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Radii minimal projections of polytopes and constrained optimization of symmetric polynomials

René Brandenberg and Thorsten Theobald

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Abstract. We provide a characterization of the radii minimal projections of polytopes onto j-dimensional subspaces in Euclidean space $\mathbb{E}^n$. Applied to simplices this characterization allows to reduce the computation of an outer radius to a computation in the circumscribing case or to the computation of an outer radius of a lower-dimensional simplex. In the second part of the paper, we use this characterization to determine the sequence of outer $(n-1)$-radii of regular simplices (which are the radii of smallest enclosing cylinders). This settles a question which arose from an error in a paper by Weißbach (1983). In the proof, we first reduce the problem to a constrained optimization problem of symmetric polynomials and then to an optimization problem in a fixed number of variables with additional integer constraints.

Key words. Polytope, projection, outer radius, enclosing cylinder, regular simplex, polynomial optimization, symmetric polynomials.

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1 Introduction

Let $\mathscr{L}_{j,n}$ be the set of all j-dimensional linear subspaces (hereafter *j-spaces*) in n-dimensional Euclidean space $\mathbb{E}^n$. The *outer j-radius* $R_j(C)$ of a convex body $C \subset \mathbb{E}^n$ is the radius of the smallest enclosing *j-ball* (j-dimensional ball) in an optimal orthogonal projection of C onto a j-space $J \in \mathscr{L}_{j,n}$, where the optimization is performed over $\mathscr{L}_{j,n}$. The optimal projections are called *R_j-minimal projections*. In this paper we show the following results:

Theorem 1. *Let $1 \leqslant j \leqslant n < m$ and $P = \operatorname{conv}\{v^{(1)}, \dots, v^{(m)}\} \subset \mathbb{E}^n$ be an n-polytope. Then one of the following is true.*

a) *In every R_j-minimal projection of P there exist $n+1$ affinely independent vertices of P which are projected onto the minimal enclosing j-sphere.*

b) *$j \geqslant 2$ and $R_j(P) = R_{j-1}(P \cap H)$ for some hyperplane $H = \operatorname{aff}\{v^{(i)} : i \in I\}$ with $I \subset \{1, \dots, m\}$.*

If $j = 1$ or if P is a regular simplex then always case a) *holds. Moreover, the number v of affinely independent vertices projected onto the minimal enclosing j-sphere is at least $n - j + 2$ and there exists a $(v - 1)$-flat F such that $R_j(P) = R_{j+v-n-1}(P \cap F)$.*

The bound $n - j + 2$ is best possible.

Theorem 1 allows to reduce the computation of an outer radius of a simplex to the computation in the circumscribing case or to the computation of an outer radius of a facet of the simplex.

Using this theorem, the second part of the paper shows the following result on the outer $(n-1)$-radius, which is the radius of a smallest enclosing cylinder.

Theorem 2. *Let $n \geqslant 2$ and T_1^n be a regular simplex in $\mathbb{E}^n$ with edge length* 1. *Then*

$$R_{n-1}(T_1^n) = \begin{cases} \sqrt{\frac{n-1}{2(n+1)}} & \text{if } n \text{ is odd}, \\ \frac{2n-1}{2\sqrt{2n(n+1)}} & \text{if } n \text{ is even}. \end{cases}$$

The case n odd has already been settled independently by Pukhov [16] and Weißbach [22], who both left the even case open in their papers. Pukhov's results also determine $R_j(T_1^n)$ for $j < n$. There also exists a later paper on $R_{n-1}(T^n)$ for even n [23], but as pointed out in [3] the proof contained a crucial error.[1] Thus Theorem 2 (re-)completes the determination of the sequence of outer j-radii of regular simplices (see also [2]).

Studying radii of polytopes is a fundamental topic in convex geometry. Motivated by applications in computer vision, robotics, computational biology, functional analysis, and statistics (see [8] and the references therein) there has been much interest from the computational point of view. See [3], [6], [17] for exact algebraic algorithms, [9], [21], [24] for approximation algorithms, and [4], [8] for the computational complexity. Reductions of smallest enclosing cylinders to circumscribing cylinders are used in exact algorithms as well as for complexity proofs (see, e.g., [3, Theorem 1] and [8, Theorems 5.3–5.5]), and have previously been given only for $j \in \{1, n\}$ as well as for dimension 3. Theorem 1 generalizes and unifies these results.

Here, we use Theorem 1 to reduce the computation of the outer $(n-1)$-radius of a regular simplex to the following optimization problem of symmetric polynomials in n variables:

$$\min \sum_{i=1}^{n+1} s_i^4 \quad \text{such that} \quad \sum_{i=1}^{n+1} s_i^3 = 0, \quad \sum_{i=1}^{n+1} s_i^2 = 1, \quad \sum_{i=1}^{n+1} s_i = 0. \tag{1.1}$$

Based on exploiting the symmetries, we solve (1.1) for any n by reducing it to an optimization problem in six variables with additional integer constraints.

[1] After [3] had been completed, Bernulf Weißbach suggested to work jointly on a new proof for the even case. Unfortunately, he died on 9th June 2003.

The paper is structured as follows. In Section 2, we introduce the necessary notation. Section 3 gives the proof of Theorem 1. Section 4 contains the derivation of the optimization problem (1.1) and the proof of Theorem 2. In Section 5 we analyze the different difficulty of the even and the odd case of (1.1) from the viewpoint of the Positivstellensatz.

2 Preliminaries

Throughout the paper we work in Euclidean space $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the usual scalar product $x \cdot y = \sum_{i=1}^n x_i y_i$ and norm $\|x\| = (x \cdot x)^{1/2}$. By $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ we denote the (closed) unit ball and unit sphere, respectively. For a set $A \subset \mathbb{E}^n$, the linear hull of A is denoted by $\mathrm{lin}(A)$, the affine hull by $\mathrm{aff}(A)$, and the convex hull by $\mathrm{conv}(A)$.

A set $C \subset \mathbb{E}^n$ is called a *body* if it is compact, convex and contains an interior point. Accordingly, we always assume that a polytope $P \subset \mathbb{E}^n$ is full-dimensional (unless otherwise stated). Let $1 \leqslant j \leqslant n$. A *j-flat* F (an affine subspace of dimension j) is *perpendicular* to a hyperplane H with normal vector h if h and F are parallel. For $p, p' \in \mathbb{E}^n$ and subspaces $E \in \mathscr{L}_{j,n}$, $E' \in \mathscr{L}_{j',n}$, a j-flat $F = p + E$ and a j'-flat $F' = p' + E'$ are *parallel* if $E \cup E' = \mathrm{lin}(E \cup E')$.

A *j-cylinder* is a set of the form $J + \rho\mathbb{B}^n$ with an $(n-j)$-flat J and $\rho > 0$. For a body $C \subset \mathbb{E}^n$, the outer j-radius $R_j(C)$ of C (as defined in the introduction) is also the radius ρ of a smallest enclosing j-cylinder of C. It follows from a standard compactness argument that this minimal radius is attained (see, e.g., [7]). Let $1 \leqslant j \leqslant k < n$. If $C' \subset \mathbb{E}^n$ is a compact, convex set whose affine hull F is a k-flat then $R_j(C')$ denotes the radius of a smallest enclosing j-cylinder $\mathscr{C}'$ relative to F, i.e., $\mathscr{C}' = J' + R_j(C')(\mathbb{B}^n \cap F)$ with a $(k-j)$-flat $J' \subset F$.

A *simplex* $S := \mathrm{conv}\{v^{(1)}, \dots, v^{(n+1)}\}$ (with $v^{(1)}, \dots, v^{(n+1)} \in \mathbb{E}^n$ affinely independent) is *regular* if all its vertices are equidistant. Whenever a statement is invariant under orthogonal transformations and translations we denote by T^n *the* regular simplex in $\mathbb{E}^n$ with edge length $\sqrt{2}$. The reason for the choice of $\sqrt{2}$ stems from the following embedding of T^n into $\mathbb{E}^{n+1}$. Let $\mathscr{H}_\alpha^n = \{x \in \mathbb{E}^{n+1} : \sum_{i=1}^{n+1} x_i = \alpha\}$. Then the *standard embedding* $\mathbf{T}^n$ of T^n is defined by

$$\mathbf{T}^n := \mathrm{conv}\{e^{(i)} \in \mathbb{E}^{n+1} : 1 \leqslant i \leqslant n+1\} \subset \mathscr{H}_1^n,$$

where $e^{(i)}$ denotes the i-th unit vector in $\mathbb{E}^{n+1}$. By $\mathscr{S}^{n-1} := \mathbb{S}^n \cap \mathscr{H}_0^n$ we denote the set of unit vectors parallel to $\mathscr{H}_1^n$.

A j-cylinder $\mathscr{C}$ containing some simplex S is called a *circumscribing* j-cylinder of S if all the vertices of S are contained in the boundary of $\mathscr{C}$.

3 Minimal and circumscribing *j*-cylinders

It is well known that the (unique) minimal enclosing ball B (i.e., the minimal enclosing n-cylinder) of a polytope $P \subset \mathbb{E}^n$ may contain only few vertices of P on its boundary (e.g., two diametral vertices) [1, p. 54]. However, in cases where less than $n+1$ vertices of P are contained in the boundary of B, it is easy to see that there exists a hyperplane H such that $P \cap \mathrm{bd}(B) \subset H$ and the center of B is contained in H.

Then the smallest enclosing ball of P and the smallest enclosing ball of $P \cap H$ relative to H have the same radius.

In [7, Theorem 1.9] the following characterization for the minimal enclosing 1-cylinder (two parallel hyperplanes defining the width of the polytope) is given:

Proposition 3. *Any minimal enclosing* 1*-cylinder of a polytope* $P \subset \mathbb{E}^n$ *contains at least* $n+1$ *affinely independent vertices of* P *on its boundary.*

We give a characterization of the possible configurations of minimal enclosing j-cylinders of polytopes for arbitrary j, unifying and generalizing the above statements.

Lemma 4. *Let* $P = \operatorname{conv}\{v^{(1)}, \dots, v^{(m)}\}$ *be a polytope in* $\mathbb{E}^n$, $1 \leqslant j \leqslant n-1$, *and* J *be an* $(n-j)$*-flat such that* $\mathscr{C} = J + R_j(P)\mathbb{B}^n$ *is a minimal enclosing* j*-cylinder of* P. *Then for every* $I \subset \{1, \dots, m\}$ *such that* $\{i : v^{(i)} \in \operatorname{bd}(\mathscr{C})\} \subset I$ *and* $H_I := \operatorname{aff}\{v^{(i)} : i \in I\}$ *is of affine dimension* $n-1$, J *is parallel to* H_I.

Proof. Suppose that there exists a hyperplane $H := H_I$ of this type with J not parallel to H. Let $\bar{n} := |\{v^{(i)} \in H : 1 \leqslant i \leqslant m\}|$. Without loss of generality we can assume $H = \{x \in \mathbb{E}^n : x_n = 0\}$ and $I = \{v^{(1)}, \dots, v^{(\bar{n})}\}$. Hence, $v^{(\bar{n}+1)}, \dots, v^{(m)} \notin H \cup \operatorname{bd}(C)$.

First consider the case that J is not perpendicular to H. Let $p, s^{(1)}, \dots, s^{(n-j)} \in \mathbb{E}^n$ such that $J = p + \operatorname{lin}\{s^{(1)}, \dots, s^{(n-j)}\}$. Since, by assumption, J is not parallel to H, we can assume $p = 0 \in J \cap H$, $s_n^{(1)} = \cdots = s_n^{(n-j-1)} = 0$ and $s_n^{(n-j)} > 0$. For every $s_n' \in (0, s_n^{(n-j)})$ and $s' := (s_1^{(n-j)}, \dots, s_{n-1}^{(n-j)}, s_n') \in \mathbb{E}^n$ let $J' = p + \operatorname{lin}\{s^{(1)}, \dots, s^{(n-j-1)}, s'\}$. Geometrically, J' results from J by rotating J towards the hyperplane H in such a way that the orthogonal projection of J onto H remains invariant (see Figure 1).

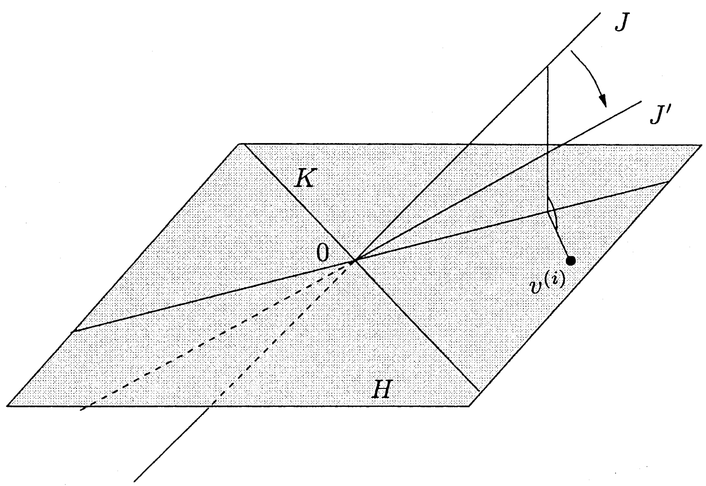

Figure 1. For $n = 3$ and $j = 2$ the figure shows how the underlying flat J of the j-cylinder $\mathscr{C}$ is rotated towards its orthogonal projection onto the plane H. The distances between the vertices $v^{(i)}$, $1 \leqslant i \leqslant \bar{n}$, and the j-cylinder axis are not increased, and decreased if $v^{(i)} \notin K$.

Since J and H are not perpendicular we obtain $J \neq J'$, and because $v^{(1)}, \dots, v^{(\bar{n})} \in H$ that

$$\operatorname{dist}(v^{(i)}, J') \leqslant \operatorname{dist}(v^{(i)}, J), \quad 1 \leqslant i \leqslant \bar{n}, \tag{3.1}$$

where $\operatorname{dist}(\cdot, \cdot)$ denotes the Euclidean distance. In (3.1), "$<$" holds whenever $v^{(i)} \notin K := J^{\perp} \cap H$. Obviously, $\dim(K) = j - 1$. If none of the $v^{(i)}$ lies in $K \cap \operatorname{bd}(\mathscr{C})$ then, by choosing s_n' sufficiently close to $s_n^{(n-j)}$, all vertices of P lie in the interior of $\mathscr{C}' = J' + R_j(P)\mathbb{B}^n$, a contradiction to the minimality of $\mathscr{C}$. Hence, there must be some vertex of P in $K \cap \operatorname{bd}(\mathscr{C})$.

Let $\bar{k} := |\{v^{(i)} \in K \cap \operatorname{bd}(\mathscr{C}) : 1 \leqslant i \leqslant m\}|$. By renumbering the vertices we can assume that $v^{(1)}, \dots, v^{(\bar{k})} \in K \cap \operatorname{bd}(\mathscr{C})$. Let $F := \operatorname{conv}\{v^{(1)}, \dots, v^{(\bar{k})}\}$ and $k := \dim F$.

Now assume $F \cap J = \emptyset$. We have shown above that for sufficiently small s_n' the rotation from J to J' keeps all vertices within the j-cylinder $\mathscr{C}'$ and $v^{(1)}, \dots, v^{(\bar{k})}$ are the only vertices on $\operatorname{bd}(\mathscr{C}')$. Let J'' be a translate of J' with $\operatorname{dist}(J'', F) < \operatorname{dist}(J', F)$, and J'' sufficiently close to J' to keep $v^{(\bar{k}+1)}, \dots, v^{(m)}$ within the interior of $\mathscr{C}'' = J'' + R_j(P)\mathbb{B}^n$. Then all vertices of P lie in the interior of $\mathscr{C}''$, again a contradiction.

It follows that $F \cap J \neq \emptyset$, and since $F \subset K = J^{\perp} \cap H$ that $F \cap J = \{p\} = \{0\}$. Since $\operatorname{dist}(p, v^{(i)}) = R_j(P)$ for all $i \in \{1, \dots, \bar{k}\}$ and since $p \in F$, it follows that p is the unique center of the smallest enclosing k-ball of F. Now let J''' result from J' by rotating J' around the origin towards a direction in $\mathbb{R}^n \backslash (\bigcup_{i=1}^{\bar{k}} (v^{(i)})^{\perp})$. This set of directions is nonempty, since every subspace in the union is only of dimension $n - 1$. For $i \in \{1, \dots, \bar{k}\}$ the property $\operatorname{dist}(v^{(i)}, J) = \operatorname{dist}(v^{(i)}, J') = \operatorname{dist}(v^{(i)}, p)$ implies $\operatorname{dist}(v^{(i)}, J''') < \operatorname{dist}(v^{(i)}, J')$. Again, by keeping the rotation sufficiently small, $v^{(\bar{k}+1)}, \dots, v^{(m)}$ remain in the interior of $\mathscr{C}''' = J''' + R_j(P)\mathbb{B}^n$. Now, all vertices lie in the interior of $\mathscr{C}'''$, once more a contradiction.

Finally, consider the case that J is perpendicular to H. Then $J \cap H$ is an optimal $(n - 1 - j)$-flat for the j-radius of $P \cap H$ (taken in $(n - 1)$-dimensional space). However, it is easy to see that in this case any small perturbation J' of J with $J' \cap H = J \cap H$ keeps $v^{(\bar{n}+1)}, \dots, v^{(m)}$ within the j-cylinder, not increasing the distances of all the other vertices to the new $(n - j)$-flat. Indeed, the case J perpendicular to H describes a local maximum. So the same argument as in the non-perpendicular case applies to show a contradiction. □

Corollary 5. *Let* $S = \operatorname{conv}\{v^{(1)}, \dots, v^{(n+1)}\}$ *be a simplex in* $\mathbb{E}^n$ *and* $S^{(i)}$ *be the facet of* S *with* $v^{(i)} \notin S^{(i)}$. *Let* $1 \leqslant j \leqslant n - 1$ *and* J *be an* $(n - j)$*-flat such that* $\mathscr{C} = J + R_j(S)\mathbb{B}^n$ *is a minimal enclosing* j*-cylinder of* S. *Then* J *is parallel to every* $S^{(i)}$ *for which* $v^{(i)} \notin \operatorname{bd}(\mathscr{C})$.

Lemma 6. *Let* $P = \operatorname{conv}\{v^{(1)}, \dots, v^{(m)}\}$ *be a polytope in* $\mathbb{E}^n$, $1 \leqslant j \leqslant n$, *and* J *be an* $(n - j)$*-flat such that* $\mathscr{C} = J + R_j(P)\mathbb{B}^n$ *is a minimal enclosing* j*-cylinder of* P. *If there exists a hyperplane* $H_I = \operatorname{aff}\{v^{(i)} : i \in I\}$ *which is parallel to* J, *then one of the following holds*:

a) *There exists a vertex* $v^{(i)} \notin H_I$ *that lies on the boundary of* $\mathscr{C}$; *or*

b) $j \geqslant 2$, $J \subset H_I$, *and* $R_j(P) = R_{j-1}(P \cap H_I)$.

Proof. By Proposition 3, for $j = 1$ always a) holds; so let $j \geqslant 2$, and suppose neither a) nor b) holds. Since b) does not hold there exist $(n-j)$-flats parallel to J and closer to H_I, and since a) does not hold, for any such $(n-j)$-flat J', such that all vertices $v^{(i)} \notin H_I$ stay within $\mathscr{C}$, the distances from the vertices $v^{(i)}$, $i \in I$, to J' are strictly smaller than their distances to J. Hence $\mathscr{C}$ cannot be a minimal enclosing cylinder. □

In the case that P is a simplex, the proof can be carried out more explicitly: Let $P^{(n+1)}$ be the facet of P not including the vertex $v^{(n+1)}$. Suppose that J is parallel to $P^{(n+1)}$, that $P^{(n+1)} \subset H := \{x \in \mathbb{E}^n : x_n = 0\}$, and that $v_n^{(n+1)} > 0$. Let $p \in J$. Since $v_n^{(n+1)} > 0$ it follows that $p_n \geqslant 0$ and obviously

$$R_j(P) \geqslant v_n^{(n+1)} - p_n. \tag{3.2}$$

On the other hand, since J is parallel to $P^{(n+1)}$,

$$R_j(P)^2 = R_{j-1}^2(P^{(n+1)}) + p_n^2. \tag{3.3}$$

Let

$$p_n^* = \frac{(v_n^{(n+1)})^2 - R_{j-1}^2(P^{(n+1)})}{2v_n^{(n+1)}} \tag{3.4}$$

be the unique minimal solution for p_n to (3.2) and (3.3). Due to $p_n \geqslant 0$, we obtain $p_n = \max\{0, p_n^*\}$. Now, we see that case a) holds if $p_n = p_n^*$ and case b) if $p_n = 0$.

Statements 3–6 almost complete the proof of Theorem 1. If the number v of affinely independent vertices of P lying on the boundary of $\mathscr{C}$ is at most n, it follows from Lemma 4 and 6 that case b) of Theorem 1 must hold. Moreover, if $v \leqslant n-1$ we can again apply these lemmas on the lower-dimensional polytope $P \cap H_I$ with H_I as in Lemma 6. Now we can iterate this argument. If during this iteration the outer 1-radius of a polytope P' has to be computed, then by Proposition 3 the minimal enclosing 1-cylinder touches at least $\dim(P') + 1$ affinely independent vertices. From the same iterative argument it follows that $R_j(P) = R_{j+v-n-1}(P \cap F)$ for some $(v-1)$-flat F.

Suppose $S = \operatorname{conv}\{v^{(1)}, \dots, v^{(n+1)}\}$ is a simplex in $\mathbb{E}^n$, and $\bar{J}$ an $(n-j)$-flat, such that

$$\begin{aligned}\operatorname{dist}(v^{(1)}, J) = \cdots = \operatorname{dist}(v^{(n-j+2)}, J) &= R_1(\operatorname{conv}\{v^{(1)}, \dots, v^{(n-j+2)}\}) \\ &> \operatorname{dist}(v^{(n-j+3)}, J) \geqslant \cdots \geqslant \operatorname{dist}(v^{(n+1)}, J).\end{aligned}$$

Then obviously $R_j(S) = R_1(\operatorname{conv}\{v^{(1)}, \dots, v^{(n-j+2)}\})$ and at most $n-j+2$ vertices are situated on the boundary of the minimal enclosing j-cylinder.

The last point which remains to show is that every minimal enclosing j-cylinder of the regular simplex T^n also circumscribes T^n. Because of Proposition 5 it suffices to show that the value p_n^* in (3.4) is positive for all $1 \leqslant j \leqslant n-1$, showing that b) in Lemma 6 never holds for T^n.

However, in almost all cases the desired circumscribing property follows already from [16], see also [2].

Proposition 7. *For* $1 \leqslant j \leqslant n$ *it holds* $R_j(T^n) \geqslant \sqrt{\frac{j}{n+1}}$. *If* n *is odd or* $j \notin \{1, n-1\}$, *then* $R_j(T^n) = \sqrt{\frac{j}{n+1}}$, *and every minimal enclosing* j*-cylinder of* T^n *is a circumscribing* j*-cylinder of* T^n.

We can easily apply Proposition 7 to compute p_n^* if n is even.

Lemma 8. *Let* $1 \leqslant j \leqslant n-1$. *If* $S = T^n$ *then always case* a) *in Theorem* 1 *holds.*

Proof. We can assume $j \geqslant 2$, since otherwise b) cannot hold. We use the notation as in Lemma 6. Because of Proposition 7 it suffices to consider the case where n is even, and as mentioned above the proof is complete if we show that p_n^* is positive. Since Proposition 7 yields $R_{n-1}(T^{n-1}) = \sqrt{(n-1)/n}$, we have $v_n^{(n+1)} = \sqrt{2-(n-1)/n} = \sqrt{(n+1)/n}$. Also by Proposition 7, $R_{j-1}(T^{n-1}) = \sqrt{(j-1)/n}$ and therefore

$$p_n^* = \frac{n-j+2}{2\sqrt{n(n+1)}} > 0. \qquad \square$$

Choosing an optimal $(n-1)$-cylinder among those parallel to a facet of T^n with $p_n^* = \frac{3}{2\sqrt{n(n+1)}}$, gives an upper bound for the outer $(n-1)$-radius of a regular simplex,

$$R_{n-1}(T^n) \leqslant \frac{2n-1}{2\sqrt{n(n+1)}}.$$

Theorem 2 states that for even n this bound is tight.

4 Reduction to an algebraic optimization problem

In this section, we provide an algebraic formulation for a minimal circumscribing j-cylinder $J + \rho(\mathbb{B}^{n+1} \cap \mathscr{H}_0^n)$ of the regular simplex $\mathbf{T}^n$ in standard embedding. Let $J = p + \operatorname{lin}\{s^{(1)}, \dots, s^{(n-j)}\}$ with pairwise orthogonal $s^{(1)}, \dots, s^{(n-j)} \in \mathscr{S}^{n-1}$, and let p be contained in the orthogonal complement of $\operatorname{lin}\{s^{(1)}, \dots, s^{(n-j)}\}$. The orthogonal projection P of a vector $z \in \mathscr{H}_1^n$ onto the orthogonal complement of $\operatorname{lin}\{s^{(1)}, \dots, s^{(n-j)}\}$ (relative to $\mathscr{H}_1^n$) can be written as $P(z) = (I - \sum_{k=1}^{n-j} s^{(k)}(s^{(k)})^T)z$, where I denotes the identity matrix. Hence, for a general polytope with vertices $v^{(1)}, \dots, v^{(m)}$ (embedded in $\mathscr{H}_1^n$) the computation of the square of R_j can be expressed by the following optimization problem. Here, we use the convention $x^2 := x \cdot x$.

$$\min \rho^2 \quad \text{such that} \quad \begin{array}{lll} \text{(i)} & (p - Pv^{(i)})^2 \leqslant \rho^2, & i = 1, \dots, m, \\ \text{(ii)} & p \cdot s^{(k)} = 0, & k = 1, \dots, n-j, \\ \text{(iii)} & s^{(1)}, \dots, s^{(n-j)} \in \mathscr{S}^{n-1}, & \text{pairwise orthogonal}, \\ \text{(iv)} & p \in \mathscr{H}_1^n. & \end{array} \tag{4.1}$$

In the case of $\mathbf{T}^n$, (i) can be replaced by

$$\text{(i}'\text{)} \quad \left(p - e^{(i)} + \sum_{k=1}^{n-j} s_i^{(k)} s^{(k)}\right)^2 = \rho^2, \quad i = 1, \ldots, n+1, \tag{4.2}$$

where the equality sign follows from Theorem 1. By (ii) and $s^{(k)} \in \mathscr{S}^{n-1}$, (i$'$) can be simplified to

$$\text{(i}''\text{)} \quad p^2 - \rho^2 = \sum_{k=1}^{n-j} (s_i^{(k)})^2 + 2p_i - 1, \quad i = 1, \ldots, n+1.$$

Summing over all i gives $(n+1)(p^2 - \rho^2) = (n-j) + 2 - (n+1)$, i.e., $p^2 - \rho^2 = \frac{1-j}{n+1}$. We substitute this value into (i$''$) and obtain $p_i = \frac{1}{2}\left(\frac{n-j+2}{n+1} - \sum_{k=1}^{n-j} (s_i^{(k)})^2\right)$. Hence, all the p_i can be replaced in terms of the $s_i^{(k)}$,

$$\begin{aligned} \rho^2 &= \frac{(2 + (n-j))(2 - (n-j))}{4(n+1)} + \frac{1}{4}\sum_{i=1}^{n+1}\left(\sum_{k=1}^{n-j} (s_i^{(k)})^2\right)^2 + \frac{j-1}{n+1}, \\ p \cdot s^{(k)} &= -\frac{1}{2}\sum_{i=1}^{n+1}\sum_{k'=1}^{n-j} (s_i^{(k')})^2 s_i^{(k)}. \end{aligned} \tag{4.3}$$

We arrive at the following characterization of the minimal enclosing j-cylinders:

Theorem 9. *Let $1 \leqslant j \leqslant n$. A set of vectors $s^{(1)}, \ldots, s^{(n-j)} \in \mathscr{S}^{n-1}$ spans the underlying $(n-j)$-dimensional subspace of a minimal enclosing j-cylinder of $\mathbf{T}^n \subset \mathscr{H}_1^n$ if and only if it is an optimal solution of the problem*

$$\begin{gathered} \min \sum_{i=1}^{n+1}\left(\sum_{k=1}^{n-j} (s_i^{(k)})^2\right)^2 \quad \textit{such that} \\ \sum_{i=1}^{n+1}\sum_{k'=1}^{n-j} (s_i^{(k')})^2 s_i^{(k)} = 0, \quad k = 1, \ldots, n-j, \\ s^{(1)}, \ldots, s^{(n-j)} \in \mathscr{S}^{n-1} \ \textit{pairwise orthogonal}. \end{gathered} \tag{4.4}$$

It is easy to see that in case $j = n-1$ the program (4.4) reduces to (1.1) stated in the introduction.

By (4.3), in order to prove $R_{n-1}(T^n) = (2n-1)/(2\sqrt{n(n+1)})$ for even n, we have to show that the optimal value of (1.1) is $1/n$. We apply the following statement from [3].

Proposition 10. *Let $n \geqslant 2$. The direction vector $(s_1, \ldots, s_{n+1})^T$ of any extreme circumscribing $(n-1)$-cylinder of $\mathbf{T}^n$ satisfies $|\{s_1, \ldots, s_{n+1}\}| \leqslant 3$.*

For completeness we repeat the short proof.

Proof. We can assume $n \geqslant 3$. Let $s \in \mathcal{S}^{n-1}$ be the axis direction of a locally extreme circumscribing $(n-1)$-cylinder. Let $f(s) := \sum_{i=1}^{n+1} s_i^4$ be the objective function from (1.1), let $g_1(s) := \sum_{i=1}^{n+1} s_i^3$, $g_2(s) := \sum_{i=1}^{n+1} s_i^2 - 1$, and $g_3(s) := \sum_{i=1}^{n+1} s_i$. A necessary condition for a local extremum is that for any pairwise different indices $a, b, c, d \in \{1, \dots, n+1\}$,

$$\det\begin{pmatrix} -\frac{\partial f}{\partial s_a} & \frac{\partial g_1}{\partial s_a} & \frac{\partial g_2}{\partial s_a} & \frac{\partial g_3}{\partial s_a} \\ -\frac{\partial f}{\partial s_b} & \frac{\partial g_1}{\partial s_b} & \frac{\partial g_2}{\partial s_b} & \frac{\partial g_3}{\partial s_b} \\ -\frac{\partial f}{\partial s_c} & \frac{\partial g_1}{\partial s_c} & \frac{\partial g_2}{\partial s_c} & \frac{\partial g_3}{\partial s_c} \\ -\frac{\partial f}{\partial s_d} & \frac{\partial g_1}{\partial s_d} & \frac{\partial g_2}{\partial s_d} & \frac{\partial g_3}{\partial s_d} \end{pmatrix} = -24 \det\begin{pmatrix} s_a^3 & s_a^2 & s_a & 1 \\ s_b^3 & s_b^2 & s_b & 1 \\ s_c^3 & s_c^2 & s_c & 1 \\ s_d^3 & s_d^2 & s_d & 1 \end{pmatrix} = 0.$$

The latter is a Vandermonde determinant, which implies $|\{s_a, s_b, s_c, s_d\}| \leqslant 3$. □

Using Proposition 10, (1.1) can be written as the following polynomial optimization problem in six variables with additional *integer* conditions.

$$\min k_1 s_1^4 + k_2 s_2^4 + k_3 s_3^4 \quad \text{such that} \quad \begin{array}{rr} \text{(i)} & k_1 s_1^3 + k_2 s_2^3 + k_3 s_3^3 = 0, \\ \text{(ii)} & k_1 s_1^2 + k_2 s_2^2 + k_3 s_3^2 = 1, \\ \text{(iii)} & k_1 s_1 + k_2 s_2 + k_3 s_3 = 0, \\ \text{(iv)} & k_1 + k_2 + k_3 = n+1, \\ & s_1, s_2, s_3 \in \mathbb{R},\ k_1, k_2, k_3 \in \mathbb{N}_0. \end{array} \tag{4.5}$$

Since the odd case of Theorem 2 is well-known [16], [22], we assume from now on that n is even. The mindful reader will notice that for odd n the optimal value of (4.5) coincides with the optimal value of the real relaxation (where the condition $k_1, k_2, k_3 \in \mathbb{N}_0$ is replaced by $k_1, k_2, k_3 \geqslant 0$).

For $k_3 = 0$ the equality constraints in (4.5) immediately yield $k_1 = k_2 = (n+1)/2 \notin \mathbb{N}$, and similarly, for $s_2 = s_3$ we obtain $k_1 = k_2 + k_3 = (n+1)/2 \notin \mathbb{N}$. Hence, we can assume that s_1, s_2, and s_3 are distinct and $k_1, k_2, k_3 \geqslant 1$. Moreover, for $s_3 = 0$ the resulting optimal value is $1/n$ which will turn out to be the optimal solution. Finally, by (iii), not all of the s_i have the same sign. Hence it suffices to show that for $s_1 < 0$ and $s_3 > s_2 > 0$ every admissible solution to the constraints of (4.5) has value at least $1/n$.

The linear system of equations in k_1, k_2, k_3 defined by (i), (ii), and (iii) is regular, which is easily seen from the Vandermonde computation

$$\det\begin{pmatrix} s_1^3 & s_2^3 & s_3^3 \\ s_1^2 & s_2^2 & s_3^2 \\ s_1 & s_2 & s_3 \end{pmatrix} = s_1 s_2 s_3 (s_1 - s_2)(s_1 - s_3)(s_2 - s_3) \neq 0.$$

Solving for k_1, k_2, k_3 yields

$$k_1 = \frac{s_2 + s_3}{-s_1(s_2 - s_1)(s_3 - s_1)}, \tag{4.6}$$

$$k_2 = \frac{s_1 + s_3}{s_2(s_2 - s_1)(s_3 - s_2)}, \tag{4.7}$$

$$k_3 = \frac{-(s_1 + s_2)}{s_3(s_3 - s_1)(s_3 - s_2)}. \tag{4.8}$$

Since all factors in the denominators are strictly positive, (4.7) and (4.8) imply in particular $s_1 + s_3 > 0$ and $s_1 + s_2 < 0$.

Using (iv) in (4.5) we can express one of the s_i by the others. Solving for s_2 gives

$$s_2 = -\frac{s_1 + s_3}{(n+1)s_1 s_3 + 1}. \tag{4.9}$$

Note that the denominator in (4.9) is strictly negative.

Our main goal now is to show $k_1 < (n+1)/2$ and then to use the integer condition to deduce $k_1 \leqslant n/2$. In order to achieve this, substitute (4.9) into the inequality $s_1 + s_2 < 0$ which (in connection with $s_1 + s_3 > 0$) allows to conclude $s_3^2 > s_1^2 > 1/(n+1)$. Then substitute (4.9) into (4.6) which yields

$$\begin{aligned} k_1 - \frac{n+1}{2} &= -\frac{((n+1)s_1^2 - 1)((n+1)s_3(s_3 - s_1) - 2)}{2(s_3 - s_1)((n+1)s_1^2 s_3 + 2s_1 + s_3)} \\ &= -\frac{((n+1)s_1^2 - 1)(((n+1)s_3^2 - 1) - ((n+1)s_1 s_3 + 1))}{2(s_3 - s_1)(((n+1)s_1 s_3 + 1)s_1 + (s_1 + s_3))} < 0, \end{aligned}$$

since all factors within the last fraction are positive. Hence, $k_1 < (n+1)/2$ and since it is an integer $k_1 \leqslant n/2$. Similarly, although we do not need this, one can show $k_3 \leqslant n/2$. However, note that this bound does not hold for k_2.

Now it follows from $k_1 \leqslant n/2$ and (4.6) that

$$\begin{aligned} 0 &\leqslant 2 - ns_3^2 - 2s_3^2 - n^2 s_1^3 s_3 + n^2 s_1^2 s_3^2 - ns_1^3 s_3 + ns_1^2 s_3^2 - 2ns_1^2 + ns_1 s_3 \\ &= -2(ns_1^2 - 1)((n+1)s_1 s_3 + 1) + s_3(s_1 + s_3)(n(n+1)s_1^2 - n - 2). \end{aligned}$$

That means at least one of the two terms of the sum must be non-negative which gives $s_1 \leqslant -1/\sqrt{n}$. Moreover, $s_1 + s_3 > 0$ also implies $s_3 \geqslant 1/\sqrt{n}$.

Finally, we show that for any admissible solution to the constraints of (4.5) the objective value is at least $1/n$. Replacing k_1, k_2, k_3 and s_2 in the objective function via (4.6)–(4.9) and using the inequalities $-s_1 < s_3$ and $s_1 \leqslant -1/\sqrt{n}$ obtained above we get

$$k_1 s_1^4 + k_2 s_2^4 + k_3 s_3^4 = \frac{1}{n+1} + \frac{((n+1)s_1^2 - 1)((n+1)s_3^2 - 1)}{(n+1)(-(n+1)s_1 s_3 - 1)}$$
$$\geqslant \frac{1}{n+1} + \frac{((n+1)s_1^2 - 1)}{n+1} \geqslant \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n}.$$

Hence, the optimal value of (4.5) is $1/n$, and by our remark before Proposition 10 this completes the proof of Theorem 2.

5 Connections to the Positivstellensatz

We close the paper by discussing the greater difficulty of the even case of computing $R_{n-1}(T^n)$ compared to the odd case, by analyzing problem (1.1) from the viewpoint of the Positivstellensatz [20]. This theorem states the existence of a certificate whenever a system of polynomial equalities and inequalities does not have a solution, and it can be regarded as a common generalization of Hilbert's Nullstellensatz and of linear programming duality. For our purposes, it suffices to consider the following version of Putinar (see [15], [18]). For $n \in \mathbb{N}$ let $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ denote the ring of polynomials in $x_1, \dots, x_n$, and let

$$\sum \mathbb{R}[x]^2 = \left\{ \sum_{j=1}^{k} b_j^2 \text{ for some } k \in \mathbb{N}, b_1, \dots, b_k \in \mathbb{R}[x] \right\}$$

be the set of all finite sums of squares of polynomials. Set $g_0 := 1$.

Proposition 11. *If the polynomials $f, g_1, \dots, g_m \in \mathbb{R}[x]$ satisfy*

$$f(x) > 0 \quad \textit{for all } x \in S := \{x \in \mathbb{R}^n : g_1(x) \geqslant 0, \dots, g_m(x) \geqslant 0\}$$

and

$$M := \left\{ \sum_{i=0}^{m} \sigma_i g_i : \sigma_0, \dots, \sigma_m \in \sum \mathbb{R}[x]^2 \right\} \tag{5.1}$$

contains the polynomial $1 - \sum_{i=1}^{n} x_i^2$, then $f \in M$.

Hence, in order to prove that the optimal value of (1.1) is bounded from below by some given value α, it suffices (by compactness of the admissible set) to show the existence of such a representation for $f(s) := \sum_{i=1}^{n+1} s_i^4 - \alpha + \varepsilon$ in terms of $g_1(s) := \sum_{i=1}^{n+1} s_i^3$, $g_2(s) := -\sum_{i=1}^{n+1} s_i^3$, $g_3(s) := \sum_{i=1}^{n+1} s_i^2 - 1$, $g_4(s) := -\sum_{i=1}^{n+1} s_i^2 + 1$, $g_5(s) := \sum_{i=1}^{n+1} s_i$, $g_6(s) := -\sum_{i=1}^{n+1} s_i$ for every $\varepsilon > 0$.

Bounding the degrees of the polynomials $\sigma_i g_i$ by a fixed constant in (5.1) serves to give lower bounds on the minimum. These relaxations can be computed by semidefinite programming (SDP) and are at the heart of current developments in SDP-based constrained polynomial optimization (see [12], [13]).

For the case n odd of (1.1) there exists a simple polynomial identity

$$\sum_{i=1}^{n+1} s_i^4 - \frac{1}{n+1} = \frac{2}{n+1}\left(\sum_{i=1}^{n+1} s_i^2 - 1\right) + \sum_{i=1}^{n+1}\left(s_i^2 - \frac{1}{n+1}\right)^2 \tag{5.2}$$

which shows that the minimum is bounded from below by $1/(n+1)$, and since this value can be attained by $s_1 = \cdots = s_{(n+1)/2} = -s_{(n+3)/2} = \cdots = -s_{n+1} = 1/\sqrt{n+1}$, the minimum is $1/(n+1)$. For any $\varepsilon > 0$, adding ε on both sides of (5.2) yields a representation of the positive polynomial on the left side as a sum of squares of the g_i. Note that for every odd n this representation uses only polynomials $\sigma_i g_i$ of (total) degree at most 4.

For the case n even (with minimum $1/n$) the situation is quite different. A computer calculation using the Software GloptiPoly [10] showed that already for $n = 4$ it is necessary to go up to degree 8 to find the Positivstellensatz-type certificate of optimality. Since from a practical point of view the computational efforts of this calculation drastically increase with the number of variables, we do not know up to which degree it is necessary to go for $n = 6$.

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R. Brandenberg, Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, D–85747 Garching bei München
Email: brandenb@ma.tum.de

T. Theobald, Institut für Mathematik, MA 6-2, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin
Email: theobald@math.tu-berlin.de