}}$ are linearly independent, clearly: $\boldsymbol{w}[x^i] = w^j \frac{\partial x^i}{\partial x^j} = w^j \delta^i{}_j = w^i$, so $\boldsymbol{w} = \boldsymbol{w}[x^i] \frac{\partial}{\partial x^i}$ is a vector. Therefore, $\left\{\frac{\partial}{\partial x^i}\right\} \equiv \left\{\boldsymbol{\frac{\partial}{\partial x^i}}\right\}$ is a vector basis for the w^i —the so-called* Gaussian basis*—, and $\mathbb{R}^{n_{\dim}}$ indeed is a vector space.*

Definition 3.2.8. Let $\mathcal{M}$ be a differentiable manifold, $\mathcal{U} \subset \mathcal{M}$ a subset and $(\mathcal{U}, x)$ a chart, then the *tangent space* $T_P\mathcal{M}$ at point $P \in \mathcal{M}$ is a vector space that is spanned by the partial derivatives of the coordinates $x(P) = \{x^i\}_P$ (the Gaussian basis) such that $\left\{\boldsymbol{\frac{\partial}{\partial x^i}}\right\}_P \in T_P\mathcal{M}$ is a basis. Formally, the tangent space can be written as

$$T_P\mathcal{M} = \{P\} \times \mathcal{V}_{n_{\dim}} \,.$$

That is, the tangent space is a vector space attached to a point of the manifold and which is independent of the tangent space at any other point. Note that the pair $\left(P, \boldsymbol{\frac{\partial}{\partial x^i}}\right)$ is the local version of an affine frame of reference on $\mathcal{M}$ that has been introduced in 2.2.6. $\diamondsuit$

Definition 3.2.9. The *tangent vector* $\boldsymbol{w} \in T_P\mathcal{M}$ associated with tensor components w^i is given by its local representative

$$w^i(P) \quad \leftrightarrow \quad \boldsymbol{w}(P) = w^i(P) \boldsymbol{\frac{\partial}{\partial x^i}}(P) \quad \in T_P\mathcal{M} \,.$$

Whenever one talks about a vector $\boldsymbol{w}$ in the manifold, the picture of the w^i in the tangent space is meant. However, to get a local representative of $\boldsymbol{w}$, the first step is to choose a chart on $\mathcal{M}$. $\diamondsuit$

Definition 3.2.10. The disjoint union

$$T\mathcal{M} = \bigcup_{P \in \mathcal{M}} T_P\mathcal{M}$$

of all tangent spaces at all points $P \in \mathcal{M}$ is called the *tangent bundle of* $\mathcal{M}$. $\diamondsuit$

Definition 3.2.11. A manifold $\mathcal{M}$ is called *metrizable*, if the tangent bundle $T\mathcal{M}$ has a *fibre metric*, that is, there exists a positive symmetric bilinear mapping

$$\begin{aligned} T_P\mathcal{M} \times T_P\mathcal{M} &\rightarrow \mathbb{R} \\ \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\right) &\mapsto \left\langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} \right\rangle_P = g_{ij}(P) \end{aligned}$$

at every point $P \in \mathcal{M}$. The $g_{ij} = g_{ji}$ are called *metric coefficients*. If $\mathcal{M}$ is also torsion-free ($\mathit{\Gamma}_k{}^j{}_i = \mathit{\Gamma}_i{}^j{}_k$, see sec. 3.6.1), then $\mathcal{M}$ is called a *Riemannian manifold*. $\diamond$

Definition 3.2.12. A *linear form* is a map

$$\begin{aligned} \boldsymbol{a}^* : \mathcal{V} &\rightarrow \mathbb{R} \\ \boldsymbol{v} &\mapsto \boldsymbol{a}^*(\boldsymbol{v}) \equiv \boldsymbol{a}^* \cdot \boldsymbol{v}\,. \end{aligned}$$

So $\boldsymbol{a}^* \in \mathcal{V}^*$, where $\mathcal{V}^*$ is the dual space of $\mathcal{V}$, as it is used in linear algebra. Using the notation $\boldsymbol{a}^* \cdot \boldsymbol{v}$, the map is also called *contraction*. Note that the $\cdot$ operator is not commutative and it is not equivalent to the inner product of vectors! $\diamond$

As there are no "traditional" vectors on manifolds, but only tangent vectors, the definition of a linear form has to be revised. On manifolds, the linear forms are called *differential 1-forms*, or shortly *1-forms*.

Definition 3.2.13. Let $\mathcal{M}$ be an n-dimensional differentiable manifold. The *cotangent space* $T_P^*\mathcal{M}$ is a vector space at point $P \in \mathcal{M}$ that can be formally written as $T_P^*\mathcal{M} = \{P\} \times \mathcal{V}_n^*$. The union $T^*\mathcal{M} = \bigcup_{P\in\mathcal{M}} T_P^*\mathcal{M}$ is referred to as the *cotangent bundle of* $\mathcal{M}$. $\diamond$

Proposition 3.2.14. *In a chart* $(\mathcal{U}, x)$ *on a manifold* $\mathcal{M}$*, where* $\mathcal{U} \subset \mathcal{M}$*, the differential forms* $\mathbf{d}x^i$ *are dual to the* $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ *such that the cotangent space is spanned by the coordinate differentials, that is,* $\{\mathrm{d}x^i\}_P \equiv \{\mathbf{d}x^i\}_P \in T_P^*\mathcal{M}$ *is a basis for every* $P \in \mathcal{U}$.

Proof. It has to be shown that the duality relation $\mathbf{d}x^i \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = \delta^i{}_j$ holds, where $\delta^i{}_j$ is the Kronecker delta on $\mathcal{M}$ and $i, j \in \{1, 2, \ldots, n_{\text{dim}}\}$. However, in $\mathbb{R}^{n_{\text{dim}}}$ contraction reduces to ordinary multiplication such that for every $P \in \mathcal{U}$,

$$\mathrm{d}x^i(P)\frac{\partial}{\partial x^j}(P) = \frac{\partial x^i}{\partial x^j}(P) = \delta^i{}_j \quad \Leftrightarrow \quad \mathbf{d}x^i(P) \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}(P) = \delta^i{}_j\,. \qquad \blacksquare$$

In mathematical and physical literature the $\frac{\partial}{\partial x^i} = \partial_i$ with lightface symbols of partial derivation typically denote the basis vectors of the tangent space, and the $\mathrm{d}x^i$ denote their duals. The notation with boldface symbols used here is a reasonable convention borrowed from [2] and [6]. Note that $w^i \frac{\partial}{\partial x^i}$ may be read as a scalar as well as a tangent vector and thus may cause confusion. The chosen notation avoids such ambiguity.

Definition 3.2.15. The *differential* of a C^1 scalar field $f : \mathcal{M} \rightarrow \mathbb{R}$ is a 1-form given by $\mathbf{d}f = \frac{\partial f}{\partial x^i}\,\mathbf{d}x^i$, in which $\frac{\partial f}{\partial x^i}$ are the components of $\mathbf{d}f$. $\diamond$

Corollary 3.2.16. *(i) By 3.2.14, the contraction of a 1-form $\boldsymbol{a}^* \in T_P^*\mathcal{M}$ and a tangent vector $\boldsymbol{v} \in T_P\mathcal{M}$ can be written as*

$$\boldsymbol{a}^* \cdot \boldsymbol{v} = \left(a_i \mathbf{d}x^i\right) \cdot \left(v^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\right) = a_i\, v^j\, \mathbf{d}x^i \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = a_i\, v^j\, \delta^i{}_j = a_i\, v^i\,,$$

where $a_i \mathbf{d}x^i$ is the local representative of $\boldsymbol{a}^*$.

(ii) With 3.2.15, the derivative 3.2.5 of a C^1 real function f in the direction $\boldsymbol{w} \in T_P\mathcal{M}$ can be written as

$$\boldsymbol{w}[f] = \frac{\partial f}{\partial x^i} w^i = \frac{\partial f}{\partial x^i} w^j \delta_j{}^i = \frac{\partial f}{\partial x^i} w^j\, \mathbf{d}x^i \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = \mathbf{d}f \cdot \boldsymbol{w}\,.$$

Note that neither the terminology ”co- and contravariant basis vectors” nor a metric for raising or lowering the indices is involved to perform $\boldsymbol{a}^* \cdot \boldsymbol{v}$, as it is typically done when using curvilinear coordinates in Euclidian spaces. This is an example of how differential geometry can offer advantages and clarification of the basic theory.

3.3 Tensor Algebra

Definition 3.3.1. A $\binom{p}{q}$*-tensor $\boldsymbol{T}(P)$ at point P of a manifold $\mathcal{M}$* is a multilinear mapping

$$\boldsymbol{T} : \underbrace{T_P^*\mathcal{M} \times \ldots \times T_P^*\mathcal{M}}_{p-\text{fold}} \times \underbrace{T_P\mathcal{M} \times \ldots \times T_P\mathcal{M}}_{q-\text{fold}} \to \mathbb{R}\,, \qquad \diamond$$

By multilinearity it is meant that

$$\boldsymbol{T}\left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \lambda\,\boldsymbol{x}, \ldots, \boldsymbol{v}_q\right) = \lambda\,\boldsymbol{T}\left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \boldsymbol{x}, \ldots, \boldsymbol{v}_q\right)\,,$$

for some $\lambda \in \mathbb{R}$, and

$$\begin{aligned}\boldsymbol{T}&\left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \boldsymbol{x}+\boldsymbol{y}, \ldots, \boldsymbol{v}_q\right) = \\ &\boldsymbol{T}\left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \boldsymbol{x}, \ldots, \boldsymbol{v}_q\right) + \boldsymbol{T}\left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \boldsymbol{y}, \ldots, \boldsymbol{v}_q\right)\,,\end{aligned}$$

where $\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^* \in T_P^*\mathcal{M}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q \in T_P\mathcal{M}$.

Definition 3.3.2. A $\binom{p\ r}{q\ s}$*-two-point tensor over map $\phi : \mathcal{M} \to \mathcal{N}$* is a multilinear mapping

$$\begin{aligned}\boldsymbol{T} : &\underbrace{T_Q^*\mathcal{N} \times \ldots \times T_Q^*\mathcal{N}}_{p-\text{fold}} \times \underbrace{T_Q\mathcal{N} \times \ldots \times T_Q\mathcal{N}}_{q-\text{fold}} \\ &\times \underbrace{T_P^*\mathcal{M} \times \ldots \times T_P^*\mathcal{M}}_{r-\text{fold}} \times \underbrace{T_P\mathcal{M} \times \ldots \times T_P\mathcal{M}}_{s-\text{fold}} \to \mathbb{R}\,,\end{aligned}$$

where $Q = \phi(P)$ and $\boldsymbol{T}$ is a function of P. Every $\binom{p\ r}{q\ s}$-tensor is an element of a $(p \times q \times r \times s)$-dimensional vector space. One refers to $(p+q+r+s)$ as the rank of the tensor.
$\diamond$

Example 3.3.3. A differential 1-form $\boldsymbol{a}^*$ is a $\binom{0}{1}$-tensor, also called a covariant rank-one tensor, and a vector $\boldsymbol{v}$ is a $\binom{1}{0}$-tensor by setting $\boldsymbol{a}^*(\boldsymbol{v}) = \boldsymbol{v}(\boldsymbol{a}^*)$.

Definition 3.3.4. The *components of a tensor*, that has been defined through 3.3.1, are obtained by delivering the basis vectors of the cotangent space and of the tangent space as arguments of the tensor:

$$T^{\mu_1,\mu_2,\ldots,\mu_p}{}_{\nu_1,\nu_2,\ldots,\nu_q} = \boldsymbol{T}\left(\mathbf{d}x^{\mu_1}, \mathbf{d}x^{\mu_2}, \ldots, \mathbf{d}x^{\mu_p}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^{\nu_1}}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^{\nu_2}}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^{\nu_q}}\right).$$

Components of two-point tensors are unwrapped analogously. ◇

If the tensor cannot be represented by a single symbol $\boldsymbol{T}$, it will be put in parentheses such that $(\boldsymbol{T})^{\mu_1,\mu_2,\ldots,\mu_p}{}_{\nu_1,\nu_2,\ldots,\nu_q}$ denote the components of the tensor. Lower case greek letters are used here as coordinate indices, whereas lower case Latin denote labels. The nested indication is necessary as $p+q$ index slots exist, and each independently pass through the numbers $1, 2, \ldots, n_{\text{dim}}$.

Corollary 3.3.5. *Let $\frac{\partial x^{\mu'}}{\partial x^{\mu}}, \frac{\partial x^{\mu}}{\partial x^{\mu'}}$ denote the Jacobian matrix and its inverse, respectively, of a chart transition $x\,(\mathcal{U}\cap\mathcal{U}') \to x'\,(\mathcal{U}\cap\mathcal{U}')$; $\mathcal{U}\cap\mathcal{U}' \neq \emptyset$ being understood. Then, by 3.2.3, the components of a $\binom{p}{q}$-tensor transform according to*

$$T^{\mu_1',\mu_2',\ldots,\mu_p'}{}_{\nu_1',\nu_2',\ldots,\nu_q'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\frac{\partial x^{\mu_2'}}{\partial x^{\mu_2}}\cdots\frac{\partial x^{\mu_p'}}{\partial x^{\mu_p}}\,\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\frac{\partial x^{\nu_2}}{\partial x^{\nu_2'}}\cdots\frac{\partial x^{\nu_q}}{\partial x^{\nu_q'}}\,T^{\mu_1,\mu_2,\ldots,\mu_p}{}_{\nu_1,\nu_2,\ldots,\nu_q}\,.$$

For the sake of completeness, some generalization of the tensorial transformation criterion is given here.

Definition 3.3.6. (See also [5], ch. vii; [6], ch. 21; [25], ch. 5; and [4]) Let $\mathfrak{T}$ be a kind of $\binom{p}{q}$-tensor whose components transform according to the rule

$$\mathfrak{T}^{\mu_1',\ldots,\mu_p'}{}_{\nu_1',\ldots,\nu_q'} = f\left[\det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right)\right]\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_p'}}{\partial x^{\mu_p}}\,\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\cdots\frac{\partial x^{\nu_q}}{\partial x^{\nu_q'}}\,\mathfrak{T}^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q}\,,$$

where $f[\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})]$ is a function of the determinant of the inverse Jacobian matrix. Then $\mathfrak{T}$ is called an *even relative tensor of weight* w, if $f[\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})] = \det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})^w$, and it is called an *odd relative tensor of weight* w, if $f[\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})] = |\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})|^w$. Moreover, $\mathfrak{T}$ is a *pseudotensor*, if $f[\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})] = \operatorname{sign}[\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})]$.

The even (odd) relative tensors of weight 1 are also called *even (odd) tensor densities*, and the even (odd) relative scalars of weight 1 are also called *even (odd) scalar densities.*

An *absolute tensor* (or ordinary tensor) is obtained by setting $f[\det(\frac{\partial x^{\mu}}{\partial x^{\mu'}})] = 1$. Except for special topics of integration theory, this paper solely deals with ordinary (two-point) tensors. ◇

Definition 3.3.7. *Contraction* reduces the rank of tensors. For example, contracting a $\binom{1}{1}$-tensor $\boldsymbol{T}$ and a $\binom{1}{0}$-tensor resp. vector $\boldsymbol{v} = v^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^i}$ yields the $\binom{1}{0}$-tensor

$$\boldsymbol{T}\cdot\boldsymbol{v} = \boldsymbol{T}(\boldsymbol{v}) = v^i\,\boldsymbol{T}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^i}\right) \qquad \text{, in components} \qquad T^i{}_j\,v^j\,.$$

The contraction of a $\binom{1}{1}$-tensor $\boldsymbol{T}$ and a $\binom{0}{1}$-tensor (1-form) $\boldsymbol{a}^*$ is

$$\boldsymbol{a}^* \cdot \boldsymbol{T} = \boldsymbol{a}^*(\boldsymbol{T}) \qquad \text{, in components} \qquad T^i{}_j\, a_i \,.$$

The contraction two tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ in the i-th covariant slot of $\boldsymbol{T}$ and the j-th contravariant slot of $\boldsymbol{S}$ are defined in a similar way, as if the covariant slot is a 1-form and the contravariant slot is a vector, respectively. The contraction of two tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ in the i-th contravariant slot of $\boldsymbol{T}$ and the j-th covariant slot of $\boldsymbol{S}$ is straightforward.

Let T^{abcd} and S_{ijkl} be the components of two tensors $\boldsymbol{T}$ and $\boldsymbol{S}$. Then, without a specification of slots,

$$\boldsymbol{T} \cdot \boldsymbol{S} \qquad \text{with} \qquad T^{abcd} S_{ijkd} = \tilde{T}^{abc}{}_{ijk}$$

denotes the contraction and

$$\boldsymbol{T} : \boldsymbol{S} \qquad \text{with} \qquad T^{abcd} S_{ijcd} = \hat{T}^{ab}{}_{ij}$$

the double contraction of $\boldsymbol{T}$ and $\boldsymbol{S}$, respectively. A single tensor is contracted analogously, provided that the tensor has as well covariant as contravariant slots. ◊

Definition 3.3.8. The contraction of a $\binom{1}{1}$-tensor is called the *trace* of the tensor:

$$\operatorname{tr} \boldsymbol{T} = T^i{}_i \,.$$ ◊

Definition 3.3.9. Let the tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ have the same rank and be compatible in terms of contraction, then $\langle \boldsymbol{T}, \boldsymbol{S} \rangle$ denotes the *contraction on all index slots* resp. the *inner product*. Taking the examples of $\boldsymbol{T}$ and $\boldsymbol{S}$ from definition 3.3.7, then

$$\langle \boldsymbol{T}, \boldsymbol{S} \rangle = T^{ijkl} S_{ijkl} \,.$$ ◊

By the chosen definition of tensors, $\boldsymbol{T}$ is an operator acting on the vector slots and 1-form slots of some other tensor $\boldsymbol{S}$, that is, $\boldsymbol{T}(\boldsymbol{S}) \neq \boldsymbol{S}(\boldsymbol{T})$ in general!

Definition 3.3.10. Let $\boldsymbol{T}$ be a $\binom{p}{q}$-tensor and $\boldsymbol{S}$ a $\binom{r}{s}$-tensor. The *tensor product* $\boldsymbol{T} \otimes \boldsymbol{S}$ yields an $\binom{p+r}{q+s}$-tensor

$$\begin{aligned}(\boldsymbol{T} \otimes \boldsymbol{S})(P) :\ & \underbrace{T_P^*\mathcal{M} \times \ldots \times T_P^*\mathcal{M}}_{p-\text{fold}} \times \underbrace{T_P\mathcal{M} \times \ldots \times T_P\mathcal{M}}_{q-\text{fold}} \\ & \times \underbrace{T_P^*\mathcal{M} \times \ldots \times T_P^*\mathcal{M}}_{r-\text{fold}} \times \underbrace{T_P\mathcal{M} \times \ldots \times T_P\mathcal{M}}_{s-\text{fold}} \to \mathbb{R}\end{aligned}$$

such that

$$\begin{aligned}(\boldsymbol{T} \otimes \boldsymbol{S}) & \left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q, \boldsymbol{b}_1^*, \ldots, \boldsymbol{b}_r^*, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_s\right) \\ & = \boldsymbol{T}\left(\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_p^*, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q\right) \boldsymbol{S}\left(\boldsymbol{b}_1^*, \ldots, \boldsymbol{b}_r^*, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_s\right) ,\end{aligned}$$

and

$$(\boldsymbol{T}\otimes\boldsymbol{S})^{\alpha_1,\dots,\alpha_p}{}_{\beta_1,\dots,\beta_q}{}^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots,\nu_s} = T^{\,\alpha_1,\dots,\alpha_p}{}_{\beta_1,\dots,\beta_q} S^{\,\mu_1,\dots,\mu_r}{}_{\nu_1,\dots,\nu_s}\,.$$

That is, the tensor product generates ordered tuples of basis vectors, and so for a $\binom{p}{q}$-tensor $\boldsymbol{T}$ one may abbreviate $\boldsymbol{T}\in T^p_q(\mathcal{M})$, where $T^p_q(\mathcal{M})=\bigotimes^p T\mathcal{M}\otimes\bigotimes^q T^*\mathcal{M}$ is the $\binom{p}{q}$*-tensor bundle*, and $\bigotimes^k$ denotes the k-th *tensor power*. ◇

Corollary 3.3.11. *By using the definitions 3.3.4, 3.3.7 and 3.3.10, a $\binom{p}{q}$-tensor has the local representative*

$$\boldsymbol{T}(P) = T^{\mu_1,\mu_2,\dots,\mu_p}{}_{\nu_1,\nu_2,\dots,\nu_q}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\mu_1}}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\mu_2}}\otimes\dots\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\mu_p}}\otimes\mathbf{d}x^{\nu_1}\otimes\mathbf{d}x^{\nu_2}\otimes\dots\otimes\mathbf{d}x^{\nu_q}\,.$$

Definition 3.3.12. The definition 3.3.6 can be generalized further by applying the tensor product. An *absolute pseudotensor* is obtained by tensor-multiplying an even and an odd relative tensor of opposite weights, and a *relative pseudotensor of weight* $w=u+v$ is the results of the tensor product of an even relative tensor of weight u (resp. v) and an odd relative tensor of weight v (resp. u). ◇

The reader should carefully distinguish the tensors on manifolds from those in ordinary vector spaces, as the latter do not carry point information. The following proposition helps to clarify this aspect.

Proposition 3.3.13. *Let $\mathcal{V},\mathcal{W}$ be vector spaces and $\{\boldsymbol{G}_I\}\in\mathcal{V}$, $\{\boldsymbol{g}_i\}\in\mathcal{W}$ appropriate bases, then every linear map $\boldsymbol{A}:\mathcal{V}\to\mathcal{W}$ corresponds with a $\binom{1}{1}$-tensor.*

Proof. Note that from the definitions 2.1.10 and 3.3.2, $\boldsymbol{AV}=V^I(\boldsymbol{AG}_I)=A^i{}_I V^I\boldsymbol{g}_i\in\mathcal{W}$, where $\boldsymbol{V}=V^I\boldsymbol{G}_I$ is a vector. On the other hand

$$A^i{}_I V^I\boldsymbol{g}_i = A^i{}_J V^I\delta^J{}_I\boldsymbol{g}_i = A^i{}_J V^I\boldsymbol{g}_i\otimes\boldsymbol{G}^J\cdot\boldsymbol{G}_I = \left(A^i{}_J\boldsymbol{g}_i\otimes\boldsymbol{G}^J\right)\cdot\left(V^I\boldsymbol{G}_I\right)\,,$$

so $\boldsymbol{A}=A^i{}_I\boldsymbol{g}_i\otimes\boldsymbol{G}^I$ is the tensor of the linear map. ■

An alternative proof of the previous statement can be obtained by showing that the $A^i{}_I$ of the linear map transform as the components of a tensor.

To simplify notation, boldface italics are used for both, linear transformations and tensors. The difference will become clear when the object is applied to a vector $\boldsymbol{v}$ of the tangent space: $\boldsymbol{Av}$ is the linear map $\boldsymbol{A}$ applied to the vector $\boldsymbol{v}$, and $\boldsymbol{A}\cdot\boldsymbol{v}$ is the contraction of the tensor $\boldsymbol{A}$ and the vector $\boldsymbol{v}$.

Definition 3.3.14. Let $\mathcal{M}$ be a manifold and $P\in\mathcal{M}$. From the metric coefficients in 3.2.11, the *metric tensor*, or shortly *metric* $\boldsymbol{g}(P)=g_{ij}\,\mathbf{d}x^i\otimes\mathbf{d}x^j$ can be defined, so that the pair $(\mathcal{M},\boldsymbol{g})$ is a *metric space* (just write $\mathcal{M}$ instead of $(\mathcal{M},\boldsymbol{g})$ if there is no danger of confusion). Through $g^{ik}\,g_{kj}=\delta^i{}_j$, an *inverse metric* $\boldsymbol{g}^{-1}(P)=g^{ij}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}$ can be obtained, such that

$$\begin{aligned}\boldsymbol{g}\cdot\boldsymbol{g}^{-1} &= \left(g_{kl}\,\mathbf{d}x^k\otimes\mathbf{d}x^l\right)\cdot\left(g^{ij}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\right)\\ &= g_{kl}g^{ij}\delta^i{}_l\mathbf{d}x^k\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = \mathbf{d}x^i\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \boldsymbol{I}_{\mathcal{M}}\end{aligned}$$

is the *second-rank unit tensor on* $\mathcal{M}$. $\diamondsuit$

Corollary 3.3.15. *Since*

$$g^{ik}\,g_{kj} = g^{ik}\left\langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\right\rangle = \delta^i_{\ j} = \mathbf{d}x^i \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}$$
$$\Leftrightarrow \qquad \mathbf{d}x^i = g^{ik}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} = g^{ik} g_{kj}\,\mathbf{d}x^j\,,$$

tensor indices can be raised by the inverse metric coefficients, and lowered by the metric coefficients.

Definition 3.3.16. Let $\boldsymbol{T} = T^i_{\ j}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j$ be a $\binom{1}{1}$-tensor, then the *associated tensors of* $\boldsymbol{T}$ are

$$\boldsymbol{T}^\flat = T_{ij}\,\mathbf{d}x^i \otimes \mathbf{d}x^j = g_{ik}T^k_{\ j}\,\mathbf{d}x^i \otimes \mathbf{d}x^j$$
$$\text{and} \qquad \boldsymbol{T}^\sharp = T^{ij}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = T^i_{\ k}g^{kj}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\,,$$

where $^\flat$ is the *index lowering operator* and $^\sharp$ is the *index raising operator*, respectively. If $\boldsymbol{S} \in T^p_q(\mathcal{M})$, then $\boldsymbol{S}^\flat \in T^0_{p+q}(\mathcal{M})$ is the tensor with all indices lowered, and $\boldsymbol{S}^\sharp \in T^{p+q}_0(\mathcal{M})$ is the tensor with all indices raised. $\diamondsuit$

Definition 3.3.17. Let $P \in \mathcal{M}$ and $\boldsymbol{v} \in T_P\mathcal{M}$ a vector, then define the linear transformation $\boldsymbol{g}^\flat(P) : T_P\mathcal{M} \to T^*_P\mathcal{M}$ by $\boldsymbol{g}^\flat(P)\cdot\boldsymbol{v} = \langle \boldsymbol{v}, \cdot\rangle_P$, and its inverse $\boldsymbol{g}^\sharp(P) : T^*_P\mathcal{M} \to T_P\mathcal{M}$, such that for $\boldsymbol{T} \in T^1_1(\mathcal{M})$, $\boldsymbol{T}^\flat = \boldsymbol{g}^\flat \cdot \boldsymbol{T}$ and $\boldsymbol{T}^\sharp = \boldsymbol{T}\cdot\boldsymbol{g}^\sharp$, respectively. $\diamondsuit$

Corollary 3.3.18. *The associated tensors of* $\boldsymbol{I}_\mathcal{M}$ *are*

$$\boldsymbol{I}^\flat = g_{ik}\delta^k_{\ j}\,\mathbf{d}x^i \otimes \mathbf{d}x^j = \boldsymbol{g} = \boldsymbol{g}^\flat \qquad \textit{and} \qquad \boldsymbol{I}^\sharp = \delta^i_{\ k}g^{kj}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = \boldsymbol{g}^{-1} = \boldsymbol{g}^\sharp\,.$$

Example 3.3.19. Raising the indices of the differential $\mathbf{d}f$ of a scalar field gives the *gradient* $\boldsymbol{\nabla} f$, which is a vector:

$$(\mathbf{d}f)^\sharp = \boldsymbol{\nabla} f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\,.$$

Associated tensors are different objects, that is, $\boldsymbol{T} \neq \boldsymbol{T}^\sharp \neq \boldsymbol{T}^\flat$! In cartesian spaces, however, the distinction is unnecessary.

Definition 3.3.20. Let $\mathcal{M}, \mathcal{N}$ be manifolds, $\boldsymbol{T}(P) : T_P\mathcal{M} \to T_Q\mathcal{N}$ a two-point tensor over a regular map $\phi : \mathcal{M} \to \mathcal{N}$, where $Q = \phi(P)$, and let $\boldsymbol{U} \in T_P\mathcal{M}$ be a vector on $\mathcal{M}$ and $\boldsymbol{a}^* \in T^*_Q\mathcal{N}$ a one form on $\mathcal{N}$. The *transpose* or *adjoint of* $\boldsymbol{T}$, $\boldsymbol{T}^*(Q) : T^*_Q(\phi(\mathcal{M})) \to T^*_{\phi^{-1}(Q)}\mathcal{M}$, is then defined through

$$\left(\boldsymbol{a}^* \cdot \boldsymbol{T}(\boldsymbol{U})\right)_Q = \left(\boldsymbol{T}^*(\boldsymbol{a}^*)\cdot\boldsymbol{U}\right)_P\,.$$

The transpose will be defined slightly different if $\mathcal{M}$ and $\mathcal{N}$ are metric spaces. Let $\boldsymbol{v} \in T_Q\mathcal{N}$ be a vector, then the *transpose* $\boldsymbol{T}^{\mathrm{T}}(Q) : T_Q(\phi(\mathcal{M})) \to T_{\phi^{-1}(Q)}\mathcal{M}$ is defined through

$$\langle \boldsymbol{v}, \boldsymbol{T}(\boldsymbol{U})\rangle_Q = \left\langle \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{v}), \boldsymbol{U}\right\rangle_P\,. \qquad \diamondsuit$$

Proposition 3.3.21. *If $g_{ij}(Q)$ are the metric coefficients on $\mathcal{N}$ and $G^{IJ}(P)$ are the coefficients of the inverse metric on $\mathcal{M}$, then (i) the components of $\boldsymbol{T}^{\mathrm{T}}$ are*

$$(\boldsymbol{T}^{\mathrm{T}})^I{}_i(Q) = g_{ij}\left(T^j{}_J \circ \phi^{-1}\right)\left(G^{IJ} \circ \phi^{-1}\right) ,$$

and (ii) $\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{G}^{\natural} \cdot \boldsymbol{T}^{*} \cdot \boldsymbol{g}^{\flat}$.

Proof. To proof (i), let $\mathcal{U}(P) \subset \mathcal{M}$, $\mathcal{V}(Q) \subset \mathcal{N}$ be neighborhoods with appropriate charts $(\mathcal{U}, X)$, $(\mathcal{V}, x)$, respectively, in which $X(\mathcal{U}) = \{X^I\}_{\mathcal{U}}$ and $x(\mathcal{V}) = \{x^i\}_{\mathcal{V}}$. Then $\left\{\mathbf{d}X^I\right\} \in T^*_P\mathcal{M}$ is a dual basis at P and $\left\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\right\} \in T_Q\mathcal{N}$ is a basis at Q, so that $\boldsymbol{T}(P) = T^i{}_I(P)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^I$ is a local representative of $\boldsymbol{T}$. Set $\boldsymbol{U} = U^I\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}$ and $\boldsymbol{v} = v^i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$, then,

$$\boldsymbol{T}(\boldsymbol{U}) = T^i{}_I U^J \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^I \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J} = T^i{}_I U^I \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$$

and

$$\langle \boldsymbol{T}(\boldsymbol{U}), \boldsymbol{v}\rangle_Q = v^j T^i{}_I U^I \left\langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} \right\rangle = v^j T^i{}_I U^I g_{ij} \, .$$

On the other hand, $\boldsymbol{T}^{\mathrm{T}}(\boldsymbol{v}) = (\boldsymbol{T}^{\mathrm{T}})^I{}_i v^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \otimes \mathbf{d}x^i \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = (\boldsymbol{T}^{\mathrm{T}})^I{}_i v^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}$, so

$$\left\langle \boldsymbol{U}, \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{v})\right\rangle_P = U^J (\boldsymbol{T}^{\mathrm{T}})^I{}_i v^i \left\langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J} \right\rangle = U^J (\boldsymbol{T}^{\mathrm{T}})^I{}_i v^i G_{IJ} \, .$$

By definition, $v^i U^J T^j{}_J g_{ij} = v^i U^J (\boldsymbol{T}^{\mathrm{T}})^I{}_i G_{IJ}$. Multiplying both sides with G^{JK}, and by noting that $\boldsymbol{U}$ and $\boldsymbol{v}$ are arbitrary, one gets

$$(\boldsymbol{T}^{\mathrm{T}})^I{}_i = g_{ij}\, T^j{}_J\, G^{IJ} \, .$$

The assertion follows by uncovering the point arguments.

For (ii), let $\boldsymbol{a}^* = \boldsymbol{v}^\flat = \boldsymbol{g}^\flat \cdot \boldsymbol{v}$. By definition, $\boldsymbol{T}^*(\boldsymbol{a}^*) = \boldsymbol{T}^*(\boldsymbol{g}^\flat \cdot \boldsymbol{v}) = (\boldsymbol{T}^* \cdot \boldsymbol{g}^\flat)(\boldsymbol{v})$ is a 1-form on $\mathcal{M}$, so $\boldsymbol{G}^\natural \cdot ((\boldsymbol{T}^* \cdot \boldsymbol{g}^\flat)(\boldsymbol{v})) = (\boldsymbol{G}^\natural \cdot \boldsymbol{T}^* \cdot \boldsymbol{g}^\flat)(\boldsymbol{v})$ is a vector on $\mathcal{M}$, where the last identity is due to linearity. Since $\boldsymbol{v}$ is arbitrary, comparison with the second definition of a transpose gives the result. ■

Definition 3.3.22. Let $\boldsymbol{T}$, $\boldsymbol{U}$ and $\boldsymbol{v}$ be as before. The operations

$$\boldsymbol{T}^{-1}(P) \cdot \boldsymbol{T}(P) = \boldsymbol{I}_{\mathcal{M}} \qquad \text{and} \qquad \boldsymbol{T}^{-1} \cdot \boldsymbol{T}(\boldsymbol{U}) = \boldsymbol{U}$$

involve the *inverse tensor* $\boldsymbol{T}^{-1}$. It is easy to verify that $\boldsymbol{T}^{-1}$ has the local representative

$$\boldsymbol{T}^{-1}(P) = (\boldsymbol{T}^{-1})^I{}_i(P)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \otimes \mathbf{d}x^i \, ,$$

where $(\boldsymbol{T}^{-1})^I{}_i$ are the components of the inverse of the matrix $(T^i{}_I)$. Define the *inverse transposed tensor* $\boldsymbol{T}^{-\mathrm{T}}(Q)$ analogously from

$$\boldsymbol{T}^{-\mathrm{T}}(Q) \cdot \boldsymbol{T}^{\mathrm{T}}(Q) = \boldsymbol{I}_{\mathcal{N}} \qquad \text{resp.} \qquad \boldsymbol{T}^{-\mathrm{T}} \cdot \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{v}) = \boldsymbol{v} \, ,$$

yielding $\boldsymbol{T}^{-\mathrm{T}}(Q) = (\boldsymbol{T}^{-\mathrm{T}})^j{}_J(Q)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^I$. ◊

The local representatives of the transpose, the inverse and the inverse transpose of (one-point) tensors can be obtained easily by choosing $P = Q$ and $\mathcal{N} = \mathcal{M}$. It is notable that $\boldsymbol{T}^{\mathrm{T}}$ calls for metrics on $\mathcal{M}$ and $\mathcal{N}$, whereas $\boldsymbol{T}^{-1}$ does not.

Definition 3.3.23. A two-point tensor $\boldsymbol{T} : T_P\mathcal{M} \to T_{\phi(P)}\mathcal{N}$ is called *orthogonal* provided that

$$\boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{T} = \boldsymbol{I}_{\mathcal{M}} \qquad \text{and} \qquad \boldsymbol{T} \cdot \boldsymbol{T}^{\mathrm{T}} = \boldsymbol{I}_{\mathcal{N}} \,.$$

$\boldsymbol{T}$ is called *proper orthogonal*, if it is orthogonal and $\det \boldsymbol{T} = +1$. ◇

Definition 3.3.24. Let $\mathcal{N}$ be a metric space. A $\binom{1}{1}$-tensor $\boldsymbol{S} : T\mathcal{N} \to T\mathcal{N}$ is called *symmetric*, if $\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}$. ◇

Note that with two-point tensors it does not make sense to talk about symmetry!

Corollary 3.3.25. *Let $\boldsymbol{S}(Q) = S^i{}_j(Q)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j$ be symmetric, $Q \in \mathcal{N}$ and g_{ij} the metric coefficients on $\mathcal{N}$. Then by 3.3.21,*

$$S^i{}_j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j = g_{jk} S^k{}_l g^{li} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j = S^i{}_j \, \mathbf{d}x^j \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \,.$$

Definition 3.3.26. Let $\boldsymbol{S} : T\mathcal{N} \to T\mathcal{N}$ be symmetric and $\dim(\mathcal{N}) = 3$, then, by Cayley-Hamilton's theorem [9, 2],

$$\boldsymbol{S}^3 - I_1(\boldsymbol{S})\boldsymbol{S}^2 + I_2(\boldsymbol{S})\boldsymbol{S} - I_3(\boldsymbol{S})\boldsymbol{I}_{\mathcal{N}} = \boldsymbol{0} \,.$$

I_1, I_2, I_3 are scalar functions of $\boldsymbol{S}$, which are rotationally invariant, i.e. invariant under transformations that belong to the special orthogonal group $\mathrm{SO}(T\mathcal{N})$. They are called the *principal invariants of* $\boldsymbol{S}$ having the properties

$$\begin{aligned} I_1(\boldsymbol{S}) &= \operatorname{tr} \boldsymbol{S} \,, \\ I_2(\boldsymbol{S}) &= \det \boldsymbol{S} \, (\operatorname{tr} \boldsymbol{S}^{-1}) = \frac{1}{2} \left((\operatorname{tr} \boldsymbol{S})^2 - \operatorname{tr}(\boldsymbol{S}^2) \right) , \\ \text{and} \quad I_3(\boldsymbol{S}) &= \det \boldsymbol{S} \,. \end{aligned}$$

◇

3.4 Bundles and Tensor Fields

3.4.1 Sections of Fibre Bundles

The previous section was about tensors at single points of the manifold. To give an precise definition of fields of vectors and tensors, the theory of fibre bundles is briefly introduced. For further studies see, for example, [26] and [1].

$$\begin{tikzcd}
\pi^{-1}(\mathcal{U}) \subset \mathcal{E} \arrow[r, "y"] \arrow[d, "\pi"'] & \mathcal{U} \times \mathcal{F} \arrow[dl, "\mathrm{pr}_1"] \\
\mathcal{U} \subset \mathcal{M} &
\end{tikzcd}$$

Figure 3.3: Projection and local trivialization.

Definition 3.4.1. Let $\mathcal{E}, \mathcal{M}$ be at least topological spaces, $\dim(\mathcal{E}) = n + m$ and $\dim(\mathcal{M}) = m$, respectively, and let

$$\pi : \mathcal{E} \to \mathcal{M}$$

be a continuous surjection, then the triple $(\mathcal{E}, \pi, \mathcal{M})$ is called *n-dimensional fibre bundle over $\mathcal{M}$* and π is called *projection* (figure 3.3). $\mathcal{E}$ and $\mathcal{M}$ are referred to as the *total space* and the *base space*, respectively. If $\mathcal{U} \subset \mathcal{M}$ is a subset, then $\pi|_{\mathcal{U}} : \mathcal{E}|_{\mathcal{U}} \to \mathcal{U}$, with $\mathcal{E}|_{\mathcal{U}} = \pi^{-1}(\mathcal{U})$, is called the *restriction of π to $\mathcal{U}$*. For $P \in \mathcal{U}$, $\pi^{-1}(P) = \mathcal{E}|_P$ is called *fibre over P*, so n is the dimension of the fibre. $\diamondsuit$

Instead of $(\mathcal{E}, \pi, \mathcal{M})$, a frequent notation for fibre bundles is $\pi : \mathcal{E} \to \mathcal{M}$, or simply $\mathcal{E}$ if the meaning is clear.

Definition 3.4.2. Let $(\mathcal{E}, \pi, \mathcal{M})$ be an n-dimensional fibre bundle and $\mathcal{U} \subset \mathcal{M}$ a subset. The pair $(\mathcal{W}, y)$ including the homeomorphism $y : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}$, where $\pi^{-1}(\mathcal{U}) = \mathcal{W} \subset \mathcal{E}$, is called *bundle chart* or *local trivialization* (figure 3.3). $\mathcal{F}$, with $\dim(\mathcal{F}) = n$, is called the *fibre space*. The term fibre space is legitimate since the fibre $\pi^{-1}(P)$ is homeomorphic to $\mathcal{F}$ through $y(\pi^{-1}(P)) = \{P\} \times \mathcal{F}$. The mappings $\mathrm{pr}_1 : \mathcal{U} \times \mathcal{F} \to \mathcal{U}$ and $\mathrm{pr}_2 : \mathcal{U} \times \mathcal{F} \to \mathcal{F}$ denote the projections onto the first and second factor of $y(\pi^{-1}(\mathcal{U}))$, respectively. $\diamondsuit$

Definition 3.4.3. A collection $(y_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathcal{F})_{i \in \mathcal{I}}$ of bundle charts of $(\mathcal{E}, \pi, \mathcal{M})$, such that $\bigcup_{i \in \mathcal{I}} \mathcal{U}_i = \mathcal{M}$ for $\mathcal{I} \subset \mathbb{N}$, is called *bundle atlas* of $(\mathcal{E}, \pi, \mathcal{M})$. $\diamondsuit$

Definition 3.4.4. Let $(\mathcal{E}, \pi, \mathcal{M})$ be a fibre bundle. If all $\pi^{-1}(\mathcal{U}), \mathcal{U} \subset \mathcal{M}$, are locally trivializable and $\pi^{-1}(\mathcal{M}) \to \mathcal{M} \times \mathcal{F}$ is a homeomorphism, then the fibre bundle is called *globally trivializable*. If the total space is the *product topology* $\mathcal{E} = \mathcal{M} \times \mathcal{F}$, and $(\mathcal{M}, y = \mathrm{Id})$ is a global bundle chart such that $\pi = \mathrm{pr}_1 : \mathcal{M} \times \mathcal{F} \to \mathcal{M}$ and $\pi^{-1}(P) = \{P\} \times \mathcal{F}$, then $(\mathcal{E}, \pi, \mathcal{M})$ is a *trivial bundle*. $\diamondsuit$

Definition 3.4.5. If the fibre space is an n-dimensional vector space $\mathcal{V}_n$, a fibre bundle $(\mathcal{E}, \pi, \mathcal{M})$ is called a *vector bundle*, and for $\mathcal{E} = \mathcal{M} \times \mathcal{V}_n$ it is called a *trivial vector bundle*. $\diamondsuit$

Definition 3.4.6. If the fibre of a fibre bundle over $\mathcal{M}$ is spanned by the tangent vectors at each $P \in \mathcal{M}$, i.e. $\pi^{-1}(P) = T_P\mathcal{M}$, then the total space is denoted by $T\mathcal{M}$, and $\pi = \tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ respectively $(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ is referred to as the *tangent bundle over $\mathcal{M}$* (cf. 3.2.10). A local trivialization is then assumed to be diffeomorphic, not only homeomorphic. $\diamondsuit$

Example 3.4.7. The simplest example of a tangent bundle is found in the affine point space $(\mathbb{R}^n, \mathbb{R}^n)$. The set of all vectors $\boldsymbol{v} = \{v^1, \ldots, v^n\}$ at all points $P \in \mathbb{R}^n$, i.e. set of all pairs $(P, \boldsymbol{v})$, is just the cartesian product $\mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n$. So $T\mathbb{R}^n$ is trivial, and $\{P\} \times \mathbb{R}^n$ is a tangent space at P.

Example 3.4.8. A tube is a trivial bundle, in which the base space is the circular cross section $\mathbb{S}^1$, and $\mathbb{S}^1 \times \mathbb{R}^1$ is the total space. The fibre $\pi^{-1}(P)$, $P \in \mathbb{S}^1$, is a line on the tube parallel to the axis. The Möbius strip is a non-trivial bundle over $\mathbb{S}^1$. Locally, tube and Möbius strip are identical.

Proposition 3.4.9. *Let $\mathcal{M}$ be an n-dimensional differentiable manifold, then (i) $T\mathcal{M}$ is a $2n$-dimensional manifold, and (ii) $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ is a vector bundle.*

PROOF. (i) Let $\mathcal{U} \subset \mathcal{M}$ be a subset and $(\mathcal{U}, x)$ a chart, where $x : \mathcal{U} \to x(\mathcal{U}) \subset \mathbb{R}^n$. Then, by recalling the definitions 3.2.8 and 3.2.10, a *natural chart* of the tangent bundle would be $(T\mathcal{U}, Tx)$ including the map

$$\begin{aligned} Tx:\ T\mathcal{M}|_{\mathcal{U}} &\to x(\mathcal{U}) \times \mathbb{R}^n \\ (P, \boldsymbol{v}) &\mapsto Tx(P, \boldsymbol{v}) = \{\xi^1(\boldsymbol{v}(P)), \xi^2(\boldsymbol{v}(P)), \ldots \\ &\qquad \ldots, \xi^n(\boldsymbol{v}(P)), v^1(\boldsymbol{v}), v^2(\boldsymbol{v}), \ldots, v^n(\boldsymbol{v})\}\,. \end{aligned}$$

Therein, $\xi^i(\boldsymbol{v}(P)) = x^i(P)$ are the local coordinates of $P \in \mathcal{M}$, $v^i(\boldsymbol{v}) = \boldsymbol{v}[x^i]$ are the components of $\boldsymbol{v}(P) \in T_P\mathcal{M}$, and $T\mathcal{M}|_{\mathcal{U}}$ is the restriction of $T\mathcal{M}$ to $\mathcal{U}$.

The *natural atlas* of $T\mathcal{M}$, then, is the collection $T\mathfrak{A} = \{(T\mathcal{U}_i, Tx_i)\}_{i\in\mathcal{I}}$ of natural charts, where $T\mathcal{M} = \bigcup_{i\in\mathcal{I}} T\mathcal{U}_i$ and $\mathcal{I} \subset \mathbb{N}$. Hence, $T\mathcal{M}$ is homeomorphic to $\mathbb{R}^{2n}$, so $\dim(T\mathcal{M}) = 2n$. As $\mathcal{M}$ is Hausdorffian and second countable by definition, and Tx is a diffeomorphism by 3.4.6, $T\mathcal{M}$ is also Hausdorffian and second countable, i.e. a manifold.

(ii) By the definition of a local chart, $x(\mathcal{U})$ is homeomorphic to $\mathcal{U}$, and $T\mathcal{M}|_{\mathcal{U}} = \bigcup_{P\in\mathcal{U}} T_P\mathcal{M} = \tau_{\mathcal{M}}^{-1}(\mathcal{U})$, so

$$\begin{aligned} y_{T\mathcal{M}}:\qquad \tau_{\mathcal{M}}^{-1}(\mathcal{U}) &\to \mathcal{U} \times \mathbb{R}^n \\ \boldsymbol{v}(P) = (P, \boldsymbol{v}) &\mapsto \{P, v^1, v^2, \ldots, v^n\}\,, \end{aligned}$$

$v^i = \boldsymbol{v}[x^i]$ being understood, is a local trivialization of $T\mathcal{M}$ and $\tau_{\mathcal{M}}^{-1}(P) = T_P\mathcal{M}$ is a fibre. By 3.2.7 and the example 2.1.7, $\mathbb{R}^n$ is also a vector space, so that $T_P\mathcal{M} \to \{P\} \times \mathbb{R}^n$ is an isomorphism. Therefore, $T\mathcal{M}$ has a vector bundle structure induced by the differentiable structure of $\mathcal{M}$. ∎

Corollary 3.4.10. *Let $\mathcal{U} \subset \mathcal{M}$, $\mathcal{M}$ being differentiable, be a subset and $(\mathcal{U}, x)$ a chart, then*

$$Tx : T\mathcal{M}|_{\mathcal{U}} \to x(\mathcal{U}) \times \mathbb{R}^n$$

defined in 3.4.9(i) is a local tangent bundle map.

Proposition 3.4.11. *(Without proof; cf. [1], p. 155.) Let $(\mathcal{U}, x)$ and $(\mathcal{U}', x')$, $\mathcal{U}\cap\mathcal{U}' \neq \emptyset$, be regular charts on a differentiable manifold $\mathcal{M}$ such that the chart transition map $x' \circ x^{-1}|_{x(\mathcal{U}\cap\mathcal{U}')} : x(\mathcal{U}\cap\mathcal{U}') \to x'(\mathcal{U}\cap\mathcal{U}')$ is a diffeomorphism, then $T(x' \circ x^{-1})$ is a local tangent bundle isomorphism —meaning that it is an isomorphism on each fibre.*

Definition 3.4.12. Let $(\mathcal{E}, \pi, \mathcal{N})$ respectively $\pi : \mathcal{E} \to \mathcal{N}$ be a fibre bundle with fibre space $\mathcal{F}$ and let $\phi : \mathcal{M} \to \mathcal{N}$ be a continuous map. The *induced bundle* or *pullback bundle* is defined through

$$\pi' : \ \phi^\star \mathcal{E} \to \mathcal{M} \,,$$

in which the total space $\phi^\star \mathcal{E}$ has the same fibres as $\mathcal{E}$, that is, $\phi^\star \mathcal{E} = \phi(\mathcal{M}) \times \mathcal{F}$ provided that $\mathcal{E} = \mathcal{N} \times \mathcal{F}$ is trivial. ◇

Vector fields are sections of vector bundles. Moreover, the tensor product of vector spaces can be transferred to vector bundles to generate tensor bundles, whose sections are then called tensor fields.

Definition 3.4.13. Let $\pi : \mathcal{E} \to \mathcal{M}$ be a fibre bundle, then a map

$$\boldsymbol{\sigma} : \mathcal{M} \to \mathcal{E} \,,$$

with $\pi(\boldsymbol{\sigma}(P)) = P$, $\forall P \in \mathcal{M}$, is called *bundle section.* If the fibre bundle is a vector bundle, then $\boldsymbol{\sigma}$ is called *vector field on* $\mathcal{M}$. The set of all sections of $\mathcal{E}$ is denoted by $\Gamma(\mathcal{E})$. ◇

In the physical and mechanical literature, instead of the correct $\boldsymbol{\sigma} \in \Gamma(\mathcal{E})$ respectively $\boldsymbol{\sigma} : \mathcal{M} \to \mathcal{E}$ it is common to write $\boldsymbol{\sigma}(P)$ for the field, indicating that $\boldsymbol{\sigma}$ depends on the points $P \in \mathcal{M}$.

Definition 3.4.14. Let $\pi : \mathcal{E} \to \mathcal{M}$ be a vector bundle with n-dimensional fibre, $y : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{V}_n$ a bundle chart and $\{\boldsymbol{g}_i\}$ a basis in $\mathcal{V}_n$, then

$$\boldsymbol{\sigma}_i(P) = y^{-1}(P, \boldsymbol{g}_i) \,, \quad i = 1, \ldots, n \,,$$

defines the *local basis sections* or *local basis vector fields* $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n : \mathcal{U} \to \mathcal{E}|_{\mathcal{U}}$ for every $P \in \mathcal{U} \subset \mathcal{M}$. ◇

The local basis sections are necessary to express vector fields and tensor fields in local coordinates, so to get a local representative of tensor fields.

Definition 3.4.15. Let $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ be a tangent bundle and $(T\mathcal{U}, Tx)$ a natural chart of $T\mathcal{M}$, with the chart map $Tx : T\mathcal{M}|_{\mathcal{U}} \to x(\mathcal{U}) \times \mathbb{R}^n$, as defined in 3.4.9(i) resp. 3.4.10. If $y \in \mathbb{R}^n$ and $\{\boldsymbol{e}_i\} : \mathbb{R}^n \to T\mathbb{R}^n$ are the canonical basis sections in $\mathbb{R}^n$ —so $\{\boldsymbol{e}_i\}$ is the canonical basis at every y—, then

$$(Tx)^{-1}(y, \boldsymbol{e}_i) = \left(x^{-1}(y), \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \right) = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}(P) \,, \quad i = 1, \ldots, n \,,$$

defines the local basis sections of $T\mathcal{M}$ for all $x^{-1}(y) = P \in \mathcal{U} \subset \mathcal{M}$, that is,

$$\left\{ \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n} \right\} : \mathcal{U} \to T\mathcal{M}|_{\mathcal{U}} \,.$$

If $\{\boldsymbol{e}^i\}$ is the dual basis of $\{\boldsymbol{e}_i\}$, then $(Tx)^{-1}\boldsymbol{e}^i = \mathbf{d}x^i$. ◇

Definition 3.4.16. Let $(\mathcal{E}_1, \mathcal{M}, \pi_1)$ and $(\mathcal{E}_2, \mathcal{M}, \pi_2)$ be vector bundles, $P \in \mathcal{M}$, and let $\mathcal{E}_1 \otimes \mathcal{E}_2$ be their tensor product. Moreover, let $\boldsymbol{\sigma}_1 \in \Gamma(\mathcal{E}_1)$, $\boldsymbol{\sigma}_2 \in \Gamma(\mathcal{E}_2)$ be sections, then

$$\boldsymbol{T}(P) = (\boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_2)(P) \in \Gamma(\mathcal{E}_1 \otimes \mathcal{E}_2)$$

is called *second-rank tensor field on $\mathcal{M}$*. Therein, the tensor product of the sections has been applied point-by-point:

$$(\boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_2)(P) = \boldsymbol{\sigma}_1(P) \otimes \boldsymbol{\sigma}_2(P) \,.$$

Tensor fields of any rank are defined analogously. A $\binom{p}{q}$-tensor field on a manifold $\mathcal{M}$ is a section of the *tensor bundle* $T^p_q(\mathcal{M})$. The set $\Gamma(T^p_q(\mathcal{M}))$, resp. $\Gamma^\infty(T^p_q(\mathcal{M}))$ on smooth manifolds, is usually written $\mathfrak{T}^p_q(\mathcal{M})$. ◊

Corollary 3.4.17. *Let $\tau_\mathcal{M} : T^1_1(\mathcal{M}) \to \mathcal{M}$ be a $\binom{1}{1}$-tensor bundle, $(\mathcal{U}, x)$ a chart of $\mathcal{U} \subset \mathcal{M}$ and $\boldsymbol{T}(P) \in T^1_1(\mathcal{M})$ a tensor at $P \in \mathcal{U}$. Then, by 3.4.2 and 3.4.9(ii),*

$$\begin{array}{rccc} y_{T\mathcal{M}}|_{\tau^{-1}_\mathcal{M}(P)} : & \tau^{-1}_\mathcal{M}(P) & \to & \{P\} \times \mathbb{R}^{n^{1+1}} \\ & \boldsymbol{T}(P) & \mapsto & (T^i{}_j)(P) \end{array}$$

is a vector space isomorphism that gives the components of $\boldsymbol{T}$ for all $P \in \mathcal{U}$, where $\mathbb{R}^{n^{1+1}} = \mathbb{R}^n \times \mathbb{R}^n$ denotes the $n \times n$-matrices (see also [4], ch. 4).

The corollary can be applied in a simmilar form to arbitrary $\binom{p}{q}$-tensor fields in order to undress the tensor components at every point where the field is defined.

Definition 3.4.18. Let $(\mathcal{E}_1, \mathcal{M}, \pi_1)$, $(\phi^\star\mathcal{E}_2, \mathcal{M}, \pi'_2)$ be vector bundles, $\boldsymbol{\sigma}_1 \in \Gamma(\mathcal{E}_1)$ and $\phi^\star\boldsymbol{\sigma}_2 \in \Gamma(\phi^\star\mathcal{E}_2)$ sections, and $\phi : \mathcal{M} \to \mathcal{N}$ a continuous map. The section

$$\boldsymbol{T}(P) = (\boldsymbol{\sigma}_1 \otimes \phi^\star\boldsymbol{\sigma}_2)(P) \in \Gamma(\mathcal{E}_1 \otimes \phi^\star\mathcal{E}_2)$$

where $P \in \mathcal{M}$, is called *second-rank two-point tensor field over ϕ on $\mathcal{M}$*, if $(\phi^\star\mathcal{E}_2, \mathcal{M}, \pi'_2)$ is the induced bundle of some $(\mathcal{E}_2, \mathcal{N}, \pi_2)$. With this,

$$\phi^\star\boldsymbol{\sigma}_2 : \mathcal{M} \to T\mathcal{N}$$

is called *induced section* or *vector field over ϕ*. Note that if $\boldsymbol{\sigma}_2(Q) \in \Gamma(\mathcal{E}_2)$ and $Q = \phi(P)$, then

$$(\phi^\star\boldsymbol{\sigma}_2)(P) = \boldsymbol{\sigma}_2(\phi(P)) \,.$$ ◊

Example 3.4.19. Two-point tensors and induced sections play an important role in continuum mechanics. A famous example of a two-point tensor field is the deformation gradient, as it acts on two different configurations of a material body. The Lagrangian or particle velocity field is an induced section $\boldsymbol{V}_t : \mathcal{B} \to T\mathcal{S}$, where $\mathcal{B}$ is the material body and $\mathcal{S}$ is the ambient space (see section 4.2).

The algebraic operations on tensors defined in section 3.3 all carry over to tensor fields, by applying the operation to each fibre of the corresponding bundle.

3.4.2 Action of Maps

The following paragraphs should investigate the action of maps on tensor fields, by starting with linear transformations and then concentrating on maps $\phi : \mathcal{M} \to \mathcal{N}$ between manifolds.

Definition 3.4.20. Let $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$ be an isomorphism of vector spaces, $\boldsymbol{T} \in \mathfrak{T}^p_q(\mathcal{V})$ a $\binom{p}{q}$-tensor field, $\boldsymbol{a}^*_1, \ldots, \boldsymbol{a}^*_p \in \mathcal{W}^*$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q \in \mathcal{W}$. Then

$$\begin{aligned}(\boldsymbol{A}_\star \boldsymbol{T})\left(\boldsymbol{a}^*_1, \ldots, \boldsymbol{a}^*_p, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q\right) \\ = \boldsymbol{T}\left((\boldsymbol{A}^* \boldsymbol{a}^*_1), \ldots, (\boldsymbol{A}^* \boldsymbol{a}^*_p), (\boldsymbol{A}^{-1}\boldsymbol{v}_1), \ldots, (\boldsymbol{A}^{-1}\boldsymbol{v}_q)\right) .\end{aligned}$$

is called the *pushforward of* $\boldsymbol{T}$ *by* $\boldsymbol{A}$. The *pullback by* $\boldsymbol{A}$ is defined through $\boldsymbol{A}^\star = (\boldsymbol{A}^{-1})_\star$. ◇

Definition 3.4.21. Let $\mathcal{M}, \mathcal{N}$ be continuous differentiable manifolds and $\phi : \mathcal{M} \to \mathcal{N}$ a differentiable map. The tangent bundle homeomorphism

$$\begin{array}{rccl} T\phi : & T\mathcal{M} & \to & T\mathcal{N} \\ & (P, \boldsymbol{V}) & \mapsto & T\phi\,(P, \boldsymbol{V}) = (\phi(P), D\phi(P) \cdot \boldsymbol{V}) \end{array}$$

is called the *tangent map* or the *differential of* ϕ. $D\phi(P)$ is the derivation of ϕ at $P \in \mathcal{M}$ and $D\phi(P) \cdot \boldsymbol{V}$ means $D\phi(P)$ applied to $\boldsymbol{V} \in T_P\mathcal{M}$ as a linear map. Write $T_P\phi : T_P\mathcal{M} \to T_{\phi(P)}\mathcal{N}$ for the restriction of $T\phi$ to P.

To get a local version of the tangent map, let $(\mathcal{U}, X)$, $(\mathcal{V}, x)$ be appropriate charts, with $\mathcal{U} \subset \mathcal{M}$ and $\mathcal{V} \subset \mathcal{N}$, respectively, so that $\phi^i = x^i \circ \phi \circ X^{-1}$ is the coordinate system on $\mathcal{N}$ arising from the coordinate functions X^I of $(\mathcal{U}, X)$ via ϕ. Let $\left\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}\right\} \in T_P\mathcal{M}$ and $\left\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\right\} \in T_{\phi(P)}\mathcal{N}$ be the related bases of the tangent spaces, then

$$\begin{array}{rccl} T\phi : & T\mathcal{M} & \to & T\mathcal{N} \\ & \left(P,\ \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}(P)\right) & \mapsto & \left(\phi(P),\ \dfrac{\partial \phi^i}{\partial X^I}(P)\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}(\phi(P))\right) , \end{array}$$

i.e. $D\phi = \frac{\partial \phi^i}{\partial X^I}$ in coordinates. ◇

Corollary 3.4.22. *From the given coordinate expression the chain rule* $T(\psi \circ \phi) = T\psi \circ T\phi$ *can be easily verified.*

Corollary 3.4.23. *Let* $\boldsymbol{V} = V^I \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}$ *be a vector, then*

$$T\phi\,(\boldsymbol{V})\,(P) = \underbrace{\left(\frac{\partial \phi^i}{\partial X^I} V^I\right)}_{=(T\phi(\boldsymbol{V}))^I}(P)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}(\phi(P)) = V^I \frac{\partial \phi^i}{\partial X^I}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} .$$

As the tangent map is linear, one may define a two-point tensor $\boldsymbol{F}$ *to obtain the same result:*

$$T\phi\,(\boldsymbol{V}) = V^I \frac{\partial \phi^i}{\partial X^I}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = V^I \underbrace{\left(\frac{\partial \phi^i}{\partial X^J}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^J\right)}_{=\boldsymbol{F}} \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} = \boldsymbol{F} \cdot \boldsymbol{V} .$$

Conclude that the tangent map acts on a vector like a linear transformation.

Definition 3.4.24. With the conventions of 3.4.21, the *inverse tangent map* over diffeomorphisms $\phi : \mathcal{N} \to \mathcal{M}$ is defined through

$$\begin{array}{rcl} T(\phi^{-1}) : \; T\mathcal{N} & \to & T\mathcal{M} \\ \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} & \mapsto & \dfrac{\partial(\phi^{-1})^I}{\partial x^i}\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}\,, \end{array}$$

with $(\phi^{-1})^I = X^I \circ (\phi^{-1}) \circ x^{-1}$. Dual to the tangent map, the *cotangent map over* ϕ is for 1-forms:

$$\begin{array}{rcl} T^*\phi : \; T^*\mathcal{N} & \to & T^*\mathcal{M} \\ \mathbf{d}x^i & \mapsto & \dfrac{\partial \phi^i}{\partial X^I}\mathbf{d}X^I\,, \end{array}$$

so $T^*\phi(\boldsymbol{a}^*)(Q) = \boldsymbol{a}^* \cdot \boldsymbol{F}$, where $\boldsymbol{a}^* \in T^*_Q\mathcal{N}$. $T^*\phi$ has the inverse $T^*(\phi^{-1})(\mathbf{d}X^I) = \frac{\partial(\phi^{-1})^I}{\partial x^i}\mathbf{d}x^i$. ♢

Note that $T\phi$ and $T^*\phi$, respectively $T(\phi^{-1})$ and $T^*(\phi^{-1})$, have the same component matrices, but are evaluated at different points!

For vector fields $\boldsymbol{V} : \mathcal{M} \to T\mathcal{M}$, the operation $T\phi(\boldsymbol{V})$ is also called the *tilt of* $\boldsymbol{V}$ *by* ϕ [2]. If $\mathcal{M}$ and $\mathcal{N}$ have different dimensions, the vector $T\phi(\boldsymbol{V})$ at point $\phi(P)$ is tangent to $\mathcal{N}$, but it need not to be tangent to $\phi(\mathcal{M})$. This is because $T\phi(T\mathcal{M}) = T(\phi(\mathcal{M})) \subset T\mathcal{N}$ is a subspace of $T\mathcal{N}$.

Definition 3.4.25. A map $\phi : \mathcal{M} \to \mathcal{N}$ is called *immersion*, if its tangent $T_P\phi : T_P\mathcal{M} \to T_{\phi(P)}\mathcal{N}$ is injective at each $P \in \mathcal{M}$. If $T_P\phi$ is surjective at each $P \in \mathcal{M}$, then ϕ is called *submersion*. ♢

Proposition 3.4.26. *(Without proof; cf. [1], p. 165.) Let $\phi : \mathcal{M} \to \mathcal{N}$ be an immersion, then there is a neighborhood $\mathcal{U}(P) \subset \mathcal{M}$ such that $\phi(\mathcal{U}) \subset \mathcal{N}$ is a submanifold (see definition 3.1.26) and $\mathcal{U} \to \phi(\mathcal{U})$ is a diffeomorphism.*

The preceding proposition does not imply that $\phi(\mathcal{M})$ is a submanifold, and even if $T\phi$ is injective, ϕ might not be. However, define the following for ϕ being injective.

Definition 3.4.27. Let $\phi : \mathcal{M} \to \mathcal{N}$ be an *injective immersion*, then $\phi : \mathcal{M} \to \phi(\mathcal{M})$ is a diffeomorphism and $\phi(\mathcal{M})$ is called an *immersed submanifold* in $\mathcal{N}$. An immersion $\phi : \mathcal{M} \to \mathcal{N}$ is called *embedding* provided that it is a homeomorphism onto $\phi(\mathcal{M})$ with the topology induced by $\mathcal{N}$. ♢

In other words, an immersion $\phi : \mathcal{M} \to \mathcal{N}$ is a local embedding, and if ϕ in an embedding, then $\phi(\mathcal{M}) \subset \mathcal{N}$ is a submanifold and $\mathcal{M} \to \phi(\mathcal{M})$ is a diffeomorphism.

Corollary 3.4.28. *An injective immersion $\phi : \mathcal{M} \to \mathcal{N}$ is an embedding, if it maps open (closed) sets in $\mathcal{M}$ onto open (closed) sets in $\mathcal{N}$, and inversely —in fact, this is what the homeomorphism in the definition 3.4.27 of an embedding requires.*

$$\begin{tikzcd} T\mathcal{M} \arrow[r, "T\phi"] & T\mathcal{N} \\ \mathcal{M} \arrow[u, "\boldsymbol{V}"] \arrow[r, "\phi"'] \arrow[ur, "T\phi(\boldsymbol{V})"'] & \mathcal{N} \arrow[u, "\phi_\star \boldsymbol{V}"'] \end{tikzcd}$$

Figure 3.4: Tilt and pushforward of a vector field $\boldsymbol{V} \in \Gamma(T\mathcal{M})$.

Example 3.4.29. The circle is an embedding $\mathbb{S}^1 \to \mathbb{R}^2$, and the "8" is an immersion $\mathbb{S}^1 \to \mathbb{R}^2$ that is not injective.

After some tangent map $T\phi : T\mathcal{M} \to T\mathcal{N}$ has been applied to the vector field $\boldsymbol{V}$: $\mathcal{M} \to T\mathcal{M}$, the vector field became a so-called *vector field over the map* $\phi : \mathcal{M} \to \mathcal{N}$ (see definition 3.4.18), which is not an honest vector field on $\mathcal{N}$. In order to transfer the field to $\mathcal{N}$ with respect to ϕ, the reference point have also to be switched. The emerging operations are referred to as the pushforward and the pullback (figure 3.4).

The reader should be warned about different uses in the literature. There is often no distinction being made between vectors and vector fields, and the tangent map is carelessly identified as the pushforward.

Definition 3.4.30. Let $g : \mathcal{N} \to \mathbb{R}$ be a scalar field on $\mathcal{N}$, and $\phi : \mathcal{M} \to \mathcal{N}$ a continuous map. The *pullback*

$$\phi^\star g = g \circ \phi : \ \mathcal{M} \ \to \ \mathbb{R}$$

has the same value at $P \in \mathcal{M}$, as g has at $Q \in \mathcal{N}$, where $Q = \phi(P)$. $\diamondsuit$

Definition 3.4.31. Let $f : \mathcal{M} \to \mathbb{R}$ be a scalar field on $\mathcal{M}$ and $\phi : \mathcal{M} \to \mathcal{N}$ a regular map. The *pushforward of* f to $\mathcal{N}$ is defined through $\phi_\star f = f \circ \phi^{-1}$, that is, $\phi_\star = (\phi^{-1})^\star$. $\diamondsuit$

Corollary 3.4.32. *(i) If $g = \phi_\star f$ arises from the pushforward of f, then the pullback is the inverse operation:*

$$\phi^\star(\phi_\star f)(P) = f \circ \phi^{-1} \circ \phi\,(P) = f(P)\,.$$

(ii) For a composition of maps ϕ and ψ, the chain rule gives

$$(\psi \circ \phi)^\star = \phi^\star \circ \psi^\star \qquad \textit{and} \qquad (\psi \circ \phi)_\star = \psi_\star \circ \phi_\star\,.$$

Definition 3.4.33. Let $\boldsymbol{V} : \mathcal{M} \to T\mathcal{M}$ resp. $\boldsymbol{V} \in \Gamma(T\mathcal{M})$ be an honest vector field on $\mathcal{M}$ and let $\phi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. The *pushforward* $\phi_\star \boldsymbol{V} : \mathcal{N} \to T\mathcal{N}$ is a vector field on $\mathcal{N} = \phi(\mathcal{M})$ given by

$$\phi_\star \boldsymbol{V} = T\phi \circ \boldsymbol{V} \circ \phi^{-1}\,.$$

$\diamondsuit$

Definition 3.4.34. Let $\boldsymbol{w} : \mathcal{N} \to T\mathcal{N}$ be a vector field on $\mathcal{N} = \phi(\mathcal{M})$. The *pullback* $\phi^\star \boldsymbol{w} : \mathcal{M} \to T\mathcal{M}$ is a vector field on $\mathcal{M} = \phi^{-1}(\mathcal{N})$ defined through

$$\phi^\star \boldsymbol{w} = T(\phi^{-1}) \circ \boldsymbol{w} \circ \phi \,. \qquad \diamondsuit$$

Note that $\phi^\star$ also marks induced sections of vector bundles, but the meaning is different! Induced sections $\phi^\star \boldsymbol{\sigma} : \mathcal{M} \to T\mathcal{N}$ are not proper vector fields on $\mathcal{M}$ (or $\mathcal{N}$).

Definition 3.4.35. The specification of the pullback and pushforward operators for fields of differential 1-forms $\boldsymbol{a}^* : \mathcal{N} \to T^*\mathcal{N}$ and $\boldsymbol{B}^* : \mathcal{M} \to T^*\mathcal{M}$, respectively, is straightforward. They are being defined through their action on vector fields. Let $\boldsymbol{V} : \mathcal{M} \to T\mathcal{M}$, then the *pullback of* $\boldsymbol{a}^*$ is given by

$$((\phi^\star \boldsymbol{a}^*) \cdot \boldsymbol{V})(P) = (\boldsymbol{a}^* \circ \phi) \cdot (T\phi \circ \boldsymbol{V}) \,,$$

for $P \in \mathcal{M}$ and $\phi(P) \in \mathcal{N}$, clearly,

$$\phi^\star \boldsymbol{a}^* = (\boldsymbol{a}^* \circ \phi) \cdot T\phi \,.$$

The *pushforward* $\phi_\star \boldsymbol{B}^* : \mathcal{N} \to T^*\mathcal{N}$ of $\boldsymbol{B}^*$ is a 1-form on $\mathcal{N}$ defined by

$$\phi_\star \boldsymbol{B}^* = (\phi^{-1})^\star \boldsymbol{B}^* = (\boldsymbol{B}^* \circ \phi^{-1}) \cdot T(\phi^{-1}) \,.$$

This can be easily obtained from the definition of the pullback of $\boldsymbol{a}^*$ by setting $\phi_\star \boldsymbol{B}^* = \boldsymbol{a}^*$. $\diamondsuit$

The pushforward and the pullback of arbitrary tensor fields can be realized through the application of the pushforward and the pullback of vector fields and fields of 1-forms, respectively, to all index slots of the tensor.

Definition 3.4.36. Let $\phi : \mathcal{M} \to \mathcal{N}$ be regular, $\boldsymbol{T} \in \mathfrak{T}^p_q(\mathcal{M})$ and $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{N})$, then

$$\begin{aligned}(\phi_\star \boldsymbol{T})(Q) & \left(\boldsymbol{a}^*_1, \ldots, \boldsymbol{a}^*_p, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_q\right) \\ &= \boldsymbol{T}(P) \left((\phi^\star \boldsymbol{a}^*_1), \ldots, (\phi^\star \boldsymbol{a}^*_p), (\phi^\star \boldsymbol{w}_1), \ldots, (\phi^\star \boldsymbol{w}_q)\right) \,,\end{aligned}$$

and

$$\begin{aligned}(\phi^\star \boldsymbol{t})(P) & \left(\boldsymbol{B}^*_1, \ldots, \boldsymbol{B}^*_p, \boldsymbol{V}_1, \ldots, \boldsymbol{V}_q\right) \\ &= \boldsymbol{t}(Q) \left((\phi_\star \boldsymbol{B}^*_1), \ldots, (\phi_\star \boldsymbol{B}^*_p), (\phi_\star \boldsymbol{V}_1), \ldots, (\phi_\star \boldsymbol{V}_q)\right) \,,\end{aligned}$$

where $P \in \mathcal{M}$ and $Q = \phi(P)$. $\diamondsuit$

Proposition 3.4.37. *If $\phi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism, then $\phi_\star : \mathfrak{T}^p_q(\mathcal{M}) \to \mathfrak{T}^p_q(\mathcal{N})$ is an isomorphism.*

Proof. This statement is fibrewisely proved by noting that $\phi_\star \circ (\phi^{-1})_\star = (\phi \circ \phi^{-1})_\star = \mathrm{Id}_\star$ is the identity on $\mathfrak{T}^p_q(\mathcal{M})$. $\blacksquare$

Corollary 3.4.38. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a map, $\boldsymbol{s} \in \mathfrak{T}^0_p(\mathcal{N})$ and $\boldsymbol{t} \in \mathfrak{T}^0_q(\mathcal{N})$, then $\phi^\star(\boldsymbol{s} \otimes \boldsymbol{t}) = \phi^\star \boldsymbol{s} \otimes \phi^\star \boldsymbol{t}$.*

Note that if $\phi = \mathrm{Id} : \mathcal{M} \to \mathcal{M}$ is the identity map, then the pushforward of a tensor reduces to the transformation under a change of coordinates. Therefore, a differentiable map ϕ of manifolds provides the natural analogon of a linear transformation of vector spaces discussed in section 2.1.

One is now able to obtain expressions of tensor fields in local coordinates.

Definition 3.4.39. Let $\boldsymbol{v} \in \Gamma(T\mathcal{N})$ be a vector field and $(\mathcal{V}, x)$ a chart on $\mathcal{N}$, $\dim(\mathcal{N}) = n$, and let $y \in \mathbb{R}^n$ and $\{\boldsymbol{e}_i\} \in \Gamma(T\mathbb{R}^n)$ the canonical basis vector field, then

$$(x_\star \boldsymbol{v})(y) = Tx \circ \boldsymbol{v}(x^{-1}(y)) = \{v^1(y), \ldots, v^n(y)\} = v^i(y)\, \boldsymbol{e}_i$$

is called the *local representative of $\boldsymbol{v}$ in the chart* at every $x^{-1}(y) \equiv P \in \mathcal{V} \subset \mathcal{N}$. The tangent map Tx has been defined in 3.4.9(i) resp. 3.4.10. ◇

Corollary 3.4.40. *Let $\mathcal{N}$, $\boldsymbol{v}$, x etc., be as before. Since Tx is an isomorphism on each fibre of $T\mathcal{N}$, $x_\star$ also is, and thus $x^\star = (x^{-1})_\star = (x_\star)^{-1}$ does exist:*

$$x^\star(v^i \boldsymbol{e}_i) = (Tx)^{-1} \circ \{v^1, \ldots, v^n\} \circ x = \boldsymbol{v}\,.$$

The local representative of 1-forms is defined analogously by adopting 3.4.35 and replacing ϕ by x.

Corollary 3.4.41. *Let $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}^i\}$ be the canonical basis sections of $T\mathbb{R}^n$ and its duals, respectively, then from 3.2.14 and 3.4.15, $x^\star(\boldsymbol{e}_i) = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ and $x^\star(\boldsymbol{e}^i) = \mathbf{d}x^i$, $i = 1, \ldots, n$.*

Proposition 3.4.42. *Let $(\mathcal{V}, x)$, $\mathcal{V} \subset \mathcal{N}$, be a chart and $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{N})$ a tensor field, then the local representative of $\boldsymbol{t}$ on $\mathcal{V}$ is*

$$\boldsymbol{t}|_{\mathcal{V}} = t^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\mu_1}} \otimes \ldots \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\mu_p}} \otimes \mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_q}\,.$$

Proof. The components of $\boldsymbol{t}$ at every $P \in \mathcal{V}$ are the scalar functions

$$t^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q} = \boldsymbol{t}\left(\mathbf{d}x^{\mu_1}, \ldots, \mathbf{d}x^{\mu_p}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu_1}}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu_q}}\right)\,.$$

Setting $y = x(P)$ and applying 3.4.39, the local representative $x_\star \boldsymbol{t}$ on the subset $x(\mathcal{V}) \subset \mathbb{R}^{n_{\dim}}$ would be

$$x_\star(\boldsymbol{t}|_{\mathcal{V}}) = (t^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q} \circ x^{-1})\, \boldsymbol{e}_{\mu_1} \otimes \ldots \otimes \boldsymbol{e}_{\mu_p} \otimes \boldsymbol{e}^{\nu_1} \otimes \ldots \otimes \boldsymbol{e}^{\nu_q}\,,$$

Application of 3.4.38 and 3.4.41 at every point then proofs the statement. ■

3.5 Exterior Algebra of Differential Forms

Exterior algebra deals with the cross product in three-dimensional vector spaces and with determinants. On manifolds, exterior algebra is a basic ingredient to detect symmetries and orientations, in order to establish measures in n-dimensional non-Euclidian spaces and to generalize integrals, balance equations and the theorems of Gauß and Stokes. Fundamental objects of exterior algebra are totally skew-symmetric covariant tensors, which are generally called *alternating multilinear forms* respectively *differential forms* on manifolds.

In this section, lower case Greeks are used for coordinate indices, and lower case Latins are used for labels.

Definition 3.5.1. A *permutation* of a set $\mathcal{S}_k = \{1, \ldots, k\} \subset \mathbb{N}$ is a map $\pi : \mathcal{S}_k \to \mathcal{S}_k$. Note that the set Π_k of all permutations $\{\pi(1), \ldots, \pi(k)\}$ on $\mathcal{S}_k$ has $k!$ elements. Define the *permutation of a k-tuple* of any elements $(a_1, \ldots, a_k)$ by

$$\pi(a_1, \ldots, a_k) = (a_{\pi(1)}, \ldots, a_{\pi(k)}) .$$

A *transposition* is a permutation that interchanges only two elements. Every permutation π is composed of even or odd numbers ν of transpositions. An *even (odd) permutation* is obtained from an even (odd) number of transpositions. Define the *signature of the permutation* by $\operatorname{sgn} : \Pi_k \to \{-1, +1\}$, then set $\operatorname{sgn}\pi = +1$ for even permutations, and $\operatorname{sgn}\pi = -1$ for odd permutations, so that $\operatorname{sgn}\pi = (-1)^\nu$. ♢

Definition 3.5.2. A permutation $\pi \in \Pi_{k,l} \subset \Pi_{k+l}$ of the set $\{1, \ldots, k, k+1, \ldots, k+l\} \subset \mathbb{N}$ is called a *shuffle permutation* or *(k,l)-shuffle*, if $\pi(1) < \ldots < \pi(k)$ and $\pi(k+1) < \ldots < \pi(k+l)$. ♢

Proposition 3.5.3. *The number of all shuffle permutations on $\{1, \ldots, k+l\}$ is*

$$\binom{k+l}{k} = \frac{(k+l)!}{k!\, l!} .$$

Proof. A shuffle permutation $\pi\{1, \ldots, k+l\} = \{\pi(1), \ldots, \pi(k+l)\}$ can be realized as follows. Choose k numbers without repetitions and put the last l numbers behind them. Reorder if the sequence of numbers does not satisfy $\pi(1) < \ldots < \pi(k)$ and $\pi(k+1) < \ldots < \pi(k+l)$.

Therefore, the first k of $k+l$ numbers chosen completely determine a shuffle permutation, and allowing for reordering means that the order does not matter. However, the number of such combinations without repetitions is just the binomial coefficient $\binom{k+l}{k}$, so the assertion follows. ■

Definition 3.5.4. Let $\boldsymbol{T}(P) \in T^0_k(\mathcal{M})$ be a $\binom{0}{k}$-tensor at $P \in \mathcal{M}$, i.e. a multilinear mapping

$$\boldsymbol{T}(P) : \underbrace{T_P\mathcal{M} \times T_P\mathcal{M} \times \ldots \times T_P\mathcal{M}}_{k-\text{fold}} \quad \to \quad \mathbb{R} .$$

The *alternation mapping* $\mathsf{Alt} : T^0_k(\mathcal{M}) \to \bigwedge^k T^*\mathcal{M}$, where $\bigwedge^k$ is called *k-th exterior power*, is pointwisely defined by

$$\mathsf{Alt}\, \boldsymbol{T}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \frac{1}{k!} \sum_{\pi \in \Pi_k} (\operatorname{sgn} \pi)\, \boldsymbol{T}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)})\,,$$

for every $P \in \mathcal{M}$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in T_P\mathcal{M}$. π denotes the permutation on $\{1, \ldots, k\}$ and the factor $1/k!$ is a convention to avoid double counts. $\diamondsuit$

It should be clear that $\bigwedge^k T^*\mathcal{M}$ is a subspace of $T^0_k(\mathcal{M})$. If n is the dimension of $\mathcal{M}$, then the dimension of $\bigwedge^k T^*\mathcal{M}$ is $\binom{n}{k}$.

Definition 3.5.5. A *(differential) k-form* $\boldsymbol{\omega}(P) \in \bigwedge^k T^*_P\mathcal{M}$ at $P \in \mathcal{M}$ is a $\binom{0}{k}$-tensor such that $\mathsf{Alt}\, \boldsymbol{\omega} = \boldsymbol{\omega}$. $\diamondsuit$

Corollary 3.5.6. *For* $\boldsymbol{T} \in T^0_k(\mathcal{M})$, $\mathsf{Alt}\, \boldsymbol{T} \in \bigwedge^k T^*\mathcal{M}$, *thus* $\mathsf{Alt}(\mathsf{Alt}\, \boldsymbol{T}) = \mathsf{Alt}\, \boldsymbol{T}$ *by 3.5.5.*

Proposition 3.5.7. *Let* $\boldsymbol{\omega}(P) \in \bigwedge^k T^*_P\mathcal{M}$ *be a k-form and* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in T_P\mathcal{M}$, *then*

$$\boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}) = (\operatorname{sgn} \pi)\, \boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)\,.$$

Proof. First, note that $(\operatorname{sgn} \pi)^2 = 1$ and the number of all permutations on $\{1, \ldots, k\}$ is $k!$, that is, $\sum_{\pi \in \Pi_k} a = k!\, a$. Then from $\mathsf{Alt}\, \boldsymbol{\omega} = \boldsymbol{\omega}$,

$$\begin{aligned} \frac{1}{k!} \sum_{\pi \in \Pi_k} (\operatorname{sgn} \pi)\, \boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}) &= \boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \\ &= \frac{1}{k!} \sum_{\pi \in \Pi_k} (\operatorname{sgn} \pi)^2\, \boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)\,. \end{aligned}$$

Comparing both sides gives the result. ∎

Definition 3.5.8. Let $\boldsymbol{T}(P) \in T^0_k(\mathcal{M})$ and $\boldsymbol{S}(P) \in T^0_l(\mathcal{M})$ be tensors, then the *exterior product* or *wedge product* $(\boldsymbol{T} \wedge \boldsymbol{S}) \in \bigwedge^{k+l} T^*_P\mathcal{M}$ is a $(k+l)$-form defined by

$$\boldsymbol{T} \wedge \boldsymbol{S} = \frac{(k+l)!}{k!\, l!}\, \mathsf{Alt}\,(\boldsymbol{T} \otimes \boldsymbol{S})\,,$$

where the point arguments have been omitted. The wedge product for differential forms $\boldsymbol{\omega}(P) \in \bigwedge^k T^*_P\mathcal{M}$ and $\boldsymbol{\beta}(P) \in \bigwedge^l T^*_P\mathcal{M}$ is defined in the same manner. $\diamondsuit$

Corollary 3.5.9. *Let* $\boldsymbol{a}^*, \boldsymbol{b}^* \in T^*_P\mathcal{M}$ *be 1-forms, then*

$$\begin{aligned} \boldsymbol{a}^* \wedge \boldsymbol{b}^* = \frac{(1+1)!}{1!1!} \frac{1}{2!} (\boldsymbol{a}^* \otimes \boldsymbol{b}^* - \boldsymbol{b}^* \otimes \boldsymbol{a}^*) &= \boldsymbol{a}^* \otimes \boldsymbol{b}^* - \boldsymbol{b}^* \otimes \boldsymbol{a}^* \\ &= -(\boldsymbol{b}^* \otimes \boldsymbol{a}^* - \boldsymbol{a}^* \otimes \boldsymbol{b}^*) = -\boldsymbol{b}^* \wedge \boldsymbol{a}^*\,. \end{aligned}$$

Proposition 3.5.10. *Let $\boldsymbol{\omega}(P) \in \bigwedge^k T_P^*\mathcal{M}$ be a k-form, $\boldsymbol{\beta}(P) \in \bigwedge^l T_P^*\mathcal{M}$ an l-form and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+l} \in T_P\mathcal{M}$ vectors, then*

$$(\boldsymbol{\omega} \wedge \boldsymbol{\beta})(P)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+l}) = \sum_{\pi \in \Pi_{k,l}} (\operatorname{sgn} \pi)\, \boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}) \boldsymbol{\beta}(\boldsymbol{v}_{\pi(k+1)}, \ldots, \boldsymbol{v}_{\pi(k+l)})\,,$$

where $\sum_{\pi \in \Pi_{k,l}}$ denotes the sum over all (k,l)-shuffles π on $\{1, \ldots, k+l\}$.

Proof. By 3.5.7, alternation of $\boldsymbol{\omega}$ (or $\boldsymbol{\beta}$) reduces to the sum with a single summand:

$$\begin{aligned}\mathsf{Alt}\,\boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) &= \frac{1}{k!} \sum_{\pi \in \Pi_k} (\operatorname{sgn} \pi)\, \boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}) \\ &= \sum_{\pi(1)<\ldots<\pi(k)} \boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)})\,.\end{aligned}$$

Therefore, the sum in $\mathsf{Alt}\,(\boldsymbol{\omega} \otimes \boldsymbol{\beta}) = \mathsf{Alt}\,(\mathsf{Alt}\,\boldsymbol{\omega} \otimes \mathsf{Alt}\,\boldsymbol{\beta})$ is only over the shuffle permutations that satisfy $\pi(1) < \ldots < \pi(k)$ and $\pi(k+1) < \ldots < \pi(k+l)$. From 3.5.3, the number of all shuffle permutations is $\frac{(k+l)!}{k!\,l!}$, so

$$\begin{aligned}&\mathsf{Alt}\,(\boldsymbol{\omega} \otimes \boldsymbol{\beta})(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+l}) \\ &= \frac{k!\,l!}{(k+l)!} \sum_{\pi \in \Pi_{k,l}} (\operatorname{sgn} \pi)\, \boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}) \boldsymbol{\beta}(\boldsymbol{v}_{\pi(k+1)}, \ldots, \boldsymbol{v}_{\pi(k+l)})\,,\end{aligned}$$

where the definition 3.3.10 of the tensor product has been used. Substitution into 3.5.8 then gives the result. ∎

Proposition 3.5.11. *(Without proof.) For $a \in \mathbb{R}$ and forms $\boldsymbol{\omega} \in \bigwedge^k T_P^*\mathcal{M}$, $\boldsymbol{\beta} \in \bigwedge^l T_P^*\mathcal{M}$ and $\boldsymbol{\gamma} \in \bigwedge^m T_P^*\mathcal{M}$, the exterior product has the properties*

(i) $a(\boldsymbol{\omega} \wedge \boldsymbol{\beta}) = a\boldsymbol{\omega} \wedge \boldsymbol{\beta} = \boldsymbol{\omega} \wedge a\boldsymbol{\beta}$ (bilinearity),

(ii) $\boldsymbol{\omega} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = (\boldsymbol{\omega} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma}$ (associativity), and

(iii) $\boldsymbol{\omega} \wedge \boldsymbol{\beta} = (-1)^{kl} \boldsymbol{\beta} \wedge \boldsymbol{\omega}$ (supercommutativity).

Corollary 3.5.12. *For a 1-form $\boldsymbol{\omega} \in T_P^*\mathcal{M}$, $\boldsymbol{\omega}\wedge\boldsymbol{\omega} = 0$ by 3.5.11(iii) (see also corollary 3.5.9). However, if $\boldsymbol{\omega} \in \bigwedge^k T_P^*\mathcal{M}$, then $\boldsymbol{\omega} \wedge \boldsymbol{\omega} \neq 0$ in general.*

Corollary 3.5.13. *Recall 3.3.4 for the components of a tensor, and let $(\mathcal{U}, x)$, with $\mathcal{U} \subset \mathcal{M}$ a subset, be a chart with coordinate functions $\{x^\mu\}$.*

(i) If $\boldsymbol{T} \in T_2^0(\mathcal{M})$ is a $\binom{0}{2}$-tensor, the components of $\mathsf{Alt}\,\boldsymbol{T}$ are

$$\begin{aligned}&(\mathsf{Alt}\,\boldsymbol{T})_{\mu\nu} = \mathsf{Alt}\,\boldsymbol{T}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\mu}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}\right) \\ &= \frac{1}{2}\left(\boldsymbol{T}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\mu}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}\right) - \boldsymbol{T}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\mu}\right)\right) = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)\,.\end{aligned}$$

(ii) Let $\boldsymbol{\alpha} \in \bigwedge^2 T_P^*\mathcal{M}$ be a 2-form, then 3.5.7 yields

$$\alpha_{\mu\nu} = \boldsymbol{\alpha}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\mu}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}\right) = -\boldsymbol{\alpha}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\mu}\right) = -\alpha_{\nu\mu}\,.$$

(iii) Let $\boldsymbol{\omega} \in \bigwedge^k T_P^*\mathcal{M}$ *and* $\boldsymbol{\beta} \in \bigwedge^l T_P^*\mathcal{M}$*, then 3.5.10 gives*

$$(\boldsymbol{\omega} \wedge \boldsymbol{\beta})_{\nu_1 \ldots \nu_{k+l}} = \sum_{\pi \in \Pi_{k,l}} (\operatorname{sgn} \pi)\, \omega_{\pi(\nu_1)\ldots\pi(\nu_k)}\, \beta_{\pi(\nu_{k+1})\ldots\pi(\nu_{k+l})}\,,$$

where $\sum_{\pi \in \Pi_{k,l}}$ *again denotes the sum over all* (k,l)*-shuffles in* Π_{k+l}.

Proposition 3.5.14. *Let* $P \in \mathcal{U} \subset \mathcal{M}$, $\dim(\mathcal{M}) = n$ *and* $(\mathcal{U}, x)$ *a chart such that* $x(P) = \{x^\nu\}_P$.

(i) A local representative of $\boldsymbol{\omega}(P) \in \bigwedge^k T_P^*\mathcal{M}$*,* $k \le n$*, is*

$$\boldsymbol{\omega}(P) = \sum_{\nu_1 < \ldots < \nu_k} \omega_{\nu_1 \ldots \nu_k}(P)\, \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}\,.$$

(ii) $\{\mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}\}$*, with* $1 \le \nu_1 < \ldots < \nu_k \le n$*, is a basis of* $\bigwedge^k T_P^*\mathcal{M}$*, which therefore has the dimension* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. (i) Through $\bigwedge^k T^*\mathcal{M} \subset T_k^0(\mathcal{M})$, every k-form is a $\binom{0}{k}$-tensor that has a local representative

$$\boldsymbol{\omega} = \omega_{\nu_1 \ldots \nu_k} \mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_k}$$

in a chart $(\mathcal{U}, x)$, $\mathcal{U} \subset \mathcal{M}$, by 3.3.11. Here $\omega_{\nu_1 \ldots \nu_k} = \boldsymbol{\omega}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu_1}}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu_k}}\right)$, and $\{\mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_k}\}$ is the tensor basis of $T_k^0(\mathcal{M})$ at $P \in \mathcal{U}$. Alternation and 3.5.8 then gives

$$\boldsymbol{\omega} = \mathsf{Alt}\, \boldsymbol{\omega} = \omega_{\nu_1 \ldots \nu_k} \mathsf{Alt}\,(\mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_k}) = \frac{1}{k!} \omega_{\nu_1 \ldots \nu_k} \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}\,.$$

Note that the Einstein summation convention is still in force and the sum runs over all choices of indices $\nu_1, \ldots, \nu_k \in \{1, \ldots, n\}$, including those where not all indices are distinct. However, in that latter case, $\omega_{\nu_1 \ldots \nu_k} = 0$ from 3.5.12. For the other cases where $\nu_1, \ldots, \nu_k$ are distinct, applying 3.5.7 for both $\boldsymbol{\omega}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu_1}}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu_k}}\right)$ and $\mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}$ yields

$$\omega_{\nu_1 \ldots \nu_k} \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k} = \omega_{\pi(\nu_1) \ldots \pi(\nu_k)} \mathbf{d}x^{\pi(\nu_1)} \wedge \ldots \wedge \mathbf{d}x^{\pi(\nu_k)}$$

for any permutation $\pi \in \Pi_k$ and by noting that $(\operatorname{sgn} \pi)^2 = 1$. Now, since rearranging the indices does not change $\omega_{\nu_1 \ldots \nu_k} \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}$, the sum needs to be only over one of the $k!$ index permutations. Choose $\nu_1 < \ldots < \nu_k$ to finally get $\boldsymbol{\omega} = \sum_{\nu_1 < \ldots < \nu_k} \omega_{\nu_1 \ldots \nu_k} \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}$ as desired.

(ii) Linear independency of the $\mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}$ can be shown by using the complementary set $\mathbf{d}x^{\nu_{k+1}}, \ldots, \mathbf{d}x^{\nu_n}$ and the condition $(\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n)\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right) = 1$ (see, for example, [1], p. 332). As the number of ways that k coordinate differentials can be chosen from n coordinate differentials regardless of order (equivalently, with one fixed order) is just $\binom{n}{k}$, the assertion follows. ∎

Corollary 3.5.15. *Let* $\boldsymbol{\omega}$ *be a* k*-form and* $\boldsymbol{\beta}$ *an* l*-form, then in a coordinate system* $\{x^\nu\}$*, the local representative of* $\boldsymbol{\omega} \wedge \boldsymbol{\beta} = (\mathsf{Alt}\, \boldsymbol{\omega}) \wedge (\mathsf{Alt}\, \boldsymbol{\beta})$ *is*

$$\boldsymbol{\omega} \wedge \boldsymbol{\beta} = \frac{1}{k!\, l!} \omega_{\nu_1 \ldots \nu_k} \beta_{\nu_{k+1} \ldots \nu_{k+l}} \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_k} \wedge \mathbf{d}x^{\nu_{k+1}} \wedge \ldots \wedge \mathbf{d}x^{\nu_{k+l}}\,.$$

Definition 3.5.16. Let $\boldsymbol{\sigma}^* : \mathcal{M} \to T^*\mathcal{M}$ denote the sections of the cotangent bundle $T^*\mathcal{M}$ of a manifold $\mathcal{M}$, then a *k-form field* is defined as a section of the k-th exterior power of the cotangent bundle,

$$\boldsymbol{\omega} = \mathsf{Alt}\,(\boldsymbol{\sigma}^*_1 \otimes \boldsymbol{\sigma}^*_2 \otimes \ldots \otimes \boldsymbol{\sigma}^*_k) \in \Gamma(\textstyle\bigwedge^k T^*\mathcal{M})\,,$$

where $\Gamma(\bigwedge^k T^*\mathcal{M})$ is usually written $\Omega^k(\mathcal{M})$. $\diamondsuit$

Note that in the literature, the term "k-form" is generally reserved for the sections of $\bigwedge^k T^*\mathcal{M}$, and not for the totally skew-symmetric covariant tensors at single points of a manifold. This is because of their predominance in differential and integral calculus. By abuse of language, "k-form" will be adopted for both elements of $\bigwedge^k T^*\mathcal{M}$ and $\Omega^k(\mathcal{M})$ in the paper, as long as the meaning will be clear from the context.

Proposition 3.5.17. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a map, $\boldsymbol{\omega} \in \Omega^k(\mathcal{N})$ and $\boldsymbol{\beta} \in \Omega^l(\mathcal{N})$, then $\phi^\star(\boldsymbol{\omega} \wedge \boldsymbol{\beta}) = \phi^\star\boldsymbol{\omega} \wedge \phi^\star\boldsymbol{\beta}$.*

Proof. Every differential form constitutes a covariant tensor, so 3.4.38 can be applied, and alternation commutes with the pullback. ■

Definition 3.5.18. For $\boldsymbol{\omega} \in \Omega^k(\mathcal{M})$, define the *interior product* or *degree-1 derivation* ($\neq$ inner product!) by contracting a vector field $\boldsymbol{u} \in \Gamma(T\mathcal{M})$ with the first index of $\boldsymbol{\omega}$ at every $P \in \mathcal{M}$:

$$\begin{aligned} \boldsymbol{i_u} :\ \Omega^k(\mathcal{M}) &\quad\to\quad \Omega^{k-1}(\mathcal{M}) \\ \boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) &\quad\mapsto\quad \boldsymbol{i_u\omega}(\boldsymbol{v}_2, \ldots, \boldsymbol{v}_k) = \boldsymbol{\omega}(\boldsymbol{u}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)\,. \end{aligned}$$

In local coordinates $\{x^\nu\}$,

$$\boldsymbol{i_u\omega} = \frac{1}{(k-1)!}\, u^\alpha \omega_{\alpha\nu_2\ldots\nu_k} \mathbf{d}x^{\nu_2} \wedge \ldots \wedge \mathbf{d}x^{\nu_k}\,. \qquad \diamondsuit$$

Proposition 3.5.19. *(Without proof.) If $\boldsymbol{\omega}$ is a k-form and $\boldsymbol{\beta}$ an l-form, then*

$$\boldsymbol{i_u}\,(\boldsymbol{\omega} \wedge \boldsymbol{\beta}) = \boldsymbol{i_u\omega} \wedge \boldsymbol{\beta} + (-1)^k \boldsymbol{\omega} \wedge \boldsymbol{i_u\beta}\,.$$

3.6 Differentiation on Manifolds

So far only the algebra of tensor fields on manifolds has been analyzed. However, in continuum mechanics, tensor fields generally depend on time and location. Due to the various geometric facilities on manifolds, beyond partial derivatives there are three possible types of differential calculi: covariant differentiation, Lie differentiation and exterior differentiation. Each is appropriate for special problems, but all types have to satisfy Leibniz' rule

$$D(\boldsymbol{AB}) = (D\boldsymbol{A})\boldsymbol{B} + \boldsymbol{A}(D\boldsymbol{B})\,.$$

D is the differential operator under consideration and $\boldsymbol{A}$, $\boldsymbol{B}$ are arbitrary tensor fields. The so-called exterior derivative $\mathbf{d}$ satisfies the general Leibniz' rule

$$\mathbf{d}(\boldsymbol{\omega} \wedge \boldsymbol{\beta}) = \mathbf{d}\boldsymbol{\omega} \wedge \boldsymbol{\beta} + (-1)^k \boldsymbol{\omega} \wedge \mathbf{d}\boldsymbol{\beta}\,,$$

where $\boldsymbol{\omega} \in \Omega^k(\mathcal{M})$ and $\boldsymbol{\beta}$ is an arbitrary form. First, the covariant derivative is being briefly introduced.

3.6.1 Covariant Derivative

Definition 3.6.1. Let $\mathcal{U} \subset \mathcal{M}$ be a subset. A curve $c : \mathcal{I} \to \mathcal{M}$, where $\mathcal{I} \subset \mathbb{R}$ is open, is called *integral curve* of a vector field $\boldsymbol{w} : \mathcal{U} \to T\mathcal{M}$, if $\boldsymbol{w}(c(t)) = \mathrm{d}c/\mathrm{d}t(t)$ is a tangent vector of the curve for every $t \in \mathcal{I}$. Clearly, $(\boldsymbol{w} \circ c)(t) = Tc(t, 1)$, where

$$\begin{aligned} Tc :\ T\mathcal{I} = \mathcal{I} \times \mathbb{R} \quad &\to \quad T\mathcal{M} \\ (t, s) \quad &\mapsto \quad Tc(t, s) = \left(c(t), \frac{\mathrm{d}c}{\mathrm{d}t}(t)\, s \right)\,, \end{aligned}$$

and $Tc(\cdot, 1) : \mathcal{I} \to T\mathcal{M}$. ◊

Corollary 3.6.2. *Let $c, \boldsymbol{w}$ be as in 3.6.1. Moreover, let $\{x^i\}_P = x(P)$ be the local coordinates of $P = c(t) \in \mathcal{U} \subset \mathcal{M}$ in a chart $(\mathcal{U}, x)$, $\boldsymbol{w} = w^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ and $c^i = x^i \circ c$. Then*

$$w^i(c(t)) = \mathbf{d}x^i \cdot \boldsymbol{w}(c(t)) = \mathrm{d}x^i(Tc(t, 1)) = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) = \dot{c}^i(t)\,,$$

by noting that $\mathrm{d}c^i/\mathrm{d}t = \mathrm{d}x^i(\mathrm{d}c/\mathrm{d}t)$ *through 3.2.5 and 3.2.7.*

Definition 3.6.3. A vector field $\boldsymbol{v} : \mathcal{M} \to T\mathcal{M}$ defined along a curve $c : \mathcal{I} \to \mathcal{M}$ is called *locally parallel transported along the curve*, if there is an *(affine) connection* $\boldsymbol{\nabla} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ on $\mathcal{M}$, so that

$$\boldsymbol{\nabla}_{\dot{c}} \boldsymbol{v}(P) = \mathbf{0}\,.$$

Vector fields $\boldsymbol{v}$ which fulfill this condition are called *covariant-constant.* ◊

Definition 3.6.4. Manifolds with connection according to 3.6.3 are called *affinely connected.* ◊

In general, parallelism is not global —this holds only for Euclidian point spaces. A locally parallel transported tangent vector of a curve remains tangent to the curve. Indeed, the main difference between Euclidian and non-Euclidian geometry rests on the fact that for the latter any geometric consideration is local, whereas the former implies the existence of a global parallelism.

The abstract definition 3.6.3 should now be motivated step by step. The total differential $\mathrm{d}\boldsymbol{v}$ of a vector field does not transform as a tensor, hence avoiding invariant results. This drawback is eliminated by defining the absolute differential.

Definition 3.6.5. Let $v^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ be the local representative of a vector field $\boldsymbol{v} \in \Gamma(T\mathcal{M})$ in a chart $(\mathcal{U}, x)$, where $\mathcal{U} \subset \mathcal{M}$. Then

$$\mathrm{D}v^i(P) = \mathrm{d}v^i + v^k \mathrm{d}x^j \mathit{\Gamma}_k{}^i{}_j$$

are referred to as the *absolute differentials of the components of* $\boldsymbol{v}$ *at* $P \in \mathcal{U}$. If $\boldsymbol{a}^* = a_i \, \mathbf{d}x^i \in \Gamma(T^*\mathcal{M})$ is a 1-form, then define

$$\mathrm{D}a_i(P) = \mathrm{d}a_i + a_j \mathrm{d}x^k \mathit{\Gamma}_k{}^j{}_i$$

in the same chart. The $\mathit{\Gamma}_k{}^j{}_i$ are called the *connection coefficients*.

In Riemannian spaces $(\mathcal{M}, \boldsymbol{g})$, the $\mathit{\Gamma}_k{}^j{}_i$ are called *Christoffel symbols of the second kind*. As they depend on the metric coefficients g_{ij} and g^{ij} in a chart $(\mathcal{U}, x)$ of $\mathcal{M}$ via

$$\mathit{\Gamma}_i{}^k{}_j = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) ,$$

which is not shown here, one has $\mathit{\Gamma}_k{}^j{}_i = \mathit{\Gamma}_i{}^j{}_k$. For that reason Riemannian spaces are called *torsion-free*, and the connection $\boldsymbol{\nabla}$ on $(\mathcal{M}, \boldsymbol{g})$ is referred to as the *Levi-Civita connection.* ◊

Definition 3.6.6. Let $(\mathcal{U}, x)$, $(\mathcal{U}', x')$ be regular charts on $\mathcal{M}$, and $x' \circ x^{-1}|_{x(\mathcal{U} \cap \mathcal{U}')}$: $x\,(\mathcal{U} \cap \mathcal{U}') \to x'\,(\mathcal{U} \cap \mathcal{U}')$ the chart transition map, then define

$$\mathit{\Gamma}_k{}^j{}_i = \frac{\partial(x^{k'} \circ x^{-1})}{\partial x^k} \frac{\partial(x^j \circ x'^{-1})}{\partial x^{j'}} \frac{\partial(x^{i'} \circ x^{-1})}{\partial x^i} \mathit{\Gamma}_{k'}{}^{j'}{}_{i'} + \frac{\partial(x^j \circ x'^{-1})}{\partial x^{m'}} \frac{\partial^2(x^{m'} \circ x^{-1})}{\partial x^k \partial x^i} \,. \qquad \diamond$$

Corollary 3.6.7. *The absolute differentials* $\mathrm{D}v^i$ *are proper tensor components.*

Definition 3.6.8. Let $\boldsymbol{v} = v^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \in \Gamma(T\mathcal{M})$ a vector field, then the

$$\nabla_i v^j(P) = \frac{(v^j + \mathrm{D}v^j) - v^j}{\partial x^i}(P) = \frac{\mathrm{D}v^j}{\partial x^i} = \frac{\partial v^j}{\partial x^i} + v^k \mathit{\Gamma}_k{}^j{}_i$$

are called the *covariant derivatives* of the components v^i at every $P \in \mathcal{M}$. It is not uncommon to write $v^i{}_{|j}$ or $v^i{}_{;j}$ instead of $\nabla_j v^i$. ◊

Definition 3.6.9. Since the $\nabla_i v^j$ are components of a $\binom{1}{1}$-tensor field, define the *covariant derivative of* $\boldsymbol{v}$ pointwisely through

$$\boldsymbol{\nabla v}(P) = \nabla_i v^j \; \mathbf{d}x^i \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} \qquad \in \mathfrak{T}^1_1(\mathcal{M}) \,.$$

Let $\boldsymbol{w}$ be another vector field, then

$$\boldsymbol{\nabla_w v} = \boldsymbol{\nabla v}(\boldsymbol{w}) = \nabla_j v^i w^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \left(\frac{\partial v^i}{\partial x^j} w^j + v^k w^j \mathit{\Gamma}_k{}^i{}_j \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$$

is called the *covariant derivative of* $\boldsymbol{v}$ *along* $\boldsymbol{w}$. Note that the latter gives a condition for the $\mathit{\Gamma}_i{}^j{}_k$:

$$\boldsymbol{\nabla}_{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} = \frac{\partial}{\partial x^j} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \right) = \mathit{\Gamma}_k{}^i{}_j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \,. \qquad \diamond$$

Descriptively, the covariant derivative is the deviation of the locally parallel transported vector from the original vector in the vicinity of a point.

Proposition 3.6.10. *For covariant-constant vector fields the absolute differentials of their components vanish, that is, in a coordinate system* $\{x^i\}$, $\mathrm{D}v^i = 0$ *and*

$$\frac{\partial v^i}{\partial x^j} = -v^k \mathit{\Gamma}_{k\ j}^{\ i} \,.$$

Proof. In 3.6.9, set $\boldsymbol{w} = \dot{c}$ and use the condition 3.6.3, then

$$\boldsymbol{\nabla}_{\dot{c}} \boldsymbol{v} = \dot{c}^j \left(\frac{\partial v^i}{\partial x^j} + v^k \mathit{\Gamma}_{k\ j}^{\ i} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \mathbf{0} \,.$$

As $\dot{c}$ is arbitrary and the $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ are linearly independent, the assertion follows. ■

Corollary 3.6.11. *Let* $\boldsymbol{a}^* : \mathcal{M} \to T^*\mathcal{M}$ *a field of 1-forms,* $\boldsymbol{T} \in \mathfrak{T}_1^1(\mathcal{M})$ *and* $P \in \mathcal{U} \subset \mathcal{M}$*. By defining*

$$\boldsymbol{\nabla} \boldsymbol{a}^*(P) = \left(\frac{\partial a_i}{\partial x^j} - a_k \mathit{\Gamma}_{i\ j}^{\ k} \right) \mathbf{d}x^i \otimes \mathbf{d}x^j$$

in a chart $(\mathcal{U}, x)$*, one has*

$$\nabla_i T^j{}_k(P) = \frac{\partial T^j{}_k}{\partial x^i} + T^l{}_k \mathit{\Gamma}_{l\ i}^{\ j} - T^j{}_l \mathit{\Gamma}_{k\ i}^{\ l} \,,$$

and

$$\boldsymbol{\nabla}_{\boldsymbol{w}} \boldsymbol{T}(P) = \nabla_k T^i{}_j w^k \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j$$
$$= \left(\frac{\partial T^i{}_j}{\partial x^k} w^k + T^l{}_j w^k \mathit{\Gamma}_{l\ k}^{\ i} - T^i{}_l w^k \mathit{\Gamma}_{j\ k}^{\ l} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j \,.$$

Proposition 3.6.12. *Let* $(\mathcal{M}, \boldsymbol{g})$, $\boldsymbol{g} = g_{ij} \mathbf{d}x^i \otimes \mathbf{d}x^j$*, be a metric space with connection* $\boldsymbol{\nabla}$*, then*

$$\boldsymbol{\nabla} \boldsymbol{g} = \mathbf{0} \,.$$

Proof. From 3.6.11 and 3.6.5,

$$\nabla_i g_{jk} = \frac{\partial g_{jk}}{\partial x^i} - 2 g_{jl} \mathit{\Gamma}_{k\ i}^{\ l} = \frac{\partial g_{jk}}{\partial x^i} - \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j} \right) = 0 \,.$$ ■

The covariant derivative can also be applied to vector and tensor fields over maps, from which the following important result can be obtained.

Theorem 3.6.13 (Induced Connection). *(See also [3], sec. 5.7) Let* $\mathcal{N}$ *be a manifold with connection* $\boldsymbol{\nabla}$ *and* $\boldsymbol{v} \in \Gamma(T\mathcal{N})$*. A regular map* $\phi : \mathcal{M} \to \mathcal{N}$ *induces a unique connection* $\boldsymbol{\nabla}^\star$ *on* $\mathcal{M}$ *such that for* $\boldsymbol{t} \in T_P\mathcal{M}$*, with* $P \in \mathcal{M}$*, one has*

$$\boldsymbol{\nabla}^\star_{\boldsymbol{t}} (\boldsymbol{v} \circ \phi) = \boldsymbol{\nabla}_{T\phi(\boldsymbol{t})} \boldsymbol{v} \quad \in \Gamma(\phi^\star T\mathcal{N}) \,,$$

where $(\boldsymbol{v} \circ \phi) : \mathcal{M} \to T\mathcal{N}$ *is the corresponding vector field over* ϕ*.*

Proof. Let $P \in \mathcal{U} \subset \mathcal{M}$ and $Q \in \mathcal{V} \subset \mathcal{N}$, and let $(\mathcal{U}, X)$, $(\mathcal{V}, x)$, respectively, be appropriate charts such that $\boldsymbol{t}(P) = t^I \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}$ and $\boldsymbol{v}(Q) = v^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$. Moreover, let $F^i{}_I(P) = \frac{\partial x^i \circ \phi \circ X^{-1}}{\partial X^I}(P)$ be the components of the tangent map $T\phi : T\mathcal{M} \to T\mathcal{N}$, and $\gamma_j{}^i{}_k(Q)$ the coefficients of $\boldsymbol{\nabla}$. By 3.6.9 and 3.4.23, locally there is

$$\boldsymbol{\nabla}_{T\phi(\boldsymbol{t})}\boldsymbol{v} = t^I F^k{}_I \left(\frac{\partial v^i}{\partial x^k} + v^j \gamma_j{}^i{}_k \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$$
$$= \left(\left(\frac{\partial v^i}{\partial x^k} \circ \phi \right) F^k{}_I t^I + \left(v^j \circ \phi \right) t^I \left(\gamma_j{}^i{}_k \circ \phi \right) F^k{}_I \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$$
$$= \left(\frac{\partial \left(v^i \circ \phi \right)}{\partial X^I} t^I + \left(v^j \circ \phi \right) t^I \gamma_j{}^i{}_I \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \boldsymbol{\nabla}^\star_{\boldsymbol{t}} (\boldsymbol{v} \circ \phi) ,$$

where $\gamma_j{}^i{}_I(P) = (\gamma_j{}^i{}_k \circ \phi) F^k{}_I$ are the coefficients of the induced connection $\boldsymbol{\nabla}^\star$ on $\mathcal{M}$. ■

Corollary 3.6.14. *Additional to the previous definitions, let $\mathit{\Gamma}_I{}^J{}_K(P)$ be the connection coefficients at every $P \in \mathcal{M}$, and $\boldsymbol{T} \in \Gamma(T^1_1(\mathcal{M}) \otimes \phi^\star T^1_1(\mathcal{N}))$ a $\binom{1\ 1}{1\ 1}$-two-point tensor field over $\phi : \mathcal{M} \to \mathcal{N}$. Use 3.6.11 and 3.6.13 to get*

(i) At $P \in \mathcal{M}$, the components of the covariant derivative of $\boldsymbol{T}$ are

$$\nabla_I T^J{}_K{}^i{}_j = \frac{\partial T^J{}_K{}^i{}_j}{\partial X^I} + T^L{}_K{}^i{}_j \mathit{\Gamma}_L{}^J{}_I - T^J{}_L{}^i{}_j \mathit{\Gamma}_K{}^L{}_I + T^J{}_K{}^k{}_j \gamma_k{}^i{}_l F^l{}_I - T^J{}_K{}^i{}_k \gamma_j{}^k{}_l F^l{}_I ,$$

where the point arguments have been omitted.

(ii) The covariant derivative of $\boldsymbol{T}$ at $P \in \mathcal{M}$ along a vector field $\boldsymbol{W} \in \mathit{\Gamma}(T\mathcal{M})$ reads

$$\boldsymbol{\nabla}_{\boldsymbol{W}} \boldsymbol{T} = \nabla_I T^J{}_K{}^i{}_j W^I \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J} \otimes \mathbf{d}X^K \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j$$
$$= \left(\frac{\partial T^J{}_K{}^i{}_j}{\partial X^I} W^I + T^L{}_K{}^i{}_j W^I \mathit{\Gamma}_L{}^J{}_I - T^J{}_L{}^i{}_j W^I \mathit{\Gamma}_K{}^L{}_I + T^J{}_K{}^k{}_j W^I \gamma_k{}^i{}_l F^l{}_I \right.$$
$$\left. - T^J{}_K{}^i{}_k W^I \gamma_j{}^k{}_l F^l{}_I \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J} \otimes \mathbf{d}X^K \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j .$$

Here the symbol $\boldsymbol{\nabla}$ should cover the covariant derivative of the $T_P\mathcal{M}$- and $T^*_P\mathcal{M}$-slots, as well as of the $T_{\phi(P)}\mathcal{N}$- and $T^*_{\phi(P)}\mathcal{N}$-slots, that is, $\boldsymbol{\nabla}$ in the above constitutes an operator for two-point-tensorial covariant differentiation. Again, the point arguments have been omitted.

(iii) Let $\boldsymbol{T}(P) = \boldsymbol{S} \circ \phi$, then, by noting that $\mathit{\Gamma}_J{}^K{}_I(P) = \mathit{\Gamma}_J{}^K{}_i(P) F^i{}_I(P)$, one has

$$\boldsymbol{\nabla}^\star_{\boldsymbol{W}} (\boldsymbol{S} \circ \phi)(P)$$
$$= W^I F^k{}_I \left(\left(\frac{\partial S^J{}_K{}^i{}_j}{\partial x^k} \circ \phi \right) + \left(S^L{}_K{}^i{}_j \circ \phi \right) \mathit{\Gamma}_L{}^J{}_k - \left(S^J{}_L{}^i{}_j \circ \phi \right) \mathit{\Gamma}_K{}^L{}_k \right.$$
$$\left. + \left(S^J{}_K{}^l{}_j \gamma_l{}^i{}_k \circ \phi \right) - \left(S^J{}_K{}^i{}_l \gamma_j{}^l{}_k \circ \phi \right) \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J} \otimes \mathbf{d}X^K \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j$$
$$= \boldsymbol{\nabla}_{T\phi(\boldsymbol{W})} \boldsymbol{S} .$$

The application to arbitrary $\binom{p}{q}$- and $\binom{p\ r}{q\ s}$-tensor fields is straightforward.

Definition 3.6.15. The *divergence* of an at least 1-fold contravariant tensor field $\boldsymbol{T}$ is defined as the contraction of the first covariant and last contravariant slots of $\boldsymbol{\nabla T}$. For example, let $\boldsymbol{T} \in \mathfrak{T}^2_0(\mathcal{M})$, then

$$(\operatorname{div}\boldsymbol{T})^i = \nabla_j T^{ij} = T^{ij}{}_{|j}\,,$$

and for a vector field $\boldsymbol{v}$,

$$\operatorname{div}\boldsymbol{v} = \nabla_i v^i = \frac{\partial v^i}{\partial x^i} + v^j \mathit{\Gamma}_j{}^i{}_i\,. \qquad \diamondsuit$$

Definition 3.6.16. A vector field $\boldsymbol{v}$ is called *solenoidal*, if $\operatorname{div}\boldsymbol{v} = 0$. $\diamondsuit$

Proposition 3.6.17. *If $(\mathcal{M}, \boldsymbol{g})$ is a Riemannian manifold with metric coefficients g_{ij}, and $(\mathcal{U}, x)$, $\mathcal{U} \subset \mathcal{M}$, a (positively oriented) chart, then*

$$\operatorname{div}\boldsymbol{v} = \frac{1}{\sqrt{\det\, g_{kl}}}\frac{\partial}{\partial x^i}\left(\sqrt{\det\, g_{kl}}\; v^i\right)\,.$$

Proof. The reader probably remembers from linear algebra that the (i,j)-th cofactor of a matrix $\boldsymbol{A} = (A_{ij})$ is defined through $\det\boldsymbol{A} = A_{ij}(\operatorname{Cof}\boldsymbol{A})^{ij}$, and that

$$\frac{\partial \det\boldsymbol{A}}{\partial A_{ij}} = (\operatorname{Cof}\boldsymbol{A})^{ij}\,,$$

assuming the determinant function to be differentiable. Therefore, $\frac{\partial \det g_{kl}}{\partial g_{ij}} = (\det g_{kl}) g^{ij}$, by noting that $\boldsymbol{g}^{-1}$ is symmetric, and

$$\frac{\partial \det g_{jk}}{\partial x^i} = \frac{\partial \det g_{jk}}{\partial g_{mn}}\frac{\partial g_{mn}}{\partial x^i} = (\det g_{kl})\, g^{mn}\frac{\partial g_{mn}}{\partial x^i}\,.$$

Substitution into $\mathit{\Gamma}_i{}^k{}_j = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$ yields

$$\mathit{\Gamma}_j{}^i{}_i = \frac{1}{2}g^{ikl}\frac{\partial g_{ik}}{\partial x^j} = \frac{1}{2\det g_{ik}}\frac{\partial \det g_{ik}}{\partial x^j} = \frac{1}{\sqrt{\det\, g_{ik}}}\frac{\partial\sqrt{\det\, g_{ik}}}{\partial x^j}\,,$$

so finally,

$$\operatorname{div}\boldsymbol{v} = \frac{\partial v^i}{\partial x^i} + v^j\frac{1}{\sqrt{\det\, g_{ik}}}\frac{\partial\sqrt{\det\, g_{ik}}}{\partial x^j} = \frac{1}{\sqrt{\det\, g_{kl}}}\frac{\partial}{\partial x^i}\left(\sqrt{\det\, g_{kl}}\; v^i\right)\,. \qquad \blacksquare$$

3.6.2 Lie Derivative

Another choice of calculus on manifolds is the Lie differentiation, which constitutes a measure for the change of tensor fields under action of maps, i.e. if the manifold evolves.

Definition 3.6.18. In a dynamical system, let the states of the system at time t be represented by points $P(t) \in \mathcal{M}$, in which $\mathcal{M}$ is then called the *state space* or *phase space*. At *starting time* s, $P(s) = P_s$ are the starting points that constitute the *initial state*. After some time $t - s$ has elapsed, the state changes to

$$\psi_{t,s}(P_s) = P(t)$$

at time t. The operator $\psi_{t,s}$ is referred to as the *time-dependent flow on* $\mathcal{M}$ provided that

$$\psi_{t,s} \circ \psi_{s,r} = \psi_{t,r} \qquad \text{and} \qquad \psi_{t,t} = \mathrm{Id}_{\mathcal{M}}\,. \qquad \diamond$$

Proposition 3.6.19. *A time-independent flow is a one-parameter group of mappings*

$$\psi : \ \mathcal{M} \times \mathcal{I} \ \rightarrow \ \mathcal{M}\,,$$

where $\mathcal{I} \subset \mathbb{R}$ is open, $\psi(P,t) = \psi_t(P)$ at fixed t and $\psi_t : \mathcal{M} \to \mathcal{M}$.

Proof. If the evolution of the dynamical system is time-independent, then the flow only depends on the difference $t - s$, i.e. $\psi_{t,s} = \psi_{t-s}$, and the terms "starting time" and "ending time" are meaningless. Setting $\psi_t = \psi_{t,0}$, 3.6.18 becomes

$$\psi_{t,0} \circ \psi_{s,0} = \psi_t \circ \psi_s = \psi_{t+s} \qquad \text{and} \qquad \psi_{t-t} = \psi_0 = \mathrm{Id}_{\mathcal{M}}\,.$$

By noting that $\psi_{t+s} = \psi_{s+t} = \psi_s \circ \psi_t$ also holds, ψ_t fulfills the group properties. ■

Recalling 3.6.1, every integral curve $c : \mathcal{I} \to \mathcal{M}$ defines a field of tangent vectors $\boldsymbol{u}(c(t)) = \dot{c}(t)$. However, in this section the converse statement is more interesting:

Proposition 3.6.20. *Let $\mathcal{U} \subset \mathcal{M}$ be n-dimensional, and $\boldsymbol{u} : \mathcal{U} \to T\mathcal{M}$ a C^k vector field, where $k \geq 1$, i.e. $\boldsymbol{u}$ has k continuous derivatives. Then there exists a $\mathbb{R} \supset \mathcal{I} \neq \emptyset$ and a unique C^{k+1} integral curve $c : \mathcal{I} \to \mathcal{M}$ of $\boldsymbol{u}$ for each $P \in \mathcal{U}$ such that $c(0) = P$.*

Proof. Let $\{x^i\}_P = x(P)$ be the local coordinates of $P = c(t) \in \mathcal{U} \subset \mathcal{M}$ in a chart $(\mathcal{U}, x)$, then from 3.6.2,

$$u^i(c(t)) = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) = \dot{c}^i(t)\,.$$

where $\boldsymbol{u} = u^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ and $c^i = x^i \circ c$. Moreover, $c(0) = P$ becomes $c^i(0) = x^i \circ c(0) = x^i(P)$ in the chart, so there is a system of n ordinary differential equations with given initial conditions. As $\boldsymbol{u}$ resp. the u^i are C^k, by Picard-Lindelöf's theorem they have a unique solution satisfying the initial conditions: the C^{k+1} functions $c^i(t)$. ■

Definition 3.6.21. By 3.6.20, the vector field $\boldsymbol{u} : \mathcal{M} \to T\mathcal{M}$ generates integral curves for every $P \in \mathcal{M}$. The totality or collection $\psi_t = \{c(t) \,|\, t \mapsto c(t) = \psi_t(P),\, \forall\, P \in \mathcal{M}\}$ of integral curves $c(t)$ through $c(0) = \psi_0(P) = P$ is a one-parameter group of mappings, thus called the *time-independent flow of* $\boldsymbol{u}$.

Note that the flow need not be defined explicitly, but it is implicitly well-established by the vector field. Therefore, $\boldsymbol{u}$ is called the *generating vector field of the flow* ψ_t. $\diamond$

Definition 3.6.22. Let $\boldsymbol{u}(P)$, $P \in \mathcal{M}$, be the (time-independent) vector field generating the flow ψ_t, then the *Lie derivative* of a possibly time-dependent tensor field $\boldsymbol{T} \in \mathfrak{T}^p_q(\mathcal{M})$ is defined by

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{T}(P) = \lim_{t\to 0} \frac{\psi_t^\star(\boldsymbol{T}(\psi_t(P))) - \boldsymbol{T}(P)}{t} = \lim_{t\to 0} \frac{\psi_t^\star \boldsymbol{T} - \psi_0^\star \boldsymbol{T}}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^\star \boldsymbol{T}\Big|_{t=0} .$$

$\psi_t^\star$ is the pullback concerning ψ_t. ◇

The Lie derivative approximately answers the question how a tensor field $\boldsymbol{T}$ changes under some flow.

Definition 3.6.22 is only for time-independent vector fields $\boldsymbol{u}$. In general, however, the generating vector field is time-dependent, i.e. $\boldsymbol{u} : \mathcal{M} \times \mathcal{I} \to T\mathcal{M}$, where $\mathcal{I} \subset \mathbb{R}$. What is the flow that will be generated? Clearly, it is asked for the solution of the differential equation

$$\dot{c}(t) = \boldsymbol{u}(c(t), t) \quad \text{and} \quad c(s) = P_s \,,$$

where s is the starting time and $P_s = c(s)$ is the initial condition. However, this is again an ordinary differential equation which has the unique solution $c(t) = \psi_{t,s}(P_s)$, where $\psi_{t,s}$ is a time-dependent flow on $\mathcal{M}$ —note this is a more general version of proposition 3.6.20. Again, a proof can be performed with the aid of Picard-Lindelöf's theorem, and uniqueness of the solution gives the properties 3.6.18 of the flow (see also [1], ch. 4; and [4], ch. 5).

Definition 3.6.23. A mapping $\psi_{t,s} : \mathcal{M} \to \mathcal{M}$, $\forall\, t, s \in \mathcal{I} \subset \mathbb{R}$, is referred to as the *time-dependent flow of* $\boldsymbol{u} : \mathcal{M} \times \mathcal{I} \to T\mathcal{M}$, if for each s and $P \in \mathcal{M}$

$$c(t) = \psi_{t,s}(P_s) = \psi(P_s, s, t)$$

is the unique integral curve of $\boldsymbol{u}$ starting at P_s at time $t = s$, i.e.

$$\dot{c}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\psi_{t,s}(P_s) = \boldsymbol{u}(\psi_{t,s}(P_s), t) \quad \text{and} \quad c(s) = \psi_{s,s}(P_s) = P_s \,.$$ ◇

Definition 3.6.24. The *Lie derivative* of a possibly time-dependent tensor field $\boldsymbol{T} \in \mathfrak{T}^p_q(\mathcal{M})$ along a time-dependent vector field $\boldsymbol{u}$ on $\mathcal{M}$ is defined by

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{T}\,(P_s, t) = \lim_{\Delta t\to 0} \frac{\psi_{t,s}^\star \boldsymbol{T}_t - \boldsymbol{T}_s}{\Delta t} = \lim_{\Delta t\to 0} \frac{\psi_{t,s}^\star \boldsymbol{T}_t - \psi_{s,s}^\star \boldsymbol{T}_s}{\Delta t} = \frac{\mathrm{d}}{\mathrm{d}t}\psi_{t,s}^\star \boldsymbol{T}_t\Big|_{t=s} ,$$

with $\Delta t = t - s$ and the pullback operator $\psi_{t,s}^\star$ concerning the flow $\psi_{t,s}$. Fixing t in $\boldsymbol{T}_t = \boldsymbol{T}(\cdot, t)$ gives the *autonomous Lie derivative*

$$\lim_{\Delta t\to 0} \frac{\psi_{t,s}^\star \boldsymbol{T}_s - \psi_{s,s}^\star \boldsymbol{T}_s}{\Delta t} = \frac{\mathrm{d}}{\mathrm{d}t}\psi_{t,s}^\star \boldsymbol{T}_s\Big|_{t=s} = \mathrm{L}_{\boldsymbol{u}}\boldsymbol{T} - \frac{\partial \boldsymbol{T}}{\partial t} = \pounds_{\boldsymbol{u}}\boldsymbol{T} \,.$$

That is, if $\boldsymbol{T}$ is time-independent, $\mathrm{L}_{\boldsymbol{u}}\boldsymbol{T} \equiv \pounds_{\boldsymbol{u}}\boldsymbol{T}$. Figure 3.5 illustrates the concept. ◇

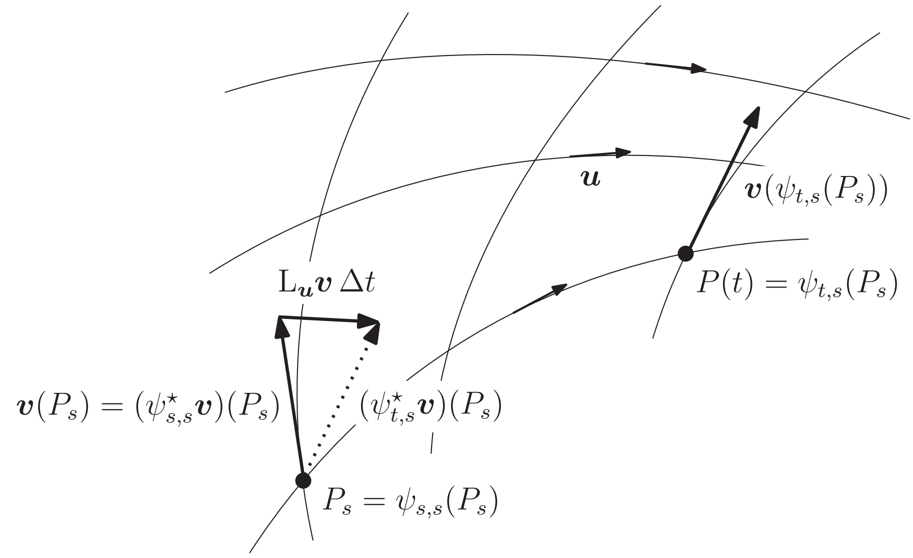

Figure 3.5: Lie derivative of a time-independent vector field $\boldsymbol{v}$ along a time-dependent vector field $\boldsymbol{u}$.

Proposition 3.6.25.

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{T} = \psi_{\star t,s}\frac{\mathrm{d}}{\mathrm{d}t}\psi^\star_{t,s}\boldsymbol{T}_t\,,$$

that is, the Lie derivative of an arbitrary tensor is obtained by pulling it back from t to s, performing the time derivative, and then pushing it forward to t again.

Proof. Let $t \neq s$ and $f : \mathcal{M} \to \mathbb{R}$ be a differentiable function on $\mathcal{M}$, then

$$\begin{aligned}\psi^\star_{t,s}(\mathrm{L}_{\boldsymbol{u}}f) = \psi^\star_{t,s}\,\frac{\mathrm{d}}{\mathrm{d}r}\psi^\star_{r,t}f_r\Big|_{r=t} &= \frac{\mathrm{d}}{\mathrm{d}r}f_r\circ\psi_{r,t}\circ\psi_{t,s}\Big|_{r=t}\\ &= \frac{\mathrm{d}}{\mathrm{d}r}(\psi_{r,t}\circ\psi_{t,s})^\star\,f_r\Big|_{r=t} = \frac{\mathrm{d}}{\mathrm{d}t}\psi^\star_{t,s}f_t\end{aligned}$$

by the property 3.4.32(ii) of pullbacks. Since this result holds for f replaced by arbitrary, and possibly time-dependent tensor fields $\boldsymbol{T}$ (see [1], sec. 5.4 for a detailed discussion), the assertion follows. ■

Example 3.6.26. Let $f : \mathcal{M}\times\mathcal{I} \to \mathbb{R}$ be a differentiable time-dependent scalar field, where $f_t(P) = f(P,t)$ and $P \in \mathcal{M}$, and let $\boldsymbol{u}(P,t)$ be a time-dependent vector field —the time-dependency of P being understood. By 3.4.30, $(\psi^\star_{t,s}f_t)(P_s) = f_t\circ\psi_{t,s} = f(\psi_{t,s}(P_s),t)$. Therefore

$$\begin{aligned}\mathrm{L}_{\boldsymbol{u}}f\,(P,t) = \frac{\mathrm{d}}{\mathrm{d}t}\psi^\star_{t,s}f_t\Big|_{t=s} &= \frac{\mathrm{d}}{\mathrm{d}t}f(\psi_{t,s}(P_s),t)\Big|_{t=s}\\ = \frac{\partial f}{\partial t}(P,t) + \frac{\partial f}{\partial x^i}(P,t)\frac{\partial\psi^i_{t,s}}{\partial t}(P) &= \frac{\partial f}{\partial t}(P,t) + \frac{\partial f}{\partial x^i}(P,t)\,u^i(P,t)\,,\end{aligned}$$

that is, $\mathrm{L}_{\boldsymbol{u}}f = \frac{\partial f}{\partial t} + \boldsymbol{u}[f]$.

Proposition 3.6.27. *(Without proof.) The Lie derivative of a vector field $\boldsymbol{v}$ is*

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + [\boldsymbol{u}, \boldsymbol{v}] \ ,$$

where $[\boldsymbol{u}, \boldsymbol{v}] = \left(\frac{\partial v^i}{\partial x^j} u^j - \frac{\partial u^i}{\partial x^j} v^j\right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ *is the so-called* Lie bracket *of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$.

Corollary 3.6.28. *If* $\boldsymbol{\nabla}$ *is a connection without torsion, i.e.* $\Gamma_k{}^i{}_j = \Gamma_j{}^i{}_k$, *and* $\boldsymbol{u}, \boldsymbol{v}$ *are vector fields, then*

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} \,.$$

Some additional properties of the (autonomous) Lie derivative are being stated without proof:

Proposition 3.6.29. *(Without proof.) (i) If* $\boldsymbol{T} \in \mathfrak{T}^p_q(\mathcal{M})$ *and* ϕ *is a diffeomorphism, then* $\phi^\star(\pounds_{\boldsymbol{u}}\boldsymbol{T}) = \pounds_{\phi^\star \boldsymbol{u}}\, \phi^\star \boldsymbol{T}$, *that is,* $\pounds_{\boldsymbol{u}}$ *is natural with the pullback. (ii) If* $\boldsymbol{u}, \boldsymbol{v}$ *are vector fields, then* $\pounds_{\boldsymbol{u}+\boldsymbol{v}} = \pounds_{\boldsymbol{u}} + \pounds_{\boldsymbol{v}}$.

Definition 3.6.30. Let $(\mathcal{M}, \boldsymbol{g})$ be a metric space, then a map $\phi : \mathcal{M} \to \mathcal{M}$ is called *isometry of* $\boldsymbol{g}$, if $\phi^\star \boldsymbol{g} = \boldsymbol{g}$. If each map $\psi_{t,s}$ of the flow generated by a vector field $\boldsymbol{u} : \mathcal{M} \times \mathcal{I} \to T\mathcal{M}$ is an isometry, then $\boldsymbol{u}$ is referred to as a *Killing vector field*. It is easy to verify that for Killing vector fields,

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{g} = \boldsymbol{0} \,.$$

◇

3.6.3 Exterior Derivative

The third type of a calculus of differentiation on manifolds involves the exterior derivative, which is restricted to fields of differential forms.

Definition 3.6.31. Let $\boldsymbol{\omega} \in \Omega^k(\mathcal{M})$ be a k-form on $\mathcal{M}$, then its *exterior derivative* is the $(k+1)$-form

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega} = \mathbf{d} \wedge \boldsymbol{\omega} &= \frac{1}{k!} \mathbf{d}\omega_{\mu_1 \ldots \mu_k} \wedge \mathbf{d}x^{\mu_1} \wedge \ldots \wedge \mathbf{d}x^{\mu_k} \\ &= \frac{1}{k!} \frac{\partial \omega_{\mu_1 \ldots \mu_k}}{\partial x^\nu} \mathbf{d}x^\nu \wedge \mathbf{d}x^{\mu_1} \wedge \ldots \wedge \mathbf{d}x^{\mu_k} \qquad \in \Omega^{k+1}(\mathcal{M}) \,. \end{aligned}$$

If $\boldsymbol{\omega} = f$ is a scalar field, then $\mathbf{d}f$ is its differential, defined by 3.2.15. ◇

Definition 3.6.32. A k-form $\boldsymbol{\omega}$ is called *exact*, if there exists a $(k-1)$-form $\boldsymbol{\alpha}$ such that $\mathbf{d}\boldsymbol{\alpha} = \boldsymbol{\omega}$, and $\boldsymbol{\omega}$ is called *closed*, if $\mathbf{d}\boldsymbol{\omega} = 0$. ◇

Corollary 3.6.33. *(i) Since* $\mathbf{d} \circ \mathbf{d} = 0$ *from Schwarz' theorem, every exact form is closed, but the converse need not hold. (ii) By 3.5.12 and 3.5.14, every* $(k = n_{\mathrm{dim}})$*-form is closed.*

Proposition 3.6.34. *(Without proof.) (i) $\phi^\star(\mathbf{d}\boldsymbol{\omega}) = \mathbf{d}(\phi^\star\boldsymbol{\omega})$, i.e. the exterior derivative commutates with the pullback. (ii) Let $\boldsymbol{u}$ be a vector field, then $\pounds_{\boldsymbol{u}}\boldsymbol{\omega} = \boldsymbol{i}_{\boldsymbol{u}}\mathbf{d}\boldsymbol{\omega} + \mathbf{d}\boldsymbol{i}_{\boldsymbol{u}}\boldsymbol{\omega}$ (Cartan's formula).*

Summarizing the preceding sections, the covariant derivative needs an additional structure on manifolds, whereas the Lie derivative restricts the direction of differentiation. The exterior derivative is exclusively for differential forms. All three types lead to proper tensors, whereas the usual partial derivative does not. The coordinate-invariant or "covariant" formulation of physical equations, therefore, is ensured when applying the covariant, Lie, and exterior calculus.

3.7 Integration on Manifolds

A rough introduction of integration calculus will be given in the following section. In order to define the integral in Riemannian and non-Riemannian spaces, one needs a more detailed study of the differential n-forms in an n-dimensional manifold, as well as a terminology for orientation.

As in section 3.5, lower case Greeks are used for coordinate indices and lower case Latins are used for labels.

3.7.1 Orientation and Determinants

Definition 3.7.1. Let n —here and in the remainder of this section— denote the dimension of the manifold $\mathcal{M}$. An n-form $\boldsymbol{\mu} \in \Omega^n(\mathcal{M})$ such that $\boldsymbol{\mu}(P) \neq \mathbf{0}$ for all $P \in \mathcal{M}$ is called *volume form*, and the set of all volume forms on $\mathcal{M}$ is the *volume bundle*. If there exists a volume bundle on $\mathcal{M}$, then $\mathcal{M}$ is called *orientable* and the pair $(\mathcal{M}, \boldsymbol{\mu})$ is referred to as a *volume manifold*. ♢

Definition 3.7.2. Two volume forms $\boldsymbol{\mu}', \boldsymbol{\mu}$ are *equivalent*, if there is some $f : \mathcal{M} \to \mathbb{R}$ with $f(P) > 0$ such that $\boldsymbol{\mu}' = f\boldsymbol{\mu}$. The equivalence classes $[\boldsymbol{\mu}]$ and $[-\boldsymbol{\mu}]$ of volume forms are called the *orientation* and the *reverse orientation* (which is also an orientation) on $\mathcal{M}$, respectively. An orientable manifold is referred to as an *oriented manifold*, if it has an orientation. ♢

Definition 3.7.3. Let $\boldsymbol{\mu}_{\mathcal{N}}$ be a volume form on the manifold $\mathcal{N}$. A differentiable map $\phi : \mathcal{M} \to \mathcal{N}$ is called *volume preserving*, if $\phi^\star\boldsymbol{\mu}_{\mathcal{N}} = \boldsymbol{\mu}_{\mathcal{M}}$ is a volume form on $\mathcal{M}$, i.e. $\phi^\star\boldsymbol{\mu}_{\mathcal{N}} \neq \mathbf{0}$ and the determinant of the Jacobian matrix of ϕ is non-zero at every $P \in \mathcal{M}$. ϕ is called *orientation preserving*, if $\phi^\star\boldsymbol{\mu}_{\mathcal{N}} \in [\boldsymbol{\mu}_{\mathcal{M}}]$, and *orientation reserving*, if $\phi^\star\boldsymbol{\mu}_{\mathcal{N}} \in [-\boldsymbol{\mu}_{\mathcal{M}}]$. ♢

Note that if ϕ is volume preserving, it can be either orientation preserving or orientation reserving, that is, the determinant of the Jacobian matrix of ϕ is either positive or negative, respectively. The sign of the determinant, however, is important when it comes to the definition of volume measures.

Example 3.7.4. The Moebius strip is a non-orientable manifold, because the equivalence classes $[\boldsymbol{\mu}]$ and $[-\boldsymbol{\mu}]$ can only be defined locally.

Proposition 3.7.5. *Volume forms (and n-forms) have a single component, that is,* $\dim(\Omega^n(\mathcal{M})) = 1$.

Proof. This follows directly from 3.5.14 by setting $k = n$. ■

Corollary 3.7.6. *(i) 3.5.14(i) states that the order of the coordinate differentials in the basis of $\bigwedge^k T_P^*\mathcal{M}$ is arbitrary. So carrying this over to each fibre of $\Omega^k(\mathcal{M})$ and using 3.7.5, one obtains a local representative of the volume form $\boldsymbol{\mu}$ at $P \in \mathcal{M}$ in a coordinate system $\{x^\nu\}_\mathcal{U}$ on $\mathcal{U} \subset \mathcal{M}$:*

$$\boldsymbol{\mu}(P) = \mu(P)\, \mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_n} \,.$$

(ii) By 3.7.5, any other volume form $\boldsymbol{\mu}' = f\boldsymbol{\mu}$ is a linear combination of the basis $\{\mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_n}\} \in \Omega^n(\mathcal{M})$, by choosing suitable $f : \mathcal{M} \to \mathbb{R}$. If $f > 0$, then $\boldsymbol{\mu}' \in [\boldsymbol{\mu}]$, and if $f < 0$, then $\boldsymbol{\mu}' \in [-\boldsymbol{\mu}]$.

Definition 3.7.7. Let $\mathcal{M}$ be orientable with orientation $[\boldsymbol{\mu}]$, $P \in \mathcal{U}$ and $\left\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}\right\} \in T_P\mathcal{M}$ a basis. If $\boldsymbol{\mu}(P)\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right) > 0$ (resp. < 0) for all $P \in \mathcal{U}$, then the basis is called *positively* (resp. *negatively*) *oriented* relative to the volume form $\boldsymbol{\mu}$. ◇

Definition 3.7.8. For every chart $(\mathcal{U}, x)$ on orientable $\mathcal{M}$ with orientation $[\boldsymbol{\mu}]$, where $x : \mathcal{U} \to \mathbb{R}^n$, define a map $x_\star$ through

$$x_\star|_\mathcal{U} : \Omega^n(\mathcal{U}) \to \Omega^n(x(\mathcal{U}))\,,$$

and set $x_\star(\mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_n}) = f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$. The chart $(\mathcal{U}, x)$ is then called *positively oriented*, if $x_\star(\boldsymbol{\mu}|_\mathcal{U})$ is equivalent to

$$\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \in \Omega^n(x(\mathcal{U}))\,,$$

the *standard volume form* on $\mathbb{R}^n$. ◇

The reader should notice that the map $x_\star$ has been already defined in a similar form to obtain the local representative of tensor fields. However, for n-forms one sets $x_\star(\mathbf{d}x^i \wedge \ldots \wedge \mathbf{d}x^n) = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ instead of taking $x_\star\mathbf{d}x^i = \boldsymbol{e}^i$ (see 3.4.41). It can be shown that $x_\star$ is an isomorphism on each fibre of $\bigwedge^n T^*\mathcal{U}$, so the inverse map $x^\star$ indeed exists.

Corollary 3.7.9. *For $\mathcal{M}$ being orientable, there is an atlas $\mathfrak{A}(\mathcal{M}) = \{(\mathcal{U}_i, x_i)\}_{i\in\mathcal{I}}$, with $\mathcal{I} \subset \mathbb{N}$, of which all charts are positively oriented, that is, the determinant of the Jacobian matrix of every two chart transitions $x_j\,(\mathcal{U}_j \cap \mathcal{U}_k) \to x_k\,(\mathcal{U}_j \cap \mathcal{U}_k)$, $j \neq k \in \mathcal{I}$, is positive.*

Theorem 3.7.10 (Transformation of n-Forms). *Let $\boldsymbol{a}_i = a^\nu{}_i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu} \in T_P\mathcal{M}$, where $1 \leq i \leq n$, be vectors (the subscript labels are for convenience), and let $\boldsymbol{\mu} \in \Omega^n(\mathcal{M})$, then at $P \in \mathcal{M}$,*

$$\boldsymbol{\mu}\left(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\right) = \det\left(a^\nu{}_i\right) \boldsymbol{\mu}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right)\,,$$

where $(a^\nu{}_i)$ is the matrix whose columns are the components of $\boldsymbol{a}_i$.

Proof. It is convenient to prepare the proof in two steps. First, it will be shown that

$$\left(\boldsymbol{a}^{*1}\wedge\ldots\wedge\boldsymbol{a}^{*n}\right)\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^n}\right)=\det\left(\boldsymbol{a}^{*i}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^\nu}\right)\right),$$

where $\boldsymbol{a}^{*1},\ldots,\boldsymbol{a}^{*n}$ are 1-forms. For the purpose of this paper it is sufficient to consider the case $n=2$; a generalization can be found in [1], sec. 6.2. Define $\boldsymbol{a}^{*i}=a^i{}_\nu\mathbf{d}x^\nu\in T^*_P\mathcal{M}$, then, by 3.5.10,

$$\begin{aligned}\left(\boldsymbol{a}^{*1}\wedge\boldsymbol{a}^{*2}\right)\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^1},\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^2}\right)&=\sum_{\pi\in\Pi_{1,1}}(\operatorname{sgn}\pi)\,\boldsymbol{a}^{*1}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^{\pi(1)}}\right)\boldsymbol{a}^{*2}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^{\pi(2)}}\right)\\&=a^1{}_1a^2{}_2-a^1{}_2a^2{}_1=\det\left(a^i{}_\nu\right).\end{aligned}$$

Here $\sum_{\pi\in\Pi_{1,1}}$ denotes the sum over the two $(1,1)$-shuffles π on $\{1,2\}$, and $(a^i{}_\nu)=\left(\boldsymbol{a}^{*i}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^\nu}\right)\right)$ is the matrix whose rows are the components of $\boldsymbol{a}^{*i}$, as desired.

Now define $\boldsymbol{\mu}=\boldsymbol{a}^{*1}\wedge\ldots\wedge\boldsymbol{a}^{*n}$. As a special case of the previous result is $\mathbf{d}x^1\wedge\ldots\wedge\mathbf{d}x^n\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^n}\right)=1$, one has

$$\mathbf{d}x^1\wedge\ldots\wedge\mathbf{d}x^n\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^n}\right)=\det\left(a^\nu{}_i\right)\boldsymbol{\mu}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^n}\right).$$

Here $(a^\nu{}_i)$ is the inverse of the matrix $(a^i{}_\nu)$, whose components are $a^\nu{}_i=\mathbf{d}x^\nu\left(\boldsymbol{a}_i\right)$ and $a^i{}_\nu=\boldsymbol{a}^{*i}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^\nu}\right)$, respectively. It follows that $\boldsymbol{a}^{*i}$ and $\boldsymbol{a}_i$ so defined are dual, such that

$$\boldsymbol{a}^{*i}(\boldsymbol{a}_i)=a^i{}_\nu a^\mu{}_i\mathbf{d}x^\nu\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^\mu}\right)=a^i{}_\nu a^\mu{}_i\delta^\nu{}_\mu=a^i{}_\nu a^\nu{}_i=1\qquad(\text{no sum over } i),$$

for all $i\in\{1,\ldots,n\}$. From this and by using $\boldsymbol{\mu}$ defined above, $\mathbf{d}x^1\wedge\ldots\wedge\mathbf{d}x^n\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^n}\right)=\boldsymbol{\mu}\left(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n\right)$, which proofs the theorem. ∎

Proposition 3.7.11. *Let $\phi:\mathcal{M}\to\mathcal{N}$ be a differentiable map, $\mathcal{U}\subset\mathcal{M}$, $\mathcal{V}\subset\mathcal{N}$ and $\phi^{-1}(\mathcal{V})\cap\mathcal{U}\neq\emptyset$. Moreover, let $(\mathcal{U},X)$, $(\mathcal{V},x)$ be appropriate charts such that $x\circ\phi\circ X^{-1}$ defines the chart map x concerning ϕ with respect to X. Let $\{x^\nu\}_P=x(P)$ and $\{X^\alpha\}_P=X(P)$ for some $P\in\phi^{-1}(\mathcal{V})\cap\mathcal{U}\subset\mathcal{M}$, and let $\boldsymbol{\mu}_\mathcal{N}=\mu\,\mathbf{d}x^1\wedge\ldots\wedge\mathbf{d}x^n\in\Omega^n(\mathcal{N})$, then*

$$\phi^\star(\mu\,\mathbf{d}x^1\wedge\ldots\wedge\mathbf{d}x^n)=\det\left(\frac{\partial x^\nu\circ\phi\circ X^{-1}}{\partial X^\alpha}\right)(\mu\circ\phi)\,\mathbf{d}X^1\wedge\ldots\wedge\mathbf{d}X^n,$$

where $\left(\frac{\partial x^\nu\circ\phi\circ X^{-1}}{\partial X^\alpha}\right)$ is the Jacobian matrix of ϕ with respect to x and X.

Proof. Because $\mu:\mathcal{N}\to\mathbb{R}$, one has $\phi^\star\mu=\mu\circ\phi$ by 3.4.30, so it is left proving

$$\phi^\star(\mathbf{d}x^1\wedge\ldots\wedge\mathbf{d}x^n)=\det\left(\frac{\partial\phi^\nu}{\partial X^\alpha}\right)\mathbf{d}X^1\wedge\ldots\wedge\mathbf{d}X^n,$$

where $\phi^\nu = x^\nu \circ \phi \circ X^{-1}$ has been set, and point arguments are omitted. Multiplying both sides with $\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^1}, \dots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^n}\right)$, then the right hand side becomes just $\det\left(\frac{\partial \phi^\nu}{\partial X^\alpha}\right)$. On the left hand side, however,

$$
\begin{aligned}
&\phi^\star(\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n)\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^1}, \dots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^n}\right) \\
&= (\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n)\left(\phi_\star\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^1}\right), \dots, \phi_\star\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^n}\right)\right) \\
&= (\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n)\left(\frac{\partial \phi^\nu}{\partial X^1}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}, \dots, \frac{\partial \phi^\nu}{\partial X^n}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}\right) \\
&= \det\left(\frac{\partial \phi^\nu}{\partial X^\alpha}\right)(\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n)\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \dots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right) \\
&= \det\left(\frac{\partial \phi^\nu}{\partial X^\alpha}\right)
\end{aligned}
$$

by 3.4.35 and theorem 3.7.10. ∎

Corollary 3.7.12. *Let $\phi = \mathrm{Id} : \mathcal{M} \to \mathcal{M}$, and $(\mathcal{U}, x)$, $(\mathcal{U}', x')$ be regular charts on $\mathcal{M}$, then by 3.7.11,*

(i) Under chart transitions $x' \circ x^{-1}|_{x(\mathcal{U}\cap\mathcal{U}')} : x(\mathcal{U}\cap\mathcal{U}') \to x'(\mathcal{U}\cap\mathcal{U}')$,

$$\mathbf{d}x^{1'} \wedge \ldots \wedge \mathbf{d}x^{n'} = \det\left(\frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^\nu}\right)\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n \,.$$

(ii) If $\boldsymbol{\mu} = \mu' \,\mathbf{d}x^{1'} \wedge \ldots \wedge \mathbf{d}x^{n'} = \mu \,\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n$, then

$$\mu' = \det\left(\frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^\nu}\right)^{-1} \mu \,.$$

Conclude that $\mu = \boldsymbol{\mu}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \dots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right)$ is not a scalar, and that n-forms are even relative scalars of weight 1, i.e. even scalar densities in terms of 3.3.6.

Corollary 3.7.13. *Recall the definitions in 3.7.11 and let $(\mathcal{U}', X')$, $(\mathcal{V}', x')$ be other charts on $\mathcal{M}$ and $\mathcal{N}$, respectively. Then, by using the abbreviation $\frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^\nu} = \frac{\partial x^{\nu'}}{\partial x^\nu}$ and applying the standard rule for the determinant of products of matrices,*

$$\det\left(\frac{\partial \phi^{\nu'}}{\partial X^{\alpha'}}\right) = \det\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial \phi^\nu}{\partial X^\alpha}\frac{\partial X^\alpha}{\partial X^{\alpha'}}\right) = \det\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)\det\left(\frac{\partial \phi^\nu}{\partial X^\alpha}\right)\det\left(\frac{\partial X^\alpha}{\partial X^{\alpha'}}\right) .$$

Thus $\det(\frac{\partial \phi^\nu}{\partial X^\alpha})$ is not a proper scalar function in general, i.e. it is not coordinate-invariant.

Definition 3.7.14. Let $\boldsymbol{A} : \mathcal{V} \to \mathcal{W}$ be an isomorphism of vector spaces and $\boldsymbol{\mu} \in \bigwedge^n \mathcal{W}$ an n-form with respect to the vector space $\mathcal{W}$, then, by 3.7.11 and 3.4.20, a coordinate-free definition of the *determinant* can be obtained:

$$\boldsymbol{A}^\star \boldsymbol{\mu} = (\det \boldsymbol{A})\,\boldsymbol{\mu} \,.$$

Note that $\det \boldsymbol{A} = \det(A^\alpha{}_i)$, where $(A^\alpha{}_i)$ is the matrix of the linear transformation (see 2.1.10). ◇

Definition 3.7.15. Let $\boldsymbol{a}^{*1}, \ldots, \boldsymbol{a}^{*n} \in T_P^*\mathcal{M}$ be 1-forms, then define the n-form-valued $\boldsymbol{\epsilon}$-*tensor* at $P \in \mathcal{M}$ through

$$\begin{aligned}
\boldsymbol{\epsilon}(P): \underbrace{T_P^*\mathcal{M} \times \ldots \times T_P^*\mathcal{M}}_{n\text{-fold}} &\to \bigwedge^n T_P^*\mathcal{M} \\
\left(\boldsymbol{a}^{*1}, \ldots, \boldsymbol{a}^{*n}\right) &\mapsto \boldsymbol{a}^{*1} \wedge \ldots \wedge \boldsymbol{a}^{*n} = \epsilon_{\nu_1 \ldots \nu_n} \boldsymbol{a}^{*\nu_1} \otimes \ldots \otimes \boldsymbol{a}^{*\nu_n},
\end{aligned}$$

in which $\epsilon_{\nu_1 \ldots \nu_n}(P)$ is the *Levi-Civita symbol* or *permutation symbol*, given by

$$\epsilon_{\nu_1 \ldots \nu_n} = \begin{cases} +1 & \text{if } \nu_1, \ldots, \nu_n \text{ is an even permutation of } 1, \ldots, n \\ -1 & \text{if } \nu_1, \ldots, \nu_n \text{ is an odd permutation of } 1, \ldots, n \\ 0 & \text{if } \nu_i = \nu_j \text{ for some } i \neq j \end{cases}$$

in every coordinate system. ◇

Proposition 3.7.16. *The $\boldsymbol{\epsilon}$-tensor is an even relative tensor of weight -1.*

Proof. In a regular chart $(\mathcal{U}, x)$ on $\mathcal{M}$, note that $\boldsymbol{\epsilon}\left(\mathbf{d}x^1, \ldots, \mathbf{d}x^n\right) = \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n = \epsilon_{\nu_1 \ldots \nu_n} \mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_n}$ by definition, and $\boldsymbol{a}^{*1} \wedge \ldots \wedge \boldsymbol{a}^{*n} = \det(a^\mu{}_\nu) \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n$ by 3.7.10, where $\boldsymbol{a}^{*\mu} = a^\mu{}_\nu \mathbf{d}x^\nu$ has been set —here lower case Greeks are also used for labels. Therefore,

$$\boldsymbol{\epsilon}\left(\boldsymbol{a}^{*1}, \ldots, \boldsymbol{a}^{*n}\right) = \boldsymbol{a}^{*1} \wedge \ldots \wedge \boldsymbol{a}^{*n} = \det\left(a^\mu{}_\nu\right) \epsilon_{\nu_1 \ldots \nu_n} \mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_n} = \det\left(a^\mu{}_\nu\right) \boldsymbol{\epsilon},$$

so that $\boldsymbol{\epsilon} = \epsilon_{\nu_1 \ldots \nu_n} \mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_n}$ is a local representative of $\boldsymbol{\epsilon}$.

Now, under a chart transition $x' \circ x^{-1}|_{x(\mathcal{U} \cap \mathcal{U}')} : x\left(\mathcal{U} \cap \mathcal{U}'\right) \to x'\left(\mathcal{U} \cap \mathcal{U}'\right)$,

$$\begin{aligned}
\mathbf{d}x^{1'} \wedge \ldots \wedge \mathbf{d}x^{n'} &= \det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)^{-1} \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n \\
&= \det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)^{-1} \epsilon_{\nu_1 \ldots \nu_n} \mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_n}
\end{aligned}$$

by 3.7.12 and using the abbreviation $\frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^\nu} = \frac{\partial x^{\nu'}}{\partial x^\nu}$. But $\boldsymbol{\epsilon}$ is a tensor, so $\epsilon_{\nu'_1 \ldots \nu'_n} \mathbf{d}x^{\nu'_1} \otimes \ldots \otimes \mathbf{d}x^{\nu'_n} = \epsilon_{\nu_1 \ldots \nu_n} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x^{\nu'_n}} \mathbf{d}x^{\nu'_1} \otimes \ldots \otimes \mathbf{d}x^{\nu'_n}$ from 3.3.5. Substitution into the preceding then yields

$$\epsilon_{\nu'_1 \ldots \nu'_n} = \det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)^{-1} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x^{\nu'_n}} \epsilon_{\nu_1 \ldots \nu_n}$$

as desired. ∎

Note that $\boldsymbol{\epsilon}$ as a tensor has n^n components, but only $n!$ are non-zero. Hence, the even relative $\boldsymbol{\epsilon}$-tensor of weight -1 boils down to the even relative scalar $\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n = \boldsymbol{\epsilon}\left(\mathbf{d}x^1, \ldots, \mathbf{d}x^n\right)$ of weight -1 when applied to the n-tuple $\left(\mathbf{d}x^1, \ldots, \mathbf{d}x^n\right)$.

3.7.2 Stokes' Theorem and Volume Measures

The n-forms on n-dimensional manifolds have a single component, thus they can be integrated over open sets by prescribing the following:

Definition 3.7.17. Let $\mathcal{X} \subset \mathbb{R}^n$ be open and $f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \in \Omega^n(\mathcal{X})$, $f = f(x^1 \ldots x^n)$ being understood, then define

$$\int_{\mathcal{X}} f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n = \int_{\mathcal{X}} f\ \mathrm{d}x^1 \ldots \mathrm{d}x^n \equiv \int_{\mathcal{X}} f\,,$$

where $\int_{\mathcal{X}} f$ is the ordinary Riemann integral of f in $\mathbb{R}^n$. ◇

Theorem 3.7.18 (Change of Variables). *(Without proof.) Let $\mathcal{M}$ be orientable, $\phi : \mathcal{M} \to \mathcal{N}$ an orientation preserving map and $\boldsymbol{\omega} \in \Omega^n(\phi(\mathcal{M}))$, then*

$$\int \phi^\star \boldsymbol{\omega} = \int_\phi \boldsymbol{\omega}\,.$$

This fundamental theorem well-known from the analysis of real functions leads to the answer of how n-forms are to be integrated over manifolds.

Proposition 3.7.19. *Let $(\mathcal{U}, x)$, $\mathcal{U} \subset \mathcal{M}$, be a positively oriented chart and $\boldsymbol{\omega} \in \Omega^n(\mathcal{M})$ an n-form, then*

$$\int_{\mathcal{U}} \boldsymbol{\omega} = \int_{x(\mathcal{U})} \omega \circ x^{-1}\,.$$

Proof. Without loss of generality assume that $\boldsymbol{\omega}(P) = \omega(P)\,\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n$ for every $P \in \mathcal{M}$, then

$$x_\star(\boldsymbol{\omega}|_{\mathcal{U}}) = (\omega \circ x^{-1})\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

by 3.7.8 together with 3.4.42, and

$$\int_{x(\mathcal{U})} x_\star \boldsymbol{\omega} = \int_{x(\mathcal{U})} \omega \circ x^{-1}$$

by definition. The result then follows by the change of variables formula. ■

In other words, n-forms get access to the calculus of integration of real functions by defining their integral in the chart as the ordinary Riemann integral in $\mathbb{R}^n$ (cf. [1], sec. 7.1). Clearly, $\mathbf{d}x^{\nu_1} \wedge \ldots \wedge \mathbf{d}x^{\nu_n}$ is replaced by its Lebesgue measure, and the 1-form valued coordinate differentials $\mathbf{d}x^i$ on $\mathcal{U}$ are read as the usual coordinate differentials $\mathrm{d}x^i$ of the chart $x(\mathcal{U}) \subset \mathbb{R}^n$ —note that this approach is comparable to the definition of a differentiable structure on manifolds presented in section 3.1.

If the atlas $\mathfrak{A}(\mathcal{M}) = \{(\mathcal{U}_i, x_i)\}_{i \in \mathcal{I}}$ of $\mathcal{M} = \bigcup_{i \in \mathcal{I}} \mathcal{U}_i$ contains of more than one chart, then the integral of $\boldsymbol{\omega}$ over $\mathcal{M}$ is defined via the sum of integrals over the subsets $\mathcal{U}_i$ by assuming that there is a so-called *partition of unity subordinate to the atlas.*

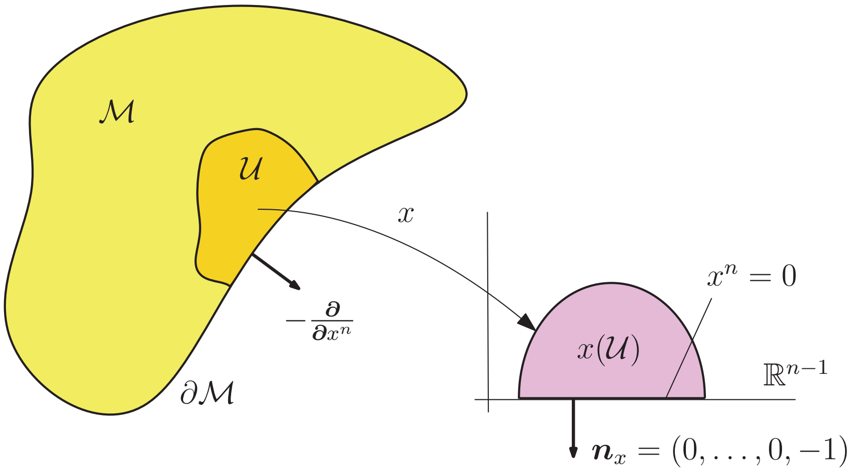

Figure 3.6: Definition of the outward normal on the boundary $\partial\mathcal{M}$.

In continuum mechanics, Stokes's theorem is of fundamental importance. For its general version on manifolds, it is necessary that the oriented manifold $\mathcal{M}$ has a compatible oriented boundary $\partial\mathcal{M}$ —for $\mathcal{M}$ being n-dimensional, note that $\partial\mathcal{M}$ is $(n-1)$-dimensional.

Remember that the classic Stokes' theorem for surfaces embedded in $\mathbb{R}^3$ applies the right-hand rule to achieve compatibility: the outward normal vector of the surface is linked to the counter-clockwise, i.e. positive orientation of the boundary. This rule is used to define the positively oriented boundary of a manifold (figure 3.6):

Definition 3.7.20. Let $\mathcal{M}$ be oriented and $(\mathcal{U}, x)$ a chart, then a basis $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{n-1}\}$ $\in T_P(\partial\mathcal{M})$ at $P \in \partial\mathcal{M}$ is *positively oriented*, if $\{-\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_{n-1}\} \in T_P\mathcal{M}$ is positively oriented in the orientation of $\mathcal{M}$. By this, $\mathcal{M}$ induces an orientation on $\partial\mathcal{M}$. $\diamond$

Theorem 3.7.21 (Stokes' Theorem). *(Without proof; see [1] or [4]) Let $\mathcal{M}$ be oriented with an oriented boundary $\partial\mathcal{M}$, and let $\boldsymbol{\omega}$ be an $(n-1)$-form, then*

$$\int_{\mathcal{M}} \mathbf{d}\boldsymbol{\omega} = \int_{\partial\mathcal{M}} \boldsymbol{\omega} \,.$$

On Riemannian manifolds, i.e. metric spaces, volume forms enable volume measurement. The volume measure should always be non-zero and positive to be consistent with the specification in ordinary $\mathbb{R}^3$.

From linear algebra, the volume of the parallelepiped spanned by the three independent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in \mathbb{R}^3$ is given by $\mathrm{Vol}(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3) = \sqrt{\det \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle}$, if $\boldsymbol{w}_1, \boldsymbol{w}_2$ and $\boldsymbol{w}_3$ are positively oriented, and by

$$\mathrm{Vol}(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3) = \sqrt{|\det \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle|} \,,$$

if $\boldsymbol{w}_1, \boldsymbol{w}_2$ and $\boldsymbol{w}_3$ are negatively oriented. $\det\langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle$ denotes the determinant of the matrix (W_{ij}), whose elements are given by the inner products $W_{ij} = \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle$. To carry this over to manifolds, one needs to define quantities that involve the absolute value of a determinant.

Definition 3.7.22. A multilinear mapping

$$\boldsymbol{\varrho} : \underbrace{\mathcal{V} \times \ldots \times \mathcal{V}}_{n\text{-fold}} \quad \to \quad \mathbb{R}$$

over a vector space $\mathcal{V}$ is called an *α-density*, if for every $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathcal{V}$ and every endomorphism $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$,

$$\boldsymbol{\varrho}(\boldsymbol{A}\boldsymbol{v}_1, \ldots, \boldsymbol{A}\boldsymbol{v}_n) = |\det \boldsymbol{A}|^{\alpha} \, \boldsymbol{\varrho}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \, .$$

The set of all α-densities over $\mathcal{V}$ is denoted by $|\bigwedge^n|^{\alpha} \, \mathcal{V}$. 1-densities (or just densites) are also called *odd* or *twisted n-forms.* ◊

α-densities can be constructed from n-forms as follows: If $\boldsymbol{\mu} \in \bigwedge^n \mathcal{V}$ is an n-form, then the α-density $|\boldsymbol{\mu}|^{\alpha} \in |\bigwedge^n|^{\alpha} \, \mathcal{V}$ is defined through $|\boldsymbol{\mu}|^{\alpha} \, (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = |\boldsymbol{\mu}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)|^{\alpha}$. Conversely, let $\mathcal{V}$ be orientable, $\{\boldsymbol{g}_i\} \in \mathcal{V}$ a positively oriented basis, $\boldsymbol{v}_i = \boldsymbol{A}\boldsymbol{g}_i$, and $\boldsymbol{\varrho} \in |\bigwedge^n| \, \mathcal{V}$ a 1-density, then

$$\boldsymbol{\mu}_{\varrho}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = (\det \boldsymbol{A}) \; \boldsymbol{\varrho}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n)$$

defines an n-form on $\mathcal{V}$. α-densities can be carried over to manifolds in order to derive the next result.

Corollary 3.7.23. *Let $\phi = \mathrm{Id} : \mathcal{M} \to \mathcal{M}$ and $(\mathcal{U}, x)$, $(\mathcal{U}', x')$ be regular charts on a differentiable manifold $\mathcal{M}$. Let $\boldsymbol{\varrho} \in \Gamma(|\bigwedge^n| \, T^*\mathcal{M})$ be a twisted n-form and $x' \circ x^{-1}|_{x(\mathcal{U} \cap \mathcal{U}')} : x\,(\mathcal{U} \cap \mathcal{U}') \to x'\,(\mathcal{U} \cap \mathcal{U}')$ the chart transition map, then 3.7.22 carries over to*

$$\boldsymbol{\varrho}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right) = \left|\det\left(\frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^{\nu}}\right)\right| \boldsymbol{\varrho}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{1'}}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{n'}}\right) \, ,$$

where $\boldsymbol{A}\boldsymbol{v}_i$ has been replaced by the tangent map $T\phi(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu}}) = \frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^{\nu}} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{\nu'}}$. Conclude that $\varrho = \boldsymbol{\varrho}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \ldots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\right)$ is not a scalar, and that twisted n-form are odd relative scalars of weight 1, i.e. odd scalar densities in the sense of 3.3.6.

With twisted n-forms one can formulate integrals and Stokes' theorem even on non-orientable manifolds.

Definition 3.7.24. Let $(\mathcal{M}, \boldsymbol{g})$ be an oriented Riemannian manifold and the set $\boldsymbol{w}_1$, $\ldots, \boldsymbol{w}_n \in \Gamma(T\mathcal{M})$ positively oriented, then the *Riemannian volume form* or *$\boldsymbol{g}$-volume* $\mathbf{d}\boldsymbol{v} \in \Omega^n(\mathcal{M})$ is defined by

$$\mathbf{d}\boldsymbol{v}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n) = \sqrt{\det \langle \boldsymbol{w}_{\mu}, \boldsymbol{w}_{\nu} \rangle} \, ,$$

i.e. $\mathbf{d}\boldsymbol{v}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ is the volume of the parallelepiped spanned by the vectors $\boldsymbol{w}_1, \ldots,$ $\boldsymbol{w}_n$. More general, and for arbitrary orientations of the $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$, define the *Riemannian α-density* or *$\boldsymbol{g}$-α-density* through

$$|\mathbf{d}\boldsymbol{v}|^{\alpha} \, (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n) = |\det \langle \boldsymbol{w}_{\mu}, \boldsymbol{w}_{\nu} \rangle|^{\frac{\alpha}{2}} \, .$$

◊

Let $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n\} \in T_P\mathcal{M}$ be a positively oriented orthonormal basis, and $\{\boldsymbol{e}^1,\ldots,\boldsymbol{e}^n\} \in T_P^*\mathcal{M}$ its dual. The volume of the parallelepiped spanned by the $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n$ is just $\mathbf{d}\boldsymbol{v}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n) = 1$, but from 3.7.10 this is achieved with $\mathbf{d}\boldsymbol{v} = \boldsymbol{e}^1 \wedge \ldots \wedge \boldsymbol{e}^n$. The more general case follows.

Corollary 3.7.25. *In a regular chart $(\mathcal{U}, x)$ on $(\mathcal{M}, \boldsymbol{g})$, let $\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\} \in T_P\mathcal{M}$ be a positively oriented basis, then $\mathbf{d}\boldsymbol{v}(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1},\ldots,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}) = \sqrt{\det \langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\mu}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^\nu}\rangle} = \sqrt{\det\, g_{\mu\nu}}$ is the component of the Riemannian volume form such that*

$$\mathbf{d}\boldsymbol{v} = \sqrt{\det\, g_{\mu\nu}}\, \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n$$

is its local representative.

Proposition 3.7.26. *Let $(\mathcal{M}, \boldsymbol{G})$, $(\mathcal{N}, \boldsymbol{g})$ be orientable, $\phi : \mathcal{M} \to \mathcal{N}$ a diffeomorphism, $\mathcal{U} \subset \mathcal{M}$, $\mathcal{V} \subset \mathcal{N}$ and $\phi^{-1}(\mathcal{V}) \cap \mathcal{U} \neq \emptyset$. Moreover, let $(\mathcal{U}, X)$, $(\mathcal{V}, x)$ be positively oriented charts and let $\phi^\nu = x^\nu \circ \phi \circ X^{-1}$ denote the coordinate functions of x concerning ϕ with respect to $\{X^\alpha\}_P = X(P)$, for every $P \in \phi^{-1}(\mathcal{V}) \cap \mathcal{U} \subset \mathcal{M}$. Let $\mathbf{d}\boldsymbol{V}, \mathbf{d}\boldsymbol{v}$ be the Riemannian volume forms on $\mathcal{M}$ and $\mathcal{N}$, respectively, then*

$$\phi^\star \mathbf{d}\boldsymbol{v} = J_\phi\, \mathbf{d}\boldsymbol{V}\,,$$

where $J_\phi(P) = \det\left(\frac{\partial \phi^\nu}{\partial X^\alpha}\right) \frac{\left(\sqrt{\det\, g_{\mu\nu}}\right)\circ\phi}{\sqrt{\det\, G_{\alpha\beta}}}$, called the Jacobian of ϕ, *is a proper scalar.*

Proof. By 3.7.11,

$$\begin{aligned}\phi^\star \mathbf{d}\boldsymbol{v} &= \left(\sqrt{\det\, g_{\mu\nu}} \circ \phi\right) \phi^\star(\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n) \\ &= \left(\sqrt{\det\, g_{\mu\nu}} \circ \phi\right) \det\left(\frac{\partial \phi^\nu}{\partial X^\alpha}\right) \mathbf{d}X^1 \wedge \ldots \wedge \mathbf{d}X^n\,.\end{aligned}$$

But $\mathbf{d}X^1 \wedge \ldots \wedge \mathbf{d}X^n = (\det\, G_{\alpha\beta})^{-\frac{1}{2}}\, \mathbf{d}\boldsymbol{V}$ by 3.7.25, hence $\phi^\star \mathbf{d}\boldsymbol{v} = J_\phi\, \mathbf{d}\boldsymbol{V}$ as desired.

To proof the second part, i.e. the invariance of $J_\phi(P)$, let $(\mathcal{U}', X')$, $(\mathcal{V}', x')$ be other regular and positively oriented charts on $\mathcal{M}$ and $\mathcal{N}$, respectively. Now, since $\det\, g_{\mu\nu} = \det\left(g_{\mu'\nu'} \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu}\right)$, where the abbreviation $\frac{\partial(x^{\nu'} \circ x^{-1})}{\partial x^\nu} = \frac{\partial x^{\nu'}}{\partial x^\nu}$ has been used, it is

$$\frac{\sqrt{\det\, g_{\mu\nu}}}{\sqrt{\det\, g_{\mu'\nu'}}} = \det\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)\,.$$

Therefore,

$$\begin{aligned}J'_\phi &= \det\left(\frac{\partial \varphi^{\nu'}}{\partial X^{\alpha'}}\right) \frac{\sqrt{\det(g_{\mu'\nu'})}}{\sqrt{\det(G_{\alpha'\beta'})}} \\ &= \det\left(\frac{\partial \varphi^{\nu'}}{\partial X^{\alpha'}}\right) \frac{\sqrt{\det(g_{\mu'\nu'})}}{\sqrt{\det(g_{\mu\nu})}} \frac{\sqrt{\det(G_{\alpha\beta})}}{\sqrt{\det(G_{\alpha'\beta'})}} \frac{\sqrt{\det(g_{\mu\nu})}}{\sqrt{\det(G_{\alpha\beta})}} \\ &= \det\left(\frac{\partial \varphi^{\nu'}}{\partial X^{\alpha'}}\right) \det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right) \det\left(\frac{\partial X^{\alpha'}}{\partial X^\alpha}\right) \frac{\sqrt{\det(g_{\mu\nu})}}{\sqrt{\det(G_{\alpha\beta})}} \\ &= \det\left(\frac{\partial \varphi^\nu}{\partial X^\alpha}\right) \frac{\sqrt{\det(g_{\mu\nu})}}{\sqrt{\det(G_{\alpha\beta})}} = J_\phi\,,\end{aligned}$$

so J_ϕ —in contrast to det $\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)$, see 3.7.13— indeed is a scalar. ■

Proposition 3.7.27. *In the light of 3.3.12, Riemannian volume forms are absolute pseudoscalars such that the formula of 3.7.26 boils down to*

$$\mathbf{d}\boldsymbol{v} \circ \phi = J_\phi \, \mathbf{d}\boldsymbol{V} \, .$$

Proof. For the moment, drop the condition that both $(\mathcal{V}, x)$ and $(\mathcal{V}', x')$ previously defined are positively oriented, then

$$\sqrt{\det\, g_{\mu\nu}} = \sqrt{\det\left(g_{\mu'\nu'}\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\right)} = \sqrt{\det\, g_{\mu'\nu'}}\left|\det\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)\right| .$$

Let $\mathbf{d}\boldsymbol{v} = \sqrt{\det\, g_{\mu\nu}}\; \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n = \sqrt{\det\, g_{\mu'\nu'}}\; \mathbf{d}x^{1'} \wedge \ldots \wedge \mathbf{d}x^{n'}$, then use 3.7.12(i) to obtain

$$\begin{aligned}
\sqrt{\det\, g_{\mu'\nu'}}\mathbf{d}x^{1'} \wedge \ldots \wedge \mathbf{d}x^{n'} &= \det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\sqrt{\det\, g_{\mu'\nu'}}\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n \\
&= \frac{\det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)}{\left|\det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\right|}\sqrt{\det\, g_{\mu\nu}}\; \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n \\
&= \operatorname{sign}\left[\det\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\right]\sqrt{\det\, g_{\mu\nu}}\; \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n \, ,
\end{aligned}$$

as desired. Now since $\mathbf{d}\boldsymbol{v}$ is a scalar, one arrives at $\phi^\star \mathbf{d}\boldsymbol{v} = \mathbf{d}\boldsymbol{v} \circ \phi$ by applying definition 3.4.30. ■

Corollary 3.7.28. $\sqrt{\det\, g_{\mu\nu}}$ *is an odd relative scalar of weight 1, i.e. an odd scalar density, and* $\mathbf{d}\boldsymbol{v} = \sqrt{\det\, g_{\mu\nu}}\; \epsilon_{\nu_1\ldots\nu_n}\mathbf{d}x^{\nu_1} \otimes \ldots \otimes \mathbf{d}x^{\nu_n} = \sqrt{\det\, g_{\mu\nu}}\; \boldsymbol{\epsilon}(\mathbf{d}x^1, \ldots, \mathbf{d}x^n)$.

Definition 3.7.29. $\bar{\boldsymbol{\epsilon}} = \sqrt{\det\, g_{\mu\nu}}\; \boldsymbol{\epsilon}$ is called the *Levi-Civita tensor.* ◇

Corollary 3.7.30. *By 3.3.12, 3.7.16 and 3.7.28 the Levi-Civita tensor is an absolute pseudotensor.*

Proposition 3.7.31. *(Without proof.) If* $\mathbf{d}\boldsymbol{v}$ *is a Riemannian volume form, then* $\pounds_{\boldsymbol{u}}\mathbf{d}\boldsymbol{v} = \mathbf{d}(\boldsymbol{i}_{\boldsymbol{u}}\mathbf{d}\boldsymbol{v}) = (\operatorname{div}\boldsymbol{u})\,\mathbf{d}\boldsymbol{v}$.

Note this characterization of divergence does not require any metric or affine connection.

Proposition 3.7.32. *Let* $\mathcal{M}$ *be a manifold with boundary* $\partial\mathcal{M}$, $\boldsymbol{w} \in \Gamma(T\mathcal{M})$ *a vector field and* $\boldsymbol{n}^* \in \Gamma(T^*\mathcal{M})$ *the "unit normals" on* $\partial\mathcal{M}$*. Then, on* $\partial\mathcal{M}$,

$$\boldsymbol{i}_{\boldsymbol{w}}\mathbf{d}\boldsymbol{v} = (\boldsymbol{n}^* \cdot \boldsymbol{w})\, \mathbf{d}\boldsymbol{a} \, ,$$

where $\mathbf{d}\boldsymbol{a}$*, defined through* $\boldsymbol{n}^* \wedge \mathbf{d}\boldsymbol{a} = \mathbf{d}\boldsymbol{v}$*, is called* area form.

PROOF. Note that $\mathbf{d}\boldsymbol{a}$ is effectively the $\mathbf{d}\boldsymbol{v}$ for $\partial\mathcal{M}$, thus it also has a single component. Now let $\{x^\nu\}$ be a coordinate system for $\mathcal{M}$ and choose $\partial\mathcal{M}$ to be the plane where $x^1 = 0$ and $\boldsymbol{n}^* = \mathbf{d}x^1$, then $\mathbf{d}\boldsymbol{a} = \sqrt{\det\, g_{ab}}\, \mathbf{d}x^2 \wedge \ldots \wedge \mathbf{d}x^n$ $(a, b = 2, \ldots, n)$. By 3.5.18, 3.5.19, and noting that $0! = 1$,

$$\begin{aligned} \boldsymbol{i_w}\mathbf{d}\boldsymbol{v} &= \boldsymbol{i_w}\left(\boldsymbol{n}^* \wedge \mathbf{d}\boldsymbol{a}\right) = \boldsymbol{i_w}\boldsymbol{n}^* \wedge \mathbf{d}\boldsymbol{a} - \boldsymbol{n}^* \wedge \boldsymbol{i_w}\mathbf{d}\boldsymbol{a} \\ &= w^1\sqrt{\det\, g_{ab}}\, \mathbf{d}x^2 \wedge \ldots \wedge \mathbf{d}x^n - \mathbf{d}x^1 \wedge \boldsymbol{i_w}\sqrt{\det\, g_{ab}}\, \mathbf{d}x^2 \wedge \ldots \wedge \mathbf{d}x^n\,. \end{aligned}$$

Evaluated on $\partial\mathcal{M}$, i.e. at $x^1 = 0 = \text{const.}$, the second term vanishes and one is left with $w^1\sqrt{\det\, g_{ab}}\, \mathbf{d}x^2 \wedge \ldots \wedge \mathbf{d}x^n = (\boldsymbol{n}^* \cdot \boldsymbol{w})\, \mathbf{d}\boldsymbol{a}$. ■

Instead of presuming the unit normals to be 1-forms, they may also be interpreted as a vector field $\boldsymbol{n}$ satisfying $\boldsymbol{i_n}\mathbf{d}\boldsymbol{v} = \mathbf{d}\boldsymbol{a}$ (cf. figure 3.6 and [2], pp. 136-137).

Theorem 3.7.33 (Divergence Theorem).

$$\int_{\mathcal{M}} (\operatorname{div} \boldsymbol{w})\, \mathbf{d}\boldsymbol{v} = \int_{\partial\mathcal{M}} (\boldsymbol{n}^* \cdot \boldsymbol{w})\, \mathbf{d}\boldsymbol{a}\,.$$

PROOF. The proof can be obtained by using Stokes' Theorem 3.7.21 and applying the propositions 3.7.31 and 3.7.32. ■

Chapter 4

Application: Continuum Mechanics

The geometric concepts introduced in the previous chapter should be applied to continuum mechanics in the following. It is being restricted here to the equations of kinematics with important strain measures, and the conservation of mass as example of balance equations. The derivations are mainly based on the contributions [2, 11, 12, 13]. The concept of an abstract material manifold follows Noll [27, 28]. In the last section, a framework is outlined so as to extend the geometric continuum mechanics by an Arbitrary Lagrangian-Eulerian (ALE) formulation [e.g. 22] of kinematics on manifolds.

4.1 Material Body and Ambient Space

Definition 4.1.1. Considering the geometrical definition of a continuum, let $\mathfrak{M}$ denote a sufficient differentiable *material manifold.* A submanifold $\mathfrak{B} \subset \mathfrak{M}$ is referred to as a *material body.* Its particles or material points possess the relevant properties of the material. The material body is placed in the ambient Riemannian space $\mathcal{S}$ via the embedding

$$\kappa : \quad \mathfrak{B} \quad \rightarrow \quad \mathcal{S} \,,$$

and changes in configurations are noticed and measured in $\mathcal{S}$. One refers to $\mathcal{B} = \kappa(\mathfrak{B}) \subset \mathcal{S}$, where $\mathfrak{B} \rightarrow \kappa(\mathfrak{B})$ is a diffeomorphism, as the *reference configuration* of the body and to $P \in \mathcal{B}$ as the *particles* of the body in the reference configuration. ♢

Definition 4.1.2. Given a reference configuration $\mathcal{B}$, the set $\mathcal{C} = \{\varphi_t \,|\, \varphi_t : \mathcal{B} \rightarrow \mathcal{S}\}$ of embeddings φ_t is called the *configuration space* —configuration is used here as a synonym for deformation. A *motion* of the body is a family of configurations dependent on time $t \in \mathbb{R}$, i.e. a curve $t \mapsto \varphi_t \in \mathcal{C}$:

$$\begin{aligned} \varphi_t : \quad \mathcal{B} \quad &\rightarrow \quad \mathcal{S} \\ P \quad &\mapsto \quad \varphi_t(P) = Q \,, \end{aligned}$$

with $\varphi_t(P) = \varphi(P,t)$ at fixed t. $\varphi_t(\mathcal{B})$ is referred to as the *current configuration* of the body (figure 4.1). The *placement* κ_t at time t, then, can be defined through $\varphi_t = \kappa_t \circ \kappa^{-1}$. ♢

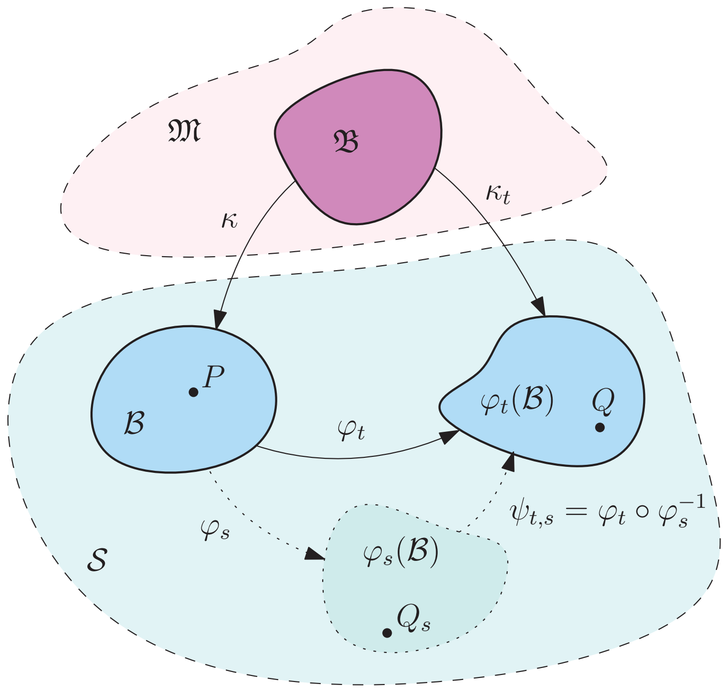

Figure 4.1: Material body $\mathfrak{B}$, reference configuration $\mathcal{B}$, current configuration $\varphi_t(\mathcal{B})$ and related mappings.

Corollary 4.1.3. *Let $\varphi_0(\mathcal{B}) = \mathcal{B}$, that is $\varphi_0 = \mathrm{Id}_{\mathcal{S}}$, then φ_t is a one-parameter group of mappings (see 3.6.19 for details), and $\mathcal{B} \equiv \mathcal{B}_0$ is called the* initial configuration *of the body at $t = 0$.*

If $\mathcal{B}$ is the initial configuration, the particles of the body are identified with its initial places in $\mathcal{S}$. Although the choice of a initial configuration is not necessary to describe the kinematics and kinetics of the body, it has historical significance in continuum mechanics.

To the neighborhoods $\mathcal{U}(P) \subset \mathcal{B}$, charts $(\mathcal{U}, X)$ with local coordinate functions X^I can be assigned. As $\mathcal{S}$ and $\mathcal{B}$ are differentiable manifolds, the partial derivatives $\left\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}\right\} \in T_P\mathcal{B}$ establish a basis of the tangent space at each P, and the coordinate differentials $\left\{\mathbf{d}X^I\right\} \in T_P^*\mathcal{B}$ is its dual in the cotangent space. Since $\mathcal{S}$ is Riemannian, $G_{IJ}(P) = \left\langle\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J}\right\rangle_P$ are the metric coefficients on $\mathcal{B}$. The current configuration contains the *current places* $Q = \varphi_t(P) \in \mathcal{S}$ of the particles of the material body. Charts of the neighborhoods $\mathcal{V}(Q) \subset \mathcal{S}$ are denoted by the pairs $(\mathcal{V}, x)$, with $\{x^i\}_Q = x(Q) \in \mathbb{R}^3$, that is, x^i are the coordinates of the ambient space. The definition of a local base $\left\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\right\} \in T_{\varphi_t(P)}\mathcal{S}$ and the dual base $\{\mathbf{d}x^i\} \in T^*_{\varphi_t(P)}\mathcal{S}$ is straightforward. The $g_{ij}(Q) = \left\langle\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j}\right\rangle_Q$ are the metric coefficients on $\mathcal{S}$.

As configurations are understood as embeddings, chart transitions from X to x and vice versa are compatible. Herein, regularity of the local coordinate systems is demanded everywhere in order to exclude breaks and cracks of the material respectively the ambient space.

The difference between Q and $\varphi_t(P)$ is important: the spatial points Q —and also $\mathcal{S}$— are independent of the motion of the body. By contrast, $\varphi_t(P)$ denotes the point occupied by the particle P at time t in the ambient space, and $\varphi_t(\mathcal{B})$ only considers those spatial points which are occupied by the body during its motion.

As a common notation, upper case Latins are used for coordinates ($I, J, K, \ldots \in \{1, 2, 3\}$), vectors and tensors of the reference configuration, or are related to the Lagrangian formulation defined later on. Lower case Latins are related to the current configuration, the ambient space, or to the Eulerian formulation. This is only a general convention, inconsistency might be useful. Moreover, in order to reduce the amount of necessary symbols within the text and in formulas, the picture $X(P)$ in the chart is often identified with the particle P and a function $f(X) = f \circ X(P)$, for example, is written $f(P)$. However, it should be emphasised that there is a fundamental difference between the object in the manifold and its picture or localization in the chart.

The embedding of the body in space enables the observation of the body, and the measurement of physical properties or deformations, as $\mathcal{B}$ benefits from the inner product in $\mathcal{S}$. In the material manifold $\mathfrak{M}$ it does not exist any metric because of the arbitrary choice of the map κ [27, 28]. Lengths and angles change permanently during a deformation process, hence it does not make any sense to define a distance and suchlike measures. The material manifold is merely a continuous assembly of particles with certain physical properties. It is indeed a useful construction in fracture mechanics and in material sciences, for example. Therein, it is used to establish microstructure of the material or forces on cracks, inclusions and dislocations of crystal lattices (cf. configuration mechanics, Eshelby mechanics).

It is assumed throughout this chapter that both the ambient space and the body are three-dimensional. However, for some concepts it is illuminative to include the case where $\dim(\mathfrak{B}) < \dim(\mathcal{S})$. For example, if $\mathfrak{B}$ is a shell, note that $\dim(\mathfrak{B}) = \dim(\mathcal{B}) = 2$, because κ is an embedding. Since φ_t is also an embedding and $\varphi_t(\mathcal{B}) \subset \mathcal{S}$ is a submanifold, $\dim(\varphi_t(\mathcal{B})) = 2$ and charts of $\varphi_t(\mathcal{B})$ and $\mathcal{S}$ are compatible in the sense of 3.1.26. Note that even if $\mathcal{B}$ and $\mathcal{S}$ have different dimensions, φ_t can be invertible, i.e. a regular map.

Definition 4.1.4. Let $\varphi_t : \mathcal{B} \to \mathcal{S}$ be the motion of a material body, and let $\mathcal{U} \subset \mathcal{B}$, $\mathcal{V} \subset \mathcal{S}$ and $\varphi_t^{-1}(\mathcal{V}) \cap \mathcal{U} \neq \emptyset$. Additionally, let $(\mathcal{U}, X)$, $(\mathcal{V}, x)$ be appropriate charts, then, analogous to definition 3.1.21,

$$x \circ \varphi_t \circ X^{-1}\big|_{X\left(\varphi_t^{-1}(\mathcal{V}) \cap \mathcal{U}\right)} : \quad X\left(\varphi_t^{-1}(\mathcal{V}) \cap \mathcal{U}\right) \;\to\; x\left(\varphi_t^{-1}(\mathcal{V}) \cap \mathcal{U}\right) .$$

defines the *chart transition concerning* φ_t or the *localization of the motion*. If $\{x^i\}_Q = x(Q)$ for $Q \in \mathcal{V}$, then abbreviate $\varphi_t^i = x^i \circ \varphi_t \circ X^{-1}$, so $x^i \circ \varphi_t = \varphi_t^i$ are the coordinates on $\varphi_t(\mathcal{B})$. ◇

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "T\varphi_t"] & T\mathcal{S} \\
\mathcal{B} \arrow[u, "\mathfrak{v}_t"] \arrow[ur, "\boldsymbol{V}_t=T\varphi_t(\mathfrak{v}_t)"'] \arrow[r, "\varphi_t"'] & \mathcal{S} \arrow[u, "\boldsymbol{v}_t=\varphi_{t\star}\mathfrak{v}_t=\boldsymbol{V}_t\circ\varphi_t^{-1}"']
\end{tikzcd}$$

Figure 4.2: Material, Lagrangian and Eulerian velocity fields.

Definition 4.1.5. Let φ_t be a C^1-motion of $\mathcal{B}$ in $\mathcal{S}$, that is, φ_t is at least 1-fold continuous differentiable, and let the spatial coordinates $x^i \circ \varphi_t = \varphi_t^i$ be given as functions of the motion φ_t according to 4.1.4, then

$$\boldsymbol{V}_t = \left.\frac{\partial \varphi_t^i}{\partial t}\right|_P \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = V_t^i(P)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}(Q)$$

describes the particle velocity field at time t, called the *Lagrangian velocity field over* φ_t. Clearly: $\boldsymbol{V}_t : \mathcal{B} \to T\mathcal{S}$, with $\boldsymbol{V}_t(P) = \boldsymbol{V}(P,t)$ at fixed t. The corresponding *spatial* or *Eulerian velocity field*,

$$\boldsymbol{v}_t = \boldsymbol{V}_t \circ \varphi_t^{-1} = \left(\left.\frac{\partial \varphi_t^i}{\partial t}\right|_P \circ \varphi_t^{-1} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = v_t^i(Q)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \qquad \in \Gamma(T\mathcal{S})\,,$$

where $Q = \varphi(P,t)$, is obtained by switching the point arguments (figure 4.2). ♢

$\boldsymbol{v}_t(Q) = \boldsymbol{v}(Q,t)$ is a proper vector field on $\mathcal{S}$ as it follows the motion on $\varphi_t(\mathcal{B})$; in other words, $\boldsymbol{v}_t$ is the "instantaneous" velocity at Q. The Lagrangian velocity field $\boldsymbol{V}$ is certainly no vector field on $\mathcal{B}$, because $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}(Q) \in T_Q\mathcal{S}$ depends on the choice of Q.

Definition 4.1.6. Let φ_t be the motion of $\mathcal{B}$ in $\mathcal{S}$, and $\boldsymbol{v}_t$ the spatial velocity, then the vector field $\mathfrak{v}_t = \varphi_t^\star \boldsymbol{v}_t \in \Gamma(T\mathcal{B})$ on the initial configuration is called the *material velocity*, that goes along with the use of so-called *convected coordinates* $\xi_t^I(x(Q)) = X^I \circ \varphi_t^{-1}$. ♢

Corollary 4.1.7. *In the chart $(\mathcal{U}, X)$, $\mathcal{U} \subset \mathcal{B}$, set $\mathfrak{v} = \mathfrak{v}^I \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}$. Then, by noting that $\varphi_\xi^I(P) = \xi_t^I \circ x \circ \varphi_t = X^I \circ \varphi_t^{-1} \circ \varphi_t = X^I(P)$,*

$$\boldsymbol{V}_t = T\varphi_t(\mathfrak{v}_t) = \frac{\partial \varphi_\xi^J}{\partial X^I} \mathfrak{v}_t^I \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \xi_t^J} = \mathfrak{v}_t^I(P) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \xi_t^I}(Q)\,.$$

The $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \xi^I}(Q,t)$ are time-dependent basis vectors that are scribed on $\mathcal{B}$ and follow its deformation. For φ_t being a one-parameter group, $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \xi_t^I} \equiv \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}$ at $t = 0$.

If $\mathcal{B}$ and $\mathcal{S}$ have different dimensions, as this is the case for shells, the Eulerian velocity field $\boldsymbol{v}_t$ is tangent to $\mathcal{S}$, but it is not necessarily tangent to $\varphi_t(\mathcal{B})$. Thus, defining the material velocity through $\boldsymbol{v}_t = \varphi_{t\star}\mathfrak{v}_t$ has no physical meaning. Note that in the literature, $\mathfrak{v}$ is often called the convective velocity. This, however, conflicts the term used for a fundamental velocity field in the ALE formulation.

Considering section 3.6.2, every proper vector field $\boldsymbol{v}_t \in \Gamma(T\mathcal{S})$ generates a flow $\psi_{t,s}$ on $\mathcal{S}$. Hence, one may ask: what is the flow of the spatial velocity field associated with the motion φ_t? The following proposition gives the answer (see also [2], p. 95).

Proposition 4.1.8. *The time-dependent flow $\psi_{t,s}$ on $\mathcal{S}$ associated with the regular motion $\varphi_t : \mathcal{B} \to \mathcal{S}$ is given by*

$$\psi_{t,s} = \varphi_t \circ \varphi_s^{-1} : \quad \mathcal{S} \supset \varphi_s(\mathcal{B}) \quad \to \quad \varphi_t(\mathcal{B}) \subset \mathcal{S}\,, \quad \exists\, \psi_{t,s}^{-1}\,, \quad t, s \in \mathbb{R}\,.$$

Proof. Note that before reaching $Q = \varphi_t(P)$ at time t, the particles pass some points $Q_s = \varphi_s(P)$ at $t = s$ (figure 4.1). For the components of the spatial velocity $\boldsymbol{v}$ at Q_s,

$$v_t^i(Q_s) = \left.\frac{\partial \varphi_t^i}{\partial t}\right|_P \circ \varphi_s^{-1} = \frac{\partial}{\partial t}\varphi_t^i\left(\varphi_s^{-1}(Q_s)\right) = \frac{\partial}{\partial t}\psi_{t,s}^i(Q_s)\,.$$

Since $\psi_{s,s}(Q_s) = Q_s$, the assertion follows. ■

Proposition 4.1.9. *The Lie derivative of arbitrary $\boldsymbol{t} \in \mathfrak{T}_q^p(\mathcal{S})$ along the spatial velocity is*

$$\mathrm{L}_{\boldsymbol{v}}\boldsymbol{t} = (\varphi_t \circ \varphi_s^{-1})_\star \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t \circ \varphi_s^{-1})^\star\, \boldsymbol{t} = \varphi_{t\star} \circ \varphi_s^\star \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_{s\star} \circ \varphi_t^\star \boldsymbol{t}) = \varphi_{t\star}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\star\, \boldsymbol{t}\,.$$

Proof. This follows from the previous proposition, by recalling the chain rule 3.4.32, using 3.6.25, and by noting that $(\varphi_s^\star)^{-1} = (\varphi_s^{-1})^\star = \varphi_{s\star}$. ■

4.2 Deformation Gradient and Strain Measures

The definitions made in section 4.1, concerning φ_t, $\mathcal{B}$, $\mathcal{S}$, coordinates X^I and x^i etc., are used throughout.

Definition 4.2.1. Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a configuration. According to 3.4.23, a two-point tensor $\boldsymbol{F}$, called the *deformation gradient*, is assigned to the tangent map $T\varphi$:

$$\begin{array}{rcl} T\varphi : \; T_P\mathcal{B} & \to & T_{\varphi(P)}\mathcal{S} \\ \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} & \mapsto & T\varphi\left(\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}\right) = \dfrac{\partial \varphi^i}{\partial X^I}\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \boldsymbol{F}\cdot\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J}\,, \end{array}$$

that is, for each $P \in \mathcal{B}$,

$$\boldsymbol{F}(P) = \frac{\partial \varphi^i}{\partial X^I}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^I = F^i{}_I(P)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^I \qquad \in \Gamma(\varphi^\star T\mathcal{S} \otimes T^*\mathcal{B})\,.$$

The expression $\varphi^\star T\mathcal{S}$ denotes the induced bundle of $T\mathcal{S}$ over the map φ according to 3.4.12, i.e. $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ attached to $\varphi(P)$ being understood. If φ_t is a motion of $\mathcal{B}$ in $\mathcal{S}$, then $\boldsymbol{F}$ depends on both $P \in \mathcal{B}$ and $t \in \mathbb{R}$, that is, $\boldsymbol{F}(P,t) = F^i{}_I(P,t)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}X^I$. ◇

Corollary 4.2.2. *(i) By 3.3.20 and 3.3.21, the transpose of the deformation gradient is*

$$\begin{aligned}\boldsymbol{F}^{\mathrm{T}} : \; T\mathcal{S} \supset T(\varphi(\mathcal{B})) &\to T\mathcal{B} \\ \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} &\mapsto \boldsymbol{F}^{\mathrm{T}} \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = (\boldsymbol{F}^{\mathrm{T}})^I{}_i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} = g_{ij} F^j{}_J G^{IJ} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}\end{aligned}$$

for each $P = \varphi^{-1}(Q)$. In fussy local notation this reads

$$\begin{aligned}\boldsymbol{F}^{\mathrm{T}}(Q) &= g_{ij} \left(\frac{\partial \varphi^j}{\partial X^J} \circ \varphi^{-1} \right) \left(G^{IJ} \circ \varphi^{-1} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J} \otimes \mathbf{d} x^j \\ &= \left(F^i{}_I \circ \varphi^{-1} \right) \mathbf{d} X^I \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} .\end{aligned}$$

(ii) Recall 3.3.22 and 3.4.24, then conclude that the inverse deformation gradient $\boldsymbol{F}^{-1}(P) : T_{\varphi(P)}\mathcal{S} \to T_P\mathcal{B}$ corresponds to the inverse tangent map with switched base points, i.e. $\boldsymbol{F}^{-1} = T(\varphi^{-1}) \circ \varphi$. Therefore,

$$\boldsymbol{F}^{-1}(P) = \left(\frac{\partial (\varphi^{-1})^I}{\partial x^i} \circ \varphi \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \otimes \mathbf{d} x^i ,$$

where $(\varphi^{-1})^I = X^I \circ \varphi^{-1} \circ x^{-1}$ and $\frac{\partial (\varphi^{-1})^I}{\partial x^i}(Q) \circ \varphi = \frac{\partial (\varphi^{-1})^I}{\partial x^i}(P)$.

(iii) From 3.3.22, also obtain the inverse transpose of the deformation gradient:

$$\begin{aligned}\boldsymbol{F}^{-\mathrm{T}}(Q) &= g^{ij} \frac{\partial (\varphi^{-1})^J}{\partial x^j} \left(G_{IJ} \circ \varphi^{-1} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d} X^I \\ &= \left((\boldsymbol{F}^{-1})^I{}_i \circ \varphi^{-1} \right) \mathbf{d} x^i \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} .\end{aligned}$$

Applying the relations to 3.4.36 proofs the following corollary concerning the pushforward and pullback of tensor fields on $\mathcal{B}$ and $\mathcal{S}$, respectively.

Corollary 4.2.3. *(i) Let $\boldsymbol{T} \in \mathfrak{T}^p_q(\mathcal{B})$, $\boldsymbol{a}^*_1, \ldots, \boldsymbol{a}^*_p \in \Gamma(T^*\mathcal{S})$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_q \in \Gamma(T\mathcal{S})$, and let $\varphi : \mathcal{B} \to \mathcal{S}$ be a regular configuration. Then, by dropping the point argument,*

$$\begin{aligned}&\varphi_\star \boldsymbol{T} \left(\boldsymbol{a}^*_1, \ldots, \boldsymbol{a}^*_p, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_q \right) \\ &\qquad = \boldsymbol{T} \left((\boldsymbol{a}^*_1 \cdot \boldsymbol{F}), \ldots, (\boldsymbol{a}^*_p \cdot \boldsymbol{F}), (\boldsymbol{F}^{-1} \cdot \boldsymbol{w}_1), \ldots, (\boldsymbol{F}^{-1} \cdot \boldsymbol{w}_q) \right) .\end{aligned}$$

(ii) On the other hand, if $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{S})$, $\boldsymbol{B}^*_1, \ldots, \boldsymbol{B}^*_p \in \Gamma(T^*\mathcal{B})$ and $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_q \in \Gamma(T\mathcal{B})$, then

$$\begin{aligned}&\varphi^\star \boldsymbol{t} \left(\boldsymbol{B}^*_1, \ldots, \boldsymbol{B}^*_p, \boldsymbol{W}_1, \ldots, \boldsymbol{W}_q \right) \\ &\qquad = \boldsymbol{t} \left((\boldsymbol{B}^*_1 \cdot \boldsymbol{F}^{-1}), \ldots, (\boldsymbol{B}^*_p \cdot \boldsymbol{F}^{-1}), (\boldsymbol{F} \cdot \boldsymbol{W}_1), \ldots, (\boldsymbol{F} \cdot \boldsymbol{W}_q) \right) .\end{aligned}$$

Example 4.2.4. Let $\boldsymbol{t} = t_{ij}\, \mathbf{d}x^i \otimes \mathbf{d}x^j \in \mathfrak{T}^0_2(\mathcal{S})$ and $\boldsymbol{W}_k = W^I_k \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \in \Gamma(T\mathcal{B})$, $k = 1, 2$, then

$$
\begin{aligned}
&\varphi^\star \boldsymbol{t}\,(\boldsymbol{W}_1, \boldsymbol{W}_2) = \boldsymbol{t}\,((\boldsymbol{F}\cdot\boldsymbol{W}_1), (\boldsymbol{F}\cdot\boldsymbol{W}_2)) \\
&= t_{ij} W^I_1 W^J_2 F^k{}_K F^l{}_L \left(\mathbf{d}x^i \cdot \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \otimes \mathbf{d}X^K \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I}\right)\right) \otimes \left(\mathbf{d}x^j \cdot \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^l} \otimes \mathbf{d}X^L \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^J}\right)\right) \\
&= t_{ij} W^I_1 W^J_2 F^k{}_I F^l{}_J \left(\mathbf{d}x^i \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k}\right)\left(\mathbf{d}x^j \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^l}\right) \\
&= t_{ij} W^I_1 W^J_2 F^i{}_I F^j{}_J\,.
\end{aligned}
$$

Since $\boldsymbol{W}_1, \boldsymbol{W}_2$ are arbitrary, $(\varphi^\star \boldsymbol{t})_{IJ} = t_{ij} F^i{}_I F^j{}_J$.

Proposition 4.2.5. *Let $\boldsymbol{t} \in \mathfrak{T}^0_2(\mathcal{S})$, then, without point arguments,*

$$\varphi^\star \boldsymbol{t} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{t} \cdot \boldsymbol{F}\,.$$

Proof. By 4.2.3(ii), $\varphi^\star \boldsymbol{t}\,(\cdot,\cdot) = \boldsymbol{t}\,(\boldsymbol{F}(\cdot), \boldsymbol{F}(\cdot))$. However, for $\boldsymbol{t} = t_{ij}\, \mathbf{d}x^i \otimes \mathbf{d}x^j$,

$$
\begin{aligned}
\boldsymbol{t}\,(\boldsymbol{F}(\cdot), \boldsymbol{F}(\cdot)) &= t_{ij}\, \mathbf{d}x^i(\boldsymbol{F}(\cdot)) \otimes \mathbf{d}x^j(\boldsymbol{F}(\cdot)) \\
&= t_{ij} g^{ik} \left\langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k}, \boldsymbol{F}(\cdot) \right\rangle \otimes \left(\mathbf{d}x^j \cdot \boldsymbol{F}(\cdot)\right) \\
&= t_{ij} g^{ik} \left\langle \boldsymbol{F}^{\mathrm{T}} \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k}, (\cdot) \right\rangle \otimes \left(\mathbf{d}x^j \cdot \boldsymbol{F}(\cdot)\right) \\
&= t_{ij}\, \boldsymbol{F}^{\mathrm{T}} \cdot \mathbf{d}x^i(\cdot) \otimes \mathbf{d}x^j \cdot \boldsymbol{F}(\cdot) \\
&= (\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{t} \cdot \boldsymbol{F})\,(\cdot,\cdot)
\end{aligned}
$$

by 3.3.16 and 3.3.20. It is important to keep in mind that for the proposition to be valid, a metric structure on $\mathcal{S}$ is required. ■

Corollary 4.2.6. *Operate analogously to show that if (i) $\boldsymbol{s} \in \mathfrak{T}^2_0(\mathcal{S})$, then $\varphi^\star \boldsymbol{s} = \boldsymbol{F}^{-1} \cdot \boldsymbol{s} \cdot \boldsymbol{F}^{-\mathrm{T}}$. (ii) Let $\boldsymbol{S} \in \mathfrak{T}^2_0(\mathcal{B})$, then $\varphi_\star \boldsymbol{S} = \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{\mathrm{T}}$, and (iii) $\varphi_\star \boldsymbol{T} = \boldsymbol{F}^{-\mathrm{T}} \cdot \boldsymbol{T} \cdot \boldsymbol{F}^{-1}$ for $\boldsymbol{T} \in \mathfrak{T}^0_2(\mathcal{B})$.*

Definition 4.2.7. In numerical applications of continuum mechanics it is the tangent map $T\varphi = \boldsymbol{F}$ of the configuration $\varphi : \mathcal{B} \to \mathcal{S}$ that is usually given, and not φ itself. Therefore, it might be convenient to borrow the notation of definition 3.4.20 and to replace $\varphi^\star$ by $\boldsymbol{F}^\star$, and $\varphi_\star$ by $\boldsymbol{F}_\star$. For example, if $T\varphi = \boldsymbol{R} : T_P\mathcal{B} \to T_Q\mathcal{S}$ is orthogonal and $\boldsymbol{t} \in \mathfrak{T}^0_2(\mathcal{S})$, then $\boldsymbol{R}^\star \boldsymbol{t} = \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{t} \cdot \boldsymbol{R}$ denotes the pullback of $\boldsymbol{t}$ over pure rotational φ, also called the ***R**-pullback of **t***. ◇

The deformation gradient includes stretches as well as rigid rotations of the body. There are different choices to split off the stretches and, therefore, different strain measures can be defined. Their applicability depends on the problem and is controversially discussed. However, all strain measures are symmetric tensors by definition.

Definition 4.2.8. The *right Cauchy-Green tensor* $\boldsymbol{C} : T\mathcal{B} \to T\mathcal{B}$, often just called the *deformation tensor*, is a so-called *Lagrangian strain measure*, and it is defined for every $P \in \mathcal{B}$ through

$$\boldsymbol{C}(P) = (\boldsymbol{F}^{\mathrm{T}} \circ \varphi) \cdot \boldsymbol{F} = G^{KI}\,(g_{ik} \circ \varphi)\, F^k{}_K F^i{}_J \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \otimes \mathbf{d}X^J\,. \qquad \diamondsuit$$

Proposition 4.2.9. *Let $\boldsymbol{g} = g_{ij}\,\mathbf{d}x^i \otimes \mathbf{d}x^j \in \mathfrak{T}^0_2(\mathcal{S})$ be the local representative of the spatial metric tensor, then*

$$\boldsymbol{C}^\flat = \varphi^\star \boldsymbol{g}\,.$$

Proof. By 4.2.5, 4.2.2(i) and 3.3.16,

$$\varphi^\star \boldsymbol{g}(P) = (\boldsymbol{F}^{\mathrm{T}} \circ \varphi) \cdot (\boldsymbol{g} \circ \varphi) \cdot \boldsymbol{F} = (g_{ij} \circ \varphi) F^i{}_I F^j{}_J\, \mathbf{d}X^I \otimes \mathbf{d}X^J = \boldsymbol{C}^\flat(P)\,. \qquad \blacksquare$$

Definition 4.2.10. The *Green-Lagrange strain tensor* or *material strain tensor* $\boldsymbol{E} : T\mathcal{B} \to T\mathcal{B}$ is also a Lagrangian strain measure. It is defined through

$$2\boldsymbol{E}\,(P) = (\boldsymbol{C} - \boldsymbol{I}_{\mathcal{B}})(P) = \left(C^I{}_J - \delta^I{}_J\right)(P) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \otimes \mathbf{d}X^J\,,$$

where $\boldsymbol{I}_{\mathcal{B}}$ is the second-rank unit tensor and $\delta^I{}_J$ is the Kronecker delta on $\mathcal{B}$, respectively. Note that $\boldsymbol{E}^\flat = \frac{1}{2}(\boldsymbol{C}^\flat - \boldsymbol{G})$, where $\boldsymbol{G} = G_{IJ}\mathbf{d}X^I \otimes \mathbf{d}X^J$ is the metric on $\mathcal{B}$. $\diamondsuit$

Definition 4.2.11. *Eulerian strain measures* can be defined as mappings $T^*\mathcal{S} \times T\mathcal{S} \to \mathbb{R}$ on the current configuration. The *left Cauchy-Green tensor* on $\mathcal{S}$ is

$$\boldsymbol{b} = \left(\boldsymbol{F} \circ \varphi^{-1}\right) \cdot \boldsymbol{F}^{\mathrm{T}}\,.$$

Suppressing the arguments, in local coordinates one has

$$\boldsymbol{b} = G^{IJ} g_{jk} F^i{}_I F^k{}_J \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j\,.$$

In contrast to the right Cauchy-Green tensor, the left Cauchy-Green tensor requires $\varphi : \mathcal{B} \to \mathcal{S}$ to be regular (i.e. invertible). $\diamondsuit$

Definition 4.2.12. Let the configuration map φ be regular, then define the *Euler-Almansi strain tensor* or *spatial strain tensor* $\boldsymbol{e} : T\mathcal{S} \to T\mathcal{S}$ through

$$2\boldsymbol{e} = \boldsymbol{i}_{\mathcal{S}} - \boldsymbol{c} = \left(\delta^i{}_j - c^i{}_j\right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}x^j\,,$$

in which $\boldsymbol{c} = \boldsymbol{b}^{-1}$ is called the *Finger tensor*, $\boldsymbol{i}_{\mathcal{S}}$ is the unit tensor and $\delta^i{}_j$ is the Kronecker delta on $\mathcal{S}$, respectively. $\diamondsuit$

Corollary 4.2.13. *By 4.2.6, $(\varphi_\star \boldsymbol{G})_{ij} = (G^{IJ})^{-1}(\boldsymbol{F}^{-1})^I{}_i(\boldsymbol{F}^{-1})^J{}_j = (\boldsymbol{g}^\flat \cdot \boldsymbol{b}^{-1})_{ij}$, so $\varphi_\star \boldsymbol{G} = \boldsymbol{c}^\flat$. From this and 4.2.9,*

$$\varphi_\star \boldsymbol{E}^\flat = \frac{1}{2}(\varphi_\star \boldsymbol{C}^\flat - \varphi_\star \boldsymbol{G}) = \frac{1}{2}(\boldsymbol{g} - \boldsymbol{c}^\flat) = \boldsymbol{e}^\flat\,.$$

An important observation on manifolds respectively in Riemannian spaces is that there are indeed strains, but no displacement fields such a kind that $\boldsymbol{u}(P) = (\boldsymbol{x} - \boldsymbol{X})(P)$, because of the loss of position vectors.

Several problems ask for the rate of strain. Because of the time-dependence of the motion $\varphi(\cdot, t) = \varphi_t$ of the material body, all strain measures are time-dependent.

Definition 4.2.14. Let the motion $\varphi_t : \mathcal{B} \to \mathcal{S}$ be differentiable and $\boldsymbol{C}$ the right Cauchy-Green tensor, then the *Lagrangian* or *material rate of deformation tensor* is defined by

$$2\boldsymbol{D}(P,t) = \frac{\partial}{\partial t}\boldsymbol{C}(P,t)\,.$$

Note that an equivalent definition is $\boldsymbol{D} = \partial \boldsymbol{E}/\partial t$. $\diamondsuit$

Proposition 4.2.15. *The components of* $\boldsymbol{D}$ *are*

$$2D^I{}_J(P,t) = G^{KI}(g_{ik}\circ\varphi_t)\left[\left(\nabla^\star_K V^k\right)F^i{}_J + \left(\nabla^\star_J V^i\right)F^k{}_K\right]\,,$$

where $\varphi_t(P) = \varphi(P,t)$, V^i *are the components of the Lagrangian velocity, and* $\boldsymbol{\nabla}^\star$ *is the connection on* $\mathcal{B}$ *induced by the connection* $\boldsymbol{\nabla}$ *on* $\mathcal{S}$ *(see section 3.6.1).*

Proof. Let $\gamma_i{}^j{}_k$ be the coefficients of $\boldsymbol{\nabla}$ on $\mathcal{S}$, then by 3.6.8 and 3.6.13,

$$\frac{\partial V^i}{\partial X^I} = \nabla^\star_I V^i - V^j\gamma_j{}^i{}_k F^k{}_I = \nabla^\star_I V^i - \frac{1}{2}V^j F^k{}_I g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)\,.$$

By definition 4.2.8, $G_{IK}(P)C^I{}_J(P,t) = (g_{ik}\circ\varphi(P,t))\,F^k{}_K(P,t)F^i{}_J(P,t)$ —note that the metric coefficients G_{IJ} are time-independent. By using the abbreviation $\varphi^i_t(P) = x^i\circ\varphi_t\circ X^{-1}$ and ommiting the arguments, differentiating in t then gives

$$2G_{IK}D^I{}_J = \frac{\partial g_{ik}}{\partial x^j}V^jF^k{}_KF^i{}_J + g_{ij}\frac{\partial V^j}{\partial X^K}F^i{}_J + g_{ij}F^j{}_K\frac{\partial V^i}{\partial X^J}\,.$$

Now substitute the identity for $\frac{\partial V^i}{\partial X^I}$ to obtain

$$\begin{aligned}
2G_{IK}D^I{}_J &= \frac{\partial g_{ik}}{\partial x^j}V^jF^k{}_KF^i{}_J \\
&+g_{ik}(\nabla^\star_K V^k)F^i{}_J - \frac{1}{2}V^jF^m{}_KF^i{}_J\left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^m} - \frac{\partial g_{jm}}{\partial x^i}\right) \\
&+g_{ik}(\nabla^\star_J V^i)F^k{}_K - \frac{1}{2}V^jF^i{}_JF^k{}_K\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k}\right) \\
&= g_{ik}\left[(\nabla^\star_K V^k)F^i{}_J + (\nabla^\star_J V^i)F^k{}_K\right] + V^jF^i{}_JF^k{}_K\frac{\partial g_{ik}}{\partial x^j} \\
&-\frac{1}{2}V^jF^i{}_JF^k{}_K\frac{\partial g_{ki}}{\partial x^j} - \frac{1}{2}V^jF^i{}_JF^k{}_K\frac{\partial g_{ik}}{\partial x^j} \\
&= g_{ik}\left[(\nabla^\star_K V^k)F^i{}_J + (\nabla^\star_J V^i)F^k{}_K\right]\,,
\end{aligned}$$

g_{ij} and $\frac{\partial g_{ij}}{\partial x^k}$ evaluated at P being understood. Multiplying both sides with G^{KL} and relabelling the indices then gives the result. ∎

Definition 4.2.16. Let φ_t be a regular motion of $\mathcal{B}$ in $\mathcal{S}$, then define the *Eulerian* or *spatial rate of deformation tensor* $\boldsymbol{d}$, where $\boldsymbol{d}_t \in \mathfrak{T}^1_1(\mathcal{S})$ for t fixed, through

$$\boldsymbol{d}^\flat = \varphi_{t\star}(\boldsymbol{D}^\flat)\,,$$

or, equivalently, $\boldsymbol{d}^\flat = \partial \boldsymbol{e}^\flat/\partial t$. ◇

Note that for arbitrary tensor fields $\boldsymbol{T}$, $\varphi_\star(\boldsymbol{T}^\flat) \neq (\varphi_\star \boldsymbol{T})^\flat$ in general!

Proposition 4.2.17. *Let $\boldsymbol{v}$ be the spatial velocity and $\boldsymbol{g}$ the spatial metric, then*

$$2\boldsymbol{d}^\flat(Q,t) = \mathrm{L}_{\boldsymbol{v}}\boldsymbol{g}\,.$$

Proof. Definition 4.2.16 implies $2\boldsymbol{D}^\flat = \frac{\partial}{\partial t}\boldsymbol{C}^\flat$, and from 4.2.14 and proposition 4.2.9 one has

$$2\boldsymbol{d}^\flat = \varphi_\star \frac{\partial}{\partial t}\varphi^\star \boldsymbol{g}\,.$$

Now setting $\frac{\partial}{\partial t}\varphi^\star = \frac{\mathrm{d}}{\mathrm{d}t}\varphi^\star_t$ makes sense, because the pullback describes a curve of tensors with parameter t. The result then follows from proposition 4.1.9. (Consult [2], p. 98, for an alternative proof.) ■

4.3 Lagrangian and Eulerian Formulations

The following section should give a precise definition of the Lagrangian and Eulerian formulations of classic resp. traditional continuum mechanics, and it points to the different use of the term "convective".

Definition 4.3.1. A $\binom{p}{q}$-tensor-valued *physical field* f on the ambient space $\mathcal{S}$, also called a *spatial field*, is a function

$$f:\ \mathcal{S}\times\mathcal{I} \ \rightarrow\ T^p_q(\mathcal{S})\,,$$

where $\mathcal{I} \subset \mathbb{R}$. Dependent on the order of p and q, the physical field thus appears as a tensor-valued function of $(Q,t) \in (\mathcal{S}\times\mathcal{I})$, i.e. a time-dependent scalar, vector or tensor field f on $\mathcal{S}$. The field is presumed to be measurable on a subset $\mathcal{V} \subset \mathcal{S}$ filled with matter; however, f would be meaningless in an "empty space". ◇

Definition 4.3.2. Let $\varphi : \mathcal{B}\times\mathcal{I} \rightarrow \mathcal{S}$ be the motion of a material body, then the physical field defined in 4.3.1, but restricted to $\varphi(\mathcal{B},t)$, can be written as

$$f\circ\varphi:\ \mathcal{B}\times\mathcal{I}\times\mathcal{I} \ \rightarrow\ T^p_q(\mathcal{S})\,.$$

Here $\mathcal{I}\times\mathcal{I} = \{(t,t)\,|\,t\in\mathcal{I}\}$ is meant to be the diagonal, i.e. both "t" in $f(\varphi(P,t),t)$ take the same value. Depending on whether $Q = \varphi_t(P) \in \mathcal{S}$ or $P \in \mathcal{B}$ serve as the independent variables, one refers to $f_t(Q) = f(Q,t)$ as the *Eulerian formulation*, and to $F_t(P) = f_t \circ \varphi_t$ as the *Lagrangian formulation* of the field at fixed t, respectively. ◇

Corollary 4.3.3. *The $\binom{p}{q}$-tensor-valued Lagrangian field F is understood as a map $F : \mathcal{B} \times \mathcal{I} \to T^p_q(\mathcal{S})$. The difference between F and $f \circ \varphi$ defined vanishes for fixed t, that is,*

$$F_t = f_t \circ \varphi_t : \mathcal{B} \;\; \to \;\; T^p_q(\mathcal{S}) \, ,$$

so $F_t = f_t \circ \varphi_t$ is a $\binom{p}{q}$-tensor field over φ_t.

It is notable that the distinction between the Lagrangian formulation and the Eulerian formulation of the same physical field is only valid if there are affine point spaces resp. general manifolds together with a point map. In ordinary vector spaces, the equation $F_t(P) = f_t \circ \varphi_t$ does not make sense, as the related tensors have no base points. However, for simplicity, the difference between f, F and their localizations $f \circ x^{-1}$ and $F \circ X^{-1}$, respectively, is dropped throughout.

Definition 4.3.4. Another way to formulate the spatial field f in terms of the reference configuration $\mathcal{B}$ is to pull it back by the motion φ_t, such that $(\varphi^\star_t f_t) \in \mathfrak{T}^p_q(\mathcal{B})$, which is then called the *convective representative of f at time t* [2, 11, 12]. However, one better uses the term "material" rather than "convective", because in the ALE community and so within this paper, "convective" has a different meaning. $\diamond$

Definition 4.3.5. The *material time derivative* of a physical field $f(Q,t)$ on $\mathcal{S}$ is defined through

$$\dot{f}(Q,t) = \left.\frac{\partial f}{\partial t}\right|_Q + \boldsymbol{\nabla}_{\boldsymbol{v}} f \qquad = \Upsilon(Q,t) \, ,$$

in which $\Upsilon(Q,t)$ denotes a source term, e.g. a constitutive equation. The convective term $\boldsymbol{\nabla}_{\boldsymbol{v}} f$ is the covariant derivative of f along the spatial velocity field $\boldsymbol{v}$, and $\boldsymbol{\nabla}$ denotes the connection on $\mathcal{S}$. $\diamond$

It should be clear that $\boldsymbol{\nabla}_{\boldsymbol{v}} f$ is due to $Q = \varphi(P,t)$, thus legitimating the term "material".

Corollary 4.3.6. *Let $f : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ be a scalar field, then*

$$\dot{f} = \left.\frac{\partial f}{\partial t}\right|_Q + \frac{\partial f}{\partial x^i} v^i = \left.\frac{\partial f}{\partial t}\right|_Q + \boldsymbol{v}[f] = \mathrm{L}_{\boldsymbol{v}} f \, .$$

Corollary 4.3.7. *Let $F_t(P) = f_t \circ \varphi_t$ be the Lagrangian representative of a spatial field $f(Q,t)$. By noting that*

$$\dot{f}(\varphi(P,t),t) = \frac{\partial F}{\partial t} \qquad \textit{resp.} \qquad \dot{f}_t = \left(\frac{\partial}{\partial t} \left(f_t \circ \varphi_t \right) \right) \circ \varphi_t^{-1}$$

through the chain rule, an important link between the Lagrangian formulation and the Eulerian formulation is

$$\frac{\partial F_t}{\partial t} \circ \varphi_t^{-1} = \left.\frac{\partial f}{\partial t}\right|_Q + \boldsymbol{\nabla}_{\boldsymbol{v}} f \qquad = \Upsilon(Q,t) \, .$$

Definition 4.3.8. Let $\boldsymbol{V}_t \in \Gamma(\varphi^\star T\mathcal{S})$ be the particle velocity resp. the Lagrangian velocity and $\varphi_t : \mathcal{B} \to \mathcal{S}$ a C^2-motion —that is, φ_t should be twice continuous differentiable—, then

$$\boldsymbol{A}(P,t) = \frac{\partial}{\partial t}\boldsymbol{V}(P,t)$$

is called the *particle acceleration* or *Lagrangian acceleration*. The *spatial* or *Eulerian acceleration*, defined by $\boldsymbol{a}_t = \boldsymbol{A}_t \circ \varphi_t^{-1}$, is a proper vector field on $\mathcal{S}$. ♢

Proposition 4.3.9. *Let $\boldsymbol{v}$ be the spatial velocity, then $\boldsymbol{a} = \dot{\boldsymbol{v}}$.*

Proof. Due to $V^i(P,t) = v^i(\varphi(P,t),t)$ by freezing t, and 4.3.7, one has

$$\frac{\partial V^i}{\partial t} = \frac{\partial v^i}{\partial t} \circ \varphi + \left(\frac{\partial v^i}{\partial x^j} \circ \varphi\right) V^j$$

for each component, and

$$\left(\frac{\partial V^i}{\partial t} + V^k V^j \left(\gamma_k{}^i{}_j \circ \varphi\right)\right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \frac{\partial \boldsymbol{V}}{\partial t} = \boldsymbol{A}(P,t)$$

by definition 3.6.9 —covariant differentiation is applied here because $\boldsymbol{A}$ should transform as a vector. Substituting $\boldsymbol{A}(P,t) = \boldsymbol{a}(\varphi(P,t),t)$, with $Q = \varphi(P,t)$, then yields

$$\boldsymbol{a}(Q,t) = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_Q + \left(\frac{\partial v^i}{\partial x^j} v^j + v^k v^j \gamma_k{}^i{}_j\right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_Q + \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{v} = \dot{\boldsymbol{v}}$$

as desired. ∎

4.4 Conservation of Mass and Piola Transformation

Proposition 4.4.1. *Let the material manifold be a measure space $(\mathfrak{M}, m)$ with a non-negative scalar measure m, called the* distributed mass, *and a* mass *$m(\mathfrak{P}) \in \mathbb{R}$ should be assigned to every part $\mathfrak{P} \subset \mathfrak{B}$ of the material body $\mathfrak{B} \subset \mathfrak{M}$. Furthermore, let the measure be handed down to every orientable placement $\mathcal{V} = \kappa_t(\mathfrak{P}) \subset \mathcal{S}$ in the ambient space at time $t \in \mathcal{I} \subset \mathbb{R}$, and let $\mathbf{d}\boldsymbol{v}$ and $\int_\mathcal{V} \mathbf{d}\boldsymbol{v}$ denote the Riemannian volume form and the volume measure (Lebesgue measure) of $\mathcal{V}$, respectively, then there is a function $\rho : \mathcal{V} \times \mathcal{I} \to \mathbb{R}$ such that*

$$m(\mathfrak{P}) = \int_\mathcal{V} \rho \, \mathbf{d}\boldsymbol{v}\,.$$

Proof. Note that $m(\mathfrak{P}) = m(\mathcal{V})$, because $\mathfrak{P}$ and $\kappa_t(\mathfrak{P})$ are assumed to have the same mass measure. The rest of the proof can be done with the aid of Radon-Nikodym's theorem, but it should be omitted here. ∎

Definition 4.4.2. $\rho(Q,t)$ is called *spatial* or *Eulerian mass density* at $Q \in \mathcal{V}$ and time t, and $\mathbf{d}\boldsymbol{m} = \rho \, \mathbf{d}\boldsymbol{v}$ is called *mass form* on $\mathcal{V}$. ♢

Definition 4.4.3. Let $\mathcal{B} = \kappa(\mathfrak{B})$ be the reference configuration of a material body, $\varphi_t = \kappa_t \circ \kappa^{-1} : \mathcal{B} \to \mathcal{S}$ a motion and $\mathcal{U} \subset \mathcal{B}$ a subset with piecewise continuous differentiable (at least C^1) boundary. Then, the spatial mass density ρ obeys *conservation of mass*, if

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\varphi_t(\mathcal{U})} \rho \, \mathbf{d}\boldsymbol{v} = 0 \,,$$

so that the mass is constant in every (orientable) configuration $\varphi_t(\mathcal{U})$ of the subset. $\diamond$

Proposition 4.4.4. *Let* $\dim(\mathcal{B}) = \dim(\mathcal{S})$, *and let the motion* $\varphi_t = \varphi(\cdot, t) : \mathcal{B} \to \mathcal{S}$ *be a one-parameter group of* C^1 *diffeomorphisms for each* $\varphi_t(\mathcal{B}) \subset \mathcal{S}$ *such that* $\varphi_0(\mathcal{B}) \equiv \mathcal{B}$ *is the initial configuration. In addition, let both* $\mathcal{B}$ *and* $\mathcal{S}$ *be orientable,* J_φ *the Jacobian and* $\boldsymbol{v}$ *the spatial velocity of* φ*, respectively, then*

$$\frac{\partial J_\varphi}{\partial t} = (\operatorname{div} \boldsymbol{v} \circ \varphi_t) \, J_\varphi \,.$$

Proof. Let $\mathbf{d}\boldsymbol{V}, \mathbf{d}\boldsymbol{v}$ be the Riemannian volume forms on $\mathcal{B}$ and $\mathcal{S}$, respectively, then $\varphi_t^\star \mathbf{d}\boldsymbol{v} = J_\varphi \mathbf{d}\boldsymbol{V}$ by proposition 3.7.26. Holding t fixed, then by 3.6.25 and 3.7.31 the time derivative becomes

$$\frac{\partial}{\partial t} J_\varphi \, \mathbf{d}\boldsymbol{V} = \varphi_t^\star \pounds_{\boldsymbol{v}} \mathbf{d}\boldsymbol{v} = \varphi_t^\star((\operatorname{div} \boldsymbol{v}) \, \mathbf{d}\boldsymbol{v}) = (\operatorname{div} \boldsymbol{v} \circ \varphi_t) \, J_\varphi \, \mathbf{d}\boldsymbol{V} \,. \qquad \blacksquare$$

Proposition 4.4.5. *Let the motion* $\varphi_t : \mathcal{B} \to \mathcal{S}$, $P \mapsto Q = \varphi_t(P)$ *be as before, then conservation of mass is equivalent to*

(i) $\rho(\varphi(P,t),t) \, J_\varphi(P,t) = \rho_{\mathrm{ref}}(P)$,

where $\rho_{\mathrm{ref}}(P) = \rho(P,0)$ *is the mass density of a subset* $\mathcal{U} \subset \mathcal{B}$*, and*

(ii) $\frac{\partial \rho}{\partial t}\big|_Q + \operatorname{div}(\rho \, \boldsymbol{v}) = 0$.

Proof. (i) By $\varphi_0(\mathcal{U}) \equiv \mathcal{U}$, conservation of mass can be written $\int_{\varphi_t(\mathcal{U})} \rho(Q,t) \, \mathbf{d}\boldsymbol{v} = \int_{\mathcal{U}} \rho_{\mathrm{ref}}(P) \, \mathbf{d}\boldsymbol{V}$. Application of the change of variables theorem 3.7.18 and the formula $\varphi_t^\star \mathbf{d}\boldsymbol{v} = J_\varphi \mathbf{d}\boldsymbol{V}$ then gives

$$\int_{\mathcal{U}} \rho(Q,t) \, J_\varphi(P,t) \, \mathbf{d}\boldsymbol{V} = \int_{\mathcal{U}} \rho_{\mathrm{ref}}(P) \, \mathbf{d}\boldsymbol{V} \,.$$

Since $Q = \varphi(P,t)$, and $\mathcal{U}$ is arbitrary, (i) follows.

(ii) Use 4.4.4 and (i) to obtain

$$\frac{\partial}{\partial t}(\rho \, J_\varphi) = \dot{\rho} \, J_\varphi + \rho \frac{\partial J_\varphi}{\partial t} = \dot{\rho} \, J_\varphi + (\rho \operatorname{div} \boldsymbol{v}) J_\varphi = 0 \,,$$

where $\dot{\rho}$ is the material time derivative of $\rho(Q,t)$. Multiplying both sides with $1/J_\varphi$ and noting that $\operatorname{div}(\rho \, \boldsymbol{v}) = \frac{\partial \rho}{\partial x^i} v^i + \rho \operatorname{div} \boldsymbol{v}$ then proofs the assertion. $\blacksquare$

Definition 4.4.6. One refers to 4.4.5(ii) as the *continuity equation*, and to $\frac{\partial}{\partial t}(\rho \, J_\varphi) = 0$ as its *conservative form*. $\diamond$

Corollary 4.4.7. *By 3.7.3, a motion φ_t is volume preserving, if $J_\varphi = 1$.*

Another important geometric concepts in Lagrangian and Eulerian continuum mechanics is the Piola transformation, which can be applied to any tensor. The Piola transform of the Cauchy stress tensor, for example, provides the first Piola-Kirchhoff stress tensor.

Definition 4.4.8. Let the initial configuration $\varphi_0(\mathcal{B}) \equiv \mathcal{B}$ and the current configuration $\varphi_t(\mathcal{B}) \subset \mathcal{S}$ of a material body be orientable and $\boldsymbol{w} \in \Gamma(T\mathcal{S})$, then the *Piola transform of* $\boldsymbol{w}$ is defined through the proper vector field

$$\boldsymbol{W} = J_\varphi\, \varphi_t^\star \boldsymbol{w} = J_\varphi\, \boldsymbol{F}_t^{-1} \cdot (\boldsymbol{w} \circ \varphi_t) \qquad \in J_\varphi \varphi_t^\star(\Gamma(T\mathcal{S})) \subset \Gamma(T\mathcal{B}),$$

where J_φ is the Jacobian of φ_t. $\diamondsuit$

Proposition 4.4.9. *(Without proof; a proof can be found in [2], p. 117.) Let $\mathbf{d}\boldsymbol{V}, \mathbf{d}\boldsymbol{v}$ be the Riemannian volume forms on $\mathcal{B}$ and $\mathcal{S}$, respectively, then $\boldsymbol{W}$ is the Piola transform of $\boldsymbol{w}$, if and only if for a motion $\varphi_t : \mathcal{B} \to \mathcal{S}$,*

$$\varphi_t^\star(\boldsymbol{i}_{\boldsymbol{w}} \mathbf{d}\boldsymbol{v}) = \boldsymbol{i}_{\boldsymbol{W}} \mathbf{d}\boldsymbol{V}.$$

Theorem 4.4.10 (Piola Identity). *Let* DIV, div *denote the divergence operators on $\mathcal{B}$ and $\mathcal{S}$, respectively, and let $\boldsymbol{W}$ be the Piola transform of $\boldsymbol{w}$ with respect to the motion $\varphi_t : \mathcal{B} \to \mathcal{S}$, then*

$$\mathrm{DIV}\, \boldsymbol{W} = (\mathrm{div}\, \boldsymbol{w} \circ \varphi_t) J_\varphi.$$

Proof. Note that by 3.6.34, $\mathbf{d}$ commutes with pullback, so by propositions 3.7.31, 4.4.9, and the formula $\varphi_t^\star \mathbf{d}\boldsymbol{v} = J_\varphi \mathbf{d}\boldsymbol{V}$,

$$\begin{aligned}(\mathrm{div}\, \boldsymbol{w} \circ \varphi_t) J_\varphi\, \mathbf{d}\boldsymbol{V} = \varphi_t^\star((\mathrm{div}\, \boldsymbol{w})\, \mathbf{d}\boldsymbol{v}) &= \varphi_t^\star \mathbf{d}(\boldsymbol{i}_{\boldsymbol{w}} \mathbf{d}\boldsymbol{v}) = \mathbf{d}\varphi_t^\star(\boldsymbol{i}_{\boldsymbol{w}} \mathbf{d}\boldsymbol{v}) \\ &= \mathbf{d}\boldsymbol{i}_{\boldsymbol{W}} \mathbf{d}\boldsymbol{V} = (\mathrm{DIV}\, \boldsymbol{W})\, \mathbf{d}\boldsymbol{V},\end{aligned}$$

that is, $\mathrm{DIV}\, \boldsymbol{W} = (\mathrm{div}\, \boldsymbol{w} \circ \varphi_t) J_\varphi$ as desired. ∎

Proposition 4.4.11 (Nanson's Formula). *Let $\mathbf{d}\boldsymbol{A}$, $\mathbf{d}\boldsymbol{a}$ be the area forms on positively oriented $\partial\mathcal{B}$ and $\partial\varphi_t(\mathcal{B})$, respectively, then*

$$\mathbf{d}\boldsymbol{a}\,(Q, t) = (J_\varphi\, \mathbf{d}\boldsymbol{A} \cdot \boldsymbol{F}_t^{-1}) \circ \varphi_t^{-1}.$$

Proof. $\mathbf{d}\boldsymbol{A}$ and $\mathbf{d}\boldsymbol{a}$ are defined through $\boldsymbol{N}^* \wedge \mathbf{d}\boldsymbol{A} = \mathbf{d}\boldsymbol{V}$ and $\boldsymbol{n}^* \wedge \mathbf{d}\boldsymbol{a} = \mathbf{d}\boldsymbol{v}$, respectively, where $\boldsymbol{N}^*$ and $\boldsymbol{n}^*$ are the related unit normals (also consider the remark after 3.7.32). By using the property 3.5.11(i), $J_\varphi \mathbf{d}\boldsymbol{V} = \boldsymbol{N}^* \wedge (J_\varphi \mathbf{d}\boldsymbol{A})$. However, setting $\boldsymbol{N}^* = \varphi_t^\star \boldsymbol{n}^*$ and applying 3.5.17 yields

$$J_\varphi \mathbf{d}\boldsymbol{V} = \varphi_t^\star(\boldsymbol{n}^* \wedge \mathbf{d}\boldsymbol{a}) = \boldsymbol{N}^* \wedge \varphi_t^\star \mathbf{d}\boldsymbol{a},$$

that is, $\varphi_t^\star \mathbf{d}\boldsymbol{a} = J_\varphi \mathbf{d}\boldsymbol{A}$. Note that $\varphi_t^\star \mathbf{d}\boldsymbol{a}$ is evaluated on $\partial\mathcal{B}$ and has a single covariant component, so pushing forward and recalling 4.2.6(iii) finally gives (point arguments are suppressed)

$$\mathbf{d}\boldsymbol{a} = \varphi_{t\star}(\varphi_t^\star \mathbf{d}\boldsymbol{a}) = \varphi_{t\star}(J_\varphi\, \mathbf{d}\boldsymbol{A}) = J_\varphi\, \mathbf{d}\boldsymbol{A} \cdot \boldsymbol{F}_t^{-1}.$$ ∎

4.5 Objectivity and Covariance

4.5.1 Motions, Framings, and the Gauge Freedom

The notion of objectivity has been introduced in section 2.2 for vectors in flat spaces, in particular Euclidian point spaces, and it should be generalized to arbitrary tensor fields on differentiable manifolds. By applying the pushforward and pullback operators for tensor fields derived in section 3.4.2, objectivity is defined as follows:

Definition 4.5.1. Let $\mathcal{S}$, $\mathcal{S}'$ be differentiable manifolds, $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{S})$ a tensor field on $\mathcal{S}$ and $\xi : \mathcal{S} \to \mathcal{S}'$ a diffeomorphism, then

$$\boldsymbol{t}' = \xi_\star \boldsymbol{t}$$

is called the *objective transformation of* $\boldsymbol{t}$ *under the map* ξ. $\boldsymbol{t}$ is called *invariant under the transformation* $\xi_\star$, if $\xi_\star \boldsymbol{t} = \boldsymbol{t}$, where it is being understood that both sides are to be evaluated at the same point. $\diamond$

In the light of definition 2.2.17 and if $\mathcal{S}' = \mathcal{S}$, then ξ can be understood actively or passively. From the passive viewpoint, ξ is understood as a chart transition so that $\boldsymbol{t}'(Q) = \xi_\star \boldsymbol{t}(Q)$ is the standard tensor transformation 3.2.3 resp. 3.3.5 at every $Q \in \mathcal{S}$. By definition, however, every honest tensor field is passively objective. Applications in continuum mechanics —especially within the theory of materials— generally involve the active form of objectivity, in which $\boldsymbol{t}$ itself will be changed.

It has been shown in section 2.2, that for Euclidian point spaces the active view on the transformation $\xi_\star$ respectively on an affine isometry ξ is twofold. It can either be interpreted as a transformation under superposed rigid motions, as done in 2.2.15, or as a transformation under change of framings, that is, a quasi-motion (cf. 2.2.16). Therefore, it is appropriate for continuum mechanical reasons to subdivide active objectivity into active objectivity under superposed motions and active objectivity with respect to changes of framings [29, 30].

Definition 4.5.2. Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a configuration of the material body $\mathcal{B}$ in the ambient space $\mathcal{S}$, then a *superposed rigid body motion* is a regular map $\xi : \mathcal{S} \to \mathcal{S}$ (time-dependency is dropped for simplicity, but the term *motion* is used throughout) such that the tangent map at $Q \in \mathcal{S}$, $T_Q\xi : T_Q\mathcal{S} \to T_{\xi(Q)}\mathcal{S}$, is proper orthogonal and the spatial metric $\boldsymbol{g} = \xi_\star \boldsymbol{g}$ is left unchanged. $\diamond$

Definition 4.5.3. Let $\xi : \mathcal{S} \to \mathcal{S}$ be a superposed rigid body motion, then a spatial tensor field $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{S})$ is called *indifferent with respect to superposed rigid body motions (IRBM)*, if it transforms (actively) objective under the map ξ, that is, $\boldsymbol{t}' = \xi_\star \boldsymbol{t}$. $\diamond$

It is necessary to clarify what is meant with a "change of framing" that is equivalent to a superposed rigid body motion, so as to define an equivalent pushforward by change of framing on the manifold. Note that in definition 4.5.3 it is not the motion ξ itself that needs to be known, but only its tangent $T\xi$ implemented into the pushforward operator $\xi_\star$ (see section 3.4.2).

Definition 4.5.4. Let $Q, Q' \in \mathcal{S}$ and $\boldsymbol{R} : T_Q\mathcal{S} \to T_{Q'}\mathcal{S}$ an orthogonal spatial two-point tensor, then define the invertible *change of framings* $(Q, \boldsymbol{v}) \mapsto (Q', \boldsymbol{R}(Q)\cdot\boldsymbol{v})$ for every spatial vector $\boldsymbol{v}$, and $(Q, \boldsymbol{a}^*) \mapsto (Q', \boldsymbol{a}^* \cdot \boldsymbol{R}^{-1}(Q))$ for every spatial one-form $\boldsymbol{a}^*$, respectively, by assuming the inverse $\boldsymbol{R}^{-1}$ to exist. ◊

Corollary 4.5.5. *Let $\xi : \mathcal{S} \to \mathcal{S}$ be a superposed rigid body motion, $\boldsymbol{R} : T_Q\mathcal{S} \to T_{Q'}\mathcal{S}$ proper orthogonal (i.e. $\det \boldsymbol{R} = +1$) and $Q' = \xi(Q)$, then*

(i) By 3.4.21, the change of framing $(Q, \boldsymbol{v}) \mapsto (Q', \boldsymbol{R}\cdot\boldsymbol{v})$ for vectors $\boldsymbol{v}$ is the tangent map of ξ,

$$\begin{array}{rccl} T\xi : & T\mathcal{S} & \to & T\mathcal{S} \\ & (Q, \boldsymbol{v}) & \mapsto & T\xi\,(Q, \boldsymbol{v}) = (\xi(Q), D\xi(Q)\cdot\boldsymbol{v}) \ , \end{array}$$

so that $\boldsymbol{R}(Q) \equiv T_Q\xi : T_Q\mathcal{S} \to T_{\xi(Q)}\mathcal{S}$.

(ii) It follows from 3.4.24 that the change of framing $(Q, \boldsymbol{a}^*) \mapsto (Q', \boldsymbol{a}^* \cdot \boldsymbol{R}^{-1})$ for one-forms $\boldsymbol{a}^*$ is the inverse cotangent map $T^*(\xi^{-1})$, and $\boldsymbol{R}^{-1}(Q) \equiv T^*_Q(\xi^{-1}) : T^*_Q\mathcal{S} \to T^*_{\xi(Q)}\mathcal{S}$.

Corollary 4.5.5 phrases that changes of framings defined point by point through 4.5.4 are compatible with superposed rigid motions ξ, that is, they can be interchanged with the differential resp. tangent $T\xi$. Moreover, changes of framings are invertible for every $Q, Q' \in \mathcal{S}$ by definition, even if $\boldsymbol{R}$ is only orthogonal, but not proper orthogonal. Therefore, changes of framings —even if $\mathcal{S}$ is Riemannian (curved) and not Euclidian (flat)— can be applied to fields of vectors and one-forms on $\mathcal{S}$ as pushforwards and pullbacks (see section 3.4.2). Finally, the pushforward $\xi_\star\boldsymbol{t}$ under a change of framing is well-defined by 3.4.36:

Definition 4.5.6. Let $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{S})$, and $\boldsymbol{t}' = \xi_\star\boldsymbol{t}$ its (active) objective transformation under a change of framing that is compatible with a superposed rigid motion ξ by means of 4.5.5, then $\boldsymbol{t}$ is called *Euclidian frame-indifferent (EFI)*. To make a notational differentiation, write $\boldsymbol{t}' = \xi_\star^{\mathrm{EFI}}\boldsymbol{t}$ for the EFI interpretation and $\boldsymbol{t}' = \xi_\star^{\mathrm{IRBM}}\boldsymbol{t}$ for the IRBM interpretation of objectivity, respectively. ◊

Definition 4.5.7. A tensor field $\boldsymbol{t}$ on the ambient space $\mathcal{S}$ which transforms objective under superposed rigid motions $\xi : \mathcal{S} \to \mathcal{S}$, i.e. under spatial isometries, is just called *objective* (the same applies to ξ viewed as a change of framing). If $\boldsymbol{t}$ transforms objectively under arbitrary spatial diffeomorphisms, then it is called *spatially covariant*. ◊

Objective resp. covariant tensors become manifest in the mathematical formulation of physical theories, e.g. the theory of materials and also in general relativity. However, the principle of objectivity applied to the tensor fields of the theory should not be confused with that of invariance of the governing equations of the theory. While the former demands the use only of those tensors which are objective with respect to some group of transformations, i.e. which are geometric objects, the latter requires the equations set up to be form-invariant (or generally covariant), i.e. to have the same form before and after a transformation belonging to the group of transformations.

Definition 4.5.8. Let a physical theory be described by some spatial tensor fields $\boldsymbol{s}, \boldsymbol{t}, \ldots \in \mathfrak{T}^p_q(\mathcal{S})$, and the governing equations of the theory have the form $f(\boldsymbol{s}, \boldsymbol{t}, \ldots) = 0$. The equations are called *generally covariant* (also called *form-invariant* or *diffeomorphism-invariant*), if for any diffeomorphism $\xi : \mathcal{S} \to \mathcal{S}$,

$$\xi_\star(f(\boldsymbol{s}, \boldsymbol{t}, \ldots)) = f(\xi_\star \boldsymbol{s}, \xi_\star \boldsymbol{t}, \ldots)\,,$$

so that f remains functionally unchanged under the map ξ. A theory is covariant if all its governing equations are covariant. Let $\xi = \mathrm{Id}_{\mathcal{S}}$, then generally covariance requires the equations to be form-invariant under arbitrary coordinate transformation, which is then referred to as their *passive diffeomorphism invariance.* $\diamondsuit$

Historically, general covariance had solely become manifest in passive diffeomorphism invariance; the advanced tensor calculus on manifolds presented in this paper was not available to Einstein when he proposed his theory of gravitation. Since nowadays it is well-known that any physical theory can be set up passively diffeomorphism-invariant by using proper tensor equations, some scientists denounced this principle as physically vacuous (see [19] for a historical survey).

The active view on 4.5.8, however, needs to be analyzed in more detail. By restricting the tensors $\boldsymbol{s}, \boldsymbol{t}, \ldots$ to a subset $\mathcal{V} \subset \mathcal{S}$, the reader will recognize that the left hand side of the equation in 4.5.8 is evaluated at some $Q \in \mathcal{V}$ before it is pushed forward by ξ, whereas the right hand side is evaluated at $Q' = \xi(Q) \in \xi(\mathcal{V})$. Therefore, definition 4.5.8 implies the following theorem to hold.

Theorem 4.5.9 (Gauge Theorem). *(See also [18]) Let $f(\boldsymbol{s}, \boldsymbol{t}, \ldots) = 0$, where $\boldsymbol{s}, \boldsymbol{t}, \ldots \in \mathfrak{T}^p_q(\mathcal{S})$ and $\mathcal{S}$ is differentiable, be an equation of a physical theory that is form-invariant with respect to any regular coordinate transformation, and let $\xi : \mathcal{S} \to \mathcal{S}$ be an arbitrary diffeomorphism, then $f(\xi_\star \boldsymbol{s}, \xi_\star \boldsymbol{t}, \ldots) = 0$ is also an equation of the theory.*

Proof. Let $(\mathcal{V}, x)$ be a regular chart of $\mathcal{V} \subset \mathcal{S}$. According to 3.1.17 and 4.5.8, the form-invariance "with respect to any regular coordinate transformation" can be identified with passive diffeomorphism-invariance for fixed $\mathcal{V}$, i.e. with invariance of $f(\boldsymbol{s}, \boldsymbol{t}, \ldots) = 0$ under arbitrary relabelling of points $x \to \tilde{\xi} \circ x$, where $\tilde{\xi}$ is a passive diffeomorphism in terms of definition 3.1.25. By this definition, then, there are infinitely many pairs (η, x') of active diffeomorphisms $\eta : \mathcal{S} \to \mathcal{S}$ that take $\mathcal{V}$ to $\mathcal{V}' = \eta(\mathcal{V})$, and chart maps $\mathcal{V}' \to x'(\mathcal{V}') \subset \mathbb{R}^{n_{\mathrm{dim}}}$ such that

$$x' \circ \eta = \tilde{\xi} \circ x\,.$$

What is the consequence? Having the new coordinates of points in $\mathcal{V} \subset \mathcal{S}$, one cannot decide whether they result from a relabelling $\tilde{\xi}$ of $\mathcal{V}$, i.e. a chart transition, or a warping η of $\mathcal{S}$ (figure 4.3). Clearly, there is some indeterminism in the physical theory under consideration, which is also called the *gauge freedom.*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\eta"] \arrow[d, "x"'] & \mathcal{V}' \arrow[d, "x'"] \\ x(\mathcal{V}) \arrow[r, "\tilde{\xi}"'] & x' \circ \eta(\mathcal{V}) = \tilde{\xi} \circ x(\mathcal{V}) \end{tikzcd}$$

Figure 4.3: Changed coordinates of $\mathcal{V} \subset \mathcal{S}$ can either result from a relabelling $\tilde{\xi}$ of $\mathcal{V}$, or a warping η of $\mathcal{S}$.

Now by arbitrariness of $\tilde{\xi}$ and $\xi : \mathcal{S} \to \mathcal{S}$ presumed, set $\xi \equiv \eta = x'^{-1} \circ \tilde{\xi} \circ x$. Then, by 3.4.21, for a spatial vector field $\boldsymbol{v} : \mathcal{S} \to T\mathcal{S}$, the tangent map consistent with, that is, comprising the relabelling of points $Q \in \mathcal{S}$, is just

$$T\xi\,(Q, \boldsymbol{v}) = (\xi(Q), D\xi(Q) \cdot \boldsymbol{v})\ ,$$

where $D\xi = D(x'^{-1} \circ \tilde{\xi} \circ x)$.

If $\{x^i\}, \{x^{i'}\}$ denote the local coordinate functions of x and x', respectively, and $\xi^{i'} = x^{i'} \circ \xi \circ x^{-1}$ is the coordinate transition concerning ξ, then $\xi^{i'} = x^{i'} \circ (x'^{-1} \circ \tilde{\xi} \circ x) \circ x^{-1} \equiv \tilde{\xi}^{i'}$. Therefore, the components of the tangent map $T\xi$ are equal to the partial derivatives of the relabelling of points $\frac{\partial \tilde{\xi}^{i'}}{\partial x^i}$, and thus for a local representative $\boldsymbol{v}(P) = v^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$ of the vector field,

$$\xi_\star \boldsymbol{v} = T\xi \circ \boldsymbol{v} \circ \xi^{-1} = \left(\left(\frac{\partial \tilde{\xi}^{i'}}{\partial x^i} v^i \right) \circ \xi^{-1} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^{i'}}$$

by 3.4.33. Because the pushforward by $\xi : \mathcal{S} \to \mathcal{S}$ of arbitrary $\binom{p}{q}$-tensor fields on $\mathcal{S}$ is realized in a similar way setting $\xi^{i'} \equiv \tilde{\xi}^{i'}$, the assertion follows. ■

The gauge freedom underlying the principle of general covariance is an important ingredient to restrict or reduce the governing equations of a physical theory considerably —which is basically the main goal of the endeavor. However, if the particular equation $f(\xi_\star \boldsymbol{s}, \xi_\star \boldsymbol{t}, \ldots) = 0$ is physically substantial or not depends on the physical theory under consideration.

4.5.2 Material Frame-Indifference

In continuum mechanics, a typical application of the principles of objectivity and general covariance can be found in the material frame-indifference of constitutive equations of materials. The remainder of the section will be devoted to this topic, but only a few key facts will be outlined using the example of simple hyperelastic material.

Definition 4.5.10. A *hyperelastic* or *Green elastic material* is an ideally elastic material that relates the configuration resp. deformation $\varphi : \mathcal{B} \to \mathcal{S}$ of the material body $\mathcal{B}$ at every particle $P \in \mathcal{B}$ (time-dependency will be again omitted) to the stress by the stored energy at P. ◇

Proposition 4.5.11. *(Without proof; cf. [2]) A hyperelastic constitutive equation can locally be written in the form*

$$\boldsymbol{P} = \mathfrak{P}(P, \boldsymbol{F}),$$

in which $\boldsymbol{F}$ is the deformation gradient at $P \in \mathcal{B}$, and $\boldsymbol{P} : T_P\mathcal{B} \to T_{\varphi(P)}\mathcal{S}$ is called the first Piola-Kirchhoff stress tensor*.*

Note that $\boldsymbol{P}$ is a two-point tensor having the one leg at P and a "spatial" leg at $Q = \varphi(P) \in \mathcal{S}$.

Normally, the theory of materials concerns the fields of stress and deformation rather than their point values. The hyperelastic constitutive equation then becomes $\boldsymbol{P} = \mathfrak{P}(\boldsymbol{F})$, in which $\boldsymbol{P} : \mathcal{B} \to T\mathcal{S} \otimes T^*\mathcal{B}$ now is the first Piola-Kirchhoff stress field, and the constitutive function is a map

$$\mathfrak{P} : \ \mathcal{E} \ \to \ \mathcal{F},$$

where $\mathcal{E}$ is the bundle over $\mathcal{B}$ whose fibre at $P \in \mathcal{B}$ is $T_{\varphi(P)}\mathcal{S} \otimes T^*_P\mathcal{B}$, and $\mathcal{F} = \varphi^\star T\mathcal{S} \otimes T^*\mathcal{B}$.

Definition 4.5.12. Let $\xi : \mathcal{S} \to \mathcal{S}$ be an arbitrary spatial diffeomorphism resp. a superposed motion taking spatial points Q to Q' (equivalently, ξ can be understood as a change of framing), then the hyperelastic constitutive equation $\boldsymbol{P} = \mathfrak{P}(\boldsymbol{F})$ is called *spatially covariant* provided that

$$\boldsymbol{P}' = \mathfrak{P}(\boldsymbol{F}'),$$

where $\boldsymbol{F}'(P) : T_P\mathcal{B} \to T_{Q'}\mathcal{S}$ and $Q' = \xi(Q) = \xi(\varphi(P)) \in \mathcal{S}$. Spatial covariance means that the constitutive equation $\mathfrak{P}$ itself is form-invariant under spatial diffeomorphisms. If the spatial diffeomorphism is replaced by a superposed rigid body motion, i.e. a spatial isometry such that $T_Q\xi : T_Q\mathcal{S} \to T_{Q'}\mathcal{S}$ is proper orthogonal and $\boldsymbol{P}' = \mathfrak{P}(\boldsymbol{F}')$ still holds, then the constitutive equation is called *material frame-indifferent (MFI).* ◊

Material frame-indifference in Euclidian point spaces has often been the matter in dispute in the last decades, and it is usually stated as an independent assumption or axiom. However, it can be derived from the more general principles of Euclidian frame-indifference and form-invariance:

Proposition 4.5.13. *(See also [29, 30].)*

(i) Both Euclidian frame-indifference (EFI) and form-invariance (FI) of constitutive equations together demonstrate material frame-indifference (MFI); conceptually, $EFI + FI = MFI$.

(ii) Indifference of constitutive equations with respect to superposed rigid body motions (IRBM) is equivalent to material frame-indifference; conceptually, $IRBM = MFI$.

PROOF. (i) $\boldsymbol{F}' = \xi_\star \boldsymbol{F} = T\xi(\boldsymbol{F}) \circ \xi^{-1}$, so $\boldsymbol{F}$ transforms objectively (clearly, the spatial leg of $\boldsymbol{F}$), and since $\boldsymbol{P}$ is a two-point tensor of the same class, it also does. Now

$\boldsymbol{P}' = \xi_\star^{\mathrm{EFI}}\boldsymbol{P}$ demands $(\mathfrak{P}(\boldsymbol{F}))' = \xi_\star^{\mathrm{EFI}}(\mathfrak{P}(\boldsymbol{F}))$ from the constitutive equation. On the other hand, form-invariance forces $\xi_\star^{\mathrm{EFI}}(\mathfrak{P}(\boldsymbol{F})) = \mathfrak{P}(\xi_\star^{\mathrm{EFI}}\boldsymbol{F}) = \mathfrak{P}(\boldsymbol{F}')$, therefore, $\boldsymbol{P}' = \mathfrak{P}(\boldsymbol{F}')$ as desired.

(ii) Indifference of the constitutive equation $\mathfrak{P}$ with respect to superposed rigid body motions means that $\xi_\star^{\mathrm{IRBM}}\boldsymbol{P} = \mathfrak{P}(\xi_\star^{\mathrm{IRBM}}\boldsymbol{F})$. As shown in (i), however, $\boldsymbol{P}$ and $\boldsymbol{F}$ transform objectively in the same manner, so IRBM and MFI are equivalent. ∎

Basically, proposition 4.5.13(i) derives material frame-indifference from the relative motion of different Euclidian observers, whereas 4.5.13(ii) involves two motions of a body —the original motion and the motion overlayed by a rigid one— with respect to the same observer. However, both viewpoints are equivalent, except for the fact that the first requires the linear transformation of both frames to be orthogonal, while the latter includes motions with proper orthogonal tangents only, since rigid motions necessarily are orientation-preserving.

In their encyclopedia ([8], appendix 19A), Truesdell and Noll summarize that the form 4.5.13(i) is historically connected with the names Zaremba and Jaumann, and 4.5.13(ii) is a form that has essentially been proposed by Hooke, Poisson and Cauchy. Truesdell and Noll [8] side themselves with Zaremba and Jaumann, as Noll [31] did already before; he called material frame-indifference at that time the "principle of objectivity of material properties". Therefore, the following notational differentiation would be convenient.

Definition 4.5.14. Refer to the fact 4.5.13(i) as the *Zaremba-Jaumann-Noll form*, and to 4.5.13(ii) as the *Hooke-Poisson-Cauchy form* of material frame-indifference. ◇

The reader probably knows that material frame-indifference is a useful concept to reduce the form of constitutive equations, but not all materials comply with it. For example, kinetic gas does not fulfill invariance with respect to superposed rigid body motions and thus it is not material frame-indifferent [29, 30].

4.6 Arbitrary Lagrangian-Eulerian Formulation

Large deformation initial boundary value problems in continuum mechanics are usually solved by applying the finite element method. In this context, however, the classical Lagrangian and Eulerian formulations have some shortcomings. In the Lagrangian finite element formulation, excessive element distortions may occur that may lead to unstable and inaccurate numerical analyses, or even terminate the calculation. From the Eulerian viewpoint, the discretized domain is fixed in space and, therefore, following free surfaces and moving material interfaces becomes a cumbersome task.

The Arbitrary Lagrangian-Eulerian (ALE) formulation succeeds in combining the advantages of both classic Lagrangian and Eulerian viewpoints by choosing the finite element mesh as a time-dependent reference domain different from the material (Lagrangian) and spatial (Eulerian) configurations [20, 21, 22]. For the last two decades the

ALE framework has been developed to a powerful analysis tool for large deformation problems, especially metal forming processes, free surface flows and fluid-deformable structure interaction.

Consistent with the geometrical overall context of the paper, the reference domain underlying the ALE formulation is introduced as a grid manifold that is embedded in the ambient space, and which is independent of the material and its configurations. The affine connection and the inner product of the ambient space are used to establish a covariant derivative on the reference domain and to enable measurement of deformation of the grid, respectively. It has to be noted that this additional functionality is not intrinsic, whether to the underlying material manifold, nor to the grid manifold, but it is induced to the latter's configurations in space.

The important convective velocity field, which is well-known in the ALE community, will be defined subsequently as a section of the tangent bundle of the ambient space. Differential geometry, then, reveals three important facts resulting from the relative motion between the body and the reference domain. First, the pullback of the convective velocity concerning the relative motion establishes the so-called referential velocity field, which is essential for the definition of a material time derivative on the reference domain. Second, the fundamental ALE operator takes advantage of an induced connection on the reference domain given by the Levi-Civita connection of the ambient space and, third, mesh regularization or optimization of the mesh, in context of the ALE finite element method, can be interpreted as the numerical representative of a time-dependent flow of the reference domain in the ambient space.

4.6.1 Grid Manifold and Velocity Fields

Definition 4.6.1. In order to describe geometry and kinematics of a material body independent of its reference configuration and current configuration, let $\mathfrak{G}$ be a manifold, called the *grid manifold.* A subset $\mathfrak{R} \subset \mathfrak{G}$ is referred to as the *reference grid of the material body* $\mathfrak{B}$, shortly: *reference grid*, provided that the map $\mathfrak{R} \to \mathcal{B}$ between the reference grid and the reference configuration is a homeomorphism. $\diamond$

The required homeomorphism ensures the one-to-one correspondence of grid nodes and particles of the body. Hence, the grid manifold $\mathfrak{G}$ is also a continuum and the reference grid $\mathfrak{R}$ inherits the topology of the body. The grid cells are infinitely small and every grid node has a neighborhood containing other grid nodes. Descriptively, the grid is a prototype of a manifold: it has no further properties besides its topology.

Due to the arbitrary choice of the reference grid, it does not make sense to define an inner product or an affine connection on the grid manifold. But in order to specify the state of the grid, that is, stretches of the grid cells and distances of the grid nodes, $\mathfrak{R}$ is embedded in the ambient space in the same way as the material body (figure 4.4):

Definition 4.6.2. Considering that the ALE formulation has its origin in computational mechanics, one may refer to $\mathcal{M}$, defined by the map $\eta : \mathfrak{G} \supset \mathfrak{R} \to \mathcal{M} \subset \mathcal{S}$ and being differentiable, as the *computational representative* or the *model of* $\mathfrak{B}$, that is, $\mathcal{M}$

is the reference configuration of $\mathfrak{R}$ in $\mathcal{S}$ —this is the same idea as mapping a set of nodes with topology onto the finite element mesh (i.e. the model) of a work piece. $\diamond$

Definition 4.6.3. A *time-dependent configuration* or *motion of* $\mathcal{M}$ is a map

$$\begin{aligned} \mu : \ \mathcal{M}\times\mathbb{R} &\rightarrow \mathcal{S} \\ (M,t) &\mapsto \mu(M,t) = \hat{Q}\,, \end{aligned}$$

with $\mu_t(M) = \mu(M,t)$ at fixed t, and μ_t an embedding of $\mathcal{M}$ in $\mathcal{S}$. The current configuration $\mu_t(\mathcal{M}) = \mathcal{R}\subset\mathcal{S}$ at time t is referred to as the *reference domain of the body* ($\neq$ reference configuration!). $\hat{Q}$ is called *reference point.* $\diamond$

Time-dependency of $\mathcal{R}$ can be dropped if desired (as in the Eulerian formulation; see below). But if existent, it is assumed that the parameter "time" is universal, i.e. all processes on $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{S}$ are synchronous. Otherwise, relativistic effects have to be considered.

To the neighborhoods $\mathcal{W}(\hat{Q})\subset\mathcal{R}$ of a reference point $\hat{Q}$, charts $(\mathcal{W},\chi)$ with regular local coordinates $\{\chi^\alpha\}_{\hat{Q}} = \chi(\hat{Q})$ can be assigned. As $\mathcal{R}\subset\mathcal{S}$ is embedded, $\mathcal{R}$ is likewise differentiable and the partial derivatives $\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\alpha}\}\in T_{\hat{Q}}\mathcal{R}$ establish a basis of the tangent space, and the coordinate differentials $\{\mathbf{d}\chi^\alpha\}\in T^*_{\hat{Q}}\mathcal{R}$ its dual in the cotangent space at $\hat{Q}$. The $\hat{g}_{\alpha\beta}(\hat{Q}) = \langle\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\alpha},\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\beta}\rangle_{\hat{Q}}$ are the metric coefficients on $\mathcal{R}$. As a general convention, coordinate indices related to the reference domain will be denoted by lower case Greeks, with $\alpha,\beta,\ldots\in\{1,2,3\}$. If appropriate, vectors, tensors etc. will be marked with a hat.

Definition 4.6.4. Let the spatial coordinates x^i be given as functions of the C^1 grid motion μ_t, that is, $\mu_t^i(M) = x^i\circ\mu_t\circ\chi^{-1}$, then

$$\boldsymbol{\Omega}_t(M) = \left.\frac{\partial\mu_t^i}{\partial t}\right|_M \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^i} = \Omega_t^i(M)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x^i}(Q)$$

defines the *grid velocity over* μ_t. The corresponding *spatial grid velocity field* $\boldsymbol{\omega}_t = \boldsymbol{\Omega}_t\circ\mu_t^{-1}\in\Gamma(T\mathcal{S}))$ is obtained by switching the point arguments. $\diamond$

Definition 4.6.5. Let $\Phi_t:\mathcal{R}\rightarrow\mathcal{S}$, $\hat{Q}\mapsto Q=\Phi_t(\hat{Q})$ be a time-dependent embedding of $\mathcal{R}$ in $\mathcal{S}$, and $\Psi_t:\mathcal{R}\rightarrow\mathcal{B}$, $\hat{Q}\mapsto P=\Psi_t(\hat{Q})$ a regular map, where $\Phi_t(\hat{Q}) = \Phi(\hat{Q},t)$ and $\Psi_t(\hat{Q}) = \Psi(\hat{Q},t)$ at fixed t. Then Φ_t and Ψ_t are called the *relative map* and the *referential map of* $\mathcal{R} = \mu_t(\mathcal{M})$, respectively, provided that

$$\varphi_t = \Phi_t\circ\Psi_t^{-1}\,,$$

that is, the *physical motion* φ_t is a superposed motion of the body relative to the reference domain. Both Φ_t and Ψ_t are in general explicitly time-dependent. $\diamond$

The next theorem states how to specify the flow of the material particles via the flow of the reference points.

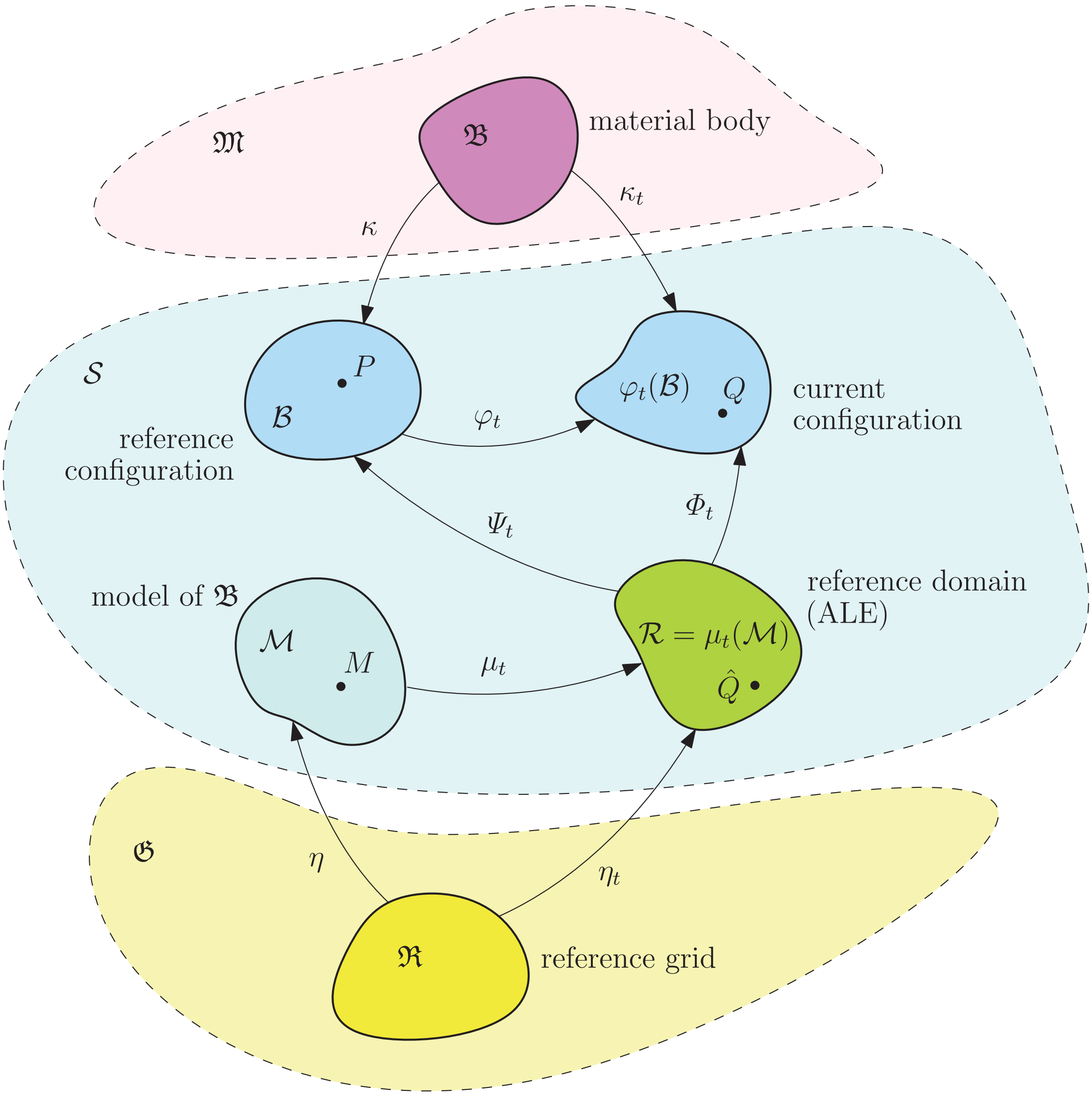

Figure 4.4: General Arbitrary Lagrangian-Eulerian formulation: reference grid, configurations and related mappings.

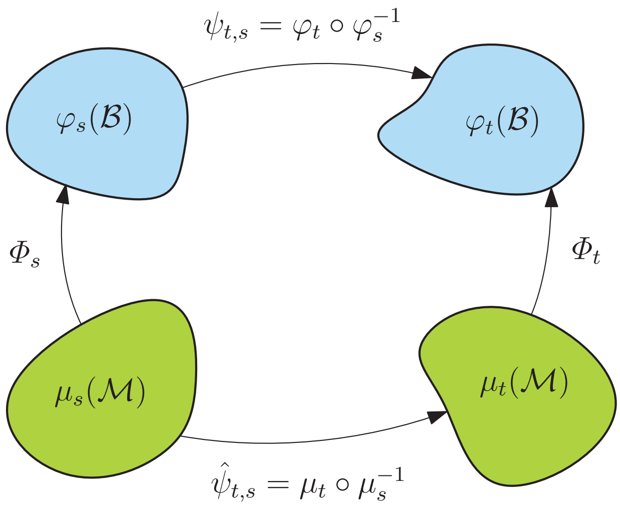

Figure 4.5: Flows associated with the motion φ_t of the body $\mathcal{B}$ and the motion μ_t of its model $\mathcal{M}$ in the ambient space.

Theorem 4.6.6 (Compatibility of Flows). *Let $\hat{\psi}_{t,s}$ be the time-dependent flow of the spatial grid velocity $\boldsymbol{\omega}$, and Φ_t, Ψ_t the relative map and referential map of $\mathcal{R} = \mu_t(\mathcal{M})$, respectively, then the flow $\psi_{t,s} = \varphi_t \circ \varphi_s^{-1}$ of the regular physical motion $\varphi_t : \mathcal{B} \to \mathcal{S}$ can be obtained from*

$$\psi_{t,s} = \Phi_t \circ \hat{\psi}_{t,s} \circ \Phi_s^{-1}\,.$$

Proof. Since $\varphi_t(\mathcal{B})$ and $\mu_t(\mathcal{M})$ evolve synchronously, compatibility of the maps requires (see figure 4.5)

$$\Phi_t \circ \mu_t \circ \mu_s^{-1} = \varphi_t \circ \varphi_s^{-1} \circ \Phi_s\,, \qquad t, s \in \mathbb{R}\,.$$

Considering 4.1.8 and assuming $\mu_t : \mathcal{M} \to \mathcal{S}$ to be regular, the time-dependent flow associated with the spatial grid velocity $\boldsymbol{\omega}_t(Q) = \boldsymbol{\omega}(Q, t)$ is

$$\hat{\psi}_{t,s} = \mu_t \circ \mu_s^{-1} : \quad \mathcal{S} \supset \mu_s(\mathcal{M}) \quad \to \quad \mu_t(\mathcal{M}) \subset \mathcal{S}\,, \qquad t, s \in \mathbb{R}\,.$$

Substitution into the previous equation and composing both sides with Φ_s^{-1} from the right then gives the result. ■

Example 4.6.7. In a numerical implementation when using the finite element method, the discretization of the domain in its current configuration, i.e. the deformed finite element mesh at time t, is a numerical representative of the reference domain $\mathcal{R} = \mu_t(\mathcal{M})$. A mesh regularization or an r-adaption of the mesh that preserves the mesh topology, then, represents the discretization of the time-dependent flow $\hat{\psi}_{t,s} : \mu_s(\mathcal{M}) \to \mu_t(\mathcal{M})$.

The localizations of the relative map $\Phi_t : \mathcal{R} \to \mathcal{S}$ and the inverse referential map $\Psi_t^{-1} : \mathcal{B} \to \mathcal{R}$ should be discussed next.

Definition 4.6.8. Keeping the definitions made in 4.1.4 in mind, let $\mathcal{W} \subset \mathcal{R}$ be an open neighborhood of $\hat{Q} \in \mathcal{R}$ having a chart $(\mathcal{W}, \chi)$, and $\mathcal{V}(\Phi_t(\hat{Q})) \subset \mathcal{S}$ a neighborhood of $\Phi_t(\hat{Q})$ with a chart $(\mathcal{V}, x)$. Moreover, let $\mathcal{W}' \subset \mathcal{R}$ be a neighborhood of $\hat{Q}' = \Psi_t^{-1}(P) \in \mathcal{R}$ having a chart $(\mathcal{W}', \chi')$, where $P \in \mathcal{U} \subset \mathcal{B}$ and $(\mathcal{U}, X)$ is a chart of the material body. Then, by presuming $\Phi_t^{-1}(\mathcal{V}) \cap \mathcal{W}$ and $\Psi_t(\mathcal{W}') \cap \mathcal{U}$ to be non-empty, define the *localizations* of Φ_t and Ψ_t^{-1} by

$$(\Phi_t)_{x\chi} = x \circ \Phi_t \circ \chi^{-1}\big|_{\chi\left(\Phi_t^{-1}(\mathcal{V}) \cap \mathcal{W}\right)} : \quad \chi\left(\Phi_t^{-1}(\mathcal{V}) \cap \mathcal{W}\right) \;\; \rightarrow \;\; x\left(\Phi_t^{-1}(\mathcal{V}) \cap \mathcal{W}\right)$$

and

$$(\Psi_t^{-1})_{\chi' X} = \chi' \circ \Psi_t^{-1} \circ X^{-1}\big|_{X(\Psi_t(\mathcal{W}') \cap \mathcal{U})} : \quad X\left(\Psi_t(\mathcal{W}') \cap \mathcal{U}\right) \;\; \rightarrow \;\; \chi'\left(\Psi_t(\mathcal{W}') \cap \mathcal{U}\right) ,$$

respectively —for those who want to make a difference between Ψ_t and $(\Psi_t^{-1})^{-1}$: the latter is meant here. If $\{x^i\}_Q = x(Q)$ are the coordinates of $Q \in \mathcal{V}$ and $\{\chi^{\alpha'}\}_{\hat{Q}'} = x(\hat{Q}')$ the coordinates of $\hat{Q}' = \Psi_t^{-1}(P) \in \mathcal{W}'$, then abbreviate

$$\Phi_t^i = x^i \circ \Phi_t \circ \chi^{-1} \qquad \text{and} \qquad (\Psi_t^{-1})^{\alpha'} = \chi^{\alpha'} \circ \Psi_t^{-1} \circ X^{-1} ,$$

where $\Phi_t^i(\hat{Q}) = \Phi^i(\hat{Q}, t)$ and $(\Psi_t^{-1})^{\alpha'}(P) = (\Psi^{-1})^{\alpha'}(P, t)$, respectively. $\diamond$

Corollary 4.6.9. *Considering the previous definitions, and let* $\varphi_t = \Phi_t \circ \Psi_t^{-1} : \mathcal{B} \rightarrow \mathcal{S}$ *be the motion of a material body and* $x \circ \varphi_t \circ X^{-1} = (\varphi_t)_{xX}$ *resp.* $\varphi_t^i = x^i \circ \varphi_t \circ X^{-1}$ *its localization. If* $\mathcal{W}' = \mathcal{W}$*, then*

$$(\varphi_t)_{xX} = (\Phi_t \circ \Psi_t^{-1})_{xX} = (\Phi_t)_{x\chi} \circ (\Psi_t^{-1})_{\chi X} ,$$

that is,

$$\varphi_t^i = x^i \circ \Phi_t \circ \Psi_t^{-1} \circ X^{-1} \qquad \textit{resp.} \qquad \varphi^i(P, t) = \Phi^i((\Psi^{-1})(P, t), t) .$$

Definition 4.6.10. Let Ψ_t^{-1} and Φ_t be at least C^1-continuous, then the *referential velocity field on* $\mathcal{R}$ is defined by

$$\left(\frac{\partial (\Psi_t^{-1})^\alpha}{\partial t}\bigg|_P \circ \Psi_t \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \chi^\alpha} = \nu_t^\alpha(\hat{Q}) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \chi^\alpha} = \boldsymbol{\nu}_t \qquad \in \Gamma(T\mathcal{R}) ,$$

and

$$\left(\frac{\partial \Phi_t^i}{\partial t}\bigg|_{\hat{Q}} \circ \Phi_t^{-1} \right) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = w_t^i(Q) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} = \boldsymbol{w}_t \qquad \in \Gamma(T\mathcal{S})$$

is called the *relative velocity field on* $\mathcal{S}$, in which $\boldsymbol{\nu}_t = \boldsymbol{\nu}(\cdot, t)$ resp. $\boldsymbol{w}_t = \boldsymbol{w}(\cdot, t)$ at fixed t. The referential velocity can be interpreted as the particle velocity measured on the reference domain, and $\boldsymbol{w}_t$ is the spatial velocity of Φ_t. $\diamond$

The relative velocity $\boldsymbol{w}_t$ might be confused with the grid velocity $\boldsymbol{\Omega}_t$, but notice there is the following relation:

Corollary 4.6.11. *The composition $(\Phi_t \circ \mu_t)(\mathcal{M})$ can be interpreted as a superposed motion of the model $\mathcal{M}$ in $\mathcal{S}$ leading to*

$$\left.\frac{\partial(\Phi_t^i \circ \mu_t)}{\partial t}\right|_M = w_t^i \circ \Phi_t \circ \mu_t + \frac{\partial \Phi_t^i}{\partial \mu_t^k} \Omega_t^k \,.$$

Corollary 4.6.12. *By 4.1.8, the time-dependent flow of $\boldsymbol{w}_t$ is $\Phi_t \circ \Phi_s^{-1}$, where $t, s \in \mathbb{R}$.*

Definition 4.6.13. The *tangent of the map* Φ_t is

$$\begin{array}{rccl} T\Phi_t: & T\mathcal{R} & \to & T(\Phi_t(\mathcal{R})) \subset T\mathcal{S} \\ & \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\alpha} & \mapsto & \dfrac{\partial \Phi_t^i}{\partial \chi^\alpha} \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \,, \end{array}$$

and its inverse is given by

$$\begin{array}{rccl} T(\Phi_t^{-1}): & T\mathcal{S} \supset T(\Phi_t(\mathcal{R})) & \to & T\mathcal{R} \\ & \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} & \to & \dfrac{\partial (\Phi_t^{-1})^\alpha}{\partial x^i} \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\alpha} \,, \end{array}$$

in which $(\Phi_t^{-1})^\alpha = \chi^\alpha \circ \Phi_t^{-1} \circ x^{-1}$, i.e. $(\Phi_t^{-1})^\alpha(x(Q)) = \chi^\alpha \circ \Phi_t^{-1}$. ◇

Again, the case should be mentioned where $\dim(\mathcal{R}) < \dim(\mathcal{S})$. Since Φ_t is an embedding and $\Phi_t(\mathcal{R}) \subset \mathcal{S}$ is a submanifold, $T\Phi_t(T\mathcal{R}) = T(\Phi_t(\mathcal{R})) \subset T\mathcal{S}$ is a subspace of $T\mathcal{S}$, so that the Jacobian matrix $\frac{\partial \Phi_t^i}{\partial \chi^\alpha}$ is invertible at $\hat{Q} \in \mathcal{R}$ (cf. definitions 3.1.22 and 3.1.26, and section 4.1).

Definition 4.6.14. The two-point tensor fields $\boldsymbol{F}_{\Psi t} \in \Gamma(\Psi_t^\star T\mathcal{B} \otimes T^*\mathcal{R})$ and $\boldsymbol{F}_{\Phi t} \in \Gamma(\Phi_t^\star T\mathcal{S} \otimes T^*\mathcal{R})$ associated with the tangent maps $T\Psi_t$ and $T\Phi_t$, respectively, are defined by

$$\boldsymbol{F}_\Psi(\hat{Q}, t) = \frac{\partial \Psi_t^I}{\partial \chi^\alpha} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} X^I} \otimes \mathbf{d}\chi^\alpha \qquad \text{and} \qquad \boldsymbol{F}_\Phi(\hat{Q}, t) = \frac{\partial \Phi_t^i}{\partial \chi^\alpha} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \otimes \mathbf{d}\chi^\alpha \,,$$

with $\Psi_t^I = X^I \circ \Psi_t \circ \chi^{-1}$. $\boldsymbol{F}_\Psi$ is called the *referential deformation gradient* and $\boldsymbol{F}_\Phi$ is called the *relative deformation gradient.* $\Psi_t^\star T\mathcal{B}$ and $\Phi_t^\star T\mathcal{S}$ are the tangent bundles on the reference domain $\mathcal{R}$ induced by the tangent bundle of $\mathcal{B}$ via Ψ_t and the tangent bundle of $\mathcal{S}$ via Φ_t, respectively. ◇

Corollary 4.6.15. *By 3.4.22 and 4.6.5, one has $T\varphi_t = T(\Phi_t \circ \Psi_t^{-1}) = T\Phi_t \circ T(\Psi_t^{-1})$. Therefore, the total deformation gradient is*

$$\boldsymbol{F}_t(P) = \left(\boldsymbol{F}_{\Phi t} \cdot \boldsymbol{F}_{\Psi t}^{-1}\right) \circ \Psi_t^{-1} \,,$$

where $\boldsymbol{F}_{\Psi t}^{-1}(\hat{Q}, t) = \frac{\partial (\Psi_t^{-1})^\alpha}{\partial X^I} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\alpha} \otimes \mathbf{d}X^I = (T(\Psi_t^{-1}) \circ \Psi_t)$ is the inverse of $\boldsymbol{F}_{\Psi t}$.

Theorem 4.6.16. *Let $\boldsymbol{v}$ be the spatial velocity of the motion of a material body, $\boldsymbol{w}$ the spatial velocity of the relative map Φ (i.e. the relative velocity) and $\boldsymbol{\nu}$ the referential velocity (see also 4.6.10 for definitions), then*

$$\boldsymbol{v}_t - \boldsymbol{w}_t = \Phi_{t\star} \boldsymbol{\nu}_t \,.$$

Proof. Substituting 4.6.9 for the components of the spatial velocity $\boldsymbol{v}_t$ (Eulerian velocity field) gives in detailed expression

$$v_t^i(Q) = \left.\frac{\partial \varphi_t^i}{\partial t}\right|_P \circ \varphi_t^{-1} = \left.\frac{\partial (x^i \circ \Phi_t \circ (\chi^{-1} \circ \chi) \circ \Psi_t^{-1})}{\partial t}\right|_P \circ \varphi_t^{-1}$$
$$= \left(\left.\frac{\partial \Phi_t^i}{\partial t}\right|_{\hat{Q}} \circ \Phi_t^{-1} \right)(Q) + \left(\frac{\partial \Phi_t^i}{\partial \chi^\alpha}(\hat{Q}) \circ \left(\left.\frac{\partial (\Psi_t^{-1})^\alpha}{\partial t}\right|_P \circ \Psi_t \right)(\hat{Q}) \circ \Phi_t^{-1} \right)(Q)$$
$$= w_t^i(Q) + \left(\left(\frac{\partial \Phi_t^i}{\partial \chi^\alpha} \nu_t^\alpha \right) \circ \Phi_t^{-1} \right)(Q) .$$

Recalling 3.4.33, it should be clear that the second term on the right hand side includes the components of the pushforward of the referential velocity field. More general, for vector fields $\boldsymbol{\xi} \in \Gamma(T\mathcal{R})$ one has

$$\Phi_{t\star}\boldsymbol{\xi} = T\Phi_t \circ \boldsymbol{\xi} \circ \Phi_t^{-1} = (\boldsymbol{F}_{\Phi t} \cdot \boldsymbol{\xi}) \circ \Phi_t^{-1} \qquad \in \Gamma(T\mathcal{S}) ,$$

hence, in direct notation, one is left with $\boldsymbol{v}_t = \boldsymbol{w}_t + \Phi_{t\star}\boldsymbol{\nu}_t$. ■

Definition 4.6.17. $\boldsymbol{c}_t = \boldsymbol{v}_t - \boldsymbol{w}_t = \Phi_{t\star}\boldsymbol{\nu}_t \in \Gamma(T\mathcal{S})$, where $\boldsymbol{c}_t(Q) = \boldsymbol{c}(Q,t)$ at fixed t, is called the *convective velocity field on $\mathcal{S}$*. ◇

The convective velocity $\boldsymbol{c}$ should not be confused with the Finger tensor (see definition 4.2.12). However, the meaning will be clear from the context.

Note that the convective velocity is a proper vector field on $\mathcal{S}$, and it provides a fundamental link between the body, its configurations and the reference domain. It denotes the relative velocity between the particles $P = \varphi_t^{-1}(Q)$ and the reference points $\hat{Q} = \Phi_t^{-1}(Q)$ as measured from the places Q.

The Lagrangian formulation and the Euler formulation are special cases of the ALE formulation. On the one hand, if $\Psi_t = \mathrm{Id}_{\mathcal{S}}$, then $\boldsymbol{w} = \boldsymbol{v}$ and $\boldsymbol{c} = \boldsymbol{0}$. The equivalence of the particle velocity and the velocity of the reference domain, however, identifies the Lagrangian formulation: the observer moves with the particles. On the other hand, if $\Phi_t = \mathrm{Id}_{\mathcal{S}}$, then $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{c} = \boldsymbol{v}$, but this is the basic principle of the Eulerian formulation: the observer permanently takes up the same spatial points.

Proposition 4.6.18. *Let $\boldsymbol{t} \in \mathfrak{T}^p_q(\mathcal{S})$ be a time-dependent spatial resp. Eulerian tensor field, then*

$$\Phi_t^\star(\mathrm{L}_{\boldsymbol{v}}\boldsymbol{t}) = \mathrm{L}_{\Phi_t^\star \boldsymbol{c}}(\Phi_t^\star \boldsymbol{t}) .$$

Proof. (See also [2], p. 101, and [12], p. 130.) By proposition 3.6.29 and theorem 4.6.16,

$$\mathrm{L}_{\boldsymbol{v}}\boldsymbol{t} = \frac{\partial}{\partial t}\boldsymbol{t} + \pounds_{\boldsymbol{w}+\Phi_\star\boldsymbol{\nu}}\boldsymbol{t} = \frac{\partial}{\partial t}\boldsymbol{t} + \pounds_{\boldsymbol{w}}\boldsymbol{t} + \Phi_\star(\pounds_{\boldsymbol{\nu}}(\Phi^\star \boldsymbol{t})) ,$$

where the "time index" is omitted for notational convenience. By 3.6.24, 4.6.10, and the corollaries 3.4.32 and 4.6.12,

$$\begin{aligned}
\mathrm{L}_{\boldsymbol{v}}\boldsymbol{t} &= \frac{\partial}{\partial t}\boldsymbol{t} + \pounds_{\boldsymbol{w}}\boldsymbol{t} + \Phi_\star(\pounds_{\boldsymbol{\nu}}(\Phi^\star\boldsymbol{t})) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t\circ\Phi_s^{-1})^\star\boldsymbol{t}_t\bigg|_{t=s} + \Phi_\star(\pounds_{\boldsymbol{\nu}}(\Phi^\star\boldsymbol{t})) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\star\circ(\Phi_s^{-1})^\star(\Phi_{t\star}(\Phi_t^\star\boldsymbol{t}_t))\bigg|_{t=s} + \Phi_\star(\pounds_{\boldsymbol{\nu}}(\Phi^\star\boldsymbol{t})) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{s\star}(\Phi_t^\star\boldsymbol{t}_t)\bigg|_{t=s} + \Phi_\star(\pounds_{\boldsymbol{\nu}}(\Phi^\star\boldsymbol{t})) \\
&= \Phi_\star\left(\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t^\star\boldsymbol{t}_t)\bigg|_{t=s} + \pounds_{\boldsymbol{\nu}}(\Phi^\star\boldsymbol{t})\right) = \Phi_\star(\mathrm{L}_{\boldsymbol{\nu}}(\Phi^\star\boldsymbol{t}))\,.
\end{aligned}$$

The result is obtained by noting that $\boldsymbol{c} = \Phi_\star\boldsymbol{\nu}$, and applying the pullback by Φ on both sides. ∎

Proposition 4.6.18 shows that it is possible to compute the Lie derivative of a spatial tensor field by performing the Lie derivative on the reference domain along the relative velocity between the body and the reference domain.

4.6.2 The General ALE Operator

Definition 4.6.19. By introducing a reference domain $\mathcal{R}$ together with a relative map Φ_t onto the ambient space $\mathcal{S}$, and a referential map Ψ_t onto the material body $\mathcal{B}$, the $\binom{p}{q}$-tensor-valued physical field $f : \mathcal{S}\times\mathcal{I}\to T^p_q(\mathcal{S})$ defined in 4.3.1 can be referred to the reference domain by setting

$$f_t = \hat{f}_t\circ\Phi_t^{-1} \qquad \text{resp.} \qquad F_t = \hat{f}_t\circ\Psi_t^{-1}$$

at fixed t. The map $\hat{f} : \mathcal{R}\times\mathcal{I}\to T^p_q(\mathcal{S})$, where $\hat{f}_t\circ\Phi_t^{-1} = f_t \in \mathfrak{T}^p_q(\mathcal{S})$, is then called the *Arbitrary Lagrangian-Eulerian (ALE) formulation* of the physical field f. In an ALE formulation, the reference points $\hat{Q}\in\mathcal{R}$ serve as the independent point variables. ◇

In section 4.3, the material time derivative of the Eulerian field $f(Q,t)$ has been defined through $\dot{f} = \frac{\partial f}{\partial t}\big|_Q + \boldsymbol{\nabla}_{\boldsymbol{v}}f$, in which the first term on the right hand side represents the local time derivative at fixed Q, and the second term (the convective term) denotes the covariant derivative of f along the spatial resp. Eulerian velocity field $\boldsymbol{v}$. In what follows, the material time derivative of the ALE field $\hat{f}(\hat{Q},t)$ should be established.

Proposition 4.6.20. *Let $f(Q,t) = f_t(Q)$ be (the Eulerian formulation of) a physical field, and Φ_t at least C^1, then the material time derivative of the ALE field $\hat{f}_t = f_t\circ\Phi_t$ on the reference domain $\mathcal{R}$ is*

$$\dot{\hat{f}} = \frac{\partial\hat{f}}{\partial t}\bigg|_{\hat{Q}} + \boldsymbol{\nabla}^\star_{\boldsymbol{\nu}}\hat{f} \qquad = \hat{\Upsilon}(\hat{Q},t)\,,$$

in which $\boldsymbol{\nu}$ is the referential velocity field from definition 4.6.10 and $\hat{\Upsilon}(\hat{Q},t)$ is a source term accounting for some corresponding response function (e.g. a constitutive equation). $\boldsymbol{\nabla}^{\star}_{\boldsymbol{\nu}}\hat{f}$ is the covariant derivative of $\hat{f}$ along $\boldsymbol{\nu}$ in terms of the Φ_t-induced connection $\boldsymbol{\nabla}^{\star}$ on $\mathcal{R}$.

Proof. Proving the local time derivative is trivial, but the convective term bares some intricacy. Recall that the Eulerian field $f_t \in \mathfrak{T}^p_q(\mathcal{S})$ has been defined in 4.3.1 as an honest tensor field on $\mathcal{S}$. Therefore, the corresponding ALE field $\hat{f}_t = f_t \circ \Phi_t$ is a tensor field over Φ_t, that is, an induced section of the tensor bundle $T^p_q(\mathcal{S})$ (see definition 3.4.18). As proposed by theorem 3.6.13 in section 3.6, the covariant derivative of an induced section calls for a connection $\boldsymbol{\nabla}^{\star}$ on $\mathcal{R}$ that is induced by the connection $\boldsymbol{\nabla}$ on $\mathcal{S}$ via the map Φ_t.

Without loss of generality, for the rest of the proof it will be considered the case where $f_t \in \Gamma(T\mathcal{S})$ is a vector field. In a chart $(\mathcal{V},x)$ on $\mathcal{S}$, f_t has the local representative $f_t(Q) = f^i_t(Q)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}$. It then follows that $\hat{f}_t : \mathcal{R} \to T\mathcal{S}$ is a vector field over Φ_t, i.e.

$$\hat{f}_t(\hat{Q}) = (f^i_t \circ \Phi_t)(\hat{Q})\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \qquad \in \Gamma(\Phi^{\star}_t T\mathcal{S})\,,$$

and $\boldsymbol{\nabla}^{\star}\hat{f}_t \in \Gamma(T^*\mathcal{R} \otimes \Phi^{\star}_t T\mathcal{S})$ is a two-point tensor field:

$$\boldsymbol{\nabla}^{\star}\hat{f}_t = \left(\frac{\partial(f^i_t \circ \Phi_t)}{\partial \chi^\alpha} + (f^j_t \circ \Phi_t)\left(\gamma_j{}^i{}_k \circ \Phi_t\right)(\boldsymbol{F}_{\Phi t})^k{}_\alpha\right)\mathbf{d}\chi^\alpha \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\,.$$

Therein, $\gamma_j{}^i{}_k(Q)$ are the coefficients of $\boldsymbol{\nabla}$ on $\mathcal{S}$, and $(\gamma_j{}^i{}_k(\boldsymbol{F}_{\Phi t})^k{}_\alpha)(\hat{Q})$, by theorem 3.6.13, are the coefficients of $\boldsymbol{\nabla}^{\star}$ on $\mathcal{R}$ induced by Φ_t. Moreover, note that $(f^i_t \circ \Phi_t)$ above functionally depends on the coordinates $\{\chi^\alpha\}_{\hat{Q}} = \chi(\hat{Q})$ of the reference domain $\mathcal{R}$.

Now $\hat{f}$ at each pair $(\hat{Q},t)$ has legs in the tangent space $T_Q\mathcal{S}$ at each $Q = \Phi(\hat{Q},t) \in \mathcal{S}$, so $\dot{\hat{f}}$, as well as $\left.\frac{\partial \hat{f}}{\partial t}\right|_{\hat{Q}}$, at each $(\hat{Q},t)$ also has. Thus $\boldsymbol{\nabla}^{\star}\hat{f} = \boldsymbol{\nabla}^{\star}\hat{f}_t$ contracted by some vector field must also be a vector field over Φ_t for the proposed material time derivative for $\hat{f}$ to make sense. This, however, can only be realized by a honest vector field on $\mathcal{R}$.

For example, taking the referential velocity field $\boldsymbol{\nu}_t = \nu^\alpha_t \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi^\alpha} \in \Gamma(T\mathcal{R})$ gives

$$\boldsymbol{\nabla}^{\star}_{\boldsymbol{\nu}}\hat{f}\;(\hat{Q}) = \left(\frac{\partial(f^i \circ \Phi_t)}{\partial \chi^\alpha}\nu^\alpha + \nu^\alpha(f^j \circ \Phi_t)\left(\gamma_j{}^i{}_k \circ \Phi_t\right)\frac{\partial \Phi^k_t}{\partial \chi^\alpha}\right)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i} \qquad \in \Gamma(\Phi^{\star}_t T\mathcal{S})\,.$$

On the other hand, for $\Phi_t = \mathrm{Id}_{\mathcal{S}}$, the relative map $\Phi_t : \mathcal{R} \to \mathcal{S}$ just becomes a coordinate transformation or relabelling of points, and the proposed material time derivative for $\hat{f}$ must reduce to the material time derivative for the Eulerian field f defined in 4.3.5. However, in this case one also has $\varphi_t = \mathrm{Id}_{\mathcal{S}} \circ \Psi_t^{-1}$ by 4.6.5, and thus $\boldsymbol{v} = (\mathrm{Id}_{\mathcal{S}})_\star\boldsymbol{\nu}$ by 4.6.10. Therefore, the referential velocity is the proper vector field to accomplish the convective term $\boldsymbol{\nabla}^{\star}_{\boldsymbol{\nu}}\hat{f}$, and to proof the assertion. ∎

Theorem 4.6.21 (General ALE Operator). *Let F, f and $\hat{f}$, respectively, be the Lagrangian, Eulerian and ALE formulation of the same time-dependent tensor-valued physical field, then*

$$\frac{\partial F}{\partial t} \circ \Psi = \left.\frac{\partial \hat{f}}{\partial t}\right|_{\hat{Q}} + \boldsymbol{\nabla}_{(\boldsymbol{c}\circ\Phi)} f \qquad = \hat{\Upsilon}(\hat{Q}, t)\,.$$

Proof. As $F_t = \hat{f}_t \circ \Psi_t^{-1}$, by the chain rule one has (see also 4.3.7)

$$\dot{\hat{f}}(\Psi^{-1}(P,t),t) = \frac{\partial F}{\partial t} \qquad \text{resp.} \qquad \dot{\hat{f}}_t = \left(\frac{\partial}{\partial t}\left(\hat{f}_t \circ \Psi_t^{-1}\right)\right) \circ \Psi_t\,.$$

In addition, from theorem 3.6.13, and noting that $\boldsymbol{c}_t \circ \Phi_t = T\Phi_t(\boldsymbol{\nu}_t)$ by 4.6.17,

$$\boldsymbol{\nabla}^{\star}_{\boldsymbol{\nu}} \hat{f} = \boldsymbol{\nabla}_{(\boldsymbol{c}\circ\Phi)} f\,.$$

Use both relations to replace the respective terms in 4.6.20, and finally obtain the desired result. ∎

Finally, the substitution of the ALE operator for material time derivatives in the balance equations of continuum mechanics enables the solution of initial boundary value problems with respect to an arbitrary time-dependent reference domain.

Proposition 4.6.22. *Let the material body $\mathcal{B}$, the reference domain $\mathcal{R}$ and the ambient space $\mathcal{S}$ have the same dimensions, and let $\hat{\rho}(\hat{Q},t)$ and $\rho(Q,t)$ be the mass densities on $\mathcal{R}$ and $\mathcal{S}$, respectively, then conservation of mass in ALE setting reads*

$$\left.\frac{\partial \hat{\rho}}{\partial t}\right|_{\hat{Q}} + \left(\frac{\partial \rho}{\partial x^i} c^i + \rho \operatorname{div} \boldsymbol{v}\right) \circ \Phi_t = 0\,.$$

Proof. The mass densities ρ and $\hat{\rho}$ are scalar fields, hence the covariant derivative is

$$(\boldsymbol{\nabla}^{\star}_{\boldsymbol{\nu}} \hat{\rho})(\hat{Q}) = \frac{\partial \hat{\rho}}{\partial \chi^\alpha} \nu^\alpha = \frac{\partial \rho}{\partial x^i} \frac{\partial \Phi^i_t}{\partial \chi^\alpha} \nu^\alpha = \left(\frac{\partial \rho}{\partial x^i} c^i\right) \circ \Phi_t = (\boldsymbol{\nabla}_{(\boldsymbol{c}\circ\Phi)} \rho)(\hat{Q})\,.$$

Also note that from proposition 4.4.5(ii) and by using the relation $\dot{f}_t = \dot{\hat{f}}_t \circ \Psi_t^{-1} \circ \varphi_t^{-1}$, continuity reads

$$\dot{\hat{\rho}}_t \circ \Psi_t^{-1} \circ \varphi_t^{-1} = -\rho \operatorname{div} \boldsymbol{v}\,,$$

By 4.6.21, one is allowed to write

$$\left(\left.\frac{\partial \hat{\rho}}{\partial t}\right|_{\hat{Q}} \circ \Psi_t^{-1} \circ \varphi_t^{-1}\right) + \frac{\partial \rho}{\partial x^i} c^i + \rho \operatorname{div} \boldsymbol{v} = 0\,,$$

where the left hand side is evaluated at spatial points $Q \in \mathcal{S}$. However, since each term is a proper scalar, the assertion follows by switching the point arguments. ∎

Chapter 5

Summary and Conclusions

After an introduction to modern differential geometry, some geometric concepts in continuums mechanics have been reviewed. In particular, the formulation of kinematics of a body and balance equations benefit from the precise geometric terminology. On the one hand, a clear and precise distinction of the Lagrangian formulation from the Eulerian formulation can be drawn. On the other hand, by using a general term of space, those issues can be revealed, for which the introduction of a metric, an affine connection or other geometric structures physically make sense, or merely lead to simplifications.

Subsequent to the classic formulations according to Lagrange and Euler, an implementation of the geometric concepts in the context of the Arbitrary Lagrangian-Eulerian formulation has been presented. As an essential component, the introduced grid manifold facilitates a consistent description of the relations between the material, the ambient space and the reference domain. This also concerns the numerical implementation by using the finite element method. The grid manifold establishes the topology of the reference domain respectively the finite element mesh, whereas the time-dependent placement of the grid in the ambient space represents the configuration of the mesh, owning the physical properties and the state of the material body. The continuous mesh regularization in the course of an ALE finite element simulation has been formulated geometrically as a flow of the grid nodes in the ambient space.

By defining mappings between the introduced manifolds —namely the material body, its configurations and the reference domain—, pushforward and pullback operators have been obtained that provide the mapping of arbitrary spatial and material tensor fields onto the reference domain. As a fundamental result, the convective velocity field on the current configuration of the body, which is well-known in the ALE community, emerges from the pushforward of the referential velocity field over the relative motion between body and grid. Finally, the paper uncovered that the fundamental ALE operator involves the notion of an induced connection, so that ALE computation can be done without defining an extra affine connection on the reference domain.

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