Infinitesimal deformations of discrete surfaces

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Abstract

We introduce Möbius invariant objects – discrete holomorphic quadratic differentials to connect discrete complex analysis, discrete integrable systems and geometric rigidity. Discrete holomorphic quadratic differentials parametrize the change of the logarithmic cross ratios of planar discrete surfaces under infinitesimal conformal deformations. We show that on planar triangulated disks, there is a one-to-one correspondence between discrete holomorphic quadratic differentials and discrete harmonic functions modulo linear functions. Furthermore, every discrete holomorphic quadratic differential yields an S^1-family of discrete minimal surfaces via a Weierstrass representation. It leads to a unified theory of discrete minimal surfaces, establishing connections between the integrable systems approach, the curvature approach and the variational approach to discrete minimal surfaces. Considering discrete holomorphic quadratic differentials on surfaces in Euclidean space results in a notion of triangulated isothermic surfaces. Triangulated isothermic surfaces can be characterized in terms of circle pattern theory and the theory of length cross ratios. This notion opens the door to develop discrete integrable systems on surfaces of arbitrary combinatorics.

Zusammenfassung

Wir führen möbiusinvariante Objekte – nämlich diskrete holomorphe quadratische Differenziale – ein, um diskrete Funktionentheorie, diskrete integrable Systeme und geometrische Starrheit zu verbinden. Diskrete holomorphe quadratische Differenziale parametrisieren die Änderung der logarithmischen Doppelverhältnisse von ebenen diskreten Flächen unter infinitesimalen konformen Verformungen. Auf ebenen triangulierten Scheiben gibt es einen eindeutigen Zusammenhang zwischen diskreten holomorphen quadratischen Differenzialen und diskret harmonischen Funktionen modulo lineare Funktionen. Jedes diskrete holomorphe quadratische Differential liefert über eine Weierstraß-Darstellung eine S^1-Familie von diskreten Minimalflächen. Dies führt zu einer einheitlichen Theorie diskreter Minimalflächen, wobei eine Verbindung zwischen dem Ansatz über integrable Systeme, dem Krümmungsansatz und dem Variationsansatz für diskrete Minimalflächen hergestellt wird. Diskrete holomorphe quadratische Differenziale auf Flächen im euklidischen Raum führen zum Begriff triangulierter isothermer Oberflächen. Triangulierte isotherme Flächen können mittels Kreismustertheorie und der Theorie der Längendoppelverhltnisse charakterisiert werden. Dieser Begriff eröffnet eine Möglichkeit, diskrete integrable Systeme auf Flächen beliebiger Kombinatorik zu entwickeln.

I dedicate this thesis to my family and Chi Man Cheung
for their support and love.

Contents

CHAPTER 1

Introduction

Discrete differential geometry concerns structure-preserving discretizations in differential geometry. It studies discrete objects, represented by a finite number of variables, that are not only approximations of smooth objects but possess similar properties. The goal of discrete differential geometry is to establish a discrete theory with rich mathematical structures in such a way that the classical smooth theory arises in the limit of refinement of the discrete one.

Structure-preserving discretizations in differential geometry has a long history. In 1813 Cauchy obtained a classical result in geometric rigidity that two convex polytopes in Euclidean space with congruent faces are equal up to a rigid motion. In 1950 Sauer studied discrete pseudospheres and their rigidity [**61**] in order to illustrate the analogy between the discrete surface theory and the smooth theory. In the meantime linear discrete complex analysis were developed by Ferrand [**23**], McNeal [**49**] and Duffin [**22**] by discretizing the Cauchy–Riemann equations. In 1985 Thurston proposed to approximate conformal mappings by circle packings, which motivated the development of nonlinear discrete complex analysis.

Discrete differential geometry is not merely of theoretical interest, but is of practical importance for numerical implementations. For example, the notion of discrete pseudospheres provides a discretization of the sine-Gordon equation [**5**], which is crucial to quantum field theory. On the other hand, discrete complex analysis is applied to the Ising model and the dimer model in statistical physics [**67**]. Furthermore discrete differential geometry motivates efficient algorithms to tackle problems in Computer Graphics [**41, 69**] and provides tools for architectural design [**57**].

However, a lot of these structure-preserving discretizations seem isolated from each other, though their continuum limits are closely related. *A natural question is whether there is a unified theory behind all these structure-preserving discretizations*, which is the guiding question of this thesis.

By considering infinitesimal deformations of discrete surfaces, we establish connections between various structure-preserving discretizations: discrete surface theory, geometric rigidity, discrete complex analysis and discrete integrable systems. A link between all these topics is via a Möbius invariant notion – discrete holomorphic quadratic differentials.

Definition 1.1. Given a non-degenerate realization $z : V \to \mathbb{C}$ of a planar mesh $M = (V, E, F)$, a function $q : E_{int} \to \mathbb{R}$ defined on unoriented interior edges is a *discrete holomorphic quadratic differential* with respect to z if for every interior vertex $i \in V_{int}$

$$0 = \sum_j q_{ij},$$
$$0 = \sum_j q_{ij}/(z_j - z_i)$$

where the summation is taken over all the vertices adjacent to vertex i.

Theorem 1.2. *Given a non-degenerate realization $z : V \to \mathbb{C}$ of a planar mesh $M = (V, E, F)$, suppose $w : V \to \mathbb{C}$ is obtained via a Möbius transformation of z, i.e.*

$$w = \frac{az + b}{cz + d} \quad \text{for some } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

Then a function $q : E_{int} \to \mathbb{R}$ is a holomorphic quadratic differential with respect to z if and only if it is a holomorphic quadratic differential with respect to w. Here we assume that no vertices of w are at infinity.

Proof. Since Möbius transformations are generated by Euclidean transformations and inversions, it suffices to consider the inversion in the unit circle or

$$w = 1/z.$$

Suppose $\sum_j q_{ij} = 0$ for an interior vertex i. Then we have

$$\sum_j q_{ij}/(w_j - w_i) = -z_i \sum_j q_{ij} - z_i^2 \sum_j q_{ij}/(z_j - z_i) = -z_i^2 \sum_j q_{ij}/(z_j - z_i).$$

Thus the claim follows. □

Discrete holomorphic quadratic differentials are closely related to discrete conformality of planar meshes, namely circle pattern theory and the theory of length cross ratios. They parametrize the change of the logarithmic cross ratios under infinitesimal conformal deformations. We briefly outline how discrete holomorphic quadratic differentials are related to several discrete theories.

Discrete minimal surfaces: Minimal surfaces are ubiquitous in nature. For example they arise as soap films. A surface in Euclidean space is minimal if it is a critical point of the total area. There are various discretizations in Computer Graphics. Particularly in the spirit of discrete differential geometry, Bobenko and Pinkall introduced circular minimal surfaces via the integrable systems approach [**4**] while Bobenko, Pottmann and Wallner suggested conical minimal surfaces via the curvature approach [**10**].

We prove that each discrete holomorphic quadratic differential on a simply connected mesh yields an $\mathbb{S}^1$-family of discrete minimal surfaces via a Weierstrass representation (Theorem 3.15). These discrete minimal surfaces include circular minimal surfaces and conical minimal surfaces as special cases.

Linear discrete complex analysis: Here discrete holomorphic functions are based on Cauchy–Riemann equations [**34**]. They are related to Kirchhoff's circuit laws and were applied to tilings of rectangles into squares [**16**].

Taking the real part of a discrete holomorphic function yields a discrete harmonic function in the sense of the cotangent Laplacian. In contrast to the smooth theory, harmonicity of functions are not preserved under Möbius transformations because of the Euclidean formulation of the cotangent Laplacian. However, it is surprising that the space of discrete holomorphic quadratic differentials is in fact isomorphic to the space of discrete harmonic functions modulo linear functions (Theorem 2.14).

Nonlinear discrete complex analysis: William Thurston proposed to use circle packings in order to approximate conformal maps. Rodin and Sullivan proved the convergence of the analogue of Riemann maps for circle packings [**60**]. There are further extensions, such as Schramm's orthogonal circle patterns [**65**].

We show that for an orthogonal circle pattern there is a canonical discrete holomorphic quadratic differential. Furthermore, the discrete minimal surfaces in the corresponding associated family are critical points of the total area (Theorem 3.36). It is for the first time we have examples that are within the variation approach and the curvature approach to discrete minimal surfaces and possess integrable structures.

Discrete integrable surfaces: Several classes of surfaces in differential geometry were shown to possess integrable structures. They are related to solitons in some partial differential equations. Bobenko and Pinkall [**4**] considered discrete Lax representations of isothermic surfaces and introduced quadrilateral isothermic surfaces, whose quadrilaterals have factorized cross ratios. This leads to a theory of discrete surfaces via an integrable systems approach.

Generalizing discrete holomorphic quadratic differentials to discrete surfaces in Euclidean space yields a notion of triangulated isothermic surfaces (Lemma 4.6). We show that by adding a diagonal to each quad, quadrilateral isothermic surfaces, introduced by Pinkall and Bobenko, satisfy our notion of triangulated isothermic surfaces. Moreover we show that triangulated isothermic surfaces can be characterized in terms of the theory of length cross ratios (Theorem 4.4) and circle pattern theory (Theorem 4.5).

Geometric rigidity: Mapping the vertices of a graph into space with prescribed edge lengths is a central problem in rigidity theory. It concerns the stability of mechanical structures, with applications in molecular structures [**73**].

One way to study the infinitesimal rigidity of a discrete surface $f : V \to \mathbb{R}^3$ is to consider its self-stresses. A self-stress of f is a function $k : E_{int} \to \mathbb{R}$ such that for every interior vertices

$$\sum_j k_{ij}(f_j - f_i) = 0,$$

which can be regarded as tension acting along edges and balanced at interior vertices. We show that if the vertices of f lie on a sphere, there is a one-to-one correspondence between self-stresses of f and discrete holomorphic quadratic differentials on the stereographic projection of f (Corollary 4.27).

In the following sections, we discuss the historical background of discrete harmonic functions. We derive discrete harmonic functions from Dirichlet's principle [**55**] and illustrate their relations to the Cauchy-Riemann equations. We then study their properties under Möbius transformations in order to motivate the notion of discrete holomorphic quadratic differentials.

1.1. Notations

We start with some notations of discrete surfaces and discrete differential forms.

Definition 1.3. A discrete surface is a cell decomposition of a surface $M = (V, E, F)$, with or without boundary. The set of vertices (0-cells), edges (1-cells) and faces (2-cells) are denoted as V, E and F. Furthermore, we write V_{int} as the set of interior vertices and E_{int} as the set of interior edges.

Given a discrete surface $M = (V, E, F)$ with boundary, the dual cell decomposition is denoted by $M^* = (V^*, E^*, F^*)$. The boundary vertices and boundary edges of M correspond to unbounded faces and unbounded edges of M^*. Without further notice we assume that all surfaces under consideration are oriented.

Definition 1.4. A *non-degenerate* realization of a discrete surface M in $\mathbb{R}^n$ is a map $f : V \to \mathbb{R}^n$ and $f_i \neq f_j$ for every edge $\{ij\} \in E$. We say f is *strongly non-degenerate* if the image of the vertices of every face is not contained in any affine line.

Given two complex numbers $z_1, z_2 \in \mathbb{C}$ we write

$$\langle z_1, z_2 \rangle := \mathrm{Re}(\bar{z}_1 z_2).$$

We make use of discrete differential forms from Discrete Exterior Calculus [**21**]. Given a discrete surface $M = (V, E, F)$, we denote $\vec{E}$ the set of oriented edges and $\vec{E}_{int}$ the set of interior oriented edges. An oriented edge from vertex i to vertex j is indicated by e_{ij}. Note that $e_{ij} \neq e_{ji}$.

Definition 1.5. A function $\omega : \vec{E} \to \mathbb{R}$ is a (primal) *1-form* if

$$\omega(e_{ij}) = -\omega(e_{ji}) \quad \forall e_{ij} \in \vec{E}.$$

It is *closed* if for every face $\phi = (v_0, v_1, \ldots, v_n = v_0)$

$$\sum_{i=0}^{n-1} \omega(e_{i,i+1}) = 0.$$

It is *exact* if there exists a function $u : V \to \mathbb{R}$ such that for $e_{ij} \in \vec{E}$

$$\omega(e_{ij}) = u_j - u_i =: du(e_{ij}).$$

One can verify that exactness implies closedness while the converse holds if the discrete surface is simply connected.

Similarly we consider a 1-form $\tau : \vec{E}^*_{int} \to \mathbb{R}$ on the dual cell decomposition $M^* = (V^*, E^*, F^*)$ of M and call τ a *dual 1-form* on M. We denote e^*_{ij} the dual edge oriented from the right face of e_{ij} to the left face. The following notions are natural if we think of a dual 1-form on M as a 1-form on M^*.

DEFINITION 1.6. A function $\tau : \vec{E}^* \to \mathbb{R}$ is a *dual 1-form* if

$$\tau(e^*_{ij}) = -\tau(e^*_{ji}) \quad \forall e^*_{ij} \in \vec{E}^*.$$

It is *closed* if for every interior vertex $i \in V_{int}$

$$\sum_j \tau(e^*_{ij}) = 0.$$

It is exact if there exists $Z : F \to \mathbb{R}$ such that

$$dZ(e^*_{ij}) := Z_{\phi_l} - Z_{\phi_r} = \tau(e^*_{ij})$$

where ϕ_l denotes the left face of e_{ij} and ϕ_r denotes the right face.

We distinguish dual 1-forms from primal 1-forms. Firstly, the closedness conditions are different. The closedness conditions are imposed on faces for primal 1-forms while they are imposed at vertices for dual 1-forms. Secondly, a discrete notion of the Hodge star operator is needed to identify 1-forms with dual 1-forms. In Discrete Exterior Calculus [**21**] one often uses the Hodge star operator, which maps a primal 1-form ω to a dual 1-form $*\omega$ via

$$*\omega(e^*_{ij}) := -(\cot\angle jki + \cot\angle ilj)\omega(e_{ij}) \quad \forall \{ij\} \in E$$

with respect to some discrete metric, which is an assignment of edge lengths for triangulated surfaces (see Figure 1.1). Given a dual 1-form τ and a primal 1-form df, we occasionally write

$$\tau(e^*_{ij}) = k_{ij} df(e_{ij})$$

for some $k : E_{int} \to \mathbb{R}$. Here we think of it as $\tau = k' * df$ for some $k' : E_{int} \to \mathbb{R}$.

1.2. Discrete harmonic functions and the cotangent Laplacian

Discrete harmonic functions were introduced on the square lattice by Ferrand [**23**], McNeal [**49**] and Duffin [**22**] in terms of discrete Cauchy-Riemann equations. This notion was later generalized to triangular meshes and led to the cotangent Laplacian, which is central to linear discrete complex analysis. A convergence result of discrete harmonic functions was recently given by Skopenkov [**66**]. It is intriguing that discrete harmonic functions appear in various contexts [**71**] and have applications in statistical mechanics [**67**].

Following [**55**], we derive discrete harmonic functions by considering the finite-element approximation of the Dirichlet energy. Then we discuss a couple of observations on discrete harmonic functions in order to motivate their associated Möbius invariants – discrete holomorphic quadratic differentials (Definition 1.1).

A smooth function u on a bounded domain $\Omega \subset \mathbb{C}$ is harmonic if it satisfies a second order differential equation

$$\triangle u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Instead of solving the partial differential equation, we can obtain harmonic functions by means of a variational approach. From Dirichlet's principle, we know that if u satisfies

$$\triangle u = 0 \quad \text{on } \Omega$$

with boundary condition

$$u = g \quad \text{on } \partial\Omega$$

then u can be obtained as the minimizer of the Dirichlet's energy

$$E_D(v) := \int_\Omega |\operatorname{grad} v|^2 dA$$

among smooth functions $v \in C^\infty(\Omega)$ such that $v = g$ on $\partial\Omega$.

Dirichlet's principle for smooth harmonic functions motivates a definition of discrete harmonic functions. Since we are interested in objects with finitely many variables, we need to discretize the domain and the functions. We assume we have a realization $z : V \to \mathbb{C}$ of a triangulated surface $M = (V, E, F)$ with finite vertices in the complex plane, as a parametrization of a bounded domain. Suppose we consider a function $v : V \to \mathbb{R}$ defined at vertices and extend it piecewise linearly to triangular faces $v : V \cup F \to \mathbb{R}$. For each face, its gradient $\mathrm{grad}_z\, v$ is constant, which yields $\mathrm{grad}_z\, v : F \to \mathbb{C}$ given by

$$(\mathrm{grad}_z\, v)_{ijk} = i\frac{v_i(z_k - z_j) + v_j(z_i - z_k) + v_k(z_j - z_i)}{2A_{ijk}}$$

satisfying

$$\langle(\mathrm{grad}_z\, v)_{ijk}, z_j - z_i\rangle = \mathrm{Re}(\overline{(\mathrm{grad}_z\, v)_{ijk}}(z_j - z_i)) = v_j - v_i.$$

With the above formula, we define the Dirichlet energy of v as in the smooth theory

$$\begin{aligned} E_D(v) :=& \sum_{ijk\in F} |(\mathrm{grad}_z\, v)_{ijk}|^2 A_{ijk} \\ =& \sum_{ijk\in F} \frac{v_i\langle z_k - z_j, (\mathrm{grad}_z\, v)_{ijk}\rangle + v_j\langle z_i - z_k, (\mathrm{grad}_z\, v)_{ijk}\rangle + v_k\langle z_j - z_i, (\mathrm{grad}_z\, v)_{ijk}\rangle}{4A_{ijk}} \\ =& \frac{1}{2}\sum_{ij\in E_{int}} (\cot\angle jki + \cot\angle ilj)|v_j - v_i|^2 + \frac{1}{2}\sum_{ij\in E_b} (\cot\angle jki)|v_j - v_i|^2 \end{aligned}$$

where A_{ijk} denotes the signed area of the triangle $\{ijk\}$ under the realization z. Note that the triangulated surface is oriented. If a non-degenerate triangle is mapped in an orientation–reversing fashion then the area A_{ijk} and the angles are considered to be negative. In particular, if neighboring triangular faces do not overlap, then all the triangle areas have the same sign.

Theorem 1.7. *Given a realization of a triangulated surface $M = (V, E, F)$ with boundary in the plane such that neighboring triangular faces do not overlap, the Dirichlet energy is either strictly convex or strictly concave on the affine space of functions $v : V \to \mathbb{R}$ with prescribed boundary values.*

Proof. Since neighboring triangular faces do not overlap, all the triangle areas have the same sign. Assuming they have positive sign, we show that the Dirichlet energy is strictly convex on the affine space of functions $v : V \to \mathbb{R}$ with some prescribed boundary values.

Let v_0 be any function with the prescribed boundary values. For any other function $v : V \to \mathbb{R}$ with the prescribed boundary values we consider $v' := v - v_0$, which vanishes on the boundary. Since the Dirichlet energy $E_D(v)$ is quadratic in v, we have

$$E_D(v) = E_D(v') + L_{v_0}(v')$$

where L_{v_0} is an affine function. Hence, E_D is strictly convex on the affine space of functions with the prescribed boundary values if and only if E_D is strictly convex on the space of functions vanishing on the boundary.

Because all the face areas are positive, we know $E_D \geq 0$. Suppose v' is a function vanishing on the boundary and $E_D(v') = 0$. Then it implies $\mathrm{grad}\, v' \equiv 0$ and hence $v' \equiv 0$. Since $E_D(v')$ is a quadratic form in v', E_D is strictly convex on the space of functions vanishing on the boundary and the claim follows. □

Corollary 1.8. *For a realization of a triangulated surface with boundary in the plane such that neighboring triangular faces do not overlap, there always exists a unique minimizer of the Dirichlet energy E_D among all discrete functions $v : V \to \mathbb{R}$ with prescribed boundary values.*

Lemma 1.9. *A function $u : V \to \mathbb{R}$ is a minimizer of $E_D(v)$ among $v : V \to \mathbb{R}$ with $v = g$ on boundary vertices V_b if and only if for each interior vertex i*

$$\sum_j (\cot\angle jki + \cot\angle ilj)(u_j - u_i) = \frac{\partial E_D}{\partial v_i}|_{v=u} = 0$$

and $u = g$ on V_b.

Definition 1.10. Given a planar triangular mesh $z : V \to \mathbb{C}$, a function $u : V \to \mathbb{R}$ is *harmonic* if for every interior vertex i

$$\sum_j (\cot\angle jki + \cot\angle ilj)(u_j - u_i) = 0$$

where $\{ijk\}$ and $\{jil\}$ are two neighboring faces sharing edge $\{ij\}$ (Figure 1.1).

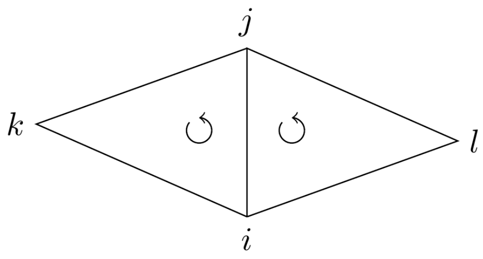


FIGURE 1.1. Two oriented triangles sharing edge $\{ij\}$.

We are going to relate discrete harmonic functions to Cauchy-Riemann equation. Suppose $z : V \to \mathbb{C}$ is a strongly non-degenerate planar triangular mesh. It induces a realization of its dual mesh $z^* : V^* \to \mathbb{C}$ where z^*_{ijk} is the circumcircle of the triangle $z_i z_j z_k$.

Theorem 1.11. *Given a strongly non-degenerate realization $z : V \to \mathbb{C}$ of a simply connected triangulated surface. A function $u : V \to \mathbb{R}$ is discrete harmonic if and only if there exists a function $v : V^* \to \mathbb{C}$ such that the discrete Cauchy-Riemann equation is satisfied*

$$\frac{u_j - u_i}{z_j - z_i} = i \frac{v_{ijk} - v_{jil}}{z^*_{ijk} - z^*_{jil}} \quad \forall \{ij\} \in E_{int}.$$

Here we call v a conjugate harmonic function of u.

PROOF. Suppose u satisfies the discrete Cauchy-Riemann equation. Then for every interior edge $\{ij\}$

$$\begin{aligned} v_{ijk} - v_{jil} =& \frac{1}{i} \frac{z^*_{ijk} - z^*_{jil}}{z_j - z_i} (u_j - u_i) \\ =& \frac{1}{2} (\cot \angle jki + \cot \angle ilj)(u_j - u_i). \end{aligned}$$

Hence

$$0 = \sum_{jk} (v_{ijk} - v_{jil}) = \frac{1}{2} \sum_j (\cot \angle jki + \cot \angle ilj)(u_j - u_i).$$

Conversely, suppose u is a discrete harmonic function. Then there exists a function $v : V^* \to \mathbb{R}$ such that

$$v_{ijk} - v_{jil} = \frac{1}{2} (\cot \angle jki + \cot \angle ilj)(u_j - u_i).$$

Consequently, the pair of functions u,v satisfy the discrete Cauchy-Riemann equation. □

1.3. Möbius transformations of planar triangular meshes

We consider *Möbius transformations*, which are conformal bijective maps of the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Every Möbius transformation $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ can be written as a fractional linear transformation

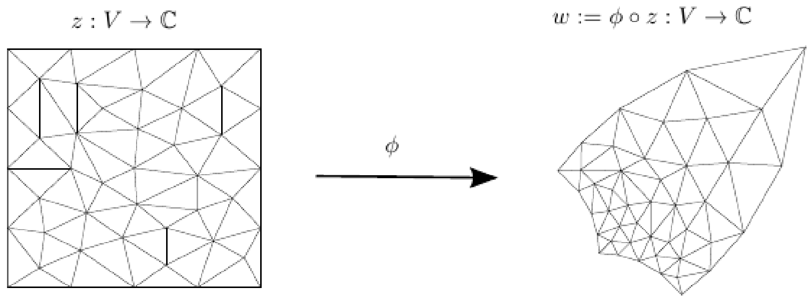
$$\phi(z) = \frac{az + b}{cz + d}$$

for some $a, b, c, d \in \mathbb{C}$ with $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Applying a Möbius transformation ϕ to the vertices of a planar triangulated surface $z : V \to \mathbb{C}$, we obtain another realization $w := \phi \circ z$. Notice that vertices of w are connected by straight line segments according to the combinatorics of z. Hence in general, angles between the edges segments are not preserved:

$$\angle z_i z_j z_k \neq \angle w_i w_j w_k \quad \forall \{ijk\} \in F.$$

In particular, *a function u is harmonic on z does not implies it is harmonic on every Möbius transformation of z*, which differs from the smooth theory that the composition of a harmonic function with a Möbius transformation is always harmonic.



However, there is an interesting observation concerning the maximum principle of discrete harmonic functions. A classical result on smooth harmonic functions is that they satisfies the maximum principle: a smooth harmonic function on a bounded domain attains its maximum and minimum on the boundary. A similar result holds for discrete harmonic functions if the triangular mesh is Delaunay.

Definition 1.12. A strongly non-degenerate planar triangular mesh $z : V \to \mathbb{C}$ is Delaunay if it satisfies the empty circle property: for every face $\{ijk\} \in F$, there is no vertex other than i, j, k lying inside the circumcircle of $\{ijk\}$.

Theorem 1.13. *Suppose the realization $z : V \to \mathbb{C}$ of a finite triangular mesh is Delaunay. Then the maximum principle holds: a discrete harmonic function on z attains its maximum and minimum on its boundary.*

Proof. Since the realization is Delaunay, two neighboring triangles do not overlap and share the same orientation. Thus without loss of generality we assume all triangles are positively oriented under the realization. Furthermore, the empty circle property implies that

$$0 < \angle jki + \angle ilj < \pi$$

for every interior edge $\{ij\}$ and hence

$$\cot \angle jki + \cot \angle ilj = \frac{\sin(\angle jki + \angle ilj)}{\sin \angle jki \, \sin \angle ilj} > 0. \tag{1.1}$$

Suppose u is harmonic and reaches a local maximum at an interior vertex i we have

$$0 = \sum_j (\cot \angle jki + \cot \angle ilj)(u_j - u_i) \leq 0$$

and hence we obtain a contradiction unless u is constant. Thus, the maximum principle holds. □

In the above argument, a crucial step to the maximum principle is that the cotangent coefficients are either all positive (1.1) or all negative. Generally such property is not preserved under Möbius transformations of the realization. However, if we consider inversion in the circles far from the mesh, then the empty circle property is preserved and the maximum principle still holds. It is intriguing that although the definition of discrete harmonic functions involves Euclidean structure, i.e. angles between straight segments, there seems to be a property related to Möbius geometry.

In Chapter 2 we show that in fact each discrete harmonic function corresponds to a Möbius invariant object – discrete holomorphic quadratic differential [**45**]. In particular, we study infinitesimal conformal deformations of planar triangular meshes in the sense of both circle pattern theory and the theory of length cross ratios. Under these infinitesimal deformations, the logarithmic changes of cross ratios give discrete holomorphic quadratic differentials. Interpreting infinitesimal conformal deformations in terms of $sl(2, \mathbb{C})$ yields a Weierstrass representation of discrete minimal surfaces.

In Chapter 3, we define two types of discrete minimal surfaces whose cell decompositions are arbitrary [**43**]. These two types of discrete minimal surfaces are conjugate to each other and generalize earlier notions of discrete minimal surfaces: circular minimal surfaces, s-isothermic minimal surfaces and conical minimal surfaces. We show that each discrete minimal surface corresponds to a discrete holomorphic quadratic differential on a planar mesh. Furthermore we show that all discrete minimal surfaces in certain associated families, obtained from isothermic quadrilateral meshes and Schramm's orthogonal circle patterns, are critical points of the area functional.

In Chapter 4, we study a class of triangulated surfaces in Euclidean space which have similar properties as isothermic surfaces in differential geometry [**46**]. It is based on a generalization of discrete holomorphic differentials on planar meshes to surfaces in Euclidean space. Isothermic triangulated surfaces can be characterized either in terms of circle patterns or conformal equivalence of triangle meshes. This definition generalizes isothermic quadrilateral meshes.

In Chapter 5, we focus on the integrable structures of P-nets, a particular class of isothermic triangulated surfaces. We derive their Christoffel transformations, Darboux transformations and Calapso transformations. The aim of this chapter is to lay a foundation for integrable structures on isothermic triangulated surfaces and a unified theory of discrete constant mean curvature surfaces.

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This dissertation continues the author's Master's thesis [**42**], where conformal deformations of triangulated surfaces in space were studied as a generalization of isometric deformations in rigidity theory. Some of the results in this thesis have been presented in an Oberwolfach report [**44**] and three recent articles [**43, 45, 46**].

CHAPTER 2

Holomorphic vector fields and quadratic differentials on planar triangular meshes

Consider an open subset U in the complex plane $\mathbb{C} \cong \mathbb{R}^2$ with coordinates $z = x+iy$ together with a holomorphic vector field

$$Y = f\frac{\partial}{\partial x}.$$

Here Y is a real vector field. It assigns to each $p \in \mathbb{R}^2$ the vector $f(p) \in \mathbb{C} \cong \mathbb{R}^2$. We do not consider objects like $\frac{\partial}{\partial z}$ which are sections of the complexified tangent bundle $(\mathrm{T}\,\mathbb{R}^2)^{\mathbb{C}}$.

Note $f : U \to \mathbb{C}$ is a holomorphic function, i.e.

$$0 = f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Let $t \mapsto g_t$ denote the local flow of Y (defined for small t on open subsets of U with compact closure in U). Then the euclidean metric pulled back under g_t is conformally equivalent to the original metric:

$$g_t^*\langle\,,\rangle = e^{2u}\langle\,,\rangle$$

for some real-valued function u. The infinitesimal change in scale $\dot{u}$ is given by

$$\dot{u} = \frac{1}{2}\operatorname{div} Y = \operatorname{Re}\left(f_z\right).$$

Note that $\dot{u}$ is a harmonic function:

$$\dot{u}_{z\bar{z}} = 0.$$

On the other hand, differentiating $\dot{u}$ twice with respect to z yields one half the third derivative of f:

$$\dot{u}_{zz} = \frac{1}{2}f_{zzz}.$$

It is well-known that the vector field Y corresponds to an infinitesimal Möbius transformation of the extended complex plane $\overline{\mathbb{C}}$ if and only if f is a quadratic polynomial. In this sense f_{zzz} measures the infinitesimal "change in Möbius structure" under Y (Möbius structures are also called "complex projective structures" [**27**]). Moreover, the holomorphic quadratic differential

$$q := f_{zzz}\,dz^2$$

is invariant under Möbius transformations Φ. This is equivalent to saying that q is unchanged under a change of variable $\Phi(z) = w = \xi + i\eta$ whenever Φ is a Möbius transformation. This is easy to see if $\Phi(z) = az + b$ is an affine transformation. In this case

$$dw = a\,dz$$
$$\frac{d}{dw} = \frac{1}{a}\frac{d}{dz}$$

and therefore

$$Y = \tilde{f}\frac{\partial}{\partial \xi}$$

with

$$\tilde{f} = a\,f.$$

Thus we indeed have

$$\tilde{f}_{www}\,dw^2 = f_{zzz}\,dz^2.$$

A similar argument applies to $\Phi(z) = \frac{1}{z}$ and therefore to all Möbius transformations.

For realizations from an open subset U of the Riemann sphere $\mathbb{C}\mathrm{P}^1$ the vanishing of the Schwarzian derivative characterizes Möbius transformations. The quadratic differential q plays a similar role for vector fields. We call q the *Möbius derivative* of Y.

An important geometric context where holomorphic quadratic differentials arise comes from the theory of minimal surfaces: Given a simply connected Riemann surface M together with a holomorphic immersion $g : M \to S^2 \subset \mathbb{R}^3$ and a holomorphic quadratic differential q on M, there is a minimal surface $F : M \to \mathbb{R}^3$ (unique up to translations) whose Gauß map is g and whose second fundamental form is $\operatorname{Re} q$.

In this chapter we provide a discrete version for all details of the above story. Instead of smooth surfaces we work with triangulated surfaces of arbitrary combinatorics. The notion of conformality will be that of conformal equivalence in terms of length cross ratios as explained in [**9**]. Holomorphic vector fields will be defined as infinitesimal conformal deformations.

There is also a completely parallel discrete story where conformal equivalence of planar triangulations is replaced by preserving intersection angles of circumcircles. To some extent we also tell this parallel story that belongs to the world of circle patterns.

This chapter is based on [**45**]. The discussion of planar triangular meshes here serves an introduction to isothermic triangulated surfaces in Chapter 4.

2.1. Discrete conformality

In this section, we review two notions of discrete conformality for planar triangular meshes.

We are interested in discrete conformality that is preserved under Möbius transformations. This requirement will certainly be met if we base our definitions on complex cross ratios: Given a triangular mesh $z : V \to \mathbb{C}$, we associate a complex number to each interior edge $\{ij\} \in E_{int}$, namely the *cross ratio* of the corresponding four vertices (See Figure 1.1)

$$\mathrm{cr}_{z,ij} = \frac{(z_j - z_k)(z_i - z_l)}{(z_k - z_i)(z_l - z_j)}.$$

Notice that $\mathrm{cr}_{z,ij} = \mathrm{cr}_{z,ji}$ and hence $\mathrm{cr}_z : E_{int} \to \mathbb{C}$ is well defined. It is easy to see that two realizations differ only by a Möbius transformation if and only if their corresponding cross ratios are the same. In order to arrive at a more flexible notion of conformality we need to relax the condition that demands the equality of all cross ratios. Two natural ways to do this is to only require equality of either the norm or alternatively the argument of the cross ratios. This leads to two different notions of discrete conformality: *conformal equivalence theory* [**48, 69**] and *circle pattern theory* [**65**].

Note that for the sake of simplicity of exposition we are ignoring here realizations in $\overline{\mathbb{C}}$ where one of the vertices is mapped to infinity.

2.1.1. Conformal equivalence.

The edge lengths of a triangular mesh realized in the complex plane provide a discrete counterpart for the induced Euclidean metric in the smooth theory. A notion of conformal equivalence based on edge lengths was proposed by Luo [**48**]. Later Bobenko et al. [**9**] stated this notion in the following form:

Definition 2.1. Two realizations of a triangular mesh $z, w : V \to \mathbb{C}$ are *conformally equivalent* if the norm of the corresponding cross ratios are equal:

$$|\,\mathrm{cr}_z\,| \equiv |\,\mathrm{cr}_w\,|,$$

i.e. for each interior edge $\{ij\}$

$$\frac{|(z_j - z_k)||(z_i - z_l)|}{|(z_k - z_i)||(z_l - z_j)|} = \frac{|(w_j - w_k)||(w_i - w_l)|}{|(w_k - w_i)||(w_l - w_j)|}.$$

This definition can be restated in an equivalent form that closely mirrors the notion of conformal equivalence of Riemannian metrics:

Theorem 2.2. *Two realizations of a triangular mesh $z, w : V \to \mathbb{C}$ are conformally equivalent if and only if there exists $u : V \to \mathbb{R}$ such that*

$$|w_j - w_i| = e^{\frac{u_i + u_j}{2}} |z_j - z_i|.$$

PROOF. It is easy to see that the existence of u implies conformal equivalence. Conversely, for two conformally equivalent realizations z, w, we define a function $\sigma : E \to \mathbb{R}$ by

$$|w_j - w_i| = e^{\sigma_{ij}} |z_j - z_i|.$$

Since z, w are conformally equivalent σ satisfies for each interior edge $\{ij\}$

$$\sigma_{jk} - \sigma_{ki} + \sigma_{il} - \sigma_{lj} = 0.$$

For any vertex i and any triangle $\{ijk\}$ containing it we then define

$$e^{u_i} := e^{\sigma_{ki} + \sigma_{ij} - \sigma_{jk}}.$$

Note the vertex star of i is a triangulated disk if i is interior, or is a fan if i is a boundary vertex. Hence the value u_i defined in this way is independent of the chosen triangle. $\square$

2.1.2. Circle patterns. Given a triangular mesh realized in the complex plane we consider the circumscribed circles of its triangles. These circles inherit an orientation from their triangles. The intersection angles of these circles from neighboring triangles define a function $\phi : E_{int} \to [0, 2\pi)$ which is the argument of the corresponding cross ratio:

$$\phi_{ij} = \operatorname{Arg}(\mathrm{cr}_{z,ij}). \tag{2.1}$$

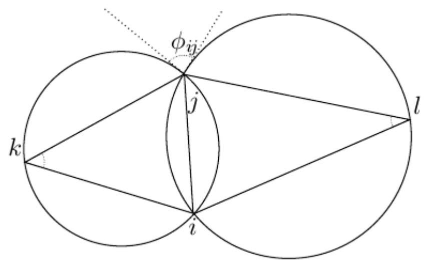

FIGURE 2.1. The intersection angle of two neighboring circumscribed circles

Based on these angles we obtain another notion of discrete conformality which reflects the angle-preserving property that we have in the smooth theory.

DEFINITION 2.3. Two realizations of a triangular mesh $z, w : V \to \mathbb{C}$ have the same *pattern structure* if the corresponding intersection angles of neighboring circumscribed circles are equal:

$$\operatorname{Arg}(\mathrm{cr}_{z,ij}) = \operatorname{Arg}(\mathrm{cr}_{w,ij}),$$

i.e. for each interior edge $\{ij\}$

$$\operatorname{Arg} \frac{(z_j - z_k)(z_i - z_l)}{(z_k - z_i)(z_l - z_j)} = \operatorname{Arg} \frac{(w_j - w_k)(w_i - w_l)}{(w_k - w_i)(w_l - w_j)}.$$

Just as conformal equivalence was related to scale factors u at vertices, having the same pattern structure is related to the existence of certain rotation angles α located at vertices:

Theorem 2.4. *Two realizations of a triangular mesh $z, w : V \to \mathbb{C}$ have the same pattern structure if and only if there exists $\alpha : V \to [0, 2\pi)$ such that*

$$\frac{w_j - w_i}{|w_j - w_i|} = e^{i\frac{\alpha_i + \alpha_j}{2}} \frac{z_j - z_i}{|z_j - z_i|}.$$

PROOF. The argument is very similar to the one for Theorem 2.2. In particular, the existence of the function α easily implies equality of the pattern structures. Conversely, assuming identical pattern structures we take any $\omega : E \to \mathbb{R}$ that satisfies

$$\frac{w_j - w_i}{|w_j - w_i|} = e^{i\omega_{ij}} \frac{z_j - z_i}{|z_j - z_i|}.$$

For any vertex i and any triangle $\{ijk\}$ containing it we define $\alpha_i \in [0, 2\pi)$ such that

$$e^{i\alpha_i} = e^{i(\omega_{ki} + \omega_{ij} - \omega_{jk})}.$$

Note the vertex star of i is a triangulated disk if i is interior, or is a fan if i is a boundary vertex. Hence having the same pattern structure implies that the value α_i is independent of the chosen triangle. □

2.2. Infinitesimal deformations and linear conformal theory

We linearize both of the above notions of discrete conformality by considering infinitesimal deformations. This allows us to relate them to linear discrete complex analysis, based on the discrete Cauchy-Riemann equation (Section 1.2).

Definition 2.5. An *infinitesimal conformal deformation* of a realization $z : V \to \mathbb{C}$ of a triangular mesh is a map $\dot{z} : V \to \mathbb{C}$ such that there exists $u : V \to \mathbb{R}$ satisfying

$$\operatorname{Re} \frac{\dot{z}_j - \dot{z}_i}{z_j - z_i} = \frac{\langle \dot{z}_j - \dot{z}_i, z_j - z_i \rangle}{|z_j - z_i|^2} = \frac{u_i + u_j}{2}.$$

We call u the *scale change* at vertices.

Definition 2.6. An *infinitesimal pattern deformation* of a realization $z : V \to \mathbb{C}$ of a triangular mesh is a map $\dot{z} : V \to \mathbb{C}$ such that there exists $\alpha : V \to \mathbb{R}$ satisfying

$$\operatorname{Im} \frac{\dot{z}_j - \dot{z}_i}{z_j - z_i} = \frac{\langle \dot{z}_j - \dot{z}_i, i(z_j - z_i) \rangle}{|z_j - z_i|^2} = \frac{\alpha_i + \alpha_j}{2}.$$

We call α the *angular velocities* at vertices.

Example 2.7. The infinitesimal deformations $\dot{z} := az^2 + bz + c$, where $a, b, c \in \mathbb{C}$ are constants, are both conformal and pattern deformations since

$$\frac{\dot{z}_j - \dot{z}_i}{z_j - z_i} = (az_i + b/2) + (az_j + b/2).$$

Infinitesimal conformal deformations and infinitesimal pattern deformations are closely related:

Theorem 2.8. *Suppose $z : V \to \mathbb{C}$ is a realization of a triangular mesh. Then an infinitesimal deformation $\dot{z} : V \to \mathbb{C}$ is conformal if and only if $i\dot{z}$ is a pattern deformation.*

Proof. Notice

$$\frac{\langle \dot{z}_j - \dot{z}_i, z_j - z_i \rangle}{|z_j - z_i|^2} = \frac{\langle i\dot{z}_j - i\dot{z}_i, i(z_j - z_i) \rangle}{|z_j - z_i|^2}$$

and the claim follows from Definition 2.5 and 2.6. □

2.2.1. Infinitesimal deformations of a triangle. Let $z : V \to \mathbb{C}$ be a realization of a triangle mesh and $\dot{z}$ an infinitesimal deformation. Up to an infinitesimal translation $\dot{z}$ is completely determined by the infinitesimal scalings and rotations that it induces on edges. These infinitesimal scalings and rotations satisfy certain compatibility conditions on each triangle. These conditions involve the cotangent coefficients well known from the theory of discrete Laplacians. As we will see in section 2.2.2, for conformal deformations (as well as for pattern deformations) the infinitesimal scalings and rotations of edges are indeed discrete harmonic functions.

We consider a non-degenerate triangle with vertices $z_1, z_2, z_3 \in \mathbb{C}$. In the following i, j, k denotes any cyclic permutation of the indexes $1, 2, 3$. The triangle angle at the vertex i is denoted by β_i. We adopt the convention that all $\beta_1, \beta_2, \beta_3$ have positive sign if the triangle z_1, z_2, z_3 is positively oriented and a negative sign otherwise. Suppose we have an infinitesimal deformation of this triangle. Then there exists $\sigma_{ij}, \omega_{ij} \in \mathbb{R}$ such that

$$\dot{z}_j - \dot{z}_i = (\sigma_{ij} + i\omega_{ij})(z_j - z_i). \tag{2.2}$$

The scalars σ_{ij} and ω_{ij} describe the infinitesimal scalings and rotations of the edges. They satisfy the following compatibility conditions.

Lemma 2.9. *Given $\sigma_{ij}, \omega_{ij} \in \mathbb{R}$ the following statements are equivalent:*

(a) There exist $\dot{z}_i$ such that (2.2) holds.

(b) We have

$$0 = (\sigma_{12} + i\omega_{12})(z_2 - z_1) + (\sigma_{23} + i\omega_{23})(z_3 - z_2) + (\sigma_{31} + i\omega_{31})(z_1 - z_3). \tag{2.3}$$

(c) There exists $\omega \in \mathbb{R}$ such that

$$\begin{aligned} i\omega &= i\omega_{23} + i\cot\beta_1(\sigma_{31} - \sigma_{12}) \\ &= i\omega_{31} + i\cot\beta_2(\sigma_{12} - \sigma_{23}) \\ &= i\omega_{12} + i\cot\beta_3(\sigma_{23} - \sigma_{31}). \end{aligned}$$

(d) There exist $\sigma \in \mathbb{R}$ such that

$$\begin{aligned} \sigma &= \sigma_{23} + i\cot\beta_1(i\omega_{31} - i\omega_{12}) \\ &= \sigma_{31} + i\cot\beta_2(i\omega_{12} - i\omega_{23}) \\ &= \sigma_{12} + i\cot\beta_3(i\omega_{23} - i\omega_{31}). \end{aligned}$$

PROOF. The relation between (a) and (b) is obvious. We show the equivalence between (b) and (c). With A denoting the signed triangle area we have the following identities:

$$\begin{aligned} 0 &= \langle i(z_j - z_i), z_j - z_i\rangle, \\ 2A &= \langle i(z_j - z_i), z_k - z_j\rangle, \\ \langle i(z_j - z_i), i(z_j - z_i)\rangle &= \langle z_j - z_i, z_j - z_i\rangle. \end{aligned}$$

Using these identities and $z_3 - z_2 \in \mathrm{span}_{\mathbb{R}}\{i(z_1 - z_3), i(z_2 - z_1)\}$ we obtain

$$z_3 - z_2 = \cot(\beta_3)i(z_2 - z_1) - \cot(\beta_2)i(z_1 - z_3). \tag{2.4}$$

Cyclic permutation yields

$$\begin{aligned} z_1 - z_3 &= \cot(\beta_1)i(z_3 - z_2) - \cot(\beta_3)i(z_2 - z_1), \\ z_2 - z_1 &= \cot(\beta_2)i(z_1 - z_3) - \cot(\beta_1)i(z_3 - z_2). \end{aligned}$$

Substituting these identities into Equation (2.3) we obtain

$$\begin{aligned} 0 = \quad & \sigma_1\big(\cot(\beta_3)i(z_2 - z_1) - \cot(\beta_2)i(z_1 - z_3)\big) + \omega_{23}i(z_3 - z_2) \\ & + \sigma_2\big(\cot(\beta_1)i(z_3 - z_2) - \cot(\beta_3)i(z_2 - z_1)\big) + \omega_{31}i(z_1 - z_3) \\ & + \sigma_3\big(\cot(\beta_2)i(z_1 - z_3) - \cot(\beta_1)i(z_3 - z_2)\big) + \omega_{12}i(z_2 - z_1) \\ = \quad & \big(\omega_1 + \cot\beta_1(\sigma_2 - \sigma_3)\big)i(z_3 - z_2) \\ & + \big(\omega_2 + \cot\beta_2(\sigma_3 - \sigma_1)\big)i(z_1 - z_3) \\ & + \big(\omega_3 + \cot\beta_3(\sigma_1 - \sigma_2)\big)i(z_2 - z_1). \end{aligned}$$

Using the fact that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ satisfy

$$\lambda_1 i(z_3 - z_2) + \lambda_2 i(z_1 - z_3) + \lambda_3 i(z_2 - z_1) = 0,$$

if and only if $\lambda_1 = \lambda_2 = \lambda_3$, we establish the equivalence of (b) and (c). The equivalence of (b) and (d) is seen similarly by eliminating $i(z_j - z_i)$ in (2.3) instead of $(z_j - z_i)$. $\square$

The quantity ω above describes the average rotation speed of the triangle. Similarly, it can be verified that σ above satisfies

$$\sigma = \frac{\dot{R}}{R}$$

where R denotes the circumradius of the triangle. Thus σ signifies an average scaling of the triangle.

2.2.2. Harmonic functions with respect to the cotangent Laplacian. In complex analysis conformal maps are closely related to harmonic functions. If a conformal map preserves orientation it is holomorphic and satisfies the Cauchy-Riemann equations. In particular, its real part and the imaginary part are conjugate harmonic functions. Conversely, given a harmonic function on a simply connected surface then it is the real part of some conformal map.

A similar relationship manifests between discrete harmonic functions (in the sense of the cotangent Laplacian) and infinitesimal deformations of triangular meshes. Discrete harmonic functions can be regarded as the real part of holomorphic functions which satisfies a discrete analogue of the Cauchy-Riemann equations (Theorem 1.11). In particular, a relation between discrete harmonic functions and infinitesimal pattern deformations was found by Bobenko, Mercat and Suris [**11**]. Integrable systems were involved in this context. We extend their result to include the case of infinitesimal conformal deformations.

Theorem 2.10. *Let $z : V \to \mathbb{C}$ be a simply connected triangular mesh realized in the complex plane and $h : V \to \mathbb{R}$ be a function. Then the following are equivalent:*

(a) h is a harmonic function with respect to the cotangent Laplacian, i.e. using the notation of Figure 1.1, for all interior vertices $i \in V_{int}$ we have

$$\sum_j (\cot \angle jki + \cot \angle ilj)(h_j - h_i) = 0. \tag{2.5}$$

(b) There exists an infinitesimal conformal deformation $\dot{z} : V \to \mathbb{C}$ with scale factors given by h. It is unique up to infinitesimal rotations and translations.

(c) There exists an infinitesimal pattern deformation $i\dot{z} : V \to \mathbb{C}$ with h as angular velocities. It is unique up to infinitesimal scalings and translations.

Proof. We show the equivalence of the first two statements. The equivalence of the first and the third follows similarly.

Suppose h is a harmonic function. Since the triangular mesh is simply connected, equation (2.5) implies the existence of a function $\tilde{\omega} : F \to \mathbb{R}$ such that for all interior edges $\{ij\}$ we have

$$i\tilde{\omega}_{ijk} - i\tilde{\omega}_{jil} = i((\cot \angle jki + \cot \angle ilj))(h_j - h_i).$$

Here $\tilde{\omega}$ is unique up to an additive constant and is the *conjugate harmonic function* of h. Using $\tilde{\omega}$ we define a function $\omega : E \to \mathbb{R}$ via

$$i\omega_{ij} = i\tilde{\omega}_{ijk} - i \cot \angle jki \, (h_j - h_i).$$

Lemma 2.9 now implies that there exists $\dot{z} : V \to \mathbb{C}$ such that

$$(\dot{z}_j - \dot{z}_i) = \left(\frac{h_i + h_j}{2} + i\omega_{ij} \right) (z_j - z_i).$$

This gives us the desired infinitesimal conformal deformation of z with h as scale factors.

To show uniqueness, suppose $\dot{z}, \dot{z}'$ are infinitesimal conformal deformations with the same scale factors. Then $\dot{z} - \dot{z}'$ preserves all the edge lengths of the triangular mesh and hence is induced from an Euclidean transformation.

Conversely, given an infinitesimal conformal deformation $\dot{z}$ with scale factors h. We write

$$\dot{z}_j - \dot{z}_i = \left(\frac{h_i + h_j}{2} + i\omega_{ij} \right) (z_j - z_i)$$

for some $\omega : E \to \mathbb{R}$. Lemma 2.9 implies that there is a function $\tilde{\omega} : F \to \mathbb{R}$ such that

$$i\tilde{\omega}_{ijk} = i\omega_{ij} + i \cot \angle jki \, (h_j - h_i).$$

We have

$$i\tilde{\omega}_{ijk} - i\tilde{\omega}_{jil} = i(\cot \angle jki + \cot \angle ilj)(h_j - h_i)$$

and

$$\sum_j (\cot \angle jki + \cot \angle ilj)(h_j - h_i) = 0.$$

Therefore h is harmonic. □

2.3. Holomorphic quadratic differentials

In this section, we show that on a simply connected triangular mesh, there is a one-to-one correspondence between discrete holomorphic quadratic differentials and discrete harmonic functions modulo linear functions. In this context, discrete holomorphic quadratic differentials are the changes of logarithmic cross ratios under infinitesimal conformal deformations. It reflects the property in the smooth theory that holomorphic quadratic differentials parametrize Möbius structures on Riemann surfaces [**27**, Ch. 9].

As introduced in Definition 1.1, a discrete holomorphic quadratic differential on a planar triangular mesh $z : V \to \mathbb{C}$ is a function $q : E_{int} \to \mathbb{R}$ defined on interior edges satisfying for every interior vertex $i \in V_{int}$

$$\sum_j q_{ij} = 0,$$
$$\sum_j q_{ij}/dz(e_{ij}) = 0$$

where $dz(e_{ij}) := z_j - z_i$.

We first show how to construct a discrete holomorphic quadratic differential from a harmonic function. Given a function $u : V \to \mathbb{R}$ on a realization of $z : V \to \mathbb{C}$ of a triangular mesh M. If we interpolate it piecewise-linearly over each triangular face, its gradient is constant on each face and we have $\operatorname{grad}_z u : F \to \mathbb{C}$ given by

$$(\operatorname{grad}_z u)_{ijk} = i\frac{u_i dz(e_{jk}) + u_j dz(e_{ki}) + u_k dz(e_{ij})}{2A_{ijk}}.$$

Note that we ignore here the non-generic case (which leads to the vanishing of the area) where the triangle degenerates in the sense that its circumcircle passes through the point at infinity. Also note that for a non-degenerate triangle that is mapped by z in $\mathbb{C}$ in an orientation reversing fashion the area A_{ijk} is considered to have a negative sign. Granted this, one can verify that the gradient of u satisfies

$$\langle (\operatorname{grad}_z u)_{ijk}, dz(e_{ij})\rangle = u_j - u_i \quad \forall \{ij\} \subset \{ijk\} \in F.$$

We define $u_z : F \to \mathbb{C}$ by

$$u_z := \frac{1}{2}\overline{\operatorname{grad}_z u}.$$

and the dual 1-form $du_z : \vec{E}^*_{int} \to \mathbb{C}$ on M by

$$du_z(e^*_{ij}) := (u_z)_{ijk} - (u_z)_{jil}$$

where $\{ijk\}$ is the left face and $\{jil\}$ is the right face of the oriented edge e_{ij}.

Lemma 2.11. *Given a function $u : V \to \mathbb{R}$ on a realization of a triangular mesh $z : V \to \mathbb{C}$, we have*

$$\begin{aligned} &du_z(e^*_{ij})dz(e_{ij}) \\ =&\frac{-i}{2}\big(\cot\angle kij\,(u_k - u_j) + \cot\angle ijk\,(u_k - u_i) + \cot\angle lji\,(u_l - u_i) + \cot\angle jil\,(u_l - u_j)\big). \end{aligned}$$

which is purely imaginary (Figure 1.1).

Proof. Since

$$\langle (\operatorname{grad}_z u)_{ijk}, dz(e_{ij})\rangle = u_j - u_i = \langle (\operatorname{grad}_z u)_{jkl}, dz(e_{ij})\rangle,$$

we have

$$\operatorname{Re}(du_z(e^*_{ij})dz(e_{ij})) = 0.$$

On the other hand, using equation (2.4) we get

$$\begin{aligned} &\operatorname{Re}(du_z(e^*_{ij})idz(e_{ij})) \\ =&\operatorname{Re}(((u_z)_{ijk} - (u_z)_{jil})idz(e_{ij})) \\ =&(\langle (\operatorname{grad}_z u)_{ijk}, \cot\angle kij\,dz(e_{jk}) - \cot\angle ijk\,dz(e_{ki})\rangle \\ &+ \langle (\operatorname{grad}_z u)_{jil}, \cot\angle lji\,dz(e_{il}) - \cot\angle jil\,dz(e_{lj})\rangle)/2 \\ =&\frac{1}{2}\big(\cot\angle kij\,(u_k - u_j) + \cot\angle ijk\,(u_k - u_i) + \cot\angle lji\,(u_l - u_i) + \cot\angle jil(u_l - u_j)\big). \end{aligned}$$

Hence the claim follows. $\square$

Lemma 2.12. *Given a realization $z : V \to \mathbb{C}$ of a triangular mesh. A function $u : V \to \mathbb{R}$ is harmonic if and only if the function $q : E_{int} \to \mathbb{R}$ defined via*

$$iq_{ij} := du_z(e^*_{ij})dz(e_{ij})$$

is a holomorphic quadratic differential.

Proof. Note q is well defined since

$$iq_{ij} = du_z(e^*_{ij})dz(e_{ij}) = du_z(e^*_{ji})dz(e_{ji}) = iq_{ji}.$$

It holds for general functions $u : V \to \mathbb{R}$ that

$$\mathrm{Re}(iq) \equiv 0$$
$$\sum_j iq_{ij}/dz(e_{ij}) = \sum_j du_z(e^*_{ij}) = 0 \quad \forall i \in V_{int}.$$

We know from Lemma 2.11 that for every interior vertex $i \in V_{int}$

$$\sum_j iq_{ij} = \sum_j du_z(e^*_{ij})dz(e_{ij}) = \frac{i}{2}\sum_j((\cot\angle jki + \cot\angle ilj))(u_j - u_i).$$

Hence, u is harmonic if and only if q is a holomorphic quadratic differential. □

Lemma 2.13. *Let $z : V \to \mathbb{C}$ be a realization of a simply connected triangular mesh. Given a function $q : E_{int} \to \mathbb{R}$ satisfying for every interior vertex $i \in V_{int}$*

$$\sum_j iq_{ij}/dz(e_{ij}) = 0,$$

there exists a function $u : V \to \mathbb{R}$ such that for every interior edge $\{ij\}$

$$iq_{ij} = du_z(e^*_{ij})dz(e_{ij}).$$

Proof. We consider a dual 1-form τ on M defined by

$$\tau(e^*_{ij}) = iq_{ij}/dz(e_{ij}).$$

Since M is simply connected and

$$\sum_j \tau(e^*_{ij}) = \sum_j iq_{ij}/dz(e_{ij}) = 0,$$

there exists a function $h : F \to \mathbb{C}$ such that

$$dh(e^*_{ij}) := h_{ijk} - h_{jil} = \tau(e^*_{ij}).$$

It implies we have $\mathrm{Re}(dh(e^*)dz(e)) = \mathrm{Re}(iq) \equiv 0$ and

$$\omega(e_{ij}) := \langle 2\bar{h}_{ijk}, dz(e_{ij})\rangle = \langle 2\bar{h}_{jil}, dz(e_{ij})\rangle$$

is a well-defined $\mathbb{R}$-valued 1-form. Since the triangular mesh is simply connected and for every face $\{ijk\}$

$$\omega(e_{ij}) + \omega(e_{jk}) + \omega(e_{ki}) = 0,$$

there exists a function $u : V \to \mathbb{R}$ such that for every oriented edge e_{ij}

$$du(e_{ij}) = u_j - u_i = \omega(e_{ij}).$$

It can be verified that

$$h = \frac{1}{2}\overline{\mathrm{grad}_z}u = u_z.$$

Hence we obtain

$$iq_{ij} = \tau(e^*_{ij})dz(e_{ij}) = dh(e^*_{ij})dz(e_{ij}) = du_z(e^*_{ij})dz(e_{ij})$$

for every interior edge $\{ij\}$. □

Theorem 2.14. *Suppose $z : V \to \mathbb{C}$ is a realization of a simply connected triangular mesh. Then any holomorphic quadratic differential $q : E_{int} \to \mathbb{R}$ is of the form*

$$iq_{ij} = du_z(e^*_{ij})dz(e_{ij}) \quad \forall e_{ij} \in \vec{E}_{int}$$

for some harmonic function $u : V \to \mathbb{R}$.

Furthermore, the space of holomorphic quadratic differentials is a vector space isomorphic to the space of discrete harmonic functions modulo linear functions.

PROOF. The first part of the statement follows from Lemma 2.12 and Lemma 2.13. In order to show the second part, it suffices to observe that

$$du_z \equiv 0 \iff \operatorname{grad} u \equiv a \iff du = \langle a, dz\rangle \iff u = \langle a, z\rangle + b$$

for some $a, b \in \mathbb{C}$. □

In previous sections, we showed that every harmonic function corresponds to an infinitesimal conformal deformation. The following shows that discrete holomorphic quadratic differentials parametrize the change in the intersection angles of circumscribed circles.

Theorem 2.15. *Let $z : V \to \mathbb{C}$ be a realization of a simply connected triangular mesh. Suppose $u : V \to \mathbb{R}$ is a discrete harmonic function and $\dot{z}$ is an infinitesimal conformal deformation with u as scale factors. Then we have*

$$du_z dz = -\frac{1}{2}\frac{\dot{\operatorname{cr}}_z}{\operatorname{cr}_z} = -\frac{i}{2}\dot{\phi}$$

where $\dot{\phi} : E_{int} \to \mathbb{R}$ denotes the change in the intersection angles of neighboring circumscribed circles.

PROOF. We write $(\dot{z}_j - \dot{z}_i) = (\frac{h_i+h_j}{2} + i\omega_{ij})(z_j - z_i)$. Applying Lemma 2.11 we have

$$\begin{aligned}
\dot{\operatorname{cr}}_{z,ij}/\operatorname{cr}_{z,ij} =& i\omega_{jk} - i\omega_{ki} + i\omega_{il} - i\omega_{lj} \\
=& i\big(\cot\angle kij(u_k - u_j) + \cot\angle ijk(u_k - u_i) + \cot\angle lij(u_l - u_i) + \cot\angle jil(u_l - u_j)\big) \\
=& -2du_z(e^*_{ij})dz(e_{ij}).
\end{aligned}$$

The equality

$$\frac{\dot{\operatorname{cr}}_z}{\operatorname{cr}_z} = i\dot{\phi}$$

follows from Equation (2.1). □

2.4. Conformal deformations in terms of End($\mathbb{C}^2$)

In this section we show how an infinitesimal conformal deformation gives rise to a discrete analogue of a holomorphic null curve in $\mathbb{C}^3$. Later we will see that the real parts of such a "holomorphic null curve" yields a Weierstrass representation of discrete minimal surfaces.

Up to now we have mostly treated the Riemann sphere $\mathbb{CP}^1$ as the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In this section we will take a more explicitly Möbius geometric approach: We will represent fractional linear transformations of $\overline{\mathbb{C}}$ by linear transformations of $\mathbb{C}^2$ with determinant one. Actually, the group of Möbius transformations is

$$\text{Möb}(\overline{\mathbb{C}}) \cong \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})/(\pm I).$$

However, since we are mainly interested in infinitesimal deformations and any map into $\mathrm{PSL}(2, \mathbb{C})$ whose values stay close to the identity admits a canonical lift to $\mathrm{SL}(2, \mathbb{C})$, we can safely ignore the difference between $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$.

Given a realization $z : V \to \mathbb{C}$ of a triangular mesh we consider its lift $\psi : V \to \mathbb{C}^2$

$$\psi := \begin{pmatrix} z \\ 1 \end{pmatrix}$$

and regard the realization as a map $\Psi : V \to \mathbb{CP}^1$ where

$$\Psi := \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} = [\psi].$$

The action of a Möbius transformation on the Riemann sphere is given by a matrix $A \in \mathrm{SL}(2,\mathbb{C})$, which is unique up to sign:

$$[\varphi] \mapsto [A\varphi].$$

Before we investigate infinitesimal deformations we first consider finite deformations of a triangular mesh $\Psi : V \to \mathbb{C}\mathrm{P}^1$. Given such a finite deformation, the change in the positions of the three vertices of a triangle $\{ijk\}$ can be described by a Möbius transformation, which is represented by $G_{ijk} \in \mathrm{SL}(2,\mathbb{C})$. They satisfy a compatibility condition on each interior edge $\{ij\}$ (see Figure 1.1):

$$[G_{ijk}\psi_i] = [G_{jil}\psi_i],$$
$$[G_{ijk}\psi_j] = [G_{jil}\psi_j].$$

Suppose now that the mesh is simply connected. Then up to a global Möbius transformation the map $G : F \to \mathrm{SL}(2,\mathbb{C})$ can be uniquely reconstructed from the *multiplicative dual 1-form* defined as

$$G(e^*_{ij}) := G^{-1}_{jil} G_{ijk}.$$

$G(e^*_{ij})$ is defined whenever $\{ij\}$ is an interior edge and we have

$$G(e^*_{ij}) = G(e^*_{ji})^{-1}.$$

Moreover, for every interior vertex i we have

$$\prod_j G(e^*_{ij}) = I.$$

The compatibility conditions imply that for interior each edge $\{ij\}$ there exist $\lambda_{ij,i}, \lambda_{ij,j} \in \mathbb{C}\backslash\{0\}$ such that

$$G(e^*_{ij})\psi_i = \lambda_{ij,i}\psi_i,$$
$$G(e^*_{ij})\psi_j = \lambda_{ij,j}\psi_j.$$

Since $\lambda_{ij,i}\,\lambda_{ij,j} = \det(G(e^*_{ij})) = 1$, we have

$$\lambda_{ij} := \lambda_{ij,i} = 1/\lambda_{ij,j}.$$

Because of $G(e^*_{ij}) = G(e^*_{ij})^{-1}$ we know

$$\lambda_{ij} = \lambda_{ij,i} = 1/\lambda_{ji,i} = \lambda_{ji}.$$

Hence λ defines a complex-valued function on the set E_{int} of interior edges.

We now show that for each interior edge λ_{ij} determines the change in the cross ratio of the four points of the two adjacent triangles. Note that the cross ratio of four points in $\mathbb{C}\mathrm{P}^1$ can be expressed as

$$\mathrm{cr}([\psi_j],[\psi_k],[\psi_i],[\psi_l]) = \frac{\det(\psi_k,\psi_j)\det(\psi_l,\psi_i)}{\det(\psi_i,\psi_k)\det(\psi_j,\psi_l)}.$$

Lemma 2.16. *Suppose we are given four points $[\psi_i],[\psi_j],[\psi_k],[\psi_l] \in \mathbb{C}\mathrm{P}^1$ and $G \in \mathrm{SL}(2,\mathbb{C})$ with*

$$G\psi_i = \lambda^{-1}\psi_i$$
$$G\psi_j = \lambda\psi_j$$

for some $\lambda \in \mathbb{C}\backslash\{0\}$. Then the cross ratio of the four new points

$$[\tilde{\psi}_i] = [G\psi_i] \quad , \quad [\tilde{\psi}_j] = [G\psi_j] \quad , \quad [\tilde{\psi}_k] = [G\psi_k] \quad , \quad [\tilde{\psi}_l] = [\psi_l]$$

is given by

$$\mathrm{cr}([\tilde{\psi}_j],[\tilde{\psi}_k],[\tilde{\psi}_i],[\tilde{\psi}_l]) = \mathrm{cr}([\psi_j],[\psi_k],[\psi_i],[\psi_l])/\lambda^2.$$

Proof.

$$\begin{aligned}\mathrm{cr}([\tilde{\psi}_j],[\tilde{\psi}_k],[\tilde{\psi}_i],[\tilde{\psi}_l]) &= \frac{\det(G\psi_k,G\psi_j)\det(\psi_l,G\psi_i)}{\det(G\psi_i,G\psi_k)\det(G\psi_j,\psi_l)}\\ &= \mathrm{cr}([\psi_j],[\psi_k],[\psi_i],[\psi_l])/\lambda^2.\end{aligned}$$

$\square$

We now can summarize the information about finite deformations of a realization as follows:

Theorem 2.17. *Let $\Psi : V \to \mathbb{C}\mathrm{P}^1$ be a non-degenerate realization of a simply connected triangular mesh. Then there is a bijection between finite deformations of Ψ in $\mathbb{C}\mathrm{P}^1$ modulo global Möbius transformations and multiplicative dual 1 forms $G : \vec{E}^*_{int} \to \mathrm{SL}(2, \mathbb{C})$ satisfying for every interior vertex i*

$$\prod_j G(e^*_{ij}) = I$$

and for every interior edge

$$\begin{aligned} G(e^*_{ij}) &= G(e^*_{ji})^{-1}, \\ G(e^*_{ij})\psi_i &= \lambda_{ij}^{-1}\psi_i, \\ G(e^*_{ij})\psi_j &= \lambda_{ij}\psi_j. \end{aligned}$$

Here $\lambda : E_{int} \to \mathbb{C}\backslash\{0\}$. We denote by $\mathrm{cr} : E_{int} \to \mathbb{C}$ the cross ratios of Ψ and $\widetilde{\mathrm{cr}} : E_{int} \to \mathbb{C}$ the cross ratios of a new realization described by G. Then

$$\widetilde{\mathrm{cr}} = \mathrm{cr}\,/\lambda^2.$$

In particular,

$$\begin{aligned} |\lambda| \equiv 1 &\implies \textit{the deformation is conformal.} \\ \mathrm{Arg}(\lambda) \equiv 0 &\implies \textit{the deformation is a pattern deformation.} \end{aligned}$$

Suppose we have a family of deformations described by dual 1-forms $G_t : \vec{E}^*_{int} \to \mathrm{SL}(2, \mathbb{C})$ with $G_0 \equiv I$. By considering $\eta := \frac{d}{dt}|_{t=0}\, G_t$ we obtain the following description of infinitesimal deformations:

Corollary 2.18. *Let $\Psi : V \to \mathbb{C}\mathrm{P}^1$ be a realization of a simply connected triangular mesh. Then there is a bijection between infinitesimal deformations of Ψ in $\mathbb{C}\mathrm{P}^1$ modulo infinitesimal Möbius transformations and dual 1 forms $\eta : \vec{E}_{int} \to \mathrm{sl}(2, \mathbb{C})$ satisfying for every interior vertex i*

$$\sum_j \eta(e^*_{ij}) = 0 \tag{2.6}$$

and for every interior edge

$$\begin{aligned} \eta(e^*_{ij}) &= -\eta(e^*_{ji}), \\ \eta(e^*_{ij})\psi_i &= -\mu_{ij}\psi_i, \\ \eta(e^*_{ij})\psi_j &= \mu_{ij}\psi_j. \end{aligned}$$

Here $\mu : E_{int} \to \mathbb{C}$. We denote by $\mathrm{cr} : E_{int} \to \mathbb{C}$ the cross ratios of Ψ and $\dot{\mathrm{cr}} : E_{int} \to \mathbb{C}$ the rate of change in cross ratios induced by the infinitesimal deformation described by η. Then

$$\mu = -\frac{1}{2}\frac{\dot{\mathrm{cr}}}{\mathrm{cr}}.$$

In particular,

$$\begin{aligned} \mathrm{Re}(\mu) \equiv 0 &\implies \textit{the infinitesimal deformation is conformal,} \\ \mathrm{Im}(\mu) \equiv 0 &\implies \textit{the infinitesimal deformation is a pattern deformation.} \end{aligned}$$

Note that given a mesh, the 1-form η is uniquely determined by the eigenfunction μ. We now investigate the constraints on μ implied by the closedness condition (2.6) of η.

Consider the symmetric bilinear form $(\,,\,) : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathrm{sl}(2, \mathbb{C})$

$$(\phi, \varphi)v := \det(\phi, v)\varphi + \det(\varphi, v)\phi.$$

For $\psi_i \neq \psi_j \in \mathbb{C}^2$ we define

$$m_{ij} := \frac{1}{\det(\psi_i, \psi_j)}(\psi_j, \psi_i) \in \mathrm{sl}(2, \mathbb{C}).$$

The matrix m_{ij} is independent of the representatives of $[\psi_i], [\psi_j] \in \mathbb{C}\mathrm{P}^1$ and we have

$$\begin{aligned} m_{ij} &= -m_{ji}, \\ m_{ij}\psi_i &= -\psi_i, \\ m_{ij}\psi_j &= \psi_j. \end{aligned}$$

Using the representatives $\psi_i = \begin{pmatrix} z_i \\ 1 \end{pmatrix}$ we obtain

$$\begin{aligned} \eta(e_{ij}^*) =& \frac{\mu_{ij}}{\det(\psi_j, \psi_i)}(\psi_i, \psi_j) \\ =& \frac{\mu_{ij}}{z_j - z_i} \begin{pmatrix} z_i + z_j & -2z_i z_j \\ 2 & -z_i - z_j \end{pmatrix}. \end{aligned}$$

Hence

$$\sum_j \eta(e_{ij}^*) = 0 \iff \sum_j \mu_{ij} = 0 \quad \text{and} \quad \sum_j \mu_{ij}/(z_j - z_i) = 0. \tag{2.7}$$

In particular, if μ is real-valued, then it is a discrete holomorphic quadratic differential.

We now consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which form a basis of $\mathrm{sl}(2, \mathbb{C})$. Then

$$\eta(e_{ij}^*) = \frac{\mu_{ij}}{z_j - z_i}((1 - z_i z_j)\sigma_1 + i(1 + z_i z_j)\sigma_2 + (z_i + z_j)\sigma_3).$$

If we now identify $\mathrm{sl}(2, \mathbb{C})$ with $\mathbb{C}^3$ via

$$\sigma_i \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \sigma_2 \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \sigma_3 \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we obtain

$$\eta(e_{ij}^*) = \frac{\mu_{ij}}{z_j - z_i} \begin{pmatrix} 1 - z_i z_j \\ i(1 + z_i z_j) \\ z_i + z_j \end{pmatrix}. \tag{2.8}$$

Thus to every infinitesimal deformation of a realized triangular mesh we can associate a closed $\mathrm{sl}(2, \mathbb{C})$-valued dual 1-form. In the special case of an infinitesimal conformal deformation (i.e. μ is real-valued) we will see in the next chapter that Equation 2.8 yields a discrete analogue of the Weierstrass representation for minimal surfaces (Theorem 3.15).

CHAPTER 3

Discrete minimal surfaces

Minimal surfaces in Euclidean space are classical in differential geometry. They arise in the calculus of variations, in complex analysis and are related to integrable systems. A surface is *minimal* if it is a critical point of the area functional, or equivalently, its mean curvature vanishes identically. The Weierstrass representation for minimal surfaces asserts that locally each minimal surface is given by a pair of holomorphic functions. New minimal surfaces can be obtained from a given minimal surface via Bonnet, Goursat and Darboux transforms.

Structure-preserving discretization is ubiquitous, particularly in discrete differential geometry. A common approach is to build upon a discrete analogue of some characterization from the smooth theory. However, several equivalent characterizations in the smooth theory might lead to different discrete theories even if their continuum limits are the same.

In a variational approach, functions are usually defined at vertices while critical points of functionals are sought via vertex-based variations. Pinkall and Polthier [**55**] considered the total area of a triangulated surface in Euclidean space and suggested to define minimal surfaces as the critical points of the total area. Conjugate minimal surfaces were introduced but they were defined on the dual meshes. Within this approach it is difficult to discuss the associated family of minimal surfaces [**56**]. Moreover it is not clear how new minimal surfaces can be obtained via Goursat and Darboux transforms.

On the other hand, many surfaces of interest arise with integrable structures, such as constant mean curvature surfaces. Bobenko and Pinkall [**4**] considered discrete Lax representations of isothermic surfaces and introduced circular minimal surfaces together with a Weierstrass representation. New minimal surfaces can be obtained via Bonnet, Goursat and Darboux transforms [**31**]. However, this notion is believed to lack the variational property of minimal surfaces.

Other characterizations of surfaces depend on notions of curvature. Smooth minimal surfaces in Euclidean space are characterized by vanishing mean curvature. Curvatures of circular quadrilateral meshes based on Steiner's formula were proposed by Schief [**62, 63**], compatible with circular minimal surfaces from the integrable systems approach. Bobenko, Pottmann and Wallner [**10**] in a similar way defined mean curvature for conical surfaces, which are polyhedral surfaces with face offsets. Conical surfaces with vanishing mean curvature are named conical minimal surfaces. However, it is unclear if conical minimal surfaces admit an analogue of the Weierstrass representation and conjugate minimal surfaces. Furthermore, their relation to the variational approach is unknown.

In this chapter we show that the theories of discrete minimal surfaces based on the variational approach, the integrable systems approach and the curvature approach are *not* disjoint. Indeed they possess interesting relation with each other.

First, we establish a new relation between the integrable systems approach and the curvature approach to discrete minimal surfaces, different from [**62, 63**]. We define two types of discrete minimal surfaces whose cell decompositions are arbitrary. One of the two types is based on the Christoffel duality of isothermic surfaces. Another is defined via vanishing mean curvature. These two types respectively generalize circular minimal surfaces (from the integrable systems approach) and conical minimal surfaces (from the curvature approach). We show that in our setting each discrete minimal surface of one type corresponds to a discrete surface of the other type, and they form a conjugate pair of minimal surfaces. These surfaces admit a Weierstrass representation. Each of them corresponds to a discrete holomorphic quadratic differential on a planar mesh.

Second, we show that those quadrilateral minimal surfaces with Weierstrass data from *nonlinear* discrete complex analysis are critical points of the area functional. Every such quadrilateral minimal surface is constructed from a *P-net* [**8**], which comes from half the vertices of a discrete isothermic net with cross ratios -1. Particular examples of P-nets are given by Schramm's orthogonal circle patterns [**65**]. Bobenko, Hoffmann and Springborn [**7**] obtained a variational construction of orthogonal circle patterns and showed that each circle pattern corresponds to a s-isothermic minimal surface, which is determined by the combinatorics of the curvature lines of a smooth surface. These discrete minimal surfaces were shown to converge to smooth ones [**7, 50, 47**]. In particular, we find that by throwing away half the vertices of a s-isothermic minimal surface, the resulting surface not only satisfies our notion of discrete minimal surfaces but is also a critical point of the area functional.

We makes use of a generalization of Christoffel duality [**45**] and the observation that mean curvature for conical surfaces can be defined without referring to face offsets [**40**]. We further define the total area of a discrete surface with non-planar faces via the vector area on faces. As a result of these notions, we obtain connections between the integrable systems approach, the curvature approach and the variational approach to discrete minimal surfaces.

In section 3.1, two types of discrete minimal surfaces are defined and shown to be conjugate to each other. Each discrete minimal surface induces an associated family of discrete surfaces with vanishing mean curvature.

In section 3.2, Goursat transforms of discrete minimal surfaces are constructed in terms of the Weierstrass representation.

In section 3.3, by vertex splitting we regard every discrete minimal surface as a trivalent surface with a triangulated Gauss map. Each discrete minimal surface is represented by a discrete harmonic function in the sense of the cotangent Laplacian, or alternatively by a self-stress in the context of the rigidity theory of frameworks.

In section 3.4, we introduce the area of a non-planar polygon and show that discrete minimal surfaces in any associated family obtained from a P-net are critical points of the total area.

This chapter is based on [**43**].

3.1. Discrete minimal surfaces and their conjugates

In this section, we frequently use discrete differential forms as introduced in Section 1.1. To simplify our discussion, we further define the following.

DEFINITION 3.1. A realization $n : V \to \mathbb{S}^2$ of a discrete surface is *admissible* if $n_i \neq -n_j$ for every edge $\{ij\}$.

Given a discrete surface $M = (V, E, F)$, the dual cell decomposition is denoted by $M^* = (V^*, E^*, F^*)$. Each vertex $i \in V$ corresponds to a dual face $i^* \in F^*$. In particular, interior vertices of M correspond to the interior faces of M^*, denoted by F^*_{int}, while the boundary vertices of M correspond to the boundary (unbounded) faces of M^*.

3.1.1. A-minimal surfaces.

We introduce two types of discrete minimal surfaces. The first one mirrors the fact that every minimal surface is a Christoffel dual of its Gauss map, as shown in Section 4.8.

DEFINITION 3.2. Given a discrete surface M and its dual M^*, a realization $f : V^* \to \mathbb{R}^3$ of M^* with an admissible realization $n : V \to \mathbb{S}^2$ is *A-minimal* with Gauss map n if for every interior edge $\{ij\}$

$$dn(e_{ij}) \times df(e^*_{ij}) = 0, \tag{3.1}$$

$$\langle n_i + n_j, df(e^*_{ij}) \rangle = 0. \tag{3.2}$$

REMARK 3.3. If n is non-degenerate, i.e. $n_i \neq n_j$, then (3.1) implies (3.2).

In other words, a discrete minimal surface is a *reciprocal-parallel mesh* of an inscribed discrete surface. The combinatorics of a discrete minimal surface is that of the dual cell complex and each dual edge is parallel to the corresponding primal edge. Figure 3.1 shows a discrete minimal surface together with its Gauss map.

REMARK 3.4. If f is A-minimal and $v \in V^*$ is a vertex of degree three, then the image of v and its three neighboring vertices lie on an affine plane.

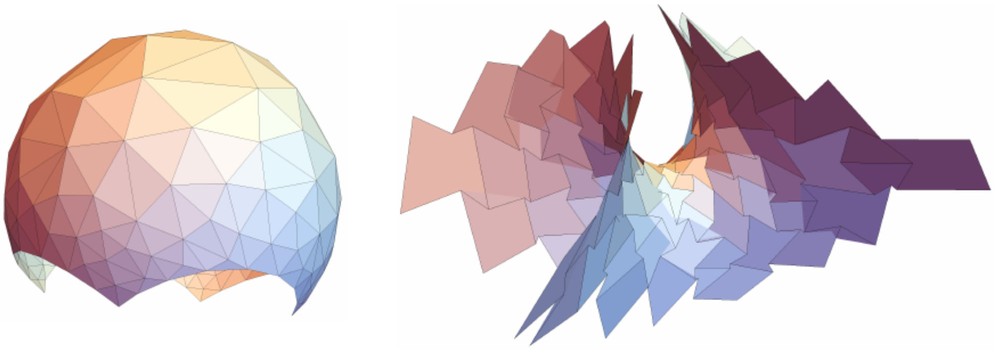

FIGURE 3.1. A triangulated surface $n : V \to \mathbb{S}^2$ (Left) as the Gauss map of an A-minimal surface $f : V^* \to \mathbb{R}^3$ (Right).

For A-minimal surfaces in the following example, their edges are in asymptotic line directions.

EXAMPLE 3.5 (Circular minimal surfaces). Bobenko and Pinkall [**4**] introduced circular minimal surfaces based on quadrilateral isothermic surfaces, which provide a discrete analogue of conformal curvature line parametrizations. A map $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ is isothermic if every elementary quadrilateral is cyclic and has factorized real cross-ratio

$$cr(f_{m,n}, f_{m+1,n}, f_{m+1,n+1}, f_{m,n+1}) = \frac{\alpha_m}{\beta_n} \quad \forall m, n \in \mathbb{Z},$$

where $\alpha_m \in \mathbb{R}$ does not depend on n and $\beta_n \in \mathbb{R}$ not depend on m. Then there exists another discrete isothermic net $f^* : V(\mathbb{Z}^2) \to \mathbb{R}^3$ satisfying

$$f^*_{m+1,n} - f^*_{m,n} = \alpha_m \frac{f_{m+1,n} - f_{m,n}}{||f_{m+1,n} - f_{m,n}||^2}, \quad f^*_{m,n+1} - f^*_{m,n} = \beta_n \frac{f_{m,n+1} - f_{m,n}}{||f_{m,n+1} - f_{m,n}||^2}.$$

Furthermore, the diagonals of f^* satisfy

$$f^*_{m+1,n} - f^*_{m,n+1} = (\alpha_m - \beta_n) \frac{f_{m+1,n+1} - f_{m,n}}{||f_{m+1,n+1} - f_{m,n}||^2}, \tag{3.3}$$

$$f^*_{m+1,n+1} - f^*_{m,n} = (\alpha_m - \beta_n) \frac{f_{m+1,n} - f_{m,n+1}}{||f_{m+1,n} - f_{m,n+1}||^2} \tag{3.4}$$

as shown in [**14**, Corollary 4.33]. In fact, every elementary quadrilateral of f^* is cyclic. If $f(V(\mathbb{Z}^2)) \subset \mathbb{S}^2$, then f^* is a *circular minimal surface.*

Now, we consider half the vertices of a quadrilateral isothermic surface. We denote $\mathbb{Z}^2_b$, $\mathbb{Z}^2_w$ the square lattices with vertices

$$V(\mathbb{Z}^2_b) := \{(m,n) \in \mathbb{Z}^2 | m + n \text{ even}\},$$
$$V(\mathbb{Z}^2_w) := \{(m,n) \in \mathbb{Z}^2 | m + n \text{ odd}\}.$$

If $f : V(\mathbb{Z}^2) \to \mathbb{S}^2$ is an isothermic net, then equations (3.3) and (3.4) imply

$$f^*|_{\mathbb{Z}^2_w} \text{ is A-minimal with Gauss map } f|_{\mathbb{Z}^2_b}.$$

In fact, one can obtain an A-minimal surface from a circular minimal surface by adding diagonals arbitrarily [**46**].

3.1.2. C-minimal surfaces. The second type of discrete minimal surfaces mimics the property that smooth minimal surfaces have vanishing mean curvature.

Given a discrete surface M and its dual M^*. Suppose we have a realization $\tilde{f} : V^* \to \mathbb{R}^3$ of M^* with planar faces. We pick a normal for each face $n : F^* \to \mathbb{S}^2$ such that n is admissible, i.e. $n_i \neq -n_j$ for every edge $\{ij\}$. We then measure its dihedral angles. If $d\tilde{f}(e^*_{ij}) \neq 0$, the sign of the dihedral angle $\alpha_{ij} \in (-\pi, \pi)$ is determined by

$$\sin \alpha_{ij} = \langle n_i \times n_j, \frac{d\tilde{f}(e^*_{ij})}{|d\tilde{f}(e^*_{ij})|} \rangle,$$
$$\cos \alpha_{ij} = \langle n_i, n_j \rangle$$

where $i^*, j^* \in F^*$ denote the left and the right face of e^*_{ij}. In the following, we are interested in the quantity $|d\tilde{f}| \tan(\alpha/2)$ for every edge. This quantity is set to zero whenever the edge is degenerate, i.e. $|d\tilde{f}| = 0$.

Definition 3.6. Suppose we have a discrete surface M and its dual M^*. A realization $\tilde{f} : V^* \to \mathbb{R}^3$ of M^* together with an admissible realization $n : V \to \mathbb{S}^2$ is *C-minimal* with Gauss map n if $\tilde{f}$ has planar faces with face normal n and the function $\tilde{H} : F^*_{int} \to \mathbb{R}$ defined by

$$\tilde{H}_i := \sum_j |d\tilde{f}(e^*_{ij})| \tan \frac{\alpha_{ij}}{2} \quad \forall i \in V_{int} \cong F^*_{int}$$

vanishes identically. We call $\tilde{H}$ the scalar mean curvature of $\tilde{f}$.

To prepare for the next section, we rewrite the scalar mean curvature in terms of face normals.

Lemma 3.7. *Given a realization $\tilde{f} : V^* \to \mathbb{R}^3$ of a discrete surface M^* with face normal $n : V \to \mathbb{R}^3$. If $n_i \neq \pm n_j$ on edge $\{ij\}$, then*

$$d\tilde{f}(e^*_{ij}) = k_{ij} n_i \times n_j$$

for some $k_{ij} = k_{ji} \in \mathbb{R}$ and

$$|d\tilde{f}(e^*_{ij})| \tan \frac{\alpha_{ij}}{2} = k_{ij}(1 - \langle n_i, n_j \rangle).$$

Proof. Suppose $\{ij\}$ is an edge of M with $n_i \neq \pm n_j$. Since $n_i, n_j \perp d\tilde{f}(e^*_{ij})$ there exists $k_{ij} \in \mathbb{R}$ such that

$$d\tilde{f}(e^*_{ij}) = k_{ij} n_i \times n_j.$$

The property $d\tilde{f}(e^*_{ji}) = -d\tilde{f}(e^*_{ij})$ implies $k_{ij} = k_{ji}$. If $d\tilde{f}(e^*_{ij}) = 0$, then $k_{ij} = 0$ and

$$|d\tilde{f}(e^*_{ij})| \tan \frac{\alpha_{ij}}{2} = 0 = k_{ij}(1 - \langle n_i, n_j \rangle).$$

If $d\tilde{f}(e^*_{ij}) \neq 0$, the dihedral angle α_{ij} satisfies

$$\sin \alpha_{ij} = \langle n_i \times n_j, \frac{d\tilde{f}(e^*_{ij})}{|d\tilde{f}(e^*_{ij})|} \rangle = \operatorname{sign}(k) |\sin \alpha_{ij}|$$

and hence

$$k_{ij}(1 - \langle n_i, n_j \rangle) = \frac{|k_{ij} n_i \times n_j|}{\sin \alpha_{ij}} \cdot 2 \sin^2 \frac{\alpha_{ij}}{2} = |d\tilde{f}(e^*_{ij})| \tan \frac{\alpha_{ij}}{2}. \qquad \square$$

In the following example, we observe that conical minimal surfaces, as a discrete analogue of curvature line parametrizations [**13, 58**], are C-minimal surfaces.

Example 3.8 (Conical minimal surfaces). A realization $\tilde{f} : V^* \to \mathbb{R}^3$ of a discrete surface M with planar faces and non-degenerate face normal $n : V \to \mathbb{S}^2$ is a *conical* surface if n has planar faces. In this case, the poles of the faces of n with respect to the unit sphere yields a realization $N : V^* \to \mathbb{R}^3$ of M^* with planar faces tangent to the unit sphere: for each dual face $i^* = (\phi^*_1, \phi^*_2, \ldots, \phi^*_n) \in F^*$

$$\langle n_i, N_{\phi_r} \rangle = 1.$$

For $t \in \mathbb{R}$ the area of each planar face under a face offset $\tilde{f} + tN$ is

$$\begin{aligned} \operatorname{Area}(\tilde{f} + tN)_i &= \frac{1}{2} \sum_r \langle (\tilde{f}_{\phi_r} + tN_{\phi_r}) \times (\tilde{f}_{\phi_{r+1}} + tN_{\phi_{r+1}}), n_i \rangle \\ &=: \operatorname{Area}(\tilde{f})_i + \operatorname{Area}(\tilde{f}, N)_i t + \operatorname{Area}(N)_i t^2. \end{aligned}$$

Bobenko, Pottmann and Wallner considered conical surfaces with face offsets and vanishing mixed area $\mathrm{Area}(\tilde{f}, N) \equiv 0$ as *conical minimal surfaces* [**10, 53**]. Karpenkov and Wallner [**40**] showed that the mixed area coincides with the scalar mean curvature in Definition 3.6. We provide a proof here. We write $\tilde{f}_{\phi_{r+1}} - \tilde{f}_{\phi_r} = k_{ir} n_i \times n_r$. Here ϕ_r, ϕ_{r+1} denote the right and the left face of e_{ir} and

$$\langle N_{\phi_r}, n_i \rangle = \langle N_{\phi_{r+1}}, n_i \rangle = \langle N_{\phi_r}, n_r \rangle = \langle N_{\phi_{r+1}}, n_r \rangle = 1.$$

Then for every interior vertex i

$$\begin{aligned}
\mathrm{Area}(\tilde{f}, N)_i &= \frac{1}{2} \sum_r \langle \tilde{f}_{\phi_r} \times N_{\phi_{r+1}} + N_{\phi_r} \times \tilde{f}_{\phi_{r+1}}, n_i \rangle \\
&= \frac{1}{2} \sum_r \langle (N_{\phi_r} + N_{\phi_{r+1}}) \times (\tilde{f}_{\phi_{r+1}} - \tilde{f}_{\phi_r}), n_i \rangle \\
&= \frac{1}{2} \sum_r \langle N_{\phi_r} + N_{\phi_{r+1}}, k_{ir}(n_r - \langle n_i, n_r \rangle n_i) \rangle \\
&= \sum_r k_{ir} (1 - \langle n_i, n_r \rangle) \\
&= \sum_r |d\tilde{f}(e^*_{ir})| \tan \frac{\alpha_{ir}}{2}.
\end{aligned}$$

Hence, conical minimal surfaces are C-minimal.

Example 3.9 (Cubic polyhedra). A cubic polyhedron is a polyhedral surface with edges exactly the same as those of the cubic lattice. Goodman-Strauss and Sullivan [**25**] showed that they are discrete minimal surfaces in the sense of the variational approach [**55**]. A cubic polyhedron has convex faces. There exists a canonical choice of normal vectors determined by the orientation. Its edge lengths are all equal and its dihedral angles are either $\frac{\pi}{2}$, 0 or $-\frac{\pi}{2}$. One can check that they are C-minimal.

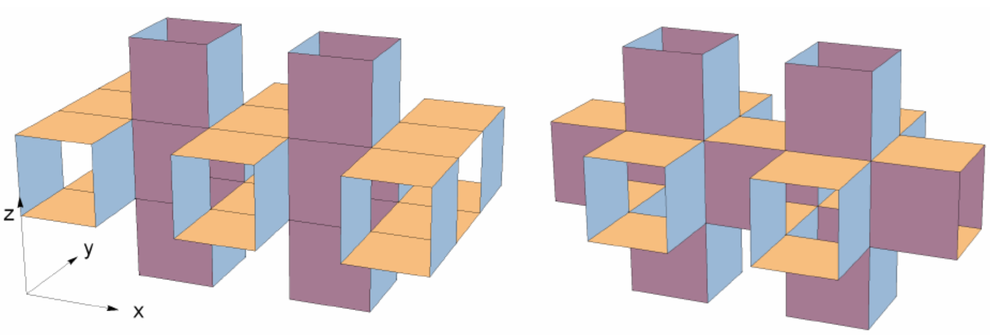

Figure 3.2. Parts of two C-minimal symmetric cubic polyhedra. Left: Merging two successive squares parallel to xy-plane and xz-plane respectively into rectangles yields vanishing scalar mean curvature. Right: Each square face has vanishing scalar mean curvature.

3.1.3. Conjugate minimal surfaces.

We show that there is a one-to-one correspondence between the two types of discrete minimal surfaces.

Theorem 3.10. *Given a simply connected discrete surface M and its dual M^*. For every admissible realization $n : V \to \mathbb{S}^2$ of M, each A-minimal surface $f : V^* \to \mathbb{R}^3$ with Gauss map n yields a C-minimal surface $\tilde{f} : V^* \to \mathbb{R}^3$ with Gauss map n via*

$$d\tilde{f}(e^*_{ij}) = n_i \times df(e^*_{ij}) = n_j \times df(e^*_{ij})$$

and vice versa. We say $(f, \tilde{f})$ form a conjugate pair of minimal surfaces.

PROOF. Suppose $f : V^* \to \mathbb{R}^3$ is A-minimal with Gauss map $n : V \to \mathbb{S}^2$. Then there exists a well defined dual 1-form $\eta : \vec{E}^*_{int} \to \mathbb{R}^3$ on M^* such that

$$\eta(e^*_{ij}) := n_j \times df(e^*_{ij}) = n_i \times df(e^*_{ij}) \tag{3.5}$$

and for every interior vertex i

$$\sum_j \eta(e^*_{ij}) = n_i \times \sum_j df(e^*_{ij}) = 0, \tag{3.6}$$

$$\langle n_i, \sum_j df(e^*_{ij})\rangle = 0. \tag{3.7}$$

Since M is simply connected, the closedness condition in (3.6) implies η is exact. Hence there exists $\tilde{f} : V^* \to \mathbb{R}^3$ such that for every interior oriented edge $e^*_{ij} \in \vec{E}_{int}$

$$d\tilde{f}(e^*_{ij}) = \eta(e^*_{ij}).$$

We now show that $\tilde{f}$ is C-minimal with Gauss map n.

Firstly, equation (3.5) implies $\tilde{f}$ has planar faces with face normal n.

Secondly, if $n_i = n_j$ for some interior edge $\{ij\}$ then

$$\langle n_i, df(e^*_{ij})\rangle = \frac{1}{2}\langle n_i + n_j, df(e^*_{ij})\rangle = 0.$$

If $n_i \neq n_j$, we write $df(e^*_{ij}) = k_{ij}(n_j - n_i)$ for some $k_{ij} = k_{ji} \in \mathbb{R}$ and thus $d\tilde{f}(e^*_{ij}) = k_{ij} n_i \times n_j$. Lemma 3.7 implies

$$\begin{aligned}\langle n_i, \sum_j df(e^*_{ij})\rangle &= \sum_{j|n_i\neq n_j} k_{ij}(\langle n_i, n_j\rangle - 1)\\ &= -\sum_{j|n_i\neq n_j} |d\tilde{f}(e^*_{ij})| \tan\frac{\alpha_{ij}}{2}\\ &= -\sum_j |d\tilde{f}(e^*_{ij})| \tan\frac{\alpha_{ij}}{2}.\end{aligned}$$

Hence equation (3.7) implies that $\tilde{f}$ has vanishing scalar mean curvature and thus is C-minimal with Gauss map n.

Conversely, we suppose $\tilde{f}$ is C-minimal with Gauss map n and define a dual 1-form $\omega : \vec{E}^*_{int} \to \mathbb{R}^3$ by

$$\omega(e^*_{ij}) := -n_i \times d\tilde{f}(e^*_{ij}) - |d\tilde{f}(e^*_{ij})| \tan\frac{\alpha_{ij}}{2}\, n_i.$$

To check ω is a 1-form on M^*, note that if $n_i = n_j$ then $\alpha_{ij} = 0$ and

$$\omega(e^*_{ji}) = -n_j \times d\tilde{f}(e^*_{ji}) = n_i \times d\tilde{f}(e^*_{ij}) = -\omega(e^*_{ij}).$$

If $n_i \neq n_j$, writing $d\tilde{f}(e^*_{ij}) = k_{ij} n_i \times n_j$ yields

$$\omega(e^*_{ij}) = -n_i \times (k_{ij} n_i \times n_j) - k_{ij}(1 - \langle n_i, n_j\rangle)n_i = k_{ij}(n_j - n_i)$$

and hence $\omega(e^*_{ji}) = -\omega(e^*_{ij})$.

In addition $\tilde{f}$ being C-minimal implies that ω is a closed dual 1-form on M^*

$$\sum_j \omega(e^*_{ij}) = 0.$$

Since M^* is simply connected, there exists $f : V^* \to \mathbb{R}^3$ such that

$$df(e^*_{ij}) = \omega(e^*_{ij}) \quad \forall \{ij\} \in E_{int}$$

and f is A-minimal with Gauss map n. □

Since we have a notion of conjugate minimal surfaces sharing the same combinatorics, we can define the associated family of minimal surfaces as in the smooth theory.

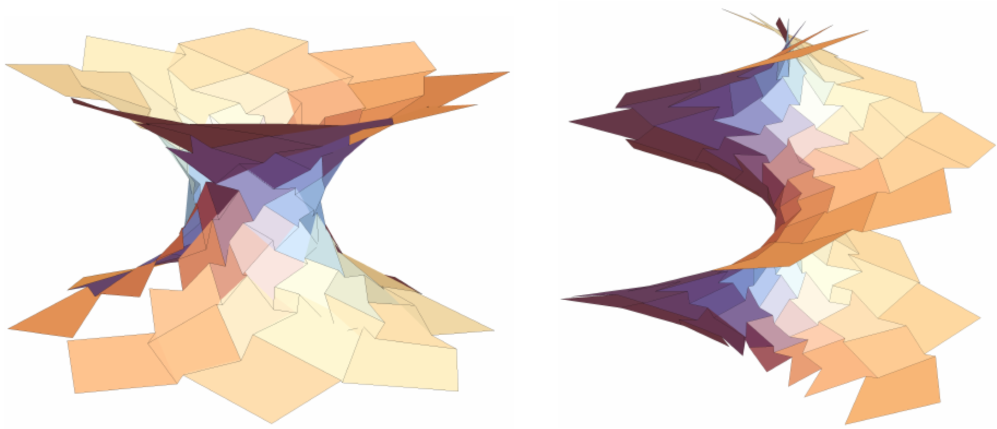

FIGURE 3.3. A conjugate pair of discrete minimal surfaces. Left: An A-minimal surface, trivalent with planar vertex stars. It is a reciprocal parallel mesh of a triangulated surface inscribed in the unit sphere. Right: A C-minimal surface, trivalent with planar faces. On each face ϕ, the *integrated mean curvature* $\sum_{e\in\phi} \ell_e \tan\frac{\alpha_e}{2}$ vanishes. Here ℓ denotes the edge lengths and α denotes the dihedral angles.

DEFINITION 3.11. For every conjugate pair of minimal surfaces $(f, \tilde{f})$ we define *its associated family of surfaces* as a S^1-family of realizations $f^\theta : V^* \to \mathbb{R}^3$

$$f^\theta := \cos\theta f + \sin\theta \tilde{f}$$

for $\theta \in [0, 2\pi]$.

3.1.4. Integrated mean curvature on faces. Mean curvature was introduced in [**62, 63, 10**] on polyhedral surfaces with vertex normals by means of Steiner's formula and further extended to the associated family of discrete minimal surfaces in [**32**].

Recall that C-minimal surfaces are defined by vanishing scalar mean curvature. In fact, the notion of scalar mean curvature can be extended naturally to the associated family of discrete minimal surfaces.

Theorem 3.12. *Every A-minimal surface $f : V^* \to \mathbb{R}^3$ with admissible Gauss map n satisfies*

$$\langle dn(e_{ij}) \times df(e^*_{ij}), \frac{n_i + n_j}{|n_i + n_j|^2}\rangle = 0 \quad \forall\{ij\} \in E_{int},$$
$$\sum_j \langle dn(e_{ij}), df(e^*_{ij})\rangle = 0 \quad \forall i \in V_{int}.$$

On the other hand, every C-minimal surface $\tilde{f} : V^* \to \mathbb{R}^3$ with admissible Gauss map $n : V \to \mathbb{S}^2$ satisfies

$$\sum_j \langle dn(e_{ij}) \times d\tilde{f}(e^*_{ij}), \frac{n_i + n_j}{|n_i + n_j|^2}\rangle = 0 \quad \forall i \in V_{int},$$
$$\langle dn(e_{ij}), d\tilde{f}(e^*_{ij})\rangle = 0 \quad \forall\{ij\} \in E_{int}.$$

If $(f, \tilde{f})$ form a conjugate pair of minimal surfaces, then the discrete minimal surfaces $f^\theta := \cos\theta f + \sin\theta \tilde{f}$ in the associated family satisfy for any interior vertex $i \in V_{int}$ (i.e. any face of f^θ)

$$H^\theta_i := \sum_j \langle dn(e_{ij}) \times df^\theta(e^*_{ij}), \frac{n_i + n_j}{|n_i + n_j|^2}\rangle = 0, \tag{3.8}$$

$$\sum_j \langle dn(e_{ij}), df^\theta(e^*_{ij})\rangle = 0. \tag{3.9}$$

Here $H^\theta : F^*_{int} \to \mathbb{R}$ is the scalar mean curvature *of f^θ. Furthermore $H^{\pi/2}$ coincides with the scalar mean curvature $\tilde{H}$ of the C-minimal surface $\tilde{f}$.*

Proof. Since f is A-minimal with Gauss map n, on every interior edge $\{ij\}$ we have

$$dn(e_{ij}) \times df(e^*_{ij}) = 0$$

and hence

$$\langle dn(e_{ij}) \times df(e^*_{ij}), \frac{n_i + n_j}{|n_i + n_j|^2} \rangle = 0.$$

We consider an interior vertex i and for each of its neighboring vertex j we write $df(e^*_{ij}) = k_{ij}(n_j - n_i)$ whenever $n_i \neq n_j$. Then the dual face i^* being a closed polygon

$$\sum_j df(e^*_{ij}) = 0$$

implies

$$\sum_j \langle dn(e_{ij}), df(e^*_{ij}) \rangle = 2 \sum_{j|n_i \neq n_j} k_{ij} \langle n_i - n_j, n_i \rangle = -2 \sum_j \langle df(e^*_{ij}), n_i \rangle = 0.$$

where we used $\langle df(e^*_{ij}), n_i \rangle = 0$ if $n_i = n_j$.

On the other hand we know

$$n_i, n_j \perp d\tilde{f}(e^*_{ij})$$

and hence

$$\langle dn(e_{ij}), d\tilde{f}(e^*_{ij}) \rangle = 0.$$

For every edge $\{ij\}$ such that $n_i \neq n_j$, we have $d\tilde{f}(e^*_{ij}) = n_i \times df(e^*_{ij}) = k_{ij} n_i \times n_j$. Hence

$$\begin{aligned} \sum_j \langle dn(e_{ij}) \times d\tilde{f}(e^*_{ij}), \frac{n_i + n_j}{|n_i + n_j|^2} \rangle &= \sum_{j|n_i \neq n_j} k_{ij} \langle n_i - n_j, n_i \rangle \langle n_i + n_j, \frac{n_i + n_j}{|n_i + n_j|^2} \rangle \\ &= \sum_j |d\tilde{f}(e^*_{ij})| \tan \frac{\alpha_{ij}}{2} \\ &= 0 \end{aligned}$$

As f^θ is a linear combination of f and $\tilde{f}$ we immediately obtain equation (3.8) and (3.9). $\square$

The following remark explains the smooth counterpart of (3.8) and (3.9).

Remark 3.13. Suppose $f : M \to \mathbb{R}^3$ is a smoothly immersed surface with Gauss map N and $X_1, X_2 \in T_pM$ form an orthonormal basis in principal directions. Then the corresponding principal curvatures $\kappa_1, \kappa_2 \in \mathbb{R}$ satisfy

$$dN_p(X_i) = \kappa_i df_p(X_i).$$

We denote J the almost complex structure induced via

$$N \times df(\cdot) = df(J \cdot).$$

For every $\theta \in \mathbb{R}$ we have

$$\begin{aligned} \langle dN_p(\cos\theta X_1 + \sin\theta X_2) \times df_p(J_p(\cos\theta X_1 + \sin\theta X_2)), N_p \rangle &= H + \frac{\kappa_1 - \kappa_2}{2} \cos 2\theta, \\ \langle dN_p(\cos\theta X_1 + \sin\theta X_2), df_p(J_p(\cos\theta X_1 + \sin\theta X_2)) \rangle &= \frac{\kappa_1 - \kappa_2}{2} \sin 2\theta \end{aligned}$$

where $H = \frac{\kappa_1 + \kappa_2}{2}$ is the mean curvature. Averaging over all directions yields

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \langle dN_p(\cos\theta X_1 + \sin\theta X_2) \times df_p(J_p(\cos\theta X_1 + \sin\theta X_2)), N_p \rangle d\theta &= H, \\ \frac{1}{2\pi} \int_0^{2\pi} \langle dN_p(\cos\theta X_1 + \sin\theta X_2), df_p(J_p(\cos\theta X_1 + \sin\theta X_2)) \rangle d\theta &= 0. \end{aligned}$$

3.2. Weierstrass representation

In this section, we relate A-minimal surfaces and C-minimal surfaces to discrete holomorphic quadratic differentials via a Weierstrass representation. In the following we consider simply connected discrete surfaces and non-degenerate Gauss maps.

The Weierstrass representation for smooth minimal surfaces in $\mathbb{R}^3$ is a classical application of complex analysis:

Theorem 3.14. *Given two meromorphic functions $g, h : U \subset \mathbb{C} \to \mathbb{C}$ such that $g^2 h$ is holomorphic. Then $f : U \to \mathbb{R}^3$ defined by*

$$df = \operatorname{Re}\left(\begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} h(z)dz \right) = \operatorname{Re}\left(\begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} \frac{q}{dg} \right)$$

is a minimal surface. Its Gauß map n is the stereographic projection of g

$$n = \frac{1}{|g|^2+1} \begin{pmatrix} 2\operatorname{Re} g \\ 2\operatorname{Im} g \\ |g|^2-1 \end{pmatrix}.$$

The holomorphic quadratic differential $q := hg_z dz^2$ is the Hopf differential of f and encodes its second fundamental form: The direction defined by a nonzero tangent vector W is

$$\begin{aligned} \textit{an asymptotic direction} \quad &\Longleftrightarrow \quad q(W) \in i\mathbb{R}, \\ \textit{a principal curvature direction} \quad &\Longleftrightarrow \quad q(W) \in \mathbb{R}. \end{aligned}$$

Locally, every minimal surface can be written in this form.

We are going to show that every discrete holomorphic quadratic differential yields an A-minimal surface and a C-minimal surface via the Weierstrass representation that appeared in the end of Section 2.4. As discussed in the Introduction (Definition 1.1), a discrete holomorphic quadratic differential on a planar mesh $z : V \to \mathbb{C}$ is a function $q : E_{int} \to \mathbb{R}$ defined on interior edges satisfying for every interior vertex $i \in V_{int}$

$$\sum_j q_{ij} = 0,$$

$$\sum_j q_{ij}/dz(e_{ij}) = 0.$$

Discrete holomorphic quadratic differentials and their relation to discrete conformality are revealed in Section 2.4 by considering infinitesimally deformations of planar triangular meshes preserving length cross ratios. In this case a holomorphic quadratic differential is simply the change of the logarithmic cross ratios, which parametrizes the change of the Möbius structure (complex projective structure) of the triangular mesh under infinitesimal conformal deformations.

Theorem 3.15. *Suppose $z : V \to \mathbb{C}$ is a non-degenerate realization of a simply connected discrete surface and $q : E_{int} \to \mathbb{R}$ is a discrete holomorphic quadratic differential. Then there exists $\mathcal{F} : V^* \to \mathbb{C}^3$ such that for every edge $\{ij\} \in E_{int}$*

$$d\mathcal{F}(e^*_{ij}) = \frac{q_{ij}}{dz(e_{ij})} \begin{pmatrix} 1 - z_i z_j \\ i(1+z_i z_j) \\ z_i + z_j \end{pmatrix}. \tag{3.10}$$

We assume the stereographic projection $n : V \to \mathbb{S}^2$ of z given by

$$n := \frac{1}{1+|z|^2} \begin{pmatrix} 2\operatorname{Re} z \\ 2\operatorname{Im} z \\ |z|^2-1 \end{pmatrix}$$

is admissible, i.e. $n_i \neq -n_j$ for $\{ij\} \in E$. We then have

(1) *$f := \operatorname{Re}(\mathcal{F}) : V^* \to \mathbb{R}^3$ is A-minimal and*
(2) *$\tilde{f} := \operatorname{Re}(i\mathcal{F}) : V^* \to \mathbb{R}^3$ is C-minimal.*

The realizations $(f, \tilde{f})$ form a conjugate pair with Gauss map n.

The converse also holds: For every conjugate pair of minimal surfaces $f, \tilde{f}$ there exists a holomorphic quadratic differential q on the stereographic projection of their Gauss map such that $\mathcal{F} := f - i\tilde{f}$ satisfies (3.10).

PROOF. We consider a dual 1-form $\eta : \vec{E}^*_{int} \to \mathbb{C}^3$ defined by

$$\eta(e^*_{ij}) := \frac{q_{ij}}{dz(e_{ij})} \begin{pmatrix} 1 - z_i z_j \\ i(1 + z_i z_j) \\ z_i + z_j \end{pmatrix}.$$

Since q is a holomorphic quadratic differential, η is closed:

$$\sum_j \eta(e^*_{ij}) = 0 \quad i \in V_{int}.$$

As M is simply connected, we can integrate η and obtain a map $\mathcal{F} : V^* \to \mathbb{C}^3$ defined on the dual vertices such that for every interior edge $\{ij\}$

$$d\mathcal{F}(e^*_{ij}) = \eta(e^*_{ij}).$$

We define $k : E_{int} \to \mathbb{R}$ by

$$k_{ij} := q_{ij}/|dz(e_{ij})|^2. \tag{3.11}$$

Inserting equation (3.11) into (3.10) we get

$$\mathrm{Re}(d\mathcal{F}(e^*_{ij})) = \frac{k_{ij}(1 + |z_i|^2)(1 + |z_j|^2)}{2}(n_j - n_i), \tag{3.12}$$

$$\mathrm{Re}(i\, d\mathcal{F}(e^*_{ij})) = \frac{k_{ij}(1 + |z_i|^2)(1 + |z_j|^2)}{2}(n_i \times n_j). \tag{3.13}$$

The converse is straightforward. Given a conjugate pair of discrete minimal surfaces $(f, \tilde{f})$, we define $k : E_{int} \to \mathbb{R}$ via (3.12). It can be shown that the function

$$q_{ij} := k_{ij}|dz(e_{ij})|^2$$

is a holomorphic quadratic differential. □

REMARK 3.16. For a real-valued holomorphic quadratic differential q, we obtain an A-minimal surface via (3.10), whose edges are regarded as in asymptotic line directions. However, in the smooth theory, the Hopf differential takes purely imaginary values along asymptotic directions. Such a difference results from the fact that in (3.10) $d\mathcal{F}$ is a dual 1-form while dz is a (primal) 1-form.

Corollary 3.17. *The associated family of discrete minimal surfaces f^θ in Definition 3.11 satisfies*

$$f^\theta = \mathrm{Re}(e^{i\theta}\mathcal{F}).$$

3.2.1. Goursat transformations. In contrast to Bonnet transformations in the smooth theory, Goursat transformations generate non-isometric minimal surfaces in general [**26**]. A conjugate pair of minimal surfaces can be regarded as a holomorphic null curve in $\mathbb{C}^3$. A Goursat transform of a conjugate pair of minimal surfaces is a complex rotation acting on the null curve. These transformations preserve respectively curvature line and asymptotic line parametrizations of minimal surfaces [**52**].

The Möbius invariant property of discrete holomorphic quadratic differentials implies that if $q : E_{int} \to \mathbb{R}$ is a discrete holomorphic quadratic differential on a non-degenerate realization $z : M \to \mathbb{C}$, then the dual 1-form defined by

$$\eta_\Phi(e^*_{ij}) := \frac{q_{ij}}{(\Phi(z_j) - \Phi(z_i))} \begin{pmatrix} 1 - \Phi(z_i)\Phi(z_j) \\ i(1 + \Phi(z_i)\Phi(z_j)) \\ \Phi(z_i) + \Phi(z_j) \end{pmatrix} \tag{3.14}$$

is closed for any Möbius transformation $\Phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. Note every Möbius transformation Φ can be represented as

$$\Phi(z) = \frac{az+b}{cz+d} \tag{3.15}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. We are going to see how a minimal surface deforms if its Gauss map is transformed under a Möbius transformation by substituting (3.15) into (3.14).

The following can be verified directly.

Lemma 3.18. *Let $\Phi(z) := \frac{az+b}{cz+d}$ with $ad - bc = 1$. Then*

$$\begin{aligned}
\Phi(z_j) - \Phi(z_i) &= \frac{z_j - z_i}{(cz_j + d)(cz_i + d)}, \\
1 - \Phi(z_j)\Phi(z_i) &= \frac{(c^2 - a^2)z_i z_j + (dc - ab)(z_i + z_j) + d^2 - b^2}{(cz_j + d)(cz_i + d)}, \\
1 + \Phi(z_j)\Phi(z_i) &= \frac{(c^2 + a^2)z_i z_j + (dc + ab)(z_i + z_j) + d^2 + b^2}{(cz_j + d)(cz_i + d)}, \\
\Phi(z_j) + \Phi(z_i) &= \frac{2ac z_i z_j + (bc + ad)(z_i + z_j) + 2bd}{(cz_j + d)(cz_i + d)}
\end{aligned}$$

and hence

$$\frac{1}{\Phi(z_j) - \Phi(z_i)} \begin{pmatrix} 1 - \Phi(z_i)\Phi(z_j) \\ i(1 + \Phi(z_i)\Phi(z_j)) \\ \Phi(z_i) + \Phi(z_j) \end{pmatrix} = \frac{1}{z_j - z_i} A_\Phi \begin{pmatrix} 1 - z_i z_j \\ i(1 + z_i z_j) \\ z_i + z_j \end{pmatrix}$$

where

$$A_\Phi := \begin{pmatrix} \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{i}{2}(a^2 + b^2 - c^2 - d^2) & -ab + cd \\ \frac{i}{2}(-a^2 + b^2 - c^2 + d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) & i(ab + cd) \\ -ac + bd & -i(ac + bd) & ad + bc \end{pmatrix}.$$

The following indicates that conjugate pairs of discrete minimal surfaces deform exactly the same way as the smooth ones under Goursat transforms.

Theorem 3.19. *Given a discrete surface M and its dual M^*, we suppose $f, \tilde{f} : V^* \to \mathbb{R}^3$ form a conjugate pair of minimal surfaces with a non-degenerate admissible Gauss map $n : V \to \mathbb{S}^2$ where f is A-minimal and $\tilde{f}$ is C-minimal. For any Möbius transformation*

$$\Phi(z) := \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1$$

such that $n_\Phi : V \to \mathbb{S}^2$ defined by

$$n_\Phi := \frac{1}{|\Phi(z)|^2 + 1} \begin{pmatrix} 2\operatorname{Re}\Phi(z) \\ 2\operatorname{Im}\Phi(z) \\ |\Phi(z)|^2 - 1 \end{pmatrix} \quad \text{where } z := \frac{n_1}{1 - n_3} + i\frac{n_2}{1 - n_3},$$

is admissible, we consider

$$A_\Phi := \begin{pmatrix} \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{i}{2}(a^2 + b^2 - c^2 - d^2) & -ab + cd \\ \frac{i}{2}(-a^2 + b^2 - c^2 + d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) & i(ab + cd) \\ -ac + bd & -i(ac + bd) & ad + bc \end{pmatrix}$$

and define $f_\Phi, \tilde{f}_\Phi : V^* \to \mathbb{R}^3$ by

$$f_\Phi - i\tilde{f}_\Phi := A_\Phi(f - i\tilde{f}).$$

Then, f_Φ is A-minimal and $\tilde{f}_\Phi$ is C-minimal. They form a conjugate pair of minimal surfaces with Gauss map n_Φ.

Proof. Note $f_\Phi, \tilde{f}_\Phi$ are well defined on M^* even if M^* is not simply connected. We denote $z : V \to \mathbb{C}$ the stereographic projection of n. From the proof the Weierstrass representation (Theorem 3.15), there exists a holomorphic quadratic differential $q : E_{int} \to i\mathbb{R}$ such that

$$df(e^*_{ij}) - i d\tilde{f}(e^*_{ij}) = \frac{q_{ij}}{dz(e_{ij})} \begin{pmatrix} 1 - z_i z_j \\ i(1 + z_i z_j) \\ z_i + z_j \end{pmatrix}.$$

Then, by lemma 3.18

$$
\begin{aligned}
df_\Phi(e_{ij}^*) - i d\tilde{f}_\Phi(e_{ij}^*) =& \frac{q_{ij}}{dz(e_{ij})} A_\Phi \begin{pmatrix} 1 - z_i z_j \\ i(1 + z_i z_j) \\ z_i + z_j \end{pmatrix} \\
=& \frac{q_{ij}}{(\Phi(z_j) - \Phi(z_i))} \begin{pmatrix} 1 - \Phi(z_i)\Phi(z_j) \\ i(1 + \Phi(z_i)\Phi(z_j)) \\ \Phi(z_i) + \Phi(z_j) \end{pmatrix}.
\end{aligned}
$$

The proof of the Weierstrass representation (Theorem 3.15) yields that f_Φ is A-minimal and $\tilde{f}_\Phi$ is C-minimal. □

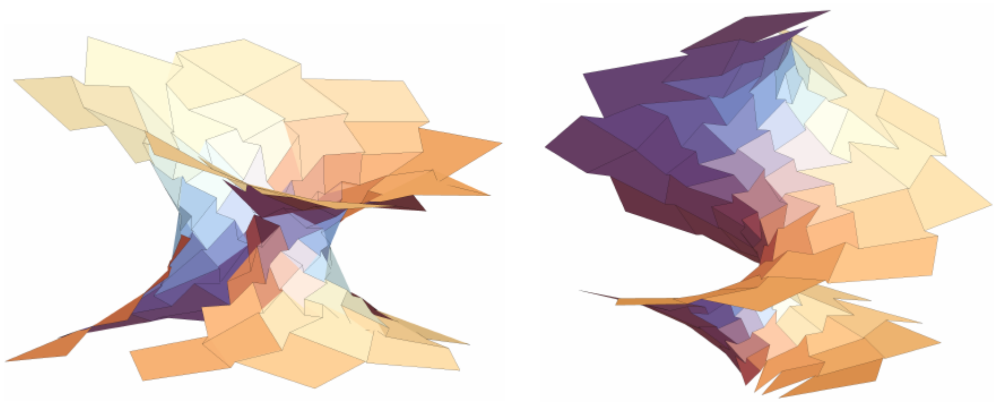

FIGURE 3.4. A conjugate pair of discrete minimal surfaces obtained via a Goursat transform (a complex rotation) of the conjugate pair in Figure 3.3.

3.3. Trivalent minimal surfaces

In this section, we argue that by vertex splitting every discrete minimal surface can be regarded as trivalent with its Gauss map triangulated. Note that our discrete minimal surfaces are allowed to have degenerate edges. With this viewpoint, each discrete minimal surface corresponds to a discrete harmonic function on a planar triangular mesh, or alternatively a self-stress on its Gauss map.

Lemma 3.20. *Suppose a triangulated surface $\hat{M} = (V, \hat{E}, \hat{F})$ is a subdivision of $M = (V, E, F)$ by adding diagonals and a trivalent surface $\hat{M}^* = (\hat{V}^*, \hat{E}^*, F^*)$ is obtained by splitting the vertices of $M^* = (V^*, E^*, F^*)$ correspondingly, which induce a map $\phi : \hat{V}^* \to V^*$. Given any conjugate pair of minimal surfaces $f, \tilde{f} : V^* \to \mathbb{R}^3$ with Gauss map $n : V \to \mathbb{S}^2$, we assume $\hat{n} := n$ yields a non-degenerate admissible realization of $\hat{M}$. Then*

(1) *$\hat{f} := f \circ \phi$ is A-minimal with Gauss map $\hat{n}$ and*
(2) *$\hat{\tilde{f}} := \tilde{f} \circ \phi$ is C-minimal with Gauss map $\hat{n}$.*

In particular, the holomorphic quadratic differential $q : E_{int} \to \mathbb{R}$ of f is extended to $\hat{q} : \hat{E}_{int} \to \mathbb{R}$ by zeros.

3.3.1. Discrete harmonic functions.

In the study of planar triangular meshes (Chapter 2), infinitesimal conformal deformations, holomorphic quadratic differentials and discrete harmonic functions in the sense of the cotangent Laplacian were related.

Combining the Weierstrass representation (Theorem 3.15), Lemma 3.20 and Theorem 2.14, we obtain the following for discrete surfaces whose cell decompositions are *arbitrary*.

Corollary 3.21. *Every simply connected A-minimal and C-minimal surface is given by a discrete harmonic function on a planar triangular mesh.*

As an example we take a triangulated square in the (x, y)-plane. Figure 3.1 (left) shows its stereographic image on the unit sphere. Notice that for a realization of a triangulated disk D in the plane such that neighboring triangular faces do not overlap, a discrete harmonic function u is uniquely determined by its boundary values (Corollary 1.8). The choice $u|_{\partial D} = xy$ results in an A-minimal surface (Figure 3.1 right) which looks like the Enneper surface.

As a second example, we consider a triangulated annulus centered at the origin with a cut along the positive x-axis. We solve for the discrete harmonic function u with boundary values given by $u|_{\partial M} = \arg z$. We obtain a conjugate pair of discrete minimal surfaces, which are seemingly the catenoid and the helicoid (Figure 3.3).

Finally, we consider a discrete harmonic function with boundary values $u|_{\partial M} = x^3 - 3xy^2$ on a triangulated disk. The corresponding A-minimal is shown in Figure 3.5, where an umbilic appears in the middle of the figure.

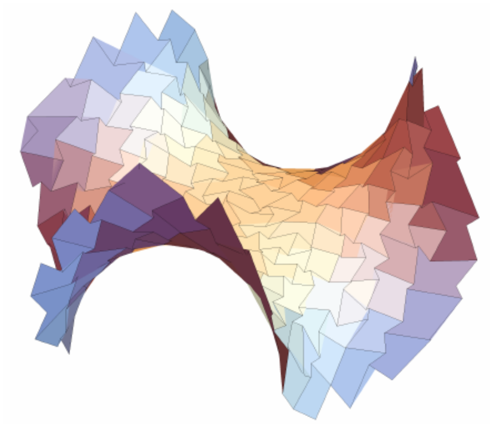

Figure 3.5. An A-minimal surface with an umbilic point.

3.3.2. Self-stresses. The Weierstrass representation asserts that a smooth minimal surface is locally determined by its Gauss map together with a holomorphic quadratic differential. A holomorphic quadratic differential in this case can be interpreted as a static stress [**68**].

In this section, we provide a discrete version of the above statement: each discrete minimal surface corresponds to a *self-stress* on its Gauss map. It is closely related to Maxwell theorem asserting that a self-stress, i.e. an assignment of forces along the edges of a realization balanced at vertices, corresponds to a *reciprocal parallel mesh* of the realization [**51**]. Here we focus on triangulated surfaces for simplicity.

The following is immediate from the definition of A-minimal surfaces.

Theorem 3.22. *Suppose we have a non-degenerate admissible realization $n : V \to \mathbb{S}^2$ of a simply connected triangulated surface M. Given a function $k : E_{int} \to \mathbb{R}$, the following are equivalent:*

(1) *There exists an A-minimal surface $f : V^* \to \mathbb{R}^3$ with Gauss map n satisfying for every interior edge $\{ij\}$*
$$df(e^*_{ij}) = k_{ij}(n_j - n_i).$$
(2) *The assignment of $k_{ij}(n_j - n_i)$ to each oriented edge e_{ij} defines equal and opposite forces to the two endpoints along every edge that is in equilibrium at every vertex, i.e. for every interior vertex i*
$$\sum_j k_{ij}(n_j - n_i) = 0.$$

There is a similar correspondence between a C-minimal surface and the polar of its Gauss map. Given a point $\hat{x} \in \mathbb{R}^3$ its *polar plane* with respect to the unit sphere is defined as
$$\bar{x} := \{y \in \mathbb{R}^3 | \langle y, \hat{x} \rangle = 1\}$$

and $\hat{x}$ is the *pole* of $\bar{x}$. If $n : V \to \mathbb{S}^2$ is a non-degenerate admissible realization of a triangulated surface in such a way that neighboring faces are not coplanar, then the pole of each face determines a non-degenerate realization $\hat{n} : V^* \to \mathbb{R}^3$ of M^* with planar faces tangent to the unit sphere and with face normal n. In particular the image of each dual edge $\{ij\}^*$ under $\hat{n}$ is parallel to $n_i \times n_j$.

Lemma 3.23. *Let $n, n_1 \dots n_r \in \mathbb{S}^2$ and $k_1, k_2 \dots k_r \in \mathbb{R}$. Then*

$$\sum_{j=1}^{r} k_j(n_j - n) = 0 \quad \Longleftrightarrow \begin{cases} \sum_{j=1}^{r} k_j n \times n_j = 0 \\ \sum_{j=1}^{r} k_j r_j \times (n \times n_j) = 0 \end{cases}$$

where $r_j := (n+n_j)/(1+\langle n, n_j\rangle)$ is a point on the line $\{x \in \mathbb{R}^3 | \langle x, n\rangle = 1\} \cap \{x \in \mathbb{R}^3 | \langle x, n_j\rangle = 1\}$.

Proof. It follows from the identities that

$$\sum_{j=1}^{r} k_j n \times n_j = n \times \sum_{j=1}^{r} k_j(n_j - n)$$

and

$$\begin{aligned}\sum_{j=1}^{r} k_j r_j \times (n \times n_j) &= \sum_{j=1}^{r} k_j \frac{-(n_j - \langle n, n_j\rangle n) + (n - \langle n, n_j\rangle n_j)}{(1 + \langle n, n_j\rangle)} \\ &= \sum_{j=1}^{r} k_j(n - n_j). \end{aligned}$$

□

Combining Theorem 3.10, Theorem 3.22 and Lemma 3.23, we have

Theorem 3.24. *Suppose we have a non-degenerate realization $n : V \to \mathbb{S}^2$ of a simply connected triangulated surface M and the polar $\hat{n} : V^* \to \mathbb{R}^3$ of n with respect to the unit sphere is non-degenerate. Given $k : E_{int} \to \mathbb{R}$, the following are equivalent:*

(1) *There exists a C-minimal surface $\tilde{f} : V^* \to \mathbb{R}^3$ with Gauss map n such that for every interior edge $\{ij\}$*
$$df(e^*_{ij}) = k_{ij} n_i \times n_j.$$
(2) *The assignment of $k_{ij} n_i \times n_j$ to each oriented edge e^*_{ij} defines equal and opposite forces to the neighboring faces along each edge of $\hat{n}$ that is in equilibrium on every face, i.e. for every dual face $i^* \in F^*_{int}$*
$$\sum_j k_{ij} n_i \times n_j = 0 \qquad \text{(forces balanced)},$$
$$\sum_j k_{ij} r_{ij} \times (n_i \times n_j) = 0 \qquad \text{(torques balanced)}$$
where $r_{ij} := (n + n_j)/(1 + \langle n, n_j\rangle)$ is a point on the image of $\{ij\}^*$ under $\hat{n}$.

3.4. Critical points of the total area

In this section, we show that among the discrete minimal surfaces that we have discussed, there is a subclass of them which possess the variational property analogous to smooth minimal surfaces.

3.4.1. Area of non-planar faces. Discrete minimal surfaces in the associated families do not have planar faces in general. In order to define area on a non-planar face, we consider its vector area. This idea has been applied in computer graphics [**2**].

Given a polygon $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$ in $\mathbb{R}^3$, its *vector area* is defined by

$$\vec{A}_\gamma = \frac{1}{2} \sum_{i=0}^{n-1} \gamma_i \times \gamma_{i+1}.$$

The vector area is invariant under translations of the polygon. The magnitude $|\vec{A}_\gamma|$ is the largest signed area over all orthogonal projections of γ to planes in space. The direction of $\vec{A}_\gamma$ indicate the normal of the plane with largest signed area. If γ is embedded in a plane, the

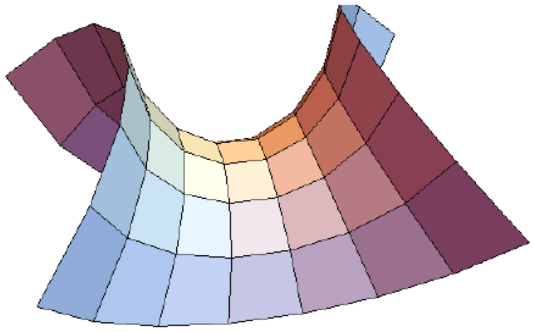

FIGURE 3.6. A C-minimal quad mesh which is a critical point of the area functional. Subdivision by adding diagonals yields a triangulated minimal surface in the sense of Pinkall and Polthier [**56**].

magnitude $|\vec{A}_\gamma|$ coincides with the usual notion of area and the direction of $\vec{A}$ is normal to the plane.

However, there is still an ambiguity to define the (signed) area A_γ of a non-planar polygon γ using the vector area, either $A_\gamma := |\vec{A}_\gamma|$ or $-|\vec{A}_\gamma|$. Such ambiguity will be fixed using the Gauss map of a discrete minimal surface later. Assuming we have picked a sign for the area of each face, we derive the gradient of the total area of a discrete surface.

DEFINITION 3.25. Suppose $f : V \to \mathbb{R}^3$ is a realization of a compact discrete surface $M = (V, E, F)$ with boundary. Let $\sigma : F \to \pm 1$ be a choice of signs. We have the vector area $\vec{A} : F \to \mathbb{R}^3$ given by

$$\vec{A}_\phi = \frac{1}{2} \sum_{i=0}^{n-1} f_i \times f_{i+1}$$

where $(v_0, v_1, \ldots, v_{n-1}) = \phi \in F$ and the ordering is determined by the orientation of the face. The *total area* is

$$\mathrm{Area}_\sigma(f) = \sum_{\phi \in F} \sigma_\phi |\vec{A}_\phi|.$$

and if $\vec{A} \neq 0$ we define the mean curvature vector field $\vec{H}_\sigma : V_{int} \to \mathbb{R}^3$ via

$$\vec{H}_{\sigma,i} := \frac{1}{2} \sum_j dn_\sigma(e^*_{ij}) \times df(e_{ij}) \quad \forall\, i \in V_{int}$$

where $n_\sigma : F \to \mathbb{S}^2$ is given by $n_\sigma := \sigma \vec{A}/|\vec{A}|$.

If f_t is a family of realizations with $f_0 = f$ and $\vec{A}_t$ is nonzero on any face, then the total area $\mathrm{Area}_\sigma(f_t)$ depends smoothly on t.

Theorem 3.26. *Suppose $f : V \to \mathbb{R}^3$ is a realization of a compact discrete surface $M = (V, E, F)$ with boundary and with non-vanishing vector area $\vec{A} : F \to \mathbb{R}^3 \backslash \{0\}$. Let $\sigma : F \to \pm 1$ be a choice of signs. Then the mean curvature vector field $\vec{H}_\sigma$ vanishes identically if and only if f is a critical point of the total area* $\mathrm{Area}_\sigma(f)$ *under infinitesimal deformation with the boundary fixed.*

PROOF. Suppose $\dot{f} : V \to \mathbb{R}^3$ is an infinitesimal deformation of f fixing the boundary. We define $n_\sigma : F \to \mathbb{S}^2$ by $n_\sigma := \sigma \vec{A}/|\vec{A}|$. We consider a face $(v_0, v_1, \ldots, v_{n-1}) = \phi \in F$ and write $f_i = f(v_i)$.

Since $\langle n_{\sigma,\phi}, \dot{n}_{\sigma,\phi} \rangle = 0$ we have

$$\langle \vec{A}_\phi, \dot{n}_{\sigma,\phi} \rangle = 0.$$

On the other hand,

$$\begin{aligned}\langle \dot{\vec{A}}_\phi, n_{\sigma,\phi}\rangle &= \frac{1}{2}\sum_{i=0}^{n-1}\langle f_{i-1}\times \dot{f}_i + \dot{f}_i \times f_{i+1}, n_{\sigma,\phi}\rangle \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\langle (f_{i+1}-f_i+f_i-f_{i-1})\times n_{\sigma,\phi}, \dot{f}_i\rangle \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\langle (f_{i+1}-f_i)\times n_{\sigma,\phi} + (f_{i-1}-f_i)\times(-n_{\sigma,\phi}), \dot{f}_i\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{\text{Area}}_\sigma(f) &= \sum_{\phi\in F}\langle \dot{\vec{A}}_\phi, n_{\sigma,\phi}\rangle + \langle \vec{A}_\phi, \dot{n}_{\sigma,\phi}\rangle \\ &= \frac{1}{2}\sum_{i\in V_{int}}\langle \sum_j dn_\sigma(e^*_{ij})\times df(e_{ij}), \dot{f}_i\rangle \\ &= \sum_{i\in V_{int}}\langle \vec{H}_{\sigma,i}, \dot{f}_i\rangle.\end{aligned}$$

It implies f is a critical point of the area functional $\text{Area}_\sigma(f)$ under infinitesimal deformations fixing the boundary if and only if its mean curvature vector field $\vec{H}_\sigma$ vanishes identically. $\square$

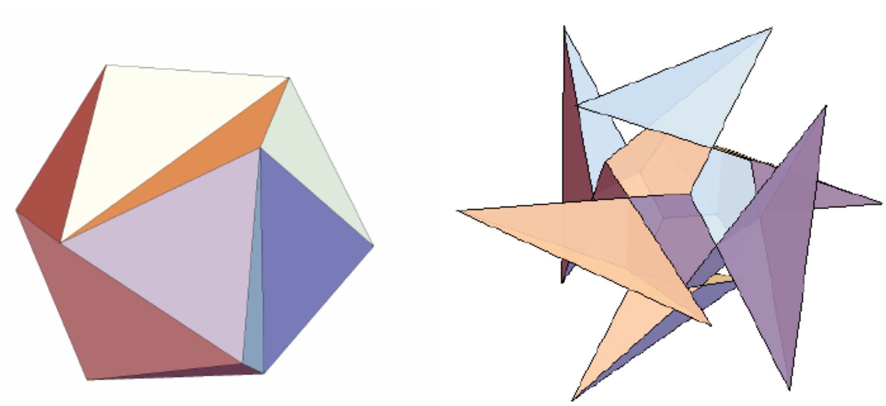

FIGURE 3.7. Jessen's orthogonal icosahedron (left) is known to be infinitesimally flexible. Its infinitesimal isometric deformation yields a *closed* C-minimal surface (right) with non-embedded planar faces. For each face of the discrete minimal surface, the vector area vanishes.

Before ending this section, we make a remark to the relation between mean curvature vector field and the cotangent formula by Pinkall and Polthier [**55**].

Corollary 3.27. *Suppose $f: V\to\mathbb{R}^3$ is a realization of a triangulated surface such that each face spans an affine plane. Then*

$$\sum_j dn(e^*_{ij})\times df(e_{ij}) = \sum_j (\cot\angle jki + \cot\angle ilj) df(e_{ij})$$

where $\{ijk\}$ and $\{jli\}$ are two neighboring faces containing the edge $\{ij\}$ and $n := \vec{A}/|\vec{A}|$ is the face normal field given by the orientation of the triangulated surface.

Hence, a realization $f: V\to\mathbb{R}^3$ of a compact triangulated surface is a critical point of the area functional $\text{Area}_\sigma(f)$, where $\sigma\equiv 1$, under infinitesimal deformations fixing the boundary if and only if for every interior vertex i

$$\sum_j (\cot\angle jki + \cot\angle ilj)(f_j - f_i) = 0.$$

PROOF. It follows from Theorem 3.26 and the identity that for every interior vertex i

$$\begin{aligned}\sum_j dn(e^*_{ij}) \times df(e_{ij}) &= \sum_{ijk} n_{ijk} \times (df(e_{ij}) - df(e_{ik})) \\ &= \sum_{ijk} n_{ijk} \times df(e_{kj}) \\ &= \sum_{ijk} \cot \angle jki \, df(e_{ij}) - \cot \angle ijk \, df(e_{ki}) \\ &= \sum_j (\cot \angle jki + \cot \angle ilj) df(e_{ij}). \end{aligned}$$

□

3.4.2. Quadrivalent meshes with the parallelogram property. In this section we focus on a special type of realizations, *P-nets*, introduced by Bobenko and Pinkall [**8**].

DEFINITION 3.28. A cell decomposition $D = (V, E, F)$ of a simply connected surface is a *P-graph* if D satisfies the followings:

(1) every interior vertex has degree 4 and
(2) every face has even number of edges.

If D is a P-graph, there exists a labeling $\mu : E \to \pm 1$ such that at each vertex, two opposite edges are labeled "+1" and the other two opposite edges are labeled "-1".

DEFINITION 3.29. A non-degenerate realization $f : V \to \mathbb{R}^3$ of a P-graph is a *P-net* if it possesses the parallelogram property: Mapping any interior vertex $f_0 := f(v_0)$ to infinity by inversion, then the image of its four neighboring vertices f_1, f_2, f_3, f_4 form a parallelogram, i.e.

$$\frac{f_1 - f_0}{|f_1 - f_0|^2} - \frac{f_2 - f_0}{|f_2 - f_0|^2} + \frac{f_3 - f_0}{|f_3 - f_0|^2} - \frac{f_4 - f_0}{|f_4 - f_0|^2} = 0. \tag{3.16}$$

(See Figure 3.8 for the indices.)

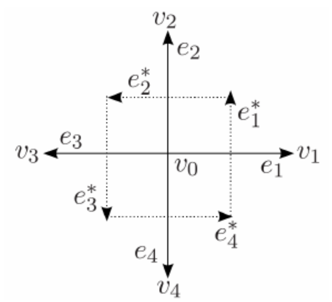

FIGURE 3.8. The edges of D around a vertex v_0 are indicated by solid lines. The edges of the dual face v_0^* are indicated by dotted lines.

EXAMPLE 3.30 (Half the vertices of a discrete isothermic net with cross ratios -1). We consider a discrete isothermic net $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ such that the cross ratio of each elementary quadrilateral is -1 (i.e. $a_m = -b_n$ in Example 3.5). It is shown in [**8**] the restriction of f to $\mathbb{Z}_b$ or $\mathbb{Z}_w$ is a P-net. Conversely, a P-net with an additional vertex can be uniquely extended to a discrete isothermic net with cross ratios -1. The proof is postponed to Corollary 5.3.

EXAMPLE 3.31 (Schramm's orthogonal circle patterns). Let D be a P-graph. Then for any orthogonal circle pattern in the complex plane with the combinatorics of D, i.e. each face of D corresponds to a circle and neighboring faces correspond to two circles that intersect orthogonally, the intersection points form a P-net [**8**] (See Figure 3.9). Indeed, if one maps any vertex to infinity by inversion, the neighboring four vertices form a rectangle.

Theorem 3.32 ([**8**]). *The parallelogram property is Möbius invariant. Hence any Möbius transform of a P-net is again a P-net.*

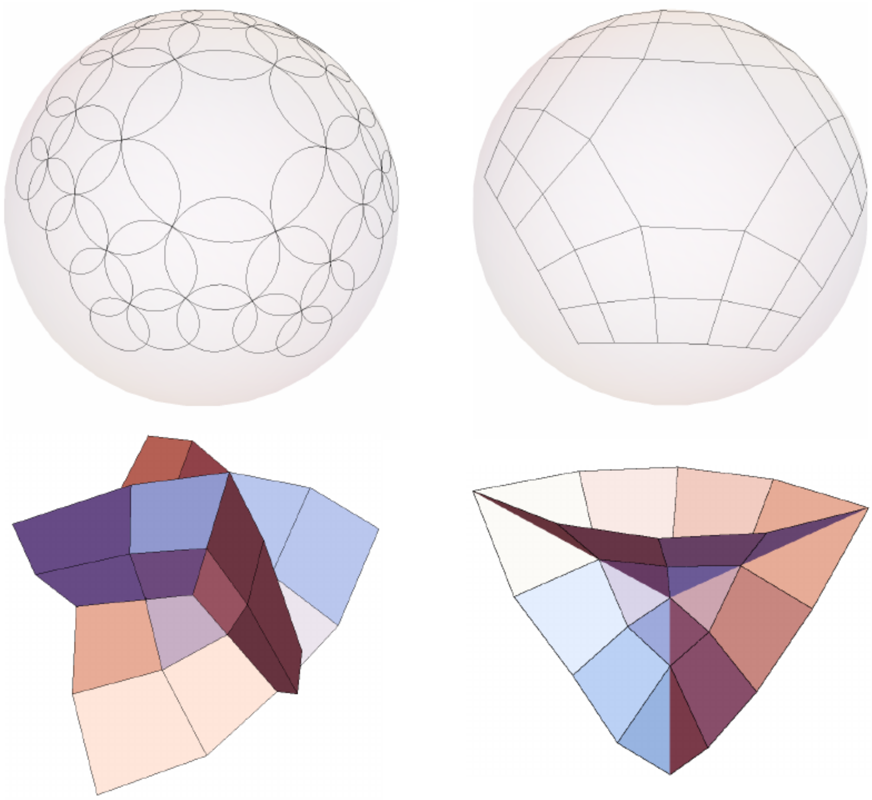

FIGURE 3.9. Top left: An orthogonal circle pattern on a sphere. Top right: A P-net formed by the intersection points of the circle pattern. The P-net yields an A-minimal surface (bottom left) and a C-minimal surface (bottom right), which are critical points of the total area.

If $F : V \to \mathbb{R}^3$ is a P-net with a labeling μ as in Definition 3.28, we define a dual 1-form $\eta : \vec{E}^*_{int} \to \mathbb{R}^3$ by

$$\eta(e^*_{ij}) := \mu_{ij} \frac{dF(e_{ij})}{|dF(e_{ij})|^2} \quad \forall \{ij\} \in E_{int}.$$

Equation (3.16) implies η is a closed dual 1-form since around any dual face $v^*_0 \in F^*_{int}$

$$\eta(e^*_1) + \eta(e^*_2) + \eta(e^*_3) + \eta(e^*_4) = 0.$$

Because D is simply connected, there exists a realization $F^* : V^* \to \mathbb{R}^3$ of D^* such that

$$dF^*(e^*_{ij}) = \eta(e^*_{ij}) = \mu_{ij} \frac{dF(e_{ij})}{|dF(e_{ij})|^2} \quad \forall \{ij\} \in E_{int}.$$

We call F^* the *Christoffel dual* of F.

3.4.3. Minimal surfaces from P-nets. Given a P-graph $D = (V, E, F)$ with a labeling μ as in Definition 3.28, we consider a P-net $n : V \to \mathbb{S}^2$ with vertices on the unit sphere. Then its Christoffel $f := n^* : V^* \to \mathbb{R}^3$ is A-minimal with Gauss map n:

$$df(e^*_{ij}) = \mu_{ij} \frac{n_j - n_i}{|n_j - n_i|^2} \tag{3.17}$$

and the C-minimal surface $\tilde{f} : V^* \to \mathbb{R}^3$ satisfies

$$d\tilde{f}(e^*_{ij}) = n_i \times \mu_{ij} \frac{n_j - n_i}{|n_j - n_i|^2} = n_j \times \mu_{ij} \frac{n_j - n_i}{|n_j - n_i|^2}. \tag{3.18}$$

REMARK 3.33. Following from Definition 3.29, a labeling μ of a P-net $n : V \to \mathbb{S}^2$ induces a discrete holomorphic quadratic differential μ on $\phi \circ n$, where ϕ is a stereographic projection.

We first consider vector area on faces. The following illustrates a discrete counterpart of the property that the area 2-form and the Gauss map of a smooth minimal surface is unchanged within the associated family.

Lemma 3.34. *Suppose D is a P-graph and $n : V \to \mathbb{S}^2$ is a P-net. We denote $f, \tilde{f} : V^* \to \mathbb{R}^3$ the corresponding conjugate minimal surfaces as given by* (3.17) *and* (3.18)*. We consider the vector area $\vec{A}^\theta : F^*_{int} \to \mathbb{R}^3$ of the discrete minimal surfaces $f^\theta := (\cos\theta) f + (\sin\theta)\tilde{f}$ in the associated family. Then $\vec{A}^\theta$ is independent of θ and parallel to n.*

Proof. We focus on a dual face $v_0^* \in F^*_{int}$, which corresponds to $v_0 \in V_{int}$ (see Figure 3.8), and decompose the neighboring edges into components

$$df(e_i^*) = df(e_i^*)^\perp + df(e_i^*)^\parallel \quad \forall\, i = 1,2,3,4$$

such that

$$df(e_i^*)^\perp \perp n_0, \quad df(e_i^*)^\parallel \parallel n_0.$$

In particular, from (3.17)

$$df(e_i^*)^\parallel = -\mu_i \frac{n_0}{2}$$

since

$$\langle df(e_i^*), n_0\rangle = \mu_i \frac{\langle n_i, n_0\rangle - 1}{|n_i - n_0|^2} = -\frac{\mu_i}{2}.$$

The C-minimal surface $\tilde{f}$ satisfies

$$d\tilde{f}(e_i^*) = n_0 \times df(e_i^*) = n_0 \times df(e_i^*)^\perp.$$

It yields

$$df^\theta(e_i^*) = df^\theta(e_i^*)^\perp + df^\theta(e_i^*)^\parallel = R_\theta(df(e_i^*)^\perp) - \mu_i \frac{\cos\theta}{2} n_0$$

where $R_\theta(v) := \cos\theta\, v^\perp + \sin\theta\, n_0 \times v^\perp$ for any $v^\perp \perp n_0$ is a rotation in the plane $n_0^\perp$. We calculate the vector area on the dual face v_0^*

$$\begin{aligned}
2\vec{A}_0^\theta =& df^\theta(e_1^*) \times df^\theta(e_2^*) + df^\theta(e_3^*) \times df^\theta(e_4^*) \\
=& df^\theta(e_1^*)^\perp \times df^\theta(e_2^*)^\perp + df^\theta(e_3^*)^\perp \times df^\theta(e_4^*)^\perp \\
&- \frac{\cos\theta}{2} n_0 \times (\mu_1 df^\theta(e_2^*)^\perp - \mu_2 df^\theta(e_1^*)^\perp + \mu_3 df^\theta(e_4^*)^\perp - \mu_4 df^\theta(e_3^*)^\perp) \\
=& df^\theta(e_1^*)^\perp \times df^\theta(e_2^*)^\perp + df^\theta(e_3^*)^\perp \times df^\theta(e_4^*)^\perp \\
&- \mu_1 \frac{\cos\theta}{2} n_0 \times (df^\theta(e_2^*) + df^\theta(e_1^*) + df^\theta(e_4^*) + df^\theta(e_3^*))^\perp \\
=& df^\theta(e_1^*)^\perp \times df^\theta(e_2^*)^\perp + df^\theta(e_3^*)^\perp \times df^\theta(e_4^*)^\perp \\
=& R_\theta(df(e_1^*)^\perp \times df(e_2^*)^\perp + df(e_3^*)^\perp \times df(e_4^*)^\perp) \\
=& df(e_1^*)^\perp \times df(e_2^*)^\perp + df(e_3^*)^\perp \times df(e_4^*)^\perp
\end{aligned}$$

which is independent of θ and parallel to n_0. Here we made use of the property of μ:

$$\mu_1 = -\mu_2 = \mu_3 = -\mu_4. \qquad \square$$

We show that discrete minimal surfaces in the associated families obtained from P-nets possess vanishing mean curvature vector fields.

Lemma 3.35.

$$\begin{aligned}
df^\theta(e_{ij}^*) \times dn(e_{ij}) &= -\mu_{ij} \sin\theta\, (n_i + n_j)/2 \\
\langle df^\theta(e_{ij}^*), dn(e_{ij})\rangle &= \mu_{ij} \cos\theta
\end{aligned}$$

Proof. We have by definition

$$df(e_{ij}^*) \times dn(e_{ij}) = 0$$

and

$$\begin{aligned}
&d\tilde{f}(e_{ij}^*) \times dn(e_{ij}) \\
=&\mu_{ij}\big((n_j \times \frac{n_j - n_i}{|n_j - n_i|^2}) \times n_j - (n_i \times \frac{n_j - n_i}{|n_j - n_i|^2}) \times n_i\big) \\
=&-\mu_{ij}\big(\frac{n_i - \langle n_i, n_j\rangle n_j}{|n_j - n_i|^2} + \frac{n_j - \langle n_i, n_j\rangle n_i}{|n_j - n_i|^2}\big) \\
=&\mu_{ij}\frac{\langle n_i, n_j\rangle - 1}{|n_j - n_i|^2}(n_i + n_j) \\
=&-\mu_{ij}(n_i + n_j)/2.
\end{aligned}$$

Since $f^\theta = \cos\theta\, f + \sin\theta\, \tilde{f}$ we get

$$df^\theta(e_{ij}^*) \times dn(e_{ij}) = -\mu_{ij}\sin\theta\,(n_i + n_j)/2.$$

On the other hand we have

$$\begin{aligned}
\langle df(e_{ij}^*), dn(e_{ij})\rangle &= \mu_{ij}, \\
\langle d\tilde{f}(e_{ij}^*), dn(e_{ij})\rangle &= 0.
\end{aligned}$$

Hence

$$\langle df^\theta(e_{ij}^*), dn(e_{ij})\rangle = \mu_{ij}\cos\theta.$$

□

Suppose D is a P-graph with boundary. We call the faces of D containing boundary edges as boundary faces. Those non-boundary faces are called interior, which form a set F_{int}. We denote the set of dual vertices which correspond to F_{int} as V_{int}^*. Applying Lemma 3.34 and 3.35, we have the following.

Theorem 3.36. *Given a P-graph $D = (V, E, F)$ and a P-net $n : V \to \mathbb{S}^2$, we assume the discrete minimal surfaces $f^\theta : V^* \to \mathbb{R}^3$ in the corresponding associated family have non-vanishing vector area $\vec{A} : F_{int}^* \to \mathbb{R}^3\backslash\{0\}$. Let $\sigma : F_{int}^* \to \pm 1$ be defined by*

$$\sigma := \langle n, \vec{A}/|\vec{A}|\rangle.$$

Then the mean curvature vector field $\vec{H}_\sigma^\theta : V_{int}^* \to \mathbb{R}^3$ vanishes identically, i.e. for each dual vertex $\phi^* = (v_1, v_2, \ldots, v_n)^* \in V_{int}^*$

$$\vec{H}_{\phi^*}^\theta := \frac{1}{2}\sum_i dn(e_{i,i+1}) \times df^\theta(e_{i,i+1}^*) = 0 \tag{3.19}$$

and furthermore

$$\sum_i \langle dn(e_{i,i+1}), df^\theta(e_{i,i+1}^*)\rangle = 0. \tag{3.20}$$

In particular, if D is compact and $\vec{A}$ is nowhere vanishing, then f^θ is a critical point of the area functional $\sum \sigma|\vec{A}|$ under infinitesimal deformations with boundary fixed.

Proof. Note the function $n_\sigma : F_{int}^* \to \mathbb{S}^2$ defined by

$$n_\sigma := \sigma\vec{A}/|\vec{A}| = n.$$

Then for each dual vertex $\phi^* = (v_1, v_2, \ldots, v_n)^* \in V_{int}^*$

$$\vec{H}_{\phi^*}^\theta = \frac{1}{2}\sum_i dn(e_{i,i+1}) \times df^\theta(e_{i,i+1}^*) = \frac{\sin\theta}{4}\sum_i(\mu_{i-1,i} + \mu_{i,i+1})n_i = 0$$

and

$$\sum_i \langle dn(e_{i,i+1}), df^\theta(e_{i,i+1}^*)\rangle = \cos\theta\sum_i \mu_{i,i+1} = 0$$

since the number of edges in a polygon ϕ is even. □

Remark 3.37. The summations in equations (3.19) and (3.20) are taken around a vertex of f^θ while those in equations (3.8) and (3.9) are taken around a face. See Remark 3.13 for their analogue in the smooth theory.

REMARK 3.38. Suppose $\tilde{f}$ is a discrete surface with planar faces. Subdivision of it by adding diagonals does not change the mean curvature vectors at vertices since $dn \equiv 0$ on diagonals. If $\tilde{f}$ is a C-minimal surface obtained from a P-net, then a subdivision of it into a triangulated surface by adding diagonals satisfies the cotangent formula (Corollary 3.27). However, the signs of the angles in the cotangent formula depend on the choice of sign σ for the area.

CHAPTER 4

Isothermic triangulated surfaces

Isothermic surfaces include all surfaces of revolution, quadrics, constant mean curvature surfaces and many other interesting surfaces [**30**]. In particular, all classes of surfaces that are describable in terms of integrable systems in some way or other seem to be related to isothermic surfaces [**18, 19**].

A smooth surface in Euclidean space is isothermic if it admits conformal curvature line parametrization around every point. Note however that there are various characterizations of isothermic surfaces that do not refer to special parametrizations.

From the viewpoint of discrete differential geometry, there are two different definitions of conformality for planar triangular meshes. One of them is the theory of *circle patterns* [**65**], where the conformal structure is defined by the intersection angles of neighboring circumcircles. It is motivated by Thurston's circle packings as a discrete analogue of holomorphic functions [**60**]. Another version of discrete conformality is based on *conformal equivalence of triangle meshes* [**48, 69**], where conformal structure is defined by the length cross ratios of neighboring triangles. Luo introduced this notion when studying a discrete Yamabe flow. Its relation to ideal hyperbolic polyhedra was investigated in [**9**].

Previous definitions of discrete isothermic surfaces were all based on quadrilateral meshes that provide a discrete version of conformal curvature line parametrizations of isothermic surface [**4, 8, 12**]. Inspired by discrete integrable systems [**14**], Bobenko and Pinkall [**4**] considered quadrilateral meshes with factorized real cross ratios, which led to further investigation of discrete minimal surfaces and constant mean curvature surfaces [**31**]. Recently, the notion of curvature was introduced to discrete surfaces with vertex normals [**10, 32**].

Here we aim for a definition of isothermic triangulated surfaces which does not involve conformal curvature line parametrizations. It is motivated by a known (although not well-known) characterization that a smooth surface in Euclidean space is isothermic if and only if locally it admits a nontrivial infinitesimal isometric deformation preserving the mean curvature. The only reference that we could find is from Cieśliński et al. [**19**], stating that this theorem was known in the 19th century.

Infinitesimal isometric deformations of triangulated surfaces have been extensively studied since Cauchy's rigidity theorem of convex polyhedral surfaces [**73, 20**]. An *infinitesimal deformation* of a triangulated surface in space is an assignment of velocity vectors to all the vertices. We can then calculate the change of edge lengths. An infinitesimal deformation is *isometric* if all the edge lengths are preserved.

Suppose we have a realization $f : V \to \mathbb{R}^3$ of a triangulated surface $M = (V, E, F)$ such that each face of f spans an affine plane. Given an infinitesimal isometric deformation $\dot{f} : V \to \mathbb{R}^3$, each triangular face $\{ijk\}$ rotates with an angular velocity given by a certain vector $Z_{ijk} \in \mathbb{R}^3$. These vectors satisfy a compatibility condition on every interior edge $\{ij\}$:

$$d\dot{f}(e_{ij}) = df(e_{ij}) \times Z_{ijk} = df(e_{ij}) \times Z_{jil}, \tag{4.1}$$

where $\{ijk\} \in F$ is the left face of e_{ij} and $\{jil\} \in F$ is the right face.

On the other hand, it is well-known that the integral $\int H\,dA$ of the mean curvature has a very canonical discrete analogue $\sum H_{ij}$. Here we define the *mean curvature* associated to edge $\{ij\}$ as

$$H_{ij} := \alpha_{ij} |df(e_{ij})|$$

where α_{ij} is the dihedral angle at the edge $\{ij\}$ [**70**]. Under the infinitesimal isometric deformation given by Z on faces (Equation (4.1)), we have

$$\dot{H}_{ij} = \dot{\alpha}_{ij} |df(e_{ij})| = \langle df(e_{ij}), Z_{ijk} - Z_{jil} \rangle.$$

If we further demanded $\dot{H}_{ij} = 0$ on every edge $\{ij\}$ then the infinitesimal isometric deformation would be trivial, i.e. an infinitesimal Euclidean deformation. Hence we consider instead the change of the *integrated mean curvature around vertices*

$$\dot{H}_i := \sum_j \dot{\alpha}_{ij} |df(e_{ij})| = \sum_j \langle df(e_{ij}), Z_{ijk} - Z_{jil} \rangle.$$

We are now ready to define isothermic triangulated surfaces. The smooth counterpart of the following formulation for isothermic surfaces is given by Smyth [**68**].

Definition 4.1. A non-degenerate realization $f : V \to \mathbb{R}^3$ of an oriented triangulated surface, with or without boundary, is *isothermic* if there exists a $\mathbb{R}^3$-valued dual 1-form $\tau : \vec{E}^*_{int} \to \mathbb{R}^3$, not identically zero, such that

$$\sum_j \tau(e^*_{ij}) = 0 \quad \forall i \in V_{int}, \tag{4.2}$$

$$df(e_{ij}) \times \tau(e^*_{ij}) = 0 \quad \forall \{ij\} \in E_{int}, \tag{4.3}$$

$$\sum_j \langle df(e_{ij}), \tau(e^*_{ij}) \rangle = 0 \quad \forall i \in V_{int}. \tag{4.4}$$

Here $\vec{E}^*_{int}$ and V_{int} denote the set of interior oriented dual edges and the set of interior vertices of M.

The following is an immediate consequence of our definition.

Corollary 4.2. *A strongly non-degenerate realization of a simply connected triangulated surface is isothermic if and only if there exists an infinitesimal isometric deformation that preserves the integrated mean curvature around vertices but is not induced from Euclidean transformations.*

We state several results about isothermic triangulated surfaces that closely reflect known theorems from the smooth theory. In Section 4.1, 4.2 and 4.3, we prove

Theorem 4.3. *The class of isothermic triangulated surfaces is Möbius invariant.*

Theorem 4.4. *For a non-degenerate realization $f : V \to \mathbb{R}^3$ of a closed genus-g triangulated surface the space of infinitesimal conformal deformations is of dimension greater or equal to $|V| - 6g + 6$. The inequality is strict if and only if f is isothermic.*

Theorem 4.5. *Suppose $f : V \to \mathbb{R}^3$ is a non-degenerate realization of a simply connected triangulated surface. Then f is isothermic if and only if there exists an infinitesimal deformation that preserves the intersection angles of neighboring circumcircles and neighboring circumspheres but is not induced from Möbius transformations.*

Note that Theorem 4.4 concerns the theory of conformal equivalence of triangle meshes [**48, 69**] while Theorem 4.5 deals with the notion of circle patterns [**65**].

In Section 4.4 we show that our definition generalizes isothermic quadrilateral surfaces [**4**]: Subdividing any isothermic quadrilateral surface in an arbitrary way we obtain an isothermic triangulated surface.

In Sections 4.5, 4.6 and 4.7 we provide examples of isothermic triangulated surfaces that are not obtained via quadrilateral isothermic surfaces. Triangulated cylinders generated by discrete groups as well as certain planar triangular meshes and triangulated surfaces inscribed in a sphere are isothermic.

In Section 4.8 we introduce Christoffel duality for isothermic triangulated surfaces. We then see that A-minimal surfaces, as introduced in Chapter 3, are Christoffel duals of triangulated surfaces (with boundary) inscribed in the unit sphere. This approach is analogous to the smooth theory that a minimal surface is a Christoffel dual of its Gauß map.

In Section 4.9, we review the smooth theory and prove new theorems that are similar to discrete results established in earlier sections.

This chapter is based on [**46**]. Throughout we use the language of discrete differential forms and quaternionic analysis as introduced by Desbrun et al. [**21**] and Pedit and Pinkall [**54**].

4.1. Möbius invariance

In this section we prove that the class of isothermic triangulated surfaces is invariant under Möbius transformations.

Given a triangulated surface $f : V \to \mathbb{R}^3$ and a Möbius transformation $\sigma : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$, we define $\sigma \circ f : V \to \mathbb{R}^3$ as the triangulated surface with vertices $(\sigma \circ f)_i := \sigma \circ f_i$. We consider only the Möbius transformations that do not map any vertex to infinity.

Taking σ to be minus the inversion in the unit sphere, we obtain a triangulated surface

$$\sigma \circ f = -\frac{f}{||f||^2} =: f^{-1}.$$

Later we identify $\mathbb{R}^3$ with imaginary quaternions, which explains the notation f^{-1}. We are going to show that f is isothermic if and only if f^{-1} is isothermic. We first relate isothermic triangulated surfaces to discrete holomorphic quadratic differentials.

Lemma 4.6. *A discrete surface $f : V \to \mathbb{R}^3$ is isothermic if and only if there exists $q : E_{int} \to \mathbb{R}$ such that for every interior vertex i*

$$\sum_j q_{ij} = 0,$$
$$\sum_j q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2} = 0.$$

We call q a discrete holomorphic quadratic differential associated to f.

Proof. It follows from a simple substitution into Definition 4.1

$$\tau(e^*_{ij}) := q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}.$$ □

Lemma 4.7. *Suppose a non-degenerate realization $f : V \to \mathbb{R}^3$ of a triangulated surface is isothermic with an associated holomorphic quadratic differential q. Then, the triangulated surface $f^{-1} : V \to \mathbb{R}^3$ is isothermic and q is again an holomorphic quadratic differential associated to f^{-1}.*

Proof. Notice that

$$|f_j^{-1} - f_i^{-1}|^2 = |f_j - f_i|^2/(|f_j|^2|f_i|^2)$$

We have

$$\begin{aligned} q_{ij} \frac{f_j^{-1} - f_i^{-1}}{|f_j^{-1} - f_i^{-1}|^2} &= q_{ij} \frac{|f_j|^2 f_i - |f_i|^2 f_j}{|f_j - f_i|^2} \\ &= q_{ij} \frac{|f_j - f_i|^2 f_i + 2\langle f_j - f_i, f_i\rangle f_i - |f_i|^2(f_j - f_i)}{|f_j - f_i|^2} \end{aligned}$$

Hence

$$\begin{aligned} \sum_j q_{ij} \frac{f_j^{-1} - f_i^{-1}}{|f_j^{-1} - f_i^{-1}|^2} &= f_i \sum_j q_{ij} + 2\langle \sum_j q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}, f_i\rangle f_i - |f_i|^2 \sum_j q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2} \\ &= 0 \end{aligned}$$ □

Proof of Theorem 4.3. It follows from the previous lemma and the fact that Möbius transformations are generated by inversions and Euclidean transformations. □

We give another formulation for isothermic triangulated surfaces, with which one can similarly discuss isothermic triangulated surfaces in various geometries, such as Laguerre geometry.

Lemma 4.8. *Given a non-degenerate realization $f : V \to \mathbb{R}^3$ of a triangulated surface, an $\mathbb{R}^3$-valued dual 1-form $\tau : \vec{E}^*_{int} \to \mathbb{R}^3$ satisfies*

$$\begin{aligned} \sum_j \tau(e^*_{ij}) = 0 \quad \forall i \in V_{int}, \\ df(e_{ij}) \times \tau(e^*_{ij}) = 0 \quad \forall \{ij\} \in E_{int}, \\ \sum_j \langle df(e_{ij}), \tau(e^*_{ij})\rangle = 0 \quad \forall i \in V_{int} \end{aligned}$$

if and only if there exists $k : E_{int} \to \mathbb{R}$ such that

$$\begin{aligned} k_{ij} df(e_{ij}) = \tau(e^*_{ij}) \quad \forall \{ij\} \in E_{int}, \\ \sum_j k_{ij} df(e_{ij}) = 0 \quad \forall i \in V_{int}, \\ \sum_j k_{ij}(|f_j|^2 - |f_i|^2) = 0 \quad \forall i \in V_{int}. \end{aligned}$$

Proof. Suppose $k : E_{int} \to \mathbb{R}$ satisfies for every interior vertex i

$$\sum_j k_{ij} df(e_{ij}) = 0.$$

Then, we have the identity

$$\sum_j \langle df(e_{ij}), k_{ij} df(e_{ij})\rangle = \sum_j k_{ij}(|f_j|^2 - |f_i|^2 - 2\langle f_j - f_i, f_i\rangle) = \sum_j k_{ij}(|f_j|^2 - |f_i|^2).$$

With it our claims can be verified. □

We consider the light cone

$$L := \{x \in \mathbb{R}^5 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = 0\}.$$

Corollary 4.9. *Suppose $f : V \to \mathbb{R}^3$ is a non-degenerate realization of a triangulated surface and $k : E_{int} \to \mathbb{R}$ is a function. Then f is isothermic with corresponding dual 1-form τ defined by*

$$\tau(e^*_{ij}) = k_{ij} df(e_{ij}) \quad \forall \{ij\} \in E_{int}$$

if and only if for every interior vertex i

$$\sum_j k_{ij} d\hat{f}(e_{ij}) = 0 \quad \forall i \in V_{int} \tag{4.5}$$

where $\hat{f} : V \to L \subset \mathbb{R}^5$ is the lift of f to $\mathbb{R}^5$ defined by

$$\hat{f}_i := (f_i, \frac{1 - |f_i|^2}{2}, \frac{1 + |f_i|^2}{2}) \in L \subset \mathbb{R}^5.$$

A function $k : E_{int} \to \mathbb{R}$ satisfying Equation (4.5) is a *self-stress* of $\hat{f}$.

It is known that the Möbius geometry of $\mathbb{R}^3 \cup \{\infty\}$ is a subgeometry of the projective geometry of $\mathbb{R}P^4$. Möbius transformations of $\mathbb{R}^3 \cup \{\infty\}$ are represented as projective transformations of $\mathbb{R}P^4$ preserving the quadric defined by the light cone L. If two non-degenerate realizations are related by a projective transformation, then the spaces of self-stresses of the two realizations are isomorphic [**35**]. Hence, we obtain another proof of Theorem 4.3.

4.2. Infinitesimal conformal deformations

We consider infinitesimal conformal deformations for a given closed triangulated surface in space. We show that a surface is isothermic if and only if it is a singular point in the space of all surfaces conformally equivalent to the original one.

4.2.1. Conformal equivalence of triangle meshes. We recall that a *discrete metric* of a triangulated surface is a function $\ell : E \to \mathbb{R}_+$ satisfying the triangle inequality on every face. A non-degenerate realization $f : V \to \mathbb{R}^3$ induces a discrete metric $\ell : E \to \mathbb{R}_+$ via

$$\ell_{ij} := |f_j - f_i| \quad \forall \{ij\} \in E.$$

As discussed in Theorem 2.2, we call two discrete metrics $\ell, \tilde{\ell} : E \to \mathbb{R}_+$ on a triangulated surface M conformally equivalent if there exists $u : V \to \mathbb{R}$ such that for every edge $\{ij\}$

$$\tilde{\ell}_{ij} = e^{\frac{u_i+u_j}{2}} \ell_{ij}.$$

Furthermore, we say two non-degenerate realizations $f, \tilde{f} : V \to \mathbb{R}^3$ are conformally equivalent if their induced discrete metrics are conformally equivalent.

It leads naturally to an infinitesimal version of conformal deformations. An infinitesimal deformation of a non-degenerate triangulated surface $f : V \to \mathbb{R}^3$ is a map $\dot{f} : V \to \mathbb{R}^3$. It is *conformal* if there exists $u : V \to \mathbb{R}$ such that the change of the induced discrete metric $\dot{\ell} : E \to \mathbb{R}$ satisfies for every edge $\{ij\}$

$$\dot{\ell}_{ij} = \frac{u_i + u_j}{2} \ell_{ij}.$$

In particular, $\dot{f}$ is an infinitesimal isometric deformation if $u \equiv 0$.

The conformal equivalence class of a triangulated surface in Euclidean space is Möbius invariant [**9**]. It can be distinguished via logarithmic length cross ratios.

DEFINITION 4.10. Given a discrete metric $\ell : E \to \mathbb{R}_+$ on a triangulated surface, its *logarithmic length cross ratio* $\log \mathrm{lcr} : \mathbb{R}^{|E|} \to \mathbb{R}^{|E_{int}|}$ is defined by

$$\log \mathrm{lcr}(\ell)_{ij} := \log \ell_{jk} - \log \ell_{ki} + \log \ell_{il} - \log \ell_{lj} \quad \forall \{ij\} \in E_{int}$$

where $\{ijk\}$ is the left face of e_{ij} and $\{jil\}$ is the right face.

Theorem 4.11 ([**9**]). *Two discrete metrics ℓ and $\tilde{\ell}$ on a triangulated surface are conformally equivalent if and only if*

$$\log \mathrm{lcr}(\ell) \equiv \log \mathrm{lcr}(\tilde{\ell}).$$

Corollary 4.12 ([**9**]). *The dimension of the space of the conformal equivalence classes of a triangulated surface is $|E| - |V|$.*

4.2.2. Infinitesimal deformations. In this section, we consider closed triangulated surfaces. Suppose $\ell : E \to \mathbb{R}_+$ is a discrete metric on a closed triangulated surface. We consider an infinitesimal change of the discrete metric $\dot{\ell}$ and write it as $\dot{\ell} = \sigma \ell$ for some infinitesimal scaling $\sigma : E \to \mathbb{R}$. Then the change of the logarithmic length cross ratio on edge $\{ij\}$ is given by

$$(\log \mathrm{lcr}(\ell))^{\cdot}_{ij} = \sigma_{jk} - \sigma_{ki} + \sigma_{il} - \sigma_{lj} =: L(\sigma)_{ij}.$$

The image of the linear map $L : \mathbb{R}^{|E|} \to \mathbb{R}^{|E|}$ is the tangent space of the space of conformal equivalence classes (which is the same space at all discrete metrics).

Lemma 4.13. *Given a closed triangulated surface. The operator L is skew adjoint with respect to the standard product $(\ ,\ )$ on $\mathbb{R}^{|E|}$ given by $(a, b) := \sum_{\{ij\} \in E} a_{ij} b_{ij}$ for any $a, b \in \mathbb{R}^{|E|}$.*

PROOF. Let $\delta^{ij} : E \to \mathbb{R}$ be the function defined by $(\delta^{ij})_{ij} = 1$ on edge $\{ij\}$ and zero on other edges. Then for any $b \in \mathbb{R}^{|E|}$, we have

$$L^*(b)_{ij} = (\delta^{ij}, L^*(b)) = (L(\delta^{ij}), b) = -b_{jk} + b_{ki} - b_{il} + b_{lj} = -L(b)_{ij}.$$

Thus we have $L^* = -L$. □

The above lemma implies that we have an orthogonal decomposition

$$\mathbb{R}^{|E|} = \mathrm{Ker}(L) \oplus \mathrm{Im}(L^*) = \mathrm{Ker}(L) \oplus \mathrm{Im}(L).$$

Lemma 4.14. *Given a closed triangulated surface. We have the following.*

$$\mathrm{Ker}(L) = \{a : E \to \mathbb{R} \,|\, \exists u \in \mathbb{R}^V \ s.t.\ \forall \{ij\} \in E, \quad a_{ij} = u_i + u_j\}$$

$$\mathrm{Im}(L) = \{a : E \to \mathbb{R} | \sum_j a_{ij} = 0 \quad \forall i \in V\}$$

Proof. It is obvious that

$$\{a : E \to \mathbb{R} | \exists u \in \mathbb{R}^V \text{ s.t. } a_{ij} = u_i + u_j \quad \forall \{ij\} \in E\} \subset \operatorname{Ker}(L).$$

Assume $a \in \operatorname{Ker}(L)$. For each vertex i of a face $\{ijk\}$ we define

$$u_i := \frac{a_{ij} + a_{ki} - a_{jk}}{2} \tag{4.6}$$

Suppose $\{ilj\}$ is the neighboring triangle sharing the edge $\{ij\}$ with $\{ijk\}$. Because of $L(a)_{ij} = 0$ we have

$$u_i = \frac{a_{ij} + a_{ki} - a_{jk}}{2} = \frac{a_{ij} + a_{il} - a_{lj}}{2} = \tilde{u}_i.$$

Since the link of each vertex is a disk (although we only need the vertex link to be a fan), Equation (4.6) in fact defines a function $u : V \to \mathbb{R}$ such that for any edge $\{ij\}$

$$a_{ij} = u_i + u_j.$$

Hence

$$\operatorname{Ker}(L) = \{a : E \to \mathbb{R} \,|\, \exists u \in \mathbb{R}^V \text{ s.t. } a_{ij} = u_i + u_j \quad \forall \{ij\} \in E\}.$$

On the other hand, it is obvious that

$$\operatorname{Im}(L) \subset \{a : E \to \mathbb{R} \,|\, \sum_j a_{ij} = 0 \quad \forall i \in V\}.$$

Since

$$\operatorname{rank}(L) = |E| - \dim \operatorname{Ker}(L) = |E| - |V|$$

the two vector spaces are indeed the same. □

Recall that conformal equivalence classes of a triangular mesh are parametrized by logarithmic length cross ratios. By the inverse function theorem the result below implies that by deforming a non-isothermic surface in space we can reach all nearby conformal equivalence classes. It is precisely in the case of an isothermic surface that the hypothesis of the inverse function theorem fails to be satisfied. Thus the space of all non-isothermic non-degenerate realizations in a fixed conformal equivalence class is a smooth manifold.

Theorem 4.15. *Suppose $f : V \to \mathbb{R}^3$ is a non-degenerate realization of a closed triangulated surface. Then f is isothermic if and only if there exists a non-trivial element $a \in \operatorname{Im}(L)$ such that*

$$(a, L(\sigma)) = 0$$

for all infinitesimal scalings $\sigma : E \to \mathbb{R}$ coming from infinitesimal extrinsic deformations in Euclidean space, i.e. for σ so that there exists $\dot{f} : V \to \mathbb{R}^3$ and $W : E \to \mathbb{R}^3$ such that $d\dot{f} = \sigma df + df \times W$.

Proof. Suppose f is isothermic with τ satisfying Definition 4.1. Let $\dot{f} : V \to \mathbb{R}^3$ be an arbitrary infinitesimal deformation and we write $d\dot{f} = \sigma df + df \times W$. Since τ is closed, i.e. $\sum_j \tau(e^*_{ij}) = 0 \ \forall i \in V$ we have

$$0 = -\sum_{i\in V}\langle \sum_j \tau(e^*_{ij}), \dot{f}_i\rangle = \sum_{\{ij\}\in E} \langle \tau(e^*_{ij}), d\dot{f}(e_{ij})\rangle = \sum_{\{ij\}} \langle \tau(e^*_{ij}), \sigma_{ij} df(e_{ij}) + df(e_{ij}) \times W_{ij}\rangle.$$

From $df(e_{ij}) \times \tau(e^*_{ij}) = 0$ we obtain

$$0 = \sum_{\{ij\}\in E} \langle \tau(e^*_{ij}), \sigma_{ij} df(e_{ij}) + df(e_{ij}) \times W_{ij}\rangle = \sum_{\{ij\}\in E} \langle \tau(e^*_{ij}), df(e_{ij})\rangle \sigma_{ij}.$$

Using

$$\langle \tau(e^*_{ij}), df(e_{ij})\rangle = \langle \tau(e^*_{ji}), df(e_{ji})\rangle$$

we see that $\langle \tau, df\rangle : E \to \mathbb{R}$ is well defined. Since we know that for every interior vertex i

$$\sum_j \langle df(e_{ij}), \tau(e^*_{ij})\rangle = 0$$

we thus have $\langle \tau, df \rangle \in \mathrm{Im}(L)$. Hence there exists an non-trivial element $a \in \mathrm{Im}(L)$ such that for every edge $\{ij\}$

$$L(a)_{ij} = -\langle \tau(e_{ij}^*), df(e_{ij}) \rangle.$$

Because $\dot{f}$ is arbitrary we conclude that

$$0 = (\langle \tau, df \rangle, \sigma) = (-L(a), \sigma) = (a, L(\sigma))$$

for all infinitesimal scaling $\sigma : E \to \mathbb{R}$ coming from infinitesimal extrinsic deformations.

On the other hand, suppose there exists a non-trivial $a \in \mathrm{Im}(L)$ such that

$$(a, L(\sigma)) = 0$$

for all infinitesimal scaling $\sigma \in \mathbb{R}^{|E|}$ coming from infinitesimal extrinsic deformations. We define a dual 1-form $\tau : \vec{E}_{int}^* \to \mathbb{R}^3$ via

$$\tau(e_{ij}^*) := -L(a)_{ij} \frac{df(e_{ij})}{|df(e_{ij})|^2}$$

which satisfies

$$\begin{aligned} df(e_{ij}) \times \tau(e_{ij}^*) &= 0, \\ \langle df(e_{ij}), \tau(e_{ij}^*) \rangle &= -L(a)_{ij} \end{aligned}$$

for every edge $\{ij\}$. Since $\langle df, \tau \rangle \in \mathrm{Im}(L)$, we have

$$\sum_j \langle df(e_{ij}), \tau(e_{ij}^*) \rangle = 0 \quad \forall i \in V.$$

In addition, for any infinitesimal deformation $\dot{f} : V \to \mathbb{R}^3$ we write $d\dot{f} = \sigma df + df \times W$ for some $\sigma : E \to \mathbb{R}$ and $W : E \to \mathbb{R}^3$. We obtain

$$-\sum_{i \in V} \langle \sum_j \tau(e_{ij}^*), \dot{f}_i \rangle = \sum_{ij} \langle \tau(e_{ij}^*), d\dot{f}(e_{ij}) \rangle = \sum_{ij} \langle \tau(e_{ij}^*), df(e_{ij}) \rangle \sigma_{ij} = (a, L(\sigma)) = 0.$$

Since $\dot{f}$ is arbitrary we conclude that τ is closed, i.e.

$$\sum_j \tau(e_{ij}^*) = 0 \quad \forall i \in V.$$

Hence, f is isothermic with dual 1-form τ. $\square$

Proof of Theorem 4.4. Consider the composition of maps

$$\{\text{infinitesimal deformations in } \mathbb{R}^3\} \xrightarrow{\sigma} \{\text{infinitesimal scalings}\} \xrightarrow{L} \{\text{change of lcrs}\}.$$

The space of infinitesimal conformal deformations is exactly $\mathrm{Ker}(L \circ \sigma)$. Moreover, we know

$$\dim(\mathrm{Ker}(L \circ \sigma)) = 3|V| - \mathrm{rank}(L \circ \sigma) \geq 3|V| - (|E| - |V|) = |V| - 6g + 6.$$

Finally we conclude: The inequality is strict $\iff$ $L \circ \sigma$ is not surjective $\iff$ f is isothermic. $\square$

Since the conformal equivalence classes of a triangle mesh are parametrized by length cross ratios, we can rephrase the previous theorems as follows.

Corollary 4.16. *Given a closed triangulated surface, isothermic realizations are precisely the points in the space of all non-degenerate realizations where the map that takes a non-degenerate realization to the conformal equivalence class of its induced metric fails to be a submersion.*

It is interesting to see how combinatorics affect geometry. It is known that the number of vertices of a closed genus-g triangulated surface satisfies the Heawood bound [**28**]

$$|V| \geq \frac{7 + \sqrt{1 + 48g}}{2}.$$

This condition is known to be sufficient for the existence of a genus-g triangulated surfaces with $|V|$ vertices except for $g = 2$. Comparing the Heawood bound with the inequality in Theorem 4.4 we obtain more examples of isothermic surfaces.

Corollary 4.17. *Every non-degenerate realization of a closed triangulated surface with $|V| < 6g + 4$ is isothermic.*

Proof. The space of infinitesimal conformal deformations contains all deformations that come from infinitesimal Möbius transformations. Therefore this space has dimension at least 10 and hence a surface must be isothermic if $10 > |V| - 6g + 6$. □

Some of these surfaces with small number of vertices can be realized in Euclidean space without self-intersection. For example, there are embedded surfaces with $g = 2$ and $|V| = 10$ as shown in [**33**].

4.3. Preserving intersection angles

Given a triangulated surface in Euclidean space, every triangle determines a circumscribed circle and every two triangles sharing an edge determine a unique circumscribed sphere if the vertices are not con-circular. Two circumscribed circles are *neighboring* if their corresponding triangles share an edge. We call two circumscribed spheres *neighboring* if they have a common vertex.

Intersection angles of circles and spheres are Möbius invariant. The intersection angles of neighboring circumcircles of a triangulated surface were used to define a discrete Willmore functional [**11**].

Proof of Theorem 4.5. Suppose we have an infinitesimal deformation $\dot{f}$ that preserves the angles between circumcircles and circumspheres but is not induced from Möbius transformations. Then it cannot be that $\dot{f}$ also preserves the length cross ratios (because it is not hard to see that in this case $\dot{f}$ is an infinitesimal Möbius transformation). We write $d\dot{f} = \sigma df + df \times W$ for some $\sigma : E \to \mathbb{R}$ and $W : E \to \mathbb{R}^3$. Then the change of logarithmic length cross ratios is $L(\sigma)$ where

$$L(\sigma)_{ij} := \sigma_{jk} - \sigma_{ki} + \sigma_{il} - \sigma_{lj} \quad \forall \{ij\} \in E_{int}.$$

(See Figure 1.1.) By our assumptions $L(\sigma)$ does not vanish identically.

We define a dual 1-form

$$\tau(e^*_{ij}) := L(\sigma)_{ij} \frac{df(e_{ij})}{|df(e_{ij})|^2}.$$

Then we have

$$\tau(e^*_{ij}) \times df(e_{ij}) = 0 \quad \forall \{ij\} \in E$$
$$\sum_j \langle \tau(e^*_{ij}), df(e_{ij}) \rangle = \sum_j L(\sigma)_{ij} = 0 \quad \forall i \in V_{int}.$$

In order to show that f is isothermic, we need to verify the closedness of τ, i.e. for every interior vertex i

$$\sum_j \tau(e^*_{ij}) = 0.$$

We identify Euclidean space $\mathbb{R}^3$ with the space $\operatorname{Im}\mathbb{H}$ of imaginary quaternions (Section 4.9). We pick any vertex v_0 and denote its neighboring vertices by $v_1, v_2, \ldots, v_n$. Then we take an inversion in the unit sphere centered at $f_0 := f(v_0)$ and denote the images of the neighboring vertices by $\tilde{f}_i$. We have the following relations:

$$\begin{aligned}
\tilde{f}_j - f_0 &= (f_j - f_0)^{-1}, \\
\tilde{f}_{j+1} - \tilde{f}_j &= -(f_j - f_0)^{-1}((f_{j+1} - f_0) - (f_j - f_0))(f_{j+1} - f_0)^{-1} \\
&= -(f_j - f_0)^{-1}(f_{j+1} - f_j)(f_{j+1} - f_0)^{-1}.
\end{aligned}$$

We define the infinitesimal scaling

$$\tilde{\sigma}_{j,j+1} := \frac{|\tilde{f}_{j+1} - \tilde{f}_j|^{\cdot}}{|\tilde{f}_{j+1} - \tilde{f}_j|}$$

By taking the logarithmic derivative of the following equation

$$\frac{|\tilde{f}_{j+1} - \tilde{f}_j|}{|\tilde{f}_j - \tilde{f}_{j-1}|} = \frac{|f_{j+1} - f_j||f_{j-1} - f_0|}{|f_{j+1} - f_0||f_j - f_{j-1}|}$$

we obtain for $j = 1, \dots, n$

$$\tilde{\sigma}_{j,j+1} - \tilde{\sigma}_{j-1,j} = (L(\sigma))_{0j}$$

where $\sigma_{ij} = |f_j - f_i|^{\cdot}/|f_j - f_i|$. On the other hand, the vertices $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n, \tilde{f}_1$ form a closed polygon in $\mathbb{R}^3$. We define

$$\tilde{\ell}_{j,j+1} := |\tilde{f}_{j+1} - \tilde{f}_j|,$$
$$\tilde{T}_{j,j+1} := \frac{\tilde{f}_{j+1} - \tilde{f}_j}{|\tilde{f}_{j+1} - \tilde{f}_j|}.$$

Since the polygon is closed, we have

$$0 = \sum_{j=1}^{n} \tilde{\ell}_{j,j+1} \tilde{T}_{j,j+1}.$$

The fact that the deformation $\dot{f}$ preserves the intersection angles of neighboring circles and neighboring spheres implies that the angles between the neighboring segments and osculating planes of the closed polygon remain constant. Thus there exists a constant vector $c \in \mathbb{R}^3$ such that

$$\begin{aligned} 0 &= \sum \dot{\tilde{\ell}}_{j,j+1} \tilde{T}_{j,j+1} + \sum \tilde{\ell}_{j,j+1} \tilde{T}_{j,j+1} \times c \\ &= \sum \tilde{\sigma}_{j,j+1} (\tilde{f}_{j+1} - \tilde{f}_j) \\ &= -\sum L(\sigma)_{0j} (f_j - f_0)^{-1} \\ &= \sum_{j=1}^{n} \tau(e^*_{0j}). \end{aligned}$$

To show that the converse is true one only has to reverse the previous argument. □

4.4. Example: Isothermic quadrilateral surfaces

We show that isothermic quadrilateral surfaces as defined by Bobenko and Pinkall [**4**] are isothermic under our definition (after an arbitrary subdivision into triangles). Isothermic quadrilateral surfaces are analogous to conformal curvature line parametrizations of smooth isothermic surfaces. They can be treated using the theory of integrable systems. New isothermic surfaces can be obtained from a given isothermic surface via the Christoffel duality and Darboux transformations [**31**]. Special discrete surfaces related to isothermic quadrilateral meshes were studied in [**3, 7**].

Questions about infinitesimal rigidity of quadrilateral meshes have been considered by [**64, 72, 36**].

We first review some results on isothermic quadrilateral surfaces from [**14**]. Then we construct an infinitesimal isometric deformation for every isothermic quadrilateral surface and show that the change of mean curvature around each vertex is zero. In this way we obtain isothermic triangulated surfaces from the earlier notion of isothermic quadrilateral surfaces.

4.4.1. Review.

Definition 4.18 ([**4**]). A discrete isothermic net is a map $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$, for which all elementary quadrilaterals are cyclic and have factorized real cross-ratios in the form

$$cr(f_{m,n}, f_{m+1,n}, f_{m+1,n+1}, f_{m,n+1}) = \frac{\alpha_m}{\beta_n} \quad \forall m, n \in \mathbb{Z},$$

where $\alpha_m \in \mathbb{R}$ does not depend on n and $\beta_n \in \mathbb{R}$ not depend on m.

Theorem 4.19 ([**4**]). *Let $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ be a discrete isothermic net. Then the discrete net $f^* : \mathbb{Z}^2 \to \mathbb{R}^3$ defined (up to translation) by the equations*

$$f^*_{m+1,n} - f^*_{m,n} = \alpha_m \frac{f_{m+1,n} - f_{m,n}}{||f_{m+1,n} - f_{m,n}||^2},$$
$$f^*_{m,n+1} - f^*_{m,n} = \beta_n \frac{f_{m,n+1} - f_{m,n}}{||f_{m,n+1} - f_{m,n}||^2}$$

is isothermic. f^* *is the* Christoffel dual *of* f.

We need a formula for the diagonals of its Christoffel dual [**14**, Corollary 4.33].

Lemma 4.20. *Given a discrete isothermic net* f, *the diagonals of any elementary quadrilateral of its Christoffel dual are given by*

$$f^*_{m+1,n} - f^*_{m,n+1} = (\alpha_m - \beta_n)\frac{f_{m+1,n+1} - f_{m,n}}{||f_{m+1,n+1} - f_{m,n}||^2},$$
$$f^*_{m+1,n+1} - f^*_{m,n} = (\alpha_m - \beta_n)\frac{f_{m+1,n} - f_{m,n+1}}{||f_{m+1,n} - f_{m,n+1}||^2}.$$

4.4.2. Infinitesimal flexibility of isothermic quadrilateral surfaces. Given a discrete isothermic net we first arbitrarily introduce a diagonal for each quadrilateral in order to get a triangulation. Then, we define infinitesimal rotations on faces as follows.

Rule: Suppose $ABCD$ is an elementary quadrilateral of a discrete isothermic net $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ and the diagonal AC is inserted. Then we get two triangles ABC and ACD. We define infinitesimal rotations $Z_{ABC} := B^*$ and $Z_{ACD} := D^*$ where B^* and D^* are the corresponding vertices of the Christoffel dual $f^* : V(\mathbb{Z}^2) \to \mathbb{R}^3$.

Theorem 4.21. *Suppose we are given a discrete isothermic net* $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ *and its Christoffel dual* $f^* : V(\mathbb{Z}^2) \to \mathbb{R}^3$. *We assume that the faces of* f *have been subdivided into triangles in an arbitrary way. Then the infinitesimal rotations given by the above rule for each triangle define an infinitesimal isometric deformation of the triangulated surface. Moreover, the infinitesimal deformation preserves the integrated mean curvature and is not induced from Euclidean transformations.*

Proof. By Theorem 4.19 and Lemma 4.20, the infinitesimal rotations of two adjacent triangles are compatible on the common edge. Therefore they define an infinitesimal isometric deformation.

It remains to show that around an arbitrary vertex the change of the integrated mean curvature around vertices is zero. For every vertex there are $2^4 = 16$ ways of inserting diagonals on the four neighboring quadrilaterals. Taking the symmetry into account we can reduce them to 6 cases. We enumerate these 6 cases and calculate the change of mean curvature on each edge in Figure 4.1. It can be checked directly that in all cases the sum around the vertex is zero. □

Remark 4.22. Although the infinitesimal rotations on faces depend on the triangulation, the deformations of the edges already present in the quad mesh do not. For example, the change of the edge $f_{m+1,n} - f_{m,n}$ is given by

$$\begin{aligned}(\dot{f}_{m+1,n} - \dot{f}_{m,n}) &= (f_{m+1,n} - f_{m,n}) \times f^*_{m+1,n}\\ &= (f_{m+1,n} - f_{m,n}) \times f^*_{m,n}\\ &= (f_{m+1,n} - f_{m,n}) \times \frac{f^*_{m+1,n} + f^*_{m,n}}{2}.\end{aligned}$$

Here we have used $(f^*_{m+1,n} - f^*_{m,n}) \parallel (f_{m+1,n} - f_{m,n})$. Moreover, the quadrilaterals do not stay con-circular under the infinitesimal deformation.

The infinitesimal isometric deformation defined above has an exact counterpart in the smooth theory. Given a simply connected isothermic surface f and its Christoffel dual f^*, there exists an infinitesimal isometric deformation $\dot{f}$ satisfying $d\dot{f} = df \times f^*$ (Section 4.9). It preserves the mean curvature but does not preserve curvature lines. If in addition the curvature lines were preserved, the shape operator would remain unchanged and the deformation would be trivial, i.e. an infinitesimal Euclidean transformation.

4.5. Example: Homogeneous discrete cylinders

In this section we show that every homogeneous triangulation of a circular cylinder in $\mathbb{R}^3$ is isothermic. Here “homogeneous” means that there is a subgroup of Euclidean transformations that acts transitively at vertices and respects the combinatorics. Note that in general none of the edges of such an isothermic discrete cylinder is aligned with the curvature line directions of the underlying smooth cylinder.

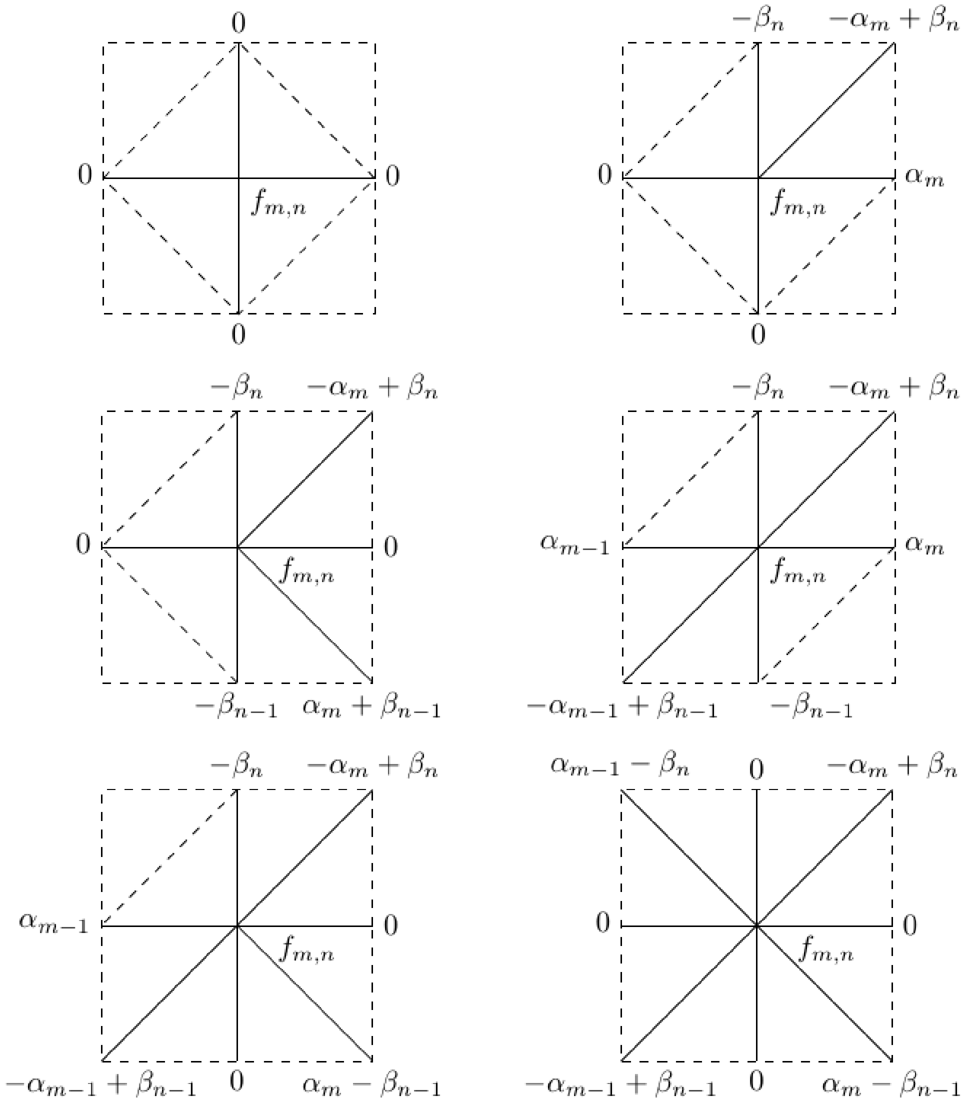

FIGURE 4.1. The six types of triangulations around a vertex $f_{m,n}$ and the corresponding change of mean curvature on the edges.

We consider the group G of all Euclidean motions that fix the z-axis. Every element $g \in G$ is of the form that acts on a point $p \in \mathbb{R}^3$ as

$$g(p) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} p + \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

where $\theta, h \in \mathbb{R}$.

We pick two elements g_1, g_2 of G in general position and consider the group H generated by g_1, g_2. For a generic choice of g_1, g_2 the group H is isomorphic to $\mathbb{Z}^2$. An element $(s,t) \in \mathbb{Z}^2$ corresponds to the element $g_1^s g_2^t \in H$.

We also consider $\mathbb{Z}^2$ as the vertex set of a triangulated surface with faces of the form $\{(s,t),(s+1,t),(s,t+1)\}$ or $\{(s+1,t),(s+1,t+1),(s,t+1)\}$ (Figure 4.2).

We now define a map $f : \mathbb{Z}^2 \to \mathbb{R}^3$ by picking $r > 0$ and setting

$$f(s,t) = g_1^s g_2^t (r,0,0).$$

For suitable $g_1, g_2 \in G$ this map f will be a non-degenerate realization. Figure 4.3 shows a piece of such a discrete surface.

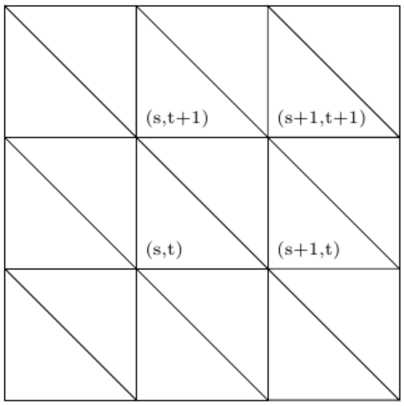

FIGURE 4.2. A triangulated surface with vertex set $\mathbb{Z}^2$

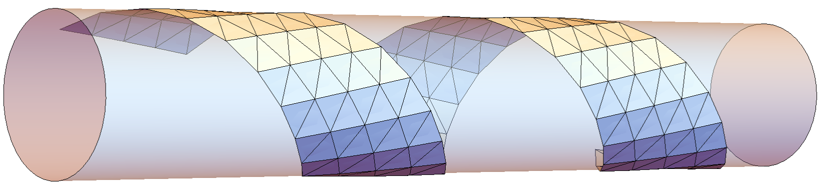

FIGURE 4.3. A strip of an isothermic triangulated cylinder

We now prove that realizations $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ constructed above are isothermic by showing that they admit a non-trivial infinitesimal isometric deformation preserving the integrated mean curvature. Note that up to symmetry there are only three types of edges, represented by $\{f(0,0), f(1,0)\}$, $\{f(1,0), f(0,1)\}$ and $\{f(0,1), f(0,0)\}$. We denote their lengths by

$$\ell_a(r, \theta_1, h_1, \theta_2, h_2) \quad , \quad \ell_b(r, \theta_1, h_1, \theta_2, h_2) \quad , \quad \ell_c(r, \theta_1, h_1, \theta_2, h_2).$$

The integrated mean curvature is the same at all vertices. We denote it by

$$H(r, \theta_1, h_1, \theta_2, h_2).$$

Now the derivative of the map $\mu := (\ell_a, \ell_b, \ell_c, H) : \mathbb{R}^5 \to \mathbb{R}^4$ has a non-trivial kernel at every point $(r, \theta_1, h_1, \theta_2, h_2) \in \mathbb{R}^5$. Moreover, it is easy to see that any non-zero element

$$(\dot{r}, \dot{\theta}_1, \dot{h}_1, \dot{\theta}_2, \dot{h}_2) \in \ker d\mu$$

corresponds to an infinitesimal deformation of f which is not induced from Euclidean transformations. This infinitesimal deformation preserves all the edge lengths and the integrated mean curvature around vertices. Therefore the triangulated cylinder f is isothermic.

4.6. Example: Planar triangular meshes

In this section we show that certain planar triangular meshes are isothermic. For a simply connected surface, we know that a realization is isothermic if and only if there exists a non-trivial infinitesimal isometric deformation preserving the integrated mean curvature (Corollary 4.2). We will see that every such deformation of a planar triangular mesh is given by a discrete harmonic function in the sense of the cotangent Laplacian [**49, 67**].

Theorem 4.23. *Suppose $f : V \to \mathbb{R}^2 \subset \mathbb{R}^3$ is a non-degenerate realization of a triangulated surface with Euler Characteristic χ and $|V_b|$ boundary vertices. Then f is isothermic if $|V_b| - 3\chi >$ 0.*

PROOF. Since each boundary component is a simple closed polygon, the number of boundary edges is $|E_b| = |V_b|$. The Euler characteristic is given by

$$|V| - |E| + |F| = \chi.$$

Since the surface is triangulated, we have

$$3|F| = 2|E| - |E_b|.$$

Hence

$$|E_{int}| - 3|V_{int}| = |V_b| - 3\chi.$$

By Lemma 4.8, a dual 1-form satisfying Definition 4.1 is equivalent to existence of a discrete holomorphic quadratic differential $q : E_{int} \to \mathbb{R}$ such that for every interior vertex i

$$\begin{gathered} \sum_j q_{ij} = 0, \\ \sum_j q_{ij}/(z_j - z_i) = 0 \end{gathered} \tag{4.7}$$

which is a system of linear equations. By simple counting and using the fact that f is planar we obtain a lower bound on the dimension of the solution space

$$\dim\{k : E_{int} \to \mathbb{R} \text{ satisfying (4.7)}\} \geq |E_{int}| - 2|V_{int}| - |V_{int}| = |V_b| - 3\chi.$$

Hence f is isothermic if $|V_b| - 3\chi > 0$. □

In particular the above theorem implies that every planar triangulated disk ($\chi = 1$) with more than 3 boundary vertices is isothermic. Since a disk is simply connected, by Corollary 4.2 there exists a non-trivial infinitesimal isometric deformation preserving the integrated mean curvature. The following indicates how to obtain such infinitesimal deformations.

Theorem 4.24. *Let $f : V \to \mathbb{R}^2 \subset \mathbb{R}^3$ be a strongly non-degenerate realization of a triangulated surface with normal $N \in \mathbb{S}^2$ and $u : V \to \mathbb{R}$ be a function. Then the infinitesimal isometric deformation $\dot{f} := uN$ preserves the integrated mean curvature if and only if u is a discrete harmonic function in the sense of the cotangent Laplacian, i.e. for every interior vertex i*

$$\sum_j (\cot \angle jki + \cot \angle ilj)(u_j - u_i) = 0$$

where triangles $\{ijk\}$ and $\{jli\}$ share an edge $\{ij\}$ (Figure 1.1).

PROOF. It is easy to see that for any function $u : V \to \mathbb{R}$ the infinitesimal deformation $\dot{f} = uN$ preserves edge lengths. The infinitesimal rotation of a face $\{ijk\}$ is given by

$$Z_{ijk} = -\frac{u_i df(e_{jk}) + u_j df(e_{ki}) + u_k df(e_{ij})}{2A_{ijk}}$$

where $A_{ijk} := \langle (df(e_{ij}) \times df(e_{ik}))/2, N\rangle$ is the (signed) area of the triangle $\{ijk\}$. This follows from

$$d\dot{f}(e_{ij}) = u_j N - u_i N = df(e_{ij}) \times Z_{ijk}.$$

To preserve the integrated mean curvature around vertices, the map $Z : F \to \mathbb{R}^3$ has to satisfy for every interior vertex i

$$0 = \sum_{\{ij\}\in E:i} \langle df(e_{ij}), Z_{ijk} - Z_{jil}\rangle = \sum_{\{ijk\}\in F:i} \langle df(e_{kj}), Z_{ijk}\rangle. \tag{4.8}$$

Note that

$$\begin{aligned} \langle df(e_{kj}), Z_{ijk}\rangle =& \langle df(e_{kj}), -\frac{u_i df(e_{jk}) + u_j df(e_{ki}) + u_k df(e_{ij})}{2A_{ijk}}\rangle \\ =& \langle df(e_{jk}), \frac{(u_j - u_i) df(e_{ki}) + (u_k - u_i) df(e_{ij})}{2A_{ijk}}\rangle \\ =& -\cot \angle jki\, (u_j - u_i) - \cot \angle ilj\, (u_k - u_i). \end{aligned}$$

Thus (4.8) is equivalent to saying that for every interior vertex i

$$0 = \sum_{\{ij\}\in E:i} (\cot \angle jki + \cot \angle ilj)(u_j - u_i).$$

Hence the infinitesimal deformation uN preserves the integrated mean curvature if and only if u is a discrete harmonic function. □

In fact, every infinitesimal isometric deformation of a strongly non-degenerate planar triangular mesh is of the form $u\,N$ for some function u modulo infinitesimal Euclidean motions.

It can be checked that an infinitesimal normal deformation $u\,N$ of a planar mesh f is an Euclidean motion if and only if u is a linear function, i.e. if there exists a vector $a \perp N$ and a constant $c \in \mathbb{R}$ such that

$$u = \langle a, f\rangle + c.$$

4.7. Example: Inscribed triangular meshes

Since the notion of isothermic triangulated surfaces is Möbius invariant (Theorem 4.3), our results for planar triangular meshes can be rephrased for triangular meshes inscribed in a sphere. Theorem 4.23 immediately implies the following.

Corollary 4.25. *Suppose $f : V \to \mathbb{S}^2$ is a non-degenerate realization of a triangulated surfaces with Euler Characteristic χ and $|V_b|$ boundary vertices. Then f is isothermic if $|V_b| - 3\chi > 0$.*

In particular, it follows that every inscribed triangulated disk with more than 3 boundary vertices is isothermic. This is analogous to the fact that disks immersed smoothly in a sphere are isothermic.

On the other hand, to show that a triangulated surface is isothermic, we look for a non-trivial infinitesimal isometric deformation preserving the integrated mean curvature. For an inscribed triangulated surface, we will see that it suffices to find an infinitesimal isometric deformation.

The following lemma shows that for any triangular mesh inscribed in a sphere a dual 1-form τ satisfying Equation (4.2) and (4.3) in Definition 4.1 will satisfy Equation (4.4) automatically.

Lemma 4.26. *Given a non-degenerate realization $f : V \to \mathbb{S}^2$ of a triangulated surface and a function $k : E_{int} \to \mathbb{R}$, then for every interior vertex i*

$$\sum_j k_{ij} df(e_{ij}) = 0 \quad \Longrightarrow \quad \sum_j k_{ij} |df(e_{ij})|^2 = 0.$$

Proof. Since $|f| \equiv 1$, we have

$$\sum_j k_{ij} |df(e_{ij})|^2 = \sum_j k_{ij}(2|f_i|^2 - 2\langle f_i, f_j\rangle) = 2\langle f_i, \sum_j k_{ij}(f_i - f_j)\rangle = 0.$$ □

Corollary 4.27. *Given a non-degenerate realization $f : V \to \mathbb{S}^2$ of a triangulated surface, a function $k : E_{int} \to \mathbb{R}$ is a self-stress of f, i.e. for every interior vertex i*

$$\sum_j k_{ij} df(e_{ij}) = 0$$

if and only if

$$q_{ij} := k_{ij} |f_j - f_i|^2$$

defines a holomorphic quadratic differential associated to f. Furthermore, there is a one-to-one correspondence between self-stresses on f and discrete holomorphic quadratic differentials on the stereographic projection of f.

Theorem 4.28. *Suppose $f : V \to \mathbb{S}^2$ is a strongly non-degenerate realization of a triangulated surface M. Then every infinitesimal isometric deformation preserves the integrated mean curvature.*

Hence if f is infinitesimally flexible, then it is isothermic. If f is isothermic and M is simply connected, then f is infinitesimally flexible.

PROOF. Suppose an infinitesimal isometric deformation is given by a rotation vector field $Z : F \to \mathbb{R}^3$. The compatibility condition implies that there exists $k : E_{int} \to \mathbb{R}$ such that on every interior edge $\{ij\}$

$$(Z_{ijk} - Z_{jil}) = k_{ij} df(e_{ij})$$

where $\{ijk\}$ denotes the left face of e_{ij} and $\{jil\}$ denotes the right face. The previous lemma yields for any vertex $i \in V_{int}$,

$$\dot{H}_i = \sum_j k_{ij} |df(e_{ij})|^2 = 0.$$

Hence the integrated mean curvature is preserved. □

EXAMPLE 4.29. Jessen's orthogonal icosahedron is obtained from a regular icosahedron by flipping 6 edges symmetrically without self intersection [**37, 24**]. Its vertices are exactly those of a regular icosahedron and hence lie on a sphere (Figure 4.4). It is known to be infinitesimally flexible and thus isothermic.

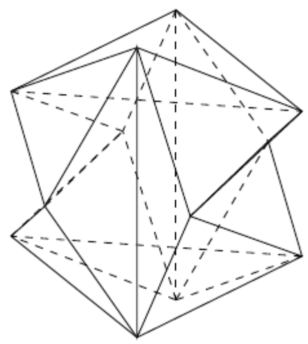

FIGURE 4.4. Jessen's orthogonal icosahedron

Note that the property of being isothermic is Möbius invariant.

Corollary 4.30. *The infinitesimal rigidity of a non-degenerate simply connected triangulated surface inscribed in a sphere is Möbius invariant.*

We can regard a Möbius transformation of a triangulated surface inscribed in a sphere as being induced from a projective transformation of the ambient space. Then the above corollary is simply a special case of the projective invariance of infinitesimal rigidity [**35**].

4.8. Christoffel duality and discrete minimal surfaces

In the smooth theory a minimal surface is the Christoffel dual of its Gauß map. Here we discuss its discrete analogue.

DEFINITION 4.31. Given a non-degenerate realization $f : V \to \mathbb{R}^3$ of a triangulated surface, a non-constant map $f^* : F \to \mathbb{R}^3$ is a *Christoffel dual* of f if

$$\begin{aligned} df(e_{ij}) \times df^*(e^*_{ij}) &= 0 \quad \forall \{ij\} \in E_{int}, \\ \sum_j \langle df(e_{ij}), df^*(e^*_{ij}) \rangle &= 0 \quad \forall i \in V_{int}. \end{aligned}$$

Since we know from the previous section that triangulated disks inscribed in a sphere with more than 3 boundary vertices are isothermic, it is natural to define their Christoffel duals as discrete minimal surfaces. In fact it leads to A-minimal surfaces as discussed in Chapter 3.

Theorem 4.32. *Given a non-degenerate realization $n : V \to \mathbb{S}^2$ of a triangulated surface such that $n_i \neq -n_j$ for every edge $\{ij\}$, then its Christoffel dual is a trivalent A-minimal surface with Gauss map n.*

4.9. Smooth analogues

The main goal of this section is to prove the smooth analogue of Theorem 4.28:

Theorem 4.33. *For every infinitesimal isometric deformation of an immersion $f : M \to S^2$, the mean curvature is preserved.*

Beyond this we also use the opportunity to review some known results on smooth isothermic surfaces that directly correspond to our discrete results. We rely on the treatment of smooth isothermic surfaces by means of quaternionic analysis as developed in [**38, 39, 59**].

Definition 4.34. Given two immersions f and $\tilde{f}$ of a surface M in $\mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$, $\tilde{f}$ is a spin transformation of f if there exists a quaternion-valued function $\lambda : M \to \mathbb{H}\backslash\{0\}$ such that

$$d\tilde{f} = \bar{\lambda} df\, \lambda.$$

In this case, the normals $N, \tilde{N}$ of $f, \tilde{f}$ are related by $\tilde{N} = \lambda^{-1} N \lambda$.

Theorem 4.35. *Given an immersion $f : M \to \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ and a function $\lambda : M \to \mathbb{H}\backslash\{0\}$. Then $\bar{\lambda} df\, \lambda$ is a closed 1-form if and only if there exists $\rho : M \to \mathbb{R}$ such that*

$$df \wedge d\lambda = -\rho\lambda |df|^2.$$

Suppose $f : M \to \mathbb{R}^3$ is a smoothly immersed surface and $X, JX \in T_pM$ form an orthonormal basis in principal directions at $p \in M$. The corresponding principal curvatures $\kappa_1, \kappa_2 \in \mathbb{R}$ is given by

$$dN(X_i) = \kappa_i df(X_i).$$

The mean curvature H_p at p then satisfies

$$df(X) \times dN(JX) - df(JX) \times dN(X) = (\kappa_1 + \kappa_2)N = 2HN.$$

Writing the above formula into quaternions yields

$$df \wedge dN = 2HN|df|^2.$$

If $\tilde{f}$ is a spin transformation of f, then the change of the metric and the mean curvature can be expressed as follows.

Theorem 4.36. *If $\tilde{f}$ is a spin transformation of f given by λ then the followings hold:*

(1) *$\tilde{f}$ and f are conformally equivalent since $|d\tilde{f}|^2 = |\lambda|^4 |df|^2$.*
(2) *We have $d(\bar{\lambda} df\, \lambda) = 0$ and hence $\exists \rho : M \to \mathbb{R}$ such that $df \wedge d\lambda = -\rho\lambda|df|^2$.*
(3) *$\tilde{H}|d\tilde{f}|^2 = (H + \rho)|\lambda|^2 |df|^2$.*

Now we use the fact that every infinitesimal conformal deformation can be expressed as an infinitesimal spin transformation.

Theorem 4.37. *Suppose $f : M \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ is a simply connected smooth surface and $\dot{\lambda} : M \to \mathbb{H}$ is a function. Then there exists an infinitesimal conformal deformation $\dot{f} : M \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ given by*

$$d\dot{f} = 2\operatorname{Im}(df\, \dot{\lambda})$$

if and only if there exists $\dot{\rho} : M \to \mathbb{R}$ such that

$$-df \wedge d\dot{\lambda} = \dot{\rho}|df|^2.$$

In particular

$$(|df|^2)^{\cdot} = 4\operatorname{Re}(\dot{\lambda})|df|^2,$$
$$(H|df|^2)^{\cdot} = (\dot{\rho} + 2\operatorname{Re}(\dot{\lambda})H)|df|^2.$$

Definition 4.38. A smooth surface $f : M \to \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ is *isothermic* if locally there exists a non-trivial $\operatorname{Im}\mathbb{H}$-valued closed 1-form τ such that

$$df \wedge \tau = 0.$$

Theorem 4.39. *A smooth surface $f : M \to \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ is isothermic if and only if locally there exists a non-trivial infinitesimal isometric deformation such that the mean curvature is unchanged.*

PROOF. Let f be isothermic and τ be a 1-form satisfying Definition 4.38. The closedness of τ implies that on any simply connected open set $U \subset M$ there exists $\dot{\lambda} : U \to \operatorname{Im}\mathbb{H}$ such that

$$\tau = d\dot{\lambda}.$$

Since $df \wedge d\dot{\lambda} = 0$, Theorem 4.37 implies that the 1-form $\operatorname{Im}(df\,\dot{\lambda})$ is closed and hence there exists an infinitesimal deformation $\dot{f} : U \to \operatorname{Im}\mathbb{H}$ satisfying

$$d\dot{f} = 2\operatorname{Im}(df\,\dot{\lambda}).$$

Because $\dot{\lambda}$ is purely imaginary and $df \wedge d\dot{\lambda} = 0$, from Theorem 4.37 we have

$$(|df|^2)^{\cdot} = 0,$$
$$(H|df|^2)^{\cdot} = 0$$

and hence $\dot{f}$ is an infinitesimal isometric deformation preserving the mean curvature. The converse is proved similarly. □

The smooth analog of Theorem 4.3 is a classical result of isothermic surfaces [**30**].

Theorem 4.40. *The class of isothermic surfaces is Möbius invariant.*

We need the global existence of τ in order to relate it to the space of immersions.

DEFINITION 4.41. A smooth surface $f : M \to \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ is *strongly isothermic* if there exists a non-trivial $\operatorname{Im}\mathbb{H}$-valued closed 1-form τ on M such that

$$df \wedge \tau = 0.$$

In addition, if τ is exact and $\tau = df^*$ for some $f^* : M \to \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$, then f^* is a *Christoffel dual* of f.

The smooth analog of Corollary 4.16 in Section 4.2 is known:

Theorem 4.42 ([**15**]). *Strongly isothermic immersions of a closed surface are the points in the space of immersions where the map from the space of immersions to Teichmüller space (which assigns to each immersion the conformal class of its induced metric) fails to be a submersion.*

Furthermore, Theorem 4.5 is analogous to the infinitesimal version of the following:

Theorem 4.43 ([**17**]). *Suppose $f, \tilde{f} : M \to S^3$ are two conformal immersions that share the same Hopf differential but do not differ by a Möbius transformation. Then f and $\tilde{f}$ are isothermic surfaces.*

Finally, we establish a smooth counterpart of Theorem 4.28 in Section 4.7. We first need a lemma.

Lemma 4.44. *Let M be a surface with a Riemannian metric and $f : M \to \mathbb{R}^3$ be an isometric immersion. Let $\lambda = g + df(Y) + hN$ be an $\mathbb{H}$-valued function on M where g, $df(Y)$ and hN are its scalar, tangential and normal components. Then we can express $df \wedge d\lambda$ in terms of standard operators from the vector calculus on M as*

$$-df \wedge d\lambda = \big[-\operatorname{curl} Y + df(J\operatorname{grad} g - \mathrm{A}\,Y + \operatorname{grad} h) - \big((\operatorname{div} Y) + 2hH\big)N\big]|df|^2.$$

PROOF. In the following we assume that $X \in T_pM$ is an unit tangent vector. We first consider the scalar component.

$$\begin{aligned}-df \wedge dg(X, JX) &= -df(X)dg(JX) + df(JX)dg(X)\\ &= N df(dg(X)X + dg(JX)JX)\\ &= N df(\operatorname{grad} g).\end{aligned}$$

Then we consider the normal component.

$$\begin{aligned}-df \wedge d(hN)(X, JX) &= \big((-df \wedge dh)N - h df \wedge dN\big)(X, JX)\\ &= N df(\operatorname{grad} h)N - h\big(df(X)dN(JX) - df(JX)dN(X)\big)\\ &= df(\operatorname{grad} h) - 2hHN.\end{aligned}$$

Finally we look at the tangential component. Notice that for an immersed surface in Euclidean space the induced Levi-Civita connection is given as follows: for any tangent vector field Y and tangent vector Z,

$$\begin{aligned} df(\nabla_Z Y) &= d(df(Y))(Z) - \langle d(df(Y))(Z), N\rangle N \\ &= d(df(Y))(Z) + \langle df(Y), df(AZ)\rangle N \\ &= d(df(Y))(Z) + \langle Y, AZ\rangle N \end{aligned}$$

where A is the shape operator of the immersion f. We recall the definition of curl and divergence operator of a tangent vector field Y:

$$\begin{aligned} \operatorname{div}(Y) :&= \langle X, \nabla_X Y\rangle + \langle X, \nabla_X Y\rangle, \\ \operatorname{curl}(Y) :&= \langle JX, \nabla_X Y\rangle - \langle X, \nabla_{JX} Y\rangle \\ &= -\langle X, \nabla_X JY\rangle - \langle JX, \nabla_{JX} JY\rangle \\ &= -\operatorname{div}(JY). \end{aligned}$$

Collecting the above information we now obtain

$$\begin{aligned} & - df \wedge d(df(Y))(X, JX) \\ =& - df(X)\big(df(\nabla_{JX}Y) - \langle Y, AJX\rangle N\big) + df(JX)\big(df(\nabla_X Y) - \langle Y, AX\rangle N\big) \\ =& \langle X, \nabla_{JX}Y\rangle - \langle JX, \nabla_X Y\rangle - \langle JX, \nabla_{JX} Y\rangle N \\ & + \langle -X, \nabla_X Y\rangle N - \langle AY, JX\rangle df(JX) - \langle AY, X\rangle df(X) \\ =& - \operatorname{curl} Y - (\operatorname{div} Y)N - df(\mathrm{A}\, Y). \end{aligned}$$

□

Proof of Theorem 4.33. Suppose $f : M \to S^2$ is an immersion and $\dot\lambda : M \to \operatorname{Im}(\mathbb{H})$ induces an infinitesimal isometric deformation of f. Then writing $\dot\lambda = df(Y) + hN$, we have

$$(H|df|^2)^{\cdot} = -df \wedge d\dot\lambda = \big(-\operatorname{curl} Y + df(-Y + \operatorname{grad} h) - ((\operatorname{div} Y + 2h))N\big)|df|^2.$$

Comparing the imaginary part yields

$$Y = \operatorname{grad} h$$

and thus

$$(H|df|^2)^{\cdot} = -\operatorname{curl}(Y) = -\operatorname{curl}(\operatorname{grad} h) = 0.$$

Since $(|df|^2)^{\cdot} = 0$, we have $\dot H = 0$. □

CHAPTER 5

Integrability of P-nets

Discrete holomorphic quadratic differentials are closely related to discrete equations of Toda type in discrete integrable systems [**1, 6**]. In this chapter, we focus on a particular class of discrete surfaces which possess a canonical discrete holomorphic differential and rich integrable structures.

We study P-nets (Definition 3.29) with the combinatorics of the square grid, i.e. $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ such that for every vertex $v_0 \in \mathbb{Z}^2$

$$\frac{f_1 - f_0}{|f_1 - f_0|^2} - \frac{f_2 - f_0}{|f_2 - f_0|^2} + \frac{f_3 - f_0}{|f_3 - f_0|^2} - \frac{f_4 - f_0}{|f_4 - f_0|^2} = 0.$$

where v_1, v_2, v_3, v_4 denote the neighboring vertices of v_0 in cyclic order. The above condition implies f satisfies the *parallelogram property*: mapping any interior vertex to infinity by inversion the neighboring vertices form a parallelogram.

P-nets appeared in several contexts of this thesis. In Chapter 3 discrete minimal surfaces from planar P-nets lie in the intersection of the integrable systems approach, the curvature approach and the variational approach to discrete minimal surfaces. Furthermore, P-nets form an important subclass of isothermic triangulated surfaces as discussed in Chapter 4. Indeed, there is a canonical discrete holomorphic quadratic differential $\mu : E(\mathbb{Z}^2) \to \pm 1$ associated to every P-net $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$, where μ is $+1$ on "horizontal" edges $\{(m, n), (m + 1, n)\}$ and -1 on "vertical" edges $\{(m, n), (m, n + 1)\}$.

The aim of this chapter is to study various transformations of P-nets, in order to lay a foundation for integrable structures on isothermic triangulated surfaces and a unified theory of discrete constant mean curvature surfaces.

Integrable structures on quadrilateral isothermic surfaces (with factorized cross ratios) were well established [**14**]. Bobenko and Pinkall [**4**] introduced quadrilateral isothermic surfaces in terms of Lax representation together with their Christoffel transformations. Hertrich-Jeromin, Hoffmann and Pinkall [**31**] studied Darboux transformations. Hertrich-Jeromin [**29**] discussed Calapso transformations of quadrilateral isothermic surfaces.

It is known that every P-net can be regarded as half a quadrilateral isothermic surface (with cross ratios -1). Labeling the vertices of a quadrilateral isothermic surface black and white alternatingly, the net consists of the black vertices is a P-net. Conversely, we can extend a P-net to a quadrilateral isothermic surfaces with cross ratios -1 by prescribing the position of a white vertex.

Since quadrilateral isothermic surfaces enjoy rich integrable structure it is expected that P-nets also exhibit some sorts of integrable structures. However, it is unclear if the transformations (Christoffel, Darboux and Calapso) depend on the choice of the completion, i.e. the "white" vertex.

We show that there is a redundancy in the theory of quadrilateral isothermic surfaces. We establish Christoffel transformations, Darboux transformations and Calapso transformations of P-nets without completing them to quadrilateral isothermic surfaces with cross ratios -1.

We first illustrate the construction of P-nets from quadrilateral isothermic surfaces with cross ratios -1 and from closed polygons in space.

Lemma 5.1. *Given a non-degenerate quadrilateral $ABCD$ in $\mathbb{R}^3$ and points a, b, c, d such that*

$$\begin{aligned} \overrightarrow{Aa} &= -\lambda \overrightarrow{aB}, \quad \overrightarrow{Bb} = -\frac{1}{\lambda}\overrightarrow{bC}, \\ \overrightarrow{Cc} &= -\lambda \overrightarrow{cD}, \quad \overrightarrow{Dd} = -\frac{1}{\lambda}\overrightarrow{dA} \end{aligned} \tag{5.1}$$

for some $\lambda \in \mathbb{R}\backslash\{0,1\}$. Then abcd is a parallelogram.

Conversely, given a parallelogram abcd, $\lambda \in \mathbb{R}\backslash\{0,1\}$ and an arbitrary point $A \neq a, d$. Then there exists a unique quadrilateral $ABCD$ satisfying (5.1).

Proof. Notice that

$$\overrightarrow{ab} = \overrightarrow{aB} + \overrightarrow{Bb} = \frac{1}{1-\lambda}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{1-\lambda}(\overrightarrow{AD} + \overrightarrow{DC}) = \overrightarrow{dD} + \overrightarrow{Dc} = \overrightarrow{dc}$$

Hence $abcd$ is a parallelogram.

On the other hand, given a parallelogram $abcd$, a number $\lambda \in \mathbb{R}\backslash\{0,1\}$ and an arbitrary point $A \neq a, d$, the points B, C, D, A' are uniquely defined by

$$\overrightarrow{Aa} = -\lambda\overrightarrow{aB}, \quad \overrightarrow{Bb} = -\frac{1}{\lambda}\overrightarrow{bC},$$
$$\overrightarrow{Cc} = -\lambda\overrightarrow{cD}, \quad \overrightarrow{Dd} = -\frac{1}{\lambda}\overrightarrow{dA'}.$$

It remains to show that $A = A'$. In fact,

$$\begin{aligned} A' &= (1-\lambda)d + \lambda D \\ &= (1-\lambda)d - (1-\lambda)c + C \\ &= (1-\lambda)d - (1-\lambda)c + (1-\lambda)b + \lambda B \\ &= (1-\lambda)d - (1-\lambda)c + (1-\lambda)b - (1-\lambda)a + A \\ &= A. \end{aligned}$$

□

Remark 5.2. If $ABCD$ lies in the complex plane, the parameter λ can be complex-valued and the result still holds [**6**].

In order to describe the cross ratio of four points in Euclidean space, we employ quaternions for convenience. Recall that the space of quaternions $\mathbb{H}$ is a 4-dimensional vector space over $\mathbb{R}$ spanned by $1, i, j, k$ with multiplicative relations

$$i^2 = j^2 = k^2 = -1,$$
$$ijk = -1.$$

We denote the space of purely real quaternions and the space of purely imaginary quaternions by

$$\mathrm{Re}(\mathbb{H}) := \mathrm{Span}_{\mathbb{R}}(1)$$
$$\mathrm{Im}(\mathbb{H}) := \mathrm{Span}_{\mathbb{R}}(i, j, k).$$

Hence we can identify the Euclidean space as the space of purely imaginary quaternions $\mathrm{Im}(\mathbb{H})$. Similar to complex numbers, the conjugation of a quaternion $q = a+bi+cj+dk$ where $a, b, c, d \in R$ is denoted by $\bar{q} = a - bi - cj - dk$. The inverse of a nonzero quaternion q is given by

$$q^{-1} := \frac{\bar{q}}{|q|^2}$$

which satisfies $qq^{-1} = q^{-1}q = 1$. In particular if $q \in \mathrm{Im}(\mathbb{H})$, then

$$q^{-1} = -\frac{q}{|q|^2}$$

is the image of q under the inversion in the unit sphere centered at the origin.

Corollary 5.3. *If $f : \mathbb{Z}^2 \to \mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H}$ is an isothermic quadrilateral mesh with cross ratios -1, i.e. for every $(m, n) \in \mathbb{Z}^2$*

$$(f_{m,n} - f_{m+1,n})(f_{m+1,n} - f_{m+1,n+1})^{-1}(f_{m+1,n+1} - f_{m,n+1})(f_{m,n+1} - f_{m,n})^{-1} = -1$$

then $f|_{\mathbb{Z}_b}$ and $f|_{\mathbb{Z}_w}$ are P-nets where

$$\mathbb{Z}_b^2 := \{(m, n) \in \mathbb{Z}^2 | m + n \text{ even}\}$$
$$\mathbb{Z}_w^2 := \{(m, n) \in \mathbb{Z}^2 | m + n \text{ odd}\}.$$

PROOF. Let $(m,n) \in \mathbb{Z}^2$. Applying an inversion in the unit sphere centered at $f_{m,n}$, the neighboring vertices are transformed as

$$f_i \mapsto \tilde{f}_i := f_{m,n} - \frac{f_i - f_{m,n}}{|f_i - f_{m,n}|^2} = f_{m,n} + (f_i - f_{m,n})^{-1}.$$

We are going to apply Lemma 5.1. Taking A, B, C, D as $\tilde{f}_{m+1,n}, \tilde{f}_{m,n+1}, \tilde{f}_{m-1,n}, \tilde{f}_{m,n-1}$ and a, b, c, d as $\tilde{f}_{m+1,n+1}, \tilde{f}_{m-1,n+1}, \tilde{f}_{m-1,n-1}, \tilde{f}_{m+1,n-1}$ yields

$$\begin{aligned}
\overrightarrow{Aa} =& \tilde{f}_{m+1,n+1} - \tilde{f}_{m+1,n} \\
=& -(f_{m+1,n+1} - f_{m,n})^{-1}(f_{m+1,n+1} - f_{m,n} + f_{m,n} - f_{m+1,n})(f_{m,n+1} - f_{m,n})^{-1} \\
=& -(f_{m+1,n+1} - f_{m,n})^{-1}(f_{m+1,n+1} - f_{m+1,n})(f_{m+1,n} - f_{m,n})^{-1}, \\
\overrightarrow{aB} =& \tilde{f}_{m,n+1} - \tilde{f}_{m+1,n+1} \\
=& -(f_{m+1,n+1} - f_{m,n})^{-1}(f_{m,n+1} - f_{m+1,n+1})(f_{m,n+1} - f_{m,n})^{-1}.
\end{aligned}$$

We have

$$(\overrightarrow{Aa})^{-1}\overrightarrow{aB} = -(f_{m,n} - f_{m+1,n})(f_{m+1,n} - f_{m+1,n+1})^{-1}(f_{m+1,n+1} - f_{m,n+1})(f_{m,n+1} - f_{m,n})^{-1} = 1$$

and hence

$$\overrightarrow{Aa} = \overrightarrow{aB}.$$

Similarly,

$$\overrightarrow{Bb} = \overrightarrow{bC}, \quad \overrightarrow{Cc} = \overrightarrow{cD}, \quad \overrightarrow{Dd} = \overrightarrow{dA}.$$

Thus by taking $\lambda = -1$ in Lemma 5.1, we obtain

$$\frac{f_{m+1,n+1} - f_{m,n}}{|f_{m+1,n+1} - f_{m,n}|^2} - \frac{f_{m-1,n+1} - f_{m,n}}{|f_{m-1,n+1} - f_{m,n}|^2} + \frac{f_{m-1,n-1} - f_{m,n}}{|f_{m-1,n-1} - f_{m,n}|^2} - \frac{f_{m+1,n-1} - f_{m,n}}{|f_{m+1,n-1} - f_{m,n}|^2} = 0.$$

Since the above holds for every $(m,n) \in \mathbb{Z}^2$, it implies $f|_{\mathbb{Z}_b}$ and $f|_{\mathbb{Z}_w}$ are P-nets. □

On the other hand, we can obtain a P-net by flowing a closed polygon.

Theorem 5.4. *Consider a closed discrete curve $S = (V, E)$. Given a realization $\gamma_0 : V \to \mathbb{R}^3$ and a vector field $v : V \to \mathbb{R}^3$, we consider a flow $\gamma : V \times Z_+ \to \mathbb{R}^3$ such that*

$$\begin{aligned}
\gamma(\cdot, 0) &:= \gamma_0 \\
\gamma(\cdot, 1) &:= \gamma_0 + v \\
\gamma(i, t+1) &:= \gamma(i,t) + R(i,t)/|R(i,t)|^2
\end{aligned}$$

where

$$R(i,t) = \frac{\gamma(i+1,t) - \gamma(i,t)}{|\gamma(i+1,t) - \gamma(i,t)|^2} + \frac{\gamma(i-1,t) - \gamma(i,t)}{|\gamma(i-1,t) - \gamma(i,t)|^2} - \frac{\gamma(i,t-1) - \gamma(i,t)}{|\gamma(i,t-1) - \gamma(i,t)|^2}.$$

Then γ is a P-net.

The parallelogram property of P-nets is imposed at every interior vertex, in contrast to quadrilateral isothermic surfaces where the cross-ratio condition is imposed on faces. Such a difference is revealed by the property that P-nets are critical points of a functional by vertex-based variations.

Theorem 5.5. *Let $D \subset \mathbb{Z}^2$ be a bounded domain. We consider the functional $\mathcal{F}$ on the space of non-degenerate realizations $f : V(D) \to \mathbb{R}^3$*

$$\mathcal{F}(f) := \log \frac{\prod_{(m,n),(m+1,n)\in D} |f_{m+1,n} - f_{m,n}|}{\prod_{(m,n),(m,n+1)\in D} |f_{m,n+1} - f_{m,n}|}$$

Then, f is a critical point of $\mathcal{F}$ under vertex-based variations with boundary fixed if and only if f is a P-net.

PROOF. For any interior vertex $v_0 \in V(D)$,

$$\frac{\partial \mathcal{F}}{\partial f_0} = \frac{f_1 - f_0}{|f_1 - f_0|^2} - \frac{f_2 - f_0}{|f_2 - f_0|^2} + \frac{f_3 - f_0}{|f_3 - f_0|^2} - \frac{f_4 - f_0}{|f_4 - f_0|^2}.$$

Hence f is a P-net if and only if f is a critical point of $\mathcal{F}$. □

5.1. Christoffel transformations

In Section 4.8, we showed that the parallelogram property of a P-net yields a realization of its dual mesh. In fact the surface obtained is again a P-net.

Theorem 5.6. *Given a P-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$, there exists a map $f^* : (\mathbb{Z}^2)^* \to \mathbb{R}^3$ such that*

$$\begin{aligned} df^*(e^*_{(m,n),(m+1,n)}) &= \frac{f_{m+1,n} - f_{m,n}}{|f_{m+1,n} - f_{m,n}|^2}, \\ df^*(e^*_{(m,n),(m,n+1)}) &= -\frac{f_{m,n+1} - f_{m,n}}{|f_{m,n+1} - f_{m,n}|^2}. \end{aligned}$$

Furthermore, f^* is a P-net. We call f^* a Christoffel transform of f.

Proof. The existence of f^* follows from the parallelogram property of f.

For convenience, we use half integer coordinates to represent the dual mesh $(\mathbb{Z}^2)^*$. Notice that

$$\begin{aligned} f^*_{m+1/2,n+1/2} - f^*_{m+1/2,n-1/2} &= \frac{f_{m+1,n} - f_{m,n}}{|f_{m+1,n} - f_{m,n}|^2}, \\ f^*_{m-1/2,n+1/2} - f^*_{m+1/2,n+1/2} &= -\frac{f_{m,n+1} - f_{m,n}}{|f_{m,n+1} - f_{m,n}|^2}. \end{aligned}$$

Then, around a vertex $(m', n') := (m + 1/2, n + 1/2) \in \mathbb{Z}^{2*}$ we have

$$\begin{aligned} &\frac{f^*_{m'+1,n'} - f^*_{m',n'}}{|f^*_{m'+1,n'} - f^*_{m',n'}|^2} - \frac{f^*_{m',n'+1} - f^*_{m',n'}}{|f^*_{m',n'+1} - f^*_{m',n'}|^2} + \frac{f^*_{m'-1,n'} - f^*_{m',n'}}{|f^*_{m'-1,n'} - f^*_{m',n'}|^2} - \frac{f^*_{m',n'-1} - f^*_{m',n'}}{|f^*_{m',n'-1} - f^*_{m',n'}|^2} \\ =&(f_{m+1,n+1} - f_{m+1,n}) - (f_{m+1,n+1} - f_{m,n+1}) + (f_{m,n} - f_{m,n+1}) - (f_{m,n} - f_{m+1,n}) \\ =&0 \end{aligned}$$

Hence, f^* satisfies the parallelogram property and thus is a P-net. □

5.2. Darboux transformations

Theorem 5.7. *Given a P-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$, for every $\lambda \in \mathbb{R}\backslash\{0, 1\}$ there exists a map $f^\dagger$: $(\mathbb{Z}^2)^* \to \mathbb{R}^3$ such that for every $(m, n) \in \mathbb{Z}^2$*

$$\begin{aligned} \lambda =&(f_{m,n} - f^\dagger_{m+\frac{1}{2},n-\frac{1}{2}})(f^\dagger_{m+\frac{1}{2},n-\frac{1}{2}} - f_{m+1,n})^{-1}(f_{m+1,n} - f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}})(f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}} - f_{m,n})^{-1} \\ =&(f^\dagger_{m-\frac{1}{2},n+\frac{1}{2}} - f_{m,n})(f_{m,n} - f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}})^{-1}(f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}} - f_{m,n+1})(f_{m,n+1} - f^\dagger_{m-\frac{1}{2},n+\frac{1}{2}})^{-1}. \end{aligned}$$

The map $f^\dagger$ is a P-net and exists uniquely whenever one of its vertex position is prescribed. We call $f^\dagger$ a Darboux transform of f.

Proof. We are going to apply Lemma 5.1 to show that by fixing $\lambda \in \mathbb{R}\backslash\{0, 1\}$ and the position of a dual vertex $(m + 1/2, n - 1/2)$, then the positions of the neighboring vertices $(m + 1/2, n + 1/2), (m - 1/2, n + 1/2), (m - 1/2, n - 1/2)$ are uniquely determined.

We take an inversion in the unit sphere centered at $f_{m,n}$

$$x \mapsto \tilde{x} := f_{m,n} - \frac{x - f_{m,n}}{|x - f_{m,n}|^2} = f_{m,n} + (x - f_{m,n})^{-1}.$$

We let a, b, c, d in Lemma 5.1 be $\tilde{f}_{m+1,n}, \tilde{f}_{m,n+1}, \tilde{f}_{m-1,n}, \tilde{f}_{m,n-1}$ and $A = \tilde{f}^\dagger_{m+1/2,n-1/2}$. Lemma 5.1 implies there exists $B, C, D \in \mathbb{R}^3$ such that Equation (5.1) holds. Defining

$$\begin{aligned} f^\dagger_{m+1/2,n+1/2} :=& f_{m,n} + (B - f_{m,n})^{-1}, \\ f^\dagger_{m+1/2,n+1/2} :=& f_{m,n} + (C - f_{m,n})^{-1}, \\ f^\dagger_{m+1/2,n+1/2} :=& f_{m,n} + (D - f_{m,n})^{-1} \end{aligned}$$

we get

$$\begin{aligned}\lambda =&(f_{m,n} - f^\dagger_{m+\frac{1}{2},n-\frac{1}{2}})(f^\dagger_{m+\frac{1}{2},n-\frac{1}{2}} - f_{m+1,n})^{-1}(f_{m+1,n} - f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}})(f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}} - f_{m,n})^{-1}\\ =&(f^\dagger_{m-\frac{1}{2},n+\frac{1}{2}} - f_{m,n})(f_{m,n} - f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}})^{-1}(f^\dagger_{m+\frac{1}{2},n+\frac{1}{2}} - f_{m,n+1})(f_{m,n+1} - f^\dagger_{m-\frac{1}{2},n+\frac{1}{2}})^{-1}\\ =&(f_{m-1,n} - f^\dagger_{m-\frac{1}{2},n-\frac{1}{2}})(f^\dagger_{m-\frac{1}{2},n-\frac{1}{2}} - f_{m,n})^{-1}(f_{m,n} - f^\dagger_{m-\frac{1}{2},n+\frac{1}{2}})(f^\dagger_{m-\frac{1}{2},n+\frac{1}{2}} - f_{m-1,n})^{-1}\\ =&(f^\dagger_{m-\frac{1}{2},n-\frac{1}{2}} - f_{m,n-1})(f_{m,n-1} - f^\dagger_{m+\frac{1}{2},n-\frac{1}{2}})^{-1}(f^\dagger_{m+\frac{1}{2},n-\frac{1}{2}} - f_{m,n})(f_{m,n} - f^\dagger_{m-\frac{1}{2},n-\frac{1}{2}})^{-1}.\end{aligned}$$

Iterating the above process for every $(m,n) \in V(\mathbb{Z}^2)$, we obtain $f^\dagger$ as required.

It remains to show that $f^\dagger$ is a P-net. Let $(m',n') := (m+1/2, n+1/2) \in V^*(\mathbb{Z}^2) \cong F(\mathbb{Z}^2)$ be a dual vertex. Then the parallelogram property of $f^\dagger$ follows from Lemma 5.1 if we take A,B,C,D to be $f_{m'+1/2,n'-1/2}, f_{m'+1/2,n'+1/2}, f_{m'-1/2,n'+1/2}, f_{m'-1/2,n'-1/2}$ and a,b,c,d to be $f^\dagger_{m'+1,n'}, f^\dagger_{m',n'+1}, f^\dagger_{m'-1,n'}, f^\dagger_{m',n'-1}$. □

5.3. Calapso transformations

In order to simplify calculations, we employ the quaternionic projective model of the 3-sphere $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ [**30**].

We regard $\mathbb{H}^2$ as a 2-dimensional right vector space over $\mathbb{H}$ and consider its projectivization $\mathbb{H}P^1$. The 3-sphere is viewed as a subset of the quaternionic projective line $\mathbb{H}P^1$:

$$S^3 = \{v\mathbb{H} \in \mathbb{H}P^1 | s(v,v) = 0\}$$

where $s(u,v) := \bar{u}_1 v_2 + \bar{u}_2 v_1$ for for $u, v \in \mathbb{H}^2$. Indeed

$$0 = s(v,v) = \bar{v_1} v_2 + \bar{v_2} v_1 \iff v_2 = 0 \quad \text{or} \quad \mathrm{Re}(v_1 v_2^{-1}) = 0$$

and we have

$$\{v\mathbb{H} \in \mathbb{H}P^1 | s(v,v) = 0\} = \{\begin{pmatrix} x \\ 1 \end{pmatrix}\mathbb{H} | x \in \mathrm{Im}(\mathbb{H})\} \cup \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{H}\} \cong \mathbb{R}^3 \cup \{\infty\} \cong S^3$$

In such a model Möbius transformations of S^3 are represented as elements in $PGL(2,\mathbb{H}) \cong GL(2,\mathbb{H})/\mathbb{R}I$ that preserve the null cone of the hermitian form s.

Given a P-net $f : V(\mathbb{Z}^2) \to \mathbb{R}^3$ with the canonical discrete holomorphic quadratic differential $\mu : E(\mathbb{Z}^2) \to \pm 1$, we consider a $gl(2,\mathbb{H})$-valued (additive) dual 1-form

$$\tau(e^*_{ij}) := \begin{pmatrix} \frac{f_i+f_j}{2} df^*(e^*_{ij}) & -f_j df^*(e^*_{ij}) f_i \\ df^*(e^*_{ij}) & -df^*(e^*_{ij}) \frac{f_i+f_j}{2} \end{pmatrix}$$

where $df^*(e^*_{ij}) = \mu_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}$. It satisfies

$$\begin{aligned}\tau(e^*_{ij})\begin{pmatrix} f_i \\ 1 \end{pmatrix} &= \frac{(f_i - f_j) df^*(e^*_{ij})}{2}\begin{pmatrix} f_i \\ 1 \end{pmatrix} = \frac{\mu_{ij}}{2}\begin{pmatrix} f_i \\ 1 \end{pmatrix},\\ \tau(e^*_{ij})\begin{pmatrix} f_j \\ 1 \end{pmatrix} &= \frac{(f_j - f_i) df^*(e^*_{ij})}{2}\begin{pmatrix} f_j \\ 1 \end{pmatrix} = -\frac{\mu_{ij}}{2}\begin{pmatrix} f_j \\ 1 \end{pmatrix}.\end{aligned}$$

Lemma 5.8. *The dual 1-form τ is closed, i.e. for every interior vertex i*

$$\sum_j \tau(e^*_{ij}) = 0.$$

Proof. Firstly, it is obvious that

$$\sum_j df^*(e^*_{ij}) = 0.$$

Secondly,

$$\begin{aligned}
&\sum_j \frac{f_i + f_j}{2} df^*(e^*_{ij}) \\
=&f_i \sum_j df^*(e^*_{ij}) + \sum_j \frac{f_i - f_j}{2} df^*(e^*_{ij}) \\
=&f_i \sum_j df^*(e^*_{ij}) + \sum_j \frac{\mu_{ij}}{2} \\
=&0.
\end{aligned}$$

The other terms follow similarly. □

For $\lambda \in \mathbb{R}$, we define a map $\eta_\lambda : \vec{E}^*(\mathbb{Z}^2) \to GL(2, \mathbb{H})$

$$\eta_\lambda(e^*_{ij}) := \cosh\lambda + 2\sinh\lambda\, \tau(e^*_{ij})$$

In fact, since the canonical holomorphic quadratic differential μ takes values ± 1, we have

$$\begin{aligned}
\eta_\lambda(e^*_{ij}) \begin{pmatrix} f_i \\ 1 \end{pmatrix} &= (\cosh\lambda + \mu_{ij}\sinh\lambda) \begin{pmatrix} f_i \\ 1 \end{pmatrix} = e^{\mu_{ij}\lambda} \begin{pmatrix} f_i \\ 1 \end{pmatrix}, \\
\eta_\lambda(e^*_{ji}) \begin{pmatrix} f_j \\ 1 \end{pmatrix} &= (\cosh\lambda - \mu_{ij}\sinh\lambda) \begin{pmatrix} f_j \\ 1 \end{pmatrix} = e^{-\mu_{ij}\lambda} \begin{pmatrix} f_j \\ 1 \end{pmatrix}
\end{aligned} \tag{5.2}$$

which implies

$$\begin{aligned}
\eta_\lambda(e^*_{ij})\eta_\lambda(e^*_{ji}) \begin{pmatrix} f_i \\ 1 \end{pmatrix} &= (\cosh^2\lambda - \sinh^2\lambda) \begin{pmatrix} f_i \\ 1 \end{pmatrix} = \begin{pmatrix} f_i \\ 1 \end{pmatrix}, \\
\eta_\lambda(e^*_{ij})\eta_\lambda(e^*_{ji}) \begin{pmatrix} f_j \\ 1 \end{pmatrix} &= (\cosh^2\lambda - \sinh^2\lambda) \begin{pmatrix} f_j \\ 1 \end{pmatrix} = \begin{pmatrix} f_j \\ 1 \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\eta_\lambda(e^*_{ij})\eta_\lambda(e^*_{ji}) &= I, \\
\eta_\lambda(e^*_{ij})^{-1} &= \eta_\lambda(e^*_{ji})
\end{aligned}$$

and η_λ is a $GL(2, \mathbb{H})$-valued multiplicative dual 1-form.

Lemma 5.9. *The multiplicative dual 1-form η_λ on a P-net is closed, i.e. for every interior vertex i*

$$\prod_j \eta_\lambda(e^*_{ij}) = I$$

and hence there exists $T_\lambda : F(\mathbb{Z}^2) \to GL(2, \mathbb{H})$ such that

$$T_{\lambda,\phi_l} = T_{\lambda,\phi_r}\eta_\lambda(e^*_{ij})$$

where ϕ_l denotes the left face of e_{ij} and ϕ_r denotes the right face of e_{ij}.

Proof. Let v_0 be an interior vertex of a P-net and v_1, v_2, v_3, v_4 be its neighboring vertices. In order to prove the claim, we are going to apply $\prod_j \eta_\lambda(e^*_{0j})$ to a basis $\{\begin{pmatrix} f_0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ of $\mathbb{H}^2$.

Firstly, it is obvious from (5.2) that

$$\eta_\lambda(e^*_{04})\eta_\lambda(e^*_{03})\eta_\lambda(e^*_{02})\eta_\lambda(e^*_{01}) \begin{pmatrix} f_0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_0 \\ 1 \end{pmatrix}.$$

since $\mu_{01} = -\mu_{02} = \mu_{03} = -\mu_{04}$. It remains to show that

$$\eta_\lambda(e^*_{04})\eta_\lambda(e^*_{03})\eta_\lambda(e^*_{02})\eta_\lambda(e^*_{01}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which is equivalent to

$$\eta_\lambda(e^*_{02})\eta_\lambda(e^*_{01}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta_\lambda(e^*_{30})\eta_\lambda(e^*_{40}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Notice we have

$$\tau(e_{02}^*)\tau(e_{01}^*)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{f_0+f_2}{2}df^*(e_{02}^*)\frac{f_0+f_1}{2}df^*(e_{01}^*)-\frac{f_0df^*(e_{02}^*)f_2df^*(e_{01}^*)}{2}-\frac{f_2df^*(e_{02}^*)f_0df^*(e_{01}^*)}{2}\\ df^*(e_{02}^*)\frac{f_0+f_1}{2}df^*(e_{01}^*)-df^*(e_{02}^*)\frac{f_0+f_2}{2}df^*(e_{01}^*)\end{pmatrix}$$
$$=\begin{pmatrix}\frac{f_2-f_0}{2}df^*(e_{02}^*)\frac{f_1-f_0}{2}df^*(e_{01}^*)+\frac{f_0df^*(e_{02}^*)(f_1-f_2)df^*(e_{01}^*)}{2}\\ df^*(e_{02}^*)\frac{f_1-f_0}{2}df^*(e_{01}^*)-df^*(e_{02}^*)\frac{f_2-f_0}{2}df^*(e_{01}^*)\end{pmatrix}$$
$$=\begin{pmatrix}\frac{\mu_{02}\mu_{01}}{4}-\frac{f_0}{2}(\mu_{01}df^*(e_{02}^*)-\mu_{02}df^*(e_{01}^*))\\ -\frac{1}{2}(\mu_{01}df^*(e_{02}^*)-\mu_{02}df^*(e_{01}^*))\end{pmatrix}$$

and similarly

$$\tau(e_{30}^*)\tau(e_{40}^*)\begin{pmatrix}1\\0\end{pmatrix}=\tau(e_{03}^*)\tau(e_{04}^*)\begin{pmatrix}1\\0\end{pmatrix}$$
$$=\begin{pmatrix}\frac{\mu_{03}\mu_{04}}{4}-\frac{f_0}{2}(\mu_{04}df^*(e_{03}^*)-\mu_{03}df^*(e_{04}^*))\\ -\frac{1}{2}(\mu_{04}df^*(e_{03}^*)-\mu_{03}df^*(e_{04}^*))\end{pmatrix}$$

which implies

$$\tau(e_{02}^*)\tau(e_{01}^*)\begin{pmatrix}1\\0\end{pmatrix}=\tau(e_{30}^*)\tau(e_{40}^*)\begin{pmatrix}1\\0\end{pmatrix}.$$

Thus

$$\eta_\lambda(e_{02}^*)\eta_\lambda(e_{01}^*)\begin{pmatrix}1\\0\end{pmatrix}=(\cosh\lambda+2\cosh\lambda\,\sinh\lambda(\tau(e_{02}^*)+\tau(e_{01}^*))+4\sinh^2\lambda\,\tau(e_{02}^*)\tau(e_{01}^*))\begin{pmatrix}1\\0\end{pmatrix}$$
$$=(\cosh\lambda+2\cosh\lambda\,\sinh\lambda(\tau(e_{30}^*)+\tau(e_{40}^*))+4\sinh^2\lambda\,\tau(e_{30}^*)\tau(e_{40}^*))\begin{pmatrix}1\\0\end{pmatrix}$$
$$=\eta_\lambda(e_{30}^*)\eta_\lambda(e_{40}^*)\begin{pmatrix}1\\0\end{pmatrix}.$$

Hence

$$\eta_\lambda(e_{04}^*)\eta_\lambda(e_{03}^*)\eta_\lambda(e_{02}^*)\eta_\lambda(e_{01}^*)=I.$$ □

For every edge $\{ij\}$ we have

$$T_{\lambda,\phi_l}\begin{pmatrix}f_i\\1\end{pmatrix}=T_{\lambda,\phi_r}\eta_\lambda(e_{ij}^*)\begin{pmatrix}f_i\\1\end{pmatrix}=T_{\lambda,\phi_r}\begin{pmatrix}f_i\\1\end{pmatrix}e^{\mu_{ij}\lambda}$$

where ϕ_l denotes the left face of e_{ij} and ϕ_r denotes the right face of e_{ij}. Hence

$$\mathtt{f}_{\lambda,i}:=[T_{\lambda,\phi_l}\begin{pmatrix}f_i\\1\end{pmatrix}]=[T_{\lambda,\phi_r}\begin{pmatrix}f_i\\1\end{pmatrix}]$$

defines a map $\mathtt{f}_\lambda:\mathbb{Z}^2\to\mathbb{H}P^1$. By suitable Möbius transformations, we assume no vertices of $\mathtt{f}_\lambda$ are at infinity. We define $f_\lambda:\mathbb{Z}^2\to\mathbb{H}$ such that

$$\mathtt{f}_\lambda=\begin{bmatrix}f_\lambda\\1\end{bmatrix}. \tag{5.3}$$

We are going to show that f_λ can be assumed to take values in $\operatorname{Im}\mathbb{H}$.

Lemma 5.10. *For any $v\in\mathbb{H}^2$*

$$s(\eta_\lambda(e_{ij}^*)v,\eta_\lambda(e_{ij}^*)v)=s(v,v).$$

Hence if we have for some $\phi_0\in F(\mathbb{Z}^2)$

$$s(T_{\lambda,\phi_0}v,T_{\lambda,\phi_0}v)=s(v,v)$$

then

$$s(T_{\lambda,\phi}v,T_{\lambda,\phi}v)=s(v,v)$$

for all $\phi\in F(\mathbb{Z}^2)$.

Proof. Notice that $\eta_\lambda(e^*_{ij})$ has eigenvectors $\mathtt{f}_i := \begin{pmatrix} f_i \\ 1 \end{pmatrix}, \mathtt{f}_j := \begin{pmatrix} f_j \\ 1 \end{pmatrix}$ which form a basis of $\mathbb{H}^2$. Let $v \in \mathbb{H}^2$. Then there exists $\alpha, \beta \in \mathbb{H}$ such that $v = \mathtt{f}_i\alpha + \mathtt{f}_j\beta$. We then have

$$s(\eta_\lambda(e^*_{ij})v, \eta_\lambda(e^*_{ij})v) = \bar{\alpha}\, s(\mathtt{f}_i, \mathtt{f}_j)\, \bar{\beta} + \bar{\beta}\, s(\mathtt{f}_j, \mathtt{f}_i)\, \bar{\alpha} = s(v, v)$$

where we have used $s(\mathtt{f}_i, \mathtt{f}_i) = s(\mathtt{f}_j, \mathtt{f}_j) = 0$. □

The above lemma implies that we can assume f_λ to take values in $\mathrm{Im}(\mathbb{H})$ by choosing a suitable T_{λ,ϕ_0}.

It remains to show that f_λ is a P-net. We first obtain a characterization of P-nets in terms of $\mathbb{H}P^1$. To every quaternionic line $v\mathbb{H} \in \mathbb{H}^2$ there is a unique line $\mathbb{H}\nu$ in the dual space $\mathbb{H}^*$ that annihilates $v\mathbb{H} \in \mathbb{H}^2$. We have $(\mathbb{H}P^1)^\perp := \{\mathbb{H}\nu | \nu \in (\mathbb{H}^2)^*\} \cong \mathbb{H}P^1$. The group $PGL(2, \mathbb{H})$ acts on $(\mathbb{H}P^1)^\perp$ via $\mathbb{H}\nu \mapsto \mathbb{H}(M \cdot \nu) = \mathbb{H}(\nu M^{-1})$.

If f is a point in $\mathbb{R}^3$, then we write $\mathtt{f} = \begin{pmatrix} f \\ 1 \end{pmatrix}$ and its annihilator $\mathtt{f}^\perp = (-1, f)$. We have $\mathtt{f}^\perp\mathtt{f} = 0$. The quaternionic cross ratio can be expressed as

$$\begin{aligned}&(f_{m,n} - f_{m+1,n})(f_{m+1,n} - f_{m+1,n+1})^{-1}(f_{m+1,n+1} - f_{m,n+1})(f_{m,n+1} - f_{m,n})^{-1}\\ =&\mathtt{f}^\perp_{m,n}\mathtt{f}_{m+1,n}(\mathtt{f}^\perp_{m+1,n+1}\mathtt{f}_{m+1,n})^{-1}\mathtt{f}^\perp_{m+1,n+1}\mathtt{f}_{m,n+1}(\mathtt{f}^\perp_{m,n}\mathtt{f}_{m,n+1})^{-1}.\end{aligned}$$

Lemma 5.11. *Let $a_i \in \mathbb{H}^2\backslash\{0\}$ be a collection of points such that $[a_i] \neq [a_{i+1}] \in \mathbb{H}P^1$. We consider*

$$Q(a_1, a_2, a_3, a_4) := a_1{}^\perp a_2(a_3{}^\perp a_2)^{-1}a_3{}^\perp a_4(a_1{}^\perp a_4)^{-1} \in \mathbb{H}.$$

Then the cross ratio

$$q([a_1], [a_2], [a_3], [a_4]) := \mathrm{Re}\, Q(a_1, a_2, a_3, a_4) + i|\,\mathrm{Im}\, Q(a_1, a_2, a_3, a_4)| \in \mathbb{C}$$

is well defined, independent of the choice of homogeneous coordinates. Similarly, we define the multi cross ratio

$$mq([a_1], [a_2], [a_3], [a_4], [a_5], [a_6]) := \mathrm{Re}\, mQ(a_1, a_2, a_3, a_4, a_5, a_6) + i|\,\mathrm{Im}\, mQ(a_1, a_2, a_3, a_4, a_5, a_6)|$$

where

$$mQ(a_1, a_2, a_3, a_4, a_5, a_6) := -a_1^\perp a_2(a_3^\perp a_2)^{-1}a_3^\perp a_4(a_5^\perp a_4)^{-1}a_5^\perp a_6(a_1^\perp a_6)^{-1}.$$

Lemma 5.12. *A map $f : \mathbb{Z}^2 \to \mathbb{R}^3$ is a P-net if for every $(m, n) \in \mathbb{Z}^2$ the multi-cross ratio*

$$mq([\mathtt{f}_{m,n}], [\mathtt{f}_{m,n+1}], [\mathtt{f}_{m+1,n}], [\mathtt{f}_{m,n}], [\mathtt{f}_{m-1,n}], [\mathtt{f}_{m,n-1}]) = -1$$

where $[\mathtt{f}] : \mathbb{Z}^2 \to \mathbb{H}P^1$ denotes the embedding of Euclidean space

$$f \mapsto [\mathtt{f}] = \begin{bmatrix} f \\ 1 \end{bmatrix} \in \mathbb{H}P^1$$

Proof. A map f is a P-net if for every $(m, n) \in \mathbb{Z}^2$, the parallelogram property is satisfied:

$$\begin{aligned}0 =&(f_{m+1,n} - f_{m,n})^{-1} - (f_{m,n+1} - f_{m,n})^{-1} + (f_{m-1,n} - f_{m,n})^{-1} - (f_{m,n-1} - f_{m,n})^{-1}\\ =& \quad (f_{m+1,n} - f_{m,n})^{-1}(f_{m,n+1} - f_{m+1,n})(f_{m,n+1} - f_{m,n})^{-1}\\ &+ (f_{m-1,n} - f_{m,n})^{-1}(f_{m-1,n} - f_{m,n-1})(f_{m,n-1} - f_{m,n})^{-1}\end{aligned}$$

It is equivalent to

$$\begin{aligned}-1 =&(f_{m,n} - f_{m,n+1})(f_{m,n+1} - f_{m+1,n})^{-1}(f_{m+1,n} - f_{m,n})(f_{m,n} - f_{m-1,n})^{-1}(f_{m-1,n} - f_{m,n-1})\\ &(f_{m,n-1} - f_{m,n})^{-1}\\ =&mQ(\mathtt{f}_{m,n}, \mathtt{f}_{m,n+1}, \mathtt{f}_{m+1,n}, \mathtt{f}_{m,n}, \mathtt{f}_{m-1,n}, \mathtt{f}_{m,n-1})\end{aligned}$$

where $\mathtt{f} = \begin{pmatrix} f \\ 1 \end{pmatrix}$. □

Theorem 5.13. *Given a P-net $f : \mathbb{Z}^2 \to \mathbb{R}^3$, there exists $f_\lambda : \mathbb{Z}^2 \to \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$ satisfying Eq. 5.3 for any $\lambda \in \mathbb{R}$. Moreover, f_λ is a P-net. We call f_λ a Calapso transformation of f.*

PROOF. To simplify the proof, we pick suitable homogeneous coordinates of $[\mathtt{f}_\lambda]$. Let $(m,n) \in \mathbb{Z}^2$. For convenience, we suppress the parameter λ of T. Then

$$\begin{aligned}
&mQ(T_{m+\frac{1}{2},n+\frac{1}{2}}\mathtt{f}_{m,n}, T_{m+\frac{1}{2},n+\frac{1}{2}}\mathtt{f}_{m,n+1}, T_{m+\frac{1}{2},n+\frac{1}{2}}\mathtt{f}_{m+1,n}, T_{m-\frac{1}{2},n+\frac{1}{2}}\mathtt{f}_{m,n}, T_{m-\frac{1}{2},n-\frac{1}{2}}\mathtt{f}_{m-1,n},\\
&\qquad T_{m+\frac{1}{2},n-\frac{1}{2}}\mathtt{f}_{m,n-1})\\
=&mQ(\mathtt{f}_{m,n}, \mathtt{f}_{m,n+1}, \mathtt{f}_{m+1,n}, \mathtt{f}_{m,n}, \mathtt{f}_{m-1,n}, \mathtt{f}_{m,n-1})\\
=&-1.
\end{aligned}$$

Hence the claim follows. $\square$

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