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I

The results in this thesis deal with two new constructions for posets, lattices, and polytopes, the *E-construction* and the *Bier construction*.

My interest in the first of these two constructions, the E-construction, originates from several questions my PhD advisor Günter M. Ziegler asked about inequality bounds for the flag vector cones of 4-polytopes and 3-spheres when I joined the Discrete Geometry Group at the Technische Universität Berlin. While working on these questions I read the paper "Fat 4-polytopes and fatter 3-spheres" by Eppstein, Kuperberg, and Ziegler and got interested in the E-construction they define. They used it for the construction of an infinite family of 2-simple and 2-simplicial 4-polytopes. Such polytopes satisfy two of the known flag vector inequalities with equality, and — at that time — were the "fattest" known 4-polytopes.

The E-construction defined in this thesis is a generalisation of the one introduced by Eppstein, Kuperberg, and Ziegler to graded posets and lattices of any length and polytopes in any dimension. By means of this construction it is now easy to produce large numbers of 2-simple and $(d-2)$-simplicial d-polytopes (and also many others) with further interesting properties.

The original E-construction requires as input simplicial 4-polytopes having their edges tangent to the unit sphere. My first important step towards the definition of its generalised version was the discovery of explicit geometric coordinates for some of the 2-simple and 2-simplicial 4-polytopes obtained from the E-construction, but without assuming edge tangency. Further working with these new polytopes, with projective transformations, and with some geometric sequences, I arrived at the first infinite sequence of *rational* 2-simple and 2-simplicial 4-polytopes. They are *not* tangent to the unit sphere, so they *cannot* be obtained with the original construction. This set off all other results about the E-construction.

Exploring properties and applications of the E-construction were my main occupation during the past three years. Many other questions and problems I have looked at were motivated by questions that arose in connection with this. A different construction that came up in this context — the *Bier construction* — is introduced in Chapter 5 of this thesis. The Bier posets and spheres defined there have a close formal similarity to those of the E-construction, but the presented results have a more topological flavour. In particular, we obtain a large number of shellable, centrally symmetric and k-nearly neighbourly PL spheres.

Some of the results presented in this thesis are already published elsewhere. The definition of the generalised E-construction, its basic properties for Eulerian lattices, for spheres and for polytopes, together with several applications of it, have appeared in a joint paper with Günter M. Ziegler in *Discrete & Computational Geometry* in November 2004 [68]. This paper is the basis for Chapter 2. Part of it has moved into Chapter 4 and was combined with some results from my recent preprint on products of polytopes [66]. Both chapters contain new material.

In particular, in Chapter 2 is a simple and useful new method for constructing polytopal realisations of spheres obtained from E-construction. Chapter 4 contains lists of known small 2-simple and 2-simplicial 4-polytopes, of known constructions for such polytopes, and a summary on higher dimensional examples.

The central results of Chapter 3 are contained in my preprint “New Polytopes from Products” [66]. It deals with a large class of polytopes to which the E-construction applies and the obtained spheres are polytopal. The main application is a new 2-parameter family of 2-simple and 2-simplicial 4-polytopes with many other remarkable properties.

Chapter 5 is independent of the previous three chapters and reflects the contents of a joint paper [21] with Anders Björner, Jonas Sjöstrand, and Günter M. Ziegler on a construction for posets leading to “Bier Spheres and Posets.” It appeared in the online version of *Discrete & Computational Geometry* in September 2004.

A rough layout of the thesis is as follows. Introductory material from the three papers is combined into Chapter 1. Chapters 2–4 are related to each other and deal with the E-construction, while Chapter 5 deals with the Bier construction. A more detailed account on the three parts is given in the next sections of the introduction. This thesis is written in British English.

F P P

The first chapter contains all notations, definitions and facts that we need from the areas of combinatorics and discrete geometry in this thesis. Even though is short, it is intended to be self-contained. However, it focuses on notions and results used in the later chapters and does not give a general introduction into these topics.

We start this introduction with two sections devoted to the four different notions of *posets*, *lattices*, *spheres*, and *polytopes*. The first two terms are of combinatorial nature; and lattices are posets with some additional structure. We deal with them in Section 1.2. The latter two are of geometric nature; and polytopes are CW spheres with some additional structure. Both are presented in Section 1.3.

Section 1.4 gives a brief introduction into the known results on flag vectors of three and four dimensional polytopes. There is a complete classification in three dimensions, while in four dimensions (and also all higher dimensions) the picture is still quite incomplete. The polytopes obtained from the E-construction lie in areas of the flag vector cone in which only very few polytopes have been known previously. However, this new information by itself does not suffice to add new structural results to the classification problem.

Finally, in Section 1.5 we give a brief introduction into hyperbolic geometry. We discuss two standard models of hyperbolic space, the upper half space model and the Klein model. We introduce geodesics, isometries, and horospheres in hyperbolic space, and show that the isometry group is transitive. These facts are necessary for one of our constructions of infinite families of 2-simple and 2-simplicial 4-polytopes in Chapter 2.

Here are some textbooks for a more detailed introduction into these topics. I learnt much of what I know about posets and lattices from the two books of Richard P. Stanley on "Enumerative Combinatorics" [83, 84]. My favourite book on topology is "Topology and Geometry" by Glen E. Bredon [28], and the basics about polytopes are in Günter M. Ziegler's book "Lectures on Polytopes" [89]. Polytope constructions are explained in detail in the book on "Convex Polytopes" [44] by Branko Grünbaum and in the classic text book of H.S.M. Coxeter on "Regular Polytopes" [30]. For hyperbolic geometry, one could look at the introductory text "Lectures on Hyperbolic Geometry" of Ricardo Benedetti and Carlo Petronio [12]. For discrete geometric questions, and some facts about flag vectors, the two volumes of the "Handbook on Convex Geometry" [43] are always a good source.

E-C

The E-construction for spheres and polytopes was introduced in a paper of Eppstein, Kuperberg, and Ziegler. They obtained the first infinite series of 2-simple and 2-simplicial 4-polytopes using this new method. Earlier claims of a construction of such a family of polytopes reported by Grünbaum in his book [44, p. 82,170], where this is attributed to Perles and Shephard, turned out to be premature.

In its original version, the E-construction applies to simplicial 4-polytopes having all their edges tangent to the unit sphere. It modifies such a simplicial polytope P by adding the vertices of its polar in a suitable way. However, the edge-tangency condition prevents most of the polytopes obtained by this construction from being realised with rational coordinates. Moreover, edge tangency is difficult to achieve for a simplicial 4-polytope, if it is possible at all. Eppstein, Kuperberg, and Ziegler use a quite intricate method for the construction of an infinite family of such 4-polytopes. Their families of polytopes are now a special case of Theorem 2.5.15.

In Chapter 2 we define a generalised version of the E-construction. It extends and modifies the original E-construction in several directions:

- The construction is extended to finite graded Eulerian posets, finite graded Eulerian lattices, PL spheres, and polytopes in any dimension.
- The special rôle played by the edges in the original version of the construction is relaxed by defining a similar construction for any dimension t of "distinguished" faces, for a parameter t between 0 and $d-1$. Here d is either the dimension of the polytope, or the rank of the poset minus one.
- Edge tangency in the case of polytopes is not anymore necessary to obtain geometric realisations.

I kept the name *E-construction* also for the generalised version. To avoid confusion, the dimension t of the special faces of the polytope is sometimes added as a subscript, so that $E_t(P)$ denotes the polytope obtained from the construction applied to faces of dimension t of a polytope P.

Defining the E-construction combinatorially on the level of graded Eulerian posets and lattices allows a much more systematic treatment of its properties than by defining it only for polytopes. Consequently, we give two different definitions of the construction, a combinatorial one in Section 2.3 for Eulerian posets and lattices, and a geometric one for PL spheres and polytopes in Section 2.4. The latter coincides with the former on the level of face lattices.

Eulerian *lattices* provide a simple model for the combinatorial properties of convex polytopes. This is a rather recent topic in combinatorics. Eulerian *posets* were formalised by Stanley [80] in 1982. Basic ideas for their definition appeared previously in Klee's paper [55] from 1964. There are some recent studies of the flag vectors of Eulerian posets (see e.g. Stanley [82]). However, there is still only little systematic knowledge and treatment of Eulerian *lattices* in the literature.

A priori, the geometric version of the generalised E-construction applies to a PL sphere S and associates a new PL sphere $E(S)$ to it. However, such PL spheres serve only an intermediate tool for our considerations. We are mainly interested in polytopes, which are PL spheres with some additional geometric structure. So we introduce in Chapters 2 and 3 several classes of polytopes with the property that the PL spheres obtained from the E-construction are in fact polytopal. For most of these classes we also provide simple methods to construct explicit geometric coordinates. In many cases these coordinates will be rational.

Here are some of the key properties of the generalised E-construction and the most important classes of polytopes obtained from it.

- By Theorem 2.3.11, the posets, lattices, and polytopes obtained from the E-construction are 2-simple and k-simplicial, for some $k \geq 2$ depending on the simplicity and simpliciality of the input.
- In Theorem 2.5.15 is the first infinite family of 2-simple and $(d-2)$-simplicial d-polytopes in any dimension $d \geq 4$.
- We obtain the first infinite family of *rational* 2-simple and 2-simplicial 4-polytopes in Corollary 2.5.11. We construct many other such families.
- The 2-simple and 2-simplicial 4-polytopes produced with this construction lie on the boundary of the cone formed by the known flag vector inequalities of 4-polytopes.
- Some of the polytopes have a high fatness. This quantity was introduced by Ziegler and is defined to be the quotient of the sum of edges and ridges of a polytope divided by the sum of its vertices and facets. Bounds on its range are of great interest in connection with the classification of flag vectors.

- For several of our families of 2-simple and 2-simplicial polytopes we are able to provide flexible geometric realisations.
- The E-construction applies to all products of polygons. We give explicit geometric realisations and examine their symmetry groups and realisation spaces.

In the case of polytopes the generalised E-construction roughly works as follows: Given is a d-polytope P and a dimension t between 0 and $d-1$. Add one new vertex beyond each facet of P in such a way, that vertices above facets sharing a common t-face lie in a common hyperplane with this t-face. If such a choice of new vertices exists, then the convex hull of P together with these new vertices is a polytope $E_t(P)$ that has precisely one facet for each t-face of P.

In Chapter 2 several properties of this construction on the level of lattices are proven. They are inspired by the properties one would expect in the geometric setting. In particular, the construction preserves the length of a poset, and the lattice $E(L)$ obtained from a lattice L via this construction is finite, graded, and Eulerian, if L has these properties.

We transfer the definitions and results into a geometric setting in Section 2.4 and apply them to PL spheres. In the rest of Chapter 2 we present several classes of polytopes to which the E-construction applies. We construct one infinite family of 2-simple and $(d-2)$-simplicial d-polytopes for any dimension $d \geq 4$, and many such families in dimension $d = 4$, most of them with rational coordinates.

Chapter 3 is entirely devoted to a quite general method for the application of the E-construction to d-polytopes in the case $t = d-2$, which does the following: Let P_0 and P_1 be two polytopes of dimensions d_0 and d_1, with $d_0 + d_1 = d$. Suppose there are polytopal realisations of $E_{d_0-2}(P_0)$ and $E_{d_1-2}(P_1)$. Theorem 3.3.1 now states that, if these realisations satisfy some additional conditions, then there is a polytopal realisation of $E_{d-2}(P_0 \times P_1)$, which can be obtained by suitably combining the coordinates of the two realisations.

The application of this construction is demonstrated by some simple examples in all dimensions ≥ 4. However, the main application is the construction of yet another infinite 2-parameter family E_{mn} of 4-polytopes. These E_{mn} are obtained from the E-construction when applied to a product of two polygons C_m and C_n with m and n vertices, for $m, n \geq 3$. For this, we first show that there is a restricted version of Theorem 3.3.1 that allows to look at the two factors separately. By Theorem 3.4.1, there are realisations of $E(C_k)$ for polygons C_k, $k \geq 3$, that satisfy these restricted conditions. The proof is constructive and we obtain simple geometric realisations for the E_{mn}. However, in general, the coordinates will not be rational, as there is one quadratic equation involved in the construction.

The E_{mn} are self-dual 2-simple and 2-simplicial 4-polytopes. E_{44} is the 24-cell. For large m and n these polytopes approach the upper bound for fatness of

E-polytopes obtained from simplicial polytopes. They have a large combinatorial symmetry group and geometric realisations that realise many on their combinatorial symmetries. However, by Theorem 3.5.6, only for $m, n = 4$ — which is the 24-cell — these two groups coincide for a geometric realisation, namely, the standard regular realisation of the 24-cell. There are a few other, only slightly less symmetric E_{mn} described in Theorem 3.5.3.

For the two small cases $m, n = 3$ and $m, n = 4$ we examine in Theorems 3.5.10 and 3.5.13 the realisation space of these polytopes. For the first example we provide an explicit way to construct all examples that satisfy the conditions in Theorem 3.4.1. For the second example we only state a 4-parameter family of realisations, as the explicit construction of all possible realisations is quite technical.

The initial idea to investigate the E-construction of products of polytopes arose from the interest in the realisability and the symmetry of the polytopes E_{mn} in the special case $m = n$. A combinatorial description of these polytopes and some symmetry properties were obtained independently by Bokowski and Gévay.

Chapter 4 is the last chapter on the E-construction. It collects results from the previous two chapters and compares them to "the outside polytope world." Namely, it addresses the relation between 2-simple and 2-simplicial polytopes and polytopes without this property. We present new results on flag vectors of 2-simple and 2-simplicial polytopes, look at their fatness, give lower bounds on the number of 2-simple and 2-simplicial polytopes and show that the flag vector does not fix the combinatorial type of such polytopes.

Further, Chapter 4 contains a summary on the known construction methods for 2-simple and 2-simplicial 4-polytopes. We provide a complete list of these up to 19 vertices, together with many more interesting examples that have a larger number of vertices. We also give a list of interesting examples in higher dimensions. This is, however, not complete and the case of d-polytopes for $d \geq 5$ is, despite the given infinite series of 2-simple and $(d-2)$-simplicial d-polytopes, still quite unexplored. We have provided a wealth of examples in dimension 4 with this thesis, but constructing explicit examples in dimensions $d \geq 5$ is still much more difficult.

B S

Chapter 5 deals with a topic different from that of the previous chapters and is joint work with Anders Björner, Jonas Sjöstrand, and Günter M. Ziegler. It introduces a second new construction for finite graded posets, the *Bier construction*. This has some formal similarity to the E-construction of the previous chapters, and some of the presented theorems and proofs will look familiar. However, it has a different origin, and the results and properties that we present about this construction have a much more combinatorial and topological flavour.

In an unpublished paper from 1992, Thomas Bier introduces a simple construction for a large number of simplicial PL spheres. His construction associates a simplicial $(n-2)$-sphere S with up to $2n$ vertices to any simplicial complex $\Delta \subset 2^{[n]}$ on n vertices by forming the *deleted join* of the complex Δ with its combinatorial Alexander dual. Bier proved that this construction does indeed produce PL spheres, by verifying that any addition of a new face to the simplicial complex Δ amounts to a bistellar flip in the sphere S.

We generalise Bier's original construction and define a *Bier poset* $\mathrm{Bier}(P, I)$ for any bounded finite poset P and any proper order ideal $I \subset P$ in this poset. The poset $\mathrm{Bier}(P, I)$ consists of all intervals in P that have their minimal element in I and their maximal element in the complement $P \setminus I$, together with an artificial maximal element $\hat{1}$. This set is ordered by reversed inclusion.

Our generalised construction contains the original construction of Thomas Bier as a special case. Namely, if P is the boolean poset B_n, then it can be viewed as the face poset of the $(n-1)$-simplex. Any proper ideal $I \subset B_n$ may be interpreted as an abstract simplicial complex Δ. The PL spheres that Bier describes in his work are spheres that have $\mathrm{Bier}(B_n, \Delta)$ as their face lattice.

We prove several new properties of this construction and the obtained posets and spheres. Here are the key results contained in Chapter 5.

- We show that the order complex of $\mathrm{Bier}(P, I)$ is PL homeomorphic to that of the poset P. We prove that this complex may be obtained by a sequence of stellar subdivisions of edges in the order complex of P.
- Like the E-construction, the Bier construction preserves several properties of the poset. In particular, if P is an Eulerian or Cohen-Macaulay poset or lattice, then $\mathrm{Bier}(P, I)$ will have that property as well.
- If L is the face lattice of a regular PL-sphere S, then the lattices $\mathrm{Bier}(L, I)$ for proper ideals $I \subset L$ are again face lattices of regular PL-spheres, the *Bier spheres* of S.
- In the case of Bier's original construction we prove that all the simplicial spheres obtained from the construction are shellable.
- The number of Bier spheres is so large, that most of the spheres $\mathrm{Bier}(B_n, \Delta)$ for large n cannot be realisable as polytopes.
  Similarly, for special choices of the abstract simplicial complex Δ in B_n, and even n, we obtain "many" nearly neighbourly and centrally symmetric $(n-2)$-spheres on $2n$ vertices.
- The g-vector of a Bier sphere $\mathrm{Bier}(B_n, \Delta)$ can be expressed explicitly in terms of the f-vector of Δ. We show that these g-vectors actually are K-sequences, and thus they satisfy a strong form of the g-conjecture for spheres. The generalised lower bound conjecture is verified for Bier spheres.

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There are many people without whom this thesis would not have been possible. First of all, I am grateful to my supervisor Günter M. Ziegler for giving me the chance to come to Berlin and do my PhD as a student in the Graduate Program "Combinatorics, Geometry, and Computation". I want to thank him for his initial questions that led to many of the results in this thesis, for his continuing support and encouragement, and for the collaboration with two papers, that are now the basis of Chapters 2 and 5. He had always time for my questions.

I want to thank Anders Björner and Jonas Sjöstrand for the work on our joint paper on "Bier Spheres and Posets", which worked out quite well via electronic mail communication and without ever meeting personally. My work on the results in Chapter 3 would never have started without the question of polytopal realisability of the spheres E_{mn} for $m = n$ asked by Jürgen Bokowski and Gabor Gévay. Prior work of Francisco Santos gave the initial idea for symmetric realisations of E_{mn} for small m and n, which then set off several of the other results.

Jürgen Richter-Gebert has accepted to co-referee this thesis and come to Berlin for this. Thanks!

In addition to financing three years of independent study without any teaching requirements, the *Graduiertenkolleg* "Combinatorics, Geometry, and Computation" (funded by the *Deutsche Forschungsgemeinschaft (DFG)*) provided a stimulating background with all the Monday lectures, the spring and fall schools, the annual workshops, and the block courses. I learned a lot during all these. Thanks go to the speaker Helmut Alt, and to the coordinators Bettina Felsner and Andrea Hoffkamp, who organised this programme so well. As a part of the graduate school, I spent a semester at ETH Zurich in the research group of Emo Welzl. Thank you, Emo, for a fruitful and productive summer in Zurich.

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Berlin, April 2005

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1.1 I

This thesis examines both combinatorial and geometric properties of polytopes that have some special properties. The constructions in the later chapters start out with combinatorial definitions on posets and lattices. These are transfered into a geometric setting by applying them to face lattices of spheres and polytopes and associating appropriate geometric structure to the result. In this chapter, we introduce the necessary combinatorial and geometric notions and give the properties needed for the later chapters. Nothing really new is included here, but some material lacks a concise source in the literature.

The chapter is split into four parts. In the first, we introduce posets and lattices, and in the second PL spheres and polytopes. In the third, we present some material about flag vectors of polytopes and the flag vector classification problem. The last part is about hyperbolic geometry, which we need for one polytope construction in the end of the second chapter.

1.2 P L

We introduce posets and lattices, which are the two important concepts of combinatorics that we need in this thesis. We restrict our attention to the properties and theorems that we need later. A more detailed treatment, and a broader view on these topics, can be found in the two books of Stanley [83] and [84] on "Enumerative Combinatorics".

1.2.1 P

Roughly, partially ordered sets are sets together with an additional structure introducing an "order" between some of its elements. The main example of a partially ordered set occurring in this thesis is the *face lattice* of a polytope P. Its underlying set is the set of all faces of P. The order relation is given by inclusion of faces. (see Section 1.3.2 for an exact statement). Here is the precise definition of a partially ordered set.

1.2.1 D [Poset]. A *partially ordered set* (or *poset* for short) P is a set (usually denoted with the same letter P) together with a binary relation $\leq$ (which can be viewed as a subset of $P \times P$) that for any $x, y, z \in P$ satisfies:

(1) $x \leq x$, (reflexivity)
(2) $x \leq y$ and $y \leq x$ imply $x = y$, (antisymmetry)
(3) $x \leq y$ and $y \leq z$ imply $x \leq z$. (transitivity)

We often write $\leq_P$ to emphasise the set on which this relation is defined.

We use $x \prec y$ to denote $x \preceq y$ and $x \neq y$. We also use $y \succeq x$ for $x \preceq y$ and $y \succ x$ for $x \prec y$. Two elements x and y of a poset P are *comparable* if either $x \preceq y$ or $y \preceq x$. Otherwise they are *incomparable*. An element y is said to *cover* an element x if $x \prec y$ and there is no $z \in P$ such that $y \prec z \prec y$.

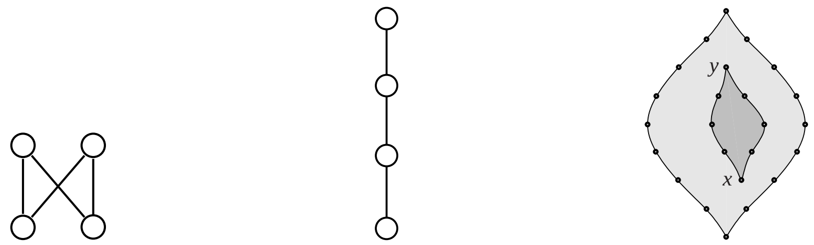
The *Hasse diagram* of a poset P is a very convenient way of visualising P in $\mathbb{R}^2$: For every element of the poset we draw a point in the x_1-x_2-plane in such a way that, for any pair $x \preceq y \in P$, the point y has a larger x_2-coordinate than the point x. We connect a pair of points x and y by an edge if y covers x. See Figures 1.1(a) and 1.1(b) for examples.

An *induced subposet* Q of a poset P is a subset of the elements of P together with the induced order relation; that is, two elements of Q are comparable in Q if and only if they are comparable as elements of P. However, the covering relations may change. An *interval* $[x, y]$ bounded by two elements $x \preceq y$ in a poset P is the induced subposet

$$[x, y] := \{z \in P \ : \ x \preceq z \preceq y\}.$$

See Figure 1.1(c) for an example. A *maximal element* in a poset P is an element $y \in P$ such that $x \preceq y$ for all $x \in P$. Similarly, one defines *minimal elements*. Note, that a poset can have several maximal and minimal elements; see Figure 1.1(a) for an illustration of this.

A poset σ is called a *chain* (or *total order* or *linear order*) if any two elements in σ are comparable. See Figure 1.1(b) for an example of a chain. The *maximal element* of a chain σ is the element $y \in \sigma$ satisfying $x \preceq y$ for all $x \in \sigma$. Similarly, the *minimal element* is the element $x \in \sigma$ satisfying $x \preceq y$ for all $y \in \sigma$. A *chain in P* is a subposet σ of P that is a chain with the induced order. A chain in P is *maximal* if there is no larger chain in P containing it.



(a) A poset with two maximal and two minimal elements.

(b) The totally ordered poset $[4] = \{1, 2, 3, 4\}$.

(c) An interval $[x, y]$ in a poset.

Figure 1.1: Examples of posets I.

1.2.2 E . Here are some examples of posets together with their Hasse diagrams. These posets will reappear frequently in the subsequent chapters.

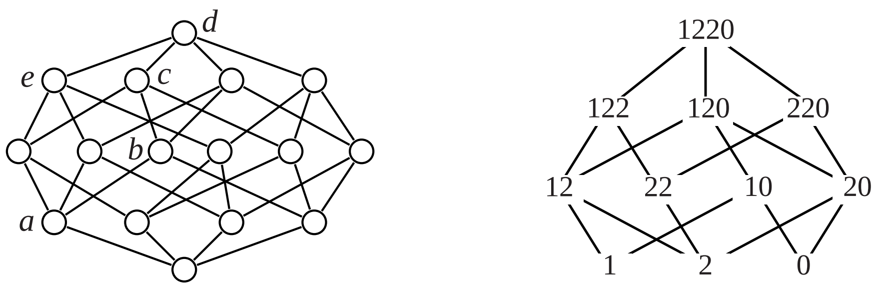
(1) Let P be the set containing the first n natural numbers $\{1, \dots, n\}$. Equip this with the order relation induced by the usual "$\leq$" in $\mathbb{N}$. This is a totally ordered set commonly denoted by $[n]$. See Figure 1.1(b) for an illustration.

(2) Let B_n for $n \in \mathbb{N}$ be the set of all subsets of $[n]$ (i.e. the power set of $[n]$) together with the relation given by inclusion (usually referred to as *"ordered by inclusion"*). This is the *Boolean poset* on n elements. Its Hasse diagram for the case $n = 4$ is shown in Figure 1.2(a).

(3) Let $\Gamma := \{0, 1, \dots, k\}$ be an alphabet with $k + 1$ letters and $\omega(\Gamma)$ the set of all words over Γ. We say that a word ω_1 is smaller than ω_2 if ω_1 is a substring of ω_2. This defines a poset with infinitely many elements. A small portion of this poset is in Figure 1.2(b).

A poset P is said to *have a zero* if there is a unique element $\hat{0} \in P$ that satisfies $\hat{0} \leq y$ for all $y \in P$. Similarly, a poset is said to *have a one* if there is an element $\hat{1}$ that satisfies $x \leq \hat{1}$ for all $x \in P$ (In a chain, these are the minimal and maximal elements). The poset in Figure 1.3(a) has a zero, but no one, and that in Figure 1.3(b) has a one, but no zero. A poset is *bounded* if it has a $\hat{0}$ and a $\hat{1}$. It is *locally finite*, if any interval contains only a finite number of elements, and it is *finite*, if the set P itself has only a finite number of elements. The Boolean poset in Example 1.2.2(2) is bounded, the one in Example 1.2.2(3) is not. Note, that a bounded poset need not be finite, and vice versa.

In a bounded poset P we say that an element is an *atom* of the poset if it covers $\hat{0}$, and it is a *coatom* if it is covered by $\hat{1}$. The set of all atoms in a poset is denoted by $\mathcal{A}(P)$ and the set of all coatoms by $C(P)$. A poset P is *connected* if its Hasse diagram is connected as a graph. A bounded poset P is *strongly connected*, if $P \backslash \{\hat{0}, \hat{1}\}$ is connected.



(a) The Boolean poset B_4.

(b) A small part of the poset $\omega(\Gamma)$: The poset of all subsequences of 1220.

Figure 1.2: Examples of posets II.

Mostly, the relation in posets that we consider is either *inclusion* or *reversed inclusion* for a set of cells in a CW sphere or of faces in a polytope. These relations are opposite to each other in the following sense.

1.2.3 D **[Opposite Poset].** The *opposite poset* P^{op} of a poset P with relation $\preceq$ is a poset with the same underlying set, but reversed order relation. That is, for all $x, y \in P^{\mathrm{op}}$, we have $y \preceq x$ in P^{op} if and only if $x \preceq y$ in P. See Figure 1.3 for a Hasse diagram of a poset and its opposite poset.

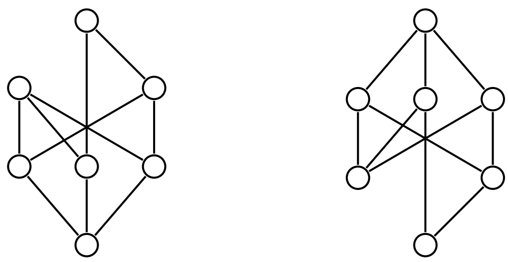
A map $m : P \to Q$ between two posets P and Q is *order-preserving* if it respects the order relation. That means, for any $x, y \in P$, their images under m should satisfy $m(x) \preceq m(y)$ if $x \preceq y$.

1.2.4 D **[Isomorphic Posets].** Two posets P and Q are *isomorphic*, if there exists an bijection $\varphi : P \to Q$ such that $x \preceq y$ if and only if $\varphi(x) \preceq \varphi(y)$. That is, φ and its inverse are order preserving. We call such a map an *automorphism* of the poset P if $Q = P$. The set of all automorphisms of P, together with composition of maps, forms a group, the *automorphism group* $\mathrm{Aut}(P)$ of P.

There are several simple constructions that produce new posets from old ones. We will later need the following two methods, which define two different products for a pair of posets.

1.2.5 D **[Product and Reduced Product].** Let P and Q be two posets with order relations $\preceq_P$ and $\preceq_Q$.

- The *(direct) product* of P and Q is the set $P \times Q := \{(x, y) : x \in P, y \in Q\}$ with order relation $(x, y) \preceq (x', y')$ if $x \preceq_P x'$ and $y \preceq_Q y'$.
- Assume that both P and Q have a zero. The *reduced product with respect to* $\hat{0}$ of P and Q is the set $P \times_{\mathrm{red}} Q := P\backslash\{\hat{0}\} \times Q\backslash\{\hat{0}\} \uplus \hat{0}$, with order relation induced from $P \times Q$, and $\hat{0} \prec (x, y)$ for all $x \in P\backslash\{\hat{0}\}$, $y \in Q\backslash\{\hat{0}\}$.



(a) A poset without $\hat{1}$. *(b)* and one without $\hat{0}$.

Figure 1.3: Examples of posets III.

In the same way, one defines a reduced product for a pair of posets with $\hat{1}$. With the obvious adaptions, all results about reduced products with respect to $\hat{0}$ are also valid for those with respect to $\hat{1}$.

Now we turn to some more powerful structures of posets. Let P be a locally finite poset. The *length* $\ell(\sigma)$ of a chain σ in P is one less than the number of elements it contains. $\ell(x,y)$ for two elements $x \preceq y$ in P denotes the length of the longest chain in the interval $[x,y]$. Similarly, the *length of P* is

$$\ell(P) := \max\{\, \ell(\sigma) \;:\; \sigma \text{ is a chain in } P \,\}.$$

We call an interval or poset *graded* (or *ranked*) if all maximal chains have the same length. Note, that this need not be true in general posets. However, if P is graded, then the same is true for any interval $[x,y] \subseteq P$. This allows the following (recursive) definition.

1.2.6 D [Rank Function]. A bounded graded poset P can be equipped with a *rank function* $\rho : P \longrightarrow \mathbb{N}$ by defining
(1) $\rho(x) := 0$ for all minimal elements of P, and
(2) $\rho(y) := \rho(x) + 1$ if y covers x.
For any $0 \leq k \leq \ell(P)$ define the *level set* of rank k in P by

$$P_k := \{\, x \in P \;:\; \rho(x) = k \,\}.$$

If $\hat{1}$ is the maximal element of P then $\rho(\hat{1})$ is the length of the poset. Observe that $\ell(x,y) = \rho(y) - \rho(x)$ for any two $x, y \in P$.

We indicate a grading of a poset in its Hasse diagram by giving elements the same height if and only if they have the same rank. Note, that an induced subposet of a graded poset need not be graded itself, see e.g. Figure 1.5(a).

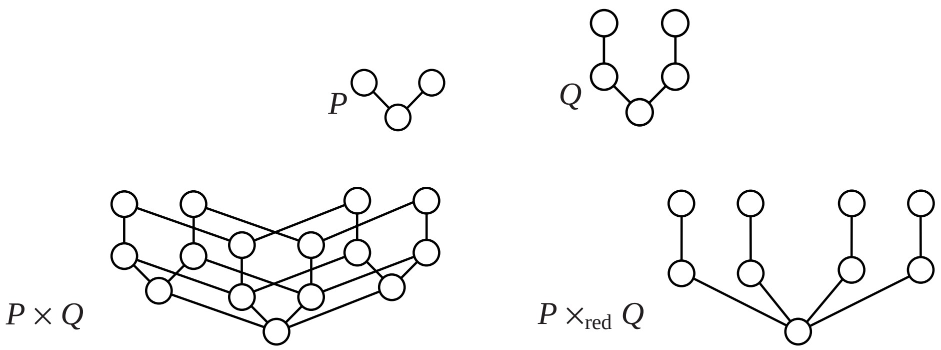


Figure 1.4: Products of two posets.

1.2.7 D **[Ideal].** Let P be a poset. An *ideal* I (also called a *down set*) in P is a subset of P that for any $y \in I$ contains all elements $x \in P$ covered by y. See Figure 1.5(b) for an example. A subset $F \subseteq P$ is called a *filter* (or *up set*) in P, if it is an ideal in the opposite poset.

The combinatorial properties of a poset are the number of its elements and – in the graded case – its rank and the covering relations between the elements. For a graded poset, the next notion captures an aggregated version of this information in a very convenient form by summing over all elements with equivalent properties.

1.2.8 D **[Flag Vector].** Let P be a finite graded poset of length ℓ. An S*-chain* for any subset $S \subseteq \{-1, \ldots, \ell-1\}$ is a chain in P that has length $|S| - 1$ and contains an element σ_j of rank $j+1$ for any $j \in S$. Let f_S be the number of all S-chains in P. The *flag vector* of P is the vector

$$\operatorname{flag}(P) := (f_S)_{S \subseteq \{-1,\ldots,\ell-1\}}$$

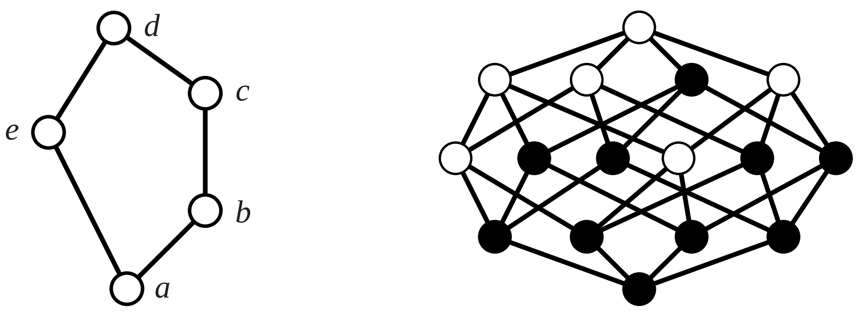
(in some previously fixed order on the power set of $\{-1, \ldots, \ell-1\}$). We usually write $f_{s_1 s_2 \ldots s_k}$ instead of $f_{\{s_1 s_2 \ldots s_k\}}$. The f*-vector* of P is the subset

$$f(P) := (f_{-1}, f_0, f_1, \ldots, f_{\ell-1})$$

of the entries of $\operatorname{flag}(P)$.

If P is bounded, then one usually drops the first and last entry in the f-vector, as they are both one. This is in particular the case for f-vectors of polytopes. In *bounded* posets of length ℓ we know that $f_{\{-1,j\}} = f_j$ and $f_{\{j,\ell-1\}} = f_j$ for any $-1 \leq j \leq \ell$ by definition. This is not true for arbitrary posets.

1.2.9 E **.**



(a) A subposet of B_4 in Figure 1.2(a). *(b)* The black elements form an ideal in B_4.

Figure 1.5: Examples of posets IV.

- The the flag vector of [4] in Figure 1.1(b) has 16 entries, which are all one.
- The f-vector of the Boolean poset in Figure 1.2(a) is $f(B_4) = (1, 4, 6, 4, 1)$. The remaining nontrivial entries of the flag vector are $f_{01} = f_{02} = f_{12} = 12$ and $f_{012} = 24$.

1.2.10 R . The index shift in the definition of the flag vector of a poset originates in the correspondence of elements of rank $j + 1$ in the face poset of a polytope to faces of dimension j in this polytope. That is, in the case of a face lattice of a polytope, f_0 counts the number of vertices, f_1 the number of edges, etc. f_{-1} is the empty set, and f_d for a d-dimensional polytope is the polytope itself.

Let $\mathcal{I}(P)$ denote the set of all intervals in P. The *Möbius function* $\mu : \mathcal{I}(P) \to \mathbb{Z}$ of a poset P is defined inductively by the following two conditions.

$$\begin{aligned} \mu([x,x]) &:= 1 && \text{for all} && x \in P \\ \mu([x,y]) &:= -\sum_{x \preceq z \prec y} \mu(x,z) && \text{for all} && x,y \in P \text{ with } x \prec y \end{aligned}$$

We usually write $\mu(x,y)$ instead of $\mu([x,y])$. The Möbius function is a special function in the algebra of all functions defined on $\mathcal{I}(P)$, which is a rather powerful tool in the theory of posets; see e.g. Stanley's book [83, pp. 113ff]. For example, the Möbius function on the poset shown in Figure 1.5(a) evaluated on the interval $[a,e]$ is 0, while all other values are either +1 or −1. In this thesis, we use the Möbius function only for the following important definition.

1.2.11 D [Eulerian Poset]. A finite and graded poset P is *Eulerian* if $\mu(x,y) = (-1)^{\ell(x,y)}$ for all $x \preceq y$ in P.

Eulerian posets are a rather recent topic in combinatorics. Basic ideas appeared in a paper of Klee in 1964 [55], while a formal definition came only in 1982 by Stanley [80]. A survey on known results for Eulerian posets is given by Stanley in [82]. To test, whether a graded poset is Eulerian or not, we will mostly use the criterion given by the following proposition.

1.2.12 P [Odd and Even Elements]. *A finite graded poset P is Eulerian if and only if all intervals $[x,y]$ of length $\ell \geq 1$ in P contain an equal number of elements of odd and even rank.*

Proof. Let ρ be the rank function on P and $x,y \in P$ two arbitrary elements with $\ell(x,y) \geq 1$. If P is Eulerian then $\mu(x,y) = (-1)^{\ell(x,y)}$ and we can compute

$$0 = (-1)^{\rho(x)}\Big[(-1)^{\ell(x,y)} + \sum_{x \preceq z \prec y} \mu(x,z)\Big]$$

$$
\begin{aligned}
&= (-1)^{\rho(x)}\Big[(-1)^{\rho(y)-\rho(x)} + \sum_{x \le z \prec y} (-1)^{\ell(x,z)}\Big] \\
&= (-1)^{\rho(y)} + (-1)^{\rho(x)} \sum_{x \le z \prec y} (-1)^{\rho(z)-\rho(x)} \\
&= (-1)^{\rho(y)} + \sum_{x \le z \prec y} (-1)^{\rho(z)} = \sum_{x \le z \le y} (-1)^{\rho(z)}.
\end{aligned}
$$

The other implication follows by induction over $\ell := \ell(x,y)$. If $\ell = 0$, then $x = y$ and $\mu(x,x) = 1 = (-1)^{\ell}$. So, if the claim is true for any $k \le \ell$ and $\ell(x,y) = \ell + 1 \ge 1$, then

$$
\begin{aligned}
(-1)^{\rho(x)}\,\mu(x,y) \overset{def.}{=} -(-1)^{\rho(x)} \sum_{x \le z \prec y} \mu(x,z) &= -(-1)^{\rho(x)} \sum_{x \le z \prec y} (-1)^{\ell(x,z)} \\
&= -\sum_{x \le z \prec y} (-1)^{\rho(z)} = (-1)^{\rho(y)}.
\end{aligned}
$$

The last equality uses that the interval $[x,y]$ has the same number of odd and even rank elements. □

1.2.13 R **.** By a result of Ehrenborg [32], it suffices to look at the intervals of even length in the proof of Proposition 1.2.12: He proves, that, if in a poset P all intervals of length up to $2k$ are Eulerian, then so are the intervals of length $2k+1$. The proof of this is a lot more involved than the arguments given for the above proof of Proposition 1.2.12.

The special case of Proposition 1.2.12 for $\ell = 2$ tells us that any interval of length two in an Eulerian poset has precisely two elements in the middle level. This is sometimes called the *diamond property* of Eulerian posets.

1.2.14 D **[Euler Equation].** Let P be a finite graded poset of length ℓ. P is said to satisfy the *Euler equation* if its f-vector satisfies

$$
f_{-1} - f_0 + f_1 \mp \cdots + (-1)^{\ell} f_{\ell-1} = 0.
$$

This reduces to the well known Euler formula for 2-spheres: If P is the poset obtained from a sphere with f facets, e edges and v vertices, then the above formula specialises to $v - e + f = 2$. Face posets of polytopes (which we define in Section 1.3.2) are Eulerian. This is proven with the help of the following simple observation.

1.2.15 P **.** *Let P be a finite graded poset. P is Eulerian if and only if any interval $[x,y] \subseteq P$ satisfies the Euler equation.* □

The property of being Eulerian is preserved by both product operations that we have defined in Definition 1.2.5.

1.2.16 T . *Let P and Q be Eulerian posets. Then both $P \times Q$ and $P \times_{red} Q$ are Eulerian.*

Proof. Clearly, both products are finite if P and Q are finite. Let ρ_P and ρ_Q be rank functions on P and Q. Then

$$\rho((x,y)) := \rho_P(x) + \rho_Q(y)$$

for $(x,y) \in P \times Q$ is a rank function on the product and

$$\rho_{red}(x,y) := \rho_P(x) + \rho_Q(y) - 1$$

for $(x,y) \in P \times_{\text{red}} Q$ and $\rho_{red}(\hat{0}) = 0$ is a rank function on the reduced product. It remains to prove that both products are Eulerian. For this we count elements of odd and even rank in intervals.

- Any interval of length at least one in $P \times Q$ has the form $[(x,y),(x',y')] = [x,x'] \times [y,y']$ for $x \preceq x'$ and $y \preceq y'$, and at least one of these relations is strict. W.l.o.g. assume that $y \prec y'$. For any $\overline{x}$ in $[x,x']$, we have equally many elements of type $(\overline{x},\overline{y})$, for $\overline{y} \in [y,y']$, with odd and even rank, as Q is Eulerian. Summing over all $\overline{x} \in [x,x']$ gives the result.
- For reduced products, the only difference to the previous argument occurs for intervals $[\hat{0},(x,y)]$. For any fixed $\overline{x} \in [\hat{0},x] \backslash \{\hat{0}\}$ of *even* rank, this interval contains all elements of type $(\overline{x},\overline{y})$ for $\overline{y} \in [\hat{0},y] \backslash \{\hat{0}\}$. As Q is Eulerian, this set has one more element of *odd* rank than elements of even rank (compared to the other product, it lacks the pair $(\overline{x},\hat{0})$). Similarly, if $\rho_P(x)$ is *odd*, then it contains one more element with *even* rank. Now $[\hat{0},x] \backslash \{\hat{0}\}$ has one more element of odd rank, so summing over all $\overline{x}$, and using that $\rho(\hat{0}) = 0$, we conclude that $[\hat{0},(x,y)]$ is Eulerian. □

The Euler equation is the only *non-trivial* linear relation between the entries of the f-vector of bounded Eulerian posets (the trivial ones are $f_{-1} \equiv 1 \equiv f_{\ell-1}$). However, for flag vectors, there are many more entries that linearly depend on some others. All such relations are subsumed in the *generalised Dehn–Sommerville Equations*. They were found by Bayer and Billera in 1985 [10].

1.2.17 T [Generalised Dehn–Sommerville–Equations]. *Let P be a finite bounded graded Eulerian poset and $S \subseteq \{0,\ldots,d-1\}$. If $\{i,k\} \subseteq S \cup \{-1,d-1\}$, $i < k-1$, and S contains no m such that $i < m < k$, then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}} = f_S (1 - (-1)^{k-i-1}).$$

All linear relations between entries of the flag vector are contained in these generalised Dehn–Sommerville–equations. *They reduce the dimension of the affine span $\mathcal{FV}(P)$ of all possible flag vectors from 2^d to $F_d - 1$, where F_d is the d-th Fibonacci number.* □

For a bounded Eulerian poset of length 4 the Dehn–Sommerville equations imply that f_0 and f_2 already carry all information contained in the flag vector:

- $f_1 = 2 - f_0 - f_2$ by the Euler equation,
- $f_{01} = 2 f_1$ by taking $S = \{1\}$, $i = -1$ and $k = 1$,
- $f_{02} = f_{01}$ by taking $S = \{0\}$, $i = 0$ and $k = 3$,
- $f_{12} = 2 f_1$ by taking $S = \{1\}$, $i = 1$ and $k = 3$, and
- $f_{012} = 2 f_{01}$ by taking $S = \{0, 1\}$, $i = 1$ and $k = 3$.

The remaining entries of the flag vector follow from these by boundedness. Similarly, for any bounded Eulerian poset of length 5 the numbers f_0, f_2, f_3, and f_{03} suffice. We call this *reduced* flag vector the *essential flag vector* of the poset. Clearly, this choice is arbitrary. We could as well take f_0, f_1, f_2 and f_{02}. If we do not explicitly state otherwise, for posets P of length 3 we will in the following always write flag(P) as (f_0, f_2) in this order, and for posets of length 5 we note $(f_0, f_2, f_3; f_{03})$, or $(f_0, f_1, f_2, f_3; f_{03})$ if we want to emphasise some symmetry in the entries.

With the following definitions we associate some simple geometric structure with posets.

1.2.18 D **[Abstract Simplicial Complex].** An *abstract simplicial complex* Δ is a finite collection of sets such that, if a set S is contained in Δ, then so is any subset of S.

With the next construction we find such an abstract simplicial complex $\Delta(P)$ for any finite poset P in such a way, that the incidence relations are given by the order relations in P.

1.2.19 D **[Order Complex].** Let P be a finite poset. Define an abstract simplicial complex $\Delta(P)$ associated to P by the following two conditions.

- The vertices of $\Delta(P)$ are the elements of P and
- a subset $\sigma \subset P$ defines a k-face in $\Delta(P)$ if and only if it is a chain of length k in the poset P.

Given a poset P, we will mostly look at the order complex of the *proper part* $\overline{P}$ of P, which is defined to be the poset P without $\hat{0}$ and $\hat{1}$ (if P has such elements).

1.2.20 P **.** *Let P be a finite graded poset. The order complex of P has a geometric realisation in some $\mathbb{R}^n$.*

Proof. We can realise $\Delta(P)$ as a subcomplex of the convex hull of the $n := |P|$ unit vectors in $\mathbb{R}^n$, which is a $(n-1)$-dimensional simplex. The order complex is a ℓ-dimensional subcomplex of it, where ℓ is the length of P. □

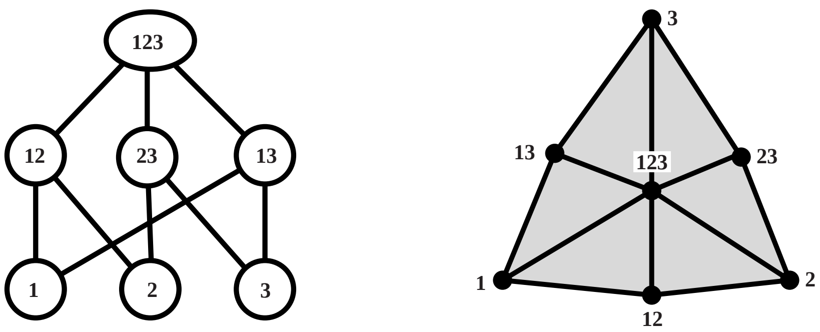
Usually, one can embed $\Delta(P)$ also in a lower dimensional space. For example, the order complex of the poset P in Figure 1.6(a) (the Boolean poset B_3 without the $\hat{0}$) is the barycentric subdivision of the full triangle shown in Figure 1.6(b).

1.2.2 L

Let P be a poset with order relation $\preceq$, and $x, y \in P$. Any element $z \in P$, such that $x \preceq z$ and $y \preceq z$, is called an *upper bound* of x and y. z is a *least upper bound* of x and y, if it is an upper bound, and any other upper bound w of x and y satisfies $z \preceq w$. In this case z is called the *join* of x and y, denoted by $z = x \vee y$. Similarly we define *lower bounds*. The greatest lower bound is called the *meet* of x and y and denoted by $x \wedge y$. The join of more than two elements can recursively be defined by $\bigvee(x_1, \ldots, x_k) := x_1 \vee (\bigvee(x_2, \ldots, x_k))$. This does not depend on the order of the x_i. Similarly, one can define the meet $\bigwedge(x_1, \ldots, x_k)$.

1.2.21 D [Lattice]. A *lattice* L is a poset in which any two elements $x, y \in L$ have a meet and a join.

See Figure 1.7 for an example of a poset, that satisfies the lattice property. The second example is a poset that is not a lattice. We consider only *finite* lattices. These are necessarily *bounded*. Furthermore, if we want to check whether a bounded poset is a lattice, by the next proposition it suffices to check either the existence of all meets or the existence of all joins. In most cases, we also assume that the lattices we consider are *Eulerian*.



(a) A poset . . .   *(b)* . . . and its order complex.

Figure 1.6: A poset and its order complex. The order complex of the proper part of P is the shown complex without the interior vertex and the full triangles.

1.2.22 P **.** *Let L be a finite bounded poset in which any two $x, y \in L$ have a meet. Then, any two elements in L also have a join.*

Proof. Assume the contrary. So there exist meets for all pairs of elements, but there is a pair $x, y \in L$ that has no join. Let F be the set of all upper bounds of x and y. This is a filter in L. Boundedness of L implies that F is not empty. As x and y have no join by assumption, F must have at least two minimal elements z_1 and z_2. Let z be their meet. Then $z \geq x, y$ as otherwise x or y would be a strictly larger lower bound for z_1 and z_2. So z is the join of x and y. □

Similarly, the existence of all joins implies the existence of meets. We are mainly interested in lattices that come from some geometric objects. These usually have several additional properties, which we now introduce.

In the same way as for posets, we can define *graded lattices L* with a *rank function* $\rho : L \to \mathbb{Z}$, giving the *length* $\ell(L)$ (or $\mathrm{rank}(L)$) of the lattice L. This allows us also to compute *f-vectors* and *flag vectors* of lattices.

A bounded lattice L is *complemented*, if for any $x \in L$ there is an element $y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. If the complement y is unique for all $x \in L$, then L is called *uniquely complemented*. If also all intervals in L are complemented, then L is *relatively complemented*.

If, in a finite lattice L, all elements except $\hat{0}$ are the join of some of its atoms, then L is called *atomic*. Similarly, if all elements of L except $\hat{1}$ are the meet of some coatoms, then L is *coatomic*. A lattice L is called *modular*, if any two elements that both cover $x \wedge y$ are covered by $x \vee y$ and vice versa.

1.2.23 E **.** In the boolean poset B_n both meets and joins exist for all pairs of elements x and y of B_n: The meet of x and y is their intersection, and the join of x and y is their union, viewed as subsets of $[n]$. It is also Eulerian, as the number of subsets of $[n]$ with even cardinality equals the number of subsets of odd cardinality.

The Boolean poset B_n is uniquely complemented: The complement of a set S is just $[n] \setminus S$. By a similar argument, it is also relatively complemented.

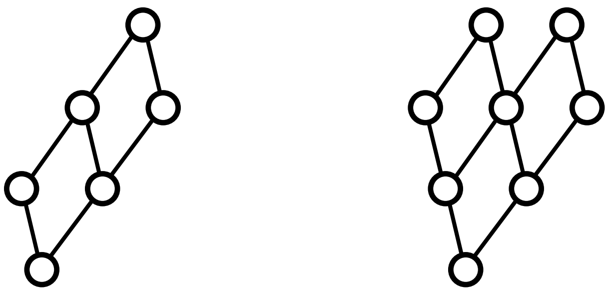


Figure 1.7: The first poset is a lattice, the second is not.

If $n \geq 3$, then, for any two subsets S_1, S_2 of $[n]$ that have empty intersection and partition $[n]$, either $|S_1| < n-1$ or $|S_2| < n-1$. Assume the first and pick $y \in S_2$. Then $S_1 \cup \{y\} \neq [n]$. So B_n for $n \geq 3$ is strongly connected. Finally, B_n is modular, as any two sets covering their intersection differ in only one element.

1.2.24 P [Atom-Coatom-Incidences]. *A finite complemented lattice L is completely defined by its atom-coatom incidence relations.*

Proof. Let $\mathcal{A}$ be the set of atoms of L and $\mathfrak{P}(\mathcal{A})$ its power set. For any $A \in \mathfrak{P}(\mathcal{A})$ let $C(A)$ be the set of coatoms that are incident to all elements of $\mathcal{A}$. We define an equivalence relation on $\mathfrak{P}(\mathcal{A})$ by $A_1 \sim A_2$ if $C(A_1) = C(A_2)$, for any $A_1, A_2 \in \mathfrak{P}(\mathcal{A})$. Let $\mathfrak{E}$ be the set of equivalence classes.

For $A \in \mathfrak{P}(\mathcal{A})$ let $J(A) := \bigvee_{a \in A} a$. Then $J(A_1) = J(A_2)$ if and only if $C(A_1) = C(A_2)$, so J is well defined on $\mathfrak{E}$. Any $x \in L$ is the join of its atoms, so there is a bijection between the elements of $L \setminus \{\hat{0}, \hat{1}\}$ and $\mathfrak{E}$. Inclusion in $\mathfrak{E}$ recovers the order relation in L. □

The following notion of simplicity and simpliciality is the central properties of lattices and polytopes for this thesis.

1.2.25 D [Simplicity and Simpliciality]. Let L be a finite graded lattice of rank ℓ.

- L is *r-simple* if all intervals $[x, \hat{1}]$ of length $r+1$ are boolean.
- It is *simple* if it is $(\ell - 2)$-simple.

- L is *s-simplicial* if all intervals $[\hat{0}, y]$ of length $s+1$ are boolean.
- It is *simplicial* if it is $(\ell - 2)$-simplicial.

Figure 1.8(a) shows a 2-simplicial lattice. r-simple and s-simplicial lattices (and polytopes) will turn up quite often in the rest of the thesis. To avoid the frequent repetition of this lengthy term in the text, we introduce a short notation if $r + s$ is one less than the length of the lattice.

1.2.26 D [(r, s)-Lattice]. Let L be a graded finite lattice of rank ℓ. We call L an (r, s)-*lattice* if $r + s = \ell - 1$ and L is r-simple and s-simplicial.

With the following sequence of propositions we show that for each ℓ there is only one lattice of rank ℓ that can have $r + s \geq \ell$. Hence, being an (r, s)-lattice is some kind of “extremal” property for a lattice.

1.2.27 P [Simple and Simplicial]. *Any simple and simplicial strongly connected Eulerian lattice L is isomorphic to a boolean poset.*

The example in Figure 1.8(b) shows that strong connectedness is necessary in this proposition.

Proof of Proposition 1.2.27. Set $\ell := \operatorname{rank}(L)$ and let $\mathcal{A}(J)$ denote the set of atoms of an interval J in L. See Figure 1.9(a) for an illustration.

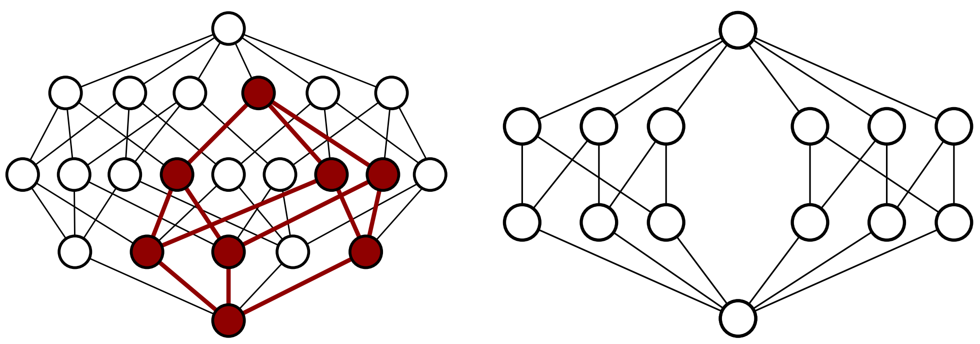
Fix a coatom c in L. By assumption, the interval $I := [\hat{0}, c]$ is boolean, so it has $\ell - 1$ atoms and coatoms. Label the coatoms by $d_1, \ldots, d_{\ell-1}$. Each d_j is covered by two coatoms, since L is Eulerian. One of these is c. Label the other by c_j. Uniqueness of meets in L implies $c_{j_1} \neq c_{j_2}$ for $j_1 \neq j_2$, $1 \leq j_1, j_2 \leq \ell - 1$. Set $I_j := [\hat{0}, c_j]$ for $1 \leq j \leq \ell - 1$. All I_j have $\ell - 1$ atoms.

Let $1 \leq j \leq \ell - 1$. Both c and c_j cover d_j, hence the intervals I_j and I intersect in $\ell - 2$ atoms of L, so there is precisely one atom in I which is not in I_j. Label this atom by a_j. Similarly, there is an atom $\tilde{a}_j$ in I_j, but not in I. I is boolean, so $a_j \lessdot d_k$ for $k \neq j$, $1 \leq k \leq \ell - 1$. Hence also $a_j \lessdot c_k$ for $k \neq j$, $1 \leq k \leq \ell - 1$. By simplicity, the coatoms c_k, $1 \leq k \leq \ell - 1$, $k \neq j$ and the coatom c are all coatoms that are comparable to the atom a_j. $d_{j_1} \neq d_{j_2}$ implies $a_{j_1} \neq a_{j_2}$ for $1 \leq j_1, j_2 \leq \ell - 1$ and $j_1 \neq j_2$. So we obtain a labelling of all atoms of I.

Fix $1 \leq j, k \leq \ell - 1$, $j \neq k$. I is boolean, so there is an element e_{jk} covered by both d_j and d_k, and $[\hat{0}, e_{jk}]$ contains $\ell - 3$ atoms. By simplicity, $J := [e_{jk}, \hat{1}]$ is boolean and has length 3, so there is one more element $\tilde{d}$ in L of rank $\ell - 2$ contained in J. L is Eulerian, so $d \notin I$, but $d \in I_j, I_k$. Hence, the atoms $\tilde{a}_j$ and $\tilde{a}_k$ must coincide. j and k were arbitrary, so $\mathcal{A}(\bigcap I_j) = \{a\}$ and the sublattice L' of L $a, a_1, \ldots, a_{\ell-1}$ and $c, c_1, \ldots, c_{\ell-1}$ is boolean.

Suppose $L' \neq L$. So there is at least one more coatom c' in L different from $c, c_1, \ldots, c_{\ell-1}$. The same argument, applied to c' instead of c, creates another boolean sublattice of L intersecting L only in $\hat{0}$ and $\hat{1}$, like the one shown in 1.8(b). By strong connectedness, this cannot happen. $\square$

1.2.28 P **[*r*-Simple and Simplicial].** *A simplicial and r-simple strongly*



(a) A 2-simplicial lattice. One Boolean interval is marked.

(b) A finite, Eulerian, simple and simplicial lattice L that is not strongly connected.

Figure 1.8: Examples of lattices.

connected Eulerian lattice L is boolean if $r \geq 3$.

Proof. Let $\ell := \operatorname{rank}(L)$. We use induction over r. The proposition is true if $r = \ell - 1$ by the previous proposition. See Figure 1.9(b) for an illustration.

Assume, that the proposition is true for any $r \geq m + 1$ and that L is m-simple. Let $x \in L$ be an element of rank $\ell - m - 2$. By assumption, the interval $[x, \hat{1}]$ is simple and simplicial. So it is boolean by the previous proposition. Hence, L is $(m + 1)$-simple. □

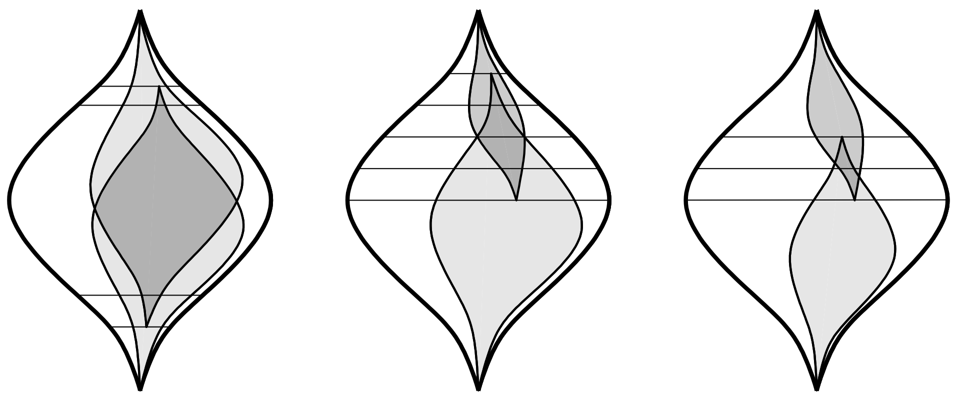
1.2.29 P [*r*-Simple and *s*-Simplicial]. *A r-simple and s-simplicial strongly connected Eulerian lattice L is boolean if $r + s \geq \operatorname{rank}(L) + 2$.*

Proof. Let $\ell := \operatorname{rank}(L)$. If $s = 2$ then $r \geq \ell$ and L is boolean. The case $r = 2$ is similar. So we can assume that $r, s \geq 3$ and proceed again by induction to prove that L is simplicial. This suffices by the previous proposition. See Figure 1.9(c) for an illustration.

The proposition is true if $r = \ell - 1$. So assume the proposition is proven for any $s \geq m + 1 \geq \ell + 2 - r$ and assume that L is m-simplicial. Let $x \in L$ with $\operatorname{rank}(x) = m + 2$. By the previous proposition the interval $[\hat{0}, x]$ is boolean. So L is $(m + 1)$-simplicial. □

If L is s-simplicial, then L^{op} is s-simple and vice versa, as the Boolean poset is isomorphic to its opposite. Lattices with this property have a special name.

1.2.30 D [Self-Duality]. A bounded lattice is *self-dual* if it is isomorphic to its opposite lattice.



(a) A simple and simplicial lattice. *(b)* An h-simple and simplicial lattice. *(c)* An h-simple and k-simplicial lattice.

Figure 1.9: Simplicity and simpliciality of lattices.

1.3 S P

Now we switch to geometric notions and introduce CW spheres and convex polytopes. Again, we focus on results used later, so in particular the treatment of spheres will be quite brief.

More on spheres can be found in the books of Bredon [28] for general, and Rourke and Sanderson [73] for PL topology. Good introductions to polytope theory are the books of Grünbaum [44] and Ziegler [89].

1.3.1 S

Here we briefly introduce CW complexes and PL manifolds. We are not too much interested in them, but we need them as intermediate tools for some constructions.

1.3.1.1 CW Complexes and Spheres

A $(d-1)$-*dimensional sphere* is a topological manifold that is homeomorphic to the *standard (or unit) sphere* $\mathbb{S}^{d-1}$ defined by $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \|x\| = 1\}$. The k-dimensional *open unit ball* is defined as $\mathring{\mathbb{D}}^k := \{\mathbf{x} \in \mathbb{R}^k : \|x\| < 1\}$, and the k-dimensional *closed unit ball* as $\mathbb{D}^k := \{\mathbf{x} \in \mathbb{R}^k : \|x\| \leq 1$. Spheres have trivial fundamental group, except for $\mathbb{S}^1$, which has $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

We define a very general form of a cell structure for topological spaces, and in particular for spheres.

1.3.1 D [CW Complex]. A *CW complex* X is a Hausdorff topological space together with a filtration $X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X^{(k)} \subseteq \ldots$ by *k-skeleta* that satisfies the following conditions:

(1) $X = \bigcup_{n \geq -1} X^{(n)}$, $X^{(-1)}$ is empty and $X^{(0)}$ is a discrete set of points, the 0-*cells*.

(2) Either $X^{(n)}{=}X^{(n-1)}$ or $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching *n-cells* in the following way:
Let B be a disjoint union of copies $\mathbb{D}^n_\sigma$ of $\mathbb{D}^n$, where σ ranges over some indexing set. These copies are the n-cells of X. Let S be the corresponding union of the boundaries of these cells and $\{f_{\partial\sigma}\}$ a family of continuous maps $f_{\partial\sigma} : \mathbb{S}^{n-1} \to X^{(n-1)}$ such that any $f_{\partial\sigma}$ touches only a finite number of $(n-1)$-cells in $X^{(n-1)}$. Let f be their union $f : S \to X^{(n-1)}$. Then $X^{(n)} = X^{(n-1)} \cup_f B$.

For each σ let f_σ be the map on $\mathbb{D}_\sigma$ defined by $f_{\partial\sigma}$. The image of f_σ in X is a *closed cell* in X, while the set $f_\sigma(\mathbb{D}^n_\sigma - \mathbb{S}^{n-1}_\sigma)$ is an *open cell*. The space X is equipped with the *weak topology*, in which a subset $U \subseteq X$ is open if and only if its intersection with $X^{(n)}$ is open for all $n \geq -1$.

A CW complex S is called a *CW* $(d-1)$-*sphere*, if the underlying topological space of S is homeomorphic to $\mathbb{S}^{d-1}$ for some $d \geq 0$.

1.3.2 D **[Star and Link].** Let X be a CW complex and σ a cell in X.

- The *star* $\mathrm{star}(\sigma)$ of σ is the subcomplex of X formed by all cells that contain σ in their closure.
- The *link* of σ is the set of all cells in the closure of $\mathrm{star}(\sigma)$ whose closure has empty intersection with σ.

We use only CW complexes that have some additional properties. All complexes we consider are *finite*, that is, they have only a finite number of cells. In particular, $X^{(n)}$ is empty for all $n \geq n_0$ and some $n_0 \in \mathbb{N}$. For a "nice" complex one clearly wants that the cells (i.e. disks) are nicely glued into $X^{(k)}$ along their boundary without any identifications, and without gluing in "unnecessary" cells. This is not part of the general definition, but is captured in the following two notions.

1.3.3 D **[Regular and Strongly Regular].** A complex X is *regular* if the attaching maps $f_{\partial\sigma}$ are embeddings, i.e. there are no identifications on the boundaries of the closed cells in X.

The complex is *strongly regular* if, in addition, any two cells in the complex intersect in a single cell (which may be empty).

For example, (geometric) simplicial complexes and polytopes (defined in the next section) have these properties. A complex X is *pure*, if all cells, whose interior does not nontrivially intersect the closure of some other cell, have the same dimension. In this case, the *dimension* of X is the dimension of these cells. If X is pure and strongly regular, then we sometimes say *faces* instead of cells, and *facets* for the top dimensional cells.

The *boundary complex* $\partial\sigma$ of a cell σ in a strongly regular complex X is the complex of all cells in X whose interior has nonempty intersection with the closure of σ. This is again a strongly regular cell complex.

A strongly regular CW complex X naturally defines the poset $\mathcal{L}(X)$ of all cells in X, together with the order relation $\sigma \preceq \tau$ between cells $\sigma, \tau \in X$ for which the closure of τ intersects the interior of σ. This is the *face poset* of X.

1.3.4 P **[Face Posets are Lattices].** *The poset $\mathcal{L}(X)$ of a finite strongly regular CW complex X is a lattice.*

Proof. The meet of two cells is given by their geometric intersection, and the join by the smallest cell containing both in its closure. Strong regularity implies that these cells exist and are unique. □

1.3.5 D **[Face Lattice].** The lattice $\mathcal{L}(X)$ of a finite strongly regular CW complex is called the *face lattice* of X.

A finite regular CW complex X is strongly regular if its face poset, augmented with an artificial $\hat{1}$, is a lattice [20, Prob. 4.47, p. 223].

1.3.6 D **[Combinatorial Equivalence].** Two strongly regular CW complexes are *combinatorially equivalent* if their corresponding face lattices are isomorphic as posets.

We introduce several methods to modify the cell structure of a CW complex. We need them in the next chapter for our construction of r-simple and s-simplicial spheres. We restrict to CW *spheres* in the following considerations.

Let $f_\sigma : \mathbb{D}^k \to \mathbb{R}^d$ be a k-cell and v a point of $\mathbb{R}^d$ not in $f_\sigma(\mathbb{S}^{k-1})$. The *cone over σ with apex v* is

$$\begin{aligned} P_v f_\sigma : \mathbb{S}^{k-1} \times [0,1] &\longrightarrow \mathbb{R}^d \\ (x,t) &\longmapsto (1-t) f_\sigma(x) + tv. \end{aligned}$$

The image is a $(k+1)$-disk whose boundary naturally carries a CW structure.

1.3.7 D **[Stacking a Cell].** Let S be a strongly regular CW sphere and σ a face of S. Stacking over σ in S is defined by

- removing all cells in $\operatorname{star}(\sigma)$ from S and
- adding a new 0-cell σ_0 and all cones over cells in $\operatorname{link}(\sigma)$ with apex σ_0,

glued with the canonical attaching maps.

Applying this construction to all facets of the sphere, or to all cells (in a suitable order), gives the following two important constructions.

1.3.8 D **[Stellar Subdivision].** Let S be a strongly regular CW $(d-1)$-sphere. The CW complex obtained from stacking all $(d-1)$-cells in S is called the *stellar subdivision* $\operatorname{sd}(S)$ of S.

1.3.9 D **[Barycentric Subdivision].** Let S be a strongly regular CW $(d-1)$-sphere. The barycentric subdivision $\mathcal{B}\mathrm{S}(S)$ of S is obtained by first stacking all $(d-1)$-cells of S, then all $(d-2)$-cells etc. down to the 1-cells.

The face lattice of the resulting CW sphere does not depend on the order in which we stack cells of the same dimension.

Here is a useful result on the connection between posets and CW spheres, which we cite from [20, Prop. 4.7.23].

1.3.10 T **.** *A bounded, graded poset P with rank function ρ is the face poset of a regular CW sphere if the order complex of every interval $[\hat{0}, x]$, $x \in P$, is homeomorphic to a sphere of dimension $\rho(x) - 2$.* □

A strongly regular CW complex X is a *simplicial complex* if the induced poset of any closed cell in the face poset $\mathcal{L}(X)$ of X is isomorphic to the Boolean poset.

This implies in particular, that any k-cell, $1 \leq k \leq d-1$, is incident to precisely $k+1$ 0-cells. If the simplicial complex X has n 0-cells, then we can describe any face of X by a subset of $[n]$. In Chapter 5 we need the following special construction for simplicial complexes.

1.3.11 D [Deleted Join]. Let S be a simplicial complex. The *deleted join* is a simplicial complex on the vertex set $V(S) \times \{0,1\}$ and is given by

$$S^*_\Delta := \{\sigma_1 \uplus \sigma_2 \;:\; \sigma_1, \sigma_2 \in S, \sigma_1 \cap \sigma_2 = \emptyset\},$$

where $\sigma_1 \uplus \sigma_2 := \sigma_1 \times \{0\} \cup \sigma_2 \times \{1\}$.

See Matoušek's book [60, Section 5.5] for a much more detailed treatment of this. A slight generalisation of a deleted join will be defined in Chapter 5 in connection with our new construction of Bier spheres and posets.

1.3.1.2 PL Spheres

This section is only a brief sketch of PL topology. Proofs for the given results can be found in the books of Hudson [47] and Rourke and Sanderson [73]. We start with the definition of several basic notions of PL topology. Some terms appear with a different meaning in the rest of the text, so we sometimes add a PL in front to avoid confusion.

A *cone* C_a for a compact set $C \subset \mathbb{R}^d$ with apex $a \in \mathbb{R}^d$ is the set

$$C_a \;:=\; \{\, ta + (1-t)x \;:\; 0 \leq t \leq 1, x \in C\,\},$$

if for all $x_1, x_2 \in C$, $x_1 \neq x_2$, the segments $ta+(1-t)x_1$ and $ta+(1-t)x_2$ intersect only in the apex a. A *cone neighbourhood* $c(p)$ of a point $p \in \mathbb{R}^d$ is a neighbourhood of p that can be written as a cone with apex p for some compact subset C of $\mathbb{R}^d$. A *polyhedron* is a set $P \subset \mathbb{R}^d$ in which all points have a cone neighbourhood. If $p \in P$, and C_p for some compact $C \subset P$ is its cone neighbourhood, then C_p is called a *star* $\mathrm{star}(p)$ of p in P and C is called a *link* of p in P.

A map $\varphi : P \to Q$ between two subsets P, Q of $\mathbb{R}^d$ is a *PL map*, if all points $p \in P$ have a cone neighbourhood C_p, for some compact basis C, on which φ has the form

$$\varphi(\lambda p + (1-\lambda)x) = \lambda\varphi(p) + (1-\lambda)\varphi(x)$$

for $0 \leq \lambda \leq 1$ and all $x \in C$.

1.3.12 D [*PL* Manifold]. A d-dimensional *PL manifold* is a topological manifold S such that all points p in S have a neighbourhood that is PL homeomorphic to an open set in $\mathbb{R}^d$.

An *n-cell* σ in PL topology is a compact convex polyhedron in $\mathbb{R}^d$. *Boundary* $\partial\sigma$ and *interior* $\mathring{\sigma}$ can be defined in the usual way. For an n-cell σ and some point $x \in \sigma$ let $l_\sigma(x)$ be the set of all lines in $\mathbb{R}^d$ that intersect σ in its interior. $l_\sigma(x) \cap \sigma$ is called a *face* σ_x of the cell σ, written $\sigma_x \leq \sigma$. If this set is empty, then we call σ_x a vertex of σ. If a face $\tau \leq \sigma$ satisfies $\tau \neq \sigma$ then this face is said to be *proper*.

1.3.13 P **.** *Let σ be an n-cell.*

(1) A cell has only finitely many vertices.
(2) If $\tau \leq \sigma$, then the set of vertices of τ is a subset of the vertices of σ. Thus, a cell has only finitely many faces.
(3) σ is the disjoint union of its open faces, and its boundary is the disjoint union of its open proper faces.
(4) The intersection of two faces of σ is again a face of σ. □

Proofs can be found in [73, Chapter 2]. A *cell complex X* in PL topology is now defined in the same way as the CW complex above: It is a finite collection of cells such that, whenever a cell is contained in the complex, then so are all its faces. The *underlying polyhedron* $|X| \subseteq \mathbb{R}^d$ is the polyhedron defined by the union of all cells in the complex X. A *subcomplex* of X is the complex formed by a subset of the cells in X together with all their faces. The *k-skeleton* of X is the subcomplex of all cells in X with dimension at most k.

The boundary of a cell clearly is a PL sphere. *Star* and *link* of a cell in a PL complex X are defined in the same way as for general CW complexes. PL spheres have the following nice property.

1.3.14 T **.** *The link of a face σ in a PL cell complex X is a PL sphere.* □

This is in contrast to general CW spheres as we defined them above, where we can only guarantee that the link is a homology sphere. See [20, Theorem 4.7.21] for a more detailed treatment, and [47, Chapter 1] for the proof.

An equivalent characterisation of this property is the following. In the face poset of the cell decomposition X, augmented by a maximal element $\hat{1}$, not only the order complexes of the lower intervals $[\hat{0}, x]$ with $x < \hat{1}$ are spheres, but the same is true for *all* intervals $[x, y]$, with the only possible exception of $[\hat{0}, \hat{1}]$, whose order complex is homeomorphic to the base space $|X|$.

1.3.1.3 Shellability

In Chapter 5 we use a very powerful concept for regular CW spheres and polytopes. For these objects, one can define a special way to build them up from their cells, adding one cell in each step, in such a way that (1) in each intermediate step we have a ball and (2) in each step the piece of the ball to which we glue the next cell is again of this type, but in one dimension lower. Here is a precise definition:

1.3.15 D [Shelling]. Let S be a pure strongly regular d-dimensional CW complex. C is *shellable* if we can find a *shelling* of C in the following way: There is a linear order $c_1, c_2, \ldots, c_k$ on its facets such that either $d = 0$, or the following two conditions are satisfied:

(1) The boundary complex ∂c_1 of the cell c_1 is shellable itself.
(2) For any other cell c_j, $2 \leq j \leq k$ the intersection of c_j with $\bigcup_{1 \leq i \leq j-1} c_i$ is a non-empty pure $(d-1)$-complex that is the beginning of a shelling of ∂c_j.

It is in general not known, whether a CW sphere, or a PL sphere, are shellable. One needs some additional properties to obtain such an ordering of the facets. One such property is polytopality, and we meet shellability in this context again in the following section. In Chapter 5 we present the class of Bier spheres derived from the Boolean poset B_n and prove that they are shellable.

1.3.2 P

This section is about *polytopes* and their properties, which are the central geometric object of this thesis. In terms of the previous sections, a polytope is roughly a strongly regular CW sphere in which all cells are realised by flat embeddings. The study of polytopes is both a quite old and a quite recent topic in mathematics.

It is old, as polytopes appear already in the mathematical work of several ancient Greek mathematicians and philosophers. They found and classified the regular and semi-regular convex polytopes in three dimensions, like the five Platonic solids, and the Archimedean semi-regular polytopes. A lot of mathematical effort went since that time into the study of regularity properties of polytopes.

It is recent, as with the emergence of modern computers, the study of *discrete* geometric objects (objects, that can be described by a finite set of input data) becomes more and more important. They now play a great rôle in a variety of mathematical areas, from combinatorial optimisation to visualisation. This way, there are now lots of "real world" applications for results about polytopes.

1.3.2.1 Basic Definitions

The following definition of a polytope has two variants, which are equivalent by the next theorem. We use both variants of the definition for describing a polytope, and switch between them without always mentioning it.

1.3.16 D [Polytope]. A *(geometric convex bounded) polytope* P is defined by one of the following two characterisations.

(1) A *polytope* is the convex hull $\mathrm{conv}(V)$ of a finite set $V = \{v_1, \ldots, v_r\}$ of r points in $\mathbb{R}^n$. We write $P = P(v_1, \ldots, v_r)$ for a polytope defined by $v_1, \ldots, v_r$.

(2) A *polytope* is the intersection of a finite set of half spaces in $\mathbb{R}^n$, if that intersection is bounded. If the half spaces are defined by m hyperplanes $\langle a_i, x\rangle = b_i$ for $1 \le i \le m$, with inwards pointing normal vectors a_i, then we denote the polytope by $P = P((a_1, b_1), \ldots, (a_m, b_m))$.

The first definition of a polytope is sometimes called the *vertex description*, the second the *hyperplane description* of a polytope. The following theorem tells us, that these definitions are equivalent from a mathematical point of view. There are, however, huge differences from an algorithmical standpoint: For general polytopes, algorithms translating one description into the other need exponential time (in the input and output).

1.3.17 T . *The two definitions (1) and (2) of a polytope in Definition 1.3.16 are equivalent; that is, the convex hull of a finite set of points can be described as a finite intersection of half spaces and vice versa.* □

A detailed proof of this can be found in Chapter 1 of Ziegler's book [89]. It is rather lengthy, so we only give the key ideas of it.

Let P be a d-polytope with s vertices and V the matrix that has the vertices of P as columns. P is given by $\{x \in \mathbb{R}^d \ : \ x = tV, \text{ for } t \in \mathbb{R}^s, \ t \ge 0, \ \sum_{j=1}^{s} t_j = 1\}$. Let H be the set $\{(x,t) \in \mathbb{R}^d \times \mathbb{R}^s \ : \ x = Vt, \ t \ge 0\}$. H is defined as an intersection of half-spaces. *Fourier-Motzkin-Elimination* allows us to project H down while maintaining a hyperplane description for it. The other direction is done by a reverse argument.

Note, that in Definition 1.3.16 (1) we do not require that the convex hull defines a set of nonzero measure in $\mathbb{R}^n$. Similarly, in the hyperplane description, we do allow half spaces that intersect only in their boundary. We define the dimension of a polytope in the following way.

1.3.18 D **[Dimension].** The *dimension* of a polytope P is the dimension d of the smallest affine subspace of $\mathbb{R}^n$ that contains P.

1.3.19 E . Here are some simple examples of polytopes. See Figure 1.10 for illustrations.

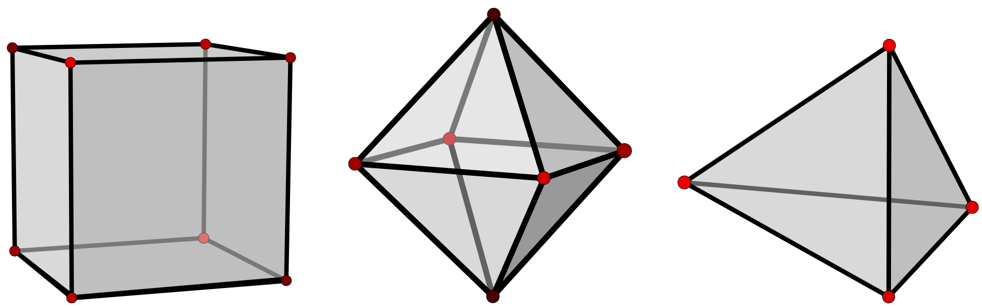
(1) The *standard d-simplex* Δ_d is the convex hull of the $d+1$ standard basis vectors $e_1, \ldots, e_{d+1}$ in $\mathbb{R}^{d+1}$. This is a d-dimensional polytope embedded in $\mathbb{R}^{d+1}$, as all points $x = (x_1, \ldots, x_{d+1})$ in the convex hull satisfy the equation $\sum_{i=1}^{d+1} x_i = 1$.
(2) The standard unit d-cube $\square_d$ is the convex hull of the points in $\{-1, +1\}^d$.
(3) The standard d-cross polytope $\boldsymbol{+}_d$ is the convex hull of the standard basis vectors and their negatives in $\mathbb{R}^d$.
(4) Let $t_d : \mathbb{R} \to \mathbb{R}^d$ be the curve $x \mapsto (x, x^2, \ldots, x^d)$. The *cyclic polytope* C_n^d is defined to be the convex hull of n distinct points on this curve. Its combinatorial properties do not depend on the actual choice of the points on the curve.

Any polytope P comes with a natural cell decomposition induced by the intersection with one or several of its defining hyperplanes (here we assume that P is given in the hyperplane description). These cells are the *faces* of the polytope. The *set of all faces* of P is denoted by $\mathrm{Faces}(P)$. Faces of dimension 0 and 1 are *vertices* and *edges*, faces of codimension 1 and 2 are *facets* and *ridges*. The *set of all vertices* of a polytope P is denoted by $\mathcal{V}(P)$, the *set of facets* by $\mathcal{F}(P)$. The polytope has exactly one cell of dimension d, the polytope itself. All other faces are called *proper*. The cell complex defined by all proper faces of the polytope is a strongly regular CW sphere, which is called the *boundary complex* ∂P of P. By construction, it can be realised in such a way, that all its cells are in fact polytopes glued along faces. Generally, any cell complex, that has such a special geometric realisation, is called *polytopal*. The *vertex figure* P/v of a vertex v in a polytope P is the polytope we obtain as the intersection of P with a hyperplane that intersects all edges incident to v in their interior.

The set of all faces, together with the order relation given by inclusion, forms a poset. We usually add an artificial *empty face* $\hat{0}$ covered by all vertices of the polytope. This is the minimal element of the poset. The maximal element $\hat{1}$ is the polytope itself. With this addition, the poset is bounded and has a natural rank function

$$\begin{aligned} \rho : \mathrm{Faces}(P) &\longrightarrow \mathbb{N} \\ \sigma &\longmapsto \dim(\sigma) + 1 \end{aligned}$$

induced by the dimension of the faces. We set $\dim(\hat{0}) = -1$. The meet of two faces of the polytope is the face given by their geometric intersection. This is a unique face by definition (possibly the empty face $\hat{0}$). Hence, this poset is a lattice.



(a) The 3-cube. *(b)* The 3-cross polytope. *(c)* The 3-simplex.

Figure 1.10: Examples of polytopes.

1.3.20 D **[Face lattice].** Let P be a polytope. The lattice $\mathcal{L}(P)$ given by $\mathrm{Faces}(P) \cup \hat{0}$ with the order induced by inclusion is the *face lattice* of P.

Polytopes are strongly regular CW spheres, so we can apply the definition of f- and flag vectors to polytopes.

1.3.21 E . Here are f-vectors for some polytopes in Examples 1.3.19.

- The simplex Δ_d has $d + 1$ vertices. Any subset F of them forms a face of dimension $|F| - 1$. So $f_j(\Delta_d) = \binom{d+1}{j+1}$, for $0 \le j \le d - 1$. See Figure 1.11(a).
- The cube $\square_d$ has 2^d vertices. For $0 \le k \le d - 1$, a subset of them is in a common k-face if they coincide in $d - k - 1$ of their entries. So $f_k(\square_d) = 2^{d-k}\binom{d}{d-k}$ for $0 \le k \le d-1$. The Hasse diagram for $d = 3$ is in Figure 1.11(b).
- The cross polytope $\maltese_d$ has $2d$ vertices. A subset of them forms a k-face if it has cardinality $k + 1$ and does not contain both a vector and its opposite. So $f_k(\maltese_d) = 2^{k+1}\binom{d}{k+1}$.

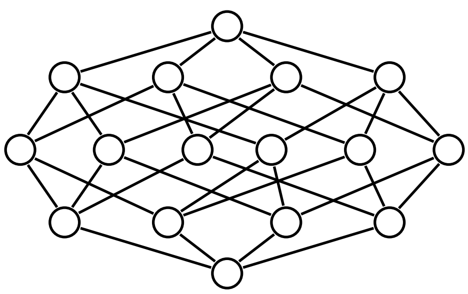
1.3.22 T . *Polytopes satisfy the Euler equation of Definition 1.2.14, that is, the f-vector of a d-polytope satisfies*

$$f_{-1} - f_0 + f_1 \mp \cdots + (-1)^d f_d = 0. \qquad \square$$

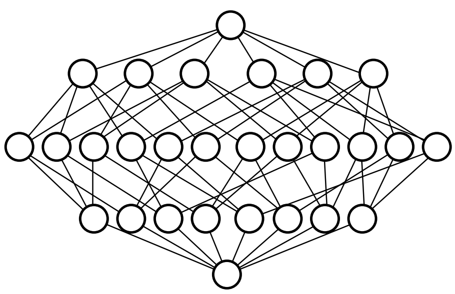
See [89, Corollary 8.17] for a proof. This theorem is a consequence of a deep Theorem of Bruggesser and Mani [29], which tells us that all polytopes are shellable. A proof of this is in [89, Section 8.2]

A d-polytope P is *centred* if the origin of $\mathbb{R}^d$ is an inner point. Clearly, any full-dimensional polytope can be transformed into a centred one — without changing its shape and combinatorial properties — by a translation.

1.3.23 D **[Dual Polytope].** A polytope P^Δ is called *dual* to a polytope P if its face lattice $\mathcal{L}(P^\Delta)$ is opposite to the face lattice $\mathcal{L}(P)$ of P.



(a) The face lattice of the 3-simplex.



(b) The face lattice of the 3-cube.

Figure 1.11: Face lattices.

For example, the simplex is dual to itself, and the cube is dual to the cross polytope. The regular versions of these polytopes satisfy this in a stronger sense, which we define now.

1.3.24 D **[Polar Polytope].** Let P be a centred polytope. The *polar polytope* is

$$P^{\diamond} := \{x \in \mathbb{R}^n \;:\; \langle x, v\rangle \leq 1 \quad \forall \quad v \in P\}\,, \tag{1.3.1}$$

which is a bounded polytope in the hyperplane description.

Clearly, the condition on P to be centred is not a severe restriction, as any polytope can be transformed into a centred one. The next proposition tells us that the polar polytope of P is dual to P. So any polytope has a dual.

1.3.25 P . *The polar polytope $P^{\diamond}$ of a centred polytope P is dual to P.* □

The proof of this proposition is tedious, but not difficult. One shows, that it suffices to consider the inequalities $\langle x, v\rangle \leq 1$ in Equation (1.3.1) only for the vertices of the polytope P, instead of all points. This reduces the description to a finite number of inequalities. Further, these inequalities constitute the hyperplane description of $P^{\diamond}$. Once knowing this, it is simple to compare the vertex–facet incidences of the two polytopes P and $P^{\diamond}$. A detailed proof of this, and the next proposition, can be found in Ziegler's book [89, Section 2.3]. For example, in the realisation given in 1.3.19, the cube is polar to the cross polytope.

If we apply the polar construction twice to a centred polytope, then we get back to the original polytope.

1.3.26 P . *Polarity for a centred polytope P is reflexive, i.e. $P^{\diamond\diamond} = P$.* □

Clearly, $P \subseteq P^{\diamond\diamond}$. Some linear algebra computations, using the fact that the vertices suffice in the definition of the polar polytope in Equation (1.3.1), show that there cannot be a point w contained in $P^{\diamond\diamond}$, but not in P.

Let P be a d-polytope in the hyperplane description. We say that a point $w \in \mathbb{R}^d$ is *beyond* a facet F of P if it lies outside the half space of $\mathbb{R}^d$ defined by F, but inside all others. It is *beneath* a facet if it is contained in the half space defined by F. More generally, a point is *beyond a k-face σ* of P, if it is outside all half spaces whose boundary contains σ, and inside all others.

Here is the central property of polytopes for this thesis. We had a combinatorial version of this already in Definition 1.2.25. We use this for the statement of the geometric version.

1.3.27 D **[Simple and Simplicial].** A d-polytope P is s-*simplicial* if its face lattice is s-simplicial, and it is *simplicial* if the face lattice is simplicial.

Similarly, a d-polytope is r-*simple* if its face lattice is r-simple, and it is *simple*, if the face lattice is simple.

1.3.28 R **.** We can characterise these conditions in a purely geometric way.

- The Boolean lattice B_{s+1} is the face lattice of the simplex Δ_s. So all s-faces in an s-simplicial d-polytope are simplices. In particular, in a simplicial d-polytope, all facets are simplices.
- r-simple d-polytopes can either be described by the fact that their dual is r-simplicial (as P^Δ has the opposite face lattice) or, without involving the dual, by the condition, that around any $(d-r-1)$-face there are $r+1$ facets. In particular, in a simple d-polytope each vertex is in d facets.

1.3.29 D **[*(r,s)*-Polytopes].** A d-polytope P is an (r, s)-*polytope* if $r+s = d$ and it is r-simple and s-simplicial.

For example, simplex and cross polytope are simplicial, while the cube is simple. Propositions 1.2.27– 1.2.29 apply — with the same proof — also in the geometric setting. Hence, in any dimension $d \geq 2$, the d-simplex Δ_d is the only r-simple and s-simplicial polytope for $r + s > d$. We present the known examples of polytopes with $2 \leq r, s \leq d - 2$ and $r + s = d$ in Chapter 4. We construct many more examples of 2-simple and $(d-2)$-simplicial polytopes in the Chapters 2 and 3.

Most polytopes that appear in this thesis, are 4-dimensional, so they cannot be visualised directly. We circumvent this problem in two different ways to provide illustrations anyway. Many of our constructions work (at least partially) also in three dimensions, so we make three dimensional drawings and point out the differences. The other way is to draw a *Schlegel diagram* of the polytope.

1.3.30 D **[Schlegel Diagram].** A *Schlegel diagram* of a 4-polytope is the three dimensional image of a central projection of the polytope onto one of its facets with centre in a point beyond that facet. See Figure 1.12 for examples.

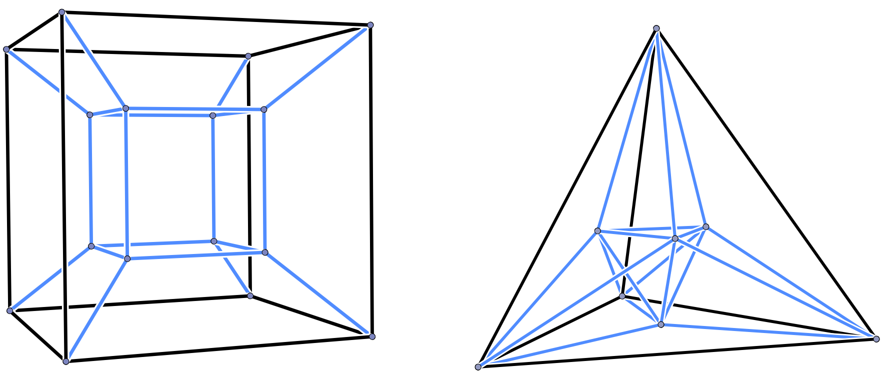


Figure 1.12: Schlegel diagrams of $\square_4$ and ✚$_4$.

Clearly, the face lattice of a polytope determines its combinatorial type completely. By Proposition 1.2.24, this is far too much information, the vertex–facet incidence relations already suffice to determine the type. In general, these incidence relations are also necessary to fix the type, but what happens, if we restrict the class of polytopes we look at? One interesting subclass of polytopes are the simplicial polytopes, or dually, the simple polytopes. For these, much less information is necessary, by a theorem of Blind and Mani [22]. A simpler proof was given by Kalai [53].

1.3.31 T [Reconstruction Theorem]. *The graph of a simple polytope determines its combinatorial type.* □

Hence, a simplicial polytope is determined by its dual graph. There are several other classes of polytopes that are reconstructible from their graph or dual graph. See e.g. Joswig [50].

1.3.2.2 Simple Polytope Constructions

For our constructions of 2-simple and 2-simplicial 4-polytopes we need some simple methods to produce new polytopes from others, by adding either new vertices or new hyperplanes to the polytope in a controlled way.

1.3.32 D [Stacking]. Let P be a polytope and F one of its faces. Choose a point w beyond F. We define a new polytope $F \backslash P$ by

$$F \backslash P := \operatorname{conv}(P \cup \{w\}) .$$

This operation is called *stacking the polytope P above the facet F*.

See Figure 1.13 for an example of this construction. It is not difficult to see that the combinatorial properties of $F \backslash P$ do not depend on the precise choice of w. We

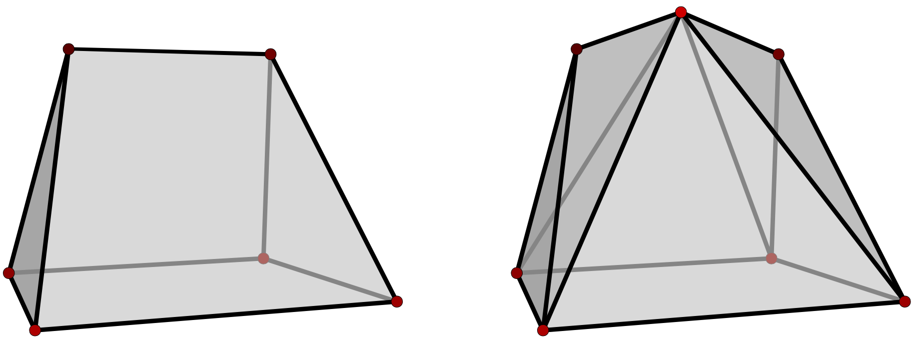


Figure 1.13: A simple 3-polytope, and the same polytope with its top edge stacked.

mainly use this construction in the case where F is a facet of P. The new f-vector of $F\backslash P$ in this case is

$$f(F\backslash P) = (f_0(P) + 1, f_1(P) + f_0(F), \ldots, f_{d-1}(P) + f_{d-2}(F)),$$

where $f(P) = (f_0(P), \ldots, f_{d-1}(P))$ and $f(F) = (f_0(F), \ldots, f_{d-2}(F))$ are the f-vectors of P and F. Iterating the construction for a sequence of facets defines the important class of *stacked polytopes*.

1.3.33 D [Stacked Polytopes]. A polytope is called *stacked*, if it can be obtained from the simplex Δ_d by successively stacking above facets.

Figure 1.14 shows two examples of stacked polytopes. The stacking construction produces many combinatorially not equivalent polytopes with the same number of vertices, as already after two stacking operations, the automorphism group on the face lattice is not anymore transitive on the facets. Stacking above facets in different orbits produces different polytopes with the same flag vector. We use a more precise count of these types for a proof of "many" distinct 2-simple and 2-simplicial 4-polytopes in Chapter 4.

There is also a polytope construction which is "dual" to stacking above facets.

1.3.34 D [Vertex Truncation]. Let P be a d-polytope for $d \geq 2$, defined by m hyperplanes $\langle a_i, x\rangle = b_i$, $1 \leq i \leq m$. Let v be a vertex of P. Choose a hyperplane $H := \{x : \langle a, x\rangle = b\}$ that intersects all edges incident to v in their interior. Orient a such that v is not in the positive half space defined by H. Define the *vertex truncation* $\mathrm{tr}(P; v)$ of P at the vertex v by

$$\mathrm{tr}(P; v) := P((a_1, b_1), \ldots, (a_m, b_m), (a, b)).$$

See Figure 1.15 for an example of a truncated cube. More generally, one may also define the truncation of k-faces for $k \geq 2$, but we do not need this. If a vertex v of the d-polytope P has degree c, then truncating v removes one vertex of P

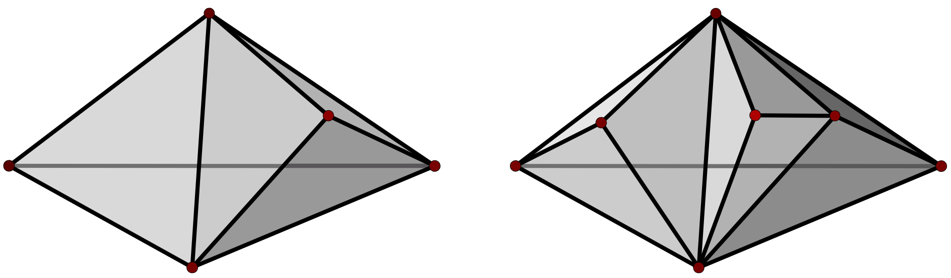


Figure 1.14: A once and a thrice stacked simplex.

and introduces c new vertices. If $c = d$, then the new facet in the truncation is a simplex. The f-vector of $\operatorname{tr}(P; v)$ does not depend on the choice of the truncation hyperplane. It is

$$f(\operatorname{tr}(P; v)) = (f_0(P) + f_0(V) - 1, f_1(P) + f_1(V), \dots, f_{d-2}(P) + f_{d-2}(V), f_{d-1}(P) + 1)$$

if $f(P) = (f_0(P), \dots, f_{d-1}(P))$ and $f(V) = (f_0(V), \dots, f_{d-2}(V))$ are the f-vectors of P and the vertex figure $V := P/v$ of the vertex v in P.

We repeat here the definition of the barycentric subdivision from the previous section to emphasise that it applies to polytopes in a particularly nice way . Namely, one can obtain polytopes from it, and not merely CW spheres.

1.3.35 D [Barycentric Subdivision]. The *barycentric subdivision* of a polytope P is the simplicial polytopal complex $\mathcal{BS}(P)$ obtained by first stacking all facets of P, then all ridges of P, etc. until we arrive at the edges.

The order, in which we stack the faces, does not matter, as long as we stack all faces of one dimension before we proceed to faces of lower dimension.

1.3.2.3 Symmetry Groups

When we say that two polytopes are *equivalent*, then we usually mean that their face lattices are isomorphic, i.e. they have the same *combinatorial* properties. This does usually not imply that — if we have a geometric realisation of both — we can also find an *affine transformation* mapping one polytope onto the other and inducing a given automorphism in the face lattice. This different behaviour of geometric realisation and combinatorial type is reflected in the following two groups.

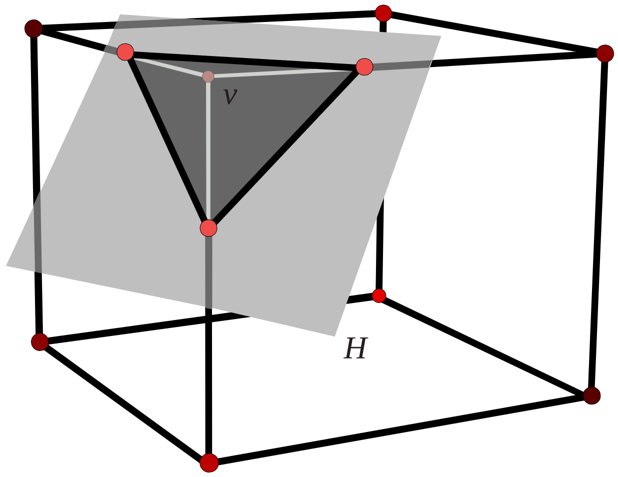


Figure 1.15: A cube with one truncated vertex.

1.3.36 D [Symmetry Groups]. Let P be a polytope with a given geometric realisation.

- Any affine transformation T of the ambient space, that preserves P set-wise, is called a *geometric symmetry transformation*. The set of all such transformations, together with composition of maps, forms a group, the *geometric symmetry group* $\mathrm{Aff}(P)$ of the polytope P.
- A *combinatorial symmetry* of P is an automorphism of the face lattice $\mathcal{L}(P)$. The group of all combinatorial symmetries is the *combinatorial symmetry group* $\mathrm{Aut}(P)$ of the polytope P.

The combinatorial symmetry group is independent of the realisation, while the geometric symmetry group clearly depends on the actual choice of coordinates for the vertices.

A geometric symmetry maps k-faces to k-faces and preserves incidences between faces. Therefore, any geometric symmetry induces a combinatorial symmetry and we obtain an injective map

$$\mathfrak{s} : \mathrm{Aff}(P) \hookrightarrow \mathrm{Aut}(P).$$

Generally speaking, this map is almost never surjective. It is also not clear, in which cases we can find — for some chosen combinatorial symmetry — a geometric realisation such that this combinatorial symmetry is in the image of $\mathfrak{s}$.

However, up to now, there are not many examples of polytopes recorded in the literature, where these two groups differ for all possible geometric realisations of a polytope. Bokowski, Ewald, and Kleinschmidt [25] (a corrected version of the coordinates was given by Bokowski and Guedes de Oliveira [26] and Altshuler [2]) have provided a 4-dimensional example on ten vertices.

Dimension 4 is smallest possible for such examples, as it is known that for 3-polytopes, and any d-polytopes with few vertices for $d \geq 3$, there are realisations in which geometric and combinatorial symmetry group are isomorphic. See the paper of Mani [59] for the first and the book of Grünbaum [44, p.120] for the second result. Grünbaum attributes it to Perles.

Asking for special properties of $\mathrm{Aff}(P)$ and $\mathrm{Aut}(P)$ leads to the large field of regular and semi-regular polytopes. The strongest property that one could look for are *flag transitive* or *regular* polytopes. These are polytopes in which the automorphism group contains, for any pair of flags in the face lattice, an automorphism that maps one flag onto the other. They are completely classified in all dimensions. See e.g. the book of Coxeter [30] for a complete treatment of these.

In dimension three there are five regular polytopes. These are (1) the simplex, (2) the cube, (3) the cross polytope, (4) the icosahedron (a polytope with 20 triangular facets, and each vertex is incident to five of them), and (5) the dodecahedron (the dual of the icosahedron).

More interesting, at least for our purposes, are the regular polytopes in higher dimensions. There are three families of such polytopes that have one member in each dimension $d \geq 2$, and three additional regular polytopes in dimension four. The families are (1) the simplex Δ_d, (2) the cube $\square_d$, and (3) its dual, the cross polytope $✚_d$. The additional regular polytopes are the following.

- The 24-cell, which is a self-dual 4-polytope with 24 vertices and facets, and 96 edges and ridges. All facets are regular $✚_3$, and we have six of them around any vertex. See Figure 1.16 for the Schlegel diagram.
- The 120-cell, which is a simple 4-polytope with 600 vertices, 1200 edges, 720 ridges, and 120 facets. The facets are regular dodecahedra, and there are four of them around any vertex.
- The 600-cell, which is the dual of the 120-cell.

Regular polyhedra will prove quite convenient as input for our constructions, and in particular for example drawings. They also make the link of our new, and quite general, constructions for 2-simple and 2-simplicial 4-polytopes to similar older constructions for regular polytopes.

A weaker property for a polytope P is *vertex transitivity* or *facet transitivity*, meaning that we can map any vertex or facet of P onto any other vertex or facet by a map in $\mathrm{Aff}(P)$.

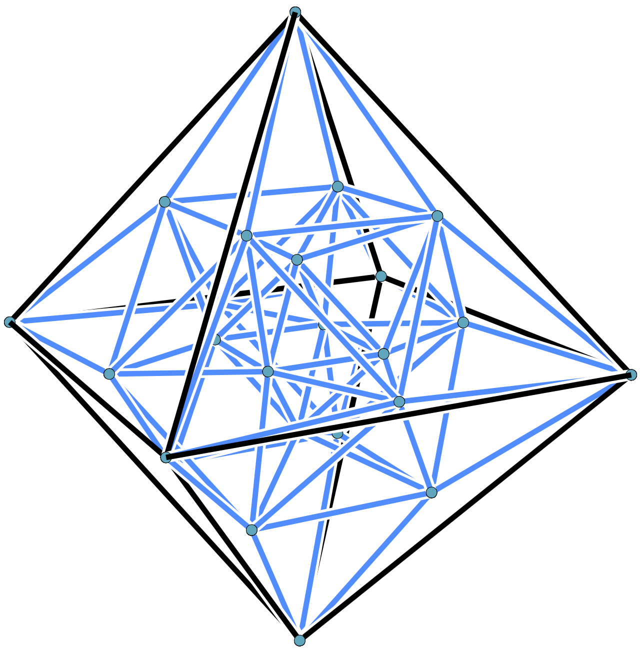


Figure 1.16: A Schlegel diagram of the 24-cell. This polytope will reappear in many variants throughout the thesis.

1.3.2.4 Realisations and Realisation Spaces

For geometric realisations of a polytope we have one more important notion that we want to mention in this introduction. It captures the flexibility that we have in the choice of coordinates.

1.3.37 D [Realisation Spaces]. The *realisation space* of a d-polytope P with n vertices is the space $\mathcal{R}(P)$ of all sets of n points in $\mathbb{R}^d$, whose convex hull is combinatorially equivalent to P. The set $\mathcal{R}(P)$ is a subset of $\mathbb{R}^{d \cdot n}$.

The *projective realisation space* $\mathcal{R}_{\text{proj}}(P)$ of a polytope is the space of all possible geometric realisations of a polytope, up to projective equivalence. It is the quotient space of $\mathcal{R}(P)$, where two realisations are defined to be equivalent if there is a projective transformation mapping one realisation onto the other.

Computing the dimension of either of these spaces is usually difficult. Compare Richter-Gebert and Ziegler [72] and the book of Richter-Gebert [71] for a more detailed treatment of realisation spaces.

Here is a simple example. The dimension of the realisation space of the simplex Δ_d is $d^2 + d$, but the dimension of its projective realisation space is 0, as any set of $d + 1$ points in $\mathbb{R}^d$ can be mapped onto any other such set by an affine map.

Not all CW spheres can be realised geometrically as polytopes. In fact, although any strongly regular 2-sphere can be realised as a polytope (this is the Theorem of Steinitz, see the next section), this fails already badly in dimension 4. Here e.g. Pfeifle and Ziegler [69] proved that there are $2^{\Omega(n^{5/4})}$ simplicial 3-spheres with n vertices. On the other hand, from a result of Goodman and Pollack [41] we know, that there are at most $2^{O(n \log n)}$ combinatorial types of simplicial 4-polytopes on n vertices. In higher dimensions, this gap between combinatorial types of CW $(d-1)$-spheres and simplicial d-polytopes is a result of Kalai [52]. We introduce the next notion to distinguish spheres that have a geometric realisation as a polytope from those that have not.

1.3.38 D [Polytopal Spheres]. A strongly regular CW sphere S is called *polytopal* if there is a polytope P that has a face lattice isomorphic to that of S. Any such polytope is a *polytopal realisation* of the sphere.

If a CW sphere is realisable as a polytope, then we can further ask for a particularly nice realisation. The following property is of interest for the computational treatment and the visualisation of polytopes and their graphs with a computer.

1.3.39 D [Rational Realisation]. A geometric realisation of a polytope is said to be *rational*, if all vertices have rational coordinates.

Clearly, all simplicial polytopes do have such a realisation, and dually also all simple polytopes. For all other polytopes, or for realisations with some additional properties, it is usually not clear whether such a realisation exists. Already in dimension two, most of the *regular* polygons lack a rational realisation. Even the d-simplex Δ_d does not have a *regular* and *rational* realisation in $\mathbb{R}^d$ for all $d \geq 2$ (observe that the standard realisation of Δ_d given in Examples 1.3.19 is regular, but in $\mathbb{R}^{d+1}$). Perles gave an example of a 8-polytope on 12-vertices that has no rational realisation [44, p. 95], and Richter-Gebert [71] later constructed examples of 4-polytopes without rational realisation.

1.4 F V F V I

We give an overview over the known (linear and nonlinear) inequalities, i.e. relations between the numbers of vertices, edges, 2-faces, etc. that hold for the entries of the f- and flag vectors of polytopes in dimensions up to four.

The cases of two and three dimensional polytopes are completely solved. In contrast, not much is known in dimension four (and in higher dimensions). However, there are some promising approaches that will hopefully shed some more light on the case of 4-polytopes.

The two dimensional case is simple: A convex polygon has as many vertices as it has edges, and there is one combinatorial type of polygon for any number $n \geq 3$ of vertices, and none for $n = 0, 1, 2$. The flag vector adds one more entry f_{01} to the f-vector, and this is just twice the number of vertices. The three dimensional case was solved by Ernst Steinitz already in 1906. We present the complete classification in the next section. In dimension 2 and 3 linear inequalities already suffice for the description of the flag vectors. In dimension 4 we also have to consider nonlinear relations between vertices, edges, ridges, and facets.

Much more is known if one looks only at simplicial (or, dually, simple) polytopes. Here the f-vectors are completely classified by the *g-Theorem* (this involves the g-vector, which is a vector obtained from the f-vector via a linear transformation) of Billera and Lee [14, 15] and Stanley [79]. Recently, a combinatorial proof of the necessity part of the g-theorem was given by McMullen [62]. The statement of the theorem, and its consequences, is given in the book of Ziegler [89, p. 270]. It requires a lot of new terminology and is based on different arguments compared to the considerations for general f-vectors. We do not need it for this thesis, so we leave it with these references to the literature.

In the following computations of flag vectors, we omit those entries f_S of the flag vector for which S contains either -1 or d. For polytopes, these entries coincide with $f_{S'}$ where $S' := S \setminus \{-1, d\}$. Hence, they do not contribute any additional information to the flag vector.

1.4.1 T D

The three dimensional case of the f-vector-classification was solved by Ernst Steinitz in 1906 [85]. The graph $\Gamma(P)$ of a three dimensional polytope P is planar, as we can apply a central projection of the graph onto one of its facets from a point lying beyond this facet (which is a *Schlegel diagram* of the polytope). $\Gamma(P)$ is 3-connected (i.e. we can remove any pair of vertices and the graph will still be connected), by a Theorem of Balinski [89, Theorem 3.14]. Steinitz proved that these two conditions already suffice for a complete characterisation [86].

1.4.1 T . *Every planar* 3*-connected graph on* $n \geq 4$ *vertices is the graph of a three dimensional polytope.* □

This is a deep theorem. We do not attempt to prove it, but refer to [89, Chapter 4] instead. Three-connectedness in particular implies that any vertex is adjacent to at least three edges. Thus $3f_0 \leq 2f_1$. Using the Euler equation we obtain

$$\begin{aligned} 3f_0 &\leq 2f_0 + 2f_2 - 4, \\ \text{and therefore} \quad f_0 &\leq 2f_2 - 4. \end{aligned} \tag{1.4.1}$$

By duality also

$$f_2 \leq 2f_0 - 4. \tag{1.4.2}$$

See Figure 1.17 for an illustration of the cone defined by the above two inequalities. These inequalities suffice: Any integral vector having three positive entries f_0, f_1 and f_2 satisfying these two conditions together with the Euler equation is in fact the f-vector of a 3-polytope. This is a consequence of the following simple considerations.

Combining (1.4.1) and (1.4.2) gives $f_0 \leq 2f_2 - 4 \leq 4f_0 - 12$. This implies $f_0 \geq 4$. Similarly, we obtain $f_2 \geq 4$. Thus, the smallest possible f-vector is $(4, 6, 4)$, which is the f-vector of the simplex. We use the two operations "stacking" and "vertex truncation" introduced in the Section 1.3.2 to produce a polytope for any f-vector in the cone of (1.4.1) and (1.4.2). Stacking above a simplicial facet adds $(1, 3, 2)$ to the f-vector and truncating a simple vertex adds $(2, 3, 1)$. Both operations produce at least one triangle face and one simple vertex. Thus, we can apply both constructions to any polytope we have obtained by an arbitrary sequence of those operations and we can construct a polytope with a given (f_0, f_2) in the flag vector cone from the following three 3-polytopes:

- the simplex with f-vector $(4, 6, 4)$,
- the pyramid over a square with f-vector $(5, 8, 5)$,
- and the pyramid over a pentagon with f-vector $(6, 10, 6)$.

For most pairs (f_0, f_2), there are many polytopes realising this f-vector. For example, the pyramid over an n-gon has f-vector $(n+1, 2n, n+1)$, but only the two smallest such, for $n = 4$ and 5, appear in the above construction.

The *flag vector* of a 3-polytope is already determined by its f-vector: Any edge has two vertices, so $f_{01} = 2f_1$ and any edge is in two facets, so $f_{12} = 2f_1$ and $f_{012} = 4f_1$. For any vertex-facet-pair there are two edges incident to both of them, so $f_{02} = 2f_1$.

1.4.2 F D

The situation is not nearly as nice for dimensions $d \geq 4$ as it is for dimensions 2 and 3. The set $\mathcal{FV}(4)$ of admissible flag vectors for 4-polytopes is not anymore defined by linear inequalities. We present a short outline of the known properties of this set. See the surveys of Bayer [9], Ziegler and Höppner [46] and Ziegler [90] for a detailed and — unfortunately — still quite accurate account of the known facts for the classification problem.

We know six linear inequalities that restrict the flag vector of a 4-polytope. Only one of these is really non-trivial, the others are two simple observations together with their duals, and one self-dual inequality. The first inequality comes

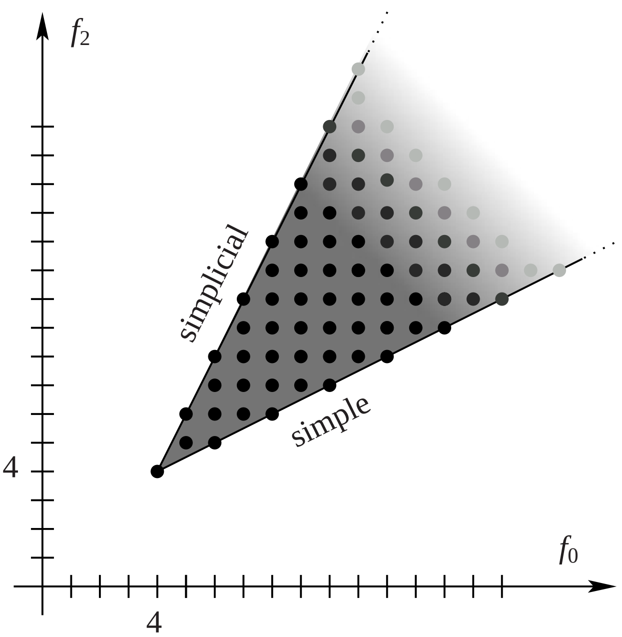


Figure 1.17: The (f_0, f_2)-projection of the f-vector-cone for 3-polytopes. f_1 is determined by the Euler equation.

from the observation that a 4-polytope must have at least five vertices and — by looking at the dual polytope — also five facets. Secondly, any 2-face of the polytope must have at least three vertices, that is, $f_{02} \geq 3f_2$. Dualising this we get $f_{13} \geq 3f_1$. Each facet is a 3-polytope, so, by using the inequalities of the last section, we know that thrice the number of 3-faces in the polytope is less than twice its number of 2-faces. Summing over all facets we get $3f_{03} \leq 2f_{02}$. The only non-trivial inequality on the entries of the flag vector of a 4-polytope so far was found by Stanley [81] for rational polytopes, and by Kalai [51] in the general case.

1.4.2 T **[Lower Bound Theorem].** *The flag vector of a* 4*-polytope satisfies*

$$0 \leq f_{03} - 3f_0 - 3f_3 + 10 \qquad \square$$

The generalised Dehn–Sommerville equations in Theorem 1.2.17 tell us that only four of the entries of the flag vector of a 4-polytope are independent. We have chosen f_0, f_2, f_3 and f_{03} for this. Transforming the six linear equations that we have derived into this set of independent entries we obtain

$$\begin{aligned}
0 &\leq f_0 - 5 && && (1.4.3)\\
0 &\leq f_3 - 5 && && (1.4.4)\\
0 &\leq f_{03} - f_2 - 2f_3 && \text{(2-simplicial)} && (1.4.5)\\
0 &\leq f_{03} + f_3 - f_2 - 3f_0 && \text{(2-simple)} && (1.4.6)\\
0 &\leq 4f_2 - 4f_3 - f_{03} && \text{(centre-boolean)} && (1.4.7)\\
0 &\leq f_{03} - 3f_0 - 3f_3 + 10 && \text{(lower bound theorem)} && (1.4.8)
\end{aligned}$$

In the same way as for the 2-faces, one could also consider the minimal number of vertices or edges a facet must have and obtain bounds for f_{03} and f_{13}. However, using Kalai's inequality, stronger bounds can be derived from the six given ones.

The cone defined by the linear equations (1.4.3) – (1.4.8) is shown in Figure 1.18. It has the seven rays, all starting from the flag vector

$$\mathrm{flag}(\Delta_4) = (5, 10, 10, 5; 20)$$

of the 4-simplex:

$$\begin{aligned}
&l_1 := \mathrm{flag}(\Delta_4) + \lambda(1,4,4,1;6), && l_2 := \mathrm{flag}(\Delta_4) + \lambda(0,1,1,0;1),\\
&l_3 := \mathrm{flag}(\Delta_4) + \lambda(0,1,2,1;4), && l_4 := \mathrm{flag}(\Delta_4) + \lambda(1,2,1,0;4), \qquad (1.4.9)\\
&l_5 := \mathrm{flag}(\Delta_4) + \lambda(1,4,6,3;12), && l_6 := \mathrm{flag}(\Delta_4) + \lambda(3,6,4,1;12),\\
&l_7 := \mathrm{flag}(\Delta_4) + \lambda(0,1,1,0;4),
\end{aligned}$$

for $\lambda \geq 0$. Dualising a polytope amounts to reflecting its corresponding point in the hyperplane orthogonal to the bottom face and running through l_2 and l_7.

All stacked polytopes lie on the ray l_5. Cyclic 4-polytopes C_d^4 have the flag vector

$$\left(n, \binom{n}{2}, 2\binom{n}{2} - 2n, \binom{n}{2} - n; 4\binom{n}{2} - 4n\right).$$

Thus, they approximate the ray l_3 and their duals the ray l_4. In addition to the simplex, also the hypersimplex (the intersection of the 5-cube with the plane $\sum x_i = 2$) lies on the ray l_1. Until very recently, no other polytopes were known that lie on this ray. Werner [88] has found a nice small 2-simple and 2-simplicial 4-polytope with f-vector $(9, 26, 26, 9)$ also lying on l_1. Further interesting facts about this polytope are presented in Section 4.3.5. In particular, we prove, that this is the smallest non-trivial 2-simple and 2-simplicial 4-polytope. No other polytopes except the simplex are known that lie on l_2 or l_7, or even just come close to one of these rays.

The set $\mathcal{FV}(4)$ cannot be closed, as there is only one polytope with five vertices, but the ray l_3 has $f_0 \equiv 5$. However, it is approximated by cyclic polytopes, so it cannot be cut off by a stronger inequality.

In the Chapters 2 – 4 we provide lots of examples that lie in the intersection of the inequalities (1.4.5) and (1.4.6). In Figure 1.18 they lie on the edge between l_1 and l_2. This is an area in the flag vector cone in which only few polytopes have been known previously.

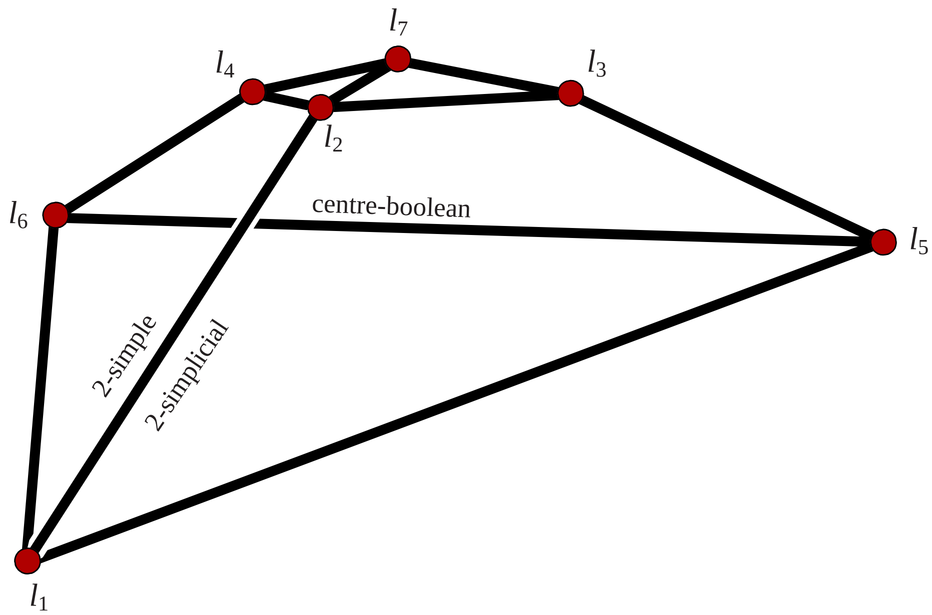


Figure 1.18: A section through the flag vector cone for 4-polytopes. The cone is symmetric with respect to the hyperplane running through l_1, l_2 and l_7.

In addition to the linear there are also some simple non-linear bounds on the flag vector:

$$\binom{f_0}{2} \geq f_0 + f_2 - 5f_3 + f_{03}, \tag{1.4.10}$$

$$\binom{f_3}{2} \geq -4f_0 + f_2 + f_{03}, \tag{1.4.11}$$

$$\binom{f_0}{2} \geq f_0 - f_2 - 5f_3 + 2f_{03}, \tag{1.4.12}$$

$$\binom{f_3}{2} \geq -6f_0 - f_2 + 2f_3 + 2f_{03}. \tag{1.4.13}$$

The first inequality is obtained by comparing the maximal number of edges between f_0 vertices and the sum of the real number of edges and the number of edges missing in the 2-faces. The second is the dual of the first. The third inequality is obtained by comparing again the number of all possible edges with the sum of the real number of edges and the number of edges missing in the facets. The forth is the dual of the third. See [9] for a detailed proof of these nonlinear inequalities.

With respect to the linear cone of possible flag vectors of 4-polytopes, the four non-linear inequalities are concave. That is, they cut out pieces of the cone. For example, the two cyclic polytopes $(5, 10, 10, 5; 20)$ and $(9, 36, 54, 27; 108)$ both satisfy (1.4.10) with equality, but the linear combination $(7, 23, 32, 16; 64)$ violates it. There are also "forbidden pairs" of entries in the f-vector. For example, Barnette [7] showed that $(f_1, f_2) \neq (18, 16)$. There are more such pairs, see Barnette and Reay [8] and the book of Grünbaum [44]. Bayer and Lee [11] give a survey on these results.

By a complete enumeration of all polytopes with at most seven vertices we know that all flag vectors with $f_0 \leq 7$ satisfying the above linear and nonlinear inequalities correspond to a polytope, with one exception. There are 29 four dimensional polytopes with seven vertices, and there is no polytope with f-vector $(7, 17, 9, 39)$. A similar approach for $f_0 = 8$ gives a list of all flag vectors of polytopes with eight vertices. There are 1294 of them, and 42 non-polytopal spheres. These were enumerated by Altshuler and Steinberg, see [3] and [4].

1.5 H G

Now we shortly leave the realm of discrete geometry and turn to a more differential geometric topic. One of the constructions for 2-simple and $(d-2)$ simplicial d-polytopes in Chapter 2 requires some tools from hyperbolic geometry. We introduce the necessary facts in this section. This is not comprehensive, and we do not attempt to prove the cited facts; the proofs, together with more background, can be found in the good text books of Benedetti and Petronio [12], Iversen [48], or Ratcliffe [70].

The *hyperbolic space* $\mathbb{H}^d$ of dimension d is a connected and simply connected d-dimensional topological space equipped with a Riemannian metric of constant sectional curvature -1. Unlike the case of the other two model spaces — those with constant sectional curvature 0 and +1, the Euclidean space and the unit sphere — there does not exist a model space for $\mathbb{H}^d$ which is an embedding into the standard Euclidean space $\mathbb{R}^m$ for some $m \geq d$ with the induced metric. However, there exist several approaches to model such a space in Euclidean space with a different metric. The most common are the *Poincaré model*, the *upper half space model*, and the *Klein model*. We discuss the last two models. Our construction can later most easily be described in the Klein model, but the upper half space model is usually more intuitive, and the classification of isometries is much simpler when one can switch between different models.

1.5.1 T U H S M

The upper half space model of hyperbolic space uses the "upper half" of the standard Euclidean space together with a conformally changed metric as a model for hyperbolic space. More precisely, consider $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \,:\, x_d \geq 0\}$ together with the metric tensor

$$g_{ij} := \begin{cases} \frac{1}{x_d^2} & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i, j \leq d$. Let $\mathbb{U}^d$ denote the upper half space equipped with this metric.

In this model of hyperbolic space it is easy to check the sectional curvature by direct computation. Let $x = (x_1, \ldots, x_d)$ be a point in $\mathbb{U}^d$ with tangent space $T_x\mathbb{U}^d$. The Christoffel symbols are are given by $\Gamma^k_{ij} = \frac{1}{x_d}$ for $i = d$ and $j = k$, $j, k \neq d$, $\Gamma^k_{ij} = -\frac{1}{x_d}$ for $i = j$ and $k = d$, and $\Gamma^k_{ij} = 0$ otherwise.

The Riemannian tensor has the entries $R_{ijij} = -R_{ijji} = -\frac{1}{x_d^4}$ for $i \neq j$ and $R_{klij} = 0$ otherwise. So, for linearly independent vectors μ and ν in the tangent space of x, the sectional curvature K is computed to be

$$K(\mu \wedge \nu) = \frac{R_{ijkl}\mu^i\mu^k\nu^j\nu^l}{(g_{ik}g_{jl} - g_{ij}g_{kl})\mu^i\mu^k\nu^j\nu^l} = \frac{\frac{1}{x_d^4}\sum_{i\neq j}\mu^i\mu^i\nu^j\nu^j}{-\frac{1}{x_d^4}\sum_{i\neq j}\mu^i\mu^i\nu^j\nu^j} = -1.$$

This transfers to the Klein model in the next section via an explicit isometry.

1.5.2 T K M

The *Klein model* of hyperbolic space uses the open unit disk

$$\mathring{\mathbb{D}}^d := \{x \in \mathbb{R}^d \ : \ \|x\|_{\mathrm{Eucl}} < 1\}$$

in $\mathbb{R}^d$ as its underlying space, together with the following metric:

$$g_{ij} = \begin{cases} \frac{1-\sum_{k\neq i} x_k^2}{(1-\|\mathbf{x}\|^2_{\mathrm{Eucl}})^2} & \text{for } i = j \\ \frac{x_i x_j}{(1-\|\mathbf{x}\|^2_{\mathrm{Eucl}})^2} & \text{otherwise,} \end{cases}$$

for $1 \leq i, j \leq d$. We denote the open unit disk equipped with this metric by $\mathbb{K}^d$. An isometry between the Klein model and the upper half space model is given by the map

$$\begin{aligned} \mathfrak{I} : \mathbb{D}^d &\longrightarrow \mathbb{R}^d \\ x &\longmapsto 2\Big[1 + \sqrt{1 - \|x\|^2_{\mathrm{Eucl}}}\Big] \frac{x + \Big[1 + \sqrt{1 - \|x\|^2_{\mathrm{Eucl}}}\Big] e_d}{\Big\| x + \Big[1 + \sqrt{1 - \|x\|^2_{\mathrm{Eucl}}}\Big] e_d \Big\|^2_{\mathrm{Eucl}}} - e_d. \end{aligned}$$

Here, e_d denotes the d-th unit vector in $\mathbb{R}^n$ with the Euclidean metric.

1.5.3 I H

For our construction we need the fact that the hyperbolic isometry group acts transitively on $\mathbb{H}^d$. This allows us to position any pair of facets of two hyperbolic polytopes in such a way that they coincide. Hence, we can glue these two polytopes along these facets geometrically.

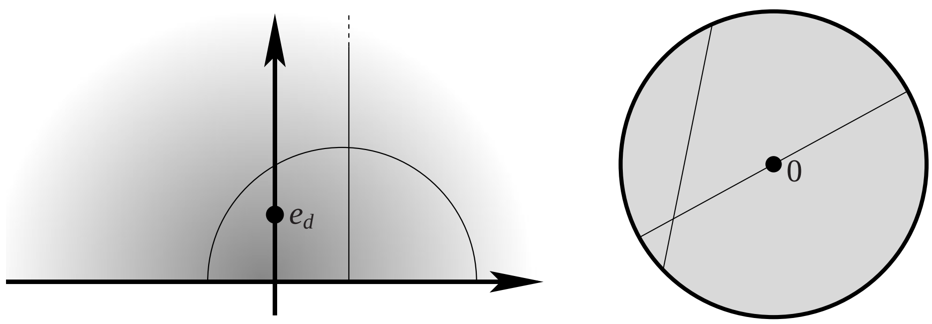


Figure 1.19: The upper half space model and the Klein model of hyperbolic space. The given isometry between the two spaces maps e_d onto 0. In each space two geodesics are indicated.

1.5.1 D [Isometry Group]. The group of isometries of the hyperbolic space $\mathbb{H}^d$ is denoted by $\mathrm{Isom}(\mathbb{H}^d)$.

1.5.2 E [Hyperbolic Isometries]. Here are some examples of isometries.

(1) Any orthogonal map $A \in O(d)$ of $\mathbb{R}^d$ restricts to an hyperbolic isometry of the Klein model. Via the isometry $\mathfrak{I}$, such a map transforms into an isometry of $\mathbb{U}^d$ fixing the tangent space $T_{e_d}\mathbb{U}^d$ at the point e_d and realising the orthogonal map A in $T_{e_d}\mathbb{U}^d$.

(2) For any $\lambda > 0$, $b \in \mathbb{R}^{d-1} \times \{0\}$ and an orthogonal map $A \in O(d)$ that preserves the e_d-axis, define the map

$$x \longmapsto \lambda A x + b.$$

Restricted to $\mathbb{U}^d$ this is an isometry.

These two types of isometries already suffice for the proof that $\mathrm{Isom}(\mathbb{H}^d)$ acts transitively on $\mathbb{H}^d$.

1.5.3 T [Isom($\mathbb{H}^d$) is transitive]. *For any two points $x, y \in \mathbb{H}^d$ and any orthogonal map $A : T_x\mathbb{H}^d \to T_y\mathbb{H}^d$ there is an hyperbolic isometry i that maps x onto y and induces A on the tangent spaces.*

Proof. By Example 1.5.2(2) there are isometries mapping x and y onto e_d in $\mathbb{U}^d$. So we can assume $x = y = e_d$. Switching to the Klein model and looking at Example 1.5.2(1), we see that A viewed as a map on $\mathbb{K}^d$ is the required isometry. □

Furthermore, the isometries given in the above two examples do already generate $\mathrm{Isom}(\mathbb{H}^d)$. Namely, suppose that $j \in \mathrm{Isom}(\mathbb{H}^d)$. Pick any $x \in \mathbb{H}^d$ and set $y := j(x)$ and $A := j_{*x} : T_x\mathbb{H}^d \to T_y\mathbb{H}^d$. By the proof of Theorem 1.5.3 we know that there is an isometry i mapping x to y and inducing A on the tangent spaces. But then, i and j must coincide.

1.5.4 D [Geodesic]. A *geodesic* in $\mathbb{H}^d$ is a continuous map $\gamma : I \to \mathbb{H}^d$ from an interval I into hyperbolic space such that, for any point $\gamma(t)$, $t \in \mathring{I}$, there is an $\varepsilon > 0$ such that for any $t_1, t_2 \in (t - \varepsilon, t + \varepsilon)$ the length of the curve $\gamma([t_1, t_2])$ is equal to the distance of $\gamma(t_1)$ and $\gamma(t_2)$. Note that $I = \mathbb{R}$ is allowed.

It can be proven that a geodesic is differentiable in its domain of definition, and in spaces of curvature ≤ 0, any geodesic can uniquely be extended to one defined on the whole real line $\mathbb{R}$ [12, p. 25].

Given a point $x \in \mathbb{H}^d$ and a unit vector $v \in T_x\mathbb{H}^d$, there is a unique geodesic γ_v such that $\gamma(0) = x$ and $\dot\gamma(0) = v$. Similarly, given two points $x, y \in \mathbb{H}^d$ there is a unique geodesic γ_{xy} connecting these two points. This allows us to define, for any $x \in \mathbb{H}^d$, the *exponential map*

$$\begin{aligned} \exp_x : T_x\mathbb{H}^d &\longrightarrow \mathbb{H}^d \\ v &\longmapsto \gamma_{v/\|v\|}(\|v\|). \end{aligned}$$

This map is a diffeomorphism between $T_x\mathbb{H}^d$ and $\mathbb{H}^d$. We equip $T_x\mathbb{H}^d$ with the usual Euclidean metric.

1.5.5 D [Hyperplane]. A *hyperplane* in hyperbolic space $\mathbb{H}^d$ is an isometric embedding of $\mathbb{H}^{d-1}$ into $\mathbb{H}^d$. These are the totally geodesic subspaces of codimension 1 in $\mathbb{H}^d$. (A subspace S of a manifold is *totally geodesic* if all geodesics, that contain a point of S and are tangent to S in this point, stay in S.)

Via the exponential map, any hyperplane in $\mathbb{H}^d$ can be described by giving a point $x \in \mathbb{H}^d$ and the set of unit vectors contained in a hyperplane in $T_x\mathbb{H}^d$. In Euclidean space any two hyperplanes can be mapped onto each other by an orthogonal map. Hence, using Theorem 1.5.3, we may conclude the following important fact.

1.5.6 T [Transitivity on Hyperplanes]. *Any two hyperbolic hyperplanes can be mapped onto each other by an hyperbolic isometry.* □

The angle between two geodesics segments γ_1 and γ_2 intersecting in $\gamma_1(0) = \gamma_2(0) = x$ is the angle between $\dot\gamma_1(0)$ and $\dot\gamma_2(0)$ in $T_x\mathbb{H}^d$. In the upper half space model, this angle coincides with the Euclidean angle that we can "measure" in the model. In the Klein model it does not. This is one reason why looking at hyperbolic phenomena in the Klein model is sometimes counterintuitive.

The preservation of angles in $\mathbb{U}^d$ and their distortion in $\mathbb{K}^d$ is is immediate from the fact that the metric tensor in $\mathbb{U}^d$ at any point is just a scalar multiple of the Euclidean one, while in $\mathbb{K}^d$ it has off-diagonal entries if x is not the origin (where also in the Klein model Euclidean and hyperbolic angles coincide).

1.5.4 H

We have already noted that any geodesic γ can be extended onto $\mathbb{R}$. From now on we assume that this is the case for all geodesics. We assume further that they are parametrised by unit speed (which means that $d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$ for all $t_1, t_2 \in \mathbb{R}$, where d is the distance function on $\mathbb{H}^d$ defined by the hyperbolic metric). Let Γ_d be the set of all (oriented) geodesics in $\mathbb{H}^d$, defined by unit speed. We define a relation $\sim$ on Γ_d by saying $\gamma_1 \sim \gamma_2$ if $d(\gamma_1(t), \gamma_2(t))$ stays bounded for $t \to \infty$. This is invariant under time shifts $t \mapsto t + a$ on one of the geodesics. The relation is clearly reflexive; Transitivity follows from the triangle inequality. So $\sim$ is an equivalence relation.

1.5.7 D [Sphere at Infinity]. The *sphere at infinity* $\mathbb{S}^{d-1}_\infty$ is the set of equivalence classes of Γ_d.

In our model spaces, we can identify the sphere at infinity
- with the boundary of the disc $\mathbb{D}^d$ in the Klein model or
- the hyperplane $\partial\mathbb{R}^{d-1} := \{x \in \mathbb{R}^d : x_d = 0\}$ together with an artificial element ∞, which represents the class of all geodesics that are parallel to the d-th coordinate axis, in the upper half space model.

Let γ be a unit speed ray $\gamma : [0,\infty) \to \mathbb{H}^d$ and

$$\begin{aligned} b_r : \mathbb{H}^d &\longrightarrow \mathbb{R} \\ x &\longmapsto d(x,\gamma(r)) - r. \end{aligned}$$

For fixed x this function is decreasing in r and $|b_r(x)|$ is bounded by $d(x,\gamma(0))$. For any $x, y \in \mathbb{H}^d$ and any $r \in \mathbb{R}_+$ the distance between $b_r(x)$ and $b_r(y)$ is bounded by the distance of x and y. Both facts follow from the triangle inequality.

These considerations allow us to define the "limit function"

$$b_\gamma := \lim_{r\to\infty} b_r.$$

It satisfies $|b_\gamma(x) - b_\gamma(y)| \leq d(x,y)$ for all $x, y \in \mathbb{H}^d$, $|b_\gamma(x)| \leq d(x,\gamma(0))$ and $b_\gamma(\gamma(r)) = -r$ for all $r \in \mathbb{R}^+$. This function is called the *Busemann function* of the ray γ.

1.5.8 D [Horospheres]. A *horosphere* H_x in $\mathbb{H}^d$ centred at a point $x \in \mathbb{S}^{d-1}_\infty$ is the level set of a Busemann function of a ray in the class of rays defining the point x.

A horosphere is independent from the actual choice of the representative in the class of x. Horospheres are particularly nice objects in hyperbolic space. Here is the one important fact that we will exploit in our construction.

1.5.9 T [Horospheres are Flat]. *The induced metric on horospheres is flat.*

Proof. We check this in the upper half space model. A horosphere H_x based at $x \in \mathbb{H}^d_\infty$ is the level set of a Busemann function b_{γ_x} corresponding to a ray γ_x diverging to x. As the isometry group acts transitively on $\mathbb{H}^d$, we can w.l.o.g assume that x is the point not in $\mathbb{R}^{d-1} \subset \mathbb{H}^d_\infty$. Any horosphere to x is then a Euclidean plane parallel to $\mathbb{R}^{d-1}$. Thus it is defined by $x_n = c$ for a constant c. The induced metric on H_x is $ds^2_{H_x} := \frac{1}{c^2}(dx_1^2 + \ldots + dx_{d-1}^2)$: It is flat. $\square$

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2.1 I

We describe and analyse a new construction that applies to graded posets, lattices, PL spheres, and polytopes. It produces new posets, lattices, PL spheres, and — with some restrictions — polytopes. Here are some of the key properties of this construction:

- For suitable input the result is r-simple and s-simplicial.
- We obtain the first infinite family of 2-simple and $(d-2)$-simplicial d-polytopes in any dimension $d \geq 4$.
- We obtain the first infinite family of *rational* 2-simple and 2-simplicial 4-polytopes.
- The 4-polytopes from this construction lie on the boundary of the cone defined by the known flag vector inequalities of 4-polytopes.
- Some of the polytopes have high "fatness" (see Definition 4.2.7).
- For any number of vertices $f_0 \geq 26$ it produces "many" combinatorially different 2-simple and 2-simplicial 4-polytopes.
- Also most of the already known 2-simple and 2-simplicial 4-polytopes are instances of this construction.

A special case of the construction was considered earlier by Eppstein, Kuperberg, and Ziegler [33], who coined the name *E-construction* for it. We use this name also for our generalised version. We sometimes add a subscript t, which gives a distinguished dimension between 0 and $d-1$, if d is the dimension of the polytope, or $d+1$ the rank of the lattice. The original version is only defined for the special case $d = 4$ and $t = 1$.

Our generalised construction comes in two different flavours, and we consequently give two different definitions of it. We start with a combinatorial definition in Definition 2.3.1, which applies to posets and lattices. We translate this into a geometric version in Definition 2.4.3. The two versions coincide on the level of face lattices of spheres and polytopes.

The present chapter introduces the construction, provides several ways to obtain geometric realisations of polytopes, and introduces basic examples. The main result is — apart from the definition and discussion of the general construction — the introduction of three infinite families of polytopes. In Section 2.5.1 we present two families of rational 2-simple and 2-simplicial 4-polytopes. The one defined in Proposition 2.5.12 was somehow the "birth" of all this: It was the first family of 2-simple and 2-simplicial 4-polytopes not contained in the original construction. Meanwhile, other families are more interesting. We introduce several simple ways for the construction of 2-simple and 2-simplicial 4-polytopes, which can produce many other infinite families. In Section 2.5.3 we present a family of 2-simple and $(d-2)$-simplicial d-polytopes in any dimension $d \geq 4$.

Chapter 3 is then entirely devoted to another large class of polytopes to which this construction applies. We work out several properties of these polytopes. In particular, we will look at their combinatorial and geometric symmetry groups, and at their realisation spaces.

Chapter 4 contains some properties of 2-simple and 2-simplicial 4-polytopes that can be proven with our methods, and it contains lists of known examples of 2-simple and 2-simplicial 4-polytopes. We also present some other methods to obtain r-simple and s-simplicial polytopes.

2.2 S R (2, 2)-P

The *E-construction* was first considered by Eppstein. In a joint work with Kuperberg and Ziegler [33] he used it to produce an infinite family of $(2, 2)$-polytopes (i.e. 2-simple and 2-simplicial 4-polytopes). Prior to this paper, only a finite number of $(2, 2)$-polytopes where known. Most of them occurred in the context of regular and semi-regular polytopes.

There is a claim in Grünbaum's book [44, Ex. 9.9.7(iii), p. 169] that Perles and Shephard had earlier obtained infinitely many $(2, d-2)$-polytopes. However, there is no known proof of this result. Eppstein, Kuperberg, and Ziegler with their work substantiated this claim in dimension 4 by constructing the first infinite family Q_n, $n \geq 1$, of $(2, 2)$-polytopes.

We finally completely prove this claim: In Theorem 2.5.15 we provide an infinite family of $(2, d-2)$-polytopes in all dimensions $d \geq 4$. This family also sheds some more light on a problem stated by Kalai in [54, 19.5.19], where he asks for values of r and s, with $r, s \geq 2$, such that there is an r-simple and s-simplicial d-polytope.

Further, also in Grünbaum's book [44, Ex. 5.2.13(ii), p. 82], one finds a claim that Shephard had proven a conjecture of Walkup, stating that $(2, 2)$-polytopes are dense among 4-polytopes. However, this was premature. The question is still open [44, p. 69b], and despite our new methods — which produce a wealth of $(2, 2)$-polytopes — we still do not seem to be close to an answer. (*Dense* in this setting means the following: Given two families $\mathcal{P}$ and $\mathcal{Q}$ of polytopes, $\mathcal{P}$ is dense in $\mathcal{Q}$, if for any $\varepsilon > 0$, and any polytope $Q \in \mathcal{Q}$, there is a polytope $P \in \mathcal{P}$ with Hausdorff distance smaller than ε from Q.)

Roughly, the construction of Eppstein, Kuperberg, and Ziegler works as follows. It uses an *edge-tangent* simplicial 4-polytope as input, and takes the convex hull of this with its polar polytope. It is not difficult to see that this results in a $(2, 2)$-polytope. The hard part in their work is the construction of sufficiently many simplicial edge-tangent polytopes. This is quite easy in three, and rather difficult in all higher dimensions.

As an application of the results of their construction, Eppstein, Kuperberg, and Ziegler [33] introduce the *fatness* parameter $\mathbf{F}(P)$ for a 4-polytope, which is the quotient of the sum of edges and ridges divided by the sum of vertices and facets. This concept was further developed by Ziegler [90], and recently in [92]. Eppstein, Kuperberg and Ziegler gave examples with fatness slightly above five. In Chapter 3 we present a family of polytopes obtained from our generalised E-construction whose fatness approaches six. Either finding an upper bound for this parameter, or proving that it is unbounded, would produce important information for the classification problem of flag vectors introduced in Section 1.4.2. It is unbounded for CW spheres, and Pfeifle and Ziegler [69] have exploited the underlying method for the construction of a large number of non-polytopal spheres.

The families of $(2,2)$-polytopes that we obtain from the E-construction in this chapter are the first infinite families of $(2,2)$-polytopes that have *rational* coordinates for their geometric realisations. This answers a question of Eppstein, Kuperberg, and Ziegler in [33], as their construction only produced examples with (in the given realisation) non-rational coordinates, due to the edge-tangency condition in their construction. In contrast to this original version, which is quite rigid, ours allows great flexibility in the choice of coordinates.

2.3 T E-C P L

We begin with a combinatorial description of our generalised version of the E-construction. It is defined on any graded poset and associates a new graded poset to it. We show that many properties of the poset are preserved by the E-construction. In fact, the result may even satisfy stronger properties than the input.

The construction depends on a parameter t specifying a level set P_{t+1} in the graded poset P. Roughly, if P is a graded poset of length $d+1$ and t a parameter between 0 and $d-1$, then the construction defines a new set $E_t(P)$ containing all elements of rank $t+1$ and all intervals $[x,z] \subseteq P$ that contain at least one element of rank $t+1$ in their interior (i.e. different from x and z). This new set can be ordered by reversed inclusion. Here is the precise definition.

2.3.1 D [The Combinatorial E-Construction]. Let P be a graded poset of length $d+1$ with order relation $\preceq_P$ and t an integer between 0 and $d-1$. Define a new poset $E_t(P)$ with order relation $\preceq$ in the following way:

(1) The elements of $E_t(P)$ are
  (a) The empty set $\emptyset$,
  (b) the elements $y \in P$ with $\operatorname{rank}(y) = t+1$, and
  (c) all intervals $[x,z] \subseteq P$ with $\operatorname{rank}(x) < t+1 < \operatorname{rank}(z)$.
(2) We order this set by reversed inclusion of sets in P. So $\emptyset$ becomes the maximal element in $E_t(P)$ and P itself the minimal.

See Figure 2.1 for an illustration. Note, that we cannot choose the bottom or top level as the “distinguished” one, as in this case no interval could “cross” the level. In later sections we consider this construction usually for a fixed parameter t. In this case, and if no confusion is possible, we omit t in the notation and write $E(P)$ for the poset obtained from P via the E-construction.

2.3.2 R . We have two remarks about the choices made in the definition.

- It would not make much difference if we order the set by inclusion instead of reversed inclusion. All posets, lattices, spheres, and polytopes constructed via this method would just turn into their opposites (or duals, respectively). We have chosen this order mainly for historical reasons: This way, it is closer to the original definition of Eppstein, Kuperberg, and Ziegler.
- The choice of $t + 1$ as the distinguished level (instead of t) in the poset is motivated by the important application of this construction to spheres and polytopes, which we discuss later in this chapter. Then, the elements of rank $t+1$ in the face lattice of a sphere or polytope correspond to cells or faces of dimension t in that sphere or polytope.

2.3.3 E . Here are some simple examples of the E-construction.

- The E_t-construction applied to a poset of length $d + 1$ is the same as E_{d-1-t} applied to its opposite, i.e.

  $$E_t(P) \cong E_{d-1-t}(P^{\mathrm{op}}).$$

  So we can derive the same posets from P and from its dual. This equality is rather immediate from the definition.

  We use this important fact quite often when constructing polytopes, and without always mentioning it. In particular, it will prove very convenient for the enumerations of $(2, 2)$-polytopes in Section 4.3.

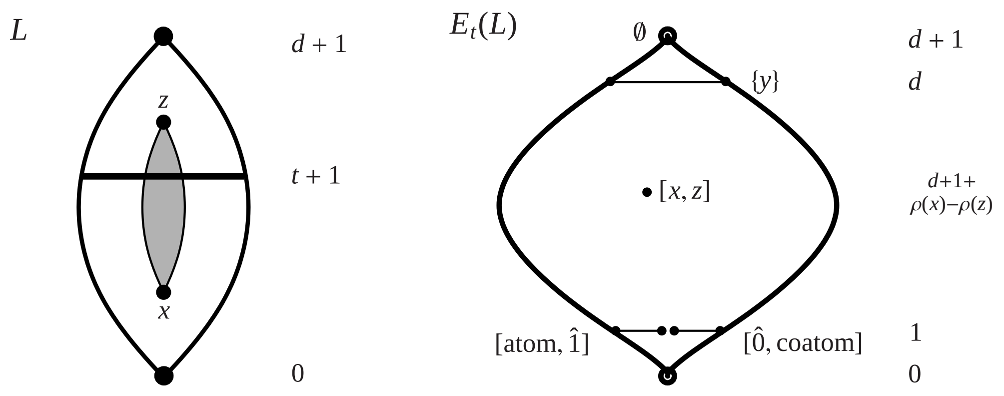


Figure 2.1: Combinatorial construction of $E_t(L)$.

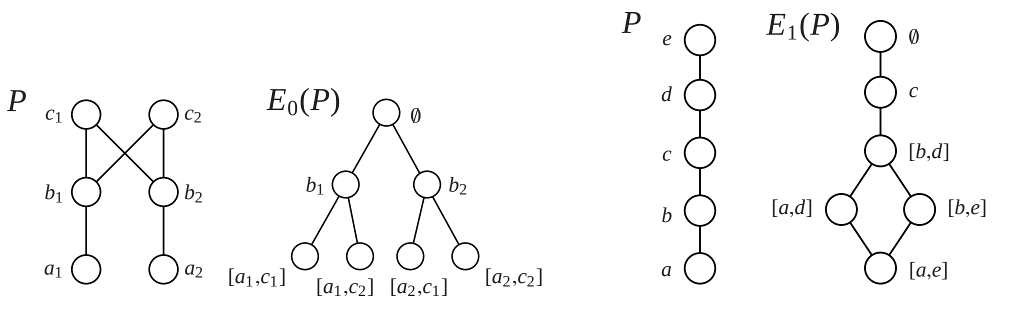
- Let P be a graded and bounded poset of length $d+1$. Then
$$E_{d-1}(P) = P.$$
Indeed, the new elements of rank d are the same as the old ones. The maximal element of an interval crossing level d is $\hat{1}_P$. An interval $[x_1, \hat{1}_P]$ contains an interval $[x_2, \hat{1}]$ if and only if $x_1 \prec_P x_2$. Dually, we also have
$$E_0(P) = P^{\mathrm{op}}.$$
We will only consider bounded graded posets, so these two cases are not really interesting for the construction, and we sometimes exclude them from our considerations.
- Figure 2.2 contains two simple, but non-trivial examples of the E-construction, shown via their Hasse diagrams.

A *chain* in the poset $E_t(P)$ is an ascending sequence of intervals in P. See Figure 2.3 for an illustration. The poset $E_t(P)$ comes with a natural *rank function* induced from inclusion of intervals in the poset P.

2.3.4 D P [The Rank Function on $E(P)$]. *Let P be a graded poset of length $d+1$ and $0 \le t \le d-1$. The poset $E_t(P)$ naturally admits a rank function ρ induced from the rank function ρ_P of the original poset P:*

$$\rho(\alpha) := \begin{cases} d+1 & \text{if } \alpha = \emptyset, \\ d & \text{if } \alpha \text{ is an element of rank } t+1 \text{ in } P, \\ (d+1) - (\rho_P(z) - \rho_P(x)) & \text{if } \alpha = [x,z] \text{ is an interval in } P \text{ crossing the level } t+1. \end{cases}$$

From the properties of the rank function ρ_P of P it is immediate that this defines a rank function ρ on $E_t(P)$. □



(a) A simple poset P and E_0 applied to it. Note, that P is not bounded.

(b) Another poset P and E_1 applied to it.

Figure 2.2: Some examples of the E-construction.

2.3.5 P [Length Preservation]. *Let P be graded poset of length $d+1$. Then P and $E_t(P)$ have the same length for any $0 \le t \le d-1$.*

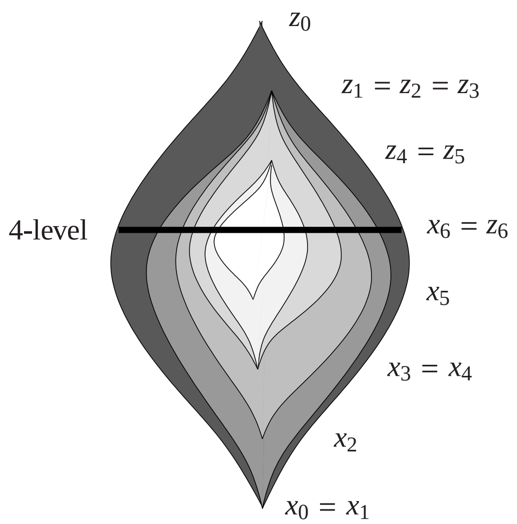
Proof. Let ρ_P be the rank function of the poset P. By Proposition 2.3.4 we know that $E_t(P)$ is graded. A maximal chain in $E_t(P)$, translated into its corresponding intervals in P, has the form

$$[x_0, z_0] \lessdot [x_1, z_1] \lessdot \ldots \lessdot [x_k, z_k] = [x_k, x_k] \lessdot \emptyset$$

where x_0 is a minimal element, z_0 a maximal element, for any $0 \le i \le k-1$ either $x_i = x_{i+1}$ and z_i covers z_{i+1}, or x_i is covered by x_{i+1} and $z_i = z_{i+1}$, and $\rho_P(z_k) = \rho_P(x_k) = t+1$. As x_i, z_i must also satisfy $\rho_P(x_i) < t+1 < \rho_P(z_i)$ for any $1 \le i \le k-1$, there are $t + ((d+1)-(t+2)) = d-1$ increasing steps. The last two elements, i.e. $[x_k, x_k]$ and $\emptyset$, were not included in this count, so in total, the above chain contains $d+2$ elements, so it has length $d+1$. □

Thus, $E_t(P)$ is again a *graded* poset of *length* $d+1$. Its *coatoms* are the one-element sets $\{y\}$, $y \in P_{t+1}$, its *atoms* are the intervals $[x, \hat{1}]$ for $1 \le t \le d-1$, and $[\hat{0}, z]$ for $0 \le t \le d-2$, where x ranges over the atoms, and z over the coatoms of P.

It is not hard to compute the *f-vector* of the new poset from the flag vector of the old, as we just have to count intervals of a certain length containing elements of rank $t+1$.



$$[x_0, z_0] \lessdot [x_1, z_1] \lessdot [x_2, z_2] \lessdot [x_3, z_3] \lessdot [x_4, z_4] \lessdot [x_5, z_5] \lessdot [x_6, x_6] \lessdot \emptyset$$

Figure 2.3: A maximal chain in $E_3(P)$, where P has rank 7.

Let P be a poset of length $d + 1$ and $0 \leq t \leq d - 1$. Then

$$f_k(E_t(P)) = \begin{cases} \sum_{i,j} f_{ij}(P) & \text{for } -1 \leq k < d-1, \\ f_t(P) & \text{for } k = d-1, \\ 1 & \text{for } k = d, \end{cases} \tag{2.3.1}$$

where the sum in the above formula ranges over all pairs i, j that satisfy

$$-1 \leq i < t < j \leq d \qquad \text{and} \qquad j - i = d - k.$$

In the same fashion, one can compute all other entries of the flag vector. However, we need this only in a few cases, where the flag vector is easier computed differently.

If P is a *lattice*, then there is a much shorter description of the poset $E_t(L)$ for any parameter t between 0 and $d - 1$. Namely, in this case, we can just define

$$E_t(L) := \Big\{ \Big[\bigwedge A, \bigvee A \Big] \;:\; A \subset L_{t+1} \Big\},$$

again ordered by reversed inclusion. Here we interpret $[\bigwedge \emptyset, \bigvee \emptyset]$ to be $\emptyset$, which defines the element $\hat{1}$ in $E_t(L)$. See Figure 2.4 for an example of the E-construction applied to a lattice. The next proposition tells us that the property of being a lattice is preserved by the E-construction.

2.3.6 P [Lattice Property Preservation]. *For any bounded and graded lattice L, and any parameter $0 \leq t \leq d - 1$, the poset $E_t(L)$ is a lattice.*

Proof. We have to show that any two elements have a join and a meet. Let ρ_L be the rank function in L and $\alpha, \beta \in E_t(L)$ be two arbitrary elements. We show that they have a meet. We distinguish two cases for this.

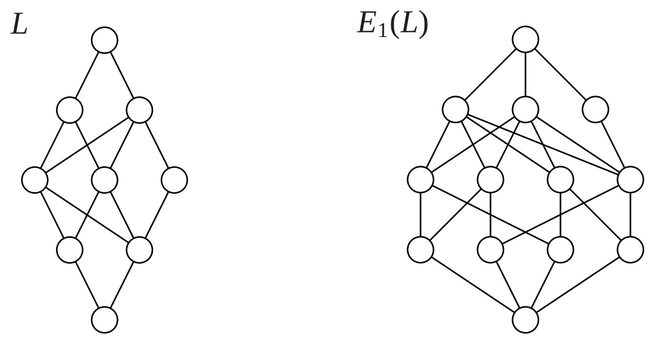


Figure 2.4: A lattice together with the E-Construction applied to the middle level.

- If $\alpha = L$, then α is the minimal element of $E_t(L)$. Clearly, in this case $\alpha \wedge \beta = \alpha$. Similarly if $\beta = L$.
- In all other cases we can assume $\alpha = [x_1, z_1]$ and $\beta = [x_2, z_2]$ for some elements $x_1, x_2, z_1, z_2 \in L$ with $\rho_L(x_1), \rho_L(x_2) \leq t+1$ and $\rho_L(z_1), \rho_L(z_2) \geq t+1$. We allow $x_1 = z_1$ or $x_2 = z_2$ in this.
  Let $a := x_1 \wedge x_2$ and $b := z_1 \vee z_2$. Then clearly $\rho_L(a) \leq t+1$ and $\rho_L(b) \geq t+1$. Consider $\zeta := [a, b]$. Then $\zeta \leq \alpha$ and $\zeta \leq \beta$, so ζ is a lower bound of α and β. Let ξ be another lower bound. Then $\xi = [a', b']$ for some elements $a', b' \in L$, and $[x_1, z_1], [x_2, z_2] \subseteq [a', b']$. So a' is a lower bound for x_1 and x_2. But a is their meet, so $a' \leq_L a$. Similarly, b' is an upper bound of z_1 and z_2. As b is their join, we have $b \leq_L b'$. This implies $[a, b] \subseteq [a', b']$ in L, and consequently $\xi \leq \zeta$ in $E_t(L)$. So ζ is the meet of α and β.

Joins can either be computed in a similar way, or one can use the fact that in bounded posets the existence of meets already implies the existence of joins, by Proposition 1.2.22. □

2.3.7 R . The converse of Proposition 2.3.6 is not true in general. $E_t(P)$ for some parameter t can be a lattice although P is not. In Figure 2.5 is a simple example of this phenomenon. It shows a poset in which two elements do not have a meet, so P is not a lattice. However, $E_1(P)$ is the face lattice of the polytope shown in Figure 2.6.

2.3.8 P . *Let P be a graded bounded poset of length $d+1$ and t a parameter between* 0 *and* $d-1$*. If P is Eulerian, then $E_t(P)$ is also Eulerian.*

Proof. This is true for $t = 0$ and for $t = d-1$, which includes all possible values for t if $d \leq 2$. We use induction on the length of P.

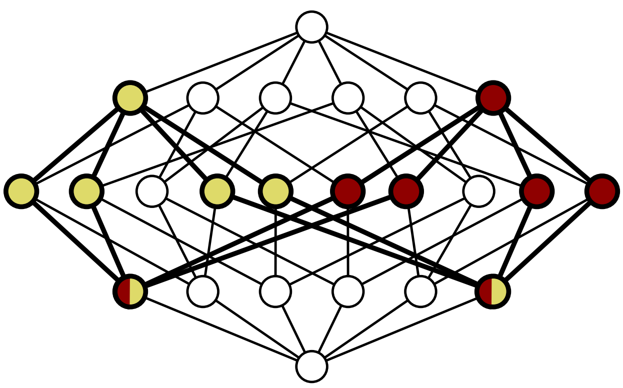


Figure 2.5: The poset P from Remark 2.3.7. Note, that the meet of the two outer elements of level three does not exist. They have two maximal lower bounds.

First we show that all proper intervals in $E_t(P)$ are Eulerian. For any element $[x,z] \in E_t(P)$, where $(x,z) \neq (\hat{0},\hat{1})$, the upper interval $[[x,z],\hat{1}\,]$ of $E_t(P)$ is isomorphic to $E_{t'}([x,z])$ for $t' = t-\rho(x)$. See Figure 2.7(a). Hence, all proper upper intervals in $E_t(P)$ are produced by the E-construction from Eulerian posets of smaller length and level t', so they are Eulerian by induction.

Similarly, if $[x,z]$ is an element of rank at most $d-1$ in $E_t(P)$, that is, an interval of P with $x < y < z$ for some $y \in P_{t+1}$, then the lower interval $[\,\hat{0},[x,z]]$ of $E_t(P)$ is isomorphic to $[\hat{0},x] \times [z,\hat{1}]^{op}$. This is Eulerian by Theorem 1.2.16, since P is Eulerian.

If $\{y\}$ is a coatom of $E_t(P)$, for $y \in P_{t+1}$, then the lower interval $[\,\hat{0},\{y\}]$ is isomorphic to $(P_{<y}) \times (P_{>y})^{op} \uplus \hat{1}$. Thus, it is the opposite of a reduced product of two Eulerian posets, which is Eulerian by Theorem 1.2.16.

Finally, we have to see that $E_t(P)$ itself has the same number of odd and even rank elements. For this we use the f-vector of $E_t(P)$, which we have already computed in (2.3.1). Every interval $[\hat{0},z]$ in P is Eulerian. Hence, for $0 \leq j \leq d-1$ and all $z \in P_{j+1}$ we have

$$\sum_{i=-1}^{j}(-1)^i f_i([\hat{0},z]) = 0,$$

which, by summing over all $z \in P_{j+1}$, yields $\sum_{i=-1}^{j}(-1)^i f_{ij} = 0$. If $j \geq t$ we can split the sum at $i = t$ to obtain

$$\sum_{i=-1}^{t-1}(-1)^i f_{ij} = -\sum_{i=t}^{j}(-1)^i f_{ij}. \qquad (\star)$$

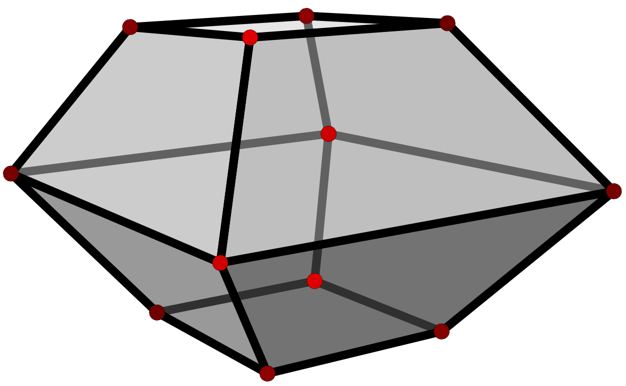


Figure 2.6: A polytope having $E_1(P)$ for the poset P in Figure 2.5 as face lattice.

This is one of the generalised Dehn–Sommerville equations, see Theorem 1.2.17. A similar argument for upper intervals shows that

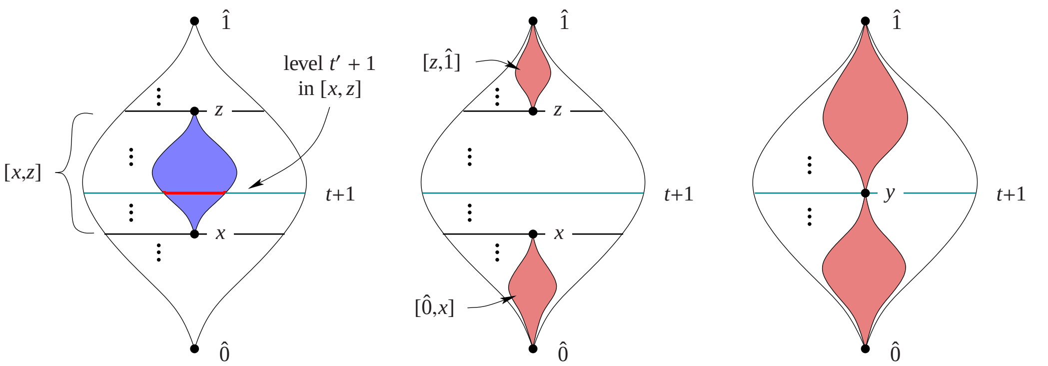
$$\sum_{j=i}^{d} (-1)^j f_{ij} = (-1)^d \delta_{id}. \tag{$\star\star$}$$

for $i \leq d$. With these two equations, we can compute

$$\begin{aligned}
\sum_{k=-1}^{d} (-1)^{d-k} f_k(E_t(P)) &= \\
&= 1 - f_t + \sum_{i=-1}^{t-1} \sum_{j=t+1}^{d} (-1)^{j-i} f_{ij} &&= 1 - f_t + \sum_{j=t+1}^{d} (-1)^j \sum_{i=-1}^{t-1} (-1)^i f_{ij} \\
&\overset{(\star)}{=} 1 - f_t - \sum_{j=t+1}^{d} (-1)^j \sum_{i=t}^{j} (-1)^i f_{ij} &&= 1 - \sum_{j=t}^{d} (-1)^j \sum_{i=t}^{j} (-1)^i f_{ij} \\
&= 1 - \sum_{i=t}^{d} (-1)^i \sum_{j=i}^{t} (-1)^j f_{ij} &&\overset{(\star\star)}{=} 1 - \sum_{i=t}^{d} (-1)^i (-1)^d \delta_{id} \\
&= 0.
\end{aligned}$$

Hence, the poset $E_t(P)$ contains as many elements of odd rank as it contains elements of even rank. This proves the claim. □

2.3.9 R . Alternatively, one may argue from Theorem 2.4.1 in the next section: Since the order complexes of P and of $E_t(P)$ are homeomorphic, they must have the same Euler characteristic, which is the Möbius function of P and $E_t(P)$, respectively. This is precisely what we need for $[x, z] = [\hat{0}, \hat{1}]$.



(a) Proper intervals. *(b)* Products of intervals. *(c)* Reduced products of intervals.

Figure 2.7: The intervals in $E_t(P)$.

The next theorem will be interesting for the comparison of the geometric and combinatorial symmetry groups of products of polygons and their E-construction in Chapter 3.

2.3.10 P [Symmetry Preservation]. *Let P be a graded poset of length $d+1$. Then there is an canonical injective map*

$$i : \mathrm{Aut}(P) \hookrightarrow \mathrm{Aut}(E_t(P))$$

for any $0 \le t \le d-1$. □

Let $\mathfrak{a}$ be a automorphism of P. Then $i(\mathfrak{a})$ is the automorphism of $E_t(P)$ sending an interval $[x, z]$ to $[\mathfrak{a}(x), \mathfrak{a}(z)]$. We will see in Section 2.5 that E_2 applied to the face lattice P of a 4-cube is isomorphic to the face lattice of the 24-cell, so the automorphism group of $E_t(P)$ can be strictly larger than that of P. Hence, i will in general not be surjective.

Now we come to the most important property of the E-construction for the remaining sections of this chapter and the following two chapters. Recall the definition of simpliciality and simplicity from Definition 1.2.25. We repeat a shorthand notation from Definition 1.2.26 that we use frequently.

D [*(r,s)*-Lattices]. Let L be an Eulerian lattice. We say that L is a *(r, s)-lattice* if it has rank $\ell = r + s + 1$ and is r-simple and s-simplicial.

We later apply the same notation also to strongly regular spheres and polytopes. So, a (r, s)-polytope is an r-simple and s-simplicial polytope of dimension $r + s$.

2.3.11 T [Simplicity and Simpliciality]. *Let L be a bounded Eulerian lattice of length $d+1$ and t a parameter between 1 and $d-2$.*

- *(1a) For $0 \le k \le d-2$, $E_t(L)$ is k-simplicial if L is r-simple and s-simplicial for $r \ge \min(k, d-t-2)$ and $s \ge \min(k, t-1)$.*
- *(1b) $E_t(L)$ is never $(d-1)$-simplicial.*
- *(2a) $E_t(L)$ is 2-simple if and only if every interval $[x, z]$ with $\mathrm{rank}(x) = t-1$ and $\mathrm{rank}(z) = t+3$ is boolean.*
- *(2b) $E_t(L)$ is never 3-simple.*

Note, that we have excluded the cases $t = 0$ and $t = d-1$ in the theorem. By Examples 2.3.3 we have $E_0(P) = P^{\mathrm{op}}$ and $E_{d-1}(P) = P$ for bounded posets. With some adjustments for the case $t-1 < 0$ or $t+3 > d+1$, the claims in $(1a)$ and $(2a)$ are still true for $t = 0$ and $t = d-1$, but $(1b)$ and $(2b)$ are clearly wrong. However, neither of these two cases is really interesting, so we omitted them.

Proof of Theorem 2.3.11. We prove all four different claims in the theorem separately. Let ρ_L be the rank function on L, and ρ that of $E_t(L)$.

(1a) The elements of rank at most $k + 1 \leq d - 1$ in $E_t(L)$ are the intervals of the form $\alpha = [x, z] \subseteq L$ with $\rho_L(x) \leq t$, $\rho_L(z) \geq t + 2$ satisfying

$$\rho_L(x) + (d + 1) - \rho_L(z) = \rho(\alpha) \leq k + 1.$$

An element $x \in L$ appears as the lower end of such an interval $[x, z]$ if and only if

$$0 \leq \rho_L(x) \leq \min\{k + 1, t\}.$$

The same is true for all elements $z \in L$ of corank

$$0 \leq d + 1 - \rho_L(z) \leq \min\{k + 1, d - t - 1\}.$$

The atoms of $E_t(L)$ below $\alpha = [x, z]$ are given by both the atoms of L below x, whose number is at least $\rho_L(x)$, and the coatoms of L above z, whose number is at least $d + 1 - \rho_L(z)$. So the interval has at least

$$\rho_L(x) + d + 1 - \rho_L(z) = k + 1$$

atoms, with equality if and only if the intervals $[\hat{0}, x]$ and $[z, \hat{1}]$ of L are both boolean.
Thus *all* lower intervals $[\hat{0}, \alpha]$ of rank $k + 1$ in $E_t(L)$ are boolean if and only if *all* intervals $[\hat{0}, x]$ and $[z, \hat{1}]$ are boolean for $\rho_L(x) \leq \min\{k + 1, t\}$ resp. $d + 1 - \rho_L(z) \leq \min\{k + 1, d - t - 1\}$.

(1b) An analysis as for (1a) shows that for any element $\{y\}$ of rank d in $E_t(L)$, i.e. for $y \in L_{t+1}$, there are at least $t + 1$ atoms $a_1, \ldots, a_m$ in L satisfying $a_i \leq y$ for $1 \leq i \leq m$ and at least $d - t$ coatoms $c_1, \ldots, c_{m'}$ in L satisfying $y \leq c_i$ for $1 \leq i \leq m'$. Hence, there are at least

$$m + m' \geq (t + 1) + (d - t) = d + 1$$

atoms below $\{y\}$ in $E_t(L)$. Too many for a $(d - 1)$-simplex.

(2a) $E_t(L)$ is 2-simple, if all intervals $[\beta, \hat{1}] \subset E_t(L)$ with $\beta = [x, z] \subset L$ and $\rho(\beta) = d - 2$ are boolean. Equivalently, these intervals must have 3 atoms or coatoms.
This is the case if and only if every interval $\beta = [x, z] \subset L$ of length three, for $\rho_L(x) < t + 1 < \rho_L(z)$, contains precisely three elements of rank $t + 1$. This is equivalent to the condition that every interval $[\overline{x}, \overline{z}]$ of length four with $\rho_L(\overline{x}) = t - 1$ and $\rho_L(\overline{z}) = t + 3$ is boolean. In terms of the usual flag vector notation, this can numerically be expressed as

$$f_{t-2,t,t+2}(L) = 6 f_{t-2,t+2}(L).$$

(2b) Similarly to the previous considerations, for $E_t(L)$ to be 3-simple we would need that every interval $[x, z]$ in L of length 4 with $\rho_L(x) < t + 1 < \rho_L(z)$ and $\rho_L(z) = \rho_L(x) + 4$ contains exactly 4 elements of rank $t + 1$.
However, this is impossible for the case where $\rho_L(x) = t-1$ and $\rho_L(z) = t+3$, where the interval $[x, z]$ has at least 6 elements in its "middle level" (that is, of rank $t + 1$) by the Eulerian condition. See Figure 2.8 for an illustration of this fact. □

2.3.12 R . The condition in $(2a)$ is in particular satisfied if L is $(t + 2)$-simplicial or $(d - t - 1)$-simple. We will only use this weaker form in the next sections, although one can construct examples of polytopes that are neither $(t+2)$-simplicial nor $(d - t - 1)$-simple, but their E-construction still is 2-simple.

Theorem 2.3.11 gives us the following important corollary. Its geometric version, which we state in the next section, will play a central rôle in the construction of our families of polytopes.

2.3.13 C [(2, *d* − 2)-Lattices]. *Let L be a bounded Eulerian lattice of length $d + 1 \geq 4$.*

(1) If L is simplicial then $E_{d-3}(L)$ is a $(2, d - 2)$-lattice.

(2) If L is simple, then $E_2(L)$ is a $(2, d - 2)$-lattice. □

We have now collected all necessary information on the combinatorial properties of the E-construction.

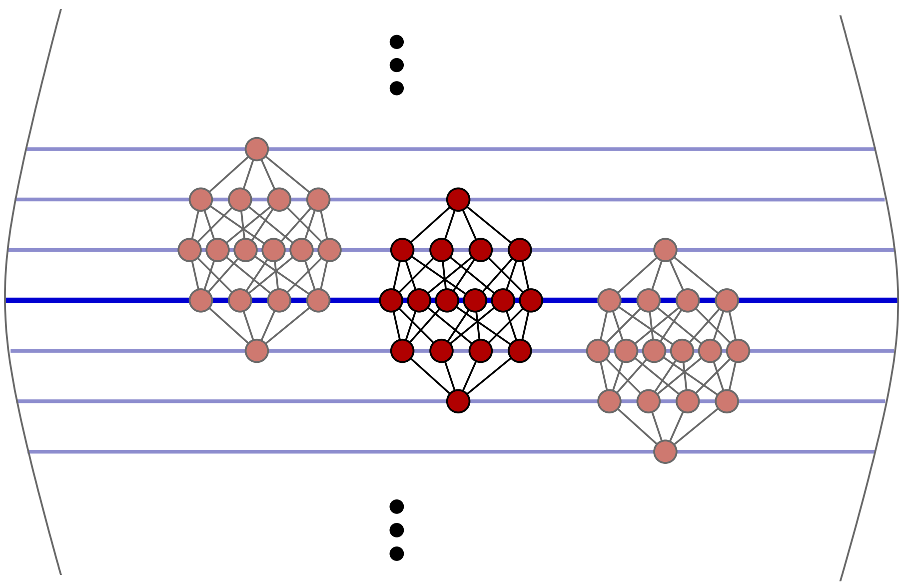


Figure 2.8: The E-construction can never produce 3-simple lattices.

2.4 T *E*-C S

There are two different ways in which we want to look at spheres: We present a more *combinatorial* and a more *geometric* version. We begin this section with a discussion of the *order complex* of a bounded Eulerian lattice and and its E-construction. We prove that these two order complexes are PL homeomorphic by constructing an explicit PL homeomorphism between the two spaces.

In the second part of this section we define — independent from the combinatorial construction — a geometric E-construction for PL spheres. We prove that this geometric construction coincides with the combinatorial Definition 2.3.1 if we look at the face lattices of the PL sphere and its E-construction.

The presentation of the PL version of the E-construction will not be concise. It should provide some intuition for the geometric properties of this construction, rather than giving a formal definition.

2.4.1 O C S

Here we look at the order complex associated to a poset P. This is the abstract simplicial complex whose vertices are the elements of P, and whose faces are subsets of the vertices that form a chain in P. See Definition 1.2.19 for a precise version. An abstract simplicial complex can be realised geometrically in some $\mathbb{R}^n$, according to Proposition 1.2.20. We work with PL topology in this section. See Section 1.3.1 for the relevant definitions and the necessary facts.

The following result is analogous to simpler results of Walker [87] and constructs an explicit PL homeomorphism between the order complex of a poset P and the order complex of its E-construction $E_t(P)$, for some admissible parameter t. Basically, this shows that the latter is a subdivision of the former. Recall, that $\overline{P}$ denotes the proper part of a poset P, i.e. the poset without its maximal and minimal elements, and $|X|$ the underlying topological space of a cell complex.

2.4.1 T [The PL Homeomorphism]. *Let P be a finite graded poset of length $d+1$ and t a parameter between 0 and $d-1$. Then P and $E_t(P)$ are PL homeomorphic:*

$$|\overline{P}| \cong |\overline{E_t(P)}|.$$

Proof. We verify that $\Delta(\overline{E_t(P)})$ is a subdivision of $\Delta(\overline{P})$, and give explicit formulas for the subdivision map and its inverse. (Compare this to Walker [87, Sects. 4,5].)

If one assumes that there is such a homeomorphism, then it is fairly clear how it should look like. So we just state the map here and prove that is has the required properties.

We define the following map:

$$\pi : |\Delta(\overline{E_t(P)})| \longrightarrow |\Delta(\overline{P})|,$$

given on the vertices by

$$\begin{aligned}
\{y\} &\longmapsto y && \text{for } y \in P_{t+1},\\
[x,z] &\longmapsto \frac{1}{2}x + \frac{1}{2}z && \text{for } \hat{0} < x < y < z < \hat{1},\ y \in P_{t+1},\\
[x,\hat{1}] &\longmapsto x && \text{for } \hat{0} < x < y < \hat{1},\qquad y \in P_{t+1},\\
[\hat{0},z] &\longmapsto z && \text{for } \hat{0} < y < z < \hat{1},\qquad y \in P_{t+1}
\end{aligned}$$

and linearly extended on the simplices. This map is well-defined and continuous. Its inverse, a subdivision map, may be described as follows: Any point of $\Delta(\overline{P})$ is an affine combination of elements on a chain in $\overline{P}$, so it may be written as

$$\mathbf{x} : \lambda_1 x_1 \ < \ \cdots \ < \ \lambda_t x_t \ < \ \lambda_{t+1} y_{t+1} \ < \ \lambda_{t+2} z_{t+2} \ < \ \cdots \ < \ \lambda_d z_d.$$

with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. We set $x_0 := \hat{0}$ and $z_{d+1} := \hat{1}$, with coefficients $\lambda_0 := 1$ and $\lambda_{d+1} := 1$.

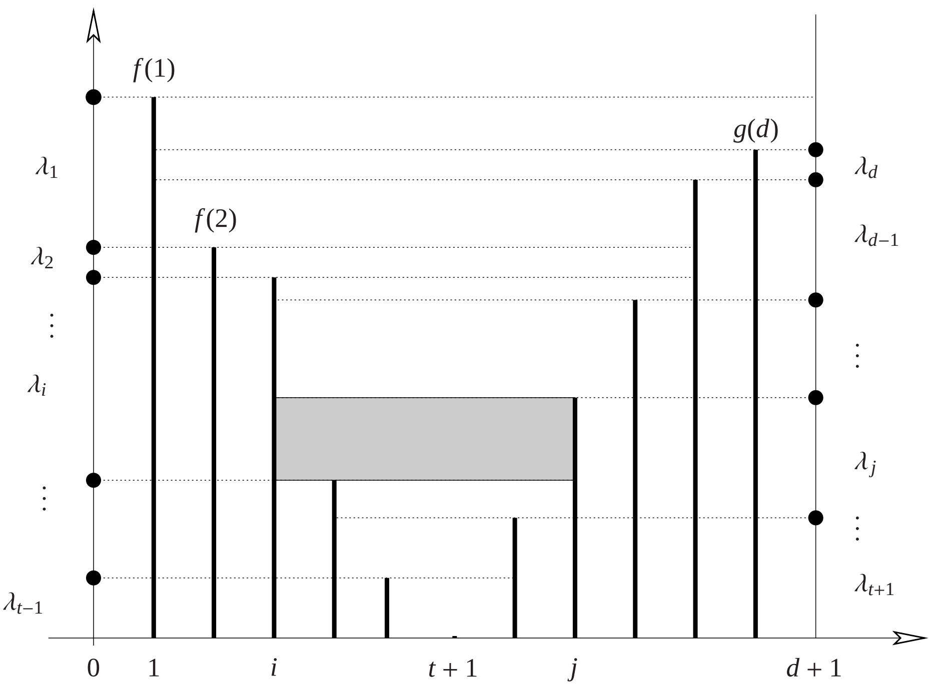


Figure 2.9: Sketch for the proof of Theorem 2.4.1. The height of the shaded rectangle indicates the size of the coefficient $\alpha_{i,j}$.

Now the above point is mapped by π^{-1} to

$$\pi^{-1}(\mathbf{x}) = \lambda_{t+1}\,\{y_{t+1}\} \;+ \sum_{1\le i<t,\; t+1<j\le d} 2\alpha_{i,j}\,[x_i, z_j] \;+ \sum_{\substack{i=0,\, t+1<j\le d \text{ or}\\ 1\le i\le t,\; j=d+1}} \alpha_{i,j}\,[x_i, z_j],$$

where the coefficients $\alpha_{i,j}$ are given by

$$\alpha_{i,j} := \begin{cases} \min\{f(i), g(j)\} \;-\; \max\{f(i+1), g(j-1)\} & \text{if this is } \ge 0,\\ 0 & \text{otherwise,} \end{cases}$$

with $f(t+1) = g(t+1) = 0$, and

$$\begin{aligned} f(i) &:= \lambda_i + \lambda_{i+1} + \ldots + \lambda_t && \text{for } 0 \le i \le t+1, \text{ and}\\ g(j) &:= \lambda_{t+2} + \ldots + \lambda_{j-1} + \lambda_j && \text{for } t+1 \le j \le d+1. \end{aligned}$$

We refer to Figure 2.9 for an illustration of the construction of the α_{ij}. □

From this theorem we can conclude the following important consequence.

2.4.2 T . *If P is the face poset of a regular PL sphere or PL manifold, then so is $E_t(P)$.*

Proof. By Proposition 2.3.8 and its proof, using the PL property, we get the cell complex. By Theorem 2.4.1, this cell complex is homeomorphic to $|\overline{P}|$. □

2.4.2 T *E*-C PL S

We translate the combinatorial E-construction of Definition 2.3.1 into the geometric setting. Nothing new is happening here, as formally this description is immediate from the combinatorial one and the homeomorphism given in the previous paragraph. However, it might provide some intuition for what the construction is doing geometrically, and for the problems we face when applying it to polytopes.

2.4.3 D **[The *E*-Construction for Spheres].** Let S be a PL sphere of dimension $d-1$ and t a parameter between 0 and $d-1$. We define a new PL sphere by the following three steps:

1. (1) Take a barycentric subdivision $\mathcal{B}\mathrm{S}(S)$ of S. This cell complex has a vertex for each cell of S and a $(d-1)$-cell for each maximal chain of faces of S.
2. (2) Merge all $(d-1)$-cells (i.e. all facets) of $\mathcal{B}\mathrm{S}(S)$ that share a vertex coming from a t-cell of S into a single new $(d-1)$-cell.
3. (3) Merge all k-dimensional cells, that become “unnecessary” by this operation in the sense that they intersect in a $(k-1)$-cell that is adjacent to no other k-cell, for $0 \le k \le d-2$.

Clearly, this construction applies to any PL sphere S and produces a new PL sphere. See Figure 2.10 for an example of E_1 applied to a 2-sphere. The geometric version of the E-construction corresponds to the combinatorial version of Definition 2.3.1 in the following way.

2.4.4 P . *Let S be a PL sphere of dimension $d-1$ and t a parameter between 0 and $d-1$. Let $\mathcal{L}(S)$ be the face lattice of S.*

Then $E_t(\mathcal{L}(S))$ (i.e. the combinatorial version of the E-construction applied to the face lattice of S) is the face lattice of $E_t(S)$.

Proof. Both $\mathcal{L}(E_t(S))$ and $E_t(\mathcal{L}(S))$ have $f_t(S)$ coatoms. By the third step in Definition 2.4.3, all vertices of $\mathcal{B}\mathrm{S}(S)$ vanish if they do not stem from a 0-cell or a $(d-1)$-cell of S. So both lattices also have the same number $f_0(S) + f_{d-1}(S)$ of atoms. Hence, it suffices to check the vertex–facet incidences.

The cells of the barycentric subdivision $\mathcal{B}\mathrm{S}(S)$ can be identified with chains in the face lattice $\mathcal{L}(S)$ of the sphere. Chains containing only one cell correspond to the vertices, and maximal chains to the facets. If two facets in $\mathcal{B}\mathrm{S}(S)$ have a cell of dimension t of the original sphere S in common then the corresponding maximal chains in the lattice intersect in an element of rank $t+1$. For any $y \in \mathcal{L}(S)$ of rank $t+1$ let C_y denote the set of all maximal chains in $\mathcal{L}(S)$ that contain y. Let σ_y be the t-cell of S represented by y.

As we merge all facets in $\mathcal{B}\mathrm{S}(S)$ that contain the same cell of dimension t, such a set C_y canonically correspond to a facet of $E_t(S)$. On the other hand, we may identify C_y with a coatom of $E_t(\mathcal{L}(S))$ by mapping C_y onto $\{y\}$.

The atoms of $E_t(\mathcal{L}(S))$ incident to $\{y\}$ are precisely the atoms and coatoms of $\mathcal{L}(S)$ contained in C_y. On the other hand, the 0-cells of the facet in $E_t(S)$ defined by C_y are the 0-cells of $\mathcal{B}\mathrm{S}(S)$ coming from $(d-1)$-cells that contain σ_y. Hence the vertex–facet incidences of the two lattices $\mathcal{L}(E_t(S))$ and $E_t(\mathcal{L}(S))$ coincide. □

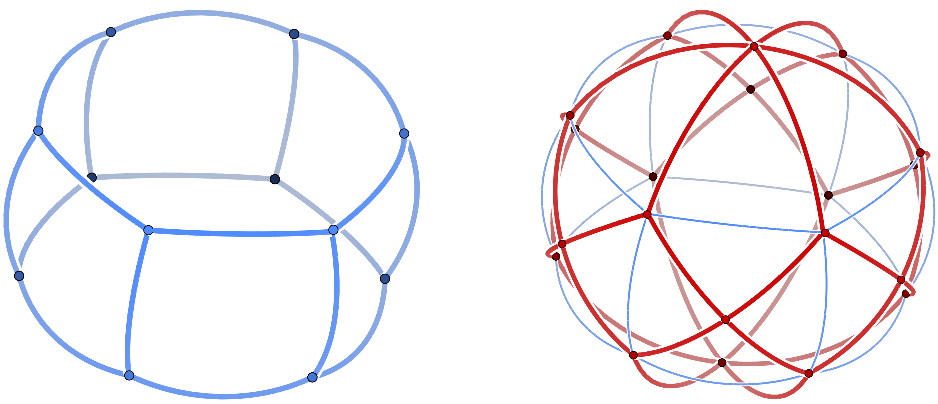


Figure 2.10: An example for the E-construction on spheres.

With this correspondence to the combinatorial setting we can restate the properties about simplicity and simpliciality of lattices obtained from the E-construction in a geometric language.

2.4.5 T . *Let S be a PL sphere of dimension $d-1$ and $1 \le t \le d-2$.*

(1a) For $0 \le k \le d-2$, $E_t(S)$ is k-simplicial if S is r-simple and s-simplicial for $r \ge \min(k, d-t-2)$ and $s \ge \min(k, t-1)$.
(1b) $E_t(S)$ is never $(d-1)$-simplicial.
(2a) $E_t(S)$ is 2-simple if and only if any $(t+2)$-cell is 3-simple.
(2b) $E_t(S)$ is never 3-simple.

Proof. This is a direct consequence of Theorem 2.4.2 and Theorem 2.3.11. □

The most important application of this are the following two possibilities to create a 2-simple and $(d-2)$-simplicial $(d-1)$-sphere. For the rest of this chapter and the whole next chapter we will work mainly with these two cases. The proof is immediate from the corresponding statement for lattices in Corollary 2.3.13. Recall, that a (r, s)-sphere is an r-simple and s-simplicial PL $(r+s-1)$-sphere.

2.4.6 C [(2, $d-2$)-Spheres]. *Let $d \ge 3$ and S any strongly regular $(d-1)$-dimensional PL sphere S.*

(1) If S is simplicial, then $E_{d-3}(S)$ is a $(2, d-2)$-sphere.
(2) Similarly, if S is simple, then $E_2(S)$ is a $(2, d-2)$-sphere. □

2.5 T E-C P

The boundary of a polytope naturally carries the structure of a strongly regular PL sphere. Hence, we can apply the E-construction of PL spheres — as defined in Section 2.4.2 — to any d-polytope P and obtain a new strongly regular PL sphere $E_t(P)$ for any $0 \le t \le d-1$.

However, it is not clear whether this sphere is polytopal, that is, whether the E-construction applied to a polytope P produces only a PL sphere or – at least in some cases – a polytope. The problem hereby clearly arises in the second step of Definition 2.4.3, where we have to merge certain cells of the sphere into a single new cell. To obtain a polytope in this step we have to ensure that the hyperplanes defined by the facets we want to merge (i.e. facets of the barycentric subdivision of P containing the same t-cell) can be deformed in such a way that they geometrically coincide, without changing the combinatorial properties of the sphere.

In the three parts of this section we present techniques that guarantee, for certain interesting classes of polytopes, that their E-construction has a polytopal realisation. We construct several infinite families of $(2, 2)$-polytopes. Lots of further examples are given in Chapter 4.

The first of our constructions, the *D-construction* or *vertex truncation*, yields polytopes (PL spheres) that are dual to the polytopes (PL spheres) obtained from the E-construction. It basically operates by truncating vertices of the polytope in a suitable way. This — surprisingly simple — construction produces the first infinite series of *rational* $(2, 2)$-polytopes.

Next, we introduce a construction that is, on the level of face lattices and for certain classes of polytopes, dual to the D-construction, and produces realisations of $E_{d-2}(P)$. However, the two constructions differ in their geometric applicability. In many cases it is difficult to construct a polytopal realisation of the sphere obtained from the construction with one of these two and rather simple with the other. This second construction will be our main tool for the realisations of $(2, 2)$-polytopes in Chapter 4.

The third construction is a direct extension of the original E-construction of Eppstein, Kuperberg, and Ziegler. It uses polytopes that have their t-faces tangent to the unit sphere as input. It applies to d-polytopes in any dimension $d \geq 4$. With its help, we are able to produce the first infinite family of $(2, d-2)$-polytopes for all $d \geq 4$.

2.5.1 A C V T

For this construction we need a special version of the vertex truncation of a vertex of a polytope P as previously described in Definition 1.3.34. The polytopes $D_1(P)$, which can be produced with this construction, are — if they are geometrically realisable — dual to $E_1(P)$.

Variants of this construction already turned up previously in the literature, in connection with the construction of regular and semi-regular polytopes from other such polytopes. Gosset [42] used it for the construction of the regular 24-cell from a regular cube. A slightly more general version of Gosset's approach appears in Coxeter's book on regular polytopes [30, pp. 145–164]. Vertex cutting techniques also appear in other parts of this book. Yet another variant is in Gévay's construction of Kepler polytopes [35].

2.5.1 D [Truncatable Polytope]. A d-polytope P is called *truncatable* if all its vertices can be truncated simultaneously and in such a way, that only one single (relatively interior) point of each edge of P remains.
The resulting *truncated polytope* is denoted by $D_1(P)$.

See Figure 2.11 for an example. The connection with the E-construction of the previous sections is given by the following theorem.

2.5.2 P [*D* is dual to *E*]. *Let P be a truncatable d-polytope and $d \geq 3$. Then $D_1(P)^\Delta$ is a polytopal realisation of $E_1(P)$.*

Proof. The polytope $D_1(P)$ has two types of facets:

(1) The facets F' obtained by vertex truncation from the facets F of P, and
(2) the "new" facets F_v obtained from truncating a vertex v of P.

We have used the condition $d \geq 3$ in the first item, as for $d = 2$ these facets would shrink to a point.

The intersection of any two new facets F_v and F_w is empty if v and w are not adjacent in P. Otherwise, the intersection $F_v \cap F_w$ consists of one single point, which is the new vertex u_e given by the edge $e = (v, w)$. A vertex u_e lies on a new facet $\overline{F}$ of the second type if and only if e is adjacent to F in P.

To check that $D_1(P)$ has the right combinatorics it suffices to check the vertex-facet incidences, by Theorem 1.2.24:

(1) A vertex u_e lies on F' if and only if e is an edge of F, and
(2) u_e lies on F_v if and only if e is adjacent to v.

This, however, is a description of the reversed atom–coatom incidences of $E_1(P)$, where we have a facet for every edge of P, and two such facets share a vertex if and only if

(1) the corresponding edges share a vertex or
(2) they are adjacent to a common facet in P. □

2.5.3 R . More general — and in the same fashion as the definition of D_1 — one could define, for a k between 2 and $d - 1$, the operator D_k assigning to a polytope P the new polytope $D_k(P)$ whose vertices are truncated in such a way that of any k-face only one relatively interior point remains. Again, for arbitrary polytopes, this need not result in a polytope with the expected combinatorics. If it

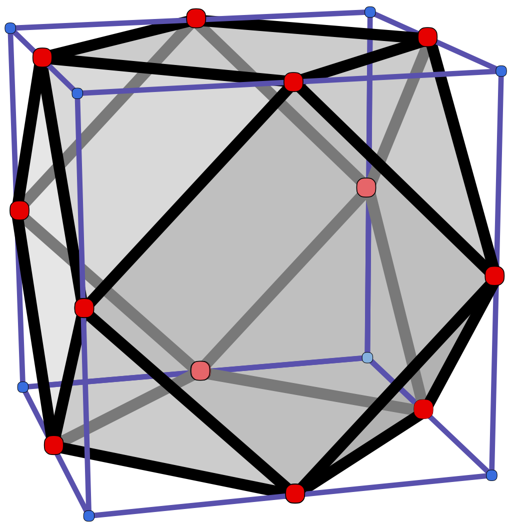


Figure 2.11: The truncation of $\square_3$ gives $D_1(\square_3)$ (which is the cuboctahedron).

does, we call the polytope P k-truncatable and denote the new polytope by $D_k(P)$. D_k produces a polytope which is dual to the one obtained by E_k.

We do not put this operator into a separate definition, as we only use this once to make a connection to some previously known examples in Chapter 4. Otherwise, we do not know of any useful new applications or constructions for this operator. D_k clearly applies to regular polytopes, and in this special setting, it appears in Coxeter's book [30].

The operator D_k has the same symmetry as E_k. So if P is k truncatable and its dual P^Δ is $(d-k-1)$-truncatable then $D_k(P) \cong D_{d-k-1}(P^\Delta)$.

Also, the operator D_k acts on the facets of a polytope in the same way as on the polytope itself: The facet of $D_k(P)$ coming from a facet F of P is $D_k(F)$. However, $D_k(P)$ has also facets coming from the vertices, which are not of this type.

In the following, we will usually omit the index in the operator and write $D(P)$ instead of $D_1(P)$.

2.5.4 E . Here are some simple examples of truncatable polytopes.

(1) The simplex Δ_d for $d \geq 3$ is truncatable. Its truncation is the well known *hypersimplex* K_2^d, which we examine in more detail in Section 4.3.1. See Figure 2.12 for a three dimensional illustration.

(2) The truncation of the four dimensional regular cross polytope ✚$_4$ yields the regular 24-cell. A Schlegel diagram of this is shown in Figure 1.16.

(3) More generally, all simple polytopes are truncatable: Any d points in general position define a unique hyperplane in $\mathbb{R}^d$. Thus, the convex hull of any choice of one interior point on each of the edges of the polytope is a realisation of the truncated polytope.

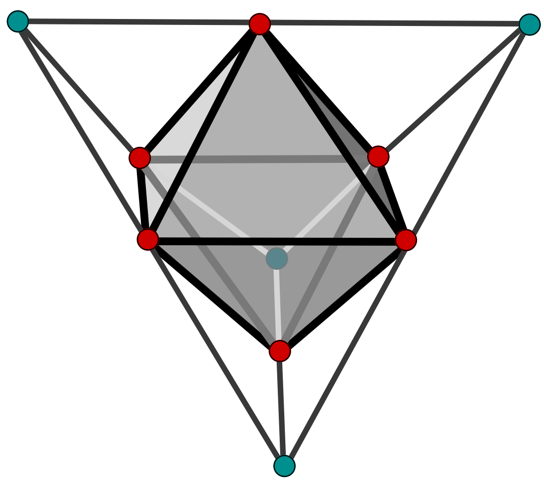


Figure 2.12: The three dimensional hypersimplex.

In the rest of this section we present classes of truncatable polytopes with different properties. We make the following definition to simplify the statements.

2.5.5 D [Edge Realisation]. Let P be a d-polytope. A geometric realisation of P, in which all edges are tangent to a (suitably centred and scaled) $(d-1)$-sphere, is called an *edge realisation* of the polytope.
A polytope that has such such a realisation is called *edge realisable*.

The next proposition shows that such polytopes are always truncatable. A similar, but more general, concept will turn up again in Section 2.5.3.

2.5.6 P [Edge Realisable Polytopes are Truncatable]. *Let P be an edge realisable d-polytope in dimension $d \geq 3$. Then it is truncatable.*

Proof. Let S be the unit sphere touching all edges of P. Choose a vertex v of P and let $e_1, \ldots, e_k$ be the incident edges of P. See Figure 2.13 for an illustration of the proof.

The cone with apex v touching S intersects S in a $(d-2)$-sphere S'. The hyperplane defined by S' contains all points in which the edges $e_1, \ldots, e_k$ touch S. Thus, the convex hull of all points in which the edges of P touch the sphere S defines the vertex truncation of P. □

We know from the Koebe–Andreev–Thurston Circle Packing Theorem (see [89, Theorem 4.12]) that any 3-polytope has an edge realisation. Hence, by Theorem 2.5.6 any 3-polytope is truncatable. However, this is not anymore true in higher dimensions. Most d-polytopes for $d \geq 4$ do not have an edge tangent geometric realisation, and, in general, it is unclear how to find such a realisation, if it exists. See also Section 4.3.4.

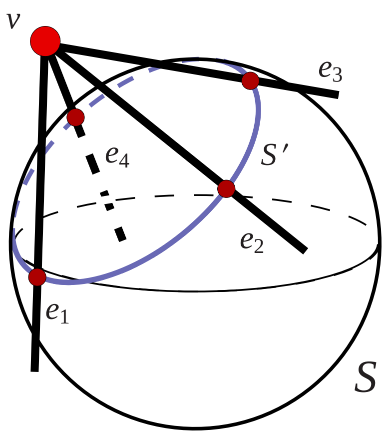


Figure 2.13: Edge tangent polytopes are truncatable.

So here is a more powerful class of truncatable polytopes, which does not need edge tangency for the realisation. Recall the definition of a stacked polytope from Definition 1.3.33.

2.5.7 T [Stacked Polytopes are Truncatable]. *Let P be a stacked polytope. Then P is truncatable.*

Proof. By definition, any stacked polytope is obtained from the simplex Δ_d by successively placing a new vertex beyond a facet and taking the convex hull.

In the following, let P_n denote a combinatorially defined n-times stacked polytope. So P_0 is the simplex. We build up a geometric realisation in the course of the proof. The actual choice of the facets that are stacked in this process, and the order in which we do this, is not important, so P_n for $n \geq 3$ may denote many combinatorially different polytopes.

We prove the theorem by induction over n. The d-simplex P_0 is truncatable by the previous proposition. So assume that P_n is truncatable and let $H_1, \ldots, H_{d+n+1}$ be the sequence of hyperplanes we have used for the truncation of the vertices of P_n to obtain $D(P_n)$.

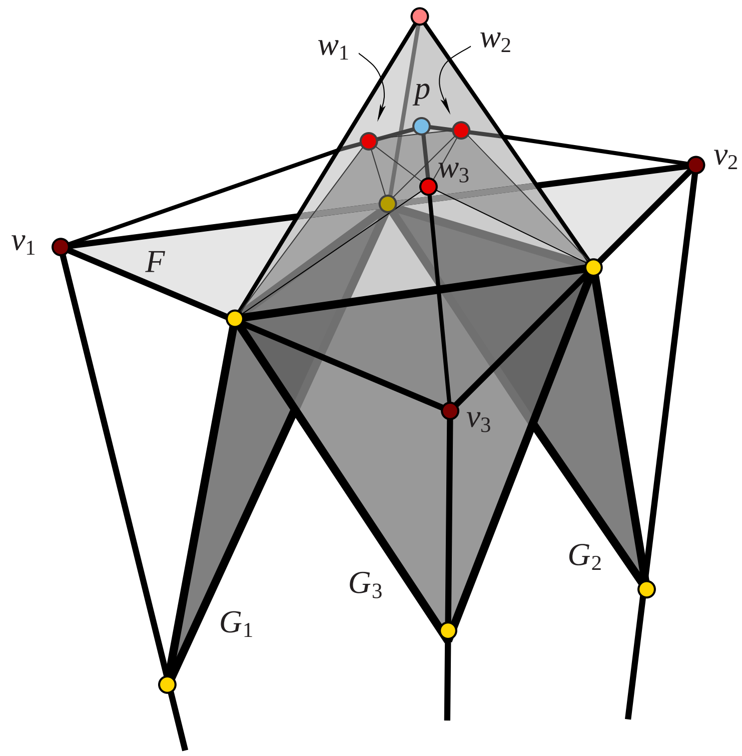


Figure 2.14: Truncation of a stacked polytope.

Choose a facet F of P_n. F is a $(d-1)$-simplex, with vertices $v_1, v_2, \dots, v_d$. See Figure 2.14 for an illustration. Precisely those d among the H_j, $1 \le j \le d+n+1$ intersect F, that truncate the vertices v_k, $1 \le k \le d$ of F. Denote this subsequence of hyperplanes by $G_1, G_2 \dots, G_d$.

Choose a new vertex p beyond the facet F and beneath all hyperplanes G_j, $1 \le j \le d$. Let P_{n+1} be the convex hull of P_n and p. The vertex p is adjacent to d edges connecting it to $v_1, v_2, \dots, v_d$. These edges intersect $G_1, \dots, G_d$ in d inner points $w_1, \dots, w_d$.

By construction, the d points $w_1, \dots, w_d$ on the edges are affinely independent. Hence, there is a unique affine hyperplane H_{d+n+2} that contains them. This hyperplane separates p from all other vertices of P_{n+1}, so it is a valid hyperplane for truncating the vertex p. Hence, the set $H_1, \dots, H_{d+n+2}$ of $d+n+2$ hyperplanes defines a vertex truncation of P_{n+1} leading to $D(P_{n+1})$. □

2.5.8 R . It is not necessary to start with a simplex in this procedure. We can start with any simplicial polytope P and a vertex truncation $D(P)$. In the same way as in the proof of the theorem, we can stack a facet F of this polytope and obtain a vertex truncation of $F \backslash P$. It even suffices that only the facet F that we stack is simplicial. We use this in the dual construction presented in the next section.

We have a lot of freedom for the choice of the new vertex p of P_{n+1} in the proof of the last theorem. In particular, we can choose it in such a way that is has rational coordinates. Hence, we can draw the following conclusion.

2.5.9 C [Rational Truncated Polytopes]. *There are infinitely many combinatorially distinct rational truncated polytopes in any dimension $d \ge 3$.*

Proof. The intersection point of a hyperplane in $\mathbb{R}^d$ with a line is the solution of a linear system of equations. Hence, it has rational coordinates, if the hyperplane is the level set of a rational normal vector at a rational level, and the line contains at least to two rational points. A hyperplane defined by d rational points can be represented by a rational normal vector and level.

Clearly, the simplex has a rational realisation and a rational truncation. The subset of $\mathbb{R}^d$ enclosed by F and $G_1, \dots, G_d$ used in the previous proof is open, so we can choose the new vertex p with rational coordinates. Thus, the d new vertices of the truncation also have rational coordinates. □

For the rest of the section we specialise to four dimensional polytopes, as these are the most interesting ones for this construction. By Proposition 2.5.2, the polytope $D(P)$ is dual to $E_1(P)$. Hence, in the case of 4-polytopes, Corollary 2.4.6 has the following immediate consequence.

2.5.10 C [(2, 2)-Polytopes]. *Let P be a simplicial truncatable 4-polytope. Then $D(P)$ is 2-simple and 2-simplicial.* □

Combining this Corollary with the previous one, Corollary 2.5.9, we conclude the following nice fact.

2.5.11 C [*D* of Stacked 4-Polytopes]. *There are infinitely many* rational 2*-simple and* 2*-simplicial* 4*-polytopes* $D(P^4_n)$*. The (essential) flag vector of these polytopes is given by*

$$\mathrm{flag}(D(P^4_n)) = (10+4n, 30+18n, 30+18n, 10+4n; 50+26n). \qquad \square$$

There are, for any fixed $n \geq 3$, many different combinatorial types of such polytopes with the same flag vector. In fact, we prove in Proposition 4.2.2, that the number of such polytopes grows exponentially in n. In Table 4.3 is an enumeration of the different combinatorial types of such polytopes for $n \leq 8$.

Only the first two instances of this family of polytopes were previously known.

- $n = 0$ produces the hypersimplex. Five of its facets are simplices and five facets are cross polytopes. The primer arise from the truncated vertices, and the latter are the truncated facets of the simplex. See Figure 2.12 for a three dimensional version.
- $n = 1$ results in a polytope first described by Braden [27] via a gluing of two cross polytopes. The coordinates of a rational geometric realisation and the corresponding Schlegel diagram are shown in Figure 2.15.

$$\begin{array}{c}
\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \\
\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} -1 & -1 & -1 & 1/3 \end{bmatrix} \\
\begin{bmatrix} -1 & -1 & 1/3 & -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1/3 & -1 & -1 \end{bmatrix} \\
\begin{bmatrix} 1/3 & -1 & -1 & -1 \end{bmatrix} \\
\begin{bmatrix} 5/3 & 1/3 & 1/3 & 1/3 \end{bmatrix} \\
\begin{bmatrix} 1/3 & 5/3 & 1/3 & 1/3 \end{bmatrix} \\
\begin{bmatrix} 1/3 & 1/3 & 5/3 & 1/3 \end{bmatrix} \\
\begin{bmatrix} 1/3 & 1/3 & 1/3 & 5/3 \end{bmatrix}
\end{array}$$

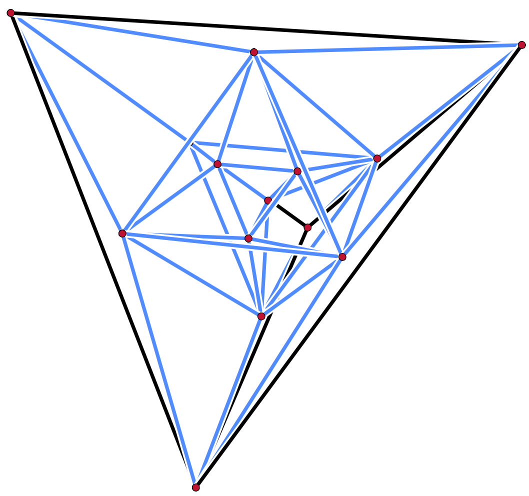


Figure 2.15: Coordinates and Schlegel diagram of a realisation of Braden's polytope $D(P^4_1)$.

Corollary 2.5.11 is a true generalisation of Proposition 2.5.6 and the original E-construction of Eppstein, Kuperberg, and Ziegler [33]: They show that a polytope of type P_n^d, i.e. a n times stacked d-polytope, has an edge realisation if and only if $n \leq 1$ [33, Prop. 8]. See also Section 4.3.4.

A similar infinite sequence of rational 2-simple and 2-simplicial 4-polytopes can be obtained from a stack of $n \geq 1$ cross polytopes. Using suitable coordinates, we obtain another family of $(2, 2)$-polytopes with *rational* coordinates, and the symmetries of a regular 3-simplex.

2.5.12 P [Glued Cross Polytopes]. *There is an infinite sequence of rational* 2*-simplicial and* 2*-simple* 4*-polytopes* $D(C_n^4)$ *for* $n \geq 1$ *with essential flag vectors*

$$\mathrm{flag}(D(C_n^4)) = (6 + 18n, 12 + 84n, 12 + 84n, 6 + 18n; 24 + 120n).$$

Proof. An illustration of the construction is given in Figures 2.16 and 2.17. Start with a *regular* cross polytope C_1 and place it in such a way that one pair of opposite facets F_0 and F_1 (which are simplices) have normal vectors $\mp[1, 1, 1, 1]$. C_1 is clearly truncatable, by symmetry.

Let $H_1, \ldots, H_4$ be the hyperplanes that truncate the vertices of F_1 in C_1. Then F_1 together with the H_i, $1 \leq i \leq 4$ encloses a subset S_1 of $\mathbb{R}^4$ with nonempty interior. Let l be the line defined by the vector $[1, 1, 1, 1]$ and running through the origin. l intersects S_1 by construction.

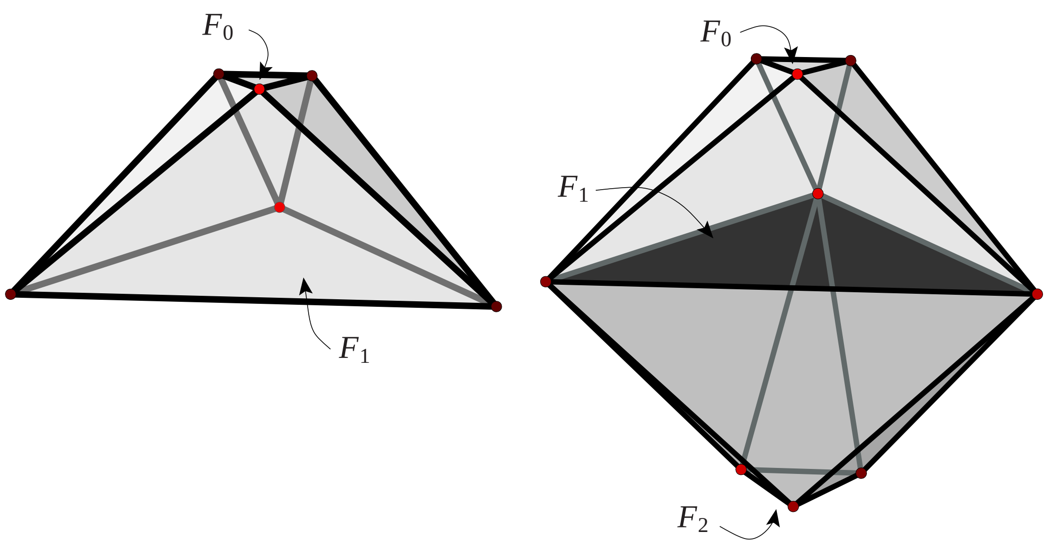


Figure 2.16: Construction of $D(C_2^3)$: The left figure shows C_1^3; the right figure is C_2^3. The axis $[1, 1, 1]$ is pointing upwards in this drawing.

Pick a point p on l in the interior of S_1. Let F_0' be a copy of F_0 translated along l and containing p. As S_1 has nonempty interior, there is a $\lambda > 0$ such that $F_2 := \lambda F_0'$ is completely contained in S_1. The convex hull C_2' of F_0 and F_2 is a cross polytope.

We can extend the truncation of the vertices of F_1 by $H_1, \ldots, H_4$ to a truncation of all vertices of C_2' by choosing four new hyperplanes $G_1, \ldots, G_4$ truncating the vertices of F_2. Clearly, this is possible in our symmetric setting. The two cross polytopes C_1 and C_2' coincide in F_1, so their common convex hull is a stack C_2 of two cross polytopes glued along F_1. The truncation of C_1 together with $G_1, \ldots, G_4$ defines a valid truncation of C_2.

We continue iteratively by considering the subset of $\mathbb{R}^4$ defined by F_2 and $G_1, \ldots, G_4$ and placing a suitably scaled and translated copy F_3 of F_1 inside this set, taking the convex hull with C_2 and choosing the new truncation hyperplanes. $\square$

Explicit coordinates for these polytopes $D(C_n^4)$ are most easily obtained by using a projective transformation for translating and scaling the F_j in the proof. Clearly, it is possible to choose *rational* coordinates for the added vertices in this construction, so we have another infinite family of *rational* $(2, 2)$-polytopes.

The same construction works in all dimensions $d \geq 3$ (we did the illustration in $d = 3$!). However, we restricted the statement of the proposition to $d = 4$, as the obtained polytopes are $(2, 2)$-polytopes only in this case.

There are clearly many more classes of polytopes to which the D-construction applies. We will meet some other such in Chapter 4.

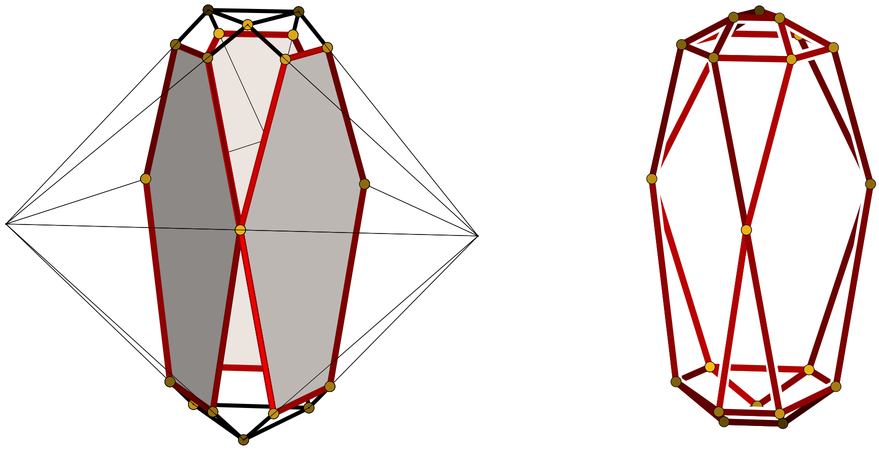


Figure 2.17: Construction of $D(C_2^3)$: The left figure is obtained when three vertices have been truncated; the right figure displays $D(C_2^3)$.

2.5.2 A S R V T

We present a translation of the concept of vertex truncation defined in the previous section from simplicial polytopes to simple polytopes. We already proved that for a stacked polytope P, if we have a realisation of $D(P)$ then we can stack any facet F of P to obtain $F \backslash P$ and construct a realisation of $D(F \backslash P)$. Here we show the dual: If we have a realisation of $E_{d-2}(P)$ for a simple polytope P, then we can truncate a vertex v of P and construct a realisation of $E_{d-2}(\mathrm{tr}(P; v))$.

This will prove quite useful for the task of producing realisations of the E-construction for some polytopes out of other already realised polytopes. We use this for the generation of our table of $(2, 2)$-polytopes in Chapter 4.

Let P be a polytope and v a vertex of P. The polytope $\mathrm{tr}(P; v)$ is obtained from P by intersecting P with a half space that contains all vertices of P in its interior, except the vertex v. See also Definition 1.3.34.

Here is a way to obtain a polytopal realisation of the E-construction of $\mathrm{tr}(P; v)$ in the case $t = d - 2$.

2.5.13 P [Truncation Preserves Realisability]. *Let P be a polytope and v a simple vertex of P. Assume that we have a realisation of $E_{d-2}(P)$ with the following property.*

(V) The vertex set of $E_{d-2}(P)$ splits into the vertex set $\mathcal{V}(P)$ of P and a set $\mathcal{V}_e$ that contains, for each facet F of P, a unique vertex beyond beyond F.

Then the PL sphere $E_{d-2}(tr(P; v))$ is polytopal. An explicit realisation of it can be computed from the realisation of $E_{d-2}(P)$.

Proof. The proof is quite similar to the one of Theorem 2.5.7. Suppose that P is a polytope having a simple vertex v that satisfies the condition of the theorem.

The facets of $E_{d-2}(P)$ are bipyramids over the ridges of P, so we can find a half space H that touches $E_{d-2}(P)$ only in v. We can perturb H slightly, so that H contains the vertex v in its interior, but all other vertices still lie outside. The boundary of H is a valid truncation hyperplane for v in P.

As v is a simple vertex, the facet F_v created by the truncation is a simplex. Let $F_1, \ldots, F_d$ be the adjacent facets of F_d in P and $v_1, \ldots, v_d$ the vertices of $E_{d-2}(P)$ beyond these. Let H_i be the hyperplane defined by v_i and the $d-1$ vertices contained in F_i, for $1 \le i \le d$ (v_i and the vertices of F_i are clearly affinely independent). $H_1 \ldots, H_d$ are affinely independent, so they intersect in a single point w.

Adding $H_1, \ldots, H_d$ to the hyperplane description of $E_{d-2}(P)$ defines a hyperplane description of $E_{d-2}(\mathrm{tr}(P; v))$. It creates the vertex w beyond the facet F_v. $\square$

2.5.3 A C H G

Here is another method to obtain polytopal realisations of the E-construction. We extend the original E-construction of Eppstein, Kuperberg, and Ziegler [33] for edge tangent simplicial 4-polytopes to all dimensions $d \geq 3$. The goal of this section are the two Theorems 2.5.14 and 2.5.15.

The first theorem shows that we obtain polytopal realisations of the E-construction for a fixed parameter t from polytopes that have their t-faces tangent to the unit sphere. For us, the interesting application of this theorem is that to a simplicial d-polytope P that has its $(d-3)$-faces tangent to the unit sphere. We obtain a polytopal realisation of $E_{d-3}(P)$. Corollary 2.4.6 tells us that this is a $(2, d-2)$-polytope, i.e. a 2-simple and $(d-2)$-simplicial d-polytope.

Hence, to obtain an infinite family of $(2, d-2)$-polytopes, we need an infinite family of simplicial d-polytopes that have their $(d-3)$-faces tangent to the unit sphere. The second theorem constructs such a family. Similarly to the proof of the original four dimensional version we use arguments from hyperbolic geometry for this. The necessary background is contained in Section 1.5.

There are more polytopes that Theorem 2.5.14 applies to then just the family given by Theorem 2.5.15. For the special case of simplicial 4-polytopes, Eppstein, Kuperberg, and Ziegler have collected several other instances. We give an overview of these in Section 4.3.4. Observe however, that edge tangent polytopes are easy to obtain in dimension three, but it is hard to come by with such polytopes in all higher dimensions.

2.5.14 T [t-Tangent Realisations]. *Let P be a d-polytope and t a parameter between 0 and $d-1$. Assume that P is realised such that its t-faces are tangent to the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. If $P^\diamond$ denotes the polar polytope of P in this realisation, then $Q := \operatorname{conv}(P \cup P^\diamond)$ is a realisation of $E_t(P)$.*

Recall the difference between polar and dual polytope in our notation. The dual polytope is just a polytope with the opposite face lattice, while the polar polytope is a special geometric realisation of the dual polytope obtained via the construction defined in equation (1.3.1).

The main application of this theorem is the following family of 4-polytopes, which is obtained from a certain way to stack cross polytopes and glue triples of simplices to all non-convex ridges of this stack.

2.5.15 T [Infinitely Many $(d-2, 2)$-Polytopes]. *For every $d \geq 3$ there are infinitely many combinatorially distinct 2-simple and $(d-2)$-simplicial d-polytopes.*

The theorem is trivial in dimension three as there are infinitely many simple 3-polytopes. Hence, we restrict to $d \geq 4$ for the proof.

Proof of Theorem 2.5.14. If $t = 0$, then the vertices of the polytope P lie on the unit sphere. Hence, the vertices lie in the facets of the polar polytope $P^\diamond$ and they vanish in Q. So $Q \cong P^\diamond$. Similarly, if $t = d-1$, then all facets are tangent to the sphere, and these contain the vertices of the polar. So Q is equivalent to P.

By Examples 2.3.3 we know that $E_0(P) = P^\Delta$ and $E_{d-1}(P) = P$, so the theorem is true for $t = 0$ and $t = d-1$. Hence, we can assume $1 \leq t \leq d-2$ for the rest of the proof.

Let F be a t-face of P, touching the unit sphere in a point p. Let $T_p\mathbb{S}^{d-1}$ be the tangent space of $\mathbb{S}^{d-1}$ in p. $T_p\mathbb{S}^{d-1}$ contains F. If $v_1, \ldots, v_k$ are the vertices of F, then the polar face $F^\diamond$ of F is the set of all points x with $\left\langle v_j, x\right\rangle = 1$ for all $1 \leq j \leq k$. All points of F are positive linear combinations of the v_i with coefficients summing up to one. So in particular $\langle p, x\rangle = 1$ and $\left\langle v_j - p, x\right\rangle = 0$ for all $x \in F^\diamond$. Thus, $F^\diamond$ is contained in $T_p\mathbb{S}^{d-1}$ and orthogonal to F. As $\dim(F) + \dim(F^\diamond) = d-1$, their convex hull $B(F)$ spans $T_p\mathbb{S}^{d-1}$. So the set of all facets of Q contains all orthogonal sums $\operatorname{conv}(F \cup F^\diamond)$ of t-faces of P. The vertices of $B(F)$ are the vertices of F and the vertices of $F^\diamond$. The latter correspond to the facets of P containing F.

We have to check that all facets of Q are of this type. For this we show that any facet of Q sharing a ridge with a facet $B(F)$ for some t-face F of P is again of this type for some other t-face F' of P.

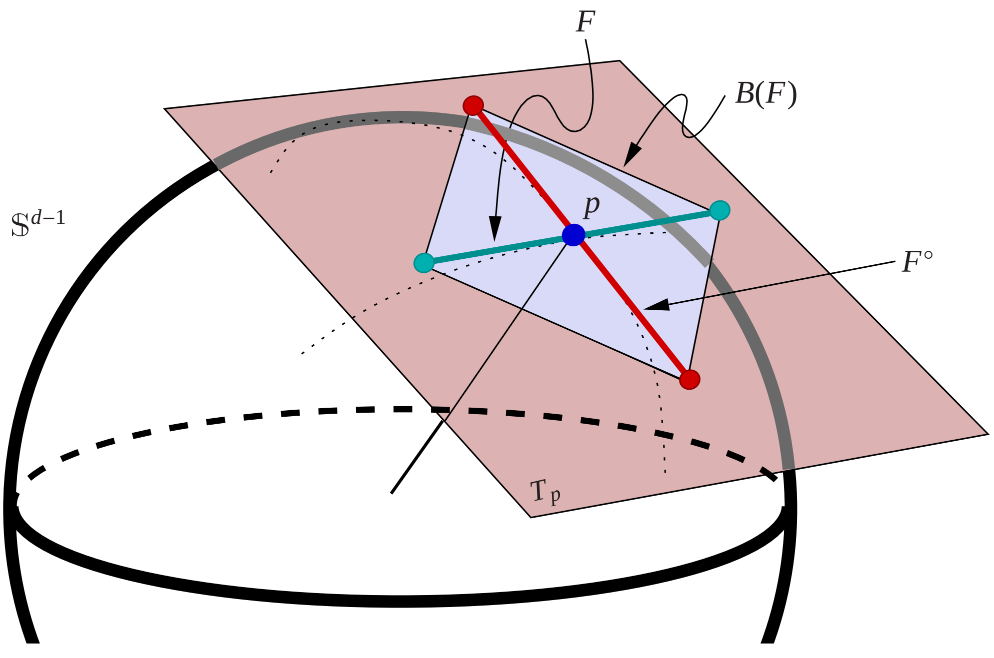


Figure 2.18: A t-face and its polar.

Any ridge R of Q contained in $B(F)$ is a facet of $B(F)$, so it is the convex hull of a facet G of F and a facet H of $F^\diamond$. G is a $(t-1)$-face, and $H^\diamond$ is a $(t+1)$-face of P. G is covered by F and $H^\diamond$ covers F in the face lattice of P. As this lattice is Eulerian, there is exactly one other t-face F' of P covered by $H^\diamond$ and covering G. $B(F')$ is the desired other facet of Q that contains the ridge R.

Thus, Q has a facet $B(F)$ for any t-face F of P, no other facets, and the same vertex-facet-incidences as $E_t(P)$. □

We want to use this theorem for the proof of Theorem 2.5.15. To do this, we have to construct an infinite family of simplicial d-polytopes whose $(d-3)$-faces are tangent to the unit sphere. We make the following definition to simplify the notation.

2.5.16 D [T^d-polytope]. A T^d-polytope is a d-polytope that has all its $(d-3)$-faces tangent to the unit sphere $\mathbb{S}^{d-1}$.

We view the interior $\mathbb{D}^d$ of the unit $(d-1)$-sphere $\mathbb{S}^{(d-1)}$ as hyperbolic space equipped with the hyperbolic metric in the Klein model $\mathbb{K}$. See Section 1.5.2 for the definitions and properties. The sphere $\mathbb{S}^{d-1}$ becomes the sphere at infinity $\mathbb{S}^{d-1}_\infty$. In the Klein model model, hyperbolic affine hyperplanes are the intersection of $\mathbb{D}^d$ with Euclidean affine hyperplanes.

The facets and ridges of a T^d-polytope P properly intersect $\mathbb{K}^d$, the $(d-3)$-faces touch the sphere at infinity, and all lower dimensional faces lie completely outside. The intersection $P^{\text{hyp}} := P \cap \mathbb{K}^d$ is therefore a convex unbounded hyperbolic polyhedron in $\mathbb{K}^d$.

Here is an important caveat: A hyperideal hyperbolic object – even a convex polytope – can be unfavourably positioned in such a way that it is unbounded as an Euclidean object (cf. [76, p. 508]). However, we have the following lemma, which we cite from [33, Lemma 6] in a version generalised to our situation (see also Springborn [78]):

2.5.17 L . *For any convex T^d-polytope P whose points of tangency do not lie in a hyperplane, there is a hyperbolic isometry h whose extension to $\mathbb{R}^d$ maps P into a bounded position.* □

For some special T^d-polytopes we will now compute the hyperbolic dihedral angle between adjacent facets. They form the basic building blocks for our infinite family of T^d-polytopes.

As facets and ridges of a T^d-polytope do at least partially lie inside $\mathbb{K}^d$, the dihedral angle is well defined as a hyperbolic angle between facets of the hyperbolic polytope P^{hyp}. By convexity, it must be strictly between 0 and π.

Clearly, the regular d-simplex, and the regular d-cross polytope, can be scaled to be T^d-polytopes. We denote their intersection with $\mathbb{K}^d$ by Δ_d^{hyp} and $✚_d^{\text{hyp}}$. The following lemma computes their dihedral angles.

2.5.18 L **[Dihedral Angles].** *The hyperbolic dihedral angles between two adjacent facets in* Δ_d^{hyp} *and* $\boldsymbol{+}_d^{\mathrm{hyp}}$ *are* $\frac{\pi}{3}$ *and* $\frac{\pi}{2}$*, respectively.*

Proof. We compute the hyperbolic dihedral angle between two facets F_1 and F_2 sharing a ridge R by intersecting R with a 2-plane $H' \cong \mathbb{H}^2$ orthogonal to that ridge in $\mathbb{H}^d$. The angle is independent of the intersection point x that we choose on R. Hence, we can in particular choose x as a point of tangency of a $(d-3)$-face E of R. Let $H \cong \mathbb{H}^3$ be the orthogonal complement to E. Then $H \supset H'$.

The intersection of the facets of P adjacent to E with $\mathbb{H}^d$ are hyperplanes in $\mathbb{H}^d$. Thus, the link of E in P is the intersection of a sufficiently small horosphere S based at the point x with the 3-plane H. See Figure 2.19 for an illustration.

The metric on horospheres induced by the hyperbolic metric is flat, by Theorem 1.5.9. Thus, the intersection will be an *Euclidean* polygon whose edges correspond to the facets adjacent to E, and the vertices to the ridges adjacent to E. The dihedral angle of the ridge R is the usual Euclidean angle at the vertex in that polygon corresponding to R.

For a regular hyperbolic simplex this polygon is clearly an equilateral triangle, and for a cross polytope it is a square. This gives the required angles. □

Now we have all necessary tools for the construction of our infinite family of T^d-polytopes in all dimensions $d \geq 4$.

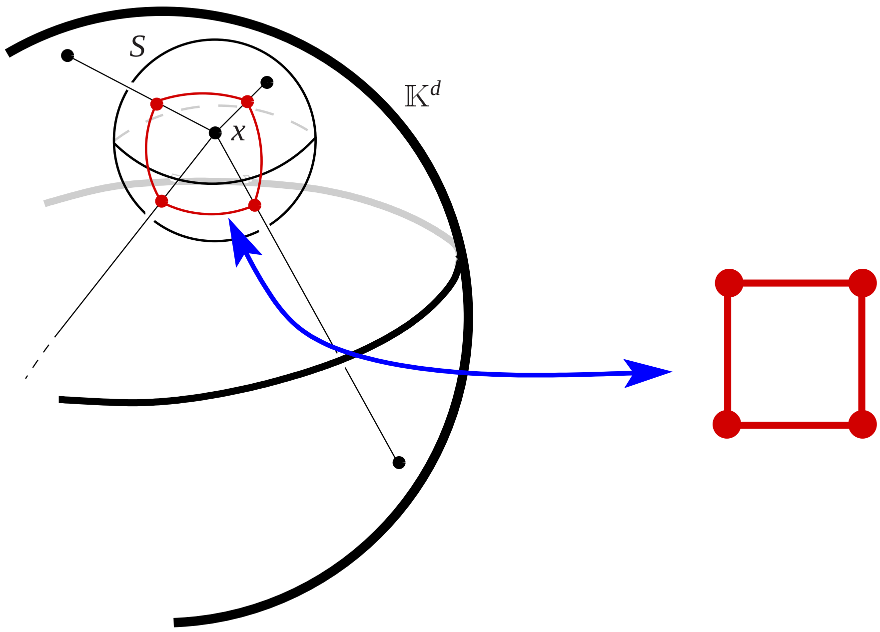


Figure 2.19: A horosphere in the Klein model, and the Euclidean square the facets cut out of it.

Proof of Theorem 2.5.15. The theorem is proven if we can provide an infinite series of T-polytopes by Theorem 2.5.14.

For the construction of this we basically glue copies of $\maltese_d^{\text{hyp}}$ at opposite faces. To do this, we have to position these copies such that the gluing facets coincide and the dihedral angles at the gluing ridges remain between 0 and π.

By Theorem 1.5.6 we can map any hyperbolic hyperplane onto any other hyperbolic hyperplane in $\mathbb{K}^d$. Thus, we can glue any facet of a simplex or cross polytope onto any facet of another simplex or cross polytope. To obtain convex polytopes we only have to care about the angles.

Gluing two cross polytopes creates — according to Lemma 2.5.18 — a dihedral angle of π at all gluing ridges. So constructing a stack C_n^d of n cross polytopes by iteratively gluing a cross polytope onto opposite facets does not produce a polytope with the expected combinatorics.

However, we can remedy this problem in the following way. Take a copy of a simplex and glue two other simplices onto two adjacent facets F and F' sharing a ridge R. See Figure 2.20 for a drawing of this compound. In this glued complex we have three different dihedral angles. At all ridges contained in F or F', except at the ridge R, the dihedral angle is $\frac{2\pi}{3}$. At the ridge R it is π, and at all other ridges it is $\frac{\pi}{3}$.

We glue a copy of this complex to all pairs of facets of our stack of cross polytopes that share a ridge with dihedral angle π in such a way, that R is glued to that ridge. The link of the $(d-3)$-faces looks like the figure shown in Figure 2.21 afterwards. Call the resulting complex SC_n^d.

Now the straight dihedral angles in C_n^d have disappeared in the interior of the

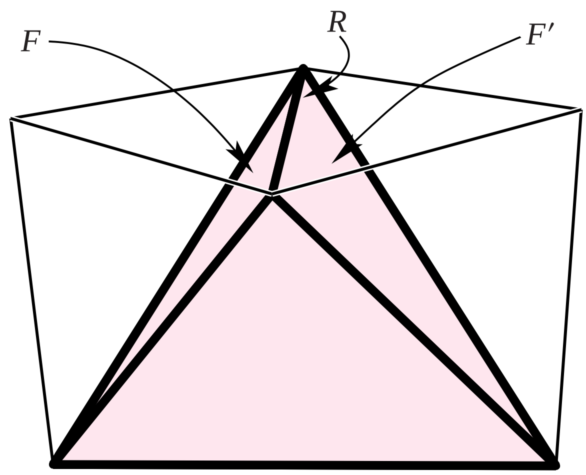


Figure 2.20: Three hyperideal tetrahedra have a bipyramidal facet (the top facet).

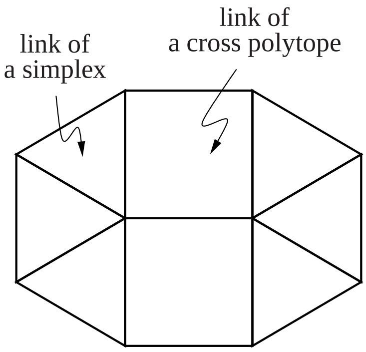


Figure 2.21: The link, with added simplices.

new complex. We have to check that we have not created new ridges with a dihedral angle of π or more. None of the ridges of the triple of simplices where two simplices meet is glued to C_n^d. Also, in C_n^d, facets adjacent to a pair of opposite facets are ridge disjoint if $d \geq 4$ (This is not true for three dimensional cross polytopes, as you can see from Figure 2.22). Thus, any ridge of C_n^d, to which such a triple of simplices is glued, has a dihedral angle of $\frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$ afterwards. So SC_n^d is a T^d-polytope.

The number of vertices of $E_{d-3}(SC_n^d)$ is the sum of the number of vertices and facets of SC_n^d. Thus, the number of vertices of $E_{d-3}(SC_n^d)$ is strictly increasing with n, and instances to different n must be combinatorially different. This finally proves the theorem. □

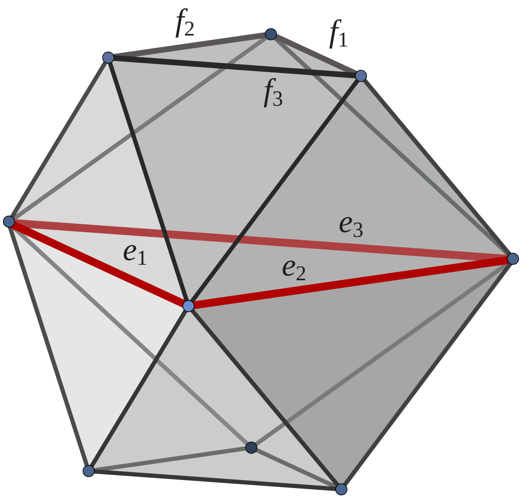


Figure 2.22: The construction fails in dimension three: Observe the three edges e_1, e_2 and e_3 in this stack of two cross polytopes. They have to be filled with a triple of simplices. If we continue by stacking the top facet with another cross polytope, then we have to fill the edges f_1, f_2, and f_3. The triple of simplices glued to f_3 shares an edge with the triples of simplices glued to e_1 and e_2.

$$f_j(SC_n^d) = \begin{cases} 2^d n - 2(n-1) + d(3d-5)(n-1) & \text{for } j = d-1, \\ 2^{d-1}dn - d(n-1) & \\ \quad + \frac{1}{2}d^2(3d-5)(n-1) & \text{for } j = d-2, \\ 2^{j+1}\binom{d}{j+1}n - \binom{d}{j+1}(n-1) & \\ \quad + d\{3\binom{d}{j} - \binom{d-1}{j}\}(n-1) & \text{for } 0 \le j \le d-2. \end{cases}$$

$$f_k(E_{d-3}(SC_n^d)) = \begin{cases} f_{d-3}(SC_n^d) & \text{for } k = d-1, \\ \binom{d-1}{2} f_{d-2}(SC_n^d) & \text{for } k = d-2, \\ \binom{d-1}{3} f_{d-2}(SC_n^d) + \binom{d}{3} f_{d-1}(SC_n^d) & \text{for } k = d-3, \\ \binom{d-1}{d-k} f_{d-2}(SC_n^d) & \\ \quad + \binom{d}{d-k} f_{d-1}(SC_n^d) + f_k(SC_n^d) & \text{for } 1 \le k \le d-4, \\ f_{d-1}(SC_n^d) + f_0(SC_n^d) & \text{for } k = 0. \end{cases}$$

Table 2.1: The flag vectors for $n \geq 1$ and — in the case of $E_{d-3}(SC_n^d)$ — dimension $d \geq 4$.

We can compute the f-vector of the polytopes SC_n^d and $E_{d-3}(SC_n^d)$, which is a rather tedious task. They are shown in Table 2.1. For $d = 4$ this formula indeed specialises to the f-vectors $(54n - 30, 252n - 156, 252n - 156, 54n - 30)$ of the 4-dimensional examples that were already constructed and computed in [33].

The coordinates obtained in this proof have non-rational coordinates, and there seems not to be an easy way to remedy this. So the problem of constructing an infinite family of *rational* 2-simple and $(d-2)$-simplicial d-polytopes for $d \geq 5$ remains open.

The family constructed in the previous proof has some similarity with the one of Proposition 2.5.12 obtained by vertex truncation. Both contain a stack of cross polytopes, but in the one of Proposition 2.5.12 it was not necessary to cover the ridges of the intersection of two cross polytopes with simplices, as we did not need an edge tangent realisation. Thus, we had much more freedom for the choice of coordinates of the stack of cross polytopes.

C 3

P P

3.1 I

This chapter is mainly devoted to a two parameter family E_{mn} of $(2,2)$-polytopes obtained from the E-construction. The family is a special case of a quite general method to obtain polytopal realisations for the E-construction applied to a product of two polytopes, if we have already a polytopal realisation of the E-construction of the two factors.

We state the general method in Theorem 3.3.1. The proof is constructive, that is, we provide an explicit way to obtain a polytopal realisation from the realisations of the two factors. If we ask for some additional property in the geometric realisation of the product and its E-construction, then the conditions given in the theorem are both necessary and sufficient.

We illustrate this with examples in all dimensions $d \geq 4$. However, the main application of the theorem is the construction of an infinite two parameter family of $(2,2)$-polytopes E_{mn}, for $m,n \geq 3$. These E_{mn} are obtained from the E-construction applied to a product of two polygons C_m and C_n with m and n vertices. We prove in Theorem 3.4.1, that such a product satisfies the conditions in Theorem 3.3.1 and give explicit geometric realisations.

The polytopes E_{mn} have several interesting properties, which we present in the second part of this chapter:

- They are self-dual, 2-simple and 2-simplicial.
- For $m,n = 4$ we obtain the 24-cell.
- There are flexible realisations of these polytopes, although a priori not with rational coordinates.
- They have a large combinatorial and geometric symmetry group.
- For small m and n we find particularly symmetric geometric realisations.
- For $m,n \geq 5$ the combinatorial symmetry group contains automorphisms that cannot be realised geometrically.
- For $m,n = 3$ and $m,n = 4$ we examine the realisation space of these polytopes. For $m,n = 3$ we provide a method to obtain all realisations allowed by our construction. For $m,n = 4$ (the 24-cell) we provide a four parameter family of realisations.
- In the next chapter, we show that the E_{mn} for $m,n \to \infty$ approach the upper bound for the fatness of E-polytopes.

The idea to look at the E-construction of arbitrary products of polytopes arose from the interest in the realisability and the symmetry of the polytopes E_{mn} in the special case $m = n$. In this case, the combinatorial description of these polytopes and some symmetry properties were described independently by Bokowski [23, 24] and Gévay [39].

3.2 P P

The main ingredient for this chapter are polytopes obtained as products of two other polytopes. Here we introduce our notation for these, provide the basic facts that we need for our constructions, and make some important distinctions between combinatorial and geometric properties.

3.2.1 D [Product of Polytopes]. Let $P_0 \subset \mathbb{R}^{d_0}$ and $P_1 \subset \mathbb{R}^{d_1}$ be two polytopes of dimensions d_0 and d_1, respectively. The *product* $P_0 \times P_1$ of P_0 and P_1 is the convex hull of the point set

$$\mathcal{V}(P_0 \times P_1) := \left\{ (v, w) \in \mathbb{R}^{d_0+d_1} \ : \ v \in \mathcal{V}(P_0), w \in \mathcal{V}(P_1) \right\}.$$

The product has dimension $d_0 + d_1$. Equivalently, one can describe the product directly by defining

$$P_0 \times P_1 := \left\{ (v, w) \in \mathbb{R}^{d_0+d_1} \ : \ v \in P_0, \ w \in P_1 \right\}.$$

See Figure 3.1 for an example of the product of a segment with a hexagon. The *faces* of a product $P_0 \times P_1$ have again product structure: Any k-face of the product is a product of a k_0-face of P_0 with a k_1-face of P_1 for some partition $k_0 + k_1 = k$ and $0 \le k_0, k_1 \le k$. In particular, vertices of $P_0 \times P_1$ are pairs of vertices of P_0 and P_1, and facets are a product of some facet of one factor with the whole other factor.

We can compute the *flag vector* of the product from the flag vectors of the factors: If $\operatorname{flag}(P_i) = (f_S(P_i))_{S \subseteq \{0,\dots,d_i-1\}}$ is the flag vector of P_i, for $i = 0, 1$, then the flag vector of the product is $\operatorname{flag}(P_0 \times P_1) := (f_S(P_0 \times P_1))_{S \subseteq \{-1,\dots,d_0+d_1\}}$ with

$$\begin{aligned} f_S(P_0 \times P_1) &:= f_{(s_1,s_2,\dots,s_k)}(P_0 \times P_1) \\ &= \sum_{u_1+v_1=s_1} \sum_{u_2+v_2=s_2} \cdots \sum_{u_k+v_k=s_k} f_{(u_1,u_2,\dots,u_k)}(P_0) f_{(v_1,v_2,\dots,v_k)}(P_1). \end{aligned}$$

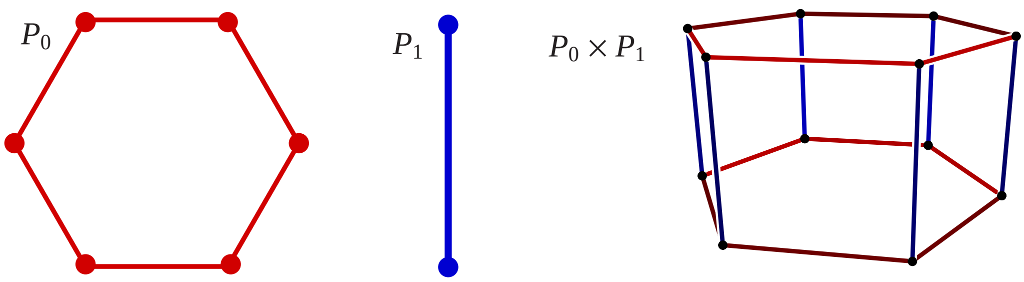


Figure 3.1: The product of a hexagon and a segment.

Here we set

$$f_{(t_1,t_2,\dots,t_k)} \equiv 0 \quad \text{unless} \quad t_1 \le t_2 \le \dots \le t_k \quad \text{and}$$
$$f_{(t_1,t_2,\dots,t_{i-1},t_i,t_{i+1},\dots,t_k)} := f_{(t_1,t_2,\dots,t_{i-1},t_{i+1},\dots,t_k)} \quad \text{if} \quad t_i = t_{i+1}.$$

The *face lattice* of $P_0 \times P_1$ can be obtained directly from the two face lattices of P_0 and P_1. It is simply the reduced product (see Definition 1.2.5 for this) of the two face lattices.

3.2.2 R [Geometric versus Combinatorial Product]. We make the following distinction that will become important later.

What we have defined in Definition 3.2.1 is the *geometric (orthogonal)* product of two polytopes as the convex hull of all pairs of geometrically given vertices. This definition requires the two input factors to be given geometrically and fixes a distinguished geometric realisation of the resulting polytope.

We sometimes need a more general definition. By a *combinatorial product* we mean a polytope, that has a face lattice which is isomorphic to the reduced product of the face lattices of the two polytopes, i.e. a polytope that is only *combinatorially equivalent* to the one obtained in the above definition.

This definition does not imply the choice of any particular geometric realisation. However, as the above definition provides a way to construct one, we know that such a combinatorial product of two polytopes is in fact always realisable.

3.2.3 C . In this chapter we consider only the special case $t = d-2$ of the E-construction, if d is the dimension of the polytope. In particular, for 4-polytopes we look at E_2. To simplify the notation we omit the index t in the rest of the chapter and write $E(P)$ instead of $E_{d-2}(P)$, for a d-polytope P.

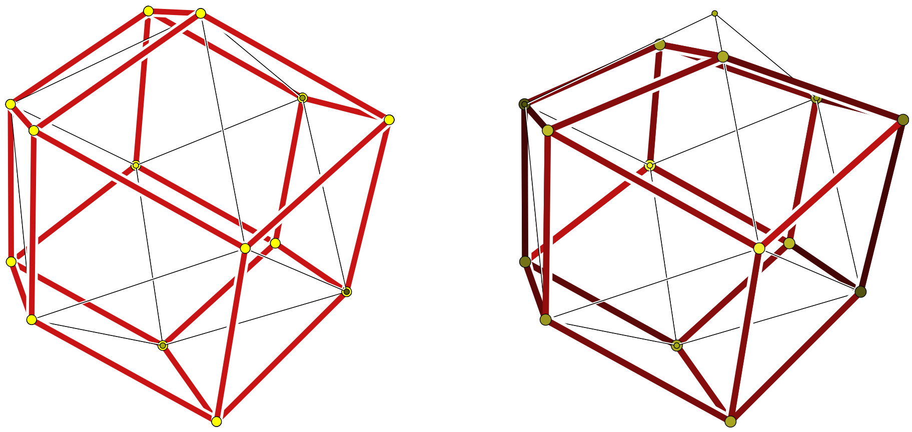


Figure 3.2: The left realisation of $E_1(\square_3)$ (drawn bold) of the cube $\square_3$ (drawn thin) is vertex-preserving, the right is not (and there is no cube for which it is): observe the top vertex.

Let P be a d-polytope. By its definition, $E(P)$ just denotes some polytope that is *combinatorially equivalent* to the sphere obtained from P via the E-construction. In this chapter we sometimes need a stricter (and more geometric) version of the connection between P and its E-polytope.

3.2.4 D [Vertex Preserving]. Let P be a geometrically given d-polytope. A polytopal realisation of $E(P)$ is *vertex preserving* if it is obtained from the realisation of P by placing one new vertex beyond any facet of P and taking the convex hull of these together with the vertices of P.

Figure 3.2 shows an example of two realisations of $E(\square_3)$. One of them is vertex preserving, the other is not. In dimension $d \geq 3$ this definition implies that the vertex set of P is a subset of the vertex set of $E(P)$. This is not true in dimension two, as here the vertices are the ridges of the polytope, and these vanish in the E-construction. See Figure 3.3 for an example. For $d \geq 3$, a vertex preserving realisation of $E(P)$ can equivalently be described by the following condition:

(VP) The vertex set of $E(P)$ splits into two disjoint sets, one of which is the vertex set $\mathcal{V}(P)$ of P, and there are no edges in $E(P)$ between any of the vertices in the other.

We denote a (convex) polygon with m vertices $v_0, \dots, v_{m-1}$ by C_m. We usually assume that the vertices are numbered consecutively and take indices modulo m.

With E_{mn} we denote the result of the E-construction applied to the product $C_m \times C_n$ of an m-gon and an n-gon, for $m, n \geq 3$. This is a 4-dimensional 2-simplicial and 2-simple *PL* sphere. The flag vectors of $C_m \times C_n$ and E_{mn} are (cf. Equation (2.3.1) for a general computation of the f-vector of E-polytopes)

$$\begin{aligned} \mathrm{flag}(C_m{\times}C_n) &= (mn, 2mn, mn + m + n, m + n; 4mn), \\ \mathrm{flag}(E_{mn}) &= (mn{+}m{+}n, 6mn, 6mn, mn{+}m{+}n; 8mn{+}2(m{+}n)), \end{aligned} \qquad (3.2.1)$$

where we have only recorded the essential values $(f_0, \dots, f_3; f_{03})$. All other entries of the flag vector follow from the generalised Dehn-Sommerville equations given in Theorem 1.2.17.

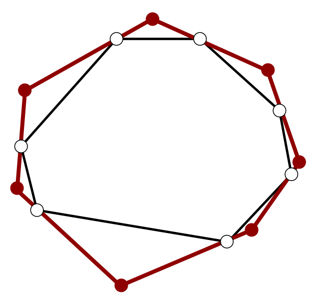


Figure 3.3: The E-construction applied to a 7-gon: The vertices vanish.

3.3 T E-C P

We proceed to the construction of $E(P_0 \times P_1)$ for a product of two polytopes of arbitrary dimension. We provide in this section a set of sufficient conditions for the existence of a polytopal realisation, if there are polytopal realisations of $E(P_0)$ and $E(P_1)$. We also show that the conditions are necessary if we want to have a vertex preserving realisation.

Let P_0 and P_1 be two geometrically realised polytopes of dimensions d_0 and d_1, respectively. Table 3.1 shows two conditions on their E-construction. For pairs P_0 and P_1 of polytopes that satisfy these conditions we define in the next theorem a point set S that is the vertex set of a vertex preserving polytopal realisation of the sphere $E(P_0 \times P_1)$.

3.3.1 T . *Let P_0, P_1 be a pair of polytopes, with* $\dim(P_0 \times P_1) \geq 3$, *that satisfies (A) and (B) in Table 3.1. Let S be the set containing the following points:*

(a) *all pairs* (p_0, p_1) *for* $p_0 \in \mathcal{V}(P_0)$, $p_1 \in \mathcal{V}(P_1)$,
(b) *all pairs* $(\alpha_0(v_0), v_0)$ *for* $v_0 \in S_0$,
(c) *all pairs* $(v_1, \alpha_1(v_1))$ *for* $v_1 \in S_1$.

Then $\mathrm{conv}(S)$ *is a vertex preserving polytopal realisation of* $E(P_0 \times P_1)$.

Moreover, for the existence of vertex preserving realisations of $E(P_0 \times P_1)$ the two conditions (A) and (B) are both necessary and sufficient.

See Figures 3.4 and 3.5 for an example of two triangles satisfying the two conditions. Both the sets S_j, for $j = 0$ and $j = 1$, and the E-polytopes of the two triangles are shown.

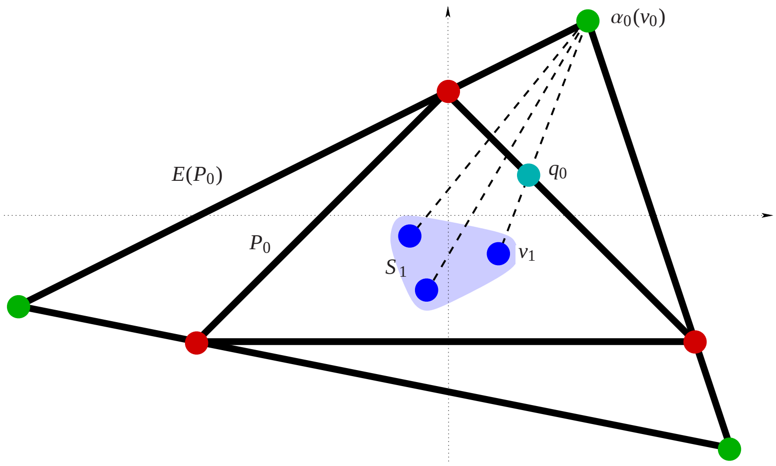


Figure 3.4: Realising the product of two triangles.

(A) There exist vertex preserving realisations of $E(P_0)$ and $E(P_1)$.
(B) For P_0, P_1, $E(P_0)$ and $E(P_1)$ there are point sets $S_1 \subset P_0$ and $S_0 \subset P_1$ that satisfy
  (1) $\#S_i = f_{d_i-1}(P_i)$ for $i = 0, 1$ (counted with multiplicities, i.e. points in S_i may coincide geometrically),
  (2) there are bijections $\alpha_i : S_i \longrightarrow \mathcal{V}(E(P_i)) - \mathcal{V}(P_i)$ for $i = 0, 1$,
  (3) for any pair $(v_0, v_1) \in S_0 \times S_1$:

$$\frac{|v_1, q_0|}{|v_1, \alpha_0(v_0)|} = \frac{|\alpha_1(v_1), q_1|}{|v_0, \alpha_1(v_1)|},$$

where q_0 is the intersection of the segment between v_1 and $\alpha_0(v_0)$ with ∂P_0, and q_1 that of the segment between v_0 and $\alpha_1(v_1)$ with ∂P_1. $|a, b|$ denotes the length of the segment between two points a and b.

Table 3.1: Sufficient conditions for the existence of $E(P_0 \times P_1)$.

Before we give a proof of this theorem we derive several properties that the polytope $E(P_0 \times P_1)$ will have, if it is vertex preserving for a given geometric realisation of $P_0 \times P_1$. This will explain the origin of (A) and (B).

Let P_0 and P_1 be two geometrically realised polytopes of dimension d_0 and d_1 with $d_0, d_1 \geq 1$ and $d_0 + d_1 \geq 3$. Suppose $E(P_0 \times P_1)$ exists and is a vertex preserving realisation of $P_0 \times P_1$. We split the vertex set of $E(P_0 \times P_1)$ into the vertex set $\mathcal{V}_p$ of $P_0 \times P_1$ and a set $\mathcal{V}_e$ of those vertices not in the product. $\mathcal{V}_e$ is a set consisting of one vertex beyond each facet of $P_0 \times P_1$.

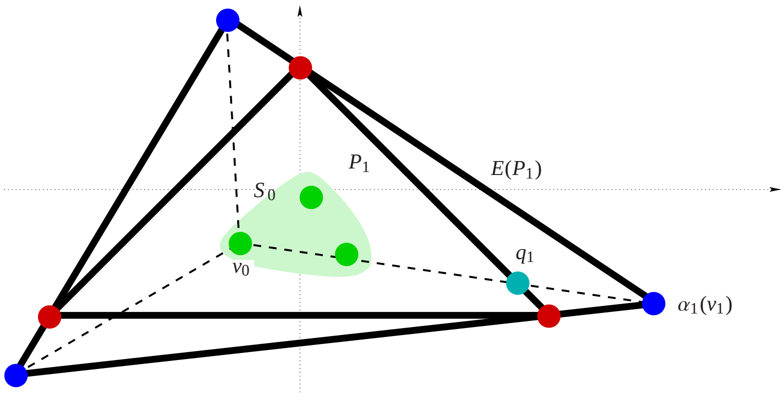


Figure 3.5: Realising the product of two triangles.

Define standard projections $\pi_j : \mathbb{R}^{d_0+d_1} \to \mathbb{R}^{d_j}$ that project $P_0 \times P_1$ onto P_j, for $j = 0, 1$. By assumption we have $\mathcal{V}(P_j) \subset \pi_j \mathcal{V}(E(P_0 \times P_1))$. We determine the images of the other vertices of $E(P_0 \times P_1)$ under the maps π_0 and π_1. The facets of the product $P_0 \times P_1$ have the form

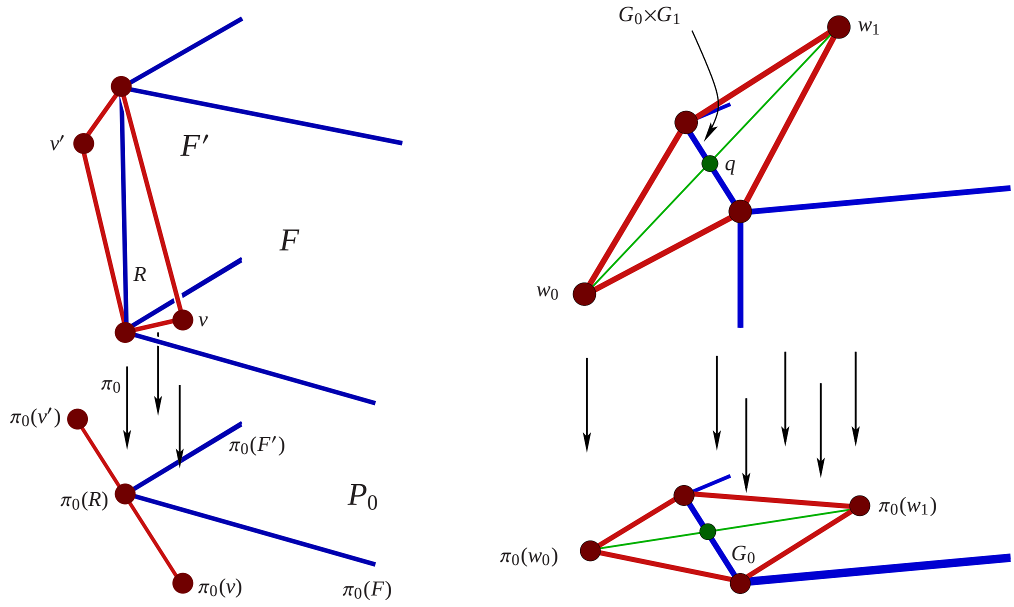
(F I) "Facet of P_0" $\times P_1$, or
(F II) $P_0 \times$ "Facet of P_1".

Accordingly, the product also has two different types of ridges:

(R I) Ridges between two facets of the same type, and
(R II) ridges between two facets of different type.

We deal with these two types of ridges separately:

(I) Let F and F' be two adjacent facets of type (F I), v, v' the two vertices of $E(P_0 \times P_1)$ beyond F and F' and R the ridge between F and F'. See Figure 3.6(a) for an illustration.
The projections $\pi_0(F)$ and $\pi_0(F')$ are adjacent facets of P_0 with common ridge $\pi_0(R)$. $\pi_0(v)$ and $\pi_0(v')$ are points beyond these facets. v, v' and R lie on a common (facet defining) hyperplane H of $E(P_0 \times P_1)$ in $\mathbb{R}^{d_0+d_1}$. So the points $\pi_0(v)$, $\pi_0(v')$ and the ridge $\pi_0(R)$ all lie on the hyperplane $\pi_0(H)$ in $\mathbb{R}^{d_0}$. Thus, $\pi_0(H)$ defines a face of $\pi_0(E(P_0 \times P_1))$, which must be a facet of the projection.



(a) The case (R I). *(b)* The case (R II).

Figure 3.6: The two cases of ridges in the vertex preserving case.

Hence, the convex hull of the image under π_0 of all vertices of $E(P_0 \times P_1)$ is the polytope $E(P_0)$. Similarly, the projection with π_1 results in $E(P_1)$.

(II) Let w_0 and w_1 be two vertices of $E(P_0 \times P_1)$, the first beyond a facet $G_0 \times P_1$, the second beyond a facet $P_0 \times G_1$ of the product $P_0 \times P_1$, where G_0 and G_1 are facets of P_0 and P_1. Let $R = G_0 \times G_1$ be the ridge between these two facets. See Figure 3.6(b).

The segment s between w_0 and w_1 intersects R in a point q. $\pi_0(q)$ is contained in G_0 and $\pi_1(q)$ is contained in G_1. So $\pi_0(w_1)$ is contained in the interior of P_0 and $\pi_1(w_0)$ in the interior of P_1. Projections preserve ratios, so

$$r := \frac{|w_1, q|}{|w_0, w_1|} = \frac{|\pi_0(w_1), \pi_0(q)|}{|\pi_0(w_1), \pi_0(w_0)|} = \frac{|\pi_1(w_1), \pi_1(q)|}{|\pi_1(w_0), \pi_1(w_1)|}.$$

To match this with (B) of Table 3.1 we set $v_1 := \pi_0(w_1) \in S_1$, $v_0 := \pi_1(w_0) \in S_0$, $\pi_0(w_0) = \alpha_1(v_1)$, $\pi_1(w_1) = \alpha_0(v_0)$, $q_0 := \pi_0(q)$, and $q_1 := \pi_1(q)$.

Thus, the projections of a vertex preserving realisation of $E(P_0 \times P_1)$ onto the two orthogonal subspaces containing P_0 and P_1 give realisations of $E(P_0)$ and $E(P_1)$ that satisfy the conditions stated in (A) and (B) of Table 3.1.

Proof of Theorem 3.3.1. The considerations of the previous paragraphs prove the necessity of the two conditions for vertex preserving realisations. It remains to prove sufficiency in the general case.

Suppose we have – according to the conditions (A) and (B) – constructed $E(P_0)$ and $E(P_1)$ together with the sets S_0 and S_1 and have formed the set S defined in the theorem. We have to show that all facets of the convex hull of S defined thereby are bipyramids over ridges of $P_0 \times P_1$ and that there is precisely one vertex of S beyond each facet of $P_0 \times P_1$. There are two different cases to consider:

(I) Let $R \times P_1$ be a ridge of $P_0 \times P_1$, where R is a ridge of P_0. See Figure 3.7 for a drawing. Let F and F' be the two facets of P_0 adjacent to R and p, p' the vertices of $E(P_0)$ above F and F' respectively. Let v be the facet normal of the facet F_E of $E(P_0)$ formed by R, p and p', and let $l := \langle v, p \rangle$.

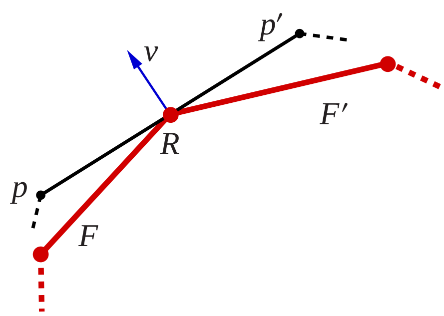


Figure 3.7: The first case in the proof of Theorem 3.3.1: Ridge × Polytope.

By construction, the points $(p, \alpha_0^{-1}(p))$, $(p', \alpha_0^{-1}(p'))$ and (r, q) for $r \in \mathcal{V}(R)$ and $q \in \mathcal{V}(P_1)$ lie in the hyperplane H defined by $\langle (v, \mathbf{0}), \cdot \rangle = l$, where $\mathbf{0}$ is the d_1-dimensional zero vector. All points of $\mathcal{V}(E(P_0)) - (\mathcal{V}(R) \cup \{p, p'\})$ are on the same side of the hyperplane defined by the facet F_E. So all points in the set

$$\mathcal{V}(E(P_0 \times P_1)) - \left(\mathcal{V}(R \times P_1) \cup \left\{(p, \alpha_0^{-1}(p)), (p', \alpha_0^{-1}(p'))\right\}\right)$$

are on the same side of the hyperplane H and

$$\operatorname{conv}\left(\mathcal{V}(R \times P_1), \left\{(p, \alpha_0^{-1}(p)), (p', \alpha_0^{-1}(p'))\right\}\right)$$

is a facet of $E(P_0 \times P_1)$. The same argument applies to ridges of type $P_0 \times R$ for any ridge R of P_1.

(II) Now consider a ridge of type $F_0 \times F_1$ for a facet F_0 of P_0 and a facet F_1 of P_1. See Figure 3.8 for an illustration. Let p_0 be the vertex of $E(P_0)$ beyond F_0 and p_1 the vertex of $E(P_1)$ beyond F_1. Let q_0 be the intersection point of the segment from p_0 to $\alpha_1^{-1}(p_1)$ and the facet F_0, and q_1 the intersection point of the segment between p_1 and $\alpha_0^{-1}(p_0)$ and the facet F_1 (i.e. in the notation of (B), $p_0 = \alpha_0(v_0)$ and $p_1 = \alpha_1(v_1)$ for some $v_0 \in S_0$ and $v_1 \in S_1$). By construction we have

$$|p_0, q_0| : |q_0, \alpha_1^{-1}(p_1)| = |\alpha_0^{-1}(p_0), q_1| : |q_1, p_1|$$

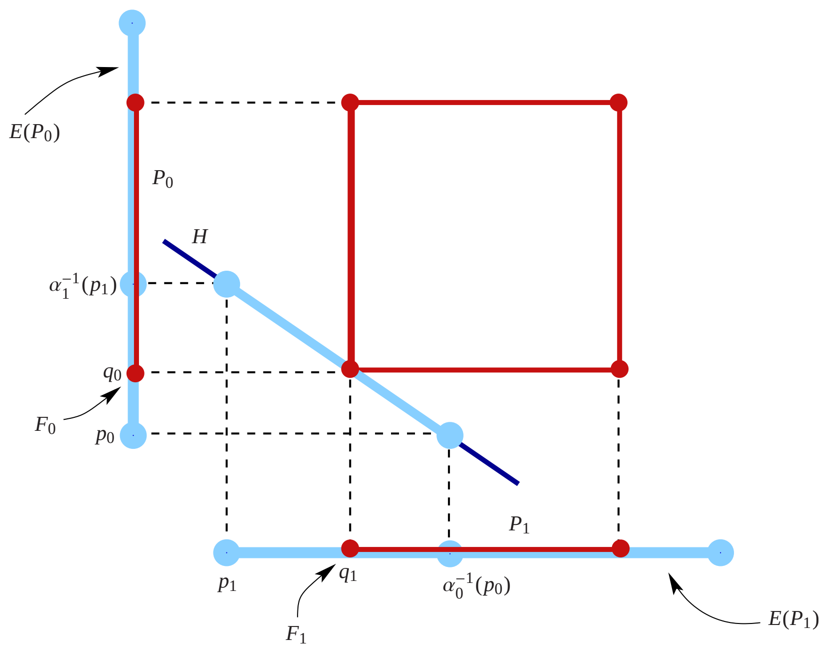


Figure 3.8: The second case in the proof of Theorem 3.3.1: Facet × Facet.

and the point $(p_0, \alpha_0^{-1}(p_0))$ is contained in the line through $(\alpha_1^{-1}(p_1), p_1)$ and (q_0, q_1). So the points in $\mathcal{V}(F_0 \times F_1)$ together with $(p_0, \alpha_0^{-1}(p_0))$ and $(\alpha_1^{-1}(p_1), p_1)$ lie on a common hyperplane H.

The product $P_0 \times P_1$ lies entirely on one side of H by construction. Suppose there is a point x of S on the other side of H. As H is a valid hyperplane for the ridge $F_0 \times F_1$, any point beyond it is also beyond either the facet hyperplane of $F_0 \times P_1$ or $P_0 \times F_1$. Assume the first. For any $z \in S$ we have by definition either

(a) $\pi_0(z) \in \mathcal{V}(E(P_0)) - \mathcal{V}(P_0)$, or
(b) $\pi_0(z) \in S_1$, or
(c) $\pi_0(z) \in \mathcal{V}(P_0)$.

$x \in S$ is beyond H, therefore only $\pi_0(x) \in \mathcal{V}(E(P_0)) - \mathcal{V}(P_0)$ is possible. $\pi_0(x)$ is beyond F_0, so $\pi_0(x)$ is the unique vertex of $E(P_0)$ beyond F_0, so $\pi_0(x) = p_0$ and $x \in H$.

This proves that the two conditions (A) and (B) are sufficient for the existence of a vertex preserving polytopal realisation of $E(P_0 \times P_1)$. □

3.3.2 R . This theorem is a quite powerful tool for obtaining polytopal realisations of the E-construction of polytopes. In Section 3.5.4 we present some general applications for it. However, in most cases it already suffices to consider a restricted version of Theorem 3.3.1. For this, replace the two conditions (A) and (B) of Table 3.1 with:

(A′) There exist vertex preserving realisations of $E(P_0)$ and $E(P_1)$ and
(B′) there are points s_0 in P_1 and s_1 in P_0, and a ratio r between 0 and 1, such that:

$$r = \frac{|s_1, q_0|}{|s_1, w_0|} = \frac{|w_1, q_1|}{|w_1, s_0|}$$

for any pair

$$(w_0, w_1) \in \Big(\mathcal{V}(E(P_0)) \; - \; \mathcal{V}(P_0)\Big) \times \Big(\mathcal{V}(E(P_1)) \; - \; \mathcal{V}(P_1)\Big),$$

where q_0 is the intersection point of the segment between s_1 and w_0 with ∂P_0 and q_1 the intersection point of the segment between s_0 and w_1 with ∂P_1.

In other words, the sets S_0 and S_1 contain only a single point with sufficient multiplicity, and all ratios occurring in condition (B) coincide. Theorem 3.3.1 now reads as follows.

3.3.3 C . *Let P_0, P_1 be a pair of polytopes with* $\dim(P_0 \times P_1) \geq 3$ *that satisfy (A′) and (B′). Let S be the set of*

(a) *all pairs* (p_0, p_1) *for* $p_0 \in \mathcal{V}(P_0)$ *and* $p_1 \in \mathcal{V}(P_1)$,
(b) *all pairs* (w_0, s_0) *for* $w_0 \in \mathcal{V}(E(P_0)) - \mathcal{V}(P_0)$,
(c) *all pairs* (s_1, w_1) *for* $w_1 \in \mathcal{V}(E(P_1)) - \mathcal{V}(P_1)$.

Then $\operatorname{conv}(S)$ *is a vertex preserving polytopal realisation of* $E(P_0 \times P_1)$. □

In this version, the only connection between the E-constructions of the two factors is the value r of the ratio. Thus, if we have

- a polytope P
- together with a vertex preserving realisation of $E(P)$ and
- a single point s in its interior such that all segments from s to the vertices of $E(P)$ not in P intersect P with ratio r,

then we can combine this with any other such instance for ratio $1 - r$ instead of r and obtain a polytopal realisation of the E-construction of the product.

3.4 E R

We apply Theorem 3.3.1 and Corollary 3.3.3 of the previous section and produce products of polytopes together with a realisation of their E-polytopes.

The main focus is on the realisation of the E-polytope E_{mn} of a product of an m-gon and an n-gon. We produce polytopal realisations for all $m, n \geq 3$. We also briefly discuss examples in dimensions $d \geq 5$ in Section 3.4.2.

3.4.1 P P

We present an explicit method to obtain a "flexible" geometric polytopal realisation of $E_{mn} := E(C_m \times C_n)$ for all $m, n \geq 3$. Degrees of freedom in this construction are discussed in Section 3.5.4 for $m, n = 3$ and $m, n = 4$.

3.4.1 T . *The CW spheres E_{mn} are polytopal for all $m, n \geq 3$.*

In the five cases when m and n satisfy $\frac{1}{m} + \frac{1}{n} \geq \frac{1}{2}$ polytopality also follows from a construction of Santos. We present this in Theorem 3.5.3.

For the proof of Theorem 3.4.1 it suffices to use the restricted version of Corollary 3.3.3. Hence, we will only construct one of the two factors of the product, together with its E-construction. We make the following definition for this.

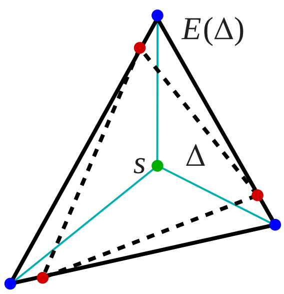


Figure 3.9: An example of a realisation of $D\left(3, \frac{1}{3}\right)$.

3.4.2 D **.** By $D(k, r)$ we denote a realisation of a k-gon C_k that

- contains the origin s and
- has a vertex-preserving E-polytope $E(C_k)$, such that segments from s to vertices of $E(C_k)$ intersect C_k with ratio r.

Figure 3.9 shows an example of a triangle and its E-construction that satisfy the conditions for $r = \frac{1}{3}$.

The construction of $D(n, r)$ has two steps. In a first step we start with the point s (the origin), one vertex of the polygon, and one edge normal of its E-polytope, and iteratively add one edge of the E-polytope and one of the polygon. The second step connects the first and last vertex of the two sequences of edges to close the polygons.

Let $l(v, b)$ for $v \neq \mathbf{0}$ denote the line in $\mathbb{R}^2$ defined by $\{x \mid \langle v, x \rangle - b = 0\}$. Here are the two steps.

S **I.** Given are three unit length vectors w_0, v_0 and v_1 such that

$$\angle(v_0, v_1) < \pi \qquad \text{and} \qquad \angle(v_0, w_0) + \angle(w_0, v_1) = \angle(v_0, v_1)$$

(i.e. w_0 is "between" v_0 and v_1). Let q_0 be a given point in the plane. See Figure 3.10. We do the following:

1. Set $g_0 := l(v_0, \langle v_0, q_0 \rangle)$ and $f_0 := l(w_0, \langle w_0, q_0 \rangle)$.
2. Set $f_0' := l(w_0, \frac{1}{r} \langle w_0, q_0 \rangle)$.
   f_0' is parallel to f_0 at distance $\frac{1}{r} \langle w_0, q_0 \rangle$ from the origin. Hence, the segment from s to any point on f_0' is intersect by f_0 with ratio r.
3. Let p_0 be the intersection point of g_0 and f_0' and set $g_1 := l(v_1, \langle v_1, p_0 \rangle)$.
4. Let q_1 be the intersection point of f_0 and g_1.

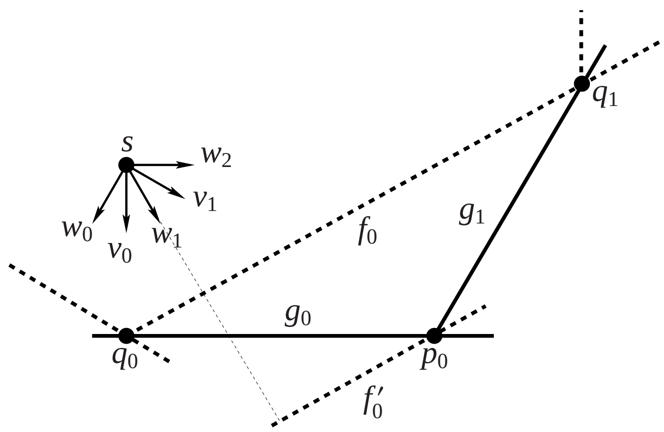


Figure 3.10: A possible configuration for Step I.

The segment between the origin and p_0 is now intersected by the line f_0 with a ratio of r.

S II. Given are unit length vectors v_0, v_1, and v_2 such that

$$\angle(v_2, v_0) > \pi \qquad \text{and} \qquad \angle(v_2, v_1) + \angle(v_1, v_0) = \angle(v_2, v_0).$$

Let q_0, q_2 be points such that

$$\langle v_0, q_0 \rangle, \langle v_2, q_2 \rangle > 0 \qquad \text{and} \qquad \langle v_0, q_2 \rangle, \langle v_2, q_0 \rangle < 0.$$

See Figure 3.11. Then we apply the following five steps:

1. Let $g_0 := l(v_0, \langle v_0, q_0 \rangle)$ and $g_2 := l(v_2, \langle v_2, q_2 \rangle)$.
2. Define a variable a and the line $g_a := l(v_1, a)$.
3. Denote the intersection points of g_a with g_0 and g_2 by p_0 and p_2 respectively, and let m_0 and m_2 be the lines running through the origin and p_0 and p_2.
4. For $j = 0, 2$ let i_j be the point on m_j dividing the segment between the origin and p_j with ratio r for $j = 0, 2$. Let f_j be the line through q_j and i_j.
5. Let $q_1 := f_0 \cap f_2$. By Lemma 3.4.3 below, there is always a value of a such that q_1 is on the line g_a.

3.4.3 L . *In the second construction step above, the variable a can be chosen in such a way that q_1 is on the line g_a.*

Proof. Let $b_j := \left\langle v_1, q_j \right\rangle$ for $j = 0, 2$ and assume $b_2 \geq b_0$. Then for $a := \frac{1}{r} b_2$ the line f_2 is parallel to g_a. Thus the point q_1 is on the same side of g_a as the origin. For large a, however, the point q_1 and the origin are on different sides of g_a. Hence, by continuity, there must be a value of a such that q_1 is on g_a. $\square$

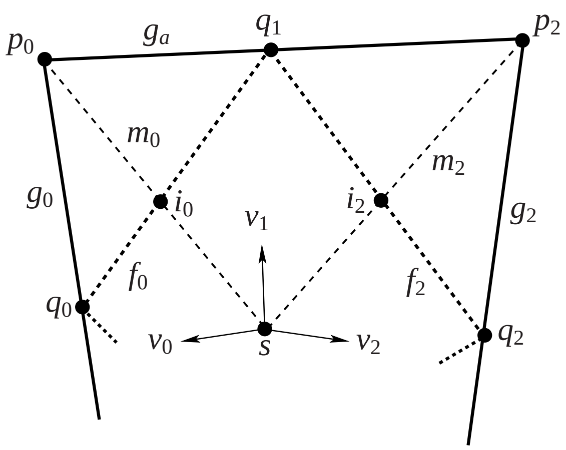


Figure 3.11: A possible configuration for Step II.

T $D(n,r)$**.** We denote vertices of C_n by q_i, vertices of $E(C_n)$ by p_j, normal vectors on the edges of C_n by w_i and normal vectors on the edges of $E(C_n)$ by v_j. To construct $D(n,r)$ from the two steps above we can use the following choice of points and normal vectors.

1. Let s be the origin, $q_0 := (-1, 0) \in \mathbb{R}^2$ and choose two sets of unit length vectors $V := \{v_0, v_1, \ldots, v_n\}$ and $W := \{w_0, w_1, \ldots, w_{n-2}\}$ such that
   (a) $\angle(v_0, v_n) < \pi$,
   (b) both sets contain n vectors, and
   (c) seen as points on the unit sphere they alternate.
   Number both sets consecutively. An illustration of such a configuration is shown in Figure 3.13.
2. Starting with Step I and q_0, v_0, v_1, and w_0 as input defines points q_1 and p_0. Continue with q_1, v_1, v_2, and w_1 to produce q_2 and p_1. Repeat this until reaching the tuple v_{n-1}, v_n, w_{n-1}, and q_{n-1}.
3. Step II applied to q_0, q_{n-1}, v_0, v_n and w_n yields $D(n,r)$.

Proof of Theorem 3.4.1. Fix $m, n \geq 3$ and choose a ratio r between 0 and 1. Construct $D(m,r)$ and $D(n, 1-r)$ according to the above algorithm. This yields realisations of the E-construction of an m-gon and an n-gon that satisfy the prerequisites of Corollary 3.3.3. This gives the desired realisation of E_{mn}. □

3.4.4 R **.** There is one more caveat in the construction of $D(n,r)$, as we have to ensure that s is really an inner point of C_n afterwards.

For each edge e_i of C_n with normal vector v_i let z_i be the point on the line defined by e_i with minimal distance to s (which we chose to be the origin).

Viewed from s we choose q_0 to the right of z_i. During the iterative addition of edges to the polygons in Step I we have to ensure that the points $q_i \in e_i$ for $i \geq 1$ lie to the left of z_i (viewed from s).

However, if we choose the normal vectors v_i in such a way that consecutive vectors enclose an angle less than π, then this property can be enforced by an appropriate choice of the normals w_i. See [66] for a detailed proof.

Figure 3.12 shows a Schlegel diagram and the coordinates of this construction applied to a triangle C_3 and and a square C_4. The ratio in this example is $r = \frac{1}{2}$.

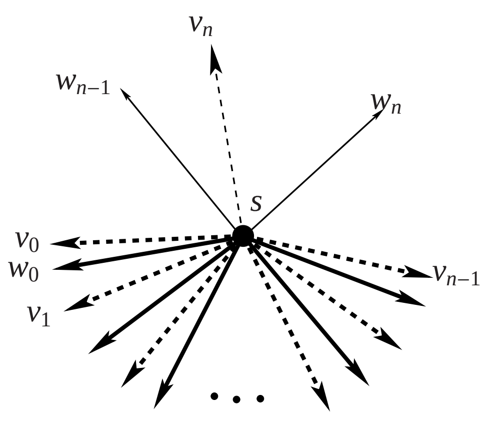


Figure 3.13: A feasible choice of vectors: W solid, V dashed. The thin vectors are constructed during Step II.

3.4.2 H -D E

Satisfying the two conditions (A) and (B) in Table 3.1, that are necessary for the application of Theorem 3.3.1, is more difficult, if the two factors P_0 and P_1 have "many" facets. Thus, in higher dimensions, and for "more complex" polytopes, it is usually hard to find appropriate sets S_0 resp. S_1, unless one can exploit some kind of symmetry.

There are, however, two obvious families of polytopes that we can choose as factors of a product polytope, the d-cube $\square_d$ and the d-simplex Δ_d. Both can be realised together with their E-construction satisfying even the more restrictive conditions of Corollary 3.3.3.

- The cube can be realised as follows: For $\square_d$ we take the standard ± 1-cube. The new vertices for the E-polytope are $\pm 2 \cdot e_i$, where e_i are the standard unit basis vectors. If we set $r = \frac{1}{2}$, then the origin is an inner point s satisfying all requirements.

1	1	2	0
1	1	−2	0
1	1	0	3
1	−1	2	0
1	−1	−2	0
1	−1	0	3
−1	1	2	0
−1	1	−2	0
−1	1	0	3
−1	−1	2	0
−1	−1	−2	0
−1	−1	0	3
0	0	7/13	1197/338
0	0	−532/195	217/845
0	0	244/91	−279/169
2	0	0	279/169
−2	0	0	279/169
0	−2	0	279/169
0	2	0	279/169

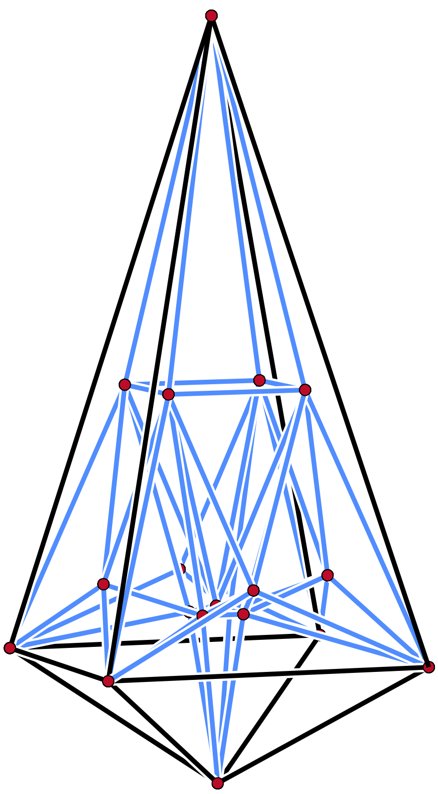


Figure 3.12: The E-construction on the product $C_3 \times C_4$. The ratio is in both factors is $\frac{1}{2}$.

- The construction for the d-simplex Δ_d is slightly more difficult. We give an inductive construction that produces realisations for any ratio $r \geq \frac{1}{2}$, that is, at least half of the segment is inside Δ (and r is the ratio appearing in the conditions in Table 3.1). We can clearly construct such a realisation for a triangle, i.e. for a simplex of dimension $d = 2$.
  For $d > 2$ we take a regular realisation Δ of the simplex and a scaled version $\Delta' := \frac{1}{r} \cdot \Delta$ with the same barycentre. We choose one facet F of Δ and the corresponding scaled facet F' in Δ'. Place the first new vertex v in the barycentre of F'. The vertices of any ridge R of F together with the point v uniquely define a hyperplane. F has d ridges, so we obtain d different hyperplanes $H_1, \ldots, H_d$ by this.
  $H_1, \ldots, H_d$ intersect all facet hyperplanes of Δ', except that to F', in codimension-2-planes that lie in a common hyperplane H. H is parallel to F.
  Project the barycentre of Δ orthogonally onto H. H cuts Δ and Δ' in two simplices $\tilde{\Delta}$ and $\tilde{\Delta}'$ of dimension $d-1$. (Recall that $r \geq \frac{1}{2}$, so H intersects Δ between the barycentre and F.) $\tilde{\Delta}'$ is (viewed in the hyperplane H) a scaled version of $\tilde{\Delta}$ with a scaling factor $\frac{1}{r'} \leq \frac{1}{r}$. By induction, we have a solution for the corresponding problem for $\tilde{\Delta}$ and $r' \geq r \geq \frac{1}{2}$ in H.
  These points, together with the one vertex v chosen before, give a realisation of $E(\Delta)$ that satisfies the conditions of Corollary 3.3.3.

With these constructions we can combine any simplex or cube with any other simplex, cube or n-gon and obtain a realisation of the E-construction of this product.

As an example, in Figure 3.14 is a Schlegel diagram of the E-construction of a product of a 3-simplex with a segment. The product structure of the original polytope is clearly visible in the result.

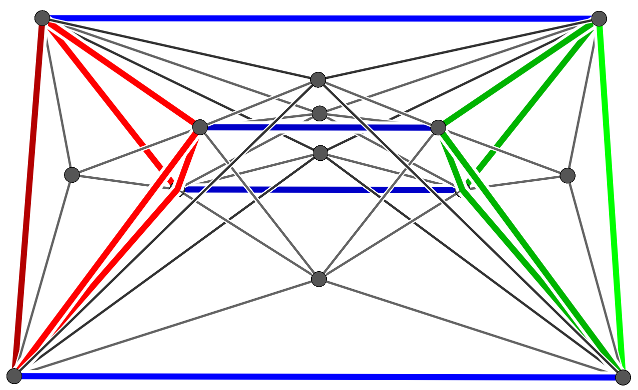


Figure 3.14: A Schlegel diagram of the E-construction of a product of a 3-simplex with a segment. The product structure is clearly visible.

3.5 P F E_{mn}

We continue our discussion of the family E_{mn} of $(2,2)$-polytopes. In particular, we count degrees of freedom for the realisation of E_{33} and prove that not all combinatorial symmetries of E_{mn} are geometrically realisable.

3.5.1 S -D

The polytopes $C_m \times C_n$ are simple. Hence, we know from Corollary 2.3.13 (or Corollary 2.4.6) that the polytope E_{mn} is 2-simple and 2-simplicial. In particular, the f-vector of E_{mn} is symmetric (cf. Equation (3.2.1)):

$$f(E_{mn}) = (mn + m + n, 6mn, 6mn, mn + m + n).$$

The polytopes E_{mn} have in fact a much stronger property: They are self-dual. This is not true for arbitrary 2-simple, 2-simplicial polytopes, which can be seen e.g. from the hypersimplex $E(\Delta)$ obtained from the 4-simplex Δ. This polytope has a facet-transitive automorphism group acting on its 10 bipyramidal facets, while the dual has 5 tetrahedral and 5 octahedral facets. The following result is due to Ziegler [91]. For $m = n$, the (combinatorial version of this) result was obtained previously also by Gévay [39].

3.5.1 T [Self-Duality]. *Each of the polytopes E_{mn} $(n, m \geq 3)$ is self-dual, with an anti-automorphism of order* 2.

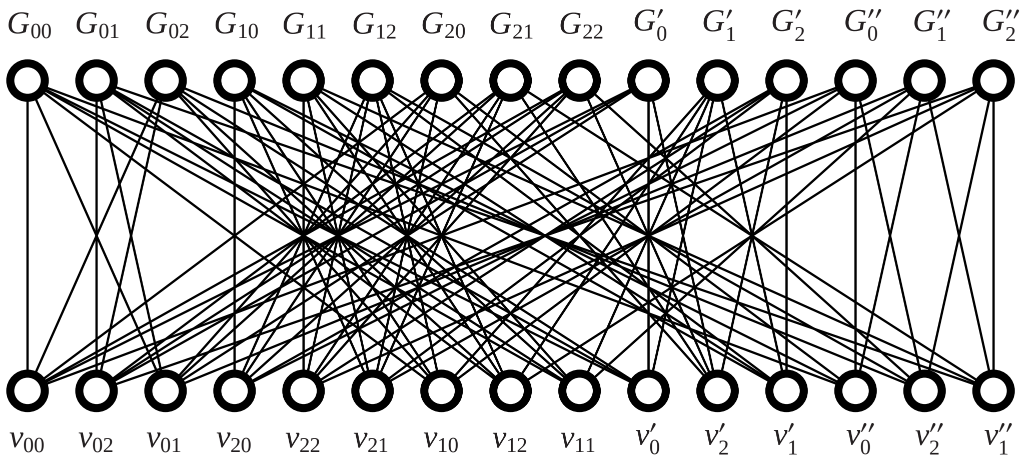


Figure 3.15: The self duality for E_{33}: Shown are the vertex-facet-incidences of E_{33}, the self duality exchanges the top and bottom row.

Proof. Number the vertices of an k-gon C_k consecutively by $v_0, \dots, v_{k-1}$. We take indices modulo k in the following. The vertices of the product $C_m \times C_n$ are $v_{i,j} := (v_i, v_j)$ for $0 \le i \le m-1$ and $0 \le j \le n-1$. We have two types of facets in the product:

$$\begin{aligned} F'_i &= \operatorname{conv}(\{v_{ij}, v_{i+1,j} \mid j = 0, \dots, n-1\}) \\ F''_j &= \operatorname{conv}(\{v_{ij}, v_{i,j+1} \mid i = 0, \dots, m-1\}) \end{aligned}$$

The E-construction adds one vertex beyond each facet of the product. We denote the new vertex beyond F'_i by v'_i and the one beyond F''_j by v''_j. The facets of E_{mn} are now of the form

$$\begin{aligned} G_{ij} &= \operatorname{conv}(v_{ij}, v_{i+1,j}, v_{i,j+1}, v_{i+1,j+1}, v'_i, v''_j), \quad \text{or} \\ G'_i &= \operatorname{conv}(\{v_{ij} \mid j = 0, \dots, n-1\}, v'_{i-1}, v'_i), \;\text{or} \\ G''_j &= \operatorname{conv}(\{v_{ij} \mid i = 0, \dots, m-1\}, v''_{j-1}, v''_j). \end{aligned}$$

From this we can read off the facets a vertex is contained in:

$$\begin{aligned} v_{ij} &\in G_{ij}, G_{i-1,j}, G_{i,j-1}, G_{i-1,j-1}, G'_{i-1}, G''_{j-1} && \text{for } i = 0, \dots, m-1, \\ & && \quad j = 0, \dots, n-1 \\ v'_i &\in G'_i, G'_{i+1}, G_{ij} && \text{for } j = 0, \dots, n-1 \\ v''_j &\in G''_j, G''_{j+1}, G_{ij} && \text{for } i = 0, \dots, m-1 \end{aligned}$$

Hence, the following correspondences give a self-duality of order 2 on the face lattice of E_{mn}:

$$G_{ij} \longleftrightarrow v_{-i,-j} \qquad G'_i \longleftrightarrow v'_{-i} \qquad G''_j \longleftrightarrow v''_{-j}. \qquad \square$$

Figure 3.15 shows an example of the self-duality on E_{33}.

3.5.2 R . There are examples of 3-polytopes that are self-dual, but that do not have a self-duality of order 2, see Ashley et. al. [5] and Jendrol’[49] for this.

3.5.2 E_{mn} C R P

There are only a few pairs (m, n) in which there are “symmetric” realisations of the polytopes E_{mn}: We show that, up to interchanging m and n, there are only five choices of pairs of regular polygons that can be taken as input for the construction defined in Corollary 3.3.3.

We will see in the next section that these five cases are also the only cases in which the product of two cyclic groups induced from rotation of the vertices in the two factors can be a subgroup of the geometric symmetry group. The next theorem was initiated by earlier work of Santos in [74, Rem. 13] and [75], where this problem occurred in a quite different context.

3.5.3 T [Symmetric Realisations]. *There are polytopal realisations of E_{mn} for which projection onto the first and last two coordinates yields*
(1) *regular polygons for C_m, C_n and their E-constructions, such that*
(2) *all intersection ratios occurring in (B) of Table 3.1 coincide in each factor,*
if and only if m and n satisfy the inequality

$$\frac{1}{m} + \frac{1}{n} \geq \frac{1}{2}.$$

See Figure 3.16 for two examples of input factors of the construction that satisfy these conditions.

Proof. The condition on the ratio implies that the sets S_0 and S_1 appearing in the construction of Theorem 3.3.1 both contain only a single point, which is counted with multiplicity n and m, respectively. These points must be the barycentres in the regular polygons C_m and C_n. By applying a translation if necessary, we may assume that these coincide with the origin.

We can now generate all configurations of a regular polygon C_m together with $E(C_m)$ in the following way: Start with a regular polygon $E(C_m)$ centred at the origin and choose a vertex for C_m on each of the edges. As C_m is regular, the vertices of C_m divide each edge with equal ratio. The segments considered in (B)(3) of Table 3.1 are the segments l between the origin and a vertex of $E(C_m)$. We are interested in the possible values of the ratio with which they are intersected by the edges of C_m.

Choosing the vertices of C_m close to those of $E(C_m)$ we see that we can have an arbitrarily high portion of l inside C_m. On the other hand, the portion inside C_m

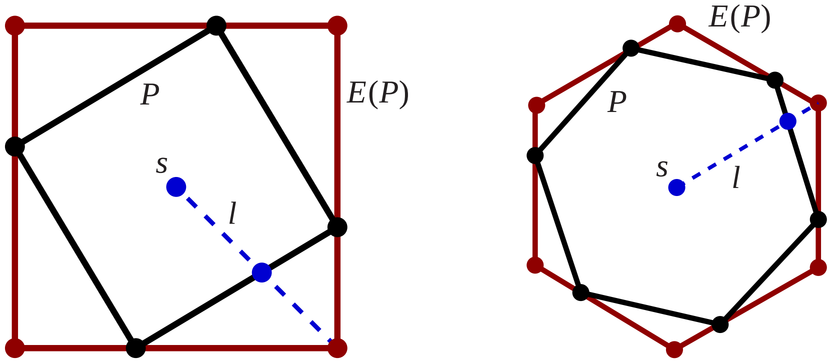


Figure 3.16: Two projections that satisfy the restrictions of Theorem 3.5.3.

is minimised when we place the vertices of C_m in the centre of the edges. In this case, the fraction of l outside C_m is $\sin^2(\frac{\pi}{n})$, see Figure 3.17.

By (B)(3) of Table 3.1, the fraction of a segment lying outside for one polygon and its E-construction has to match the fraction of a segment lying inside for the other polygon. This gives the following inequalities:

$$1 - \sin^2\left(\frac{\pi}{m}\right) \leq \sin^2\left(\frac{\pi}{n}\right) \qquad \text{and} \qquad 1 - \sin^2\left(\frac{\pi}{n}\right) \leq \sin^2\left(\frac{\pi}{m}\right),$$

which are equivalent to the condition given in the theorem. □

We can of course determine all possible values for the inequality in Theorem 3.5.3 explicitly.

3.5.4 C . *There are realisations of E_{mn} from regular polytopes only for the following pairs* (m, n) *(up to interchanging m and n):*

$$(3,3)\ ,\ (3,4)\ ,\ (3,5)\ ,\ (3,6)\ ,\ (4,4)$$ □

3.5.5 R . We made assumption (2) in Theorem 3.5.3 mainly because this is the case we need in the next section. A less restrictive version of "symmetry" would only require the points in S_0 and S_1 to also form a regular polygon (if we take the vertices in the order induced by the E-construction of the other factor). For small $m = n$ this has solutions where all points in S_0 and S_1 are different. See Table 3.2 for an example of such an E_{44}. Note however, that this severely reduces the number of geometric symmetries compared to the case of the theorem.

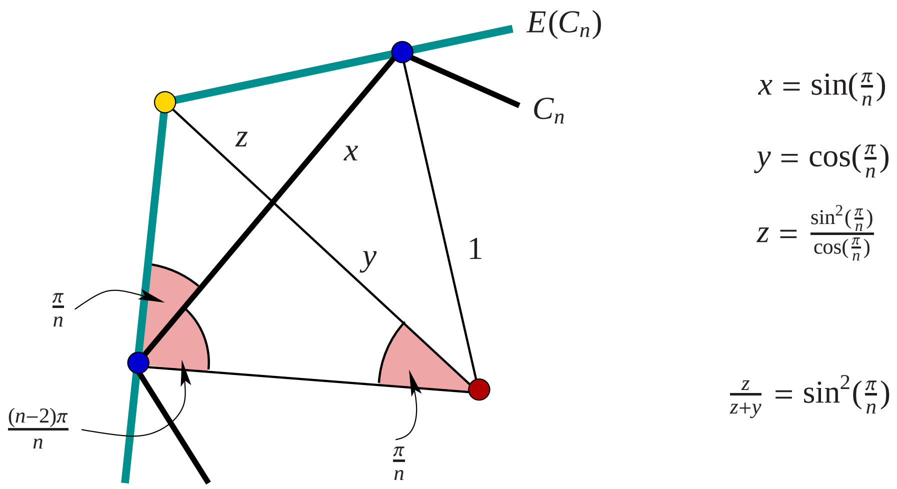


Figure 3.17: The length computation in the proof of Theorem 3.5.3.

3.5.3 C G S

There are two different notions of *symmetry* for a polytope P. We can look at *combinatorial symmetries* and *geometric symmetries*. The former are automorphisms of the face lattice of P, the latter are affine transformations that preserve a given geometric realisation of P set-wise. See Section 1.3.2.3 for a precise definition and some more background on symmetry groups.

Usually, these groups differ for a given geometric realisation of a polytope. However, there are not many polytopes known for which these groups differ for all possible geometric realisations. Bokowski, Ewald, and Kleinschmidt [25] have constructed a 4-dimensional polytope with ten vertices having a combinatorial symmetry that does not correspond to an affine transformation in any geometric realisation of the polytope. The given coordinates are erroneous, see [26] and [2] for a corrected version of the coordinates and a simpler proof. On the other hand, it is known that 3-polytopes, and d-polytopes with at most $d+3$ vertices for $d \geq 3$, always have a realisation in which geometric and combinatorial symmetry group coincide. The primer was proven by Mani [59], the latter by Perles, see [44, p.120].

We show that our family E_{mn} of 4-polytopes contains an infinite subfamily with non-realisable combinatorial symmetries. To this end, we explicitly construct such a combinatorial symmetry.

Previously, it was already observed by Gévay that no polytopal realisation of the spheres E_{nm} for $m = n$ can realise the full combinatorial symmetry group, except in the case $m = n = 4$. This is also a consequence of Corollary 3.5.8 below.

1	1	1	1	−1	−1	1	1
1	1	1	−1	−1	−1	1	−1
1	1	−1	1	−1	−1	−1	1
1	1	−1	−1	−1	−1	−1	−1
1	−1	1	1	3/5	9/5	−3/5	−3/5
1	−1	1	−1	9/5	−3/5	−3/5	3/5
1	−1	−1	1	−3/5	−9/5	3/5	3/5
1	−1	−1	−1	−9/5	3/5	3/5	−3/5
−1	1	1	1	−3/5	3/5	3/5	9/5
−1	1	1	−1	−3/5	−3/5	−9/5	3/5
−1	1	−1	1	3/5	−3/5	−3/5	−9/5
−1	1	−1	−1	3/5	3/5	9/5	−3/5

Table 3.2: The vertices of an E_{44} from regular squares, but not satisfying (2) of Theorem 3.5.3.

3.5.6 T [Non-Realisable Symmetry]. *For relatively prime $m, n \geq 5$, all E_{mn} have a combinatorial symmetry that cannot be realised as a geometric symmetry in any geometric realisation of E_{mn}.*

Note, that in this theorem we do not require that the realisation of E_{mn} is obtained by the construction in Theorem 3.4.1. It can be any geometric realisation which is combinatorially equivalent to E_{mn}. In the following, we denote such a more general realisation by P_{mn} to emphasise this distinction.

Proof of Theorem 3.5.6. We define a combinatorial symmetry T of P_{mn}. Let C_m and C_n denote polygons with vertices $v_0, \ldots, v_{m-1}$ and $w_0, \ldots, w_{n-1}$ respectively, numbered in cyclic order. We take indices modulo m and n, respectively. Let S be the combinatorial symmetry of a polygon that maps the j-th to the $(j+1)$-th vertex. See Figure 3.18 for an illustration. S induces a combinatorial symmetry S_m on $C_m \times C_n$ by mapping a vertex (v_i, w_j) to (v_{i+1}, w_j) for any $0 \leq j \leq m-1$. Similarly, S seen in the polygon C_n induces a symmetry S_n of $C_m \times C_n$ shifting the vertices of C_n.

Both symmetries uniquely extend to combinatorial symmetries $\tilde{S}_m$ and $\tilde{S}_n$ of $E(C_m \times C_n)$. Let T be the combinatorial symmetry of P_{mn} obtained by first applying $\tilde{S}_m$ and then $\tilde{S}_n$. See Table 3.3 for an example of this symmetry on P_{34} on the combinatorial level, and Figure 3.19 for a Schlegel diagram of $C_4 \times C_6$ with these symmetries indicated by arrows.

The geometric realisation of P_{mn} need not have the product structure induced by the construction of Theorem 3.3.1. However, by looking at vertex degrees, and for $m, n \geq 5$, we can split the vertex set of E_{mn} into a set $\mathcal{V}_p$ of vertices that "come from" the product and a set $\mathcal{V}_e$ of vertices that are "added" by the E-construction (as combinatorially, P_{mn} can still be viewed as an instance of the E-construction): A vertex of the product $C_m \times C_n$ always has degree 8 in E_{mn}, as $C_m \times C_n$ is simple, so any vertex has four neighbours and is in four facets. The added vertices have degree $2m$ or $2n$, which are both greater than 8 for $m, n \geq 5$.

The proof is roughly as follows. Suppose there is a geometric realisation T_g of T for some P_{mn}. In the first step we prove that any P_{mn} with the geometric sym-

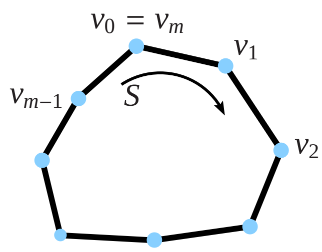


Figure 3.18: The symmetry S of a polygon.

metry T_g has the form of the construction in Theorem 3.3.1. Then, the existence of this symmetry implies that both factors are of the form defined in Theorem 3.5.3. Corollary 3.5.4 finally tells us that for $m, n \geq 5$ there are no such realisations.

As T_g set-wise fixes the vertices of P_{mn}, it also fixes their centroid. After a suitable translation we can assume that T_g is a linear transformation. As m and n are relatively prime, there is a $k_m \in \mathbb{N}$ such that $T_m := T_g^{k_m}$ restricted to the set $\mathcal{V}_p$ acts as $\tilde{S}_m$. Similarly there is a k_n such that $T_n := T_g^{k_n}$ reduces to a realisation of $\tilde{S}_n$. Both T_m and T_n are again linear transformations.

By construction, P_{mn} has two different combinatorial types of facets:

(I) Bipyramids over an m-gon and

(II) bipyramids over an n-gon.

For any facet we call the vertices of the polygon (i.e. those vertices of the facet belonging to $\mathcal{V}_p$) the base vertices.

Let F be a facet of P_{mn} of the first type. The symmetry T_m shifts the base vertices by one and fixes the two apices. Thus, T_m also fixes the centroid c_F of the base vertices of F. Restricted to the hyperplane H_F defined by F, the map T_m is a linear transformation T_m^F in H_F (if we place the origin of H_F in c_F). Now T_m fixes the two apices of F and thus fixes the whole line through the apices. So T_m^F splits into a map fixing the axis and a linear transformation of a two dimensional transversal subspace U. The axis necessarily contains c_F, and U contains the base vertices of F. So the base vertices of F lie in a common two dimensional affine subspace of $\mathbb{R}^4$. Similarly, the base vertices of any other bipyramidal facet with a base equivalent to C_m lie in a common 2-plane. These 2-planes are set-wise preserved by T_m. Hence, they are parallel.

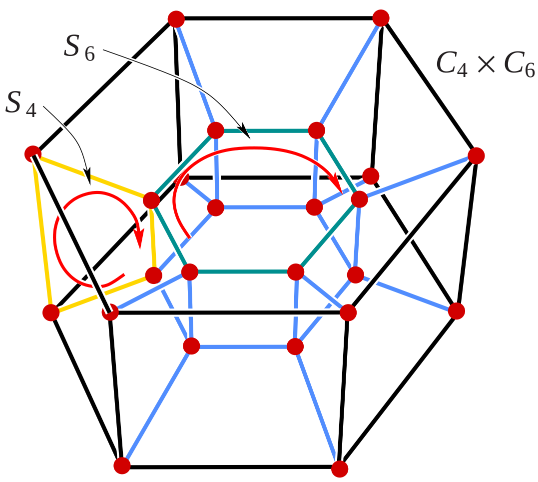


Figure 3.19: The product of S_4 and S_6 on $C_4 \times C_6$.

An example for the symmetries involved in the proof of Theorem 3.5.6
Notation:

- v_0, v_1, v_2: vertices of C_3
- w_0, w_1, w_2, w_3 vertices of C_4
- $e(j)$: edge from vertex number j to $j+1$ (mod 3 or 4) in both polygons.

Number the vertices p_k of P_{34} in the following way:

$0 \le k \le 11$: vertices $(v_{k \text{ div } 4}, w_{k \text{ mod } 4})$
$12 \le k \le 14$: vertices added above $e(k-12) \times C_4$
$15 \le k \le 19$: vertices added above $C_3 \times e(k-15)$

Then the combinatorial symmetries are given as (permutation notation, vertex numbers of p_k):

$$\tilde{S}_3 := (0,4,8)\,(1,5,9)\,(2,6,10)\,(3,7,11)\,(12,13,14)\,(15)\,(16)\,(17)\,(18)$$
$$\tilde{S}_4 := (0,1,2,3)\,(4,5,6,7)\,(8,9,10,11)\,(12)\,(13)\,(14)\,(15,16,17,18)$$
$$T := (0,5,10,3,4,9,2,7,8,1,6,11)\,(12,13,14)\,(15,16,17,18)$$

Table 3.3: The combinatorial symmetries $\tilde{S}_3$, $\tilde{S}_4$, and T acting on P_{34}.

The same argument proves that all bases of facets of the second type do lie in parallel 2-planes. These 2-planes must be transversal to the 2-planes containing the m-gons: Otherwise the vertices in $\mathcal{V}_p$ all lie in a three dimensional subspace. As P_{mn} is 4-dimensional, at least one of the vertices of $\mathcal{V}_e$ has to lie outside this 3-space. But there are no edges between vertices in $\mathcal{V}_e$.

Applying an appropriate linear transformation to P_{mn}, we can assume that the 2-spaces containing the m-gons are parallel to the x_1-x_2-plane and the ones containing the C_n are parallel to the x_3-x_4-plane. T rotates the copies of C_m in P_{mn}, so they must all be equivalent. Similarly, all the copies of the polygon C_n are affinely equivalent. So P_{mn} is an instance of Theorem 3.3.1.

Consider again the facet F with base equivalent to C_m and the restricted map T_m^F. Further restricting T_m^F to the subspace containing the base vertices defines a linear map T_b on $\mathbb{R}^2$ shifting the vertices of a polygon by one. So T_b generates a finite subgroup of $Gl(2,\mathbb{R})$ and therefore must be conjugate to an element of $O(2,\mathbb{R})$ (cf. [77] or [61] for a simple argument proving this). The same argument applies to facets with base C_n. As the copies of C_m and C_n lie in transversal subspaces of $\mathbb{R}^4$, we can apply the conjugation for C_m and C_n simultaneously. Therefore both polygons are regular up to an affine transformation.

Finally look at the n vertices added above facets of P_{mn} of type $C_m \times e$ for an edge e of C_n. Projected onto the 2-space of C_m they lie inside C_m (they form the set S_1 in the construction of Theorem 3.3.1). They are fixed by the symmetry $\tilde{S}_m$. As this map has only one fixed point the points in S_1 must coincide. The same applies to the added vertices above facets of type $e \times C_n$. (Note that, even though T is a symmetry of the E_{44} in Table 3.2, the map $\tilde{S}_4$ is not, and cannot be obtained as a power of T. The set S_0 does not consist of a single point in this polytope.)

Now we are in the situation described in Section 3.5.2. But, according to Corollary 3.5.4, this can only be the case if at least one of m and n is less than 5. This completes the proof of Theorem 3.5.6. □

3.5.7 R . This is not the strongest possible form of this theorem. It applies also to many products in which one of the polygons has four or less vertices, or m and n are not relatively prime. However, in this case, we need to be a bit more careful in the proof, as it is not always possible to uniquely split the vertices into "product vertices" and "E-construction-vertices." Furthermore, one has to argue that *one* symmetrically realisable factor does not suffice to give a realisation of the whole polytope with a geometric realisation of the constructed combinatorial symmetry.

With a similar argument as the one used in the proof of the theorem, one also proves that Corollary 3.5.4 describes all possible cases in which P_{mn} can have the product $\mathbb{Z}_m \times \mathbb{Z}_n$ of two cyclic groups induced by the rotation of the vertices in the two polygons as a subgroup of its geometric symmetry group. In this case, we do not need m and n to be relatively prime, as, in addition to their product, the two symmetries $\tilde{S}_m$ and $\tilde{S}_n$ itself are contained in $\mathbb{Z}_m \times \mathbb{Z}_n$ acting on P_{mn}.

3.5.8 C . *The combinatorial symmetry group of E_{mn} contains a subgroup G isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$ induced by rotation in the two polygon factors.*

The geometric symmetry group of a polytope P_{mn} combinatorially equivalent to E_{mn} can contain a subgroup inducing G on the face lattice only for

$$(m, n) \in \{(3, 3), (3, 4), (3, 5), (3, 6), (4, 4)\}$$

(up to interchanging m and n). □

3.5.9 R . Gévay [39] pointed out that along the lines of Theorem 3.5.6 one can also prove that the only "perfect" polytopes among the realisations of the E_{mn} are the regular 24-cell and E_{33}, constructed as in Corollary 3.3.3 from regular triangles with intersection ratio $r = 1/2$.

A rough definition of perfectness is as follows: A geometric realisation P of a polytope is *perfect* if all other geometric realisations having, up to conjugation with an isometry, the same subset of the affine transformations as symmetry group, are already similar (in the geometric sense) to P. See [36] for a precise definition.

3.5.4 R S E_{33} E_{44}

We determine the degrees of freedom that we have in the choice of coordinates for E_{33}. To achieve this, we are interested in the dimension of the realisation space $\mathcal{R}(E_{33})$ and the projective realisation space $\mathcal{R}_{\text{proj}}(E_{33})$ of this polytope. See Section 1.3.2.4 for a precise definition and some background.

Further, we present a simple 4-parameter family of $E_{44}s$ in the following section. This proves that the projective realisation space of E_{44}, which is the 24-cell, is at least four dimensional. It is possible to exhibit more degrees of freedom, but this is rather technical. As this would still not yield *all* possible degrees of freedom, we are content with the simpler version.

3.5.4.1 The Realisation Space of E_{33}

The vertex sets of all realisations of E_{33} that one can obtain from Theorem 3.3.1 contain the vertex set of an orthogonal product $C_3 \times C_3$ of two triangles. This reduces the number of degrees of freedom that we can obtain by analysing the construction in Theorem 3.4.1, compared to arbitrary geometric realisations. The next theorem determines the dimension of the space of all realisations of E_{33} that are projectively equivalent to a realisation containing the orthogonal product $C_3 \times C_3$.

3.5.10 T . $\dim(\mathcal{R}_{\text{proj}}(E_{33})) \geq 9$.

Before we prove this theorem we introduce a special way to construct realisations of two triangles and their E-polytopes satisfying the conditions (A) and (B) of Table 3.1. This will make it easy to count the degrees of freedom afterwards.

3.5.11 T . *Given two (arbitrary) triangles Δ and Δ', there is an open subset R in $\mathbb{R}^9$ such that, if we take the nine entries of a vector in R as the nine ratios appearing in (B)(3) of Table 3.1 (in some previously fixed order), then there is a realisation of E_{33} having these intersection ratios.*

Proof. This is basically proven by describing a realisation as a solution of a set of linear equations. We have to introduce some notation to write down these equations. It is convenient to note the ratios in a slightly different way as before. We transform any ratio r into $\frac{r}{1+r}$. With this, a pair of inverse ratios is r and $\frac{1}{r}$. Denote the nine ratios involved in the construction by r_{xy} for $x \in \{a, b, c\}$ and $y \in \{a', b', c'\}$. See Figure 3.20 and Figure 3.21 for illustrations of the following definitions.

Fix two triangles Δ and Δ' and let a, b, c be the sides of Δ and a', b', c' the sides of Δ'. By translating the triangles if necessary, we can assume that they both contain the origin.

Let g_a, g_b and g_c define lines outside Δ parallel to a, b, and c at distance δ_a, δ_b, and δ_c, respectively. These will afterwards contain the vertices of $E(\Delta)$, which is a triangle containing the vertices of Δ in its edges. Similarly, define lines $g_{a'}$, $g_{b'}$ and $g_{c'}$ at distances δ'_a, δ'_b, resp. δ'_c for Δ'.

Let $l_{aa'}$, $l_{ab'}$, and $l_{ac'}$ define lines parallel to a on the other side of a as g_a, at distances $r_{aa'}\delta_a$, $r_{ab'}\delta_a$, and $r_{ac'}\delta_a$ from a. Similarly, define the lines $l_{ba'}$, $l_{bb'}$, $l_{bc'}$, $l_{ca'}$, $l_{cb'}$, and $l_{cc'}$ parallel to b and c. Thus, any segment starting on g_a and ending on $l_{aa'}$ is divided by a with a ratio of $r_{aa'}$, and similar for the other eight segments.

For the triangle Δ' we define lines $l'_{aa'}$, $l'_{ab'}$, and $l'_{ac'}$ at distances $1/r_{aa'}$, $1/r_{ba'}$ and $1/r_{ca'}$, respectively, parallel to a' and on the other side as $g_{a'}$. Similarly, we define lines parallel to b' and c'. Finally, we define (outward pointing) normal vectors n_a, n_b, n_c, $n_{a'}$, $n_{b'}$, $n_{c'}$ and levels λ_a, λ_b, λ_c, $\lambda_{a'}$, $\lambda_{b'}$, $\lambda_{c'}$ such that points $x \in a$ satisfy $\langle n_a, x\rangle - \lambda_a = 0$, and similarly for the other edges.

Consider now e.g. the ratio $r_{ab'}$. Choose a vertex v_a of $E(\Delta)$ on g_a, a point w_a on the line $l'_{ab'}$ and in the interior of Δ', a vertex $v'_{b'}$ of $E(\Delta')$ lying on $g_{b'}$ and a point $w'_{b'}$ in the interior of Δ on the line $l_{ab'}$. See Figure 3.22 for a enlarged cutout of the relevant parts of Figure 3.20 and Figure 3.21. The points w_a and $w'_{b'}$ will become the corresponding points to v_a and $v'_{b'}$ under the maps α_0 and α_1 of (B)(2) in Table 3.1. The segment $s_{ab'}$ between v_a and $w'_{b'}$ is intersected by Δ with a ratio of $r_{ab'}$, and the segment $s_{b'a}$ between $v'_{b'}$ and w_a is intersected by Δ' with ratio $1/r_{ab'}$.

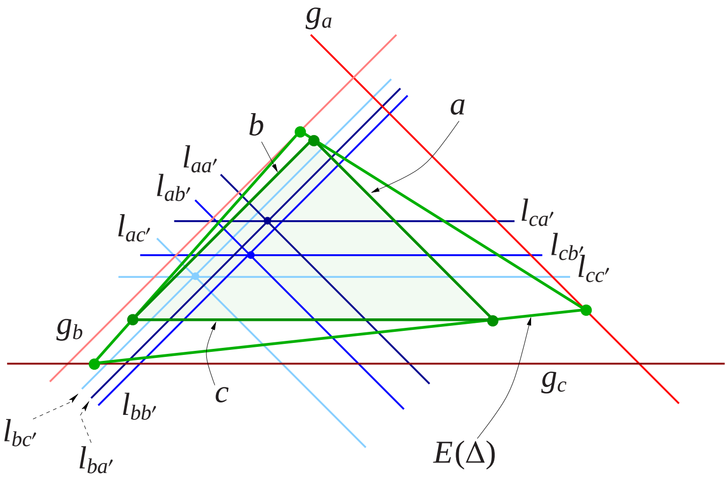


Figure 3.20: Construction of the triangles: The first factor.

So the condition set by the ratio $r_{ab'}$ will be satisfied by this choice of w_a and $w'_{b'}$.

To satisfy all conditions on the ratios that involve w_a, we have to choose w_a such that it lies as well on the lines $l'_{aa'}$, $l'_{ab'}$ and $l'_{ac'}$ and in the interior of Δ'. Similar conditions hold for the two other points inside Δ and for the three points inside Δ'.

Therefore, finding a feasible solution amounts to finding a solution to the following set of 18 linear equations:

$$
\begin{aligned}
\lambda_a &= \langle n_a, w'_{a'}\rangle + r_{aa'}\delta_a & \qquad \lambda_{a'} &= \langle n_{a'}, w_a\rangle + 1/r_{aa'}\delta_{a'}\\
\lambda_b &= \langle n_b, w'_{a'}\rangle + r_{ba'}\delta_b & \lambda_{a'} &= \langle n_{a'}, w_b\rangle + 1/r_{ba'}\delta_{a'}\\
\lambda_c &= \langle n_c, w'_{a'}\rangle + r_{ca'}\delta_c & \lambda_{a'} &= \langle n_{a'}, w_c\rangle + 1/r_{ca'}\delta_{a'}\\
\lambda_a &= \langle n_a, w'_{b'}\rangle + r_{ab'}\delta_a & \lambda_{b'} &= \langle n_{b'}, w_a\rangle + 1/r_{aa'}\delta_{b'}\\
\lambda_b &= \langle n_b, w'_{b'}\rangle + r_{bb'}\delta_b & \lambda_{b'} &= \langle n_{b'}, w_b\rangle + 1/r_{ba'}\delta_{b'}\\
\lambda_c &= \langle n_c, w'_{b'}\rangle + r_{cb'}\delta_c & \lambda_{b'} &= \langle n_{b'}, w_c\rangle + 1/r_{ca'}\delta_{b'}\\
\lambda_a &= \langle n_a, w'_{c'}\rangle + r_{ac'}\delta_a & \lambda_{c'} &= \langle n_{c'}, w_a\rangle + 1/r_{aa'}\delta_{c'}\\
\lambda_b &= \langle n_b, w'_{c'}\rangle + r_{bc'}\delta_b & \lambda_{c'} &= \langle n_{c'}, w_b\rangle + 1/r_{ba'}\delta_{c'}\\
\lambda_c &= \langle n_c, w'_{c'}\rangle + r_{cc'}\delta_c & \lambda_{c'} &= \langle n_{c'}, w_c\rangle + 1/r_{ca'}\delta_{c'}
\end{aligned}
$$

Here the coordinates of the points w_a, w_b, w_c, $w'_{a'}$, $w'_{b'}$ and $w'_{c'}$ and the distances δ_a, δ_b, δ_c, $\delta_{a'}$, $\delta_{b'}$, and $\delta_{c'}$ are the free variables, and the ratios are the parameters. The

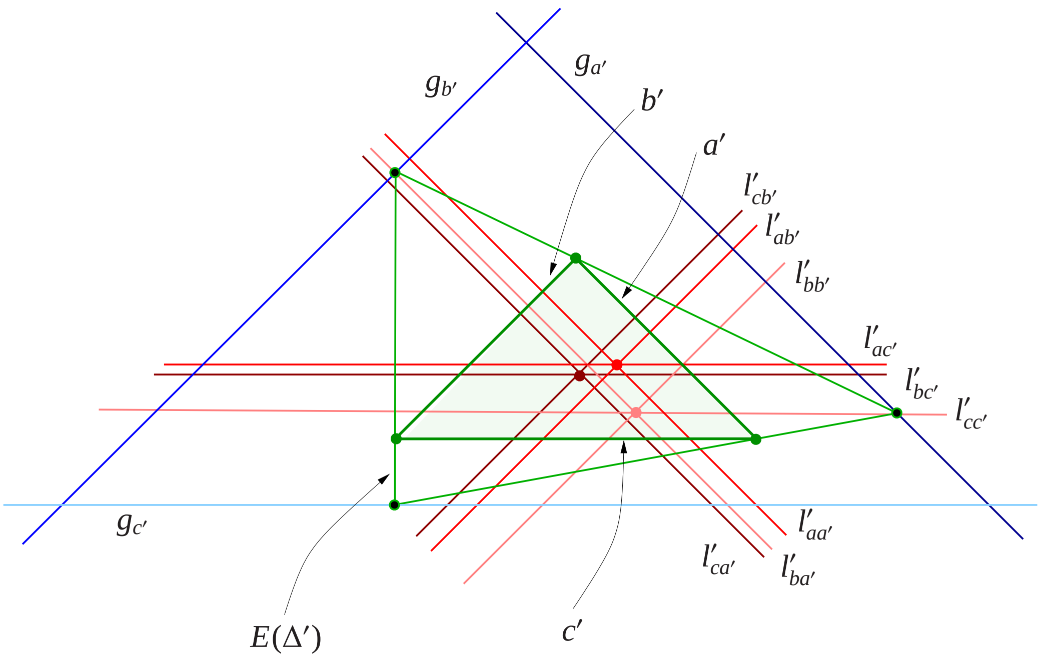


Figure 3.21: Construction of the triangles: The second factor.

0	21	0	21	9	− 6	60	− 19
42	−21	0	21	−3	−12	−12	29
−42	−21	0	21	−6	−3	−48	−31
0	21	42	−21	24	33	−10	−9
42	−21	42	−21	−72	−15	2	−1
−42	−21	42	−21	48	−39	8	−11
0	21	−42	−21				
42	−21	−42	−21				
−42	−21	−42	−21				

Table 3.4: The coordinates of a feasible non-degenerate solution. See Figures 3.4 and 3.5 for a drawing of the two factors.

first and the second set of equations are connected via the ratios.

As the equations depend smoothly on the nine parameters it suffices to show that there exists at least one feasible solution of this system. Such a solution is shown in the Figures 3.20 and 3.21 and in Table 3.4 (for some fixed product of two triangles, but this can be projectively transformed to any other).

Finally, to obtain $E(\Delta)$, we have to choose vertices on the lines g_a, g_b, and g_c such that the edges contain the vertices of Δ. Unless the distances δ_a, δ_b, and δ_c are too large compared to the size of Δ there are always two solutions to this problem.

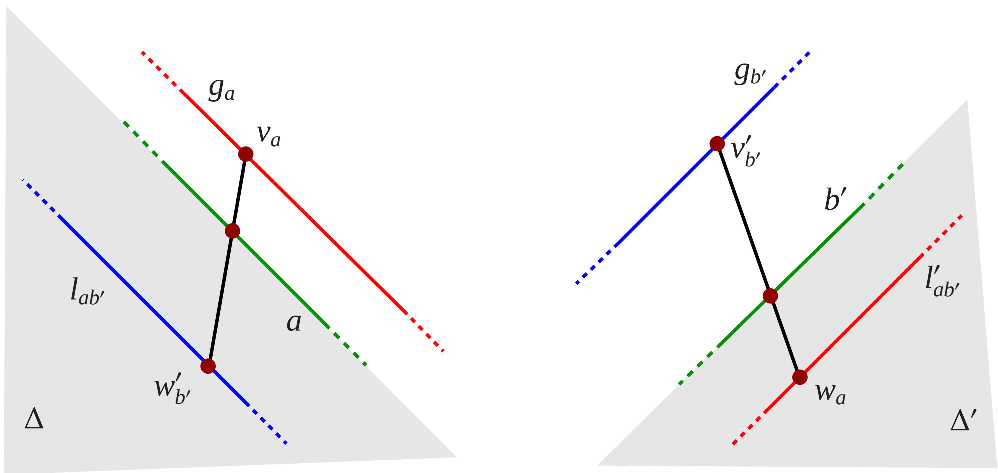


Figure 3.22: The condition for one corresponding pair of vertices.

Similarly, we can construct $E(\Delta')$. □

Now the proof of Theorem 3.5.10 is straightforward:

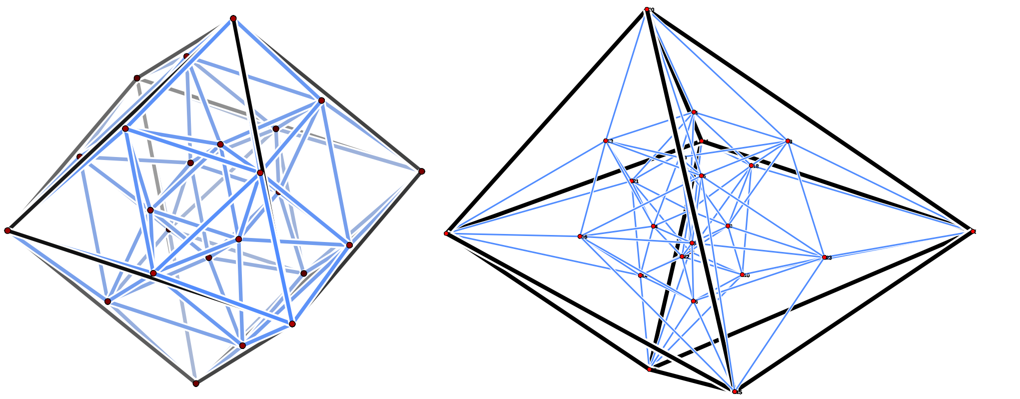
Proof of Theorem 3.5.10. All triangles in $\mathbb{R}^2$ are projectively equivalent. Therefore, up to projective equivalence, there is only one geometric realisation of an orthogonal product of two triangles. So we can fix our preferred orthogonal product of two triangles and count the degrees of freedom for adding the remaining vertices without having to worry about projective equivalence anymore. But according to the previous Theorem 3.5.11 we have, for any choice of two triangles, nine degrees of freedom for the choice of the remaining vertices. □

3.5.12 R . There might still be geometric realisations of a polytope combinatorially equivalent to E_{33} that are not projectively equivalent to a polytope containing an orthogonal product of two triangles. Thus, a priori, Theorem 3.5.10 describes only a subset of the whole realisation space $\mathcal{R}_{\text{proj}}(E_{33})$.

3.5.4.2 The 24-Cell

Our method for the realisation of the E-construction of products of polygons also provides new (non-regular) geometric realisations of the 24-cell.

For $m, n > 3$ we cannot determine the degrees of freedom in the above way anymore. Taking the mn ratios as input we obtain $2mn$ equations for only $3(m+n)$ variables. This is not merely a problem of the method; there are explicit additional



(a) Schlegel diagram of the regular 24-cell.

(b) Schlegel diagram of a polytope in the family of Table 3.5: $a_1, b_1 = \frac{1}{2}$.

Figure 3.23: 24-cells I.

restrictions on a realisation. However, also for the 24-cell it is not difficult to construct some projectively non-equivalent geometric realisations.

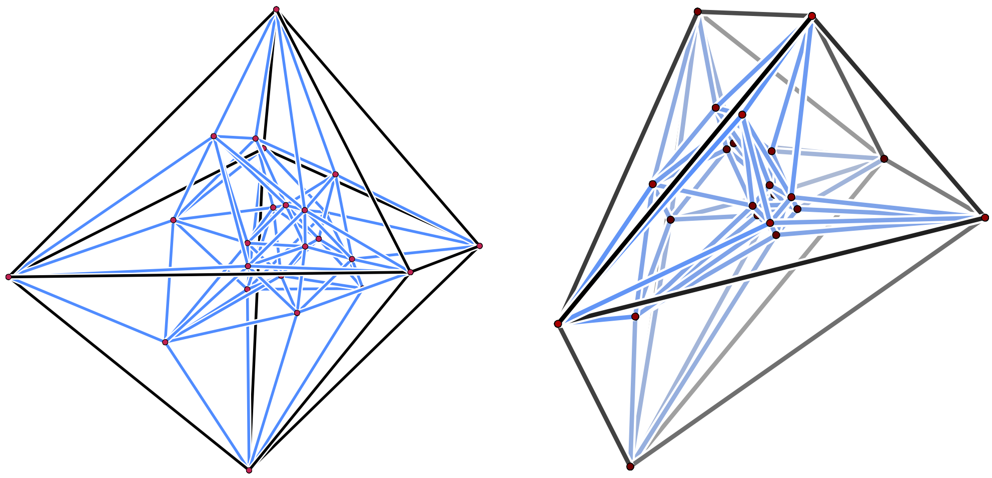
Table 3.5 shows a simple example of a 4-parameter family of 24-cells, where all four parameters range in the open interval from −1 to 1. This family spans a 4-dimensional subset of the projective realisation space, which can be seen in the following way.

−1	−1	−1	−1
1	1	−1	−1
1	−1	1	−1
−1	1	1	−1
1	−1	−1	1
−1	1	−1	1
−1	−1	1	1
1	1	1	1
1	−1	−1	−1
−1	1	−1	−1
−1	−1	1	−1
1	1	1	−1
−1	−1	−1	1
1	1	−1	1
1	−1	1	1
−1	1	1	1
a_1	b_1	a_2	$-2-b_2$
a_1	b_1	$2-a_2$	b_2
a_1	b_1	a_2	$2-b_2$
a_1	b_1	$-2-a_2$	b_2
a_1	$2-b_1$	a_2	b_2
$-2-a_1$	b_1	a_2	b_2
a_1	$-2-b_1$	a_2	b_2
$2-a_1$	b_1	a_2	b_2

Table 3.5: Vertices of a 4-parameter family of 24-cells. For $a_1, a_2, b_1, b_2 = 0$ this is the well known regular realisation.

−1	5/4	−1	1
−1	5/4	−1	−1
−1	5/4	1	−1
−1	5/4	5/3	1
−1	−1	−1	1
−1	−1	−1	−1
−1	−1	1	−1
−1	−1	5/3	1
1	−1	−1	1
1	−1	−1	−1
1	−1	1	−1
1	−1	5/3	1
1	23/12	−1	1
1	23/12	−1	−1
1	23/12	1	−1
1	23/12	5/3	1
−1/2	−1/2	−3/2	1/2
−1/2	−1/2	−5/6	−3/2
−1/2	−1/2	17/6	−1/2
−1/2	−1/2	1/2	5/2
−3/2	−5/6	−1/2	−1/2
1/2	−3/2	−1/2	−1/2
5/2	1/2	−1/2	−1/2
−1/2	10/3	−1/2	−1/2

Table 3.6: A 24-cell without any projective automorphisms.



(a) Schlegel diagram of a polytope in the family of Table 3.5: $a_1, a_2 = \frac{1}{2}$. *(b)* Schlegel diagram of the 24-cell in Table 3.6.

Figure 3.24: 24-cells II.

The vertex set of the regular 24-cell contains the vertex sets of three different regular cubes: If you set all parameters to zero, then (in the order given in Table 3.5) the first sixteen, the last sixteen and the first and last eight vertices each form a regular cube. Their 2-faces (which are squares) are not anymore present as 2-faces in the 24-cell, but their vertices still lie on a codimension-2-subspace (see e.g. the vertices $15, 16, 17, 18$ in Table 3.5). Subspaces are preserved by any projective transformation.

If we let the parameters diverge from zero then we destroy some of these "internal" squares. This necessarily results in projectively different 24-cells. Hence, we have proven the following theorem.

3.5.13 T **[24-cells].** $\dim(\mathcal{R}_{\text{proj}}(E_{44})) \geq 4$. □

The "broken" squares can also be seen in the Schlegel diagrams in Figures 3.23 and 3.24: Observe the three squares contained in the octahedral face onto which the polytope is projected.

3.5.14 R **.** Not all possible realisations of the 24-cell are contained in this 4-parameter family. The 24-cell in Table 3.6 is also a result of the construction in Theorem 3.3.1 and has no projective automorphisms.

C 4

F V E

4.1 I

Until recently, 2-simple and 2-simplicial 4-polytopes (i.e. $(2,2)$-polytopes) seemed hard to construct. With our new constructions we have now achieved quite some flexibility, and a wealth of examples. In particular, the vertex truncation method of Section 2.5.1 — and its counterpart for simple polytopes in Section 2.5.2 — make it easy to construct explicit instances, and easily test properties of $(2,2)$-polytopes.

This chapter combines the results obtained in the previous two chapters and relates them to the "outside polytope world" containing all the polytopes that are not 2-simple and 2-simplicial. We present applications, discuss several other approaches to the construction of $(2,2)$-polytopes and collect lots of examples.

The first part of this chapter is a list of results about the flag vectors of $(2,2)$-polytopes and their influence on the flag vector classification problem. In particular, we show that a $(2,2)$-polytope is not determined by its flag vector. There are (usually really many) combinatorially different $(2,2)$-polytopes with the same flag vector. We give a lower bound on the number of $(2,2)$-polytopes and show relations between the flag vector of $(2,2)$-polytopes and general polytopes.

The second part summarises other known methods (both older and more recent) for obtaining $(2,2)$-polytopes. We work out which of the examples can be seen as instances of the E-construction and which cannot.

There are some old known (r,s)-polytopes. In particular, there is the well known class of hypersimplices. Furthermore, several regular and semi-regular polytopes are r-simple and s-simplicial for $r,s \geq 2$. We present Wythoff's construction, which uses special Coxeter groups for the description of a regular polytope, and we discuss some regular polytopes obtained by Gévay.

Until recently, only finitely many $(2,2)$-polytopes were known. The first infinite family of $(2,2)$-polytopes is that of Eppstein, Kuperberg, and Ziegler, which is now a special case of Theorem 2.5.15.

A recent approach via "reverse shellings" by Werner has produced a new self-dual $(2,2)$-polytope on 9 vertices, which is not included in our construction. We show that no non-trivial (i.e. different from the simplex) $(2,2)$-polytope with eight or less vertices can exist.

The third and last part of this chapter contains tables with a list of polytopes obtained from the E-construction with up to 50 vertices (complete up to 19 vertices), the previously known $(2,2)$-polytopes, the known examples of (r,s)-polytopes for $r,s \geq 2$ in higher dimensions, some infinite families of $(2,2)$-polytopes, a couple of particularly interesting examples of $(2,2)$-polytopes with larger number of vertices, and some known polytopes that can be obtained from the E-construction, but are not 2-simple and 2-simplicial.

For many of the examples, explicit rational coordinates and combinatorial data is available in the `polymake` format.

4.2 P (2, 2)-P

We give several results about flag vectors and combinatorial types of $(2, 2)$-polytopes. The f-vector of a $(2, 2)$-polytope is necessarily symmetric, that is

$$f_0 = f_3 \qquad \text{and} \qquad f_1 = f_2.$$

f_0 and f_1 also determine the flag vector, as we have only one additional independent entry in dimension four, by the Dehn–Sommerville Equations of Theorem 1.2.17. It is given by $f_{02} = 3f_0$, that is, $f_{03} = 2f_0 + f_1$.

The dual of a 2-simplicial 4-polytope is 2-simple, and vice versa. Hence, the dual of a $(2, 2)$-polytope is again a $(2, 2)$-polytope. This gives the following result.

4.2.1 P **.** *Any vertex of a* $(2, 2)$-*polytope P has even degree in the vertex–edge graph of P.*

Proof. The dual of a $(2, 2)$-polytope is 2-simplicial. Hence, its facets are simplicial. But any simplicial 3-polytope has an even number of 2-faces. □

In general, $(2, 2)$-polytopes are not self-dual. The hypersimplex is the smallest example: it has five simplex facets, and five octahedral facets, while the ten facets of the dual are bipyramids over a triangle. Hence, in general, there are at least two combinatorial types of $(2, 2)$-polytopes for a given number of vertices. For larger number of vertices, there are more than this. Our constructions allow us to produce exponentially many different (even rational) $(2, 2)$-polytopes. We demonstrate this with one of the families obtained in Section 2.5.

4.2.2 P **[Exponentially Many (2, 2)-Polytopes].** *The number of combinatorially distinct* $(2, 2)$-*polytopes* $D(P_n^4)$ *constructed in Corollary 2.5.11 with flag vector*

$$\text{flag}(D(P_n^4)) = (10 + 4n, 30 + 18n, 30 + 18n, 10 + 4n; 50 + 26n)$$

grows exponentially with n.

Proof. There are exponentially many stacked 4-polytopes with $n+5$ vertices. This follows from the fact that there are exponentially many (unlabelled) trees of maximal degree 5 on $n + 1$ vertices.

Hence we are done, if we show that the combinatorial type of any stacked 4-polytope P_n^4 can be reconstructed from its vertex truncation $D(P_n^4)$. The facets of $D(P_n^4)$ are on the one hand truncated simplices F', which are octahedra, and on the other hand the vertex figures F_v of P_n^4, which are stacked. Furthermore, two of the octahedra F' and G' are adjacent if and only if the corresponding facets F and G of P_n^4 are adjacent. Hence, we obtain the dual graph of P_n^4 from $D(P_n^4)$. This determines the combinatorial type of P_n^4 by the Reconstruction Theorem of Blind and Mani, see Theorem 1.3.31. □

More generally, there are exponentially many (in the number of vertices) $(2,2)$-polytopes. The choice of the family from Corollary 2.5.11 for the proof of this proposition was arbitrary. Similar arguments show the same growth rate also for other families of $(2,2)$-polytopes. Included in this proposition is also the following.

4.2.3 C [Non-Isomorphic $(2,2)$-Polytopes]. *There are exponentially many non-isomorphic $(2,2)$-polytopes with the same flag vector.* □

The first known pair of examples that are not dual to each other are the E-construction applied to a product of two triangles and to a twice truncated simplex. Both $(2,2)$-polytopes have 19 vertices (cf. Section 4.4).

f-vectors of $(2,2)$-polytopes have at most two independent parameters by the above computations. The next propositions show that we do indeed need both.

4.2.4 P . *For any $f_0 \geq 26$ there is a $(2,2)$-polytope with f_0 vertices.*

Proof. Truncating a vertex of a simple 4-polytope adds $(3,6,4,1)$ to the f-vector. By Proposition 2.5.13, if we have the E-construction of a simple polytope P, then we can extend this to one for the truncation $\mathrm{tr}(P;v)$ of P at any vertex v. This adds $(4,18,18,4)$ to the f-vector of the E-construction.

(1) The truncation of a prism $\mathrm{Pr}(\Delta_3)$ over the simplex Δ_3 at one vertex, (2) the product $C_3 \times C_6$, (3) the truncation of $\square_4$ at one vertex, and (4) the product $C_4 \times C_5$ are four different simple polytopes. Their E-polytopes have 26, 27, 28, and 29 vertices, respectively. □

There are lots of $(2,2)$-spheres with less vertices, but for some of them it is still unknown whether they are polytopal.

4.2.5 P . *There are $(2,2)$-polytopes that have the same numbers of vertices (and facets), but different numbers of edges (and ridges).*

Proof. The E-construction of a product of a square and an hexagon has a polytopal realisation, by Theorem 3.4.1. Its f-vector is $(34,144,144,34)$ (cf. the computation in (3.2.1)). On the other hand, vertex truncation of a stack of six simplices as in Proposition 2.5.7 yields $D(P^4_6)$ with f-vector $(34,138,138,34)$. □

Similarly, there are $(2,2)$-polytopes with the same number of edges, but a different number of vertices. For example, $E_2(C_4 \times C_5)$ and $E_2(\mathrm{tr}(\Delta_3; 5 \text{ vertices}))$ have both 120 edges, but the first has 29 and the second 30 vertices.

One can tell from the flag vector whether a polytope is 2-simplicial, since this is equivalent to the condition $f_{02} = 3f_2$. Similarly, 2-simplicity can be read off. This is so, because there cannot be any 2-faces with less than three vertices, so the aggregated value of the flag vector already determines each single case. Our next proposition shows that there is no similar criterion to derive 2-simplicity or 2-simpliciality already from the f-vector.

4.2.6 P . *A* $(2,2)$*- and a non-*$(2,2)$*-polytope can have the same* f*-vector.*

Proof. Using a hyperbolic gluing construction for a stack of n 600-cells, Eppstein, Kuperberg and Ziegler produced a family of simplicial edge-tangent 4-polytopes Q_n with f-vectors $(106n+14, 666n+54, 666n+54, 106n+14)$; see Section 4.3.4. Applying the E-construction via Theorem 2.5.14, one obtains a family of $(2,2)$-polytopes $E_1(Q_n)$ with f-vector $f(E_1(Q_n)) = (54+666n, 240+3360n, 240+3360n, 54+666n)$.

Set $n = 13$. Then we have a $(2,2)$-polytope with f-vector

$$f(E_1(Q_{13})) = (8712, 43920, 43920, 8712).$$

This polytope has lots of facets that are bipyramids over pentagons, and lots of "regular" vertices that are contained in exactly 12 such bipyramids, with a dodecahedral vertex figure.

We truncate 80 such "regular vertices" and stack pyramids over the resulting dodecahedral facets. One such truncation operation adds $(19, 30, 12, 1)$, and one stacking operation adds $(1, 20, 30, 11)$ to the f-vector. So in total, we add $(1600, 4000, 3360, 960)$ by this.

Furthermore, we stack 80 of the bipyramidal facets that were not involved in the previous operation. One such stacking operation adds $(1, 7, 15, 9)$, so in total we add another $(80, 560, 1200, 720)$. Hence, the polytope P we obtain has the f-vector

$$f(P) = (10392, 48480, 48480, 10392).$$

It is not a $(2,2)$-polytope anymore.

On the other hand, the $(2,2)$-polytope $D(C^4_{577})$ of Proposition 2.5.12 has exactly the same f-vector. □

In connection with their E-construction, Eppstein, Kuperberg, and Ziegler [33] proposed a new quantity that might be interesting with respect to the classification problem.

4.2.7 D **[Fatness].** Let P be a 4-polytope with f-vector (f_0, f_1, f_2, f_3) different from the simplex. The *fatness* $\mathbf{F}(P)$ of P is defined as the quotient

$$\mathbf{F}(P) = \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}.$$

A similar quantity was considered earlier by Avis, Bremner, and Seidel [6]. Eppstein, Kuperberg, and Ziegler showed that this quotient is unbounded for 3-dimensional CW spheres. For polytopes, they provided an infinite series with fatness around five and gave an example of a polytope with fatness approximately 5.048. See Section 4.3.4 for this. There is also a good review by Ziegler [90].

Looking at the known inequalities for flag vectors in equations (1.4.3)–(1.4.8), we see that (2, 2)-polytopes satisfy two of these with equality, namely the third and the forth. Fatness "measures" where on this line the polytopes lie exactly. Namely, the higher the fatness of a polytope is, the closer it lies to the ray l_2 in (1.4.9).

The fatness of polytopes produced from the E-construction applied to simple 4-polytopes is bounded by six. This is immediate from the f-vector computation in (2.3.1) of Section 2.3.

The family E_{mn} of E-polytopes obtained from products of polygons in Theorem 3.4.1 has essential flag vector

$$\mathrm{flag}(E_{mn}) \;=\; (mn+m+n, 6mn, 6mn, mn+m+n; 8mn+2m+2n),$$

so their fatness is

$$\mathbf{F}(E_{mn}) \;=\; \frac{12mn-20}{2mn+2m+2n-10} \longrightarrow 6 \qquad \text{for } m,n\to\infty.$$

Thus, for $m, n \geq 10$, our polytopes are "fatter" than the example of Eppstein, Kuperberg, and Ziegler. As products of polygons are simple, our family of polytopes is also "best possible" within this setting.

However, recently Ziegler [92] constructed a family of much fatter polytopes by a method completely unrelated to the E-construction. They are neither 2-simple nor 2-simplicial. The fatness is bounded by 9.

Until recently it seemed, that (2, 2)-polytopes are among the fattest polytopes, if the number of vertices is fixed. In the next proposition we produce two 4-polytopes with the same number of vertices, one of which is not a (2, 2)-polytope, where the (2, 2)-example is less fat.

4.2.8 P . *There is a (2, 2)-polytope with the same number of vertices and facets as a non-(2, 2)-polytope, but fewer edges and ridges.*

Proof. The "bipyramidal 720-cell" is defined as $E_2(\text{120-cell}) = D(\text{600-cell})^\Delta$. It has f-vector (720, 3600, 3600, 720), see Section 4.3.3.

We perform some operations on this polytope that destroy 2-simplicity and 2-simpliciality: We truncate two vertices with dodecahedral vertex figure, and stack pyramids on the resulting dodecahedral facets, and we also stack pyramids onto two bipyramidal facets. We obtain a new polytope E' with the f-vector (762, 3714, 3714, 762).

On the other hand, vertex truncation applied to a stack of 42 cross polytopes yields the (2, 2)-polytope $D(C^4_{42})$ with f-vector (762, 3540, 3540, 762). □

It would be nice to know more about fatness of 4-polytopes and implications for the flag vector classification. In particular, a proof of the (un-)boundedness would be helpful.

4.3 F C S P

Practically all examples of 2-simple and $(d-2)$-simplicial d-polytopes for $d \geq 4$ (or, $(2, d-2)$-polytopes for short) that appear in the literature may be seen as special instances of the E-construction. They can be realised by one of the constructions presented in the previous two chapters.

In particular, the examples of Eppstein, Kuperberg, and Ziegler arise in our construction as the special case where P is a simplicial 4-polytope with an edge-tangent realisation (for parameter $t = 1$ in the construction), or equivalently a simple 4-polytope with a ridge-tangent realisation (for parameter $t = 2$ in the construction).

Prior to this E-construction, only finitely many $(2, d-2)$-polytopes were known in each fixed dimension $d \geq 4$. All but one arise from regular and semi-regular polytopes, where the t-face tangency conditions can be enforced simply by scaling — but will typically yield irrational coordinates for the vertices, and the realisation has no apparent degrees of freedom. These examples include in particular the simplex, the 24-cell, and the hypersimplices. Braden gave the only example of a non-uniform $(2, 2)$-polytope. It can now be obtained as the vertex truncation of a stacked simplex, see Corollary 2.5.11 for more on this polytope.

We discuss all previously known examples of, and constructions leading to $(2, d-2)$-polytopes in the next sections. Some polytopes turn up several times in this collection, as they can be obtained in several different ways.

There is a recent approach to finding small $(2, 2)$-polytopes by Werner, which led to a new $(2, 2)$-polytope with 9 vertices and facets and 26 edges and ridges. We discuss this in Section 4.3.5 and prove that it is the smallest non-trivial $(2, 2)$-polytope. Only the simplex has fewer vertices.

In the end we have included some basic ideas towards a generalised version of the E-construction for a d-polytope P and parameter $t = d-2$. In this approach, we do not necessarily stack above all facets of P anymore, and we allow that bipyramids over the ridges break into two pyramids. So far, we produced one further $(2, 2)$-polytope with this method, which has 16 vertices and 56 edges. However, otherwise this construction still lacks a systematic treatment.

A couple of other polytopes, which are neither 2-simple nor $(d-2)$-simplicial, but have some other interesting properties previously described in the literature occur now also as instances of the E-construction. We list some of them in the context of the construction they occur.

In the last section of this chapter, Section 4.4, we subsume all these polytopes into several tables.

4.3.1 G ¨ ' E

We start with (r, s)-polytopes that can be found in Grünbaum's book [44, p. 65,66]. There are three interesting families of polytopes mentioned, two of which can be obtained via the E-construction. The polytopes in the third family are 3-simple and $(d-3)$-simplicial, which is not possible for a polytope from the E-construction (except, of course, in dimension five by duality). However, we can apply the E-construction to it and obtain one of the other families.
Here are the three families:

$$K_k^d := \left\{ x \in \mathbb{R}^{d+1} \ : \ 0 \le x_i \le 1\,,\ \sum\nolimits_{i=1}^{d+1} x_i = k \right\} \qquad \text{for } 1 \le k \le d,\quad d \ge 2$$

"hypersimplices"

$$M^d := \left\{ x \in \mathbb{R}^d \ : \ |x_i| \le 1\,,\ \sum\nolimits_{i=1}^{d} |x_i| \le d-2 \right\}$$

$$N^d := \left\{ x \in \mathbb{R}^d \ : \ \sum\nolimits_{i=1}^{d} \varepsilon_i x_i \le d-2\,,\ \varepsilon_i = \pm 1\,,\ \#\{\varepsilon_i = 1\} \text{ is odd} \right\}$$

"dual half-cubes."

The polytopes K_k^d are called *hypersimplices*. Geometrically, they are obtained as the intersection of the standard $(d+1)$-dimensional $0/1$-cube C^{d+1} with the hyperplane

$$H_k := \left\{ x \in \mathbb{R}^{d+1} \ : \ \sum\nolimits_{i=1}^{d+1} x_i = k \right\}. \tag{4.3.1}$$

for $1 \le k \le d$. From this description it is immediate that K_k^d is combinatorially equivalent to K_{d-k+1}^d for any $1 \le k \le d$. Hence, there are only $\lfloor \frac{d}{2} \rfloor$ combinatorial types of hypersimplices in each dimension $d \ge 3$.

The polytopes M^d are obtained from the cube with vertices $\{-1, +1\}^d$ by truncating all vertices with a hyperplane whose normal vector is the point vector of that vertex in such a way, that from any 2-face only a single (inner) point remains.

The duals of the polytopes N^d constitute the family of *half cubes*. Recall, that we defined the d-cube $\square_d$ with vertices $\{-1, +1\}^d$. Geometrically, N^d is obtained as the dual of the convex hull of all "odd" vertices of $\square_d$, that is, all vertices that have an odd number of 1's in their vector. (Equivalently, one could also take the "even" vertices.)

See Figure 4.1 for two examples of hypersimplices. The three families of polytopes have the f-vectors shown in Table 4.1. We collect their properties in the following proposition.

$$f_j(K_k^d) = \begin{cases} \binom{d+1}{k} & \text{for } j = 0 \\ \binom{d+1}{j+1} \sum_{i=1}^{k} \binom{d-j}{d-j-k+i} & \text{otherwise} \end{cases}$$

$$f_j(M^d) = \begin{cases} \binom{d}{2} 2^{d-2} & \text{for } j = 0 \\ \binom{d}{2} (d-2)2^{d-1} & \text{for } j = 1 \\ \binom{d}{3} (d-1)2^{d-1} & \text{for } j = 2 \\ 2^{d-1}(d-j+1) \binom{d}{d-j-1} + 2^{d-j} \binom{d}{d-j} & \text{for } 3 \le j \le d-1 \end{cases}$$

$$f_j(N^d) = \begin{cases} \binom{d}{j+1} 2^{j+1} + \binom{d}{j} 2^{d-1} & \text{for } 0 \le j \le d-4 \\ 2^{d-1} \binom{d}{3} & \text{for } j = d-3 \\ 2^{d-2} \binom{d}{2} & \text{for } j = d-2 \\ 2^{d-1} & \text{for } j = d-1 \end{cases}$$

Table 4.1: The f-vectors of K_k^d, M^d, and N^d.

4.3.1 P [The Grünbaum Polytopes].

(1) The polytopes K_k^d are 2-simplicial and $(d-2)$-simple d-polytopes for $d \ge 3$. K_1^d is a simplex for all d.

(2) The polytopes M^d for $d \ge 3$ are 2-simplicial and $(d-2)$-simple d-polytopes. M^3 is the octahedron.

(3) The polytopes N^d for $d \ge 3$ are 3-simple and $(d-3)$-simplicial. N^3 is a simplex and N^4 is a cube.

Proof. K_1^d and K_d^d have $d+1$ vertices. Hence, they are simplices. The facets of K_k^d arise as intersection of the hyperplane H_k in (4.3.1) with a facet of C^{d+1} (the 0/1-cube), so they are combinatorially equivalent to either K_{k-1}^{d-1} or K_k^{d-1}. Iterating this we conclude that K_k^d is at least 2-simplicial. An edge of K_k^d is obtained from the intersection of H_k with a 2-face of C^{d+1}. C^{d+1} is simple, so the K_k^d are at least $(d-2)$-simple. Hence, the polytopes K_k^d for $2 \le k \le d-1$ are precisely 2-simplicial and $(d-2)$-simple.

There are two different types of facets in M^d, those that come from the hyperplanes truncating the vertices of $\square_d$ and those coming from a facet of $\square_d$. The vertices of M^d are the centroids of the 2-faces of $\square_d$. They are connected by an edge if the corresponding 2-faces share an edge.

Let e be such an edge of M^d between vertices v_1 and v_2. Let t_1 and t_2 be the two 2-faces of $\square_d$ corresponding to v_1 and v_2 and $\overline{e}$ their common edge in $\square_d$. e is contained in all facets of M^d that originate

(1) from facets of $\square_d$ that have t_1 and t_2 as 2-faces and
(2) from truncating the endpoints of $\overline{e}$.

$\square_d$ is simple, so $d-3$ of its facets contain t_1 and t_2. $\overline{e}$ has two endpoints, which define two facets containing e. So e is contained in $d-1$ facets, and M^d is at least $(d-2)$-simple.

Any ridge in M^d is adjacent to at least one facet coming from a facet of $\square_d$. These are of type M^{d-1}. M^3 is simplicial, so by induction M^d is at least 2-simplicial. M^d is not a simplex, so it is precisely 2-simplicial and $(d-2)$-simple.

The polytopes N^d are 3-simple and $(d-3)$-simplicial. The latter follows from the fact that the facets of N^d are combinatorially equivalent to the dual of K_2^{d-1}, which is $(d-3)$-simplicial. The 3-simplicity follows from the fact that the facets of $(N^d)^\Delta$ are either simplices or dual to N^{d-1}. $\square$

Now let us see how these polytopes fit into the E-construction. Here is the complete classification:

4.3.2 T . *The following combinatorial equivalences hold:*

$$\begin{aligned}
(K_k^d)^\Delta &\cong E_{k-1}(\Delta_d) \quad \textit{that is,} \quad & K_k^d &\cong D_{k-1}(\Delta_d) \qquad \textit{for } 1 \le k \le d\\
(M^d)^\Delta &\cong E_{d-2}(N^d) \quad \textit{that is,} \quad & M^d &\cong D_{d-2}(N^d)\\
&& &\cong D_1((N^d)^\Delta).
\end{aligned}$$

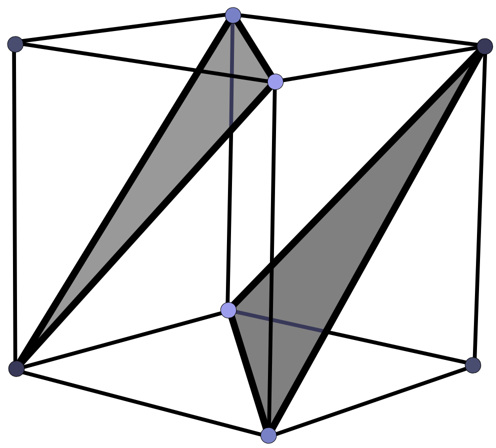


Figure 4.1: The two polytopes K_1^2 and K_2^2, which are both triangles.

Here we have used the generalisation D_k of the vertex truncation operator D_1, which we have introduced in Remark 2.5.3. The polytope N^d itself is not a result of the E-construction. The dual of N^5 is a 2-simple and 3-simplicial 5-polytope, but it has simplex facets. For $d \geq 6$ both N^d and its dual are at least 3-simple.

Proof of Theorem 4.3.2. We start with the hypersimplices K_k^d and show that their duals are $E_{k-1}(\Delta_d)$. Δ_d clearly has a geometric realisation in which all $(k-1)$-faces are tangent to the unit sphere, for any $1 \leq k \leq d$. So we can apply E_k to it, by Theorem 2.5.14. To check that this has the right combinatorics it suffices to check the vertex-facet-incidences, by Proposition 1.2.24.

Let Δ_d be in the standard representation of Examples 1.3.19. $E_{k-1}(\Delta_d)$ has $\binom{d+1}{k}$ facets. Two facets have a common vertex if and only if the corresponding $(k-1)$-faces in Δ_d

- have a common vertex or
- lie in the same facet.

If we encode a $(k-1)$-face of Δ_d by a $0/1$-vector with k ones and $(d-k+1)$ zeros, then two facets of $E_{k-1}(\Delta_d)$ share a vertex if either the two vectors component wise combined with the binary *and* is not the zero vector or their binary *or* is not the vector containing only ones. If we interpret these vectors as vertices of the $0/1$-cube C^{d+1}, then this is precisely the vertex-facet-incidence description of K_k^d.

The polytope N_d in the given geometric realisation has its $(d-2)$-faces tangent to a sphere. Thus, we can apply the E_{d-2}-construction to it, by Theorem 2.5.14. We check the vertex-facet-incidences.

Any ridge of N_d, that is, any facet of $E_{d-2}(N_d)$, can be encoded by a vector with entries in $\{0, \pm 1\}$, precisely two of which are zero. In the given realisation, these vectors are the vertices of M_d.

Let v be such a vector, representing a facet of $E_{d-2}(N_d)$ and a vertex of M_d. The vertices of a facet in $E_{d-2}(N_d)$ are the vertices of the corresponding ridge in N_d and the two vertices beyond the facets adjacent to this ridge. These two facets can be represented by the two ± 1-vectors that one obtains by replacing the two zeros in v in such a way, that the vector contains an odd number of $+1$'s.

The vertices of N_d have two types. In the ridge v they are given by

- all ± 1 vectors that replace the two zeros in such a way that the number of $+1$'s becomes even, and
- those vectors that have $\mp(d-2)$ in one entry where v has a 0 or ± 1 (observe the sign), and zeros otherwise.

But these are just the facet normals of those facets of M_d that contain the vertex v. So $E_{d-2}(N_d)$ has the opposite face lattice of M_d. $\square$

4.3.2 T G –E P

Among the regular and uniform polytopes are several that are 2-simple and $(d-2)$-simplicial. We discussed some already. Here we describe another large class of uniform polytopes which are interesting in connection with our construction.

McMullen observed that the Gosset–Elte polytopes r_{st} are $(r+2)$-simplicial and $(r+t-1)$-simple $(r+s+t+1)$-polytopes. See the review of Kalai [40, p. 344] for this. This family of polytopes is described in detail in the textbook of Coxeter [30, Ch. 11.7-8]. They arise as a special case of the Wythoff construction, which we describe briefly in the next paragraph. This construction produces polytopes with *Coxeter groups* as their symmetry group.

4.3.2.1 Wythoff's Construction

Here is the definition of a *Wythoff polytope*. Let $\mathbb{S}$ be the unit $(d-1)$-sphere. We consider d hyperplanes $H_1, H_2, \ldots, H_d$ that contain the origin. Let α_{ij} be the dihedral angle between H_i and H_j for all $1 \leq i, j \leq d$.

The hyperplanes enclose a spherical simplex T on $\mathbb{S}$. Place a point x in one of the vertices of T and consider repeated reflections of this point at the d hyperplanes. For special choices of the angles α_{ij} we obtain a finite point set on S (in particular the angles must be rational multiples of π). The convex hull is a finite bounded polytope. This is the *Wythoff polytope* associated to the hyperplanes $H_1, \ldots, H_d$.

If we have a choice of rational angles α_{ij} generating a finite point set then we can reduce the angles to the form $\frac{\pi}{p}$ for $p \in \mathbb{N}$, $p \geq 2$, as for any $\frac{j\pi}{p}$ with j and p coprime, there is an integral multiple of this angle that differs from $\frac{\pi}{p}$ by an integral multiple of π. Define integers r_{ij} for $1 \leq i, j, \leq d$ by $\alpha_{ij} = \frac{\pi}{r_{ij}}$. The tuple of integers $[r_{ij}]_{ij}$ is the *Wythoff symbol* of the Wythoff polytope. Observe, that the hyperplane arrangement $H_1, \ldots, H_d$ is uniquely defined by the Wythoff symbol, up to affine transformations in $\mathbb{R}^d$.

The hyperplane arrangement fixes the symmetry group of the polytope. It is the *Coxeter group* generated by reflexions in the hyperplanes. Reversely, there are several Wythoff polytopes associated to a finite Coxeter group.

We have a quite convenient graphical representation of the hyperplane arrangement defining a Wythoff polytope. The hyperplanes define a spherical simplex T on $\mathbb{S}$, so each hyperplane H_i has a unique opposite vertex v_i of T not contained in H_i, for $1 \leq i \leq d$. To represent the arrangement, we draw a graph with one node for each such hyperplane-vertex-pair, and we connect two nodes in the graph if the two corresponding hyperplanes enclose an angle less than $\frac{\pi}{2}$. We mark this edge by the integer r_{ij} corresponding to the two hyperplanes if $r_{ij} \geq 4$. We distinguish the chosen vertex of T in this graph by drawing a ring around it. This graph contains all necessary information to reconstruct the spherical tiling. See Figure 4.2 for an example. It shows the Wythoff graph, the Wythoff graph of its facets, the spherical tiling with the spherical simplex highlighted, and the resulting Wythoff polytope inscribed (which is a cube).

The types of facets of a Wythoff polytope can easily be derived from its graph: They are Wythoff polytopes for the graphs that we obtain by removing an unringed node with its adjacent edge, if the graph remains connected. That is, we obtain the graphs of the facets by deleting a node from one of the free ends (if the graph has any). In particular, the facets of a Wythoff polytope are Wythoff polytopes themselves. Iterating this procedure gives us the facets of the facets, i.e. the ridges of the Wythoff polytope, etc. Hence, all combinatorial types of faces of a Wythoff polytope follow from the graph.

A more careful analysis of this procedure lets one also derive the symmetry group of the polytope and its f-vector. If the ringed node is the final node on a free end, then removing the ringed node and shifting the ring to an adjacent vertex produces the graph of the vertex figure.

Note, that the reverse direction is not true in general: Not all diagrams satisfying the above conditions do indeed define a Wythoff polytope. In particular, the arrangement need not be finite, but may lead to a tiling of Euclidean space instead of a bounded polytope.

We see more examples of the Wythoff construction in the next section. Gévay [35] considers a construction of *Kepler hypersolids*, see Section 4.3.3. The duals of these polytopes are a special case of the Wythoff construction.

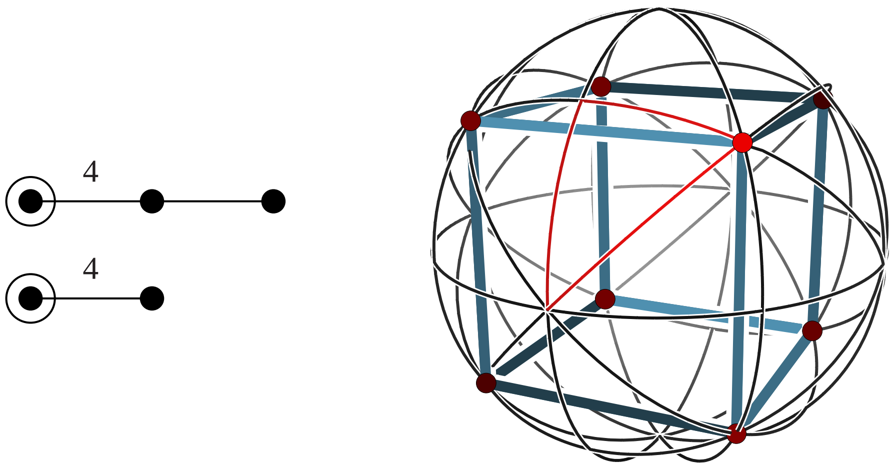


Figure 4.2: The Wythoff polytope with symbol $[3,4]$ is the cube. Its graph is shown in the upper left figure. Below is the graph of its facets.

4.3.2.2 Gosset-Elte Polytopes

The *Gosset-Elte-Polytopes* r_{st} for $r, s, t \geq 1$ are a special type of Wythoff polytopes, defined by the following Wythoff graph.

Consider the group of reflections corresponding to the diagram in Figure 4.3, where we have r nodes on the right end, s nodes on the left end and t nodes on the lower end. The Coxeter group is finite if and only if r, s and t satisfy $1/(r+1) + 1/(s+1) + 1/(t+1) > 1$. (see [30, Chapter 11.8] for this).

Note that the graph is symmetric in t and s. The inequality together with the symmetry leaves us with only three infinite series for the parameters r, s and t and a finite number of other choices.

The three infinite series are the following. We relabel the polytopes r_{st} by replacing one of the parameters by the dimension $d = r + s + t + 1$ of the polytope.

- $0_{d-k,k-1}$, for $1 \leq k \leq d$. Their diagram is shown in Figures 4.4(a) and 4.4(b). They are equivalent to the hypersimplices K_k^d from Section 4.3.1: $0_{1,1}$ is the octahedron and $0_{2,0} = 0_{0,2}$ is the simplex. We obtain the facets of $0_{d-k,k-1}$ by removing unringed end nodes. They have consequently the two different types $0_{d-k-1,k-1}$ (if $k < d$) and $0_{d-k,k-2}$ (if $k > 1$). Continuing this until $d = 3$ shows that this polytope has the same combinatorial structure as K_k^d.
- $1_{d-3,1}$. See Figure 4.4(c) for the Wythoff diagram. This is equivalent to N_d, which can again be deduced from the facet structure.
- $(d-3)_{1,1}$ is the cross polytopes $\maltese_d$. See Figure 4.4(d) for the diagram. The equivalence follows by induction over the dimension d: 0_{11} is the octahedron. All facets of $(d-3)_{1,1}$ are simplices and the vertex figures are all of the type $(d-4)_{1,1}$, that is, they are cross polytopes.

The remaining finite number of other choices for r, s and t are

- in dimension 6: 1_{22}, 2_{21} (*Schläfli polytope*),
- in dimension 7: 1_{32}, , 2_{31}, 3_{21} (*Hesse polytope*),
- in dimension 8: 1_{42}, 2_{41}, and 4_{21} (*Gosset polytope*).

Among those, only 2_{21}, 3_{21}, and 4_{21} are 2-simple and $(d-2)$-simplicial.

Of all these polytopes, only the infinite series $0_{d-k,k-1}$ for $1 \leq k \leq d$ is contained in our construction (if we exclude the trivial cases $t = d - 1$ and $t = 0$, which map a

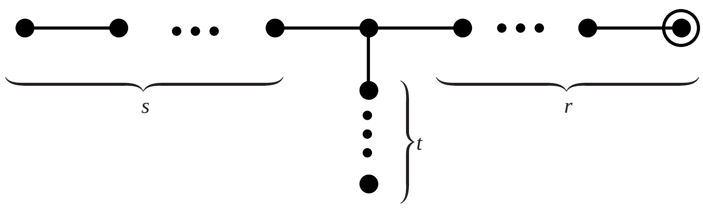


Figure 4.3: The general graph for Gosset–Elte polytopes.

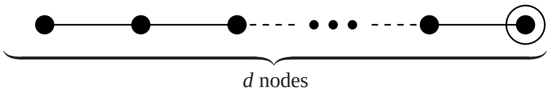
polytope to itself and its dual, respectively). The other three 2-simple and $(d-2)$-simplicial polytopes have simplices among their facets, which is impossible for a polytope resulting from the E-construction.

$1_{3,1}$ is a 3-simple and 3-simplicial 6-polytope with 44 vertices, and 2_{41} is a 4-simple and 4-simplicial 8 polytope with 2160 vertices. So for $k = 2, 3$ and 4 we know $2k$-polytopes that are k-simple and k-simplicial. There are no non-trivial polytopes known with this property for $k \geq 5$.

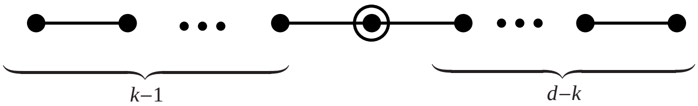
4.3.3 G´ ’ P

Gévay [35, 36] constructs a number of interesting polytopes from a construction that uses a similar idea as we use for ours. However, he considers it with a completely different intention, as he is interested in symmetry and regularity properties of the constructed polytopes. He starts out from spherical tilings generated by Coxeter groups, and considers polytopes obtained by joining some of the spherical simplices generated by this group.

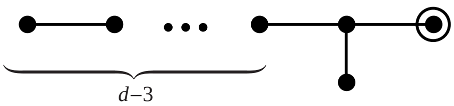
In [35], Gévay considers a construction similar to the Wythoff construction of the previous section. For a Coxeter group C, one looks at the spherical simplex and the spherical tiling of the sphere defined by it, and obtains a (semi-regular) polytope by the following method: Take a point P in this simplex and all its reflections. Define a polytope as the intersection of all half spaces defined by the tangent planes to these points. By regularity, this gives a polytope for which the given Coxeter

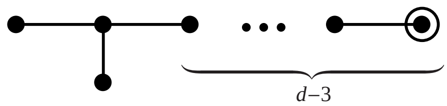


(a) The Wythoff graph of the d-simplex $0_{0,d-1}$.



(b) The figure for $0_{d-k,k-1}$.





(c) The figure for $1_{d-3,1}$.

(d) The figure for $(d-3)_{1,1}$.

Figure 4.4: The infinite Gosset–Elte series.

group is transitive on the facets. This construction method is "dual" to the Wythoff construction.

Further polytopes with the same symmetry group can be constructed by considering the "factor tessellation": Take a point x in the relative interior of some face of the spherical simplex. Let C_0 be its stabiliser. Define an equivalence relation on the spherical tiles by saying that two tiles are equivalent if they lie in the same orbit with respect to C_0. Define a new tessellation by taking the union of the equivalence classes as new tiles. Clearly, in this regular setting, a hyperplane description of the corresponding polytope is given by the tangent planes to x and its translates.

If we choose the point x in the relative interior of a k-face of the spherical simplex, then we identify all tiles in the factor tessellation that contain this k-face. So this construction can be viewed as a special case of our E_k-construction for CW spheres in Definition 2.4.3, for spherical tilings generated by reflections. Polytopality of the spheres is in this case guaranteed by the symmetry contained in the construction.

Gévay considers the transitivity properties of this construction for facets of k-faces, where $0 \leq k \leq d-1$. In general, the group C will not be facet transitive on these k-faces anymore.

They are, however, transitive for the special class of *Kepler polytopes*. These are "factor tessellations" of tilings obtained from a Coxeter group C of a regular polytope P. In this case, there is a simple way to realise the tessellation geometrically: Take P and scale it, such that its k-faces are tangent to the unit sphere. The corresponding Kepler polytope is the convex hull of P and its polar. Compare this process to Theorem 2.5.14.

In accordance with Coxeter's original notation, we denote the symmetry groups of regular polytopes by

- A_n for the the simplex,
- B_n for the cube and cross polytope,
- F_4 for the 24-cell and
- H_4 for the 120-cell.

The resulting Kepler polytopes are denoted by $X(d, k+1)$, where $X = A, B, F$ or G according to the symmetry group of P, d is the dimension of P and k is the dimension of the face containing the point x from above.

The (r, s)-polytopes for $r, s \geq 2$ among those are listed in Table 4.4 for $d = 4$ and Table 4.6 otherwise, the others in Table 4.8. We do not repeat them here.

One specifically interesting instance is the "dipyramidal 720-cell" $G(4, 2) = f_1 H_4$, which reappears as an instance of the E-construction of Eppstein, Kuperberg, and Ziegler. In our notation it is E_2(120-cell), which can be geometrically realised with the methods of Theorem 2.5.14.

These polytopes and the above described construction are further explored by

Gévay in [36], where he looks at *perfect* polytopes, which are geometrically realised polytopes with the property that all symmetry equivalent polytopes are already similar. The Kepler polytopes defined above are perfect. See Remark 3.5.9 for two more.

For some more information on perfect polytopes see also [38] and [37], containing a new class of perfect 4-polytopes constructed by truncating certain vertices of bipyramidal Kepler polytopes, and some constructions of perfect non-Wythoffian polytopes by truncations.

4.3.4 T E E , K , Z

The examples of 2-simple and 2-simplicial 4-polytopes of Eppstein, Kuperberg, and Ziegler in [33] can all be obtained via our method of realising the E-construction described in Section 2.5.14.

They constructed a large number of such polytopes, so we cannot present all of them in this summary. Moreover, for most examples they needed rather intricate arguments, which are now, with the help of our new methods, unnecessary. So we restrict to some particularly interesting examples, and to examples with a small number of vertices, which we include in our tables in the next section.

Basically, in their paper they prove Theorem 2.5.14 for dimension $d = 4$ and parameter $t = 1$. Hence, to apply their theorem for the construction of $(2, 2)$-polytopes, they need simplicial 4-polytopes that have their edges tangent to the unit sphere. For this, they switch to hyperbolic geometry and examine possible *edge links* in polytopes obtained by gluing regular simplicial 4-polytopes along facets, in the same way as we did for the proof of Theorem 2.5.15. They consider the edge links of the simplex, the cross polytope, and the 600-cell for this. In this regular and edge tangent version, these links are a regular triangle, a regular square, and a regular pentagon. So the dihedral angles are $\frac{\pi}{3}$, $\frac{\pi}{2}$ and $\frac{3\pi}{5}$. See Lemma 2.5.18 for the computation of the first two angles. The third can be obtained similarly.

Edge tangent simplicial 4-polytopes can now be obtained by gluing simplices, cross polytopes, and 600-cells along tetrahedra in such a way that

- either the resulting dihedral angle at an edge where two or more of these building blocks meet remains strictly between 0 and π, or
- the edge vanishes completely in the interior.

They give lists of possible edge links satisfying these conditions. If using only simplices, then there are only three such links, which are shown in Figure 4.5. From these links, one can construct only three different polytopes [33, Prop. 8]:

- The simplex Δ_4. This leads to the hypersimplex $E_1(\Delta_4) \cong E_2(\Delta_4) \cong K_2^4$ in the E-construction.
- The stacked simplex, which leads to a special gluing of two hypersimplices

in the E-construction. This polytope B_{14} was already described previously by Braden [27]. In our context, the simplest way to obtain a realisation is via vertex truncation, as $D(P_1^4)$, see Section 2.5.1. A possible set of coordinates and a Schlegel diagram are shown in Figure 2.15.

- The sum of a triangle and a hexagon. In our context, it is easiest described via its dual, which is the product of a triangle and a hexagon. A realisation of this polytope $\mathrm{EKZ}_1 \cong E(C_3 \times C_6)$ was obtained in Section 3.4.1.

Further, Eppstein, Kuperberg, and Ziegler classify all possible edge links, if one allows gluings of simplices and cross polytope. They find eleven different links [33, Sect. 3.2]. Three of them contain only one square. These are shown in Figure 4.6.

They classify all possible edge tangent simplicial polytopes using only these three links in their Proposition 10 and obtain 21 edge tangent simplicial polytopes, which are glued from one cross polytopes and k simplices for $k \geq 0$. Due to their angle sum restrictions, they can only glue simplices onto facets of the cross polytope that do not share a ridge. With our methods, we are not bound to this anymore, so already from these simple building blocks we obtain many more different $(2,2)$-polytopes. See Table 4.2 for a comparison. The first type in the list, i.e. the cross polytope without any glued simplices, leads to the 24-cell in the E-construction. In our setting, all the different types can most easily be constructed via Proposition 2.5.13, with a realisation of $E_2(\square_4)$ as input.

Note also, that the construction of Eppstein, Kuperberg, and Ziegler is limited to gluing simplices onto a facet of the cross polytope. We can produce further, combinatorially different, $(2,2)$-polytopes by gluing simplices onto facets of simplices glued in some earlier step (i.e. in the dual language, truncating vertices that are the result of some previous truncation operation). This leads to already 4877 different types in the sixth step, compared to the 50 when only gluing simplices onto facets of the cross polytope, and 2 in the original construction.

Here are the small instances with up to 32 vertices among these polytopes.

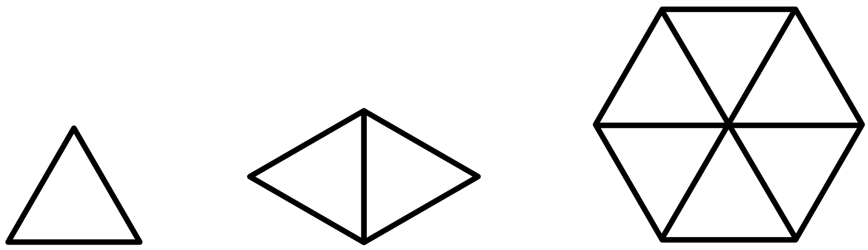


Figure 4.5: The three edge links involving only simplices. Below are the polytopes they appear in.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\sum$
f_0	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	
EKZ	1	1	3	3	6	3	2	1	1	0	0	0	0	0	0	0	0	21
A	1	1	4	6	19	27	50	56	74	56	50	27	19	6	4	1	1	402
B	1	1	5	16	102	628	4877			...	many							

Table 4.2: The number of combinatorially different possibilities of gluing $k \geq 0$ simplices onto a facet of $✚_4$, such that the E-construction can be applied to the result.
The first row lists the possibilities allowed in the construction of Eppstein, Kuperberg, and Ziegler, the second row (types A) the possibilities for our construction if we only glue simplices onto facets of $✚_4$, and the third row (types B) shows the number of different types if we successively glue simplices without this restriction.

(1) the 24-cell for $k = 0$,
(2) the E-construction EKZ_2 of a stacked cross polytope,
(3) and the E-construction $\mathrm{EKZ}_3^{(j)}$ of a cross polytope with two stacked facets. There are three variants $j = 1, 2, 3$ of this polytope obtained by Eppstein, Kuperberg, and Ziegler. By our construction we obtain one more — combinatorially different — variant $\square_4^2$.

In Theorem 11 and Section 3.3 of their paper, Eppstein, Kuperberg, and Ziegler construct two infinite families of (2, 2)-polytopes, which were the first of their kind.

The first series of (2, 2)-polytopes contains polytopes with flag vector

$$(54n - 30, 252n - 156, 252n - 156, 54n - 30; 360n - 216),$$

for $n \geq 1$. They are a special case of our family in Theorem 2.5.15 for $d = 4$. It uses the edge link shown in Figure 2.21 at the gluing ridges.

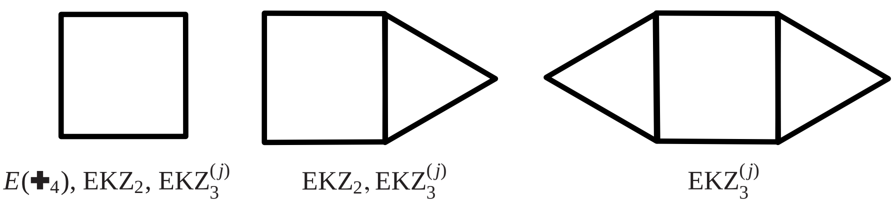


Figure 4.6: The three possible edge links involving one square, and the polytopes they appear in. The polytopes have also links involving only the triangle of Figure 4.5.

The second family stems from a gluing of 600-cells. For this, they modify the 600-cells by truncating vertices, as otherwise the angles at the gluing edges are $\frac{2\pi}{5}$ — which is too large. The series of E-polytopes obtained from this has flag vector

$$(666n + 54, 3360n + 240, 3360n + 240, 666n + 54; 4692n + 348),$$

for $n \geq 0$. There are no polytopes similar to the 600-cell in higher dimensions. Hence, this family cannot be lifted to dimensions $d > 4$ to produce a family of $(2, d-2)$-polytopes.

They further elaborate this gluing construction with 600-cells. By using properties of the symmetry group and semi-regular polytopes they identify subsets of the vertices that can be truncated and the resulting facet glued with a copy of another truncated 600-cell. This results in a $(2, 2)$-polytope $\mathrm{EKZ}_{\mathrm{fat}}$ with 459360 vertices and 2319120 edges. At that time, $\mathrm{EKZ}_{\mathrm{fat}}$ was the fattest (in the sense of Definition 4.2.7) known polytope, with a fatness of roughly 5.048. This has sparked a small race for fatter 4-polytopes, which produced several results of this thesis as a "side effect". With $E_{10,10}$ it contains a fatter polytope, but there are already other, even fatter, polytopes found by Ziegler [92].

4.3.5 W ' E

Recently, Werner [88] found a new small and highly symmetric self-dual $(2, 2)$-polytope W_9 with 9 vertices and 26 edges. See Figure 4.7 for the coordinates and a Schlegel diagram. W_9 is a self-dual, 2-simple and 2-simplicial 4-polytope.

Its facets are one octahedron, six stacked simplices (bipyramids over a triangle), and two simplices. The octahedron is incident to all other facets, and the two simplices are on two opposite faces of the octahedron. All stacked simplices meet in a vertex of degree eight (the octahedron in the dual polytope).

4.3.3 R [W_9 is on l_1]. The polytope W_9 lies on the ray $l_1 := \mathrm{flag}(\Delta_4) + \lambda(1, 4, 4, 1; 6)$ of the list of seven rays (1.4.9) in the boundary of the flag vector cone of 4-polytopes. See also Figure 1.18. There are only two further polytopes known lying on this ray, the *simplex*, and the *hypersimplex*. There is, however, some hope to find more. See the next section for this.

None of the polytopes obtained from the E-construction can lie on this ray, except the hypersimplex. This follows from the following simple observation. Any polytope on l_1 must be 2-simple and 2-simplicial. So, if it is a result of the E-construction, then it is (at least combinatorially) obtained via E_1 from a simplicial 4-polytope. A simplicial 4-polytope P has an f-vector of the form $(a, a+b, 2b, b)$. Hence, $E_1(P)$ has the f-vector $(a+b, 6a, 6a, a+b)$. Lying on l_1 implies

$$\frac{6a - 10}{a + b - 5} = 4 \qquad \text{and therefore} \qquad a + 5 = 2b.$$

The only simplicial 4-polytope satisfying this, is the simplex. E_1 applied to it yields the hypersimplex.

Werner found this polytope via a "reverse" shelling approach. This approach attempts to find new candidates for face lattices of $(2,2)$-polytopes by trying to build up such a lattice along the inverse of a shelling. The search is done with a computer using a client for the `polymake` package. It enumerates $(2,2)$-lattices obtainable with a previously fixed set of simplicial facet types, i.e. a fixed list of simplicial 3-polytopes.

The discovery of this polytope lead us to the following nice (non-)existence theorem. See [67] for a more thorough treatment.

4.3.4 T . *The only* 2*-simple and* 2*-simplicial* 4*-polytope with eight or less vertices is the simplex.*

In other words, nine is the minimal number of vertices a non-trivial 2-simple and 2-simplicial 4-polytope must have, and this number is attained by the polytope W_9. Compare this also to the next theorem.

Proof of Theorem 4.3.4. Assume there is a 2-simple and 2-simplicial 4-polytope P with less than nine vertices different from the simplex. The only 4-polytope with five vertices is the simplex, so it must have 6, 7 or 8 vertices.

P cannot be simplicial by Proposition 1.2.27, so it has at least one facet with five or more vertices. Let F be such a facet of P. By 2-simplicity, F is simplicial with f-vector $(a, 3a-6, 2a-4)$, for $a \geq 5$.

$$\begin{matrix} \begin{bmatrix} -2 & -2 & -2 & 2 \end{bmatrix} \\ \begin{bmatrix} -6 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -6 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & -6 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 3 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 6 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 6 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 6 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix} \end{matrix}$$

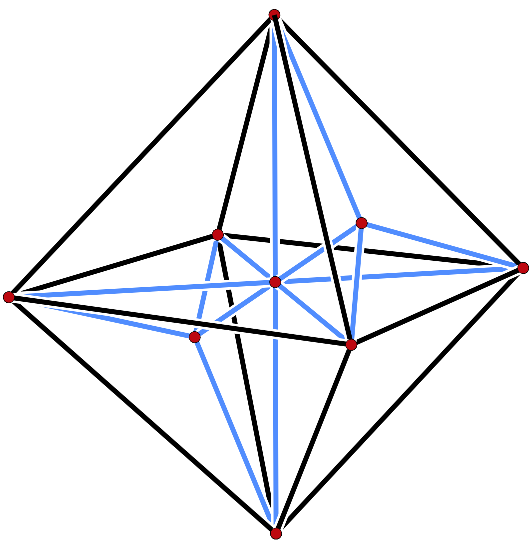


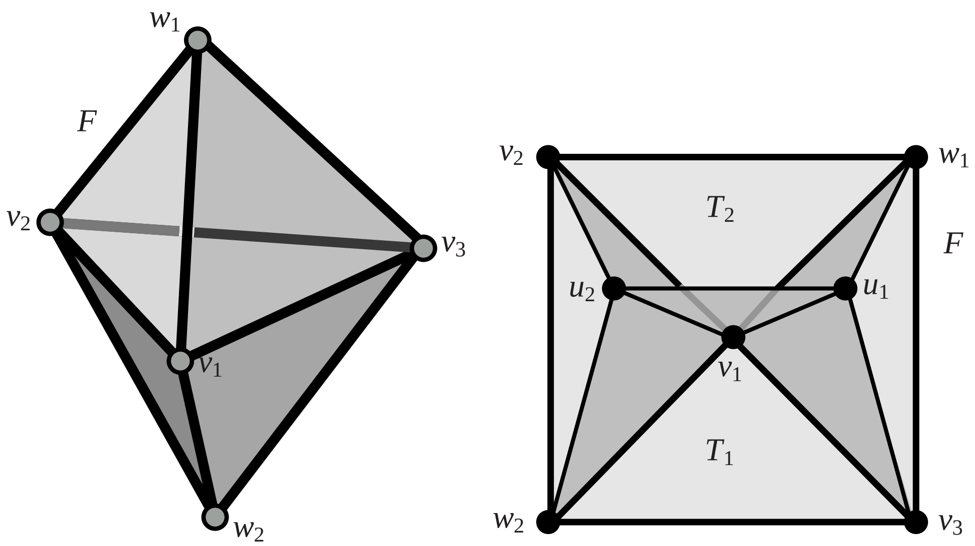
Figure 4.7: The second smallest $(2,2)$-polytope, and the first non-trivial such. Only the simplex has less vertices.

Hence, the dual polytope P^Δ has a vertex of degree $2a - 4 \geq 6$, so P^Δ has at least $2a-3 \geq 7$ vertices, and P has at least $2a-3 \geq 7$ facets. A $(2,2)$-polytope has as many vertices as it has facets, so P has at least $2a - 3$ vertices. By assumption, this is less or equal to eight, so any facet of P has at most 5 vertices. There are only two simplicial 3-polytopes with five or less vertices, which are the simplex and the stacked simplex. We denote these two possible facet types by Δ_1 and Δ_2.

F is not the simplex, so F is of type Δ_2. F has three vertices v_1, v_2 and v_3 of degree four, and two vertices w_1 and w_2 of degree three. See Figure 4.8(a) for an illustration. The dual facet to v_1 has a vertex of degree four and thus is of type Δ_2, by 2-simplicity. Similarly for v_2 and v_3.

So around each of v_1, v_2, and v_3 there are five facets — one of which is F — and two incident edges which are not edges of F. So v_1, v_2, and v_3 are each incident to two further vertices not contained in F. See Figure 4.8(b) for an illustration of the vertex link of v_1 (up to symmetry). By 2-simplicity, two adjacent vertices of degree four can share one of the two additional vertices in their neighbourhood, but not both. So we have at least $5 + 3 = 8$ vertices in P.

In the star of v_1 are two facets of type Δ_2 glued on the triangles T_1 and T_2 in such a way, that v_1 is a vertex of degree 4 in them. If we consider the triangle spanned by v_1, v_2, and v_3 as the "equator" of F, then either T_1 is above and T_2 below the equator, or vice versa. The same is true for v_1 and v_2, so the three facets of P



(a) The facet F of P.

(b) The link of a vertex v of degree 4 in a Δ_2 with other vertices $v_1, \ldots, v_4$.

Figure 4.8: Facets of small $(2,2)$-polytopes I.

adjacent to the three triangles of F above the equator must contain three vertices of degree 4. The same holds for the three facets adjacent to F below the equator.

A facet of type Δ_2 has two vertices of degree 4 in each of its 2-faces (see Figure 4.9), so there are at least two such facets F_a and F'_a above, and two such facets F_b and F'_b below the equator.

Now F_a and F'_a each have two of its vertices of degree 4 adjacent to F, but only three of them can be incident to the equator, so either F_a or F'_a must have w_1 as a vertex of degree 4. Similarly, w_2 is a vertex of degree 4 in either F_b or F'_b.

So in P^Δ the vertex w_1 of F corresponds to a facet of type Δ_2. Hence it is incident to five facets in P. Up to now, we have identified only four of them: These are F and the three facets adjacent to a triangle of F containing w_1. So there is one more facet F' of P intersecting F only in w_1. The same holds for w_2.

Counting facets, we have six facets intersecting F in a triangle, F itself, and two facets intersecting F in w_1 and w_2. But P has only eight facets. □

Any simplicial 3-polytope S with seven vertices has ten triangular faces. So, if S is a facet of a 4-polytope P, then P^Δ has a vertex of degree ten. $(2,2)$-polytopes have as many facets as they have vertices. Hence, a $(2,2)$-polytope P with only nine vertices cannot have a facet with seven or more vertices. Thus, a facet F of P can only have four (the simplex), five (the stacked simplex), or six vertices (the twice stacked simplex or the cross polytope).

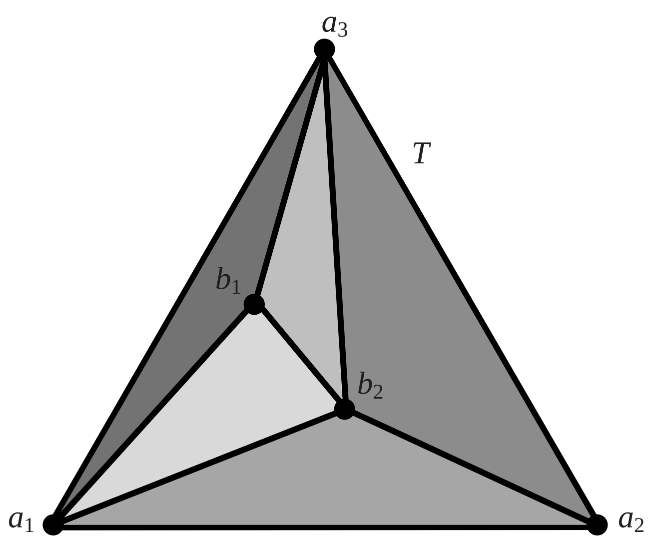


Figure 4.9: Facets of small $(2,2)$-polytopes II: A facet of type Δ_2 glued on a triangle T with vertices a_1, a_2 and a_3. The vertices b_1 and b_2 are added.

4.3.5 P . *W_9 is the only $(2,2)$-polytope with nine vertices having an octahedral facet and no other facet with six vertices.*

Proof. Let O be the octahedral facet of P. P has eight further facets, and O has eight 2-faces, so all other facets of P are incident to O. All vertices of O have degree four, so the vertex links all look like the one shown in Figure 4.8(b), where v_1, v_2, v_3, w_1 and w_2 are vertices of O.

Hence, around any vertex v of O we have to have two facets in which v is a vertex of degree four (i.e. stacked simplices, as there is no other octahedron by assumption) at diagonally opposed triangles.

Each triangle in a stacked simplex is adjacent to two vertices of degree 4. Further, O has six vertices, so there are precisely six stacked simplices adjacent to O, and the remaining two facets are simplices. Up symmetry, there is only one possibility to distribute these six stacked simplices onto the eight triangles of the octahedron, see Figure 4.10. This is the choice realised in W_9. □

In [67] we prove that W_9 is in fact the only $(2,2)$-polytope with nine vertices. The proof uses similar arguments as the ones used in the above two proofs, but is much more involved.

4.3.6 T . *The polytope W_9 is the only $(2,2)$-polytope with nine vertices.* □

In addition to Theorem 4.3.4 and Proposition 4.3.5 one has to show for the proof, that a $(2,2)$-polytope with nine vertices has precisely one facet with six vertices, and that this facet cannot be the twice stacked simplex.

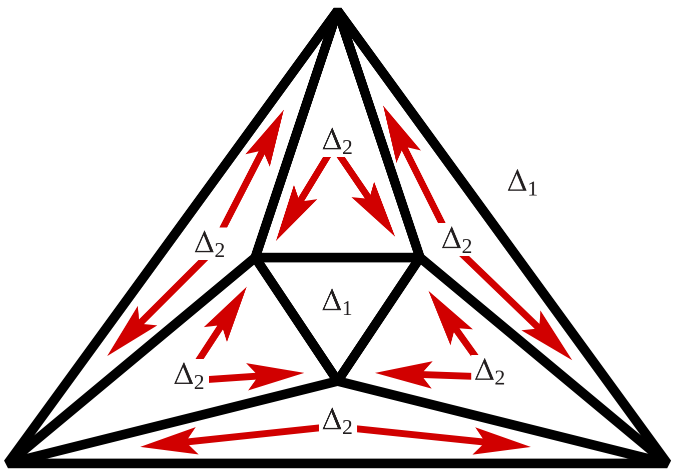


Figure 4.10: The distribution of 6 stacked simplices Δ_2 and 2 simplices Δ_1 onto the octahedral facet (unique up to symmetry). The incidences of vertices of degree 4 in Δ_2 are indicated with arrows.

4.3.6 S I T G *E*-C

There are some promising steps towards a generalisation of the E-construction. These were inspired by the polytope W_9 from the previous section.

The idea for the generalised version is the following. We restrict here to the case $t = d - 2$ for a d-polytope P. The E-Construction produces a polytope whose facets are bipyramids over the ridges of P. We can take a slightly different (local) view on this. Assume that the polytope P has a facet F that is a simplex. Choose a subset $R_1, \ldots, R_s$ of the ridges of P adjacent to F and place a new vertex v above F in such a way, that v lies in the facet hyperplanes of the facets adjacent to F in $R_1, \ldots, R_s$, and below all others. If we choose v such that it is not contained in an affine subspace defined by a k-dimensional face of P for some $k \leq d - 2$ (so the chosen ridges, or some subset of them, should better not intersect in a vertex of degree d of P), then we have achieved the following:

- All facets adjacent to F via one of the ridges $R_1, \ldots, R_s$ are turned into facets that are stacked above that ridge.
- For all ridges of F not in $R_1, \ldots, R_s$ we obtain one new facet for P, which is a simplex.

See Figure 4.11 for an example. The degree of an $(d - 3)$-face e of F changes by $r - 1$, where r is the number of adjacent ridges not among $R_1, \ldots, R_s$. The polytope W_9 is obtained by this construction from a pyramid over ✚$_3$ by choosing a pair F_1 and F_2 of opposite simplices (i.e. facets that are pyramids over a facet of ✚$_3$ and intersect only in the apex). For both facets, the list of ridges should contain those, that are pyramids over an edge of ✚$_3$, but not the bottom 2-face contained in ✚$_3$.

In principle, we are not restricted to applying this construction to simplicial facets, and we have applications in which F is not a simplex. However, the facet F remains a facet of the new polytope, and if we want to obtain a geometric realisa-

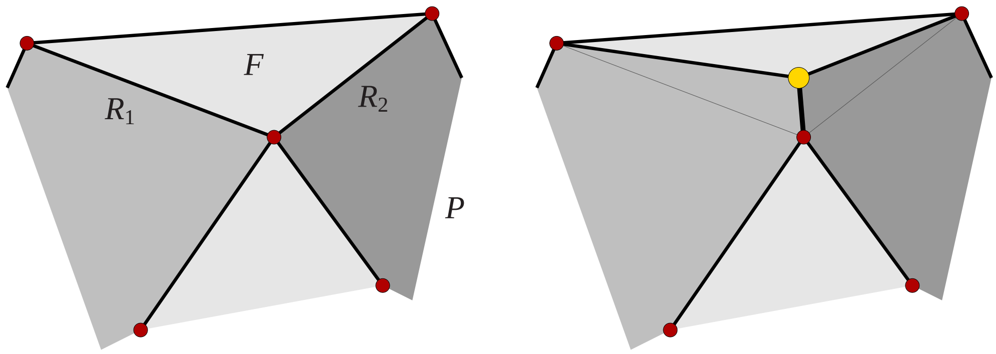


Figure 4.11: A generalised version of the E-construction: The top facet is the chosen facet F, and the big vertex has been added in the hyperplanes of the left and right facet.

tion of the polytope from this construction we have to adjust the normal vector of F. Hence, in the case that F is not a simplex, we have to assure, that the combinatorial properties of the polytope do not change at vertices of F that are not involved in the actual construction. This is in particular possible if these vertices are simple vertices in P.

So far, with the help of this construction we produced one new $(2,2)$-polytope with 16 vertices and 56 edges. It is obtained from ✚$_4$ by choosing eight of its facets, and in each facet three adjacent ridges. If one chooses the facets suitably, then one has only in the last step to deal with a facet which is not a simplex, but a stacked simplex from a previous step in the construction. In this case, the vertex of the facet not involved in the construction is simple. A Schlegel diagram of the dual of this polytope is shown in Figure 4.12. We can produce several face lattices of PL spheres that lie — just like the polytope W_9 of the previous section — on the ray l_1 of the flag vector cone of 4-polytopes (see Section 1.4.2 for the relevant definitions).

A precise description of this construction will be given elsewhere, together with criteria for polytopal realisability of the resulting spheres.

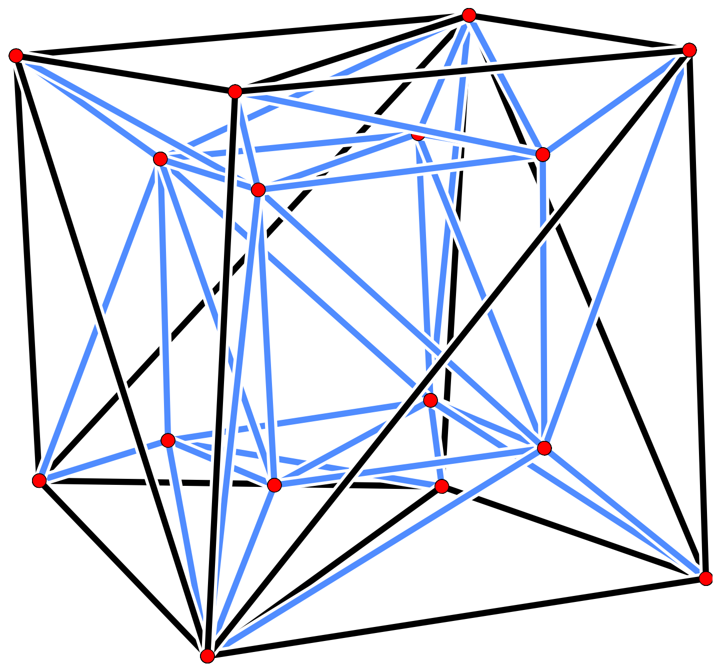


Figure 4.12: A Schlegel diagram of the "broken cube", which is the dual of the polytope with 16 vertices obtained from the generalised E-construction applied to ✚$_4$. It has 16 vertices and 56 edges.

4.4 S K E

Here is a summary on the sizes and types of $(2,2)$-polytopes obtained in Chapters 2-4, together with some computational data.

We collect small examples of our constructions, all the presented examples of previously known (r,s)-polytopes, $(2,2)$-polytopes, and other polytopes with special properties now obtainable from the E-construction in five tables:

- Table 4.4 lists all known examples of $(2,2)$-polytopes up to 50 vertices.
- Table 4.5 lists some previously known examples (with more than 50 vertices).
- Table 4.6 lists (r,s)-polytopes in dimension $d > 4$ that have $r, s \geq 2$.
- Table 4.7 lists some examples of infinite series of $(2,2)$-polytopes.
- Table 4.8 lists some examples of polytopes that can be obtained via our construction, but that are neither r-simple nor s-simplicial for some $r, s \geq 2$.

Any 2-simple and 2-simplicial 4-polytope has a flag vector of the special form

$$(f_0, f_1, f_1, f_0; f_1 + 2f_0).$$

Hence f_0 and f_1 suffice, and we list them in the flag vector column of the tables.

Let P be a simple polytope. If we have a *vertex preserving* (in the sense of Definition 3.2.4) geometric realisation of $E_2(P)$, then we can compute a realisation of $E_2(\overline{P})$, where $\overline{P}$ is obtained from P by truncating a vertex. This follows from Proposition 2.5.13. Dually, if we know $D(P)$ for a simplicial polytope, then we can compute $D(\overline{P})$ for any polytope $\overline{P}$ obtained from P by stacking one of its facets.

Truncation preserves simplicity, and stacking preserves simpliciality, so we can apply these two operations recursively. There are usually several combinatorially different ways of truncating a vertex or stacking a facet of a polytope. These lead to combinatorially different E-polytopes with the same flag vector. We use the following list of small simple 4-polytopes to generate the first table.

Δ_4	$C_3 \times C_6$	$C_5 \times C_5$	$C_4 \times C_8$
$C_3 \times C_3$	$C_4 \times C_5$	$C_3 \times C_9$	$C_3 \times C_{11}$
$C_3 \times C_4$	$C_3 \times C_7$	$C_4 \times C_7$	$C_5 \times C_7$
$C_3 \times C_5$	$C_4 \times C_6$	$C_5 \times C_6$	$C_6 \times C_6$
$\square_4$	$C_3 \times C_8$	$C_3 \times C_{10}$	$C_4 \times C_9$

The dual of a $(2,2)$-polytope is also a $(2,2)$-polytope. We include only one of the two variants in the table.

The list in Table 4.4 is a complete list of $(2,2)$-polytopes that result from the E-construction with up to 19 vertices. This can be seen by looking at small simplicial 4-polytopes. (Dually, and more in the flavour of our construction, one could as well look at simple polytopes. However, historically, the classification results for 4-polytopes used simpliciality.) By the g-Theorem of Billera and Lee (see [14] and [15]), a simplicial 4-polytope with $5 + n$ vertices has at least $5 + 3n$ facets. E_1 applied to it has therefore $f_0 \geq 5 + n + 5 + 3n = 10 + 4n$ vertices and the same number of facets. Hence, it suffices to look at simplicial 4-polytopes with up to seven vertices. These were classified by Grünbaum [44]:

(1) The one simplicial 4-polytope with five vertices is the simplex, and we obtain the hypersimplex from it.
(2) The two simplicial 4-polytopes with six vertices are the bipyramid over a triangle, and the sum of two triangles. Dually, these are the prism over a triangle, and the product of two triangles. The first leads to Braden's example B_{14}, the second to $E_{33} := E_2(C_3 \times C_3)$.
(3) Among the five simplicial 4-polytopes with 7 vertices, we only have to look at those with up to twelve facets. These are the twice stacked simplex, the join of a triangle and a square, and the dual of a truncated product of two triangles. The first leads to a $(2, 2)$-polytope with 18 vertices, which is $E_2(\mathrm{tr}(\Delta; 2 \text{ vertices}))$. The last two result in two different $(2, 2)$-polytopes with 19 vertices, which are $E_{34} = E_2(C_3 \times C_4)$ and $E_2(\mathrm{tr}(C_3 \times C_3; v))$.

There are two more simplicial 4-polytopes with seven vertices, leading to $(2, 2)$-spheres with 20 and 21 vertices. However, polytopality is unknown for both of them. These two are the two smallest such examples, and can be described as follows:

- The dual of a wedge over a triangular face of a truncated prism over a triangle is a simplicial 4-polytope with f-vector $(7, 20, 26, 13)$. The E_1-construction applied to this produces a CW $(2, 2)$-sphere with 20 vertices and 78 edges.
- The cyclic 4-polytope on seven vertices has f-vector $(7, 21, 28, 14)$, resulting in a CW $(2, 2)$-sphere with 21 vertices and 84 edges.

The 37 simplicial 4-polytopes on 8 vertices were classified by Grünbaum and Sreedharan [45]. They lead to $(2, 2)$-spheres that have between 22 and 25 vertices. There are many more simplicial 4-polytopes with 9 vertices, leading to $(2, 2)$-spheres with 26 and more vertices. For most of them it is unknown whether they can be realised as polytopes.

Recently, Werner has found a new $(2, 2)$-sphere with 14 vertices and 49 edges, that is not a result of the E-construction. 7 of its facets are stacked simplices, and 7 are octahedra. Polytopality is unknown. More small examples of $(2, 2)$-spheres with interesting properties result from the generalisation of the E-construction. See Section 4.3.6 for this.

Using Proposition 2.5.13 for a simple polytope and its E-construction we can create *really* large numbers of combinatorially different $(2, 2)$-polytopes with the same flag vector. We have met this already in Section 4.3.4, where we computed

Δ_4:

k	0	1	2	3	4	5	6	7	8
f_0	10	14	18	22	26	30	34	38	42
#	1	3	7	30	131	795	5152		

$C_3 \times C_3$:

k	0	1	2	3	4	5	6
f_0	15	19	23	27	31	35	39
#	1	1	3	14	82	570	4401

Table 4.3: The same numbers as in the last row of Table 4.2, but for the simplex and the product $C_3 \times C_3$ instead of the cross polytope: The numbers of combinatorially different $(2,2)$-polytopes obtained by applying the E_2-construction to a k-fold truncation.

the number of possible ways of successively stacking facets of the cross polytope. With six stacking operations, we obtain 4877 different $(2,2)$-polytopes. Truncating six vertices of the polytope $C_3 \times C_4$ leads to 14301 different polytopes, and consequently to the same number of different $(2,2)$-polytopes with 43 vertices.

We have computed similar numbers for the simplex, and the product of two triangles: These are the two smallest simple polytopes (the second simple polytope with six vertices is a truncation of the simplex). Table 4.3 lists the numbers of combinatorially different types of $(2,2)$-polytopes obtained by truncating k vertices, for $k \leq 8$ in the case of the simplex, and $k \leq 6$ in the case $C_3 \times C_3$. All are realisable.

4.4.1 C D

Files in the `polymake` data format for most of the examples listed in Table 4.4 (all those where explicit numbers of different types are given) and several othery are at `http://www.math.tu-berlin.de/~paffenho/polytopes/2s2s/`. Many of the files contain geometric coordinates (in particular for all instances up to 22 vertices) in addition to the combinatorial description.

The `polymake` package is a computer system by Gawrilow and Joswig [34] that provides powerful routines for the combinatorial and geometric treatment of polytopes. It can be obtained at `http://www.math.tu-berlin.de/polymake` and is free software for academic use. A client for the polymake package that implements the application of the combinatorial version of the E-construction — that is, as presented in Definition 2.3.1 — is available from the author. It produces the vertex-facet-incidence matrix, it does not check polytopality.

(f_0, f_1)	*E-*		
$(5, 10)$	—	1	simplex
$(9, 26)$	—	1	W ’s polytope W_9
$(10, 30)$	$E_2(\Delta_4)$	1	hypersimplex
$(10, 30)$	$D_1(\text{simplex})$	1	hypersimplex$^\Delta$
$(14, 48)$	$E_2(\text{tr}(\Delta_4; v)) \cong E_2(\text{Pr}(\Delta_3))$	1	B ’s polytope B_{14}
$(15, 54)$	$E_2(C_3 \times C_3)$	1	
$(16, 56)$		1	The “broken cube”
$(18, 66)$	$E_2(\text{tr}(\Delta_4; 2 \text{ vertices}))$	1	
$(19, 72)$	$E_2(C_3 \times C_4)$	1	
	$E_2(\text{tr}(C_3 \times C_3; v))$	1	
$(22, 84)$	$E_2(\text{tr}(\Delta_4; 3 \text{ vertices}))$	3	
$(23, 90)$	$E_2(C_3 \times C_5)$	1	
	$E_2(\text{tr}(C_3 \times C_4; v))$	1	
	$E_2(\text{tr}(C_3 \times C_3; 2 \text{ vertices}))$	3	
$(24, 96)$	$E_2(\square_4)$	1	24-cell
$(26, 102)$	$E_2(\text{tr}(\Delta_4; 4 \text{ vertices}))$	7	
$(27, 108)$	$E_2(C_3 \times C_6)$	1	EKZ_1
$(27, 108)$	$E_2(\text{tr}(C_3 \times C_5; v))$	1	
	$E_2(\text{tr}(C_3 \times C_4; 2 \text{ vertices}))$	7	
	$E_2(\text{tr}(C_3 \times C_3; 3 \text{ vertices}))$	14	
$(28, 114)$	$E_2(\text{tr}(\square_4; v))$	1	EKZ_2
$(29, 120)$	$E_2(C_4 \times C_5)$	1	
$(30, 120)$	$E_2(\text{tr}(\Delta_4; 5 \text{ vertices}))$	30	
$(31, 126)$	$E_2(C_3 \times C_7)$	1	
	$E_2(\text{tr}(C_3 \times C_6; v))$	1	
	$E_2(\text{tr}(C_3 \times C_5; 2 \text{ vertices}))$	7	
	$E_2(\text{tr}(C_3 \times C_4; 3 \text{ vertices}))$	33	
	$E_2(\text{tr}(C_3 \times C_3; 4 \text{ vertices}))$	82	
$(32, 132)$	$E_2(\text{tr}(\square_4; v_1, v_2))$	3	EKZ_3^j, $j = 1, 2, 3$
			for v, v_2 non-adjacent
$(32, 132)$	$E_2(\text{tr}(\square_4; v_1, v_2))$	1	$\square_4^2$ for v, v_2 adjacent
$(33, 138)$	$E_2(\text{tr}(C_4 \times C_5; v))$	1	

Table 4.4: Known $(2, 2)$-polytopes up to 50 vertices.

(f_0, f_1)	E-	
$(34, 138)$	$E_2(\mathrm{tr}(\Delta_4; 6 \text{ vertices}))$	131
$(34, 144)$	$E_2(C_4 \times C_6)$	1
$(35, 144)$	$E_2(C_3 \times C_8)$	1
	$E_2(\mathrm{tr}(C_3 \times C_7; v))$	1
	$E_2(\mathrm{tr}(C_3 \times C_6; 2 \text{ vertices}))$	9
	$E_2(\mathrm{tr}(C_3 \times C_5; 3 \text{ vertices}))$	39
	$E_2(\mathrm{tr}(C_3 \times C_4; 4 \text{ vertices}))$	239
	$E_2(\mathrm{tr}(C_3 \times C_3; 5 \text{ vertices}))$	570
$(35, 150)$	$E_2(C_5 \times C_5)$	1
$(36, 150)$	$E_2(\mathrm{tr}(\square_4; 3 \text{ vertices}))$	16
$(37, 156)$	$E_2(\mathrm{tr}(C_4 \times C_5; 2 \text{ vertices}))$	10
$(38, 156)$	$E_2(\mathrm{tr}(\Delta_4; 7 \text{ vertices}))$	795
$(38, 162)$	$E_2(\mathrm{tr}(C_4 \times C_6; v))$	1
$(39, 162)$	$E_2(C_3 \times C_9)$	1
	$E_2(\mathrm{tr}(C_3 \times C_8; v))$	1
	$E_2(\mathrm{tr}(C_3 \times C_7; 2 \text{ vertices}))$	9
	$E_2(\mathrm{tr}(C_3 \times C_6; 3 \text{ vertices}))$	50
	$E_2(\mathrm{tr}(C_3 \times C_5; 4 \text{ vertices}))$	305
	$E_2(\mathrm{tr}(C_3 \times C_4; 5 \text{ vertices}))$	1751
	$E_2(\mathrm{tr}(C_3 \times C_3; 6 \text{ vertices}))$	4401
$(39, 168)$	$E_2(C_4 \times C_7)$	1
	$E_2(\mathrm{tr}(C_5 \times C_5; v))$	1
$(40, 168)$	$E_2(\mathrm{tr}(\square_4; 4 \text{ vertices}))$	102
$(41, 174)$	$E_2(\mathrm{tr}(C_4 \times C_5; 3 \text{ vertices}))$	57
$(41, 180)$	$E_2(C_5 \times C_6)$	1
$(42, 174)$	$E_2(\mathrm{tr}(\Delta_4; 8 \text{ vertices}))$	5152
$(42, 180)$	$E_2(\text{✚}_4 + \text{✚}_4)$	1
$(42, 180)$	$E_2(\mathrm{tr}(C_4 \times C_6; 2 \text{ vertices}))$	13
$(43, 180)$	$E_2(C_3 \times C_{10})$	1
	$E_2(\mathrm{tr}(C_3 \times C_9; v))$	1
	$E_2(\mathrm{tr}(C_3 \times C_8; 2 \text{ vertices}))$	11
	$E_2(\mathrm{tr}(C_3 \times C_7; 3 \text{ vertices}))$	57

Table 4.4: Known $(2, 2)$-polytopes up to 50 vertices.

(f_0, f_1)	***E*-**	
	$E_2(\mathrm{tr}(C_3 \times C_6; 4 \text{ vertices}))$	423
	$E_2(\mathrm{tr}(C_3 \times C_5; 5 \text{ vertices}))$	2485
	$E_2(\mathrm{tr}(C_3 \times C_4; 6 \text{ vertices}))$	14301
	$E_2(\mathrm{tr}(C_3 \times C_3; 7 \text{ vertices}))$	many
$(43, 186)$	$E_2(\mathrm{tr}(C_4 \times C_7; v))$	1
	$E_2(\mathrm{tr}(C_5 \times C_5; 2 \text{ vertices}))$	6
$(44, 186)$	$E_2(\mathrm{tr}(\square_4; 5 \text{ vertices}))$	628
$(44, 192)$	$E_2(C_4 \times C_8)$	1
$(45, 192)$	$E_2(\mathrm{tr}(C_4 \times C_5; 4 \text{ vertices}))$	517
$(45, 198)$	$E_2(\mathrm{tr}(C_5 \times C_6; v))$	1
$(46, 192)$	$E_2(\mathrm{tr}(\Delta_4; 9 \text{ vertices}))$	many
$(46, 198)$	$E_2(\mathrm{tr}(C_4 \times C_6; 3 \text{ vertices}))$	75
$(47, 198)$	$E_2(C_3 \times C_{11})$	1
	$E_2(\mathrm{tr}(C_3 \times C_{10}; v))$	1
	$E_2(\mathrm{tr}(C_3 \times C_9; 2 \text{ vertices}))$	11
	$E_2(\mathrm{tr}(C_3 \times C_8; 3 \text{ vertices}))$	69
	$E_2(\mathrm{tr}(C_3 \times C_7; 4 \text{ vertices}))$	525
	$E_2(\mathrm{tr}(C_3 \times C_6; 5 \text{ vertices}))$	3567
	$E_2(\mathrm{tr}(C_3 \times C_5; 6 \text{ vertices}))$	many
	$E_2(\mathrm{tr}(C_3 \times C_4; 7 \text{ vertices}))$	many
	$E_2(\mathrm{tr}(C_3 \times C_3; 8 \text{ vertices}))$	many
$(47, 204)$	$E_2(\mathrm{tr}(C_4 \times C_7; 2 \text{ vertices}))$	88
	$E_2(\mathrm{tr}(C_5 \times C_5; 3 \text{ vertices}))$	38
$(47, 210)$	$E_2(C_5 \times C_7)$	1
$(48, 204)$	$E_2(\mathrm{tr}(\square_4; 6 \text{ vertices}))$	4877
$(48, 210)$	$E_2(\mathrm{tr}(C_4 \times C_8; v))$	1
$(48, 216)$	$E_2(C_6 \times C_6)$	1
$(49, 210)$	$E_2(\mathrm{tr}(C_4 \times C_5; 5 \text{ vertices}))$	4498
$(49, 216)$	$E_2(C_4 \times C_9)$	1
	$E_2(\mathrm{tr}(C_5 \times C_6; 2 \text{ vertices}))$	13
$(50, 210)$	$E_2(\mathrm{tr}(\Delta_4; 10 \text{ vertices}))$	many
$(50, 216)$	$E_2(\mathrm{tr}(C_4 \times C_6; 4 \text{ vertices}))$	746

Table 4.4: Known $(2,2)$-polytopes up to 50 vertices.

Table 4.5: Some of the previously known $(2,2)$-polytopes discussed in the text, and those not from the E-construction.

		G –E	**G ¨**	**G´**	***E*-C**
simplex	$(5, 10)$	$0_{3,0} = 0_{0,3}$	$K_4^4 = K_1^4$	$A(4,1) = A(4,4)$	
W_9	$(9,26)$				
hypersimplex	$(10, 30)$			$A(4,2) = A(4,3)$	$E_2(\Delta_4)$
hypersimplex$^\Delta$	$(10, 30)$	$0_{1,2} = 0_{2,1}$	$K_2^4 = K_3^4$		D_1(simplex)
B_{14}	$(14, 48)$				E_2(prism over Δ_3)
24-cell	$(24, 96)$		M_4	$B(4,2)$	$E_2(\square_4)$
EKZ_1	$(27, 108)$				$E_2(C_3 \times C_6)$
EKZ_2	$(28, 114)$				E_2(Stacked $✚_4$)
$\mathrm{EKZ}_3^{(j)}$, $j = 1, 2, 3$	$(32, 132)$				E_2(Twice Stacked $✚_4$)
Dipyramidal 720-cell	$(720, 3600)$			$G(4,2)$	E_2(120-cell)
EKZ_{fat}	$(459360, 2319120)$				

Table 4.6: A list of all previously known examples of (r, s)-polytopes in dimension $d > 4$. The column for the type contains the pair (r, s).

			G –E	**G ¨**	**G´**	***E*-C**
simplex		(d,d)	$0_{d-1,0} = 0_{0,d-1}$	$K^d_1 = K^d_d$	A(d,1)=A(d,d)	$E^d_{d-1}(\Delta_d)^\Delta$
hypersimplices		(d-2,2)	$0_{d-k-1,k}, 2 \leq k \leq d-1$	K^d_k	A(d,k+1)=A(d,d-k-1)	$E^d_{k-1}(\Delta_d)^\Delta$
dual half cubes		$(d-3, 3)$	$1_{d-3,1}$	N^d		
		(d-2,2)		M^d		$E_{d-2}(N_d)^\Delta = E_2(\Box_d)^\Delta$
		$(3, 3)$	$1_{2,2}$			
		$(4, 3)$	$1_{3,2}$			
		$(5, 3)$	$1_{4,2}$			
S ¨ polytope	$(27, 216, 720,$ $1080, 648, 99)$	$(2, 4)$	$2_{2,1}$			
		$(3, 4)$	$2_{3,1}$			
H polytope	$(56, 756, 4032, 10080,$ $12096, 6048, 702)$	$(2, 5)$	$3_{2,1}$			
G polytope	$(240, 6720, 60480,$ $241920, 483840,$ $483840, 207360, 19440)$	$(2, 6)$	$4_{2,1}$			
E_8-polytope	$(2160, \ldots, 17520)$	$(4, 4)$	$2_{4,1}$			

Table 4.7: The infinite series of (2, 2)-polytopes.

Stacked polytopes	$(10+4n, 30+18n, 30+18n, 10+4n; 50+26n)$	Corollary 2.5.11
Stack of cross polytopes	$(6+18n, 12+84n, 12+84n6+18n; 24+120n)$	Proposition 2.5.12
stack of cross polytopes with glued simplices	$(54n-30, 252n-156, 252n-156, 54n-30; ???)$	Theorem 2.5.15
Stack of cut 600-cells	$(666n+54, 3360n+240, 3360n+204, 666n+54)$	[33, Sec. 3.3]
Products of polygons	$(nm+m+n, 6mn, 6mn, mn+m+n; 8mn+2(m+n))$	Theorem 3.4.1

Table 4.8: A (not exhaustive) list of previously known polytopes that can be obtained via the E-construction or appear otherwise in this context, but that are neither r-simple nor s-simplicial for $r, s \geq 2$.

		G´	***E*-C**
hypercube	(16, 32, 24, 8; 64)	$E_3(\square_4) = E_0(\boldsymbol{+}_4)$	
	(24, 96, 88, 32; 160)	$B(4, 3)$	$E_1(\square_4) = E_2(\boldsymbol{+}_4)$
cross polytope	(8, 24, 32, 16; 64)	$B(4, 4)$	
	(48, 240, 288, 96; 480)	$F(4, 2) = F(4, 3)$	E_1(24-cell) = E_2(24-cell)
120-cell	(600, 1200, 720, 120; 2400)	$G(4, 1)$	
Dipyramidal 720-cell	(720, 3120, 3600, 1200; 6000)	$G(4, 3)$	E_2(600-cell) = E_1(120-cell)
600-cell	(120, 720, 1200, 600; 2400)	$G(4, 4)$	
hypercube		$B(d, d)$	$E_{d-1}(\square_d) = E_0(\boldsymbol{+}_d)$
cross polytope		$B(d, 1)$	
		$B(d, k+1)$	$E_k(\square_d) = E_{d-k-1}(\boldsymbol{+}_d)$

C 5

B S

(A B ¨ , J S ¨ , G¨ M. Z)

5.1 I

This chapter is independent of the previous three chapters on the E-construction. We move to a new combinatorial construction, the *Bier construction*, which is a construction defined for arbitrary finite bounded posets. The results presented here are joint work with Anders Björner, Jonas Sjöstrand, and Günter M. Ziegler [21].

The Bier construction has some formal similarity with the E-construction of Definition 2.3.1: It takes all those intervals of a poset P as the new elements of a poset $\mathrm{Bier}(P, I)$, that have their minimal elements in a given ideal I and their maximal elements outside this ideal. This poset is ordered by reversed inclusion. However, the construction serves a completely different aim, and the results we present in this chapter have a much more combinatorial flavour.

Starting point of our construction are unpublished notes of Thomas Bier [13], where he describes a simple construction for a large number of simplicial PL spheres. His construction associates a simplicial $(n-2)$-sphere S with $2n$ vertices to any simplicial complex $\Delta \subset 2^{[n]}$ on n vertices, by forming the *deleted join* of the complex Δ with its combinatorial Alexander dual $\Delta^* := \{\sigma \subset [n] : [n] \setminus \sigma \notin \Delta\}$.

Thomas Bier verified that any addition of a new face to the simplicial complex Δ amounts to a bistellar flip in the sphere S defined above. A short published account of this proof is given in Jiři Matoušek's book [60, Sect. 5.6]. Mark de Longueville [31] recently found a simple alternative proof. We show that this original construction is a special case of ours.

For our generalised version of Bier's construction we obtain several new properties. This includes in particular the following results.

- We extend Bier's construction and define more general *Bier posets* $\mathrm{Bier}(P, I)$, where P is an arbitrary bounded poset of finite length and $I \subset P$ is a proper order ideal.
- The order complex of $\mathrm{Bier}(P, I)$ is PL homeomorphic to the order complex of P. It may be obtained by a sequence of stellar subdivisions of edges in the order complex of P.
- If P is an Eulerian or Cohen-Macaulay poset or lattice, then $\mathrm{Bier}(P, I)$ will have that property as well.
- If P is the face lattice of a regular PL-sphere $\mathcal{S}$, then the lattices $\mathrm{Bier}(P, I)$ are again face lattices of regular PL-spheres, the "Bier spheres" of $\mathcal{S}$.
- If we take P to be the Boolean algebra B_n, then this may be interpreted as the face lattice of the $(n-1)$-simplex, and the ideal I in B_n may consequently be interpreted as an abstract simplicial complex Δ. This is the special setting of the original construction described by Bier.
- The simplicial PL spheres $\mathrm{Bier}(B_n, \Delta)$ are shellable.
- The number of these spheres is so large, that for $n \gg 1$ most of the Bier

spheres Bier(B_n, Δ) are not realisable as polytopes. Thus, Bier's construction provides "many shellable spheres" in the sense of Kalai [52] and Lee [56].

- Similarly, for special choices of the simplicial complex Δ in B_n, and even n, we obtain a large number of nearly neighbourly and centrally symmetric $(n-2)$-spheres on $2n$ vertices.
- The g-vector of a Bier sphere Bier(B_n, Δ) can be expressed explicitly in terms of the f-vector of Δ. These g-vectors are K-sequences. Hence, they satisfy a strong form of the g-conjecture for spheres. Additionally, the generalised lower bound conjecture is verified for Bier spheres.

The study of posets of intervals in a given poset, ordered by inclusion, goes back to a problem posed by Lindström in [57]. See Björner's work [16, 19] for some more results on interval posets.

5.2 B P P

All posets that we consider in this chapter are bounded and have finite length. Recall, that an *ideal* in P is a subset $I \subseteq P$ such that $x \preceq y$ for $x \in P$ and $y \in I$ implies that $x \in I$. An ideal is *proper* if neither $I = P$ nor $I = \emptyset$. In the following we usually denote elements of the ideal $I \subset P$ by x, x_i or x'_i and elements in the complement $P \setminus I$ by y, y_j, or y'_j.

Let P be a finite bounded poset and $I \subset P$ a proper ideal. Roughly, the Bier poset Bier(P, I) is a poset consisting of all intervals $[x, y] \subset P$ that start "inside" the ideal I and end "outside" of it. We can order this set by reversed inclusion. Here is the precise definition.

5.2.1 D [Bier poset]. Let P be a bounded poset of finite length and $I \subset P$ a proper ideal. Define a new poset Bier(P, I) as follows:
Its elements are

- all intervals $[x, y] \subseteq P$ such that $x \in I$ and $y \notin I$,
- together with an additional top element $\hat{1}$.

The order is given by $\alpha \preceq \hat{1}$ for all $\alpha \in$ Bier(P, I) and reversed inclusion of intervals in P, which means $[x', y'] \preceq [x, y]$ in Bier(P, I) if and only if $x' \preceq x \prec y \preceq y'$, for all $x, y, x', y' \in P$ with $x, x' \in I$ and $y, y' \in P \backslash I$.

The interval $I = [\hat{0}, \hat{1}]$ is the unique minimal element of Bier(P, I). Hence, the poset Bier(P, I) is *bounded*. The construction of Bier posets has a some formal similarity to the E-construction as defined in Definition 2.3.1, in the sense that it also forms a new poset out of intervals in a poset P and orders them by reversed inclusion. However, the following results on Bier posets have a more combinatorial and topological flavour, while the central aim of the E-construction was the geometric realisation of certain interesting polytopes.

5.2.2 L [Basic Properties of Bier Posets]. *Let P be a bounded finite poset of length n and $I \subset P$ a proper ideal.*

(1) The posets P and $\mathrm{Bier}(P, I)$ *have the same length.*

(2) $\mathrm{Bier}(P, I)$ *is graded if and only if P is graded.*
In that case, and if ρ_P is a rank function on P, then a rank function on $\mathrm{Bier}(P, I)$ *is given by*

$$\rho(\alpha) := \begin{cases} n + \rho_P(x) - \rho_P(y) & \text{for } \alpha = [x, y],\ x \in I,\ y \in P \setminus I \\ n & \text{for } \alpha = \hat{1}\,. \end{cases}$$

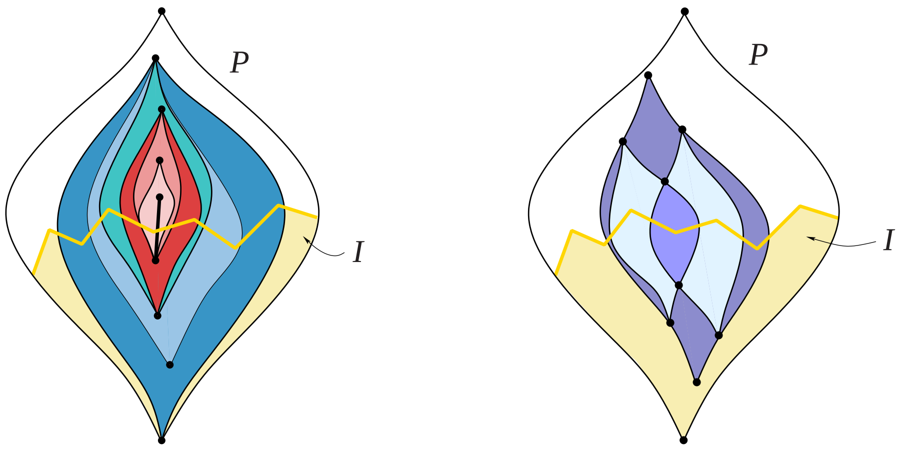
(3) The intervals of $\mathrm{Bier}(P, I)$ *have the following two types:*

$$\begin{aligned} [[x, y], \hat{1}] &\cong \mathrm{Bier}([x, y], I \cap [x, y]) \\ [[x', y'], [x, y]] &= [x', x] \times [y, y']^{\mathrm{op}}, \end{aligned}$$

where $[y, y']^{\mathrm{op}}$ *denotes the interval* $[y, y']$ *with the opposite order.*

(4) If P is a lattice then $\mathrm{Bier}(P, I)$ *is a lattice.*

Proof. (1) A maximal chain in the poset $\mathrm{Bier}(P, I)$ is a sequence of intervals in P such that any two consecutive intervals $[x, y] \subset [x', y']$ satisfy either $x = x'$ and y' covers y, or $y = y'$ and x covers x'. Hence, if we have a chain of length n in P, then we obtain a chain of $n - 1$ intervals in $\mathrm{Bier}(P, I)$ from it. Adding $\hat{1}$ gives the claim. See Figure 5.1(a) for an illustration.



(a) Bier posets preserve the length.

(b) Meet and join of the two light shaded intervals are the two dark shaded intervals, interpreted as elements of the Bier poset.

Figure 5.1: The proof of Lemma 5.2.2, claims (1) and (4).

(2) and (3) are immediate from the definition of a Bier poset.

(4) $\mathrm{Bier}(P, I)$ is bounded. Hence, it suffices to show that meets exist in $\mathrm{Bier}(P, I)$. These are given by $[x, y] \wedge [x', y'] = [x \wedge x', y \vee y']$ and $[x, y] \wedge \hat{1} = [x, y]$. Figure 5.1(b) shows an example. $\square$

5.3 B P S S

For any bounded poset P we denote by $\overline{P} := P \backslash \{\hat{0}, \hat{1}\}$ the *proper part* of P and by $\Delta(\overline{P})$ the order complex of $\overline{P}$, that is, the abstract simplicial complex of all chains in $\overline{P}$. See Definition 1.2.18 for more background.

In this section we give a geometric interpretation of $\mathrm{Bier}(P, I)$, by specifying how its order complex may be derived from the order complex of P via stellar subdivisions. For this, we need an explicit description of stellar subdivisions for abstract simplicial complexes.

5.3.1 D [Stellar Subdivision]. Let Δ be a finite dimensional abstract simplicial complex and $F \in \Delta$ an non-empty face of Δ.

The *stellar subdivision* $\mathrm{sd}_F(\Delta)$ of Δ with respect to F is obtained by removing from Δ all faces that contain F and adding new faces $G \cup \{v_F\}$ (with a new apex vertex v_F) for all faces G that do not contain F, but such that $G \cup F$ is a face in the original complex.

For an edge $e = \{v_1, v_2\}$ of Δ this means that in the stellar subdivision of Δ with respect to e each face $G \in \Delta$ that contains e is replaced by three new faces, namely $(G \backslash \{v_1\}) \cup \{v_e\}$, $(G \backslash \{v_2\}) \cup \{v_e\}$, and $(G \backslash \{v_1, v_2\}) \cup \{v_e\}$. Observe, that stellar subdivision does not change the Euler characteristic.

The stellar subdivisions in a sequence of faces $F_1, \ldots, F_N$ of the complex Δ commute, and thus may be performed in any order — or simultaneously — if and only if no two F_i, F_j are contained in a common face G of the complex, that is, if $F_i \cup F_j$ is not a face for $i \neq j$.

5.3.2 T . *Let P be a bounded poset of finite length and $I \subset P$ a proper ideal. The order complex of $\overline{\mathrm{Bier}(P, I)}$ is obtained from the order complex of $\overline{P}$ by stellar subdivision on all edges of the form $\{x, y\}$, for $x \in \overline{I}$, $y \in \overline{P} \backslash \overline{I}$, $x < y$. These stellar subdivisions must be performed in order of increasing length $\ell(x, y)$.*

Proof. Let n be the length of P. In the following, the elements denoted by x_i or x'_i will be vertices of $\overline{P}$ that are contained in $\overline{I} := I \backslash \{\hat{0}\}$, while elements denoted by y_j or y'_j are from $\overline{P} \backslash \overline{I}$. By (x'_i, y'_i) we will denote the new vertex created by the subdivision of the edge $\{x'_i, y'_i\}$.

We have to verify that subdivision of all edges of $\Delta(\overline{P})$ collected in the sets

$$E_k := \left\{ \{x, y\} \ : \ x < y, \ \ell(x, y) = k, \ x \in \overline{I}, \ y \in \overline{P} \backslash \overline{I} \right\}$$

for $k = 1, \ldots, n-2$ (in this order) results in $\Delta(\overline{\mathrm{Bier}(P,I)})$. To prove this, we will explicitly describe the simplicial complexes Γ_k that we obtain at intermediate stages, i.e. after subdivision of the edges in $E_1 \cup \cdots \cup E_k$. The complexes Γ_k are *not* in general order complexes for $0 < k < n-2$.

C . *After stellar subdivision of the edges of $\Delta(\overline{P})$ in the edge sets $E_1, \ldots, E_k$ (in this order), the resulting complex Γ_k has the faces*

$$\Big\{x_1, x_2, \ldots, x_r, (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), y_1, y_2, \ldots, y_s\Big\} \tag{5.3.1}$$

where

(1) $$x_1 < x_2 < \cdots < x_r < y_1 < y_2 < \cdots < y_s \qquad (r, s \geq 0)$$

must be a strict chain in $\overline{P}$ that may be empty, but has to satisfy $\ell(x_r, y_1) \geq k+1$ if $r \geq 1$ and $s \geq 1$, while

(2) $$[x'_t, y'_t] < \cdots < [x'_2, y'_2] < [x'_1, y'_1] \qquad (t \geq 0)$$

must be a strict chain in $\overline{\mathrm{Bier}(P,I)}$ that may be empty, but has to satisfy $\ell(x'_t, y'_t) \leq k$ if $t \geq 1$, and finally

(3) $$x_r \leq x'_t \quad \text{and} \quad y'_t \leq y_1$$

must hold if both r and t are positive resp. if both s and t are positive.

The conditions (1)–(3) of the claim together imply that the chains of Γ_k are supported on (weak) chains in $\overline{P}$ of the form

$$\hat{0} < x_1 < x_2 < \cdots < x_r \leq x'_t \leq \ldots \leq x'_2 \leq x'_1 < y'_1 \leq y'_2 \leq \ldots \\ \leq y'_t \leq y_1 < y_2 \ldots < y_s < \hat{1}\,.$$

In condition (3) not both inequalities can hold with equality, because of the length requirements for (1) and (2), which for $r, s, t \geq 1$ require that

$$\ell(x'_t, y'_t) \leq k < \ell(x_r, y_1),$$

and thus $[x'_t, y'_t] \subset [x_r, y_1]$.

We verify immediately that for $k = 0$ the description of Γ_0 given in the claim yields $\Gamma_0 = \Delta(\overline{P})$, since for $k = 0$ the length requirement for (2) does not admit any subdivision vertices.

For $k = n-2$ the simplices of Γ_{n-2} as given by the claim cannot contain both x_r and y_1, that is, they all satisfy either $r = 0$ or $s = 0$ or both, since otherwise we would get a contradiction between the length requirement for (1) and the fact that any interval $[x_r, y_1] \subseteq \overline{P}$ can have length at most $n-2$. Thus, we obtain that $\Gamma_{n-2} = \Delta(\overline{\mathrm{Bier}(P,I)})$, if we identify the subdivision vertices (x'_i, y'_i) with the intervals $[x'_i, y'_i]$ in P, the elements x_i with the intervals $[x_i, \hat{1}]$ and the elements $y_j \in \overline{P} \backslash \overline{I}$ with the intervals $[\hat{0}, y_j]$.

Finally, we prove the claim by verifying the induction step from k to $k+1$. It follows from the description of the complex Γ_k that no two edges in E_{k+1} lie in the same facet. Thus we can stellarly subdivide the edges in E_{k+1} in arbitrary order. Suppose the edge (x_r, y_1) of the simplex

$$\{x_1, \ldots, x_{r-1}, x_r, (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), y_1, y_2, \ldots, y_s\}$$

is contained in E_{k+1}. Then stellar subdivision yields the three new simplices

$$\begin{array}{llll}
\{x_1, \ldots, x_{r-1}, & (x_r, y_1), (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), y_1, y_2, \ldots, y_s, \}, & & \\
\{x_1, \ldots, x_{r-1}, x_r, & (x_r, y_1), (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), & y_2, \ldots, y_s, \}, & \text{and} \\
\{x_1, \ldots, x_{r-1}, & (x_r, y_1), (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), & y_2, \ldots, y_s, \}. &
\end{array}$$

All three sets then are simplices of Γ_{k+1}, satisfying all the conditions specified in the claim (with t replaced by $t+1$ and r or s or both reduced by 1). Also all simplices of Γ_{k+1} arise this way. This completes the induction step. $\square$

The subdivision map of the previous proof can be given explicitely. For this, we just define the map

$$\pi : \|\Delta(\overline{\mathrm{Bier}(P, I)})\| \longrightarrow \|\Delta(\overline{P})\|$$

which is given on the vertices of $\Delta(\overline{\mathrm{Bier}(P, I)})$ by

$$[x, y] \longmapsto \begin{cases} \frac{1}{2}x + \frac{1}{2}y & \hat{0} < x < y < \hat{1}, x \in I, y \notin I \\ x & \hat{0} < x < y = \hat{1}, x \in I, y \notin I \\ y & \hat{0} = x < y < \hat{1}, x \in I, y \notin I \end{cases}$$

and is extended linearly on the simplices of $\Delta(\overline{\mathrm{Bier}(P, I)})$. We have the following simple corollaries of Theorem 5.3.2.

5.3.3 C . $\|\Delta(\overline{\mathrm{Bier}(P, I)})\|$ *and* $\|\Delta(\overline{P})\|$ *are PL homeomorphic.* $\square$

In the case where P is the face poset of a regular PL sphere or manifold, this implies that the barycentric subdivision of $\mathrm{Bier}(P, I)$ may be derived from the barycentric subdivision of P by stellar subdivisions. In particular, in this case $\mathrm{Bier}(P, I)$ is again the face poset of a PL-sphere or manifold.

5.3.4 C . *If P is the face lattice of a strongly regular PL sphere then so is* $\mathrm{Bier}(P, I)$. $\square$

5.3.5 C . *If P is Cohen-Macaulay then so is* $\mathrm{Bier}(P, I)$.

Proof. Being Cohen-Macaulay is a topological property (see Munkres [64] for this), so this is immediate from the homeomorphism defined after the proof of Theorem 5.3.2. $\square$

5.4 E B P

From now on we assume that P is a *graded* poset of length n. We compute the f-vector $f(\mathrm{Bier}(P, I)) := (f_0, f_1, \ldots, f_n)$, where f_i denotes the elements of *rank i* in the poset $\mathrm{Bier}(P, I)$.

5.4.1 R . Observe, that this notation is off by 1 from the convention in the previous chapters. However, in the rest of this thesis no polytopes will appear anymore, so that the index shift we have used previously would just make the following computation more complicated without any benefit at a later place.

The following computation of the f-vector is immediate from the definition of the Bier poset of P.

5.4.2 P [f-Vector]. *Let P be a finite graded poset of length n and $I \subset P$ a proper ideal. Let ρ_P be the rank function of P. The f-vector of* $\mathrm{Bier}(P, I)$ *is*

$$f_i(\mathrm{Bier}(P, I)) = \begin{cases} 1 & \text{for } i = n \\ \#\Big\{ [x, y] : x \in I, y \notin I, n + \rho_P(x) - \rho_P(y) = i \Big\} & \text{otherwise.} \end{cases}$$

In particular, $f_0(\mathrm{Bier}(P, I)) = 1$. □

5.4.3 T [Eulerian Bier Posets]. *Let P be an Eulerian poset and $I \subset P$ a proper ideal. Then* $\mathrm{Bier}(P, I)$ *is also an Eulerian poset.*

Proof. Let ρ be the rank function on the poset P. $\mathrm{Bier}(P, I)$ is a graded poset of the same length as P by Lemma 5.2.2. Hence, it suffices to prove that all intervals of length ≥ 1 in $\mathrm{Bier}(P, I)$ contain equally many odd and even rank elements, by Theorem 1.2.12.

This can be done by induction. For length $\ell(P) \leq 1$ the claim is true. Proper intervals of the form $[[x, y], \hat{1}]$ are, in view of Lemma 5.2.2, Eulerian by induction. Proper intervals of the form $[[x', y'], [x, y]]$ are Eulerian, since any product of Eulerian posets is Eulerian, by Theorem 1.2.16.

Finally the whole poset $\mathrm{Bier}(P, I)$ contains the same number of odd and even rank elements by the following computation:

$$\begin{aligned} \sum_{i=0}^{n} (-1)^{n-i} f_i(\mathrm{Bier}(P, I)) &= 1 + \sum_{i=0}^{n-1} (-1)^{n-i} f_i(\mathrm{Bier}(P, I)) \\ &= 1 + \sum_{y \notin I} \sum_{\substack{x \in I \\ x \leq y}} (-1)^{\rho(y) - \rho(x)} \\ &= 1 + \sum_{y \notin I} \sum_{x \leq y} (-1)^{\rho(y) - \rho(x)} - \sum_{y \notin I} \sum_{\substack{x \notin I \\ x \leq y}} (-1)^{\rho(y) - \rho(x)} \qquad (5.4.1) \end{aligned}$$

$$= 1 + 0 - \sum_{x \notin I} \sum_{x \leq y} (-1)^{\rho(y)-\rho(x)} \qquad (5.4.2)$$
$$= 1 + 0 - 1 = 0$$

where the first double sum in (5.4.1) is 0 as $[\hat{0}_P, y]$ is Eulerian and $\rho(y) \geq 1$, and the double sum in (5.4.2) is -1, as $[x, \hat{1}_P]$ is Eulerian and trivial only for $x = \hat{1}_P$. □

Alternatively, the result of the computation in this proof also follows from the topological interpretation of $\mathrm{Bier}(P, I)$ in the previous section.

5.5 S B S

Now we specialise to Bier's original setting, where $P = B_n$ is the Boolean lattice. In the following, we denote with $[x, y]$ and $(x, n]$ closed and half-open sets of integers in $[n]$, respectively.

Any non-empty ideal in the Boolean algebra B_n can be interpreted as an abstract simplicial complex with at most n vertices. We denote such a complex by Δ throughout the rest of this chapter. We can restate the definition of a Bier poset in this special setting as follows:

$$\mathrm{Bier}(B_n, \Delta) \ := \left\{(B, C) \ : \ \emptyset \subseteq B \subset C \subseteq [n], B \in \Delta, C \notin \Delta\right\} \cup \left\{\ \hat{1}\ \right\},$$

again ordered by reversed inclusion of intervals (and $\hat{1}$ as the maximal element). Facets of $\mathrm{Bier}(B_n, \Delta)$ correspond in this notation to pairs (B, C) in which the set B differs from the set C by only one element. Hence, we can denote the facets of $\mathrm{Bier}(B_n, \Delta)$ by

$$(A; x) := (A, A \cup \{x\}) \in \mathrm{Bier}(B_n, \Delta). \qquad (5.5.1)$$

We write $\mathcal{F}(\Delta)$ for the set of all facets.

The poset $\mathrm{Bier}(B_n, \Delta)$ is the face lattice of a simplicial PL $(n-2)$-sphere, by Corollary 5.3.4. With the following theorem we obtain that $\mathrm{Bier}(B_n, \Delta)$ is shellable. This is a much stronger property, as it is known that shellability implies the PL-sphericity for pseudo-manifolds (see Björner [18] for this result).

5.5.1 T [Shellable Bier Spheres]. *Let $\Delta \subset B_n$ be a proper ideal in B_n. Then the $(n-2)$-sphere $\mathrm{Bier}(B_n, \Delta)$ is shellable.*

Proof. The shellability proof is in two steps. First we show that the rule

$$\begin{aligned} R : \mathcal{F}(\Delta) &\to \mathrm{Bier}(B_n, \Delta) \\ (A; x) &\mapsto (A \cap (x, n], A \cup [x, n]). \end{aligned} \qquad (5.5.2)$$

defines a *restriction operator* on the poset. That means, it induces a partition

$$\mathrm{Bier}(B_n, \Delta) \quad = \quad \biguplus_{(A;x)\in\mathcal{F}(\Delta)} [R(A;x), (A;x)],$$

and the precedence relation forced by this restriction operator is acyclic. Thus, any linear extension of the precedence relation yields a shelling order. Compare this definition also to Björner [17].

Such a restriction operator indeed defines a partition. This can be seen as follows: Take any element $(B, C) \in \mathrm{Bier}(B_n, \Delta)$. Set

$$\begin{aligned} x := \min\{y \in C\backslash B \,:\, B \cup (C \cap [1, y]) \notin \Delta\} \\ \max\{y \in C\backslash B \,:\, B \cup (C \setminus [y, n]) \in \Delta\} \end{aligned}$$

and $A := B \cup (C \cap [1, x))$. Then we have

$$A \cap (x, n] \subseteq B \subseteq A \subset A \cup \{x\} \subseteq C \subseteq A \cup [x, n]$$

and thus (B, C) is contained in $[R(A; x), (A; x)]$.

To see that the intervals in the partition do not intersect we have to show that if both $R(A; x) \preceq (A'; x')$ and $R(A'; x') \preceq (A; x)$, then $(A; x) = (A'; x')$. This is a special case of a more general fact we establish next, so we do not give the argument here.

For any shelling order "$\lhd$" that would induce R as its "unique minimal new face" restriction operator we are forced to require that if $R(A; x) \preceq (A'; x')$ for two facets $(A; x)$ and $(A'; x')$, then $(A; x) \unlhd (A'; x')$.

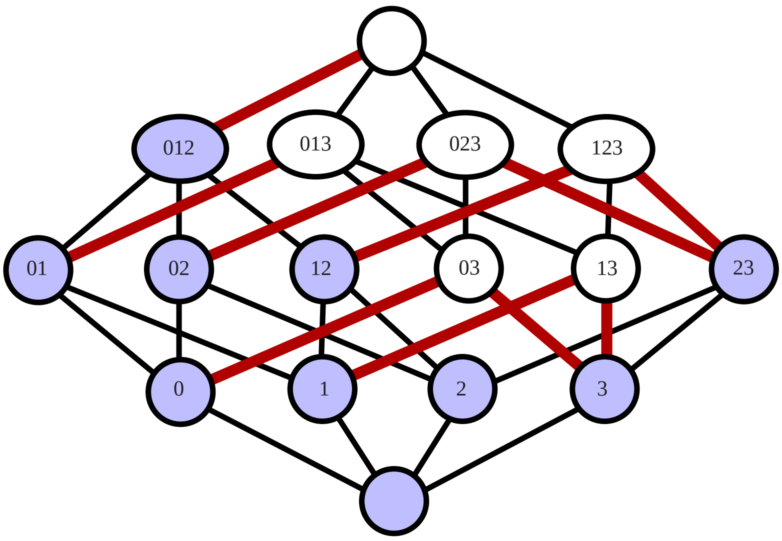


Figure 5.2: The face lattice the 3-simplex. The shaded elements form an abstract simplicial complex Δ. The bold edges define the facets of $\mathrm{Bier}(B_4, \Delta)$.

By definition, $R(A;x) \preceq (A';x')$ means that

$$A \cap (x,n] \subseteq A' \subset A' \cup \{x'\} \subseteq A \cup [x,n], \tag{5.5.3}$$

which may be reformulated as

$$(A \cup \{x\})_{>x} \subseteq A' \qquad \text{and} \qquad (A' \cup \{x'\})_{<x} \subseteq A. \tag{5.5.4}$$

We now *define* the relation $(A;x) \lhd (A';x')$ to hold if and only if (5.5.4) holds together with

$$(A \cup \{x\})_{\leq x} \nsubseteq A' \qquad \text{and} \qquad (A' \cup \{x'\})_{\geq x} \nsubseteq A. \tag{5.5.5}$$

Note that our sets A, A' belong to an ideal which does not contain $A \cup \{x\}$ and $A' \cup \{x'\}$, so (5.5.5) applies if (5.5.4) does.

By the *support* of $(A;x)$ we mean the set $A \cup \{x\}$. The element x of the support is called its *root element*.

We interpret a relation $(A;x) \lhd (A';x')$ as a *step* from $(A;x)$ to $(A';x')$. The first conditions of (5.5.4) and (5.5.5) say that

> In each step, the elements that are deleted from the support are $\leq x$; moreover, we must either loose some element $\leq x$ from the support, or we must choose x' from $(A \cup \{x\})_{\leq x}$, or both. (5.5.6)

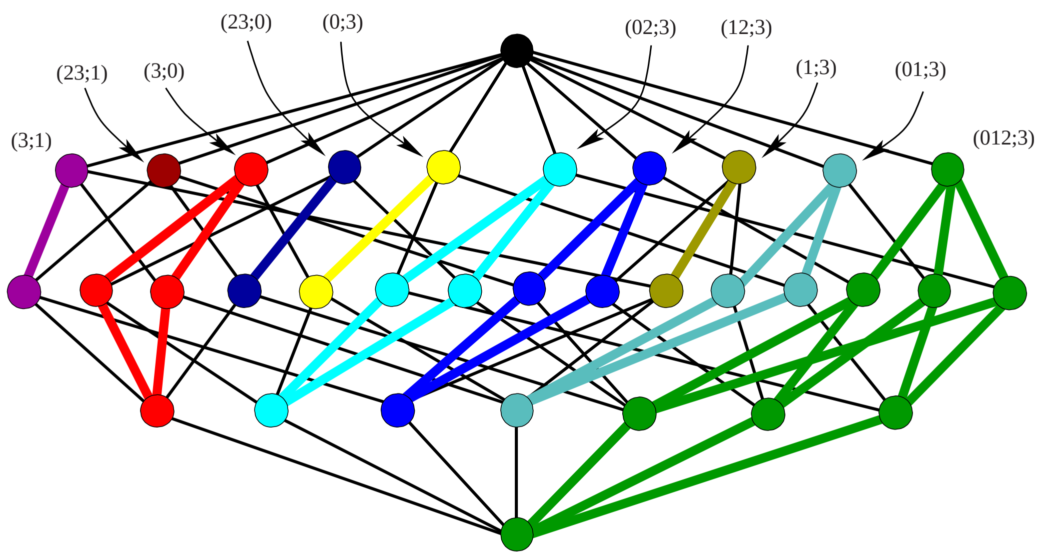


Figure 5.3: The restriction operator applied to the poset and the ideal shown in Figure 5.2. The intervals of the partition are drawn differently shaded and with thick edges. The facets are marked in the notation of (5.5.1)

Similarly, the second conditions of (5.5.4) and (5.5.5) say that

> In each step, the elements that are added to the support are $> x$; moreover, we must either add some element $> x$ to the support, or we must keep x in the support, or both. (5.5.7)

Now we show that the transitive closure of the relation $\lhd$ does not contain any cycles. So, suppose that there is a cycle,

$$(A_0; x_0) \lhd (A_1; x_1) \lhd \ldots \lhd (A_k; x_k) = (A_0; x_0).$$

First assume that not all root elements x_i in this cycle are equal. Then by cyclic permutation we may assume that x_0 is the smallest root element that appears in the cycle, and that $x_1 > x_0$. Thus x_1 is clearly not from $(A \cup \{x_0\})_{\leq x_0}$, so by Condition (5.5.6) we loose an element $\leq x_0$ from the support of $(A_0; x_0)$ in this step. But in all later steps the elements we add to the support are $> x_i \geq x_0$, so the lost element will never be retrieved. Hence we cannot have a cycle.

The second possibility is that all root elements in the cycle are equal, that is, $x_0 = x_1 = \cdots = x_k = x$. Then by Conditions (5.5.6) and (5.5.7), in the whole cycle we loose only elements $< x$ from the support, and we add only elements $> x$. The only way this can happen is that, when we traverse the cycle, no elements are lost and none are added, so $A_0 = A_1 = \cdots = A_k$. Consequently, there is no cycle. □

The relation defined on the set of all pairs $(A; x)$ with $A \subset [n]$ and $x \in [n]\backslash A$ by (5.5.4) alone does have cycles, such as

$$(\{1,4\},2) \lhd (\{1,4\},3) \lhd (\{4\},1) \lhd (\{1,4\},2).$$

This is the reason why we also require condition (5.5.5) in the definition of "$\lhd$".

The shelling order implied by the proof of Theorem 5.5.1 may also be described in terms of a linear ordering. For that we associate with each facet $(A; x)$ a vector $\chi(A; x) \in \mathbb{R}^n$, defined as follows:

$$\chi(A; x)_a := \begin{cases} -1 & \text{for } a \in (A \cup \{x\})_{\leq x}, \\ \ \ 0 & \text{for } a \notin A \cup \{x\}, \\ +1 & \text{for } a \in (A \cup \{x\})_{> x}. \end{cases}$$

With this assignment, we get that $(A; x) \lhd (A'; x')$, as characterised by the conditions in (5.5.6) and (5.5.7), implies that $\chi(A; x) <_{\text{lex}} \chi(A'; x')$. Thus we have that lexicographic ordering on the χ-vectors induces a shelling order for every Bier sphere obtained from the Boolean poset B_n.

5.6 g-V B P

In this section we derive the basic relationship between the f-vector of a Bier sphere $\mathrm{Bier}(B_n, \Delta)$ and the f-vector of the underlying simplicial complex Δ. The results in this section are completely due to Anders Björner and Jonas Sjöstrand.

In extension of the notation of Section 5.4 let $f_i(\Delta)$ denote the number of sets of cardinality i in a complex Δ. The *f-vector* of a proper subcomplex $\Delta \subset B_n$ is $f(\Delta) = (f_0, f_1, \ldots, f_n)$, with $f_0 = 1$ and $f_n = 0$.

Now let Γ be a finite simplicial complex that is pure of dimension $d = n - 2$, that is, such that all maximal faces have cardinality $n - 1$. We will apply this to $\Gamma = \mathrm{Bier}(B_n, \Delta)$. We define $h_i(\Gamma)$ by

$$h_i(\Gamma) \;:=\; \sum_{j=0}^{n-1} (-1)^{i+j} \binom{n-1-j}{n-1-i} f_j(\Gamma) \tag{5.6.1}$$

for $0 \le i \le n-1$, and $h_i(\Gamma) := 0$ outside this range. Then, conversely

$$f_i(\Gamma) \;=\; \sum_{j=0}^{n-1} \binom{n-1-j}{n-1-i} h_j(\Gamma).$$

Finally, for $0 \le i \le \lfloor \frac{n-1}{2} \rfloor$ let $g_i(\Gamma) := h_i(\Gamma) - h_{i-1}(\Gamma)$, with $g_0(\Gamma) = 1$.

Now we consider the f-, h- and g-vectors of the sphere $\Gamma = \mathrm{Bier}(B_n, \Delta)$. This is an $(n-2)$-dimensional shellable sphere on $f_1(\Delta) + n - f_{n-1}(\Delta)$ vertices. So in the usual case of $f_1 = n$ and $f_{n-1} = 0$, that is, when Δ contains all the 1-element subsets but no $(n-1)$-element subset of $[n]$, we get a sphere on $2n$ vertices.

In terms of the facets $(A; x) \in \mathcal{F}(\Delta)$ we have the following simple description of its h-vector:

$$h_i(\mathrm{Bier}(B_n, \Delta)) \;=\; \#\Big\{ (A; x) \in \mathcal{F}(\Delta) \;:\; |A \cap (x, n]| + |[1, x) \setminus A| = i \Big\} \tag{5.6.2}$$

for $0 \le i \le n-1$. This is a consequence of the fact that we can write the h-vector of a shellable complex in terms of the restriction operator defined in (5.5.2) as

$$h_i(\mathrm{Bier}(B_n, \Delta)) \;=\; \#\Big\{ (A; x) \in \mathcal{F}(\Delta) \;:\; \rho(R(A; x)) = i \Big\},$$

cf. Björner's work [17, p. 229]. Using the rank function ρ of $\mathrm{Bier}(B_n, \Delta)$ computed in Lemma 5.2.2(2) transforms this into the equation in (5.6.2).

5.6.1 L **[Dehn–Sommerville equations].** *For $0 \le i \le n-1$,*

$$h_{n-1-i}(\mathrm{Bier}(B_n, \Delta)) \;=\; h_i(\mathrm{Bier}(B_n, \Delta)).$$

Proof. This is a direct consequence of equation (5.6.2). Namely, neither the definition of the h-vector nor the construction of the Bier sphere depends on the ordering of the ground set. Hence, we can reverse the order of the ground set $[n]$ and obtain

$$h_i(\mathrm{Bier}(B_n,\Delta)) \;=\; \#\Big\{(A;x)\in\mathcal{F}(\Delta)\;:\;|A\cap[1,x)|+|(x,n]\setminus A|=i\Big\}. \tag{5.6.3}$$

Thus, a set A contributes to $h_i(\mathrm{Bier}(B_n,\Delta))$ according to (5.6.2) if and only if the complement of A with respect to the $(n-1)$-element set $[n]\setminus\{x\}$ contributes to $h_{n-1-i}(\mathrm{Bier}(B_n,\Delta))$ according to (5.6.3). □

The g-vector of $\mathrm{Bier}(B_n,\Delta)$ has the following nice form.

5.6.2 T **[g-Vector].** *For all $i=0,\dots,\lfloor\frac{n-1}{2}\rfloor$,*

$$g_i(\mathrm{Bier}(B_n,\Delta)) \;=\; f_i(\Delta)-f_{n-i}(\Delta).$$

Proof. Let Δ^{aug} be the same complex as Δ, but viewed as sitting inside the larger Boolean lattice B_{n+1}. We claim that

$$h_i(\mathrm{Bier}(B_{n+1},\Delta^{\mathrm{aug}})) \;=\; h_{i-1}(\mathrm{Bier}(B_n,\Delta))+f_i(\Delta) \tag{5.6.4}$$

for $0\le i\le n$. This can be seen from equation (5.6.3) as follows. The facets $(A;x)$ of $\mathrm{Bier}(B_{n+1},\Delta^{\mathrm{aug}})$ that contribute to $h_i(\mathrm{Bier}(B_{n+1},\Delta^{\mathrm{aug}}))$ are of two kinds: either $x\neq n+1$ or $x=n+1$. There are $h_{i-1}(\mathrm{Bier}(B_n,\Delta))$ of the first kind and $f_i(\Delta)$ of the second.
Using both equation (5.6.4) and Lemma 5.6.1 twice we compute

$$\begin{aligned}
g_i(\mathrm{Bier}(B_n,\Delta)) &= h_i(\mathrm{Bier}(B_n,\Delta))-h_{i-1}(\mathrm{Bier}(B_n,\Delta))\\
&= h_{n-1-i}(\mathrm{Bier}(B_n,\Delta))-h_{i-1}(\mathrm{Bier}(B_n,\Delta))\\
&= h_{n-i}(\mathrm{Bier}(B_{n+1},\Delta^{\mathrm{aug}}))-f_{n-i}(\Delta)-h_{i-1}(\mathrm{Bier}(B_n,\Delta))\\
&= h_i(\mathrm{Bier}(B_{n+1},\Delta^{\mathrm{aug}}))-f_{n-i}(\Delta)-h_{i-1}(\mathrm{Bier}(B_n,\Delta))\\
&= f_i(\Delta)-f_{n-i}(\Delta).
\end{aligned}$$
□

5.6.3 C **.** *The face numbers $f_i(\mathrm{Bier}(B_n,\Delta))$ of the Bier sphere depend only on n and the differences $f_i(\Delta)-f_{n-i}(\Delta)$.*

Proof. The g-vector determines the h-vector (via Lemma 5.6.1), which in turn determines the f-vector. □

For example, if $n=4$ and $f(\Delta)=(1,3,0,0,0)$ or $f(\Delta)=(1,4,3,1,0)$, then we get $g(\mathrm{Bier}(B_4,\Delta))=(1,3)$ and $f(\mathrm{Bier}(B_4,\Delta))=(1,7,15,10)$.

5.6.4 T . *Every simplicial complex $\Delta \subseteq B_n$ has a subcomplex Δ' such that*

$$f_i(\Delta') = f_i(\Delta) - f_{n-i}(\Delta)$$

for $0 \le i \le \lfloor \frac{n}{2} \rfloor$ *and* $f_i(\Delta') = 0$ *for* $i > \lfloor \frac{n}{2} \rfloor$.

Proof. For any simplicial complex Δ in B_n, define the d-vector by $d_i(\Delta) = f_i(\Delta) - f_{n-i}(\Delta)$ for $0 \le i \le \lfloor \frac{n}{2} \rfloor$ and $d_i(\Delta) = 0$ for greater i. We construct a subcomplex $\Delta' \subseteq \Delta$ with $f_i(\Delta') = d_i(\Delta)$ for all i.

Choose Δ' as a minimal subcomplex of Δ with the same d-vector. We must show that $f_i(\Delta') = 0$ for all $\lfloor \frac{n}{2} \rfloor < i \le n$. Suppose that there is a set $C \in \Delta'$ with $|C| > \frac{n}{2}$. Then there is an involution $\pi : [n] \to [n]$, i. e. a permutation of the ground set of order two, such that

$$\pi(C) \supseteq [n] \setminus C, \tag{5.6.5}$$

where $\pi(C)$ is the image of C. Define a map $\varphi : B_n \to B_n$ by $\varphi(B) = [n] \setminus \pi(B)$ for all $B \subseteq [n]$. Observe that φ satisfies the following three assertions for all $B \subseteq [n]$:

(1) $\varphi(\varphi(B)) = B$,
(2) $B' \subseteq B \Rightarrow \varphi(B') \supseteq \varphi(B)$,
(3) $|B| + |\varphi(B)| = n$.

Let $K := \{B \in \Delta' : \varphi(B) \in \Delta'\}$. We claim that $\Delta' \setminus K$ is a simplicial complex with the same d-vector as Δ'.

First, we show that $\Delta' \setminus K$ is a complex. Let $B' \subseteq B \in \Delta' \setminus K$. Then $B' \in \Delta'$ so we must show that $B' \notin K$. Property (b) gives $\varphi(B') \supseteq \varphi(B)$, so we get $B \notin K \Rightarrow \varphi(B) \notin \Delta' \Rightarrow \varphi(B') \notin \Delta' \Rightarrow B' \notin K$.

Let $K_i = \{B \in K : |B| = i\}$ for $0 \le i \le n$. We have $d_i(\Delta' \setminus K) = (f_i(\Delta') - |K_i|) - (f_{n-i}(\Delta') - |K_{n-i}|) = d_i(\Delta') - (|K_i| - |K_{n-i}|)$ for $0 \le i \le \lfloor \frac{n}{2} \rfloor$. We must show that $|K_i| = |K_{n-i}|$ for all i. Property (a) gives that $B \in K \Leftrightarrow \varphi(B) \in K$. Finally, property (c) gives that φ is a bijection between K_i and K_{n-i} for all i.

$K \neq \emptyset$ since $\varphi(C) = [n] \setminus \pi(C) \subseteq C$ by (5.6.5), whence $\varphi(C) \in \Delta'$ and $C \in K$. Thus we have found a strictly smaller subcomplex of Δ' with the same d-vector. This is a contradiction against our choice of Δ'. □

5.6.5 C . *There is a subcomplex Δ' of Δ such that*

$$g_i(\mathrm{Bier}(B_n, \Delta)) = f_i(\Delta')$$

for $0 \le i \le \lfloor \frac{n-1}{2} \rfloor$ *and* $f_i(\Delta') = 0$ *for* $i > \lfloor \frac{n-1}{2} \rfloor$. □

It is a consequence of Corollary 5.6.5 that the g-vector $(g_0, g_1, \ldots, g_{\lfloor (n-1)/2 \rfloor})$ of $\mathrm{Bier}(B_n, \Delta)$ is a *K-sequence*, i. e., it satisfies the Kruskal-Katona theorem. This is

of interest in connection with the so called g-conjecture for spheres, which suggests that g-vectors of spheres are M-sequences (satisfy Macaulay's theorem). K-sequences are a very special subclass of M-sequences, thus g-vectors (and hence f-vectors) of Bier spheres are quite special among those of general triangulated $(n-2)$-spheres on $2n$ vertices. See [89, Ch. 8] for details concerning K-sequences, M-sequences and g-vectors.
These results also imply the following.

5.6.6 C **[K-Sequence].** *Every K-sequence $(1, n, \ldots, f_k)$ with $k \leq \lfloor \frac{n-1}{2} \rfloor$ can be realised as the g-vector of a Bier sphere with $2n$ vertices.* □

We need the notion of a *bistellar flip* for this. This is defined as follows. Let Γ be a simplicial d-manifold. If A is a $(d-i)$-dimensional face of Γ, $0 \leq i \leq d$, such that $\mathrm{link}_\Gamma(A)$ is the boundary ∂B of an i-simplex B that is not a face of Γ, then the operation Φ_A on Γ defined by

$$\Phi_A(\Gamma) := (\Gamma \backslash (A * \partial B)) \cup (\partial A * B)$$

is called a *bistellar i-flip*. $\Phi_A(\Gamma)$ is itself a simplicial d-manifold which is homeomorphic to Γ. If $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$, then

$$\begin{aligned} g_{i+1}(\Phi_A(\Gamma)) &= g_{i+1}(\Gamma) + 1 \\ g_j(\Phi_A(\Gamma)) &= g_j(\Gamma) \quad \text{for all} \quad j \neq i+1. \end{aligned} \tag{5.6.6}$$

Furthermore, if d is even and $i = \frac{d}{2}$, then $g_j(\Phi_A(\Gamma)) = g_j(\Gamma)$ for all j. See also Pachner [65, p. 83].

It follows from Corollary 5.6.5 that $g_k(\mathrm{Bier}(B_n, \Delta)) \geq 0$. The case of equality is characterised as follows.

5.6.7 C **.** *For $2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, the following are equivalent:*
(1) $g_k(\mathrm{Bier}(B_n, \Delta)) = 0$,
(2) $f_k(\Delta) = 0$ *or* $f_{n-k}(\Delta) = \binom{n}{i}$,
(3) $\mathrm{Bier}(B_n, \Delta)$ *is obtained from the boundary complex of the* $(n-1)$*-simplex via a sequence of bistellar i-flips, with $i \leq k-2$ at every flip.*

Proof. (1) $\Rightarrow$ (2) : Consider the bipartite graph $G_{n,k}$ whose edges are the pairs (A, B) such that A is a k-element subset, B is an $(n-k)$-element subset of $[n]$, and $A \subset B$, where the inclusion is strict since $k < n-k$. Then $G_{n,k}$ is a regular bipartite graph (all vertices have the same degree), so by standard matching theory $G_{n,k}$ has a complete matching. The restriction of such a matching to the sets B in Δ gives an injective mapping $\Delta_{n-k} \to \Delta_k$ from faces of cardinality $n-k$ in Δ to those of cardinality k.

Equality $f_{n-k}(\Delta) = f_k(\Delta)$ implies that $G_{n,k}$ consists of two connected components, one of which is induced on $\Delta_{n-k} \cup \Delta_k$. A nontrivial such splitting cannot happen since $G_{n,k}$ is connected, so either Δ_{n-k} and Δ_k are both empty, or they are both the full families of cardinality $\binom{n}{k}$.

(2) $\Rightarrow$ (3) : As shown in [13] and [60, Sect. 5.6], adding an i-dimensional face to Δ produces a bistellar i-flip in $\mathrm{Bier}(B_n, \Delta)$. Now, Δ can be obtained from the empty complex by adding i-dimensional faces. It must hold that all $i \leq k-2$ if $f_k(\Delta) = 0$ (meaning that there are no faces of dimension $k-1$ in Δ). The case when $f_{n-k}(\Delta) = \binom{n}{i}$ is the same by symmetry.

(3) $\Rightarrow$ (1) : This follows directly from (5.6.6), since the boundary of the $(n-1)$-simplex has g-vector $(1, 0, \ldots, 0)$. □

A convex polytope whose boundary complex is obtained from the boundary complex of the $(n-1)$-simplex via a sequence of bistellar i-flips, with $i \leq k-2$ at every flip, is called *k-stacked*. The *generalised lower bound conjecture* for polytopes claims that $g_k = 0$ for a polytope if and only if it is k-stacked. This is still open for general polytopes. See McMullen [63] for a recent discussion. Corollary 5.6.7 shows that it is valid for those polytopes that arise via the Bier sphere construction.

5.7 Many Spheres

Here we show that the number of Bier spheres associated to the Boolean poset and an abstract simplicial complex therein is so large, that most of them cannot have a convex realisation, by sheer number.

5.7.1 Proposition [Many Non-Polytopal Spheres]. *Let B_n be the Boolean poset.*

Most of the Bier spheres $\mathrm{Bier}(B_n, \Delta)$ *associated to an abstract simplicial complex Δ on the ground set* $[n]$ *have no realisation as a polytope.*

Proof. For the proof it suffices to consider Bier spheres $\mathrm{Bier}(B_n, \Delta)$ for abstract simplicial complexes Δ that contain

- all sets $A \subset [n]$ of size $|A| \leq \lfloor \frac{n-1}{2} \rfloor$,
- a subcollection of the sets of size $|A| = \lfloor \frac{n-1}{2} \rfloor + 1 = \lfloor \frac{n+1}{2} \rfloor$, and
- no larger faces.

Equivalently, Δ is a complex of dimension at most $\lfloor \frac{n-1}{2} \rfloor$ that contains the complete $(\lfloor \frac{n-1}{2} \rfloor - 1)$-skeleton of the simplex Δ_{n-1}.

There are $\binom{n}{\lfloor (n+1)/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor}$ elements in the $\lfloor \frac{n+1}{2} \rfloor$-level of B_n. Hence, there are at least

$$\frac{2^{\binom{n}{\lfloor n/2 \rfloor}}}{(2n)!} \sim \frac{2^{2^n/\sqrt{n}}}{\left(\frac{2n}{e}\right)^{2n}}$$

combinatorially non-isomorphic such Bier spheres (where our rough approximation ignores polynomial factors).

On the other hand, there are at most $2^{8n^3+O(n^2)}$ combinatorially non-isomorphic simplicial polytopes on $2n$ vertices. This follows from results of Goodman and Pollack [41] and Alon [1, Theorem 5.1]. □

The work of Kalai [52] and Lee [56] contains other constructions for *many* shellable spheres.

These "numerous" spheres are quite special in various ways. Namely, they are shellable, their g-vectors are K-sequences, and for even n we obtain in fact a large number of "nearly neighbourly" examples, which we discuss in the next section.

We have defined the construction of a Bier poset for arbitrary posets and have shown that the construction produces face lattices of PL spheres from face lattices of PL spheres. However, it remains an open problem how we can extend the Bier construction to obtain numerous simplicial or shellable $(n-2)$-spheres with more than $2n$ vertices.

5.8 C S N N S

In this section we show that, if the abstract simplicial complex Δ satisfies some additional restrictions, the Bier sphere $\mathrm{Bier}(B_n, \Delta)$ is centrally symmetric or k-nearly neighbourly, respectively.

Here are the relevant definitions. Let Γ be a triangulated $(n-2)$-sphere on $2m$ vertices. The sphere Γ is *centrally symmetric* if it has a symmetry of order two which fixes no face of Γ. That means, there is a fixed point free involution α on its vertex set $\mathcal{V}$ such that

(1) for every face A of Γ also $\alpha(A)$ is a face, and
(2) $\{x, \alpha(x)\}$ is not a face, for all $x \in \mathcal{V}$.

A subset $A \subseteq \mathcal{V}$ is *antipode free* if it contains no pair $\{x, \alpha(x)\}$, for $x \in \mathcal{V}$.

A centrally symmetric sphere Γ with involution α is *k-nearly neighbourly* if all antipode-free sets $A \subseteq V$ of size $|A| \leq k$ are faces of Γ. Equivalently, Γ must contain the $(k-1)$-skeleton of the m-dimensional cross-polytope. Γ is *nearly neighbourly* if it is $\lfloor \frac{n-1}{2} \rfloor$-nearly neighbourly. For $k \geq 2$ the involution α is uniquely determined by the condition $\{x, \alpha(x)\} \notin \Gamma$.

The concept of nearly neighbourliness for centrally symmetric spheres has been studied for centrally symmetric $(n-1)$-polytopes, where α is of course the map $x \mapsto -x$. For instance, work of Grünbaum, McMullen and Shephard, Schneider, and Burton shows that there are severe restrictions to k-nearly neighbourliness in the centrally symmetric polytope case, while existence of interesting classes of nearly neighbourly spheres was proved by Grünbaum, Jockusch, and Lutz. See [89, p. 279] and [58, Chap. 4] for more background on this.

The next two propositions provide a way to obtain centrally symmetric and nearly neighbourly Bier spheres. In the following, only the special case of an $(n-2)$-sphere with $2n$ vertices occurs (i.e. $m = n$ in the above definitions).

5.8.1 P **[Centrally Symmetric Spheres].** *Let Δ be an abstract simplicial complex on the ground set $[n]$. If*

$$A \in \Delta \iff [n]\backslash A \notin \Delta,$$

then $\mathrm{Bier}(B_n, \Delta)$ *is centrally symmetric.*

Proof. The involution α is given by the pairing $[\{x\}, \hat{1}] \longleftrightarrow [\hat{1}, [n]\backslash\{x\}]$. □

5.8.2 P **[k-Nearly Neighbourly Spheres].** *Let $1 < k \le \lfloor \frac{n-1}{2} \rfloor$. The Bier sphere* $\mathrm{Bier}(B_n, \Delta)$ *is a k-nearly neighbourly $(n-2)$-sphere with $2n$ vertices if and only if*

(1) $A \in \Delta \iff [n]\backslash A \notin \Delta$, for all $A \subseteq [n]$,
(2) $B \in \Delta$, for all $B \subseteq [n]$, $|B| \le k$
(and thus $C \notin \Delta$ for all $C \subseteq [n]$, $|C| \ge n-k$).

Proof. The Bier sphere $\mathrm{Bier}(B_n, \Delta)$ has $2n$ vertices if and only if $\Delta \subset 2^{[n]}$ is a simplicial complex that contains all subsets of cardinality 1 and no subsets of cardinality $n-1$.

The antipode-free vertex sets of cardinality k in $\mathrm{Bier}(B_n, \Delta)$ then correspond to intervals $[B, C] \subseteq B_n$ such that $|B| + (n - |C|) = k$. A set B is the minimal element of such an interval if and only if $|B| \le k$, while C is a maximal element for $|C| \ge n-k$. □

Combining the two Propositions 5.8.1 and 5.8.2 we obtain a large number of even-dimensional nearly neighbourly centrally symmetric Bier spheres. Indeed, in the case of even n we get at least

$$\frac{2^{\frac{1}{2}\binom{n}{\lfloor n/2 \rfloor}}}{(2n)!}$$

non-isomorphic spheres. These are obtained from the simplicial complexes Δ which contain all sets of size $A < \frac{n}{2}$, and exactly one set from each pair of sets A and $[n]\backslash A$ of size $|A| = \frac{n}{2}$.

The case of odd n corresponds to an odd-dimensional sphere, or an even-dimensional polytope. In this case, the “nearly neighbourliness condition” is a stronger restriction and hence more interesting. However, for odd n only one instance of a nearly neighbourly centrally symmetric Bier $(n-2)$-sphere with $2n$ vertices is obtained. It occurs for the simplicial complex

$$\Delta = \{A \subset [n] : |A| \le \lfloor \frac{n}{2} \rfloor\}.$$

B

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S I

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Z

In der vorliegenden Arbeit werden zwei neue Konstruktionsmethoden für partiell geordnete Mengen eingeführt und untersucht. Einige Teile dieser Arbeit sind gemeinsam mit Anders Björner, Jonas Sjöstrand und Günter M. Ziegler entstanden.

Die erste Konstruktion — die sogenannte E-Konstruktion — wurde von Eppstein, Kuperberg, und Ziegler für simpliziale 4-Polytope eingeführt. Hier wird sie auf beliebige gradierte partiell geordnete Mengen P erweitert. Sie hängt von einem Parameter t zwischen 0 und $\ell-2$ ab, wobei ℓ die Länge von P bezeichnet, und sie weist P eine neue partiell geordnete Menge $E_t(P)$ zu.

Im zweiten Kapitel der Arbeit werden grundlegende Eigenschaften dieser Konstruktion bewiesen. Sie bildet Eulersche Verbände wieder auf solche ab und erhält ihre Länge. Für Eulersche Verbände L wird gezeigt, dass für $r, s \geq 2$ — unter bestimmten zusätzlichen Voraussetzungen an L — $E_t(L)$ ein r-einfacher und s-simplizialer Verband ist. Insbesondere ist $E_{d-2}(L)$ 2-einfach und $(d-2)$-simplizial, wenn L simplizial ist.

Aus der verallgemeinerten E-Konstruktion erhält man mehrere unendliche Familien von 2-einfachen und 2-simplizialen Polytopen (im folgenden $(2,2)$-Polytope genannt). Hierzu wird die E-Konstruktion auf Seitenverbände von Polytopen angewendet. Es werden mehrere Polytopklassen angegeben, für die die aus der Anwendung der Konstruktion resultierenden Verbände wieder Seitenverbände von Polytopen sind. Insbesondere wird gezeigt, dass sich die Konstruktion auf vierdimensionale Stapelpolytope anwenden läßt und sich daraus eine unendliche Familie von *rationalen* $(2,2)$-Polytopen ergibt. Außerdem erhält man eine unendliche Familie von 2-einfachen und $(d-2)$-simplizialen d-Polytopen in *jeder* Dimension $d \geq 4$. Sie sind die ersten explizit konstruierten Polytopfamilien mit diesen Eigenschaften. Bisher waren nur eine unendliche Familie von $(2,2)$-Polytopen mit *irrationalen* Koordinaten sowie eine *endliche* Anzahl weiterer 2-einfacher und $(d-2)$-simplizialer d-Polytope für $d \geq 4$ bekannt.

Im dritten Kapitel werden hinreichende Kriterien dafür angegeben, dass sich die Konstruktion auf Produkte von zwei Polytopen anwenden läßt. Für Produkte von Polygonen erhält man eine unendliche Familie E_{mn}, $m, n \geq 3$, von selbstdualen $(2,2)$-Polytopen. E_{44} ist das 24-Zell. Für die E_{mn} kann man sehr flexible geometrische Realisierungen angeben. Für E_{33} und E_{44} wird eine explizite untere Schranke an die Dimension des Realisierungsraumes bestimmt. Wenn $m, n \geq 5$ und koprim sind, dann besitzt der Seitenverband von E_{mn} Automorphismen, die nicht geometrisch realisierbar sind.

Das vierte Kapitel enthält Resultate über $(2,2)$-Polytope im Zusammenhang mit der Klassifikation von Fahnenvektoren von 4-Polytopen sowie über $(2,2)$-Polytope im Verhältnis zu anderen Polytopen. Außerdem werden ältere Konstruktionen für $(2,2)$-Polytope vorgestellt und ein Überblick über alle bekannten Beispiele von $(2,2)$-Polytopen mit wenigen Ecken und über unendliche Familien solcher Polytope gegeben.

Im fünften Kapitel wird eine zweite neue Konstruktion auf partiell geordneten Mengen eingeführt. Sie basiert auf einer Konstruktion, die Thomas Bier für Boolesche Verbände beschrieben hat. Hier wird sie auf allgemeine gradierte partiell geordnete Mengen P erweitert. Sie assoziiert zu einer solchen und einem eigentlichen Ideal I in P eine neue partiell geordnete Menge $\mathrm{Bier}(P, I)$. Wenn P Seitenverband einer PL Sphäre S ist, dann ist auch $\mathrm{Bier}(P, I)$ ein solcher, und zwar zu einer PL Sphäre, die durch stellare Unterteilung von Seiten in S erhalten werden kann. Wenn P ein Boolescher Verband ist, dann sind die erhaltenen Sphären schälbar. Die Anzahl kombinatorisch verschiedener Bier-Sphären ist so groß, dass die meisten von ihnen nicht polytopal sein können. Für spezielle Wahlen des Ideals I sind die Sphären fast nachbarschaftlich und zentralsymmetrisch.

Berlin, im April 2005