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A BERNSTEIN PROPERTY OF AFFINE MAXIMAL HYPERSURFACES

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A Bernstein Property of Affine Maximal Hypersurfaces

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Abstract. Let $x : M \to A^{n+1}$ be the graph of some strictly convex function $x_{n+1} = f(x_1, \cdots, x_n)$ defined in a convex domain $\Omega \subset A^n$. We introduce a Riemannian metric $G^{\#} = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$ on M. In this paper, we investigate the affine maximal hypersurface which is complete with respect to the metric $G^{\#}$, and prove a Bernstein property for the affine maximal hypersurfaces.

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Introduction

Affine maximal hypersurfaces are extremals of the interior variation of the affinely invariant volume. The corresponding Euler-Lagrange equation is a fourth order PDE. Originally, these hypersurfaces are called " affine minimal hypersurfaces". Calabi calculated the second variation and proposed to call them " affine maximal". For affine maximal surfaces, there are different versions of so called affine Bernstein conjectures, stated by Calabi and Chern (see [CA-1] and [CH]). The conjectures differ in the assumptions on the completeness of the affine maximal surface considered. While Chern assumed that the surface is a convex graph over R^2, which means that the surface is Euclidean complete, Calabi assumed that the surface is complete with respect to the Blaschke metric. In [T-W] the authors present a proof of Chern's conjecture. In [L-J] we solved Calabi's conjecture.

Let $x : M \to A^{n+1}$ be an affine maximal hypersurface given by a strictly convex function

$$x_{n+1} = f(x_1, \cdots, x_n)$$

defined in a convex domain $\Omega \subset A^n$. $x(M)$ is an affine maximal hypersurface if and only if f satisfies the following PDE of 4-th order:

$$\Delta \left[det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right]^{-1/(n+2)} = 0,$$

where Δ denotes the Laplacian with respect to the Blaschke metric G, which is defined by

$$\Delta = \frac{1}{(det(G_{kl}))^{1/2}} \sum \frac{\partial}{\partial x_i} \left(G^{ij} (det(G_{kl}))^{1/2} \frac{\partial}{\partial x_j} \right).$$

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Following E. Calabi and A.V. Pogorelov (see [CA-3] and [P]), we consider the Riemannian metric $G^{\#}$ on M, defined by

$$G^{\#} = \sum f_{ij} dx_i dx_j,$$

where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. This is a very natural metric for a convex graph. We want to raise the following conjecture:

Conjecture. Let $x : M \to A^{n+1}$ be the graph of some strictly convex function

$$x_{n+1} = f(x_1, \cdots, x_n)$$

defined in a convex domain $\Omega \subset A^n$. If $x(M)$ is an affine maximal hypersurface and if $x(M)$ is complete with respect to the metric $G^{\#}$, then M must be an elliptic paraboloid.

In this paper we give an affirmative answer to this conjecture for 2 and 3 dimensions. Precisely, we prove the following theorem:

Theorem. *Let $x_{n+1} = f(x_1, \cdots, x_n)$ be a strictly convex function defined in a convex domain $\Omega \subset A^n$. If $M = \{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) | (x_1, \cdots, x_n) \in \Omega\}$ is an affine maximal hypersurface, and if M is complete with respect to the metric $G^{\#}$, then, in the case $n = 2$ or $n = 3$, M must be an elliptic paraboloid.*

For the proof of the Theorem, we first show that if the maximal hypersurface M is complete with respect to the metric $G^{\#}$ and if the norm of its Ricci curvature $||Ric^{\#}||_{G^{\#}}$ is bounded, then M must be an elliptic paraboloid. Next, we use Hofer's Lemma to prove that $||Ric^{\#}||_{G^{\#}}$ must be bounded.

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1. Preliminaries

Let A^{n+1} be the unimodular affine space of dimension $n+1$, M be a connected and oriented C^∞ manifold of dimension n, and $x : M \to A^{n+1}$ a locally strongly convex hypersurface. We choose a local unimodular affine frame field $x, e_1, e_2, ..., e_n, e_{n+1}$ on M such that

$$e_1, ..., e_n \in T_x M,$$

$$det(e_1, ..., e_n, e_{n+1}) = 1,$$

$$G_{ij} = \delta_{ij},$$

$$e_{n+1} = Y,$$

where we denote by G_{ij} and Y the Blaschke metric and the affine normal vector field, respectively. Denote by U, A_{ijk} and B_{ij} the affine conormal vector field, the Fubini-Pick tensor and the affine Weingarten tensor with respect to the frame field $x, e_1, ..., e_n$, and by R_{ij} denote the Ricci curvature. We have the following local formulas (see [L-S-Z]):

$$U_{,ij} = -\sum A_{ijk} U_{,k} - B_{ij} U, \tag{1.1}$$

$$\Delta U = -nL_1 U, \tag{1.2}$$

$$\sum A_{iik} = 0, \tag{1.3}$$

$$R_{ij} = \sum A_{mli} A_{mlj} + \frac{n-2}{2} B_{ij} + \frac{n}{2} L_1 \delta_{ij}, \tag{1.4}$$

where L_1 denotes the affine mean curvature, and " , " denotes the covariant differentiation with respect to the Blaschke metric. A locally strongly convex hypersurface is called an affine maximal hypersurface if $L_1 = 0$ everywhere. Let $x : M \to A^{n+1}$ be given by a strictly convex function

$$x_{n+1} = f(x_1, ..., x_n).$$

We choose the following unimodular affine frame field:

$$e_1 = \left(1, 0, ..., 0, \frac{\partial f}{\partial x_1}\right)$$

$$e_2 = \left(0, 1, 0, ..., 0, \frac{\partial f}{\partial x_2}\right)$$

$$..........$$

$$e_n = \left(0, 0, ..., 1, \frac{\partial f}{\partial x_n}\right)$$

$$e_{n+1} = (0, 0, ..., 0, 1).$$

The Blaschke metric is given by (see [L-S-Z])

$$G = \left[det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j.$$

The affine conormal vector field U can be identified with

$$U = \left[det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} \left(-\frac{\partial f}{\partial x_1}, ..., -\frac{\partial f}{\partial x_n}, 1\right).$$

The formula $\Delta U = -nL_1 U$ implies that x is an affine maximal hypersurface if and only if f satisfies the following p.d.e.

$$\Delta \left[det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} = 0, \tag{1.5}$$

where Δ denotes the Laplacian with respect to the Blaschke metric, which is defined by

$$\Delta = \frac{1}{(det(G_{kl}))^{1/2}} \sum \frac{\partial}{\partial x_i}\left(G^{ij}(det(G_{kl}))^{1/2}\frac{\partial}{\partial x_j}\right).$$

Denote

$$\rho = [det(\frac{\partial^2 f}{\partial x_i \partial x_j})]^{-1/(n+2)},$$

and

$$\Phi = \frac{||\nabla \rho||_G^2}{\rho}. \tag{1.6}$$

Then (1.5) gives

$$\Delta \rho = 0. \tag{1.7}$$

Now we calculate $\Delta\Phi$. Let $p \in M$. We choose an orthonormal frame field around $p \in M$. Then

$$\Phi = \frac{\sum \rho_{,j}^2}{\rho},$$

$$\Phi_{,i} = \frac{2\sum \rho_{,j}\rho_{,ji}}{\rho} - \frac{\rho_{,i}\sum \rho_{,j}^2}{\rho^2},$$

$$\Delta\Phi = \frac{2\sum \rho_{,ji}^2 + 2\sum \rho_{,j}\rho_{,jii}}{\rho} - 4\frac{\sum \rho_{,j}\rho_{,i}\rho_{,ji}}{\rho^2} + 2\frac{||\nabla \rho||^4}{\rho^3}.$$

In the case $\Phi(p) = 0$, it is easy to get at p

$$\Delta\Phi \geq \frac{2\sum \rho_{,ij}^2}{\rho}. \tag{1.8}$$

Now we assume that $\Phi(p) \neq 0$ and choose an orthonormal frame field such that, at $p \in M$, $\rho_{,1} = ||\nabla \rho||$, $\rho_{,i} = 0 \quad \forall i > 1$. Then

$$\Delta\Phi = \frac{2\sum \rho_{,ji}^2 + 2\sum \rho_{,j}\rho_{,jii}}{\rho} - 4\frac{\rho_{,1}^2\rho_{,11}}{\rho^2} + 2\frac{\rho_{,1}^4}{\rho^3}, \tag{1.9}$$

The Schwarz inequality gives

$$\sum \rho_{ij}^2 \geq \rho_{,11}^2 + \sum_{i>1} \rho_{,ii}^2 + 2\sum_{i>1} \rho_{,1i}^2 \geq \rho_{,11}^2 + \frac{1}{n-1}\left(\sum_{i>1} \rho_{,ii}\right)^2 + 2\sum_{i>1} \rho_{,1i}^2 \tag{1.10}$$

$$= \frac{n}{n-1}\rho_{,11}^2 + 2\sum_{i>1} \rho_{,1i}^2.$$

Similarly,

$$\sum A_{ml1}^2 \geq \frac{n}{n-1} A_{111}^2. \tag{1.11}$$

Taking the $(n+1)$-th component of $U_{ij} = -\sum A_{ijk}U_{,k} - B_{ij}U$ we have

$$\rho_{,11} = -A_{111}\rho_{,1} - B_{11}\rho. \tag{1.12}$$

Using the formula (1.4) we get

$$2\sum \rho_{,j}\rho_{jii} = 2R_{11}\rho_{,1}^2 = 2\sum A_{ml1}^2\rho_{,1}^2 + (n-2)B_{11}\rho_{,1}^2 \tag{1.13}$$

$$= 2\sum A_{ml1}^2\rho_{,1}^2 - (n-2)\frac{\rho_{,11}\rho_{,1}^2}{\rho} - (n-2)A_{111}\frac{\rho_{,1}^3}{\rho}$$

$$\geq -(n-2)\frac{\rho_{,11}\rho_{,1}^2}{\rho} - \frac{(n-2)^2(n-1)}{8n}\frac{\rho_{,1}^4}{\rho^2}.$$

Substituting (1.10)–(1.13) into (1.9) we get

$$\Delta\Phi \geq \frac{2n}{n-1}\frac{\rho_{,11}^2}{\rho} + 4\frac{\sum_{i>1}\rho_{,1i}^2}{\rho} \tag{1.14}$$

$$-(n+2)\frac{\rho_{,1}^2\rho_{,11}}{\rho^2} + \left(2 - \frac{(n-2)^2(n-1)}{8n}\right)\frac{\rho_{,1}^4}{\rho^3}.$$

Note that

$$\frac{\sum \Phi_{,i}^2}{\Phi} = 4\frac{\rho_{,11}^2}{\rho} + 4\frac{\sum_{i>1}\rho_{,1i}^2}{\rho} + \frac{\rho_{,1}^4}{\rho^3} - 4\frac{\rho_{,1}^2\rho_{,11}}{\rho^2}. \tag{1.15}$$

Then (1.14) and (1.15) together give us

$$\Delta\Phi \geq \frac{n}{2(n-1)}\frac{\sum \Phi_{,i}^2}{\Phi} - \frac{n^2-n-2}{2(n-1)}\sum \Phi_{,i}\frac{\rho_{,i}}{\rho} \tag{1.16}$$

$$+\left(2-\frac{(n-2)^2(n-1)}{8n}-\frac{n^2-2}{2(n-1)}\right)\frac{\rho_{,1}^4}{\rho^3}.$$

For $n=2$

$$\Delta\Phi \geq \frac{\sum \Phi_{,i}^2}{\Phi}+\frac{1}{\rho}\Phi^2. \tag{1.17}$$

For $n=3$

$$\Delta\Phi \geq \frac{3}{4}\frac{\sum \Phi_{,i}^2}{\Phi}+\frac{1}{6}\frac{\Phi^2}{\rho}-\sum \Phi_{,i}\frac{\rho_{,i}}{\rho}. \tag{1.18}$$

Denote by $\Delta^\#$ and $R_{ij}^\#$ the Laplacian and the Ricci curvature with respect to the metric $G^\#$, resp.. By definition of the Laplacian and a direct calculation we get

$$\frac{||\nabla\rho||_{G^\#}^2}{\rho}=||\nabla\rho||_G^2, \tag{1.19}$$

$$\Delta^\# r=\rho\Delta r-\frac{n-2}{2}\frac{<\nabla\rho,\nabla r>_{G^\#}}{\rho}, \tag{1.20}$$

where r is the geodesic distance function with respect to the metric $G^\#$ on M. Denote $f_{ijk}=\frac{\partial^3 f}{\partial x_i\partial x_j\partial x_k}$. We have the following formula (see [P], p.38):

$$R_{ik}^\#=\frac{1}{4}\sum f^{jl}f^{hm}(f_{hil}f_{mjk}-f_{hik}f_{mjl}), \tag{1.21}$$

where (f^{ij}) denote the inverse matrix of (f_{ij}).

2. Proof of the Theorem

We first prove the following lemma:

Lemma 1: *Let $x: M\to A^{n+1}$ be a locally strong convex affine maximal hypersurface, which is given by a strictly convex function*

$$x_{n+1}=f(x_1,\cdots,x_n).$$

If M is complete with respect to the metric $G^\#$, and if there is a constant $N>0$ such that $||Ric^\#||_{G^\#}^2\leq N$ everywhere, then, in the case $n=2$ or $n=3, M$ must be an elliptic paraboloid.

Proof Let $p_0 \in M$. By adding a linear function and taking a parameter transformation we may assume that p_0 has coordinates $(0, \cdots, 0)$ and

$$f(0) = 0, \quad f_i(0) = 0, \quad f_{ij}(0) = \delta_{ij}.$$

Denote by $r(p_0, p)$ the geodesic distance function from p_0 with respect to the metric $G^{\#}$. For any positive number a, let $B_a(p_0) = \{p \in M | r(p_0, p) \leq a\}$. Consider the function

$$F = (a^2 - r^2)^2 \Phi$$

defined on $B_a(p_0)$. Obviously, F attains its supremum at at some interior point p^*. We may assume that r^2 is a C^2-function in a neighborhood of p^*, and $\Phi > 0$ at p^*. Then at p^*

$$F_{,i} = 0,$$

$$\sum F_{,ii} \leq 0,$$

where "," denotes the covariant differentiation with respect to the Blaschke metric. We calculate both expression explicitly

$$\frac{\Phi_{,i}}{\Phi} - \frac{2(r^2)_{,i}}{a^2 - r^2} = 0, \tag{2.1}$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum \Phi_{,i}^2}{\Phi^2} - \frac{2\|\nabla r^2\|_G^2}{(a^2 - r^2)^2} - \frac{2\Delta r^2}{a^2 - r^2} \leq 0. \tag{2.2}$$

Inserting (2.1) into (2.2) and noting that

$$\|\nabla r^2\|_G^2 = 4r^2 \|\nabla r\|_G^2,$$

$$\Delta r^2 = 2\|\nabla r\|_G^2 + 2r\Delta r,$$

we get

$$\begin{aligned} \frac{\Delta\Phi}{\Phi} &\leq \frac{6\|\nabla r^2\|_G^2}{(a^2 - r^2)^2} + \frac{2\Delta r^2}{a^2 - r^2} \\ &= \frac{24r^2\|\nabla r\|_G^2}{(a^2 - r^2)^2} + \frac{4\|\nabla r\|_G^2}{a^2 - r^2} + \frac{4r\Delta r}{a^2 - r^2}. \end{aligned} \tag{2.3}$$

Since

$$\frac{4r\Delta r}{a^2 - r^2} = \frac{4r}{a^2 - r^2} \frac{\Delta^{\#} r}{\rho} + \frac{2(n-2)r}{a^2 - r^2} \frac{< \nabla\rho, \nabla r >_{G^{\#}}}{\rho^2}$$

$$\leq \frac{4r}{a^2 - r^2} \cdot \frac{\Delta^{\#} r}{\rho} + \frac{1}{24} \frac{\|\nabla \rho\|^2_{G\#}}{\rho^3} + \frac{24(n-2)^2 r^2}{(a^2 - r^2)^2} \frac{\|\nabla r\|^2_{G\#}}{\rho},$$

and

$$\frac{\|\nabla \rho\|^2_{G\#}}{\rho} = \|\nabla \rho\|^2_G,$$

$$\frac{\|\nabla r\|^2_{G\#}}{\rho} = \|\nabla r\|^2_G,$$

$$\|\nabla r\|^2_{G\#} = 1,$$

we have

$$\frac{\Delta \Phi}{\Phi} \leq \frac{24 r^2 \|\nabla r\|^2_{G\#}}{(a^2 - r^2)^2 \rho} + \frac{4\|\nabla r\|^2_{G\#}}{(a^2 - r^2)\rho} + \frac{4r\Delta^{\#} r}{(a^2 - r^2)\rho} + \frac{1}{24}\frac{\Phi}{\rho} + \frac{24(n-2)^2 r^2}{(a^2 - r^2)^2 \rho} \|\nabla r\|^2_{G\#}. \tag{2.4}$$

$$= \frac{24(1 + (n-2)^2) r^2}{(a^2 - r^2)^2 \rho} + \frac{4}{(a^2 - r^2)\rho} + \frac{4r\Delta^{\#} r}{(a^2 - r^2)\rho} + \frac{1}{24}\frac{\Phi}{\rho}.$$

Recall that $(M, G^{\#})$ is a complete Riemanninn manifold with Ricci curvature bounded from below by a constant $-K, K > 0$. We have

$$r\Delta^{\#} r \leq (n-1)(1 + \sqrt{K} r).$$

Consequently, from (2.4), it follows that

$$\frac{\Delta \Phi}{\Phi} \leq \frac{24(1 + (n-2)^2) r^2}{(a^2 - r^2)^2 \rho} + \frac{4n}{(a^2 - r^2)\rho} + \frac{4(n-1)\sqrt{K} \cdot r}{(a^2 - r^2)\rho} + \frac{1}{24}\frac{\Phi}{\rho}. \tag{2.6}$$

For $n = 3$, we have by (1.18)

$$\frac{\Delta \Phi}{\Phi} \geq \frac{3}{4} \sum \frac{\Phi_i^2}{\Phi^2} + \frac{1}{6}\frac{\Phi}{\rho} - \sum \frac{\Phi_i}{\Phi} \cdot \frac{\rho_i}{\rho} \tag{2.7}$$

$$\geq \left(\frac{3}{4} - 3\right) \sum \frac{\Phi_i^2}{\Phi^2} + \frac{1}{12}\frac{\Phi}{\rho}$$

$$= -\frac{36 r^2}{(a^2 - r^2)^2 \rho} + \frac{1}{12}\frac{\Phi}{\rho},$$

where we used (2.1). Inserting (2.7) into (2.6) we get

$$\Phi \leq \frac{2016 r^2}{(a^2 - r^2)^2} + 288 \frac{1}{a^2 - r^2} + \frac{192\sqrt{K} \cdot r}{a^2 - r^2}. \tag{2.8}$$

Multiply both sides of (2.80) by $(a^2 - r^2)^2$. We obtain, at p^*,

$$(2.9) \qquad (a^2 - r^2)^2 \Phi \le 2304 a^2 + 192\sqrt{K} a^3$$
$$= b_1 a^2 + b_2 a^3$$

where $b_1 = 2304$ and $b_2 = 192\sqrt{K}$. It is obvious from (1.17) that (2.9) holds also for $n = 2$. Hence at any interior point of $B_a(p_0)$ we have

$$\Phi \le b_1 \frac{1}{a^2(1 - \frac{r^2}{a^2})^2} + b_2 \frac{1}{a(1 - \frac{r^2}{a^2})^2}.$$

Let $a \to \infty$, we conclude

$$(2.10) \qquad \Phi \equiv 0.$$

It follows that

$$det(\frac{\partial^2 f}{\partial x_i \partial x_j}) = 1,$$
$$G^\# = G.$$

This means that M is an affine complete parabolic affine hypersphere. By a result of E. Calabi (see [L-S-Z]) we conclude that M must be an elliptic paraboloid.

We now want to show that there is a constant $N > 0$ such that $||Ric^\#||^2_{G\#} \le N$ everywhere. We need the following lemma (see [H], p. 635, Lemma 26), which was applied several times in symplectic geometry.

Lemma 2 (Hofer). *Let (X, d) be a complete metric space with metric d, and $B_a(p) = \{x | d(p, x) \le a\}$ be a ball with center p and radius a. Let H be a positive continuous function defined on $B_{2a}(p)$. Then there is a point $q \in B_a(p)$ and a positive number $\epsilon \le \frac{a}{2}$ such that*

$$H(x) \le 2H(q) \quad for \ all \quad x \in B_\epsilon(q) \quad and \quad \epsilon H(q) \ge \frac{a}{2} H(p).$$

Now we assume that $||Ric^\#||^2_{G\#}$ is not bounded above. Then there is a sequence of points $p_\ell \in M$ such that $||Ric^\#||^2_{G\#}(p_\ell) \to \infty$. Let $B_1(p_\ell)$ be the geodesic ball with center p_ℓ and radius 1. Consider a family $\Psi(\ell) : B_2(p_\ell) \to \mathbb{R}$ of functions, $\ell \in \mathbb{N}$, defined by

$$(2.11) \qquad \Psi(\ell) = ||Ric^\#||_{G\#} + \Phi + L,$$

where Φ is defined by (1.6) and

$$L = \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}.$$

Using Hofer's Lemma with $H = \Psi^{1/2}$ we find a sequence of points q_ℓ and positive numbers ϵ_ℓ such that

$$\Psi^{\frac{1}{2}}(x) \leq 2\Psi^{\frac{1}{2}}(q_\ell), \quad \forall\ x \in B_{\epsilon_\ell}(q_\ell), \tag{2.12}$$

$$\epsilon_\ell \Psi^{\frac{1}{2}}(q_\ell) \geq \frac{1}{2}\Psi^{\frac{1}{2}}(p_\ell) \to \infty. \tag{2.13}$$

The restriction of the hypersurface x to the balls $B_{\epsilon_\ell}(q_\ell)$ defines a family $M(\ell)$ of maximal hypersurfaces. For every ℓ, we normalize $M(\ell)$ as follows:

Step 1. By adding a linear function we may assume that, at $q_l, (x_1, \cdots, x_n) = (0, \cdots, 0)$ and

$$f(0) = 0, \quad f_i(0) = 0.$$

We take a parameter transformation:

$$\widehat{x}_i(\ell) = \sum a_i^j(\ell) x_j(\ell), \tag{2.14}$$

where $a_i^j(\ell)$ are constants. Choosing $a_i^j(\ell)$ appropriately and using an obvious notation $\widehat{f}, \widehat{\Psi}$, we may assume that, for every ℓ, we have $\widehat{f}_{ij}(0) = \delta_{ij}$. Note that, under the parameter transformation (2.14), $\widehat{\Psi}$ is invariant.

Step 2. We take an affine transformation by

$$\widetilde{x}_i(\ell) = a(\ell)\widehat{x}_i(\ell), \quad 1 \leq i \leq n,$$

$$\widetilde{x}_{n+1}(\ell) = \lambda(\ell)\widehat{x}_{n+1}(\ell).$$

where $\lambda(\ell)$ and $a(\ell)$ are constants. It is easy to verify that each $\tilde{M}(\ell)$ again is a locally strongly convex maximal hypersurface. We now choose $\lambda(\ell) = a(\ell)^2$, $\lambda(\ell) = \widehat{\Psi}(q_\ell)$. Using again an obvious notation $\tilde{f}$, $\tilde{\Psi}$, one can see that

$$\tilde{f}_{ij}(\ell) = \widehat{f_{ij}}(\ell), \qquad \tilde{\Psi}(\ell) = \frac{1}{\lambda(\ell)}\widehat{\Psi}(\ell).$$

The first equation is trivial. We calculate the second one. From (1.6), (1.21) we can easily get

$$\widetilde{\Phi} = \frac{1}{\lambda}\widehat{\Phi},$$

$$||\widetilde{Ric}^{\#}||_{\tilde{G}^{\#}} = \frac{1}{\lambda}||\widehat{Ric}^{\#}||_{G^{\#}},$$

$$\widetilde{L} = \frac{1}{\lambda}\widehat{L}.$$

Then the second equality follows.

We denote $\tilde{B}_a(q_\ell) = \{x \in \tilde{M}(\ell) \mid \tilde{r}(\ell)(x, q_\ell) \leq a\}$, where $\tilde{r}(\ell)$ is the geodesic distance function with respect to the metric $\tilde{G}^{\#}$ on $\tilde{M}(\ell)$. Then $\tilde{\Psi}(\ell)$ is defined on the geodesic ball $\tilde{B}_{d(\ell)}(q_\ell)$ with $d(\ell) = \epsilon_\ell \Psi^{\frac{1}{2}}(q_\ell) \geq \frac{1}{2}\Psi^{\frac{1}{2}}(p_\ell) \to \infty$. From (2.12) we have

$$\tilde{\Psi}(q_\ell) = 1, \tag{2.15}$$

$$\tilde{\Psi}(x) \leq 4, \quad \forall x \in \tilde{B}_{d(\ell)}(q_\ell). \tag{2.16}$$

We may identify the parametrization as $(\xi_1, \cdots, \xi_n)$ for any index ℓ. Then $\widetilde{f(\ell)}$ is a sequence of functions defined in a domain $\Omega(\ell), 0 \in \Omega(\ell)$. Thus we have a sequence $\widetilde{M(\ell)}$ of maximal hypersurfaces given by $\widetilde{f}(\ell)$. We have

$$\widetilde{f}(\ell)(0) = 0, \quad \frac{\partial \widetilde{f}(\ell)}{\partial \xi_i}(0) = 0, \quad \frac{\partial^2 \widetilde{f}(\ell)}{\partial \xi_i \xi_j}(0) = \delta_{ij}. \tag{2.17}$$

$$\widetilde{\Psi}(\ell)(0) = 1, \tag{2.18}$$

$$\widetilde{\Psi}(\ell)(x) \leq 4, \quad \forall x \in \widetilde{B}_{d(\ell)}(0). \tag{2.19}$$

$$d(\ell) \to \infty \quad as\ \ell \to \infty. \tag{2.20}$$

We need the following lemma:

Lemma 3 Let M be an affine maximal hypersurface defined in a neighborhood of $0 \in \mathbb{R}^n$. Suppose that, with the notations from above,

$$f_{ij}(0) = \delta_{ij}, \tag{i}$$

$$||Ric^{\#}||_{G^{\#}} + \Phi + \sum f^{il} f^{jm} f^{km} f_{ijk} f_{lmn} \leq 4. \tag{ii}$$

Denote $D := \{(\xi_1, \cdots, \xi_n) | \sum \xi_i^2 \leq \frac{1}{4n^2}\}$. Then there is constant $C_1 > 0$ such that, for $(\xi_1, \cdots, \xi_n) \in D$, the following estimates hold:

(1)

$$\sum f_{ii} \leq 4n.$$

(2)

$$\frac{1}{C_1} \leq det(f_{ij}) \leq C_1.$$

(3) Define d_o by $d_o^2 = \frac{1}{7n^2(4n)^{n-1}C_1}$. Then $\overline{\Omega}_{d_o} \subset \{\sum \xi_i^2 < \frac{1}{7n^2}\} \subset D$, where Ω_{d_o} is the geodesic ball with center 0 and radius d_o with respect to the metric $G^{\#}$.

Proof of Lemma 3.

(1). Consider an arbitary curve $\Gamma = \{\xi_1 = a_1 s, \cdots, \xi_n = a_2 s, | \sum a_i^2 = 1, s \geq 0\}$. By assumption we have

$$\sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn} \leq 4, \quad \sum f_{ii}(0) = n.$$

Since $\sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}$ is independent of the choice of coordinates $\xi_1, ..., \xi_n$, for any point $\xi(s)$ we may assume that $f_{ij} = \lambda_i \delta_{ij}$. Then

$$\sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn} = \sum \frac{1}{\lambda_i \lambda_j \lambda_k} f_{ijk}^2 \geq \frac{\sum f_{iik}^2}{(\sum f_{ii})^3}.$$

It follows that

$$\frac{\sum f_{iik}^2}{(\sum f_{ii})^3} \leq \frac{1}{(\sum f_{ii})^3} \sum f_{ijk}^2 \leq 4,$$

and hence

$$\frac{1}{(\sum f_{ii}(x(s)))^{3/2}} \frac{d(\sum f_{ii}(x(s)))}{ds} = \frac{1}{(\sum f_{ii}(x(s)))^{3/2}} \sum f_{iik}(x(s)) a_k$$

$$\leq \sqrt{n} \left(\frac{\sum f_{iik}^2(x(s))}{(\sum f_{ii}(x(s)))^3} \right)^{\frac{1}{2}} (\sum a_k^2)^{\frac{1}{2}} \leq 2\sqrt{n}.$$

Solving this differential inequality with $\sum f_{ii}(0) = n$, we get

$$\frac{1}{\sqrt{n}} - s\sqrt{n} \leq \frac{1}{(\sum f_{ii}(x(s)))^{\frac{1}{2}}}.$$

From the assumption we have $s \leq \frac{1}{2n}$ then **(1)** follows.
(2). Consider an arbitary curve

$$\Gamma = \{\xi_1 = a_1 s, \cdots, \xi_n = a_n s | \sum a_i^2 = 1, s \geq 0\}$$

again. By assumption we have

$$\frac{\sum f^{ij}\rho_i\rho_j}{\rho^2} \leq 4.$$

It follows that

$$\frac{\sum \rho_i^2}{(\sum f_{ii})\rho^2} \leq 4.$$

By (1) we get

$$\frac{1}{\rho}\frac{d\rho(x(s))}{ds} \leq 4\sqrt{n}.$$

Solving this differential inequality with $\rho(0) = 1$ we obtain

$$-4n^{\frac{1}{2}}s \leq ln\rho(x(s)) \leq 4n^{\frac{1}{2}}s.$$

Recall that $s \leq \frac{1}{2n}$, then (2) follows.
(3). Denote by λ_{min}, λ_{max} the minimal and maximal eigenvalues of (f_{ij}). Then, from **(1)** and **(2)**, we have $\lambda_{max} \leq 4n$ and

$$\frac{1}{C_1} \leq det(f_{ij}) \leq \lambda_{min}\lambda_{max}^{n-1} \leq (4n)^{n-1}\lambda_{min}.$$

It follows that

$$4n\sum x_i^2 \geq r^2 \geq \frac{1}{C_1(4n)^{n-1}}\sum x_i^2.$$

We immediately get **(3)**.

We continue with the proof of the Theorem. Since $d(\ell) \to \infty$, we have $D \subset \Omega(\ell)$ for ℓ big enough. In fact, by **(1)**, the geodesic distance from 0 to the boundary of D with respect to the metric $\widetilde{G}^{\#}$ on $\widetilde{M}(\ell)$ is less than $\frac{1}{\sqrt{n}}$. By Lemma 3 and bootstrapping, we may get a C^k- estimate,independent of ℓ, for any k. It follows that there is a ball $\{\sum \xi_i^2 \leq C_2\}$ and a subsequence (still indexed by ℓ) such that $\widetilde{f}(\ell)$ converges to $\widetilde{f}$ on the ball and correspondingly all derivaties, where $C_2 < \frac{1}{4n^2}$ is very close to $\frac{1}{4n^2}$. Thus, as limit, we get a maximal hypersurface $\widetilde{M}$, defined on the ball, which contains a geodesic ball $\overline{\Omega}_{d_0}$. We now extend the hypersurface $\widetilde{M}$ as follows: For every boundary point $p = (\xi_{10}, \cdots, \xi_{n0})$ of the geodesic ball $\overline{\Omega}_{d_0}$ we first make a parameter transformation

$$\widetilde{\xi}_i = \sum \widetilde{a}_i^j \xi_j$$

such that at p, $(\widetilde{\xi}_1, \cdots, \widetilde{\xi}_n) = (0, \cdots, 0)$ and for the limit hypersurface $\widetilde{M}$, we have $\widetilde{f}_{ij}(0) = \delta_{ij}$. We have

$$(i') \qquad \widetilde{f}_{ij}(\ell)(p) \to \widetilde{f}_{ij}(0) = \delta_{ij}, \qquad as\ \ell \to \infty.$$

It is easy to see that under the conditions (i′) and (ii) in Lemma 3, the estimates **(1)**, **(2)** and **(3)** in Lemma 3 remain true. By the same argument as above we conclude that there is a ball aroud p and a subsequence ℓ_k, such that $\tilde{f}(\ell_k)$ converges to $\tilde{f}'$ on the ball, and correspondingly all derivatives. As limit, we get a maximal hypersurface $\tilde{M}'$, which contains a geodesic ball of radius d_o around p. Then we return to the original parameters. Note that the geodesic distance is independent of the choice of the parameters. It is obvious that $\tilde{M}$ and $\tilde{M}'$ agree on the common part. We repeat this procedure to extend $\tilde{M}$ to be defined on $\overline{\Omega}_{2d_o}$, etc. In this way we may extend $\tilde{M}$ to be a maximal hypersurface defined in a domain $\Omega \subset \mathbb{R}^n$, which is complete with respect to the metric $\tilde{G}^{\#}$. Using (2.18) and (2.19) we get

$$||\tilde{R}^{\#}||_{\tilde{G}^{\#}} \leq 4, \qquad \tilde{\Psi}(0) = 1.$$

By Lemma 1, $\tilde{M}$ must be an elliptic paraboloid. For a paraboloid we have $\tilde{\Psi} = 0$, identically, Thus, we get a contradiction. So $||R^{\#}||^2_{G^{\#}}$ must be bounded above on M. By Lemma 1 M is an elliptic paraboloid. This complete the proof of Theorem.

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