Willmore Surfaces in $\mathbb{S}^n$: Transforms and Vanishing Theorems

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Dedicated to Min

for your love and faith

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Zusammenfassung

Das Hauptinteresse des Autors gilt der Konstruktion neuer Transformationen von Willmoreflächen und dem Studium ihrer globalen Eigenschaften für geschlossene Flächen. Insbesondere sollen die Konstruktion von dualen Willmoreflächen und die bekannten Klassifikationsresultate für Willmore 2-Sphären verallgemeinert werden auf den Fall beliebiger Kodimension.

Bei den untersuchten Fragen handelt es sich um Fragen der Möbiusgeometrie. Für deren Untersuchungen wird das Lichtkegelmodell benutzt und eine Theorie von Paaren konform immersierter Flächen entwickelt. Als Anwendung dieser wird eine Lösung des Blaschke-Problems im n–dim Raum gegeben, welches Darboux Paare von Isothermflächen und duale Willmoreflächen charakterisiert. Diese Diskussion führt in natürlicher Weise auf die Begriffe "berühren" und "ko–berühren", welche wiederum die Definition der adjungierten Transformation von Willmoreflächen motivieren. Obwohl diese Transformationen im Allgemeinen nicht eindeutig sind, bleibt die Eigenschaft eine duale Willmorefläche zu sein in einem gewissen Sinne erhalten. Insbesondere gibt es gute Verallgemeinerungen der existierenden Dualitätssätze. Ein zentrales Ergebnis der Theorie ist, dass adjungierte Willmoreflächen charakterisiert werden können mittels konformer harmonischer Abbildungen.

Sehr starke globale Resultate können erhalten werden f"ur den Fall, dass die zugrundeliegende Flächen eine Sphäre ist. Wie üblich muss man dabei im Wesentlichen zwei Fälle unterscheiden: den streng m–isotropen Fall und den total isotropen Fall. In dieser Arbeit wird hauptsächlich der erste Fall diskutiert und es wird gezeigt, dass für solche Willmoreflächen eine kanonische adjungierte Transformation konstruiert werden kann, welche wieder eine verzweigte immersierte Willmore 2–Sphäre und außerdem streng $(m+1)$–isotrop ist. Dieses Resultat wird bewiesen über die Konstruktion isotroper Unterbündel und holomorpher Formen und folgt aus algebraisch–geometrischer Verschwindungssätzen. Interessanterweise spielt hier auch ein nicht–Verschwindungsresultat eine wichtige Rolle. Dieses Resultat ist möglicherweise ein Ansatz für die Klassifikation aller Willmore 2–Spähren.

Abstract

In this work, the main concern of the author is the construction of new transforms of some given Willmore surface and their global effect when the underlying surface is closed. Especially, we want to generalize the construction of dual Willmore surface and the known classification results on Willmore 2-spheres to arbitrary co-dimensional spaces.

Since these objects are Möbius invariant, we adopt the light-cone model, and establish a theory on pairs of conformally immersed surfaces. As an application, Blaschke's Problem in n-space is solved, which characterizes Darboux pairs of isothermic surfaces and dual Willmore surfaces. The notion *touch* and *co-touch* arise naturally from this discussion. This further inspires the definition of adjoint transforms of a given Willmore surface. Although such transforms are not unique in general, they are also Willmore and dual to the original surface in certain sense. So they are good generalizations of the existing duality theorems. Finally adjoint Willmore surfaces are characterized in terms of conformal harmonic maps, which seems to be the central result in our general theory.

When the underlying surface is assumed to be a 2-sphere, very strong global results are obtained. As usual, the discussion should be divided into two cases: the strict m-isotropic case and the totally isotropic case. We concentrate on the first case and show, for such a Willmore surface, one can construct a canonical adjoint transform, which is again a branched conformally immersed Willmore 2-sphere, yet is strict $(m+1)$-isotropic. This Ascending Theorem is obtained by constructing isotropic sub-bundles and holomorphic forms, then invoking the vanishing theorem. Interestingly not-vanishing result also plays a crucial role at here. This points out a possible way to classify all Willmore 2-spheres.

Introduction

In 1984, a paper under the title *A duality theorem for Willmore surfaces*, appeared on Journal of Differential Geometry ([8]). In this milestone work, Bryant studied Willmore surfaces, a generalization of minimal surfaces, in 3-dim space, and obtained many important results. First he defined conformal Gauss map of such a surface and established the relationship with harmonic maps. Then he observed that every Willmore surface in $\mathbb{S}^3$ admits a dual Willmore surface (might degenerate to a single point). Finally it was proved that any Willmore immersion $S^2 \to \mathbb{S}^3$ must be Möbius equivalent to a minimal surface in $\mathbb{R}^3$. These discoveries further inspired the interest in Willmore surfaces, and initiated the research on their transforms and on classification of Willmore 2-spheres, which become the subject of this thesis.

Willmore surface: A historic review

The recent interest in Willmore surfaces started from Willmore's paper ([73]) nearly 40 years ago. He suggested to study the total squared mean curvature $\frac{1}{2\pi}\int_M H^2 \mathrm{d}M$ associated to an immersed compact surface $f : M \to \mathbb{R}^3$ with mean curvature H and induced area form $\mathrm{d}M$. Compared to the similarly defined $\frac{1}{2\pi}\int_M K\mathrm{d}M$ which yields only a topological invariant, the Euler number of M, the new functional reflects the geometry of this surface, thus is more interesting. By this observation, equivalently one can consider

$$W(f) := \int_M (H^2 - K)\mathrm{d}M$$

which will be called the *Willmore functional* at here. The critical surface with respect to $W(f)$ is called *Willmore surface*. Before long people realized ([72]) that the 2-form $(H^2 - K)\mathrm{d}M$ is invariant under conformal transformations of $\mathbb{R}^3$, so the Willmore functional as well as Willmore surfaces are natural objects of Möbius differential geometry.

Thus it is not surprising that Willmore surface has already been discovered by Blaschke and Thomson, the classical authors on Möbius geometry ([4]). Willmore surface was called by them *Möbius minimal surface*, and Bryant's conformal gauss map corresponds to the classical *central sphere* *congruence* (or *mean curvature spheres*). The fact that minimal surfaces in space forms are Willmore, as well as the duality theorem, was also known to them . (Hertrich-Jeromin pointed out in [26] an even earlier predecessor.)

Although that, Blaschke's school restricted themselves to the local geometry and to the 3-dim space. Only 50 years later geometers began to look at the same objects from the global viewpoint. Willmore ([73]) raised the problem of finding the infimum of Willmore functional among closed surfaces of the same topological type, solved this for 2-spheres, and posed the famous *Willmore conjecture* for tori.

Since that time, there has been about 100 papers published on the study of Willmore functional and Willmore surfaces. The early works focused mainly on the estimation of total mean curvature and its generalization for higher dimensional submanifolds ([22, 30, 32, 33, 34, 45, 57, 71, 76, 77]). Especially, geometers have got important progress on solving Willmore conjecture under various additional conditions ([14, 28, 35, 45, 47, 62, 66, 70]). Yet it is still open (although M. Schmidt announced a proof in [64]). From the variational viewpoint, the first and second variation formulas have been derived, and the existence of minimizer are discussed ([51, 52, 53, 54, 65, 67, 69, 70]). Turning to the investigation of Willmore surfaces there are also a lot of work, including their properties and new examples, even classification results in some special cases ([3, 8, 15, 19, 20, 24, 38, 40, 42, 44, 46, 49, 56, 68]). Generalizations of Willmore surface, ranging from Willmore submanifold ([23, 39, 41, 59]) to similar notions in other geometry ([1, 2, 11, 18]) and to its discretization ([5]), were considered as well.

To have a better understanding of the surface theory in 3-dim and 4-dim spaces, the Berlin school introduced quaternions and established a new function theory with application to Willmore surfaces ([7, 10, 21, 31, 36, 37, 38, 55, 60]). It has been proved to be a fruitful theory and one of the most important progress in the past ten years on Willmore surfaces. Another paper ([13]) adopted the classical moving frame method independent to quaternionic set-up; it treated a lot of interesting topics like Möbius invariant flows. The influence of Pinkall and his collaborators' work will be seen in this thesis.

Our account is a sketch rather than a complete list of works has been done. For example, the author ignored the topic of Willmore flow and some constructions of Willmore tori. Regretfully there is not a survey article giving insightful comments on all these methods and results. Alternatively, I can recommend the expository article [58] on Mathematical Intelligencer or the short survey [75] by Willmore himself. In [74] there is a whole chapter dedicated to this topic, thus a very useful account on the development until 1993. The recently published monograph [26] on Möbius differential geometry contains some historic remark in Chapter 3. I hope the bibliography provided at here might be a useful guide in this fruitful research field.

Transforms of Willmore surfaces

The duality theorem (re-)discovered by Bryant ([8]) is a remarkable fact about Willmore surfaces in 3-space. To generalize this result to higher codimension case, Ejiri ([20]) introduced *S-Willmore surface equation.* One solution to this equation is automatically a solution to the *Willmore condition*, i.e. the Euler-Lagrange equation of the Willmore functional. These solutions correspond to a special class of Willmore surfaces under the name of *S-Willmore surfaces*, for which there holds a similar duality theorem. On the other hand, Ejiri pointed out that in $\mathbb{S}^n (n > 3)$ there are a lot of Willmore surfaces which are not S-Willmore. For such surfaces the duality theorem is not true.

Although that, there is still a rich theory about transforms of Willmore surfaces in $\mathbb{S}^4$, mainly developed in [10]. The quaternionic function theory enables us to construct Bäcklund transforms and Darboux transforms from a given Willmore surface in $\mathbb{S}^4 \cong \mathbb{HP}^1$ by solving certain differential equations in quaternions. Especially, the so-called *forward/backward two-step Bäcklund transform* are uniquely defined without involving integration, and there is again some kind of duality in between. They might reasonably be called *the left/right dual Willmore surface.*

These constructions are beautiful enough. They are rather interesting in view of the well-known relationship between integrable systems and surface theory. For example, there was already a systematic study of transforms of isothermic surfaces one century ago. Such a rich theory indicates a description by underlying integrable system, which was revealed only recently (see [17, 12]). This discovery provoked a series of works by Hertrich-Jeromin and other ones ([6, 9, 12, 25, 26, 27]). Their work provided not only an elegant treatment of the classical transform theory, but also generalizations to higher codimension space and to discrete case. Naturally, one would expect a similar theory for Willmore surfaces, because they can be characterized in terms of harmonic maps, and the constructions in [10] are based on analogy to isothermic case.

Upon close examination of Hertrich-Jeromin's work, we find that for a given isothermic surface, all other transforms (the *Goursat transform*, the *Bianchi transform*, and the *Darboux transform*) are defined via the original surface and its dual surface (which is also known as *the Christoffel transform*). The dual isothermic surface always exists, and the existence of such a duality is a characterization of being isothermic ([50]). Thus we may say this duality plays a central role in the transform theory of isothermic surfaces. From this viewpoint, one immediately recognizes the importance of Bryant's duality theorem and its generalization. But as pointed out before, we can not expect such a duality except the S-Willmore case. This becomes the first obstruction to our exploration.

Instead of attacking this problem directly, the author started by considering the characterization of S-Willmore case. It is well-known that a pair of dual S-Willmore surfaces share the same mean curvature sphere at corresponding points and are conformal to each other. Conversely, this property indeed gives a characterization. Yet this is a characterization by a condition unnecessarily strong, as pointed out by Pinkall, who directed the author's attention to Hertrich-Jeromin's monograph [26] (unpublished manuscript that time). From the latter I got to know *Blaschke's Problem* and its solutions in 3-dim space. Simply to say, it characterizes isothermic surfaces and Willmore surfaces as the only non-trivial solutions to a geometric problem in the category of Möbius geometry. Such a unification is out of one's expectation, at the same time an encouragement to the author's hope for a transform theory on Willmore surfaces parallel to isothermic case. Indeed its generalization to higher codimension case is also true:

Theorem A. *Suppose two distinct immersed surfaces in $\mathbb{S}^n$ envelop a 2-sphere congruence and are conformal at corresponding points. Then besides trivial cases, locally it must be among either of the following two classes:*

(i) A pair of isothermic surfaces forming Darboux transform to each other;

(ii) A pair of dual S-Willmore surfaces.

In the first case, the correspondence between the two surfaces is orientation preserving. In the second case, the correspondence is anti-conformal.

As a corollary we obtain the following characterization of CMC-1 surfaces in hyperbolic 3-space, which is also closely connected with Bryant's name.

Theorem B. *Let f be an surface immersed in $\mathbb{S}^n$, whose mean curvature sphere congruence has $\hat{f}$ as the second envelope with the same conformal structure. Then either f is Möbius equivalent to a CMC-1 surface in hyperbolic 3-space, or f is S-Willmore.*

It is possible to give a proof to Theorem B in the quaternionic model when $n = 4$. Yet the moving frame method developed in [13] is more general and direct at here. The author thus adopt the latter method and try to develop a theory for pairs of conformally immersed surfaces. There emerges two important invariants ρ and θ associated with such a pair. In the context of Blaschke's Problem, ρ vanishes in the isothermic case, θ vanishes in S-Willmore case. Generally, the vanishing of ρ and θ has concrete geometric meaning, which we will call *touching condition* and *co-touching condition* separately. Indeed, the touching condition were introduced by Pinkall and Pedit when trying to generalize classic Darboux transforms of isothermic surfaces to arbitrary surfaces in $\mathbb{S}^4$ via $\mathbb{HP}^1$ model (see [7]). In the similar sense, we find that for a given Willmore surface in $\mathbb{S}^4$ and its left/right dual, the co-touching condition is satisfied.

These observations, along with the *parity* between isothermic surfaces and S-Willmore surfaces, inspired the author to construct a new kind of transforms of Willmore surfaces in $\mathbb{S}^n$. It is called *the adjoint transform* and defined by the co-touching and conformal condition. They are again Willmore, and coincide with the left/right dual Willmore surfaces when $n = 4$. Moreover, a duality result holds, too. So this is a reasonable generalization. Although it is not unique in general, such transforms always exist locally. As the first transform defined for arbitrary Willmore surface in $\mathbb{S}^n$, this might be considered a remarkable achievement in this thesis.

Classification of Willmore 2-spheres

After discussing the transforms of Willmore surfaces, which is essentially a local theory, we turn to the global aspects. The first interesting topic is the classification of close Willmore surfaces of given genus. The genus-zero case, i.e. Willmore 2-spheres, comes out not only the first, but also the easiest and the finest case to consider. This might be seen in the following way.

There is a beautiful theory about minimal surfaces in $\mathbb{S}^n$ or $\mathbb{CP}^n$. Starting from such a surface, one can construct a sequence of minimal surfaces in the same space, which is known as harmonic sequence. Especially, when the underlying surface is a 2-sphere, due to the vanishing of all holomorphic forms on Riemann sphere, we find that any two elements from the sequence must be orthogonal to each other, and the sequence must terminate at a holomorphic curve in the complex projective space. Thus we obtain a classification of these minimal 2-spheres.

Minimal surfaces form a subclass of Willmore surfaces. The latter may be viewed as *conformal minimal surfaces* in Möbius geometry. By experience from the study of harmonic maps, we would expect that Willmore surfaces can be described by complex data and vanishing results might be obtained for 2-spheres as well.

In fact, people has obtained in this way a classification of Willmore 2-spheres in $\mathbb{S}^3$ and $\mathbb{S}^4$ ([8, 20, 46, 49]). Their results may be summarized as follows:

Theorem C. *For a Willmore 2-sphere immersed in $\mathbb{S}^4$, it comes either from a minimal surface in $\mathbb{R}^4$ via inverse stereographic projection, or from a rational curve in $\mathbb{CP}^3$ via the Penrose twistor projection $\mathbb{CP}^3 \to \mathbb{HP}^1$. If this 2-sphere is contained in some round 3-sphere, then it comes from a minimal surface in $\mathbb{R}^3$ via inverse stereographic projection.*

Although the complete result (including the quantization theorem and the infimum in a given homotopy class) was given in [46], it is Ejiri who obtained the main classification theorem at first ([20]). More than that, Ejiri initiated the study in higher codimension spaces. The main theorem in [20] gave a similar classification for 2-spheres immersed in $\mathbb{S}^n$ which is S-Willmore:

Theorem D. *Let M be an immersed S-Willmore surface of genus 0 in $\mathbb{S}^{2m}$. Then it corresponds uniquely to a totally isotropic, holomorphic, full immersion of M into $\mathbb{CP}^{2m+1}$, or it is a smooth compactification of a complete minimal surface in $\mathbb{R}^{2m}$ by the inverse stereographic projection.*

When $m = 2$, Willmore 2-spheres must be S-Willmore (the first vanishing result!), hence we deduce Theorem C from here. Note that Ejiri used term *S-Willmore* in a slightly modified sense other than this thesis. Leaving this technical problem aside, we find that Ejiri solved only the classification problem of S-Willmore 2-spheres, not for all Willmore 2-spheres. We don't know whether Willmore 2-spheres must always be S-Willmore; We don't know whether there exist new examples of Willmore 2-spheres. In this sense, Ejiri's theorem is unsatisfactory.

Another imperfect point in Ejiri and Montiel's treatment is that they worked in the framework of the metrical geometry of the n-sphere, not in the more suitable context of conformal geometry. In this aspect, [10] gave an elegant proof to Theorem C. After establishing a whole theory about Willmore surfaces in $\mathbb{HP}^1 \cong \mathbb{S}^4$, it was showed that for a Willmore 2-sphere which is not the image of a holomorphic or anti-holomorphic curve in $\mathbb{CP}^3$, its left dual and right dual are both well-defined almost everywhere. By a simple degree argument, they must coincide, hence a well-defined dual surface. (In other words, the original surface must be S-Willmore.) The differentiation of the dual surface yields a holomorphic section in some bundle, and it must vanish due to degree argument again. This shows that the dual surface is a single point, and the original surface is equivalent to a minimal surface in some affine space.

The proof sketched above is conceptual and simple. One might doubt that it could be generalized to higher codimension case due to quaternion involved in. Despite that, we have constructed adjoint transforms, the generalization of left/right dual surface, for any Willmore surfaces in $\mathbb{S}^n$. If the given Willmore surface is a 2-sphere, we can still consider the effect of the adjoint transforms on it, in expectation of vanishing results. This is the main idea in this thesis towards a classification for all Willmore 2-spheres in $\mathbb{S}^n$.

As discovered by Ejiri, here we should divide our discussion into two situations. In the *totally isotropic* case, the Hopf differential κ and its derivatives up to any order are always isotropic (note we use the light-cone model in the Minkowski space). It is conjectured that they correspond exactly to (anti-)holomorphic curves in the twistor space of a even-dimensional sphere $\mathbb{S}^{2r}$. On the other hand, if the Willmore 2-sphere is not totally isotropic, it must be *strict m-isotropic* for some m, i.e.

$$\langle \kappa, \kappa \rangle = \langle D_z\kappa, D_z\kappa \rangle = \cdots = \langle D_z^{m-1}\kappa, D_z^{m-1}\kappa \rangle = 0 \not\equiv \langle D_z^m\kappa, D_z^m\kappa \rangle.$$

For such a surface we establish the following *Ascending Theorem*:

Theorem E. *Let $M \cong S^2$, immersion $F : M \to \mathbb{S}^n$ is strict m-isotropic and Willmore. Suppose F is not the inverse image of any minimal surface in $\mathbb{R}^n$ under the stereographic projection. Then there is a canonically defined adjoint Willmore surface $\tilde{F}$, which is a strict $(m+1)$-isotropic Willmore surface. $\tilde{F}$ extends to the whole $M \cong S^2$ as a branched conformal immersion, which we call* the derived Willmore surface.

This ascending phenomenon is surprising. Furthermore, if we can derive such a chain of adjoint transforms with the ascending result always holding, this chain must terminate finally, because there can not be too many linearly independent isotropic vectors in a finite dimensional space. The only possibility is that the chain terminates at a minimal surface in $\mathbb{R}^n$, for whom the further derived Willmore surface is just the single point at infinity. Conversely, starting from a complete minimal surface with finite total curvature and embedded planar ends in $\mathbb{R}^n$, which is of genus 0 and strict m-isotropic for some $m \geq 1$, if we can find a non-trivial global adjoint transform, it will produce a Willmore 2-sphere in $\mathbb{S}^n$ (might be not S-Willmore). In general, the author conjecture the final classification result as follows:

Conjecture. *Suppose M is a Willmore 2-sphere immersed in $\mathbb{S}^{2r}$, then it corresponds to a holomorphic curve in the twistor space of $\mathbb{S}^{2r}$ when M is totally isotropic, or it comes from a minimal surface in $\mathbb{R}^{2r}$ via a series of adjoint transforms when M is strict m-isotropic.*

From the technical viewpoint, the Willmore condition enables us to construct holomorphic forms, and the assumption of 2-sphere leads to a series of vanishing results. But here is one more obstruction. Our transform might produce branch points on the new surface, and the associated forms would therefore have singularities. When the original surface is immersed, we can prove that these singularities are isolated and removable by an interesting *Extension Lemma*, which could be seen as a generalization of Riemann's Removable Singularity Theorem. In this way we proved the Ascending Theorem. Yet it works only for the first step. If we want to go further, there arises more difficulties in controlling the behavior of the singularities. Although that, the author is confident in his conjecture, because there are other results supporting it, especially a construction of orthogonal and isotropic frames, which we accomplished by building holomorphic forms and applying vanishing theorem once again!

Organization of this thesis

The explanation above should have helped the reader to gain some insight into the main methods and results of this thesis. For detailed discussion, we establish the surface theory in Chapter 2, and introduce Willmore surfaces as well as isothermic surfaces. The treatment is after [13], yet with an important supplement on pairs of conformally immersed surfaces, which we present in Chapter 3. As an application, we solve the generalized Blaschke's Problem. The proofs to Theorem A and Theorem B are contained at here as well as some discussions about isothermic surfaces and Willmore surfaces. This work inspires the definition of adjoint transforms of Willmore surfaces in Chapter 4, where we will show that the transform produce a new Willmore surface. A deeper result characterizes adjoint Willmore surfaces via conformal harmonic maps is also obtained at here. Apply the adjoint transform to Willmore 2-spheres, we will obtain our main result, the Ascending Theorem, in Chapter 6. In order that the reader may have a clear understanding, we discuss a special case in Chapter 5, which invokes the definition of isotropic conditions, the construction of holomorphic forms, and the statement of the Ascending Theorem. Some technical lemmas (including the Extension Lemma) as well as the notion of touch and co-touch are left to the Appendix.

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Chapter 1

Surface theory in Möbius differential geometry

In [13] we find the most suitable framework for the Möbius invariant surface theory in n-space. Following their treatment, in the first section we briefly review the basic concepts and equations, then characterize two special surface classes in the next section: Willmore surfaces and isothermic surfaces.

1.1 The surface theory by moving frames

1.1.1 The light-cone model

As usual, we adopt the classical light-cone model in Möbius geometry. Let $\mathcal{L}$ be the light cone in the Minkowski space $\mathbb{R}^{n+1,1}$ with quadratic form $\langle y, y\rangle = -y_1^2 + \sum_{i=1}^{n+1} y_i^2$. Then the unit sphere $\mathbb{S}^n$ in Euclidean space is identified with our projectivized light cone via

$$\mathbb{S}^n \cong \mathbb{P}(\mathcal{L}) : x \leftrightarrow [1 : x].$$

The projective action of the Lorentz group on $\mathbb{P}(\mathcal{L})$ yields a representation of the Möbius group.

The most elementary objects in Möbius geometry are spheres . In our model, points correspond to light-like vectors (lines), and hyperspheres correspond to space-like vectors. In general, k-spheres in n-space are represented by space-like $(n-k)$-dim subspaces, because the orthogonal complement to the latter is a subspace with a Minkowski metric, whose null lines form a small sphere.

For a general submanifold $f : M \to \mathbb{S}^n \cong \mathbb{P}(\mathcal{L})$, we also want to understand it through its representation in the Minkowski space. A *(local) lift* of f is just a map F from M into the light cone such that the null line spanned by $F(p)$ is $f(p)$. Two different local lifts differ by a scaling, so the metric induced from them are conformal to each other.

In our study we often associate a sphere (congruence) to a submanifold and are interested in the relationship between them. Let space-like $(n-k)$-dim subspace U stand for such a k-sphere. This sphere passes through $f(p)$ iff $\langle F(p), U\rangle = 0$. Suppose this is satisfied, then the sphere is tangent to f at p iff $\langle \mathrm{d}F(p), U\rangle = 0$. If we identify the k-sphere with $U^{\perp}$, then it is tangent to f at p iff $F(p)$ (the map itself) and $\mathrm{d}F(p)$ (all tangent vectors) are all contained in $U^{\perp}$.

1.1.2 Frames and structure equations

From now on, we concentrate on immersed surfaces. Consider a conformal map $f : M \to \mathbb{S}^n \cong \mathbb{P}(\mathcal{L})$ of Riemann surface M with local lift F. The conformality condition is equivalent to $\langle F_z, F_z\rangle = 0$ for any F and any coordinate z on M; it is immersed iff $\langle F_z, F_{\bar z}\rangle > 0$. Now given conformal immersion f, there is a Möbius invariant decomposition $M \times \mathbb{R}^{n+1,1} = V \oplus V^{\perp}$, where

$$V = \mathrm{Span}\{F, dF, F_{z\bar z}\}$$

is a rank 4 subbundle defined via local lift F and complex coordinate z (one readily checks that V is independent to such choices, thus well-defined). The restriction of the Lorentzian metric on V is of signature $(3,1)$ (check that $\langle F, F_{z\bar z}\rangle = -\langle F_z, F_{\bar z}\rangle < 0$), so $V^{\perp}$ is a space-like subbundle, called the *Möbius normal bundle*. The connection D on $V^{\perp}$ defined by orthogonal projection of the derivative in $\mathbb{R}^{n+1,1}$ is the usual *normal connection* in metric geometry, which is already known to be Möbius invariant. On the other hand, V determines a Möbius invariant 2-sphere $\mathbb{P}(V \cap \mathcal{L})$ at every point of this immersed surface. we call it the *mean curvature sphere* or *central sphere congruence*. Later on we will often deal with the complexification of V and $V^{\perp}$. They are denoted respectively as $V_{\mathbb{C}}, V_{\mathbb{C}}^{\perp}$.

Remark 1.1. The name *mean curvature sphere* comes from the remarkable property that it is tangent to the surface and has the same mean curvature vector as the surface at the tangent point, where the ambient space is endowed with a metric of Euclidean space (or any space form). This relationship with metric subgeometry is very important and useful in later discussions.

To facilitate the treatment of surface theory via moving-frame method, it is convenient to normalize our frame at first. Fix a local coordinate z, among various choice of local lifts there is a canonical one into the forward light cone, which is denoted by Y and determined by

$$|\mathrm{d}Y|^2 = |\mathrm{d}z|^2.$$

Such a canonical lift is Möbius invariant. This enable us to find a Möbius invariant frame of $V \otimes \mathbb{C}$ given by

$$\{Y, Y_z, Y_{\bar{z}}, N\},$$

where $N \in \Gamma(V)$ is chosen so that these frame vectors are orthogonal to each other except $\langle Y_z, Y_{\bar{z}}\rangle = \frac{1}{2}$, $\langle Y, N\rangle = -1$. Such a section N is also unique.

Since Y_{zz} is orthogonal to Y, Y_z and $Y_{\bar{z}}$, There must be a complex function s and a section $\kappa \in \Gamma(V_{\mathbb{C}}^{\perp})$ so that the following Hill's equation holds:

$$Y_{zz} + \frac{s}{2} Y = \kappa. \tag{1.1}$$

This defines two basic invariants. We leave the discussion of their meaning to next subsection, and go on to derive the structure equations ($\psi \in \Gamma(V^{\perp})$ denote an arbitrary section of the normal bundle):

$$\begin{cases} Y_{zz} = -\dfrac{s}{2} Y + \kappa, \\ Y_{z\bar{z}} = -\langle \kappa, \bar{\kappa}\rangle Y + \dfrac{1}{2} N, \\ N_z = -2\langle \kappa, \bar{\kappa}\rangle Y_z - s Y_{\bar{z}} + 2 D_{\bar{z}}\kappa, \\ \psi_z = D_z \psi + 2\langle \psi, D_{\bar{z}}\kappa\rangle Y - 2\langle \psi, \kappa\rangle Y_{\bar{z}}. \end{cases} \tag{1.2}$$

The computation is straightforward, hence omitted at here. The conformal Gauss, Codazzi and Ricci equations as integrable conditions are given as below:

$$\frac{1}{2} s_{\bar{z}} = 3\langle D_z \bar{\kappa}, \kappa\rangle + \langle \bar{\kappa}, D_z \kappa\rangle, \tag{1.3a}$$

$$\mathrm{Im}\Big(D_{\bar{z}} D_{\bar{z}} \kappa + \frac{1}{2} \bar{s} \kappa\Big) = 0, \tag{1.3b}$$

$$R^D_{\bar{z}z}\psi := D_{\bar{z}} D_z \psi - D_z D_{\bar{z}} \psi = 2\langle \psi, \kappa\rangle \bar{\kappa} - 2\langle \psi, \bar{\kappa}\rangle \kappa. \tag{1.3c}$$

1.1.3 Hopf differential and Schwarzian

The fundamental equation in our surface theory is (1.1):

$$Y_{zz} + \frac{s}{2} Y = \kappa.$$

The quantities κ and s are Möbius invariants depending on coordinate z. When the local coordinate changes from z to w, the new invariants κ' and s' are given by

$$\kappa' = \kappa \left(\frac{\partial z}{\partial w}\right)^{\frac{3}{2}} \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{-\frac{1}{2}}, \tag{1.4a}$$

$$s' = s \left(\frac{\partial z}{\partial w}\right)^{2} + S_w(z). \tag{1.4b}$$

where $S_w(z)$ is the classical Schwarzian derivative of z with respect to w.

Remark 1.2. According to the observation in [13], s is interpreted as the *Schwarzian* of immersion f, and κ may be identified with the normal valued *Hopf differential* up to a suitable scaling. To see that, notice if κ vanishes, according to the structure equations (1.2) given below, frame vectors $\{Y, Y_z, Y_{\bar{z}}, N\}$ will satisfy a homogeneous linear PDE system. So they are contained in a fixed Minkowski 4-space. This means the original map f is onto a totally umbilic 2-sphere, which is usually characterized by the vanishing of the classical Hopf differential. Next, as a conformal map to $S^2 \cong \mathbb{CP}^1$, f may be regarded as a meromorphic function, whose Schwarzian derivative

$$S_z(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

is easily shown to be s. It determines the meromorphic function f up to Möbius transformation of $\mathbb{CP}^1$. This is the naive case of the general conformal surface theory.

In $\mathbb{S}^3$ these invariants determine the immersion up to Möbius transformation (in higher codimension case we should count on the normal connection D). This is the fundamental theorem of conformal surface theory:

Theorem 1.3 ([13, Theorem 3.1]). *Suppose $f_1, f_2 : M \to \mathbb{S}^3$ are two conformal immersions inducing the same Hopf differentials and Schwarzians, then there is a Möbius transformation $T : \mathbb{S}^3 \to \mathbb{S}^3$ with $Tf_1 = f_2$.*

Conversely, let κ and s be given data transforming according to (1.4), *which also satisfy the conformal Gauss and Codazzi equations. Then there exists a conformal immersion $f : M \to \mathbb{S}^3$ with Hopf differential κ and Schwarzian s.*

1.1.4 Conformal Gauss map and Willmore functional

Definition 1.4. For a conformally immersed surface $f : M \to \mathbb{S}^n$ with decomposition $M \times \mathbb{R}^{n+1,1} = V \oplus V^\perp$ as before, we define

$$G := Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_z \wedge Y_{\bar{z}} \wedge N, \qquad z = u + iv.$$

This is a map from M to the Grassmannian $G_{3,1}(\mathbb{R}^{n+1,1})$, called the *conformal Gauss map* of f. This Grassmannian consists of all 4-dimensional Minkowski subspaces. [1]

Proposition 1.5. *For conformal immersion $f : M \to \mathbb{S}^n$, G induces a positive definite metric*

$$g = \frac{1}{4}\langle \mathrm{d}G, \mathrm{d}G\rangle = \langle \kappa, \bar{\kappa}\rangle |\mathrm{d}z|^2$$

[1] Since V, $V^\perp$ and G determine each other, either of them could be regarded as the conformal Gauss map. Indeed, the other way to define the conformal Gauss map is by $G' = \psi_1 \wedge \cdots \wedge \psi_{n-2}$ where $\{\psi_1, \ldots, \psi_{n-2}\}$ form an orthonormal frame of the Möbius normal bundle $V^\perp$ (compatible with a fixed orientation).

on M except at umbilic points, which is called the Möbius metric. *Especially this is a conformal metric, thus justifies the name of conformal Gauss map.*

Proof. We have from (1.2) that

$$G_z = -2i\,(Y \wedge \kappa \wedge Y_{\bar z} \wedge N + Y \wedge Y_z \wedge Y_{\bar z} \wedge 2D_{\bar z}\kappa)\,.$$

The scalar products between these multivectors are found to be

$$\langle G_z, G_z\rangle = 0, \qquad \langle G_z, G_{\bar z}\rangle = 2|\kappa|^2. \qquad \square$$

Definition 1.6. The *Willmore functional* of f is defined at here as the area of M with respect to the Möbius metric:

$$W(f) := \frac{i}{2}\int_M |\kappa|^2 \mathrm{d}z \wedge \mathrm{d}\bar z.$$

It differs from the usual definition up to multiplication by a constant.

To study the geometry of Grassmannian $G_{3,1}(\mathbb{R}^{n+1,1})$, we regard it as a submanifold embedded in $\wedge^4\mathbb{R}^{n+1,1}$ with the usual scalar product between multivectors. The embedding is given by

$$\mathrm{Span}\{\psi_i, i = 1, \ldots, 4\} \quad \mapsto \quad G = \lambda \cdot \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4,$$

whereas the factor λ is chosen so that

$$\langle G, G\rangle = -1.$$

Let $\{\psi_\alpha, \alpha = 5, \ldots, n+2\}$ be an orthogonal basis of the orthogonal complement of $\mathrm{Span}\{\psi_i, i = 1, \ldots, 4\}$ in $\mathbb{R}^{n+1,1}$. Then there is an orthogonal decomposition of $\wedge^4\mathbb{R}^{n+1,1}$ into radial, tangential, and normal parts:

$$\wedge^4\mathbb{R}^{n+1,1} = \mathrm{Span}\{G\} \oplus G^\top \oplus G^\perp, \tag{1.5}$$

$$\text{where} \quad G^\top := \mathrm{Span}\{\psi_i \wedge \psi_j \wedge \psi_k \wedge \psi_\alpha,\ i,j,k \le 4;\ \alpha \ge 5\},$$
$$G^\perp := \mathrm{Span}\{\psi_r \wedge \psi_t \wedge \psi_\alpha \wedge \psi_\beta,\ r,t = 1, \ldots, n+2;\ \alpha, \beta \ge 5\}.$$

We may directly consider a conformal map G from a Riemann surface M to $G_{3,1}(\mathbb{R}^{n+1,1})$. M is endowed with a complex structure $J : TM \to TM, J^2 = -id$, and the $*$-operator acts on 1-form ω by $*\omega = \omega \circ J$. We define the energy of G to be

$$E(G) := \int_M \langle \mathrm{d}G \wedge *\mathrm{d}G\rangle.$$

Critical points G of this functional with respect to variations of G are called *harmonic maps* from M to $G_{3,1}(\mathbb{R}^{n+1,1})$. Note $\langle \mathrm{d}G \wedge *\mathrm{d}G\rangle$ is a 2-form determined by pairing with tangent vectors (X, JX):

$$\langle \mathrm{d}G \wedge *\mathrm{d}G\rangle(X, JX) = -\langle \mathrm{d}G(X), \mathrm{d}G(X)\rangle - \langle \mathrm{d}G(JX), \mathrm{d}G(JX)\rangle.$$

Set complex coordinate $z = u + \mathrm{i}v$, we have demonstrated that

$$\langle \mathrm{d}G \wedge *\mathrm{d}G \rangle = \langle G_z, G_{\bar{z}} \rangle \cdot (-2\mathrm{i})\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Therefore, the Willmore functional of a surface f is related to the energy of its conformal Gauss map via

$$W(f) = -\frac{1}{8}E(G).$$

In the same way one can show

$$\begin{aligned} \mathrm{d} * \mathrm{d}G\left(\frac{\partial}{\partial u}, J\frac{\partial}{\partial u}\right) &= -G_{uu} - G_{vv} = -4G_{z\bar{z}}. \\ \implies \quad \mathrm{d} * \mathrm{d}G &= G_{z\bar{z}} \cdot (-2\mathrm{i})\mathrm{d}z \wedge \mathrm{d}\bar{z}. \end{aligned}$$

Because any 2-form over M is determined by such a pairing, we will follow the convention in [10] and identify the 2-form with the quadratic form so produced when this is convenient. For example, we can check that for any two vector-valued 1-form A, B,

$$\langle A \wedge *B \rangle = -\langle A, B \rangle - \langle *A, *B \rangle = \langle B \wedge *A \rangle. \tag{1.6}$$

Proposition 1.7. *Conformal map $G : M \to G_{3,1}(\mathbb{R}^{n+1,1})$ is harmonic iff the tangential part of $\mathrm{d} * \mathrm{d}G$ vanishes:*

$$(\mathrm{d} * \mathrm{d}G)^{\top} = 0.$$

Proof. Consider G_t as a variation of G, $G_0 = G$, with the variational vector filed $\dot{G} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} G_t$. We have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(G_t) &= \int_M \left(\langle \mathrm{d}\dot{G} \wedge *\mathrm{d}G \rangle + \langle \mathrm{d}G \wedge *\mathrm{d}\dot{G} \rangle\right) \\ &= 2\int_M \langle \mathrm{d}\dot{G} \wedge *\mathrm{d}G \rangle \\ &= -2\int_M \langle \dot{G}, \mathrm{d} * \mathrm{d}G \rangle. \end{aligned}$$

by (1.6) and Stokes theorem. Since $\langle G_t, G_t \rangle = -1$ implies $\langle \dot{G}, G \rangle = 0$, $\dot{G}$ can be arbitrary element in $G^{\top}$. We conclude that G is harmonic iff $(\mathrm{d} * \mathrm{d}G)^{\top} = 0$. □

1.2 Special surface classes

After establishing the general surface theory we turn to special surface classes. Two such classes, isothermic surfaces and Willmore surfaces, first appearing as objects in Euclidean space, turned out to be invariant under Möbius transformation, hence the natural objects of our study.

1.2.1 Willmore surfaces

Definition 1.8. Let M be a topological surface. Any immersion $f : M \to \mathbb{S}^n$ automatically induces a conformal structure over M, hence defines the Willmore functional $W(f)$. If f is a critical point of W with respect to any variations of the map and the induced conformal structures, it is called a *Willmore surface*. Since Willmore functional is just the area of the Möbius metric, Willmore surfaces were also historically called *Möbius minimal surface*. If we consider only variations of f through conformal mappings with respect to the original conformal structure, such a critical point is called *constrained Willmore surface*.

Theorem 1.9 ([13, 20]). *For a conformally immersed surface f in $\mathbb{S}^n$, the following three conditions are equivalent:*

(i) f is Willmore.

(ii) The Hopf differential and Schwarzian of f satisfy

$$D_{\bar z}D_{\bar z}\kappa + \frac{1}{2}\bar{s}\kappa = 0. \quad \textit{(Willmore condition)} \tag{1.7}$$

This is a condition stronger than the conformal Codazzi equation (1.3b).

(iii) The conformal Gauss map G is a harmonic map into the Grassmannian $G_{3,1}(\mathbb{R}^{n+1,1})$.

Proof. Let Y_t be a variation of the lift Y, and G_t its conformal Gauss map, $Y_0 = Y$. To ensure that Y_t stay conformal, generally we have to allow the complex structure J (equivalently the $*$ operator) vary, too. This is the main difficulty in our proof. To deal with this problem, suppose $z = u + \mathrm{i}v$ is a complex coordinate for the original J_0. We take $(\frac{\partial}{\partial u}, J_t\frac{\partial}{\partial u})$ to be the conformal frame adapted to the varying complex structure J_t, and denote Δ_t to be the varying Laplacian with $\Delta_0 = \partial^2/\partial z\partial\bar z$. Following the convention in last section and keeping in mind that $Y_{z\bar z} = \frac{1}{2}N - \langle\kappa,\bar\kappa\rangle Y$, there is

$$\begin{aligned} G_t &= 2\cdot Y_t \wedge \mathrm{d}Y_t\left(\frac{\partial}{\partial u}\right)\wedge \mathrm{d}Y_t\left(J_t\frac{\partial}{\partial u}\right)\wedge\Delta_t Y_t. \\ G_0 &= G = -2\mathrm{i}\cdot Y\wedge Y_z\wedge Y_{\bar z}\wedge N. \end{aligned}$$

Since we are concerning with the underlying map $f_t : M \to \mathbb{S}^n$, without loss of generality we may choose Y_t so that $\langle (Y_t)_u, (Y_t)_u\rangle = 1$. This guarantees

$$\langle G_t, G_t\rangle = -1, \quad \langle\dot G, G\rangle = 0.$$

$$\begin{aligned} \text{Here } \dot G &:= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} G_t \\ &= 2\,\dot Y\wedge Y_u\wedge Y_v\wedge\Delta_0 Y + Y\wedge\dot Y_u\wedge Y_v\wedge\Delta_0 Y \\ &\quad + Y\wedge Y_u\wedge(\cdots)\wedge\Delta_0 Y + Y\wedge Y_u\wedge Y_v\wedge(\cdots) \\ &= -2\mathrm{i}\cdot\dot Y\wedge Y_z\wedge Y_{\bar z}\wedge N + (\text{radial and other tangential components}) \end{aligned}$$

according to the decomposition in (1.5). Here radial component means G-component, other tangential components consist of terms like $Y\wedge\psi_\alpha\wedge Y_{\bar z}\wedge N, Y\wedge Y_z\wedge\psi_\alpha\wedge N, Y\wedge Y_z\wedge Y_{\bar z}\wedge\psi_\alpha$, with $\psi_\alpha\in\Gamma(V^\perp)$. The variational vector field $\dot Y:=\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}Y_t$ must satisfy $\langle\dot Y,Y\rangle=0$, so its general form is

$$\dot Y=aY+bY_z+\bar bY_{\bar z}+\psi,\quad a\in\mathbb{R},b\in\mathbb{C},\psi\in\Gamma(V^\perp)$$

Substitute this into the last expression of $\dot G$, and note there always holds $\langle\dot G,G\rangle=0$, the result should be

$$\dot G=-2\mathrm{i}\cdot\psi\wedge Y_z\wedge Y_{\bar z}\wedge N+(\text{other tangential components}).\tag{1.8}$$

On the other hand, by structure equations (1.2) it is easy to derive

$$\begin{aligned}\frac{\mathrm{i}}{2}G_{z\bar z}&=-2\langle\kappa,\bar\kappa\rangle Y\wedge Y_z\wedge Y_{\bar z}\wedge N-2\;Y\wedge Y_z\wedge Y_{\bar z}\wedge\Big(D_{\bar z}^2\kappa+\frac{\bar s}{2}\kappa\Big)\\&\quad+2\;Y\wedge\kappa\wedge Y_{\bar z}\wedge D_z\bar\kappa+2\;Y\wedge Y_z\wedge\bar\kappa\wedge D_{\bar z}\kappa+Y\wedge\kappa\wedge\bar\kappa\wedge N\\&=-2\;Y\wedge Y_z\wedge Y_{\bar z}\wedge\Big(D_{\bar z}^2\kappa+\frac{\bar s}{2}\kappa\Big)+(\text{radial and normal components}).\end{aligned}$$

Now we compute the first variation of the Willmore functional. Since this functional is a constant times of the energy of G, we need only to consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}E(G_t)=\underbrace{\int_M\langle\mathrm{d}\dot G\wedge*\mathrm{d}G\rangle}_{I}+\underbrace{\int_M\langle\mathrm{d}G\wedge\dot{*}\mathrm{d}G\rangle}_{II}+\underbrace{\int_M\langle\mathrm{d}G\wedge*\mathrm{d}\dot G\rangle}_{III}.$$

As in the proof to Proposition 1.7,

$$I=III=-\int_M\langle\dot G,\mathrm{d}*\mathrm{d}G\rangle=-\int_M\langle\dot G,(\mathrm{d}*\mathrm{d}G)^\top\rangle.$$

Observe that in the right hand side of (1.8), except the term $\psi\wedge Y_z\wedge Y_{\bar z}\wedge N$, any radial or tangential component is orthogonal to

$$(\mathrm{d}*\mathrm{d}G)^\top=8\;Y\wedge Y_z\wedge Y_{\bar z}\wedge\Big(D_{\bar z}^2\kappa+\frac{\bar s}{2}\kappa\Big)\cdot\mathrm{d}z\wedge\mathrm{d}\bar z.\tag{1.9}$$

So

$$\begin{aligned}I&=16\mathrm{i}\cdot\int_M\langle\psi\wedge Y_z\wedge Y_{\bar z}\wedge N,Y\wedge Y_z\wedge Y_{\bar z}\wedge\Big(D_{\bar z}^2\kappa+\frac{\bar s}{2}\kappa\Big)\rangle\cdot\mathrm{d}z\wedge\mathrm{d}\bar z\\&=-4\mathrm{i}\cdot\int_M\langle\psi,D_{\bar z}^2\kappa+\frac{\bar s}{2}\kappa\rangle\cdot\mathrm{d}z\wedge\mathrm{d}\bar z.\end{aligned}$$

Next we claim

$$\langle\mathrm{d}G\wedge\dot{*}\mathrm{d}G\rangle=0,\quad\text{hence}\;\;II=0.$$

This is proved completely the same as in [10, Proof to Theorem 3]. Denote linear operator $B = \dot{J}$, i.e. $*\omega(X) = \omega(BX)$. It must satisfy $BJ + JB = 0$. So

$$\begin{aligned}\langle \mathrm{d}G \wedge *\mathrm{d}G\rangle(X, JX) &= \langle \mathrm{d}G(X), *\mathrm{d}G(JX)\rangle - \langle \mathrm{d}G(JX), *\mathrm{d}G(X)\rangle \\ &= \langle \mathrm{d}G(X), \mathrm{d}G(BJX)\rangle - \langle \mathrm{d}G(JX), \mathrm{d}G(BX)\rangle \\ &= -\langle \mathrm{d}G(X), \mathrm{d}G(JBX)\rangle - \langle \mathrm{d}G(JX), \mathrm{d}G(BX)\rangle\ .\end{aligned}$$

Recalling that G is conformal, hence

$$\begin{aligned}&\langle \mathrm{d}G(X), \mathrm{d}G(JX)\rangle = 0, \quad \forall\ X \\ \Longrightarrow\quad &\langle \mathrm{d}G(X), \mathrm{d}G(JY)\rangle + \langle \mathrm{d}G(Y), \mathrm{d}G(JX)\rangle = 0 \quad \forall\ X, Y.\end{aligned}$$

Taking $Y = BX$ proves our claim.

Sum together, we have shown

$$\dot{W}(f_t) = -\frac{1}{8}\dot{E}(G_t) = \mathrm{i} \cdot \int_M \langle \psi, D_{\bar z}^2\kappa + \frac{\bar s}{2}\kappa\rangle \cdot \mathrm{d}z \wedge \mathrm{d}\bar z.$$

It vanishes for any ψ iff

$$D_{\bar z}^2\kappa + \frac{\bar s}{2}\kappa = 0,$$

which is also equivalent to the vanishing of $(\mathrm{d} * \mathrm{d}G)^\top$ according to (1.9). By Proposition 1.7, our proof is done. □

As a corollary, the integrability conditions for a Willmore surface turn out to be

$$\begin{cases} \frac{1}{2}s_{\bar z} = 3\langle D_z\bar\kappa, \kappa\rangle + \langle \bar\kappa, D_z\kappa\rangle, \\ D_{\bar z}D_{\bar z}\kappa + \frac{1}{2}\bar s\kappa = 0, \\ R^D_{\bar z z}\psi = 2\langle\psi, \kappa\rangle\bar\kappa - 2\langle\psi, \bar\kappa\rangle\kappa. \end{cases}$$

This system admits the symmetry

$$\kappa_\lambda = \lambda\kappa, \quad s_\lambda = s,$$

for unitary $\lambda \in S^1$, which describes the associated family of Willmore surfaces. The existence of such transforms is a consequence of the underlying integrable system.

Remark 1.10. It has been shown [13] that a surface is constrained Willmore iff

$$D_{\bar z}D_{\bar z}\kappa + \frac{1}{2}\bar s\kappa = \mathrm{Re}(\bar q\kappa) \tag{1.10}$$

for some holomorphic quadratic differential $q\mathrm{d}z^2$. This allows a similar construction of associated constrained Willmore surfaces via

$$\kappa_\lambda = \lambda\kappa, \quad s_\lambda = s + (\lambda^2 - 1)q, \quad q_\lambda = \lambda^2 q.$$

Remark 1.11. The characterization of Willmore surfaces in terms of (1.7) was introduced in [13] without proof. Alternatively, as a by-product of the new theory on conformal invariant flows, (1.10) was derived for constrained Willmore surfaces. This is understandable, since the equivalence between conditions (ii) and (iii) of Theorem 1.9 are well-known to experts in this field, whereby (i) is obtained easily. On the other hand, if one tries to give a rigorous proof as above, it will be found almost parallel to the work done before (e.g. [10]), hence not economical to the purpose of those authors. In our place, since the Willmore condition (1.7) is widely used, it is preferable to supplement a proof for the sake of completeness. Moreover, later we will give a similar characterization of adjoint Willmore surfaces in terms of harmonic maps into another Grassmannian. To compare these results together with their proofs might be interesting.

1.2.2 Isothermic surfaces

Definition 1.12. An *isothermic* surface is a surface whose Hopf differential κ is real valued for some holomorphic coordinate z.

Remark 1.13. Classically, a surface in $\mathbb{R}^3$ is called isothermic if it can be conformally parameterized by its curvature lines (away from umbilics). This notion is indeed conformally invariant. It has been generalized ([9, 50, 63]) to higher codimension spaces by an equivalent characterization that the classical Hopf differential is real-valued in a suitable complex coordinate. Since the classical Hopf differential differs from our κ by a real factor $|z|$, our definition is equivalent to their characterization.

Given an isothermic surface in $\mathbb{S}^3$ with Hopf differential κ and Schwarzian s, we observe that the conformal Gauss and Codazzi equations still hold under deformation

$$s_r = s + r, \quad \kappa_r = \kappa,$$

where $r \in \mathbb{R}$ is real parameter. By Theorem 1.3 we see that there is an associated family of isothermic surfaces, which are exactly the T-transforms of Calapso and Bianchi.

Remark 1.14. Since the Schwarzians s_r are distinct, those surfaces in the associated family are non-congruent. So the Hopf differential alone could not determine isothermic surface. But except this case, in general two surfaces sharing the same Hopf differentials must be congruent. In other words, isothermic surfaces may be characterized by the existence of such a non-trivial deformation [13, Theorem 3.3].

Chapter 2

Pair of conformally immersed surfaces

Transforms of certain surface classes is an important topic in surface theory. Generally for a given surface we construct the second immersion by solving certain differential equations. Such constructions usually admit kinds of duality between the old surface and the new one. In view of that, it is also natural to follow another line, namely to characterize such a pair of surfaces by some geometrical conditions. These considerations lead to the study of pairs of conformal immersions, which seemed to be a rather interesting theme after dealing with single surfaces.

2.1 General theory

2.1.1 Pairs of surfaces

Let us start with Riemann surface M and two arbitrary conformal immersions $f, \hat{f} : M \to \mathbb{S}^n$. Given coordinate z, set Y to be the canonical lift of f, with Schwarzian s and Hopf differential κ. A specific local lift of $\hat{f}$, denoted by $\widehat{Y}$, is chosen so that

$$\langle Y, \widehat{Y} \rangle = -1. \tag{2.1}$$

(For convenience, assume f and $\hat{f}$ are always distinct.) With the canonical frame $\{Y, Y_z, Y_{\bar{z}}, N\}$ at hand, we may express $\widehat{Y}$ explicitly:

$$\widehat{Y} = \lambda Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N + \xi, \tag{2.2}$$

where λ and μ are real-valued and complex-valued functions separately, and the real $\xi \in \Gamma(V^{\perp})$. Since $\widehat{Y}$ is isotropic, there must be $\lambda = \frac{1}{2}(|\mu|^2 + \langle \xi, \xi \rangle)$. Substituting this back into (2.2) and taking derivatives on both sides, a straightforward calculation with the help of (1.2) yields the fundamental equation for such a pair of immersions:

$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \theta\left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + \rho\left(Y_z + \frac{\mu}{2}Y\right) + \langle\xi,\zeta\rangle Y + \zeta, \tag{2.3}$$

where

$$\theta = \mu_z - \frac{1}{2}\mu^2 - s - 2\langle\xi,\kappa\rangle, \tag{2.4a}$$

$$\rho = \bar{\mu}_z - 2\langle\kappa,\bar{\kappa}\rangle + \frac{1}{2}\langle\xi,\xi\rangle, \tag{2.4b}$$

$$\zeta = D_z\xi - \frac{\mu}{2}\xi + 2\left(D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa\right) \ \in\ \Gamma(V_{\mathbb{C}}^{\perp}). \tag{2.4c}$$

It is easy to check that θ and ρ corresponds to a $(2,0)$ form and a $(1,1)$ form separately. They may be defined in another way:

$$\frac{\theta}{2}\,\mathrm{d}z^2 = \frac{\langle Y\wedge Y_z, \widehat{Y}\wedge\widehat{Y}_z\rangle}{\langle Y\wedge\widehat{Y}, Y\wedge\widehat{Y}\rangle}\,\mathrm{d}z^2, \tag{2.5a}$$

$$\frac{\rho}{2}\,\mathrm{d}z\mathrm{d}\bar{z} = \frac{\langle Y\wedge Y_{\bar{z}}, \widehat{Y}\wedge\widehat{Y}_z\rangle}{\langle Y\wedge\widehat{Y}, Y\wedge\widehat{Y}\rangle}\,\mathrm{d}z\mathrm{d}\bar{z} \tag{2.5b}$$

for arbitrary local lifts $Y, \widehat{Y}$ and coordinate z. Both are independent to such choices, hence well-defined. Here the inner product between bi-vectors is as usual. Note that if we interchange between Y and $\widehat{Y}$, ρ turns to be $\bar{\rho}$, and θ keeps invariant. They are invariants associated with such a pair of immersed surfaces.

2.1.2 Pairs of contact elements

What's the meaning of θ and ρ? From (2.3) we see that they are first order invariants. In this view, $Y\wedge Y_z$ and $\widehat{Y}\wedge\widehat{Y}_z$ may be interpreted as *complex contact elements* of f and $\hat{f}$ at corresponding points, and $\theta,\ \rho$ are just the inner products between them or their conjugates.

More concretely, a 2-dim contact element (always assumed to be oriented) is just a 2-dim oriented subspace in $T_p\mathbb{S}^n, p\in\mathbb{S}^n$, which corresponds to a 3-dim oriented subspace of signature $(2,0)$ in the Minkowski space $\mathbb{R}^{n+1,1}$. In the discussion below we will always represent such a contact element by frame $\{X, X_1, X_2\}$ with scalar product matrix $\mathrm{diag}(0,r,r), r>0$, and X is in the forward light cone. Two such frames differ by a 3×3 matrix, whose determinant is positive when they induce the same orientation . They determine (up to multiplication by a complex number) the *complex contact element* represented by $X\wedge(X_1 - \mathrm{i}X_2)$. Its conjugate corresponds to the real contact element with reversed orientation.

Consider two contact elements $\Sigma = \{Y, Y_1, Y_2\}$ and $\widehat{\Sigma} = \{\widehat{Y}, \widehat{Y}_1, \widehat{Y}_2\}$. The null lines spanned separately by $Y, \widehat{Y}$ are distinct, so $\langle Y, \widehat{Y}\rangle \neq 0$. Inspired by formulae (2.5a)(2.5b), we define two Möbius invariants associated with such a pair of 2-dim contact elements at separate points:

$$\theta = \frac{1}{2}\frac{\langle Y \wedge (Y_1 - \mathrm{i}Y_2), \widehat{Y} \wedge (\widehat{Y}_1 - \mathrm{i}\widehat{Y}_2)\rangle}{\langle Y \wedge \widehat{Y}, Y \wedge \widehat{Y}\rangle}, \tag{2.6a}$$

$$\rho = \frac{1}{2}\frac{\langle Y \wedge (Y_1 + \mathrm{i}Y_2), \widehat{Y} \wedge (\widehat{Y}_1 - \mathrm{i}\widehat{Y}_2)\rangle}{\langle Y \wedge \widehat{Y}, Y \wedge \widehat{Y}\rangle}. \tag{2.6b}$$

Note they are independent to the choice of frames (local lifts and coordinates).

How about two contact elements at the same point p? Follow the notations above and suppose Y is parallel to $\widehat{Y}$. Intuitively we need only to consider the 2-planes $\mathrm{Span}\{Y_1, Y_2\}$ and $\mathrm{Span}\{\widehat{Y}_1, \widehat{Y}_2\}$. The following two quantities

$$\underline{\theta} = \frac{1}{2}\langle Y_1 + \mathrm{i}Y_2, \widehat{Y}_1 - \mathrm{i}\widehat{Y}_2\rangle, \tag{2.7a}$$

$$\underline{\rho} = \frac{1}{2}\langle Y_1 - \mathrm{i}Y_2, \widehat{Y}_1 - \mathrm{i}\widehat{Y}_2\rangle. \tag{2.7b}$$

are similarly well-defined, i.e. they are independent to the choice of frames of $\Sigma, \widehat{\Sigma}$. (Compared to (2.6a) (2.6b), here the $\pm$ sign is reversed in two places. Why this convention will be clear in next subsection.)

2.1.3 Touch and co-touch

To better understand the geometric meaning of θ and ρ, let's consider the special case when either of them vanishes.

Definition 2.1. Two contact elements Σ and $\widehat{\Sigma}$ at one point are said to *touch* each other if $\underline{\rho} = 0$ and *co-touch* each other if $\underline{\theta} = 0$.

Consider two oriented surfaces immersed in $\mathbb{S}^n$ intersecting at p. We say they *touch (co-touch)* each other if the contact elements given by their tangent spaces at p touch (co-touch).

Example 2.2. For two surfaces tangent to each other at the same point, it is easy to see that they either touch each other at this point when their orientations are compatible, or co-touch when the orientations are opposite.

Example 2.3. Given two complex lines in $\mathbb{C}^n, n \geq 2$, and regard them as real 2-planes with the induced orientation (via the complex structure) in $\mathbb{R}^{2n}$, then they touch each other. In Appendix A we will see that in certain sense the converse is also true.

To clarify the geometric meaning of $\theta = 0$ and $\rho = 0$ for a pair of conformal immersions with lifts $Y, \widehat{Y}$, observe that given coordinate $z = u+\mathrm{i}v$, contact element $\Sigma = \{Y, Y_u, Y_v\}$ at $Y(p)$, and single point $\widehat{Y}(p)$, there is an unique oriented 2-sphere passing through $Y(p), \widehat{Y}(p)$ and tangent to Y with compatible orientation. It is given by the 4-dim subspace of signature $(3,1)$ spanned by $\{Y, Y_u, Y_v, \widehat{Y}\}$, with the orientation fixed by the oriented contact element $\Sigma = \{Y, Y_u, Y_v\}$ or the complexification $Y \wedge Y_z$. Denote it as $S(p)$. Now we may state

Proposition 2.4. *Given two conformal immersions $f, \hat{f}$, the invariant $\rho(p) = 0$ iff the 2-sphere $S(p)$ touches $\hat{f}$ at $\widehat{Y}(p)$, and $\theta(p) = 0$ iff $S(p)$ co-touches $\widehat{Y}$ at $\hat{f}(p)$.*

Proof. Since we have the freedom of choice of lifts, we may take the normalized lifts $Y, \widehat{Y}$ as before. As given by (2.2),

$$\widehat{Y} = \frac{1}{2}\Big(|\mu|^2 + \langle \xi, \xi \rangle\Big) Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N + \xi$$

is orthogonal to $Y_z + \frac{\mu}{2} Y$. Note that under the reflection with respect to $Y - \widehat{Y}$, $S(p)$ is invariant with reversed orientation, and the complex contact element $\Sigma = Y \wedge (Y_z + \frac{\mu}{2} Y)$ is mapped to $\widehat{Y} \wedge (Y_z + \frac{\mu}{2} Y)$. Thus the complex contact element given by $S(p) = \mathrm{Span}\{Y, Y_u, Y_v, \widehat{Y}\}$ at $\widehat{Y}(p)$ should be $\Sigma' = \widehat{Y} \wedge (Y_{\bar{z}} + \frac{\bar{\mu}}{2} Y)$. On the other hand, the complex contact element given by immersion $\widehat{Y}$ at $\widehat{Y}(p)$ is $\widehat{\Sigma} = \widehat{Y} \wedge \widehat{Y}_z$. Thus at $\widehat{Y}(p)$ the invariants associated with Σ' and $\widehat{\Sigma}$ are computed by the fundamental equation (2.3):

$$\underline{\theta} = 2\langle Y_z + \frac{\mu}{2} Y, \widehat{Y}_z \rangle = \theta, \quad \underline{\rho} = 2\langle Y_{\bar{z}} + \frac{\bar{\mu}}{2} Y, \widehat{Y}_z \rangle = \rho.$$

The conclusion now follows from the definition of touch and co-touch. □

2.2 Blaschke's Problem and its solutions

We have introduced Willmore surfaces and isothermic surfaces in last chapter. Although so distinct to each other, they may be characterized as the non-trivial solutions to the following problem.

Blaschke's Problem. *Let S be a sphere congruence with two envelopes $f, \hat{f} : M^2 \to \mathbb{S}^3$, such that these envelopes induce the same conformal structure. Characterize such sphere congruences and enveloping surfaces.*

Blaschke asked this question and solved it in [4]. His conclusion is

Theorem 2.5. *The non-trivial solution to Blaschke's Problem is either a pair of isothermic surfaces forming Darboux transform to each other together with the Ribaucour sphere congruence in between, or a pair of dual Willmore surfaces with their common mean curvature spheres. (Here* non-trivial *means the two envelopes are not congruent under Möbius transformations.)*

Since Blaschke and his school concerned themselves only with surfaces in 3-space, the same problem in higher co-dimension case was left untouched. It is known that the construction of Darboux pair of isothermic surfaces as well as dual Willmore surfaces can be generalized to $\mathbb{S}^n$ (see [9, 26] and [20]), and they still constitute solutions to the generalized Blaschke's Problem. Here we will show that they are exactly the only nontrivial solutions as before. In 3-space every Willmore surface is S-Willmore, so our theorem reduces to Blaschke's result. The main theorem in this section reads as follows:

Theorem 2.6. *Suppose two distinct immersed surfaces in $\mathbb{S}^n$ envelop a 2-sphere congruence and their induced metrics are conformal at corresponding points. Then besides trivial cases, locally it must be among either of the following two classes:*

1. *A Darboux pair of isothermic surfaces;*
2. *An S-Willmore surface with its dual surface.*

In the first case, the correspondence between the two surfaces is orientation preserving. In the second case, the correspondence is anti-conformal.

2.2.1 Reformulation of the problem

The theory established in previous section is an ideal framework for studying Blaschke's Problem. Assume that f and $\hat{f}$ form a pair of solution surfaces, Y and $\widehat{Y}$ are the normalized lifts according to (2.1), and $z = u + \mathrm{i}v$ is a local complex coordinate. As derived in Section 2.1,

$$\widehat{Y} = \frac{1}{2}\Big(|\mu|^2 + \langle \xi, \xi \rangle\Big) Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N + \xi$$

for some complex function μ and normal section ξ of $V^{\perp}$. Its derivative is given by (2.3):

$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \theta\left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + \rho\left(Y_z + \frac{\mu}{2}Y\right) + \langle \xi, \zeta \rangle Y + \zeta,$$

where θ, ρ, ζ are associated invariants defined in (2.4).

The 2-sphere congruence tangent to Y and passing through $\widehat{Y}$ is given by $\mathrm{Span}\{Y, \mathrm{d}Y, \widehat{Y}\}$. According to the assumptions, such 2-spheres are also tangent to $\widehat{Y}$, thus $\widehat{Y}_z$ is contained in $\mathrm{Span}\{Y, \mathrm{d}Y, \widehat{Y}\}$. This implies $\zeta \in \mathrm{Span}_{\mathbb{C}}\{Y, \mathrm{d}Y, \widehat{Y}\} \cap V_{\mathbb{C}}^{\perp}$. Since $\mathrm{Span}\{Y, \mathrm{d}Y, \widehat{Y}\} \oplus V^{\perp}$ is also a decomposition of $M \times \mathbb{R}^{n+1,1}$, we assert

$$\zeta = D_z\xi - \frac{\mu}{2}\xi + 2\left(D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa\right) = 0. \tag{2.8}$$

Under this condition (2.3) might be simplified:

$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \theta\left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + \rho\left(Y_z + \frac{\mu}{2}Y\right). \tag{2.9}$$

We calculate further that

$$\langle \widehat{Y}_z, \widehat{Y}_z \rangle = \langle \widehat{Y}_z - \frac{\mu}{2}\widehat{Y}, \widehat{Y}_z - \frac{\mu}{2}\widehat{Y} \rangle = \theta \cdot \rho.$$

By assumption, $\widehat{Y}$ is conformal to Y, so $\theta \cdot \rho = 0$, i.e. $\theta = 0$ or $\rho = 0$. Thus Blaschke's Problem is equivalent to finding all pairs of conformal immersions for which $\zeta = 0$ and $\theta \cdot \rho = 0$.

Observe that follow from (2.8)(2.4a),

$$\begin{aligned}
&D_{\bar z} D_{\bar z}\kappa + \frac{1}{2}\bar s\kappa \\
=& D_{\bar z}\left(-\frac{1}{2}D_z\xi + \frac{\mu}{4}\xi - \frac{\bar\mu}{2}\kappa\right) + \frac{\bar s}{2}\kappa \\
=& -\frac{1}{2}D_{\bar z}D_z\xi + \frac{\mu}{4}D_{\bar z}\xi + \frac{\mu_{\bar z}}{4}\xi - \frac{\bar\mu}{2}D_{\bar z}\kappa - \frac{\bar\mu_{\bar z}}{2}\kappa + \frac{\bar s}{2}\kappa \\
=& -\frac{1}{2}D_{\bar z}D_z\xi + \frac{\mu}{4}D_{\bar z}\xi + \frac{\mu_{\bar z}}{4}\xi - \frac{\bar\mu}{2}\left(-\frac{1}{2}D_z\xi + \frac{\mu}{4}\xi - \frac{\bar\mu}{2}\kappa\right) - \frac{\bar\mu_{\bar z}}{2}\kappa + \frac{\bar s}{2}\kappa \\
=& \left(-\frac{1}{2}D_{\bar z}D_z\xi - \langle \xi, \bar\kappa\rangle\kappa\right) + \left(\frac{\mu}{4}D_{\bar z}\xi + \frac{\bar\mu}{4}D_z\xi\right) - \frac{1}{8}|\mu|^2\xi + \frac{\mu_{\bar z}}{4}\xi - \frac{\bar\theta}{2}\kappa.
\end{aligned} \tag{2.10}$$

Codazzi and Ricci equations (1.3b)(1.3c) now imply

$$\mathrm{Im}(\mu_{\bar z}\xi - 2\bar\theta\kappa) = 0 \tag{2.11}$$

On the other hand, $\zeta = 0$ together with Gauss equation (1.3a) and (2.4a)(2.4b) yields

$$\begin{aligned}
\theta_{\bar z} &= (\mu_{\bar z})_z - \mu\mu_{\bar z} - s_{\bar z} - 2\langle \xi, \kappa\rangle_{\bar z} \\
&= \Big(\bar\rho + 2\langle \kappa, \bar\kappa\rangle - \frac{1}{2}\langle \xi, \xi\rangle\Big)_z - \mu\Big(\bar\rho + 2\langle \kappa, \bar\kappa\rangle - \frac{1}{2}\langle \xi, \xi\rangle\Big) \\
&\quad - 6\langle D_z\bar\kappa, \kappa\rangle - 2\langle \bar\kappa, D_z\kappa\rangle - 2\langle \xi, \kappa\rangle_{\bar z} \\
&= \bar\rho_z - \mu\bar\rho - 2\langle D_{\bar z}\xi + 2D_z\bar\kappa + \mu\bar\kappa, \kappa\rangle + \langle \xi, -D_z\xi + \frac{\mu}{2}\xi - 2D_{\bar z}\kappa\rangle \\
&= \bar\rho_z - \mu\bar\rho - \langle \bar\mu\xi, \kappa\rangle + \langle \xi, \bar\mu\kappa\rangle \\
&= \bar\rho_z - \mu\bar\rho.
\end{aligned} \tag{2.12}$$

These formulas will be very useful in later discussion.

Remark 2.7. Also note that when $\theta = \rho = 0$ on an open subset of M, there will be $\widehat{Y}_z = \frac{\mu}{2}\widehat{Y}$. This time $\widehat{Y}$ corresponds to a fixed point in $\mathbb{S}^n$. After a stereographic projection from this point, f turns out to be an immersion into $\mathbb{R}^n$ with its tangent planes passing through $\hat f \equiv \infty$. Such a trivial and degenerate case is excluded from our consideration.

2.2.2 Three cases of the solutions

1. Isothermic case: Let's consider the first case, $\rho = 0 \neq \theta$. By (2.4b) this implies $\bar{\mu}_z$ is real-valued, and (2.11) ensures $\bar{\theta}\kappa$ to be real, too. Now (2.12) implies θ is holomorphic. So $\kappa = \theta \cdot \frac{\bar{\theta}\kappa}{|\theta|^2}$, where the first term is holomorphic, and the next term is real. We may define a new holomorphic coordinate w by $\mathrm{d}w = \sqrt{\theta}\,\mathrm{d}z$. Due to the transformation rule of κ given by (1.4a), we get $\kappa' = \frac{\bar{\theta}\kappa}{|\theta|^{3/2}}$, a real vector-valued form. So our analysis shows f must be isothermic in this case.

Furthermore, if we consider $\hat{f}$ as the first (original) one in such a pair, on account of (2.5a) and (2.5b), we have the same invariant ρ up to complex conjugation, and the same θ. Similarly $\bar{\theta}\hat{\kappa}$ is real, and when the coordinate z is chosen such that κ is real, $\hat{\kappa}$ will be real at the same time. That means both surfaces are isothermic and their curvature lines correspond. By the characterization given in [9, 26], such two surfaces are said to envelop a conformal Ribaucour sphere congruence, and they form Darboux transform to each other. Especially, given such an coordinate z and real κ, the holomorphic θ must be also real-valued, thus constant. This constant may be identified with the real parameter appearing in the construction of Darboux transforms up to a choice of certain holomorphic 2-form over the underlying M (compared with [9, 29]).

Remark 2.8. After close examination, the definition of Darboux transform of an isothermic surface given in [9, Section 2.2.2] may be regarded as a purely geometric characterization that *such a pair envelops a 2-sphere congruence with conformal metric and the same orientation induced by those 2-spheres.* (As to the orientation issue, we have clarified by Proposition 2.4 and Example 2.2.) Regretfully, both Burstall and Hertrich-Jeromin only mentioned the characterization in terms of *Ribaucour* sphere congruence, yet overlooked our simpler version.

2. S-Willmore case: The second possibility is $\theta = 0 \neq \rho$ on an open subset. In this part, we first consider the seemingly *trivial* case, $\xi = 0$. Put this into (2.10), we see immediately that

$$D_{\bar{z}}D_{\bar{z}}\kappa + \frac{1}{2}\bar{s}\kappa = 0.$$

So f is a Willmore surface. Moreover, $\xi = 0$ and $\zeta = 0$ implies

$$D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa = 0. \tag{2.13}$$

In other words, κ and its differentiation $D_{\bar{z}}\kappa$ are linearly dependent. Such Willmore surfaces are called *S-Willmore* surfaces.

Note there is some kind of duality between the pair Y and $\widehat{Y}$. If we exchange between them (or equivalently speaking, if we use the frame of $\widehat{Y}$), similar result holds for $\hat{\kappa}$. So both surfaces are S-Willmore.

How about the sphere congruence in between? Substitute $\theta = 0$ into the expression of $\widehat{Y}$ and $\widehat{Y}_z$, there will be

$$\widehat{Y} = \frac{|\mu|^2}{2}Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N,$$
$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \rho\left(Y_z + \frac{\mu}{2}Y\right).$$

This implies $\mathrm{Span}\{Y, \mathrm{d}Y, Y_{z\bar{z}}\} = \mathrm{Span}\{\widehat{Y}, \mathrm{d}\widehat{Y}, \widehat{Y}_{z\bar{z}}\}$. So Y and $\widehat{Y}$ envelop a common mean curvature sphere congruence. But this sphere congruence induce opposite orientations on either surface, because $\theta = 0$ implies the 2-sphere touch one surface and co-touch the other. See Proposition 2.4 and Example 2.2.

Conversely, there holds the following duality theorem.

Theorem 2.9. *Let f be a given S-Willmore surface, then its mean curvature spheres has a second enveloping surface $\hat{f}$, which is also a S-Willmore surface. When $\hat{f}$ does not degenerate to a single point, it share the same mean curvature sphere congruence with f yet with opposite orientation.*

Proof. Suppose Y is the canonical lift of a S-Willmore surface. Locally we may assume it has no umbilic points, i.e. $\kappa \neq 0$. Thus (2.13) holds for some μ. We construct a second surface via this μ and the frames of f by

$$\widehat{Y} = \frac{|\mu|^2}{2}Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N.$$

At this moment we see the associated invariant $\theta = 0$ directly due to the Willmore condition and (2.13). So (2.3) is reduced to

$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \rho\left(Y_z + \frac{\mu}{2}Y\right).$$

If $\rho = 0$ in a small open subset, the underlying map $\hat{f}$ degenerate to a point, and the conclusion is trivially true. Otherwise, suppose locally $\rho \neq 0$. By the formula above, $\widehat{Y}$ is conformal, too. Differentiation yields

$$\mathrm{Span}\{Y, \mathrm{d}Y, Y_{z\bar{z}}\} = \mathrm{Span}\{\widehat{Y}, \mathrm{d}\widehat{Y}, \widehat{Y}_{z\bar{z}}\}.$$

So Y and $\widehat{Y}$ share the same mean curvature sphere (yet with opposite orientation by the same reason). Now they form a solution to Blaschke's Problem in the case discussed above, and the previous argument applies at here as well. So we conclude that $\widehat{Y}$ is also S-Willmore. □

Remark 2.10. The concept of a *S-Willmore* surface has appeared in [43] as *strong Willmore* surface. More earlier, it was given by Ejiri ([20]) with some slight modification. Namely, since (2.13) holds automatically for surfaces in 3-space, any Willmore surface in $\mathbb{S}^3$ is S-Willmore. This case is excluded from Ejiri's definition. In our opinion, including the codim-1 case into the definition is more natural and favorable, because for them holds the same *duality theorem* stated above.

General Willmore surfaces may not be S-Willmore, and we can hardly expect any duality results for them. For such examples see the remark in the final section of [20] and references therein.

3. Trivial case and conclusion: Finally we come to the discussion of the *non-trivial* part of case $\theta = 0 \neq \rho$, the occasion that $\xi \neq 0$. This time (2.11) implies that $\bar{\mu}_z$ as well as ρ must be real. Together with (2.12) we obtain $\rho_z = \mu\rho$. Since $\rho \neq 0$, without loss of generality we may assume it to be positive and use it to scale $\widehat{Y}$. Let $\widetilde{Y} = (1/\rho)\widehat{Y}$ (this is indeed the canonical lift of $\hat{f}$). Define $X := \widetilde{Y} - Y$ which is real-valued. Then $\rho_z = \mu\rho, \theta = 0$ and (2.9) yields

$$X_z = \left(\frac{1}{\rho}\widehat{Y} - Y\right)_z = -\frac{\mu}{\rho}\widehat{Y} + \frac{1}{\rho}\left(\frac{\mu}{2}\widehat{Y} + \rho\left(Y_z + \frac{\mu}{2}Y\right)\right) - Y_z = -\frac{\mu}{2}X.$$

Thus the real line spanned by X is constant. Normalize X by defining $\widetilde{X} = \sqrt{\rho}\cdot X$. Then we will have $\widetilde{X}_z = 0$. Thus $\widetilde{X}$ is a fixed vector in $R^{n+1,1}$. Observe that

$$\langle X, X\rangle = \langle \frac{1}{\rho}\widehat{Y} - Y, \frac{1}{\rho}\widehat{Y} - Y\rangle = \frac{2}{\rho},$$
$$\langle Y, X\rangle = \langle Y, \frac{1}{\rho}\widehat{Y} - Y\rangle = -\frac{1}{\rho},$$

we find

$$\widetilde{Y} = Y + X = Y - \frac{2\langle Y, X\rangle}{\langle X, X\rangle}X = Y - \frac{2\langle Y, \widetilde{X}\rangle}{\langle \widetilde{X}, \widetilde{X}\rangle}\widetilde{X}.$$

So $\widetilde{Y}$ is a reflection of Y with respect to $\widetilde{X}$, and the underlying map $\hat{f}$ must be congruent to f.

Sum together, we have proved Theorem 2.6. □

Remark 2.11. We are interested in such a characterization not only because this problem is natural and interesting in itself, but also due to our concern about transforms of surfaces. Note there are already many beautiful results about transformations of isothermic surfaces, e.g. dual isothermic surfaces, Bianchi and Darboux transforms together with the permutability theorems ([9, 26]). As to Willmore surfaces, there are only some partial results ([10]), which seemed similar to isothermic case yet more subtle. We think the relationship between these two surface classes is worthy of more exploration.

2.2.3 Further remarks

In the discussion of S-Willmore case above, we have shown that the mean curvature spheres of a S-Willmore surface admit a second enveloping surface. This is the direct consequence of (2.13). Indeed we can say more:

Proposition 2.12 ([20, Lemma 1.3, 3.1]). *For a conformal immersion $f : M \to \mathbb{S}^n$ with Hopf differential κ, the following three conditions are equivalent:*

(i) $D_{\bar{z}}\kappa$ is parallel to κ; *(*Ejiri's condition*)*

(ii) The mean curvature spheres of f admit another enveloping surface;

(iii) f is a S-Willmore surface, or contained in some 3-space.

Remark 2.13. Note Ejiri's condition is well-defined, i.e. independent to the choice of coordinates, because κ is a vector-valued differential form. It first appeared in [20] as $\Psi = 0$, where Ψ was an exterior form defined by the wedge product of the Hopf differential and its derivative. This condition gives a good characterization of surfaces with property (ii). Yet the part (iii) of our proposition shows that there are few of such surfaces.

Proof to Proposition 2.12.

(i)⇔(ii) Assume that the mean curvatrue spheres of f admit $\hat{f}$ as the second envelope. As before, let Y be the canonical lift of f, $\widehat{Y}$ be the lift of $\hat{f}$ satisfying $\langle Y, \widehat{Y}\rangle = -1$. Our assumption implies that

$$\widehat{Y}, \widehat{Y}_z \ \in \ \mathrm{Span}_{\mathbb{C}}\{Y, Y_z, Y_{\bar{z}}, Y_{z\bar{z}}\}.$$

Write $\widehat{Y} = \left(|\mu|^2/2\right) Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N$. By (2.3)(2.4c) there must be

$$\zeta = 2D_{\bar{z}}\kappa + \bar{\mu}\kappa = 0.$$

So Ejiri's condition holds true. Conversely, suppose locally there is μ so that the equation above is satisfied. One can verify directly that the map $\hat{f}$ with lift $\widehat{Y} = \left(|\mu|^2/2\right) Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N$ satisfies condition (ii).

(i)⇔(iii) For a surface in $\mathbb{S}^3$, codim-1 implies Ejiri's condition, because the (Möbius) normal bundle is a holomorphic bundle with respect to $D_{\bar{z}}$. S-Willmore surfaces have been discussed before. So "(iii)⇒(i)" is obvious. Conversely, suppose Ejiri's condition holds for f which is not S-Willmore. We want to show it is contained in a 3-sphere. For some μ holds $2D_{\bar{z}}\kappa + \bar{\mu}\kappa = 0$. We substitute this into the conformal Codazzi equation, which yields

$$\mathrm{Im}(\bar{\theta}\kappa) = 0, \quad \text{where } \ \theta := \mu_z - \frac{1}{2}\mu^2 - s, s \text{ is the Schwarzian.}$$

f is not Willmore, so $\theta \neq 0$. This implies $\mathrm{Re}(\kappa)$ and $\mathrm{Im}(\kappa)$ are linearly dependent over $\mathbb{R}$. Without loss of generality, suppose locally $\kappa \neq 0$; moreover, $\mathrm{Re}(\kappa) \neq 0$. Then the real derivatives of $\mathrm{Re}(\kappa)$ is parallel to itself. If we set $z = u + \mathrm{i}v$ for real coordinates (u, v), one will find the real frames

$$\{Y, Y_u, Y_v, N, \mathrm{Re}(\kappa)\}$$

satisfy a PDE system given by (1.2) and Ejiri's condition. Integration shows that Y is contained in a 5-dim Minkowski subspace. So f is an immersion into $\mathbb{S}^3$. □

Although Ejiri's condition guarantees the mean curvature spheres of f has a second surface envelope $\hat{f}$, in general the correspondence between f and $\hat{f}$ is not conformal, and the sphere congruence might not be the mean curvature sphere of $\hat{f}$. So it is also interesting to characterize the special case when f and $\hat{f}$ are conformal to each other. The theorem below contains the answer.

Theorem 2.14. *Let f be an surface immersed in $\mathbb{S}^n$, whose mean curvature sphere congruence has $\hat{f}$ as the second envelope with the same conformal structure. Then either f is S-Willmore, or Möbius equivalent to a CMC-1 (constant mean curvature 1) surface in hyperbolic 3-space.*

Proof. f and $\hat{f}$ together with the mean curvature spheres of f form a solution to Blaschke's Problem. Here it can not be the trivial case in the discussion above, because $\xi \neq 0$ implies the mean curvature spheres of f never pass through $\hat{f}$. If f is S-Willmore, we are done. Otherwise there must be $\rho = 0 \neq \theta$ on an open subset, i.e. we need only to consider the isothermic case. By Proposition 2.12 we can restrict to the codim-1 case.

Suppose f is an isothermic surface in $\mathbb{S}^3$ with Darboux transform $\hat{f}$ satisfying our assumptions. Their associated invariants θ and ρ are defined as before. As the second enveloping surface of the mean curvature spheres of f, $\hat{f}$ must also be contained in the same 3-space. The canonical lift of $\hat{f}$ is given via $\widetilde{Y} = \frac{1}{\theta}\widehat{Y}$ due to (2.9) and $\rho = 0$. Keeping in mind that θ is real and holomorphic, hence constant, and that $\bar{\mu}_z - 2\langle\kappa, \bar{\kappa}\rangle = \rho - \frac{1}{2}\langle\xi, \xi\rangle = 0$, we differentiate $\widetilde{Y}$:

$$\begin{aligned}
\widetilde{Y}_z &= \frac{1}{\theta}\widehat{Y}_z = \frac{\mu}{2}\widetilde{Y} + \left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right), \\
\widetilde{Y}_{zz} &= \frac{\mu}{2}\widetilde{Y}_z + \frac{\mu_z}{2}\widetilde{Y} + \left(Y_{\bar{z}z} + \frac{\bar{\mu}}{2}Y_z + \frac{\bar{\mu}_z}{2}Y\right) \\
&= \frac{\mu}{2}\left[\frac{\mu}{2}\widetilde{Y} + \left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right)\right] + \frac{\mu_z}{2}\widetilde{Y} + \left(\frac{1}{2}\bar{\mu}_z - \langle\kappa, \bar{\kappa}\rangle\right)Y + \frac{\bar{\mu}}{2}Y_z + \frac{1}{2}N \\
&= \left(\frac{\mu_z}{2} + \frac{\mu^2}{4}\right)\widetilde{Y} + \frac{1}{2}\left(\frac{1}{2}|\mu|^2 Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N\right) \\
&= (\cdots)\widetilde{Y}.
\end{aligned}$$

According to Remark 1.2, $\hat{f}$ has vanishing Hopf differential, thus a round 2-sphere. $\hat{f}$ encloses a 3-ball in this $\mathbb{S}^3$. Endow this 3-ball with the hyperbolic metric of constant curvature -1, then $\hat{f}$ stands as the boundary at infinity of this hyperbolic space. Each mean curvature sphere of f is tangent to this boundary 2-sphere, thus a horo-sphere with curvature 1. As pointed out in Remark 1.1, that means f is of constant mean curvature 1 with respect to the ambient hyperbolic metric, whereby $\hat{f}$ is exactly its hyperbolic Gauss map. □

Remark 2.15. In fact we have obtained a characterization of CMC-1 surfaces in hyperbolic 3-space as the only surfaces in $\mathbb{S}^n$ whose mean curvature spheres has a second enveloping surface which is conformal with compatible orientation. This is inspired by the main result in [29]. In that paper the authors characterized these surfaces as isothermic surfaces in 3-space whose mean curvature spheres give rise to a second enveloping surface and they form a Darboux pair.

We observe that general CMC surfaces in hyperbolic n-space do not share the same property, mainly due to the fact that in higher codimension case they fail to be isothermic.

Chapter 3

Adjoint transforms of Willmore surfaces

3.1 The notion of adjoint transform

3.1.1 Definition

As explained in the introduction, we are interested in the transforms of Willmore surfaces. Especially, we would like to generalize the construction of dual Willmore surfaces. The discussion of Blaschke's Problem has shown that such a duality only exists for S-Willmore surfaces. As a special case, this holds for all Willmore surfaces in $\mathbb{S}^3$, which is known to Blaschke [4] and Bryant [8].

When the codimension is 2, although the duality theorem 2.9 is false in general, there is a construction of *forward/backward two-step Bäcklund transforms* for Willmore surfaces [10, Section 9.2]. They can be viewed as left and right dual Willmore surfaces. To generalize them, we notice that either of them falls on the mean curvature sphere of the original surface and co-touch it from left or right. (See Subsection 2.1.3 and Appendix A, then compare with the geometrical characterization of Darboux transforms for general surfaces in [7, Lemma 34].) This inspires the following

Definition 3.1. A map $\hat{f} : M \to \mathbb{S}^n$ is called the adjoint transform of Willmore surface $f : M \to \mathbb{S}^n$ if it is conformal and co-touches the mean curvature sphere of f at corresponding point. Especially, $\hat{f}$ must locate on the corresponding mean curvature sphere of f. Note that $\hat{f}$ is allowed to be a degenerate point.

This definition gives the conditions characterizing an adjoint transform. Yet we prefer to a more explicit description. As in Subsection 2.1.1, consider surface pair $f, \hat{f}$ with adapted lifts $Y, \widehat{Y}$, satisfying $\langle Y, \widehat{Y}\rangle = -1$. Furthermore suppose $\hat{f}$ is on the mean curvature sphere of f. Then equations (2.2)(2.3) take the form

$$\widehat{Y} = \frac{1}{2}|\mu|^2 Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N, \tag{3.1}$$

$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \theta\left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + \rho\left(Y_z + \frac{\mu}{2}Y\right) + 2\eta. \tag{3.2}$$

Here $\mu\mathrm{d}z$ is a complex connection 1-form determined by $\mu = 2\langle \widehat{Y}, Y_z\rangle$. It further defines those invariants associated with the pair $f, \hat{f}$ as in Subsection 2.1.1:

$$\theta := \mu_z - \frac{1}{2}\mu^2 - s, \tag{3.3a}$$

$$\rho := \bar{\mu}_z - 2\langle \kappa, \bar{\kappa}\rangle, \tag{3.3b}$$

$$\eta := D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa. \tag{3.3c}$$

There follows

$$\langle \widehat{Y}_z, \widehat{Y}_z\rangle = \langle \widehat{Y}_z - \frac{\mu}{2}\widehat{Y}, \widehat{Y}_z - \frac{\mu}{2}\widehat{Y}\rangle = 4\langle \eta, \eta\rangle + \theta\cdot\rho\ , \tag{3.4}$$

$$\langle \widehat{Y}_z, \widehat{Y}_{\bar{z}}\rangle = \langle \widehat{Y}_z - \frac{\mu}{2}\widehat{Y}, \widehat{Y}_{\bar{z}} - \frac{\bar{\mu}}{2}\widehat{Y}\rangle = 4\langle \eta, \bar{\eta}\rangle + \frac{1}{2}|\theta|^2 + \frac{1}{2}|\rho|^2\ . \tag{3.5}$$

That f is Willmore implies

$$0 = D_{\bar{z}}D_{\bar{z}}\kappa + \frac{1}{2}\bar{s}\kappa = D_{\bar{z}}(\eta - \frac{\bar{\mu}}{2}\kappa) + \frac{\bar{s}}{2}\kappa = D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta - \frac{\bar{\theta}}{2}\kappa \tag{3.6}$$

Definition 3.2. The map into $\mathbb{S}^n$ represented by (3.1) is an *adjoint transform* of Y iff μ satisfies the following conditions:

$$\textbf{Co-touching:}\quad 0 = \theta = \mu_z - \frac{1}{2}\mu^2 - s, \tag{3.7a}$$

$$\textbf{Conformality:}\quad 0 = \langle \eta, \eta\rangle = \langle D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa, D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa\rangle. \tag{3.7b}$$

Example 3.3. When f is a S-Willmore surface, its dual surface $\hat{f}$ is obviously an adjoint transform of f. The corresponding μ is given by the Ejiri's condition: $2D_{\bar{z}}\kappa + \bar{\mu}\kappa = 0$; the associated η vanishes. If $\langle \kappa, \kappa\rangle \neq 0$, then this is the unique adjoint transform.

Later we will show that any adjoint transform of a Willmore surface is also Willmore, and dual to the original one in certain sense. But before that we should treat the problem of existence.

3.1.2 Existence

Our definition of adjoint transforms leads to the natural problem of existence and uniqueness of solutions to system (3.7a)(3.7b). Note that when $\langle \kappa, \kappa\rangle \neq$

0, (3.7b) is a quadratic equation about μ and much easier to solve. In such a situation, at every point we have two roots for

$$0 = \langle \eta, \eta \rangle = \langle D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa, D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa \rangle.$$

Fix either of such a root μ and differentiate this equation. By (3.6),

$$0 = \langle \eta, \eta \rangle_{\bar{z}} = 2\langle D_{\bar{z}}\eta, \eta \rangle = 2\langle \frac{\bar{\mu}}{2}\eta + \frac{\bar{\theta}}{2}\kappa, \eta \rangle = \bar{\theta}\langle \kappa, \eta \rangle.$$

If $\langle \kappa, \eta \rangle \neq 0$, we have $\theta = 0$ as desired. Otherwise, suppose $\langle \kappa, \eta \rangle = 0$ on an open subset and take derivative, one obtains

$$\begin{aligned} 0 = \langle \kappa, \eta \rangle_{\bar{z}} &= \langle D_{\bar{z}}\kappa, \eta \rangle + \langle \kappa, D_{\bar{z}}\eta \rangle \\ &= \langle \eta - \frac{\bar{\mu}}{2}\kappa, \eta \rangle + \langle \kappa, \frac{\bar{\mu}}{2}\eta + \frac{\bar{\theta}}{2}\kappa \rangle = \frac{\bar{\theta}}{2}\langle \kappa, \kappa \rangle. \end{aligned}$$

By assumption, $\langle \kappa, \kappa \rangle \neq 0$, so $\theta = 0$. Hence we see that the Willmore condition (3.6) guarantees a solution μ of (3.7) and the existence of adjoint transforms.

How about the case when $\langle \kappa, \kappa \rangle = 0$ on an open subset? By Willmore condition (1.7) it follows $0 = \langle D_{\bar{z}}\kappa, \kappa \rangle = \langle D_{\bar{z}}\kappa, D_{\bar{z}}\kappa \rangle$. That means (3.7b) holds automatically for any μ. So we need only to solve the PDE (3.7a)

$$\mu_z - \frac{1}{2}\mu^2 - s = 0$$

independently. This is a Riccati equation about μ with respect to the given Schwarzian s. In S-Willmore case this is solved in Example 3.3. When immersion $f : M \to \mathbb{S}^n$ is Willmore but not S-Willmore, the Willmore condition (1.7) implies that κ and $D_{\bar{z}}\kappa$ span a holomorphic subbundle of $V_{\mathbb{C}}^{\perp}$ [1]. Let ψ be any non-trivial holomorphic section of this subbundle. It may be written explicitly as

$$\psi = t \cdot \left(D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa \right) = t\eta$$

for some function t and μ. We assert that this μ is a solution of (3.7a). As pointed out in (3.6), Willmore condition implies

$$D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta - \frac{\bar{\theta}}{2}\kappa = 0.$$

[1] This is true at least on the open subset where $\kappa \wedge D_{\bar{z}}\kappa \neq 0$. Later we will show that this is also true around zeros of $\kappa \wedge D_{\bar{z}}\kappa$. Note that $V_{\mathbb{C}}^{\perp}$ is a holomorphic bundle with respect to the operator $D_{\bar{z}}$.

On the other hand, η span the same holomorphic line subbundle as ψ, hence $D_{\bar{z}}\eta$ linearly depends on η. The assumption that f is not S-Willmore ensures that η is not parallel to κ. So we conclude that the $\theta = \mu_z - \frac{1}{2}\mu^2 - s$ above has to be zero as desired. Conversely, any solution μ can be obtained via such a holomorphic section ψ. So there are infinitely many solutions μ. Note that for fixed initial value problem $\mu(z_0) = \mu_0$ the uniqueness is still false, because there are many holomorphic subbundles containing $D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa$.

Remark 3.4. Riccati equation is discussed in ODE text books either for real variable, or in the holomorphic category. The latter means they treat only equations with holomorphic functions as the coefficients and concern with holomorphic solutions. According to the author's knowledge there are seldom discussions about the existence and uniqueness of solutions to Riccati equations on complex domain. Generally speaking, in our case (3.7a) is a underdetermined equation, thus should admit (infinitely) many solutions. Here we would like to present a more concrete discussion.

As is well-known there is a correspondence between Riccati equations and linear systems. In our case, the Riccati equation (3.7a)

$$\mu_z - \frac{1}{2}\mu^2 - s = 0$$

corresponds to linear system

$$\frac{\partial}{\partial z}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & s \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.8}$$

One easily checks that for a solution $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ of (3.8), $\mu := \frac{x}{y}$ satisfies (3.7a). Conversely, for a given solution μ, we first solve the elliptic equation about y

$$y_z = -\frac{\mu}{2}y.$$

Then $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mu y \\ y \end{smallmatrix}\right)$ solves (3.8). Moreover, any solution of (3.8) corresponds to a solution of

$$y_{zz} + \frac{s}{2}y = 0 \tag{3.9}$$

in the obvious way. In the following we will show that for (3.9) one can construct infinitely many solutions from a given one.

Suppose y and $\tilde{y}$ are non-trivial solutions of (3.9). Let $\tilde{y} = \lambda y$, we want to see which condition λ must satisfy. From $y_{zz} + \frac{s}{2}y = 0 = (\lambda y)_{zz} + \frac{s}{2}\lambda y$ we deduce

$$0 = \lambda_{zz}y + 2\lambda_z y_z = \frac{(\lambda_z y^2)_z}{y}.$$

Therefore, if start with a given solution $y \neq 0$, one need only to solve a ∂-problem for λ: $\lambda_z = h/y^2$, where h is an anti-holomorphic function locally. Such λ's give rise to general solutions $\tilde{y} = \lambda y$.

This conclusion implies that even for a S-Willmore surface with $\langle\kappa,\kappa\rangle \equiv 0$, there are still infinitely many adjoint transforms locally.

3.1.3 Global aspects

To consider a globally defined adjoint transform of a given surface, we have to know behavior of its invariants under the change of coordinates. Let w be another complex coordinate and denote the corresponding quantities as Y', N', κ', s' etc. By the normalization condition that $\langle Y_z, Y_{\bar{z}}\rangle = \frac{1}{2} = \langle Y'_w, Y'_{\bar{w}}\rangle$, we see the new lifts of f and $\hat{f}$ are given by

$$\begin{aligned} Y' &= Y\left(\frac{\partial z}{\partial w}\right)^{-\frac{1}{2}}\left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{-\frac{1}{2}}, \\ \widehat{Y}' &= \widehat{Y}\left(\frac{\partial z}{\partial w}\right)^{\frac{1}{2}}\left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\frac{1}{2}}. \end{aligned}$$

Note $\mu = 2\langle Y_z, \widehat{Y}\rangle$, so the new data μ' with respect to coordinate w is

$$\mu' = 2\langle Y'_w, \widehat{Y}'\rangle = \mu\left(\frac{\partial z}{\partial w}\right) + \left(\frac{\partial^2 z}{\partial w^2}\right) \Big/ \left(\frac{\partial z}{\partial w}\right). \tag{3.10}$$

This means $\mu \mathrm{d}z$ is a connection 1-form of K^{-1}, where K denotes the canonical bundle of Riemann surface M. Conversely, given any connection 1-form of K^{-1}, if it satisfies (3.7) with respect to any local coordinate, then it defines a global adjoint transform. By the nature of μ we conclude that $\eta = D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa$ defines a vector-valued differential form $\eta(\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar{z})^{\frac{1}{2}}$.

In the discussion above, we ignored the problem of singularities. First, the map underlying $\widehat{Y}$ may not be immersion when $\sigma = 0$, hence $\widetilde{Y}$ as well as the underlying map $\tilde{f} : M \to \mathbb{S}^n$ might be singular around such points. Next, the connection 1-form $\mu \mathrm{d}z$ may have *poles*, which corresponds to the coincidence case of f and $\hat{f}$. These difficulties will be addressed later in the study of Willmore 2-spheres.

3.2 The property of adjoint transforms

3.2.1 Duality theorem

Fix the original Willmore surface with lift Y. Assume there is a μ solving (3.7a) and (3.7b), which defines an adjoint transform $\hat{f}$. Therefore, (3.2) is simplified to

$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \rho\left(Y_z + \frac{\mu}{2}Y\right) + 2\eta. \tag{3.11}$$

Note $\theta = 0$ also implies

$$\rho_{\bar{z}} = \bar{\mu}_{\bar{z}z} - 2\langle \kappa, \bar{\kappa}\rangle_{\bar{z}} = \bar{s}_z + \bar{\mu}\bar{\mu}_z - 2\langle \kappa, \bar{\kappa}\rangle_{\bar{z}} = \bar{\mu}\rho + 4\langle \eta, \bar{\kappa}\rangle \tag{3.12}$$

by Gauss equation (1.3a), and

$$D_{\bar{z}}\eta = \frac{\bar{\mu}}{2}\eta \tag{3.13}$$

by (3.6).

Consider the canonical lift of the adjoint transform, denoted as $\widetilde{Y}$. Let $\langle \widetilde{Y}, Y\rangle = -1/\sigma$ be a real function defined on M. Equivalently speaking, $\widetilde{Y}$ is obtained from $\widehat{Y}$ via $\widetilde{Y} = \frac{1}{\sigma}\widehat{Y}$. So

$$\frac{1}{2} = \langle \widetilde{Y}_z, \widetilde{Y}_{\bar{z}}\rangle = \frac{1}{\sigma^2}\langle \widehat{Y}_z, \widehat{Y}_{\bar{z}}\rangle.$$

Combined with (3.5) and $\theta = 0$, we get

$$\sigma^2 = 2\langle \widehat{Y}_z, \widehat{Y}_{\bar{z}}\rangle = 8\langle \eta, \bar{\eta}\rangle + |\rho|^2. \tag{3.14}$$

Thus (3.11) may be written as

$$\widetilde{Y}_z = -\frac{\tilde{\mu}}{2}\widetilde{Y} + \frac{\rho}{\sigma}\left(Y_z + \frac{\mu}{2}Y\right) + \frac{2}{\sigma}\eta, \tag{3.15}$$

where

$$\tilde{\mu} := \frac{2\sigma_z}{\sigma} - \mu. \tag{3.16}$$

To find $\widetilde{N}$ we should calculate $\widetilde{Y}_{z\bar{z}}$. Observe

$$\begin{aligned}
\langle \eta_{\bar{z}}, N\rangle &= -\langle \eta, N_{\bar{z}}\rangle = -\langle \eta, 2D_z\bar{\kappa}\rangle = \langle \eta, -2\bar{\eta} + \mu\bar{\kappa}\rangle,\\
\langle \eta_z, N\rangle &= -\langle \eta, N_z\rangle = -\langle \eta, 2D_{\bar{z}}\kappa\rangle = \bar{\mu}\langle \eta, \kappa\rangle,\\
\langle \eta_{\bar{z}}, Y_{\bar{z}}\rangle &= -\langle \eta, Y_{\bar{z}\bar{z}}\rangle = -\langle \eta, \bar{\kappa}\rangle,\\
\langle \eta_z, Y_z\rangle &= -\langle \eta, Y_{zz}\rangle = -\langle \eta, \kappa\rangle,\\
\langle \eta_{\bar{z}}, Y\rangle &= \langle \eta_z, Y\rangle = \langle \eta_{\bar{z}}, Y_z\rangle = \langle \eta_z, Y_{\bar{z}}\rangle = 0
\end{aligned}$$

by (1.2) and (3.7b). So

$$\eta_{\bar{z}} = 2\langle \eta, \bar{\eta}\rangle Y - 2\langle \eta, \bar{\kappa}\rangle \left(Y_z + \frac{\mu}{2}Y\right) + \frac{\bar{\mu}}{2}\eta, \tag{3.17a}$$

$$\eta_z = -2\langle \eta, \kappa\rangle \left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + D_z\eta. \tag{3.17b}$$

After these preparations, now we can differentiate both sides of (3.15) by $\bar{z}$

and obtain

$$\begin{aligned}
\widetilde{Y}_{z\bar{z}} =& -\frac{\tilde{\mu}_{\bar{z}}}{2}\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} + \Big(\frac{\rho}{\sigma}\Big)_{\bar{z}}\Big(Y_z + \frac{\mu}{2}Y\Big) \\
& + \Big(\frac{\rho}{\sigma}\Big)\Big(Y_z + \frac{\mu}{2}Y\Big)_{\bar{z}} + \Big(\frac{2}{\sigma}\Big)_{\bar{z}}\eta + \Big(\frac{2}{\sigma}\Big)\eta_{\bar{z}} \\
=& -\frac{\tilde{\mu}_{\bar{z}}}{2}\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} + \Big(\frac{\rho}{\sigma}\Big)_{\bar{z}}\Big(Y_z + \frac{\mu}{2}Y\Big) \\
& + \Big(\frac{\rho}{\sigma}\Big)\Big[\big(-\langle\kappa,\bar{\kappa}\rangle Y + \frac{1}{2}N\big) + \frac{\mu_{\bar{z}}}{2}Y + \frac{\mu}{2}Y_{\bar{z}}\Big] + \Big(\frac{2}{\sigma}\Big)_{\bar{z}}\eta + \Big(\frac{2}{\sigma}\Big)\eta_{\bar{z}} \\
=& -\frac{\tilde{\mu}_{\bar{z}}}{2}\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} + \Big(\frac{\rho}{\sigma}\Big)_{\bar{z}}\Big(Y_z + \frac{\mu}{2}Y\Big) \\
& + \Big(\frac{\rho}{2\sigma}\Big)\Big[\bar{\rho}Y + \big(\widehat{Y} - \frac{1}{2}|\mu|^2 Y - \bar{\mu}Y_z\big)\Big] \\
& + \Big(\frac{2}{\sigma}\Big)_{\bar{z}}\eta + \Big(\frac{2}{\sigma}\Big)\Big[2\langle\eta,\bar{\eta}\rangle Y - 2\langle\eta,\bar{\kappa}\rangle\big(Y_z + \frac{\mu}{2}Y\big) + \frac{\bar{\mu}}{2}\eta\Big] \\
=& \frac{1}{2}\big(\rho - \tilde{\mu}_{\bar{z}}\big)\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} + \frac{1}{2\sigma}\Big(|\rho|^2 + 8\langle\eta,\bar{\eta}\rangle\Big)Y + \Big[\big(\frac{2}{\sigma}\big)_{\bar{z}} + \frac{\bar{\mu}}{\sigma}\Big]\eta \\
& \Big[\big(\frac{\rho}{\sigma}\big)_{\bar{z}} - \frac{\rho}{2\sigma}\bar{\mu} - \frac{4}{\sigma}\langle\eta,\bar{\kappa}\rangle\Big]\Big(Y_z + \frac{\mu}{2}Y\Big) \\
=& \frac{1}{2}\big(\rho - \tilde{\mu}_{\bar{z}}\big)\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} + \frac{1}{2\sigma}\sigma^2 Y + \frac{1}{\sigma}\Big(\bar{\mu} - 2\frac{\sigma_{\bar{z}}}{\sigma}\Big)\eta \\
& + \Big[\frac{1}{\sigma}\big(\bar{\mu}\rho + 4\langle\eta,\bar{\kappa}\rangle\big) - \rho\frac{\sigma_{\bar{z}}}{\sigma^2} - \frac{\rho}{2\sigma}\bar{\mu} - \frac{4}{\sigma}\langle\eta,\bar{\kappa}\rangle\Big]\Big(Y_z + \frac{\mu}{2}Y\Big) \\
=& \frac{1}{2}\big(\rho - \tilde{\mu}_{\bar{z}}\big)\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} - \frac{\bar{\tilde{\mu}}}{\sigma}\frac{\rho}{2}\Big(Y_z + \frac{\mu}{2}Y\Big) + \frac{\sigma}{2}Y - \frac{\bar{\tilde{\mu}}}{\sigma}\eta \\
=& \frac{1}{2}\big(\rho - \tilde{\mu}_{\bar{z}}\big)\widetilde{Y} - \frac{\tilde{\mu}}{2}\widetilde{Y}_{\bar{z}} - \frac{\bar{\tilde{\mu}}}{2}\Big(\widetilde{Y}_z + \frac{\tilde{\mu}}{2}\widetilde{Y} - \frac{2}{\sigma}\eta\Big) + \frac{\sigma}{2}Y - \frac{\bar{\tilde{\mu}}}{\sigma}\eta \\
=& \frac{1}{2}\big(\rho - \tilde{\mu}_{\bar{z}}\big)\widetilde{Y} + \frac{1}{2}\Big(-\frac{1}{2}|\tilde{\mu}|^2\widetilde{Y} - \tilde{\mu}\widetilde{Y}_{\bar{z}} - \bar{\tilde{\mu}}\widetilde{Y}_z + \sigma Y\Big).
\end{aligned}$$

on account of equations (1.2)(3.1)(3.3b)(3.12) (3.14)(3.16)(3.17a). Define

$$\widetilde{N} := -\frac{1}{2}|\tilde{\mu}|^2\widetilde{Y} - \tilde{\mu}\widetilde{Y}_{\bar{z}} - \bar{\tilde{\mu}}\widetilde{Y}_z + \sigma Y. \tag{3.18}$$

We may verify

$$\begin{aligned}
\langle\widetilde{N},\widetilde{Y}_z\rangle &= \langle-\tilde{\mu}\widetilde{Y}_{\bar{z}} + \sigma Y, \widetilde{Y}_z\rangle = -\frac{\tilde{\mu}}{2} + \sigma\langle Y, -\frac{\tilde{\mu}}{2}\widetilde{Y}\rangle = 0; \\
\langle\widetilde{N},\widetilde{Y}\rangle &= \langle\sigma Y, \widetilde{Y}\rangle = -1; \\
\langle\widetilde{N},\widetilde{N}\rangle &= \langle\widetilde{N} + \tilde{\mu}\widetilde{Y}_{\bar{z}} + \bar{\tilde{\mu}}\widetilde{Y}_z, \widetilde{N} + \tilde{\mu}\widetilde{Y}_{\bar{z}} + \bar{\tilde{\mu}}\widetilde{Y}_z\rangle - |\tilde{\mu}|^2 \\
&= \langle-\frac{1}{2}|\tilde{\mu}|^2\widetilde{Y} + \sigma Y, -\frac{1}{2}|\tilde{\mu}|^2\widetilde{Y} + \sigma Y\rangle - |\tilde{\mu}|^2 = 0.
\end{aligned}$$

So $\{\widetilde{Y}, \widetilde{Y}_z, \widetilde{Y}_{\bar{z}}, \widetilde{N}\}$ is the canonical frame as we desired. Compare the following structure equation of $\widetilde{Y}$

$$\widetilde{Y}_{z\bar{z}} = -\langle\tilde{\kappa},\bar{\tilde{\kappa}}\rangle\widetilde{Y} + \frac{1}{2}\widetilde{N}$$

with our previous calculated result, we may similarly define

$$\tilde{\rho} := \bar{\tilde{\mu}}_z - 2\langle \tilde{\kappa}, \bar{\tilde{\kappa}} \rangle. \tag{3.19}$$

Then there must be

$$\tilde{\rho} = \bar{\rho}. \tag{3.20}$$

How about the corresponding invariants $\tilde{\kappa}$ and $\tilde{s}$? According to structure equations of $\widetilde{Y}$, $\tilde{s}$ is determined by $\tilde{\kappa} = \widetilde{Y}_{zz} + \frac{\tilde{s}}{2}\widetilde{Y} \in \tilde{V}^{\perp}$, where $\tilde{V} := \mathrm{Span}\{\widetilde{Y}, \widetilde{Y}_z, \widetilde{Y}_{\bar{z}}, \widetilde{Y}_{z\bar{z}}\} = \mathrm{Span}\{\widetilde{Y}, \widetilde{Y}_z, \widetilde{Y}_{\bar{z}}, Y\}$ (the second equality is by (3.18)). Note that $\widetilde{Y}_{zz}$ and $\widetilde{Y}$ are always orthogonal to $\widetilde{Y}, \widetilde{Y}_z, \widetilde{Y}_{\bar{z}}$, We need only to find $\tilde{s}$ so that

$$\begin{aligned}
0 = \langle \widetilde{Y}_{zz} + \frac{\tilde{s}}{2}\widetilde{Y}, Y\rangle &= \langle \widetilde{Y}_z, Y\rangle_z - \langle \widetilde{Y}_z, Y_z\rangle - \frac{\tilde{s}}{2\sigma} \\
&= \langle -\frac{\tilde{\mu}}{2}\widetilde{Y}, Y\rangle_z - \langle -\frac{\tilde{\mu}}{2}\widetilde{Y}, Y_z\rangle - \frac{\tilde{s}}{2\sigma} \\
&= (\frac{\tilde{\mu}}{2\sigma})_z + \frac{\tilde{\mu}}{2}\cdot\frac{1}{\sigma}\cdot\frac{\mu}{2} - \frac{\tilde{s}}{2\sigma} \\
&= \frac{1}{2\sigma}\left(\tilde{\mu}_z - \frac{1}{2}\tilde{\mu}^2 - \tilde{s}\right).
\end{aligned}$$

Therefore

$$\tilde{s} = \tilde{\mu}_z - \frac{1}{2}\tilde{\mu}^2. \tag{3.21}$$

Denote the normal connection of $\widetilde{Y}$ as $\widetilde{D}$. We have the third structure equation

$$2\widetilde{D}_{\bar{z}}\tilde{\kappa} = \widetilde{N}_z + 2\langle \tilde{\kappa}, \bar{\tilde{\kappa}}\rangle \widetilde{Y}_z + \tilde{s}\widetilde{Y}_{\bar{z}}.$$

An immediate consequence is

$$\tilde{\eta} := \widetilde{D}_{\bar{z}}\tilde{\kappa} + \frac{\bar{\tilde{\mu}}}{2}\tilde{\kappa} = -\frac{\tilde{\rho}}{2}\left(\widetilde{Y}_z + \frac{\tilde{\mu}}{2}\widetilde{Y}\right) + \frac{\sigma}{2}\left(Y_z + \frac{\mu}{2}Y\right) \tag{3.22}$$

by (3.18)(3.21)(3.16) and the structure equations of $\widetilde{Y}$. We omit the calculation here, and concentrate on the most important issue: to verify the Willmore condition (3.6) for $\widetilde{Y}$. Differentiate the structure equation mentioned above, and modulo components inside $\tilde{V}$, one obtains

$$\begin{aligned}
2\big(\widetilde{D}_{\bar{z}}\tilde{\kappa}\big)_{\bar{z}} + \bar{\tilde{s}}\tilde{\kappa} &\equiv \widetilde{N}_{z\bar{z}} + \tilde{s}\widetilde{Y}_{\bar{z}\bar{z}} + \bar{\tilde{s}}\tilde{\kappa} \\
&\equiv \left(-\frac{1}{2}|\tilde{\mu}|^2\widetilde{Y} - \tilde{\mu}\widetilde{Y}_{\bar{z}} - \bar{\tilde{\mu}}\widetilde{Y}_z + \sigma Y\right)_{z\bar{z}} + \tilde{s}\widetilde{Y}_{\bar{z}\bar{z}} + \bar{\tilde{s}}\tilde{\kappa} \\
&\equiv \big(\tilde{s} - \tilde{\mu}_z\big)\widetilde{Y}_{\bar{z}\bar{z}} - \bar{\tilde{\mu}}_{\bar{z}}\widetilde{Y}_{zz} - \tilde{\mu}\widetilde{Y}_{\bar{z}\bar{z}z} - \bar{\tilde{\mu}}\widetilde{Y}_{zz\bar{z}} + (\sigma Y)_{z\bar{z}} + \bar{\tilde{s}}\tilde{\kappa} \\
&\equiv \big(\tilde{s} - \tilde{\mu}_z\big)\bar{\tilde{\kappa}} + \big(\bar{\tilde{s}} - \bar{\tilde{\mu}}_z\big)\tilde{\kappa} - \tilde{\mu}\bar{\tilde{\kappa}}_z - \bar{\tilde{\mu}}\tilde{\kappa}_{\bar{z}} + (\sigma Y)_{z\bar{z}} \\
&\equiv -\frac{1}{2}\tilde{\mu}^2\bar{\tilde{\kappa}} - \frac{1}{2}\bar{\tilde{\mu}}^2\tilde{\kappa} - \tilde{\mu}\widetilde{D}_z\bar{\tilde{\kappa}} - \bar{\tilde{\mu}}\widetilde{D}_{\bar{z}}\tilde{\kappa} + (\sigma Y)_{z\bar{z}} \\
&\equiv -\frac{1}{2}\tilde{\mu}^2\bar{\tilde{\kappa}} - \frac{1}{2}\bar{\tilde{\mu}}^2\tilde{\kappa} - \frac{\tilde{\mu}}{2}\widetilde{N}_{\bar{z}} - \frac{\bar{\tilde{\mu}}}{2}\widetilde{N}_z + (\sigma Y)_{z\bar{z}} \\
&\equiv -\frac{1}{2}\tilde{\mu}^2\bar{\tilde{\kappa}} - \frac{1}{2}\bar{\tilde{\mu}}^2\tilde{\kappa} - \frac{\tilde{\mu}}{2}\left[-\tilde{\mu}\widetilde{Y}_{\bar{z}\bar{z}} + (\sigma Y)_{\bar{z}}\right] \\
&\qquad - \frac{\bar{\tilde{\mu}}}{2}\left[-\bar{\tilde{\mu}}\widetilde{Y}_{zz} + (\sigma Y)_z\right] + (\sigma Y)_{z\bar{z}} \\
&\equiv -\frac{\tilde{\mu}}{2}(\sigma Y)_{\bar{z}} - \frac{\bar{\tilde{\mu}}}{2}(\sigma Y)_z + (\sigma Y)_{z\bar{z}} \\
&\equiv \left(\sigma_z - \frac{\tilde{\mu}}{2}\sigma\right)Y_{\bar{z}} + \left(\sigma_{\bar{z}} - \frac{\bar{\tilde{\mu}}}{2}\sigma\right)Y_z + \sigma\cdot\frac{1}{2}N \\
&\equiv \frac{\sigma}{2}\big(\mu Y_{\bar{z}} + \bar{\mu}Y_z + N\big) \\
&\equiv \frac{\sigma}{2}\left(\sigma\widetilde{Y} - \frac{1}{2}|\mu|^2 Y\right) \\
&\equiv 0.
\end{aligned}$$

Thus we have proved that the Willmore condition (3.6) is also satisfied for $\widetilde{Y}$. Furthermore, equation (3.18) shows that Y may be viewed as an adjoint transform of $\widetilde{Y}$. First $\tilde{\mu}$ satisfies (3.21) which amounts to say that $\tilde{\theta}$ defined similarly vanishes. Second we already know the *conformality* between $\widetilde{Y}$ and Y. Indeed we can verify $\langle\tilde{\eta},\tilde{\eta}\rangle = 0$ directly by (3.22). Note also that (3.22) is the dual form of (3.15). This remarkable *duality* is just what we expected, since we already know such a relationship between a Willmore surface in $\mathbb{S}^4$ and its *forward/backward two-step Bäcklund transforms*([10]). Sum together, we get

Theorem 3.5. *An adjoint transform $\widetilde{Y}$ of a Willmore surface Y is also Willmore, which is called an* adjoint Willmore surface of Y *or* a Willmore surface adjoint to Y. *Vice versa, Y is also an adjoint transform of $\widetilde{Y}$. The relationship between their corresponding invariants are given by*

$$-\frac{1}{\sigma} = \langle\widetilde{Y}, Y\rangle, \quad \frac{2\sigma_z}{\sigma} = \tilde{\mu} + \mu, \quad \tilde{\rho} = \bar{\rho}.$$

Remark 3.6. According to discussions in Subsection 3.1.2, if $\langle\kappa,\kappa\rangle \neq 0$, there will be two such adjoint transforms. An interesting problem is to determine when such two solutions coincide. In such a case there must be $\langle\eta,\kappa\rangle = 0$ for the commom μ and η. Conversely, the condition that $\bar{\mu} := -\big(\ln\langle\kappa,\kappa\rangle\big)_{\bar{z}}$ satisfies (3.7a) characterizes the coincidence case. Moreover, suppose the adjoint Willmore surface $\widetilde{Y}$ also satisfies $\langle\tilde{\kappa},\tilde{\kappa}\rangle \neq 0$. Then it is easy to check that the original Willmore surface Y is also the unique adjoint transform of $\widetilde{Y}$.

In our discussion later, no matter whether $\langle\kappa,\kappa\rangle \neq 0$, the case that $\langle\eta,\kappa\rangle = 0$ is important. This time we deduce from (3.17b) that $\eta_z = D_z\eta$. Since we know the representation of $\widetilde{Y}$ and $\tilde{s}$, $\tilde{\kappa}$ can be written down explicitly via $\tilde{\kappa} = \widetilde{Y}_{zz} + \frac{\tilde{s}}{2}\widetilde{Y}$. We give the result below and omit the calculation.

$$\tilde{\kappa} = \frac{1}{\sigma}\Big[(\rho_z - \tilde{\mu}\rho)\left(Y_z + \frac{\mu}{2}Y\right) + 2D_z\eta - (\mu + 2\tilde{\mu})\eta + \rho\kappa\Big]. \tag{3.23}$$

3.2.2 Characterization by conformal harmonic maps

In last chapter, we have developed a theory of pairs of conformally immersed surfaces $f, \hat{f}$ from a Riemann surface M into $\mathbb{S}^n$. Another way to look at them is considering the 2-plane spanned by their lifts $Y, \widehat{Y}$. This defines a map

$$\begin{aligned} H : M &\to G_{1,1}(\mathbb{R}^{n+1,1}), \\ p &\mapsto Y(p) \wedge \widehat{Y}(p). \end{aligned}$$

Similar to the description of the conformal Gauss map, here the Grassmannian $G_{1,1}(\mathbb{R}^{n+1,1})$ consists of all 2-dim Minkowski subspaces, and we regard it as a submanifold embedded in $\wedge^2\mathbb{R}^{n+1,1}$. The bi-vector is uniquely determined if we put the restriction

$$\langle Y, \widehat{Y}\rangle = -1, \quad \Longrightarrow \quad \langle H, H\rangle = -1.$$

Conversely, such a map corresponds to a pair of surfaces in $\mathbb{S}^n$.

Associated with $Y, \widehat{Y}$ are invariants θ, ρ defined via (2.4). They appear also as invariants of H. It turns out [2]

$$\langle H_z, H_z\rangle = \theta, \quad \langle H_z, H_{\bar{z}}\rangle = \frac{1}{2}(\rho + \bar{\rho}).$$

$$\Longrightarrow \quad \langle \mathrm{d}H, \mathrm{d}H\rangle = \theta\mathrm{d}z^2 + \frac{1}{2}(\rho + \bar{\rho})(\mathrm{d}z\mathrm{d}\bar{z} + \mathrm{d}\bar{z}\mathrm{d}z) + \bar{\theta}\mathrm{d}\bar{z}^2. \tag{3.24}$$

So the co-touching condition is equivalent to the conformality of H, thus seems to be a natural assumption.

On the other hand, (3.24) also helps us to compute the energy of H:

$$E(H) := \int_M \langle \mathrm{d}H \wedge *\mathrm{d}H\rangle = -\mathrm{i}\int_M (\rho + \bar{\rho}) \cdot \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

[2] In the computation, without loss of generality we may assume Y is the canonical lift of f, and using the formulae in Subsection 2.1.1.

Now comes another natural question: What is the condition that H being conformal harmonic? Note that H is similar to the conformal Gauss map in that each of them is into some Grassmannian associated with $\mathbb{R}^{n+1,1}$. The latter being harmonic iff the original surface is Willmore (Theorem 1.9). By analogy one would expect some similar result for H. Of course we should assume that the underlying maps $f, \hat{f}$ are also conformal. Surprisingly, these simple conditions give a nice characterization of adjoint Willmore surfaces. Moreover, we have

Theorem 3.7. *Let M be a Riemann surface. Assume $Y, \widehat{Y}$ are local lifts of immersions $f, \hat{f} : M \to \mathbb{S}^n$ satisfying $\langle Y, \widehat{Y}\rangle = -1$, which induce map $H = Y \wedge \widehat{Y} : M \to G_{1,1}(\mathbb{R}^{n+1,1})$. Then the three conditions below are equivalent:*

(i) $f, \hat{f}$ are a pair of Willmore surfaces forming adjoint transform to each other.

(ii) $f, \hat{f}$ and H are conformal maps, and $f, \hat{f}$ locate on the mean curvature sphere of each other.

(iii) $f, \hat{f}$ are conformal to each other and $H = Y \wedge \widehat{Y} : M \to G_{1,1}(\mathbb{R}^{n+1,1})$ is conformal harmonic.

Proof. By the same reason in the previous footnote, the theory in Subsection 2.1.1 applies to this case as well. So Y might be chosen to be the canonical lift of f and we can write

$$\widehat{Y} = \frac{1}{2}\Big(|\mu|^2 + \langle\xi,\xi\rangle\Big)Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N + \xi,$$
$$\widehat{Y}_z = \frac{\mu}{2}\widehat{Y} + \theta\left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + \rho\left(Y_z + \frac{\mu}{2}Y\right) + \langle\xi,\zeta\rangle Y + \zeta,$$

where θ, ρ, ζ are associated invariants given in (2.4).

Let $H_t = Y_t \wedge \widehat{Y}_t$ be a variation of $H = H_0 = Y \wedge \widehat{Y}$, so that

$$\langle Y_t, Y_t\rangle = \langle \widehat{Y}_t, \widehat{Y}_t\rangle = 0, \langle Y_t, \widehat{Y}_t\rangle = -1, \quad \Longrightarrow \quad \langle H_t, H_t\rangle = -1.$$

As before, we abbreviate $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$. Thus the only restrictions on the variational vector field $\dot{H}$, or equivalently on $\dot{Y}, \dot{\widehat{Y}}$, are

$$\langle \dot{H}, H\rangle = 0,$$

and

$$\langle \dot{Y}, Y\rangle = \langle \dot{\widehat{Y}}, \widehat{Y}\rangle = \langle \dot{Y}, \widehat{Y}\rangle + \langle Y, \dot{\widehat{Y}}\rangle = 0. \tag{3.25}$$

The first variation of the energy of H_t is computed almost the same as in the proof to Proposition 1.7:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(H_t) = -2\int_M \langle \dot{H}, \mathrm{d} * \mathrm{d}H\rangle = 4\mathrm{i}\int_M \langle \dot{H}, H_{z\bar{z}}\rangle \cdot \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

So H is conformal harmonic iff

$$\theta = 0 \quad \text{and} \quad \langle \dot{H}, H_{z\bar{z}}\rangle = 0, \ \forall\ \dot{H}.$$

First we show **(iii)⇒(ii)**. Take special variational vector fields

$$\dot{Y} = 0, \quad \dot{\widehat{Y}} = \langle \xi, \xi\rangle Y + \xi.$$

It is easy to verify that they satisfy (3.25) by checking $\langle \dot{\widehat{Y}}, Y\rangle = 0 = \langle \dot{\widehat{Y}}, \widehat{Y}\rangle$. Computation shows

$$\begin{aligned}\langle \dot{H}, H_{z\bar{z}}\rangle &= \frac{1}{2}\langle \dot{\widehat{Y}}, \bar{\mu}Y_z + \mu Y_{\bar{z}} + N\rangle \\ &= \frac{1}{2}\langle \dot{\widehat{Y}}, \widehat{Y} - \frac{1}{2}(|\mu|^2 + \langle \xi, \xi\rangle)Y - \xi\rangle \\ &= \frac{1}{2}\langle \dot{\widehat{Y}}, -\xi\rangle = -\frac{1}{2}\langle \xi, \xi\rangle.\end{aligned}$$

Since the restriction of the Minkowski metric on the Möbius normal bundle $V^{\perp}$ is positive definite, $\dot{E} = 0$ implies $\xi = 0$, [3] i.e. $\hat{f}$ is on the mean curvature sphere of f. But there is no bias for f or $\hat{f}$ in the assumptions, so these two surfaces should be dual to each other. Hence f is also on the mean curvature sphere of $\hat{f}$.

Next we prove **(ii)⇒(i)**. With $\xi = 0, \theta = 0$ we have the simplified formulae below:

$$\begin{aligned}\widehat{Y} &= \frac{1}{2}\Big(|\mu|^2 + \langle \xi, \xi\rangle\Big)Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N, \\ \widehat{Y}_z &= \frac{\mu}{2}\widehat{Y} + \rho\left(Y_z + \frac{\mu}{2}Y\right) + 2\eta,\end{aligned}$$

where

$$\rho := \bar{\mu}_z - 2\langle \kappa, \bar{\kappa}\rangle, \quad \eta := D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa.$$

As in last subsection, $\theta := \mu_z - \frac{1}{2}\mu^2 - s = 0$ further implies

$$\rho_{\bar{z}} = \bar{\mu}\rho + 4\langle \eta, \bar{\kappa}\rangle$$

by Gauss equation (1.3a), and

$$D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta = D_{\bar{z}}D_{\bar{z}}\kappa + \frac{\bar{s}}{2}\kappa,$$

[3] Intuitively, this is because the expression of ρ (2.4b) contains the term $\langle \xi, \xi\rangle$, hence ξ must vanish if the integral of $\rho + \bar{\rho}$ is critical.

which is real-valued by Codazzi equation (1.3b). Also note

$$\eta_{\bar{z}} = D_{\bar{z}}\eta + 2\langle\eta,\bar{\eta}\rangle Y - 2\langle\eta,\bar{\kappa}\rangle\left(Y_z + \frac{\mu}{2}Y\right)$$

by (1.2). Now the differentiation of $\widehat{Y}_z$ can be computed out with the ourcome

$$\widehat{Y}_{z\bar{z}} = \frac{\mu}{2}\widehat{Y}_{\bar{z}} + \frac{\bar{\mu}}{2}\widehat{Y}_z + \left(\frac{\mu_{\bar{z}}}{2} + \frac{\rho}{2} - \frac{|\mu|^2}{4}\right)\widehat{Y} + \left(\frac{1}{2}|\rho|^2 + 4\langle\eta,\bar{\eta}\rangle\right)Y + 2\left(D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta\right). \tag{3.26}$$

Since $\hat{f}$ is also on the mean curvature sphere of f, Y is a linear combination of

$$\{\widehat{Y},\widehat{Y}_z,\widehat{Y}_{\bar{z}},\widehat{Y}_{z\bar{z}}\}.$$

The $\widehat{Y}_{z\bar{z}}$-component of Y is not zero. (Otherwise Y is a combination of $\widehat{Y},\widehat{Y}_z,\widehat{Y}_{\bar{z}}$, hence $\langle Y,\widehat{Y}\rangle = 0$, a contradiction.) So $\widehat{Y}_{z\bar{z}}$, as well as $D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta$, is contained in

$$\mathrm{Span}\{\widehat{Y},\widehat{Y}_z,\widehat{Y}_{\bar{z}},Y\}$$

By the expressions of $\widehat{Y},\widehat{Y}_z$, this is true only if

$$0 = D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta = D_{\bar{z}}D_{\bar{z}}\kappa + \frac{\bar{s}}{2}\kappa,$$

i.e. f is Willmore. Again by the duality between f and $\hat{f}$ we know $\hat{f}$ is also Willmore. The assumptions directly imply that they form adjoint transform to each other.

Finally one should verify **(i)⇒(iii)**. This case $\theta = 0$, $\xi = 0$, $D_{\bar{z}}\eta - \frac{\bar{\mu}}{2}\eta = 0$, and (3.26) takes the following form:

$$\widehat{Y}_{z\bar{z}} = \frac{\mu}{2}\widehat{Y}_{\bar{z}} + \frac{\bar{\mu}}{2}\widehat{Y}_z + \left(\frac{\mu_{\bar{z}}}{2} + \frac{\rho}{2} - \frac{|\mu|^2}{4}\right)\widehat{Y} + \left(\frac{1}{2}|\rho|^2 + 4\langle\eta,\bar{\eta}\rangle\right)Y.$$

We compute $\langle\dot{H},H_{z\bar{z}}\rangle$ for arbitrary $\dot{H}$, or equivalently, for any variational vector fields $\dot{Y},\dot{\widehat{Y}}$. Invoking the restrictions (3.25), there follows

$$\begin{aligned}\langle\dot{H},H_{z\bar{z}}\rangle &= \langle\dot{Y},\widehat{Y}_{z\bar{z}} - \frac{\mu}{2}\widehat{Y}_{\bar{z}} - \frac{\bar{\mu}}{2}\widehat{Y}_z + \left(\frac{|\mu|^2}{4} - \langle\kappa,\bar{\kappa}\rangle\right)\widehat{Y}\rangle \\ &\quad + \langle\dot{\widehat{Y}},\frac{\mu}{2}Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y_z + \frac{1}{2}N - \left(\langle Y,\widehat{Y}_{z\bar{z}}\rangle + \langle\kappa,\bar{\kappa}\rangle\right)Y\rangle \\ &= \langle\dot{Y},\left(\frac{\mu_{\bar{z}}}{2} + \frac{\rho}{2} - \langle\kappa,\bar{\kappa}\rangle\right)\widehat{Y} + \left(\frac{1}{2}|\rho|^2 + 4\langle\eta,\bar{\eta}\rangle\right)Y\rangle \\ &\quad + \langle\dot{\widehat{Y}},\frac{1}{2}\widehat{Y} - \frac{|\mu|^2}{4}Y + \left(\frac{\mu_{\bar{z}}}{2} + \frac{\rho}{2} + \frac{|\mu|^2}{4} - \langle\kappa,\bar{\kappa}\rangle\right)Y\rangle \\ &= 0.\end{aligned}$$

So H is harmonic. This completes our proof. □

Remark 3.8. Indeed, our discussion above provides another proof to the Duality Theorem 3.5.

Chapter 4

Willmore 2-spheres: motivation from a special case

We have defined *adjoint transforms* of Willmore surfaces in $\mathbb{S}^n$, which generalize the *two-step Bäcklund transforms* of Willmore surfaces in $\mathbb{S}^4$. As explained in the introduction, next we want to apply these transforms to Willmore 2-spheres and expect to obtain classification theorem. This simple idea will lead us to interesting discoveries. To give the motivation of some important definitions and constructions in the following, we start with a special case, namely when the Hopf differential satisfies

$$\langle \kappa, \kappa \rangle \not\equiv 0.$$

We will show that such a Willmore 2-sphere has a unique adjoint transform, which is a branched conformal immersion. Further discussion yields some vanishing and not-vanishing results contrasting to each other. These surprising results indicate a generalization to much wider and stronger vanishing theorems and a reduction procedure, which are left to the next chapter.

4.1 The unique adjoint transform

Let $f : M \to \mathbb{S}^n, n \geq 3$ be an arbitrary immersed Willmore surface, and assume the Hopf differential of f satisfies $\langle \kappa, \kappa \rangle \not\equiv 0$. This allows us to consider the solutions to quadratic equation

$$0 = \langle \eta, \eta \rangle = \langle D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa, D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa \rangle \tag{4.1}$$

on M_0, the subset of M on which $\langle \kappa, \kappa \rangle \neq 0$, and those two adjoint Willmore surfaces defined by such μ's.

Since we concern ourselves mainly about the case $M \cong S^2$, it is natural to try to construct holomorphic forms on M and to obtain vanishing results. The most direct way to do that is by taking scalar product between two vector-valued forms. Observe that either solution
$\eta_i = D_{\bar z}\kappa + (\bar\mu_i/2)\kappa, i = 1,2$ defines a vector-valued form $\eta_i(\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar z)^{\frac{1}{2}}$ as we know in Subsection 3.1.3. Its scalar product with $\kappa(\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar z)^{-\frac{1}{2}}$ yields a holomorphic $(3,0)$-form outside singularities:

$$\langle\kappa,\eta_i\rangle_{\bar z} = \langle D_{\bar z}\kappa,\eta_i\rangle + \langle\kappa, D_{\bar z}\eta_i\rangle = \langle\eta_i - \frac{\bar\mu_i}{2}\kappa,\eta_i\rangle + \langle\kappa,\frac{\bar\mu_i}{2}\eta_i\rangle = 0.$$

As a heuristic argument, let us ignore the singularities arising either from the zeros of $\langle\kappa,\kappa\rangle$ or from the double roots of (4.1), and assume $M \cong S^2$. There follows that $\langle\eta_i,\kappa\rangle(\mathrm{d}z)^3, i = 1,2$ are two holomorphic forms on S^2. They must vanish, hence

$$0 \equiv \langle\eta_1,\kappa\rangle - \langle\eta_2,\kappa\rangle = \frac{(\bar\mu_1 - \bar\mu_2)}{2}\langle\kappa,\kappa\rangle.$$

This suggests that the adjoint transform is unique under these assumptions.

Despite the obvious gaps, this argument lends inspiration to us. Generally, after assuming $\langle\kappa,\kappa\rangle \neq 0$, we might ask: Whether those two solutions of (4.1) are really distinct? Do they extend continuously to the whole S^2? These observations lead our attention to the discriminant of equation (4.1). A close examination shows

Lemma 4.1. *This discriminant globally defines a holomorphic form*

$$\Delta = \Big(\langle D_{\bar z}\kappa,\kappa\rangle^2 - \langle\kappa,\kappa\rangle\langle D_{\bar z}\kappa, D_{\bar z}\kappa\rangle\Big)(\mathrm{d}z)^6.$$

Proof. It is straightforward to verify that this $(6,0)$-form is globally defined by (1.4a). Next we verify it is holomorphic:

$$\begin{aligned}
&\left(\langle D_{\bar z}\kappa,\kappa\rangle^2 - \langle\kappa,\kappa\rangle\langle D_{\bar z}\kappa, D_{\bar z}\kappa\rangle\right)_{\bar z}\\
=\;& 2\langle D_{\bar z}\kappa,\kappa\rangle\left(\langle D_{\bar z}D_{\bar z}\kappa,\kappa\rangle + \langle D_{\bar z}\kappa, D_{\bar z}\kappa\rangle\right)\\
&-2\langle D_{\bar z}\kappa,\kappa\rangle\langle D_{\bar z}\kappa, D_{\bar z}\kappa\rangle - 2\langle\kappa,\kappa\rangle\langle D_{\bar z}D_{\bar z}\kappa, D_{\bar z}\kappa\rangle\\
=\;& \langle D_{\bar z}\kappa,\kappa\rangle\langle -\bar s\kappa,\kappa\rangle - \langle\kappa,\kappa\rangle\langle -\bar s\kappa, D_{\bar z}\kappa\rangle = 0\ . \qquad \square
\end{aligned}$$

Now we have a rigorous proof of the following fact mentioned above.

Corollary 4.2. *Suppose the Willmore surface $M \cong S^2$, then Δ vanishes identically. If $\langle\kappa,\kappa\rangle \not\equiv 0$, we have a unique solution to equation* (4.1) *and a unique adjoint Willmore surface. The double root μ is given by*

$$\bar\mu = -\frac{\langle\kappa,\kappa\rangle_{\bar z}}{\langle\kappa,\kappa\rangle}\ , \tag{4.2}$$

and $\eta = D_{\bar z}\kappa + \frac{\bar\mu}{2}\kappa$ satisfies $\langle\eta,\kappa\rangle = 0$.

From now on we concentrate on Willmore 2-spheres, so it will be always assumed that $M \cong S^2$ unless claimed otherwise. In case that $\langle\kappa,\kappa\rangle \not\equiv 0$ holds for the original Willmore 2-sphere, we have found a Willmore surface adjoint to f. This remarkable new surface is well-defined on M_0 and denoted as $\widetilde{F}$, which is assumed not to degenerate to a single point. We wish to show that $\tilde{f}$ can be extended continuously to the whole M, thus a branched conformal immersion of S^2.

Since Willmore immersion is analytic([48]), the zero points of $\langle\kappa,\kappa\rangle$ must form a closed subset of measure zero, and M_0 is an open dense subset. To refine our analysis we may obtain more information from Corollary 4.2.

Lemma 4.3.

1. *$g := \langle\kappa,\kappa\rangle$ satisfies the following differential equation with respect to a local coordinate z, where s stands for the Schwarzian:*

$$\frac{1}{2}g_{\bar z}^2 - g g_{\bar z\bar z} - \bar{s}g^2 = 0. \tag{4.3}$$

2. *Let $p \in M \setminus M_0$ be a zero point of g, and take a coordinate z with $z(p) = 0$. Then there is another analytic solution $\hat{g}$ to equation (4.3), whose Taylor expansion at $z = 0$ is either $\hat{g} = (az + b\bar{z})^2 + o(|z|^2)$ with $b \neq 0$, or $\hat{g} = c + o(|z|)$ with $c \neq 0$, so that $g = z^r \cdot \hat{g}$ for a non-negative integer r.*

The first conclusion is equivalent to equation $\mu_z - \frac{1}{2}\mu^2 - s = 0$, which we know already. The second result is a direct consequence of equation (4.3). It is stated as Lemma B.1 in the Appendix B with a straightforward algebraical proof. Now we can address the extension problem of $\tilde{f}$.

Corollary 4.4. *The adjoint Willmore surface $\tilde{f}$ extends analytically to the whole M as a branched conformal immersion.*

Proof. Under a given local coordinate z, a lift of the adjoint surface $\tilde{f}$ is given by

$$\widehat{Y} = \frac{1}{2}|\mu|^2 Y + \bar{\mu}Y_z + \mu Y_{\bar z} + N. \tag{4.4}$$

Following Lemma 4.3, suppose $z = 0$ is a zero of g. We find

$$\bar{\mu} = -\frac{g_{\bar z}}{g} = -\frac{\hat{g}_{\bar z}}{\hat{g}},$$

where $g = \langle\kappa,\kappa\rangle = z^r\hat{g}$, and the lowest order term in the expansion of $\hat{g}$ is either a non-zero constant, or $(az + b\bar{z})^2$ with $b \neq 0$. In the first case, μ is non-singular. In the second case, we construct

$$\bar{\lambda} := \bar{\mu} \cdot \hat{g}_{\bar z} = -\frac{\hat{g}_{\bar z}^2}{\hat{g}} = -2\hat{g}_{\bar z\bar z} - 2\bar{s}\hat{g}.$$

This is an analytic function defined around $z = 0$. It takes value $-4b^2 \neq 0$ at $z = 0$, hence does not vanish locally. So we may scale properly and obtain another lift

$$|\hat{g}_{\bar{z}}|^2 \widehat{Y} = \frac{1}{2}|\lambda|^2 Y + (\bar{\hat{g}}_z \bar{\lambda}) Y_z + (\hat{g}_{\bar{z}}\lambda) Y_{\bar{z}} + |\hat{g}_{\bar{z}}|^2 N.$$

This new lift is locally non-zero and analytic as we desired. □

4.2 Vanishing results

The discussion above shows that $\tilde{f}$ is a conformally immersed Willmore 2-sphere with branch points. We would expect the Hopf differential of this second surface is not isotropic, so that we can construct a sequence of such adjoint transforms. The first exploration seems a mixture of disappointment and excitement.

Proposition 4.5. *The Hopf differential of $\tilde{f} : S^2 \to \mathbb{S}^n$ is isotropic everywhere.*

This is the consequence of following lemmas. As a preparation, let $\widetilde{Y}$ and $\tilde{\kappa}$ denote the canonical lift and the Hopf differential of $\tilde{f}$ under a given coordinate z. Note here holds $\langle \eta, \kappa \rangle = 0$. The expression of $\tilde{\kappa}$ was contained in (3.23):

$$\tilde{\kappa} = \frac{1}{\sigma}\Big[(\rho_z - \tilde{\mu}\rho)\big(Y_z + \frac{\mu}{2}Y\big) + 2\xi - (\mu + 2\tilde{\mu})\eta\Big]. \tag{4.5}$$

Here μ and η are given as in Corollary 4.2, and

$$\xi := D_z\eta + \frac{\rho}{2}\kappa, \quad \rho := \bar{\mu}_z - 2\langle \kappa, \bar{\kappa} \rangle, \quad \sigma^2 := 8\langle \eta, \bar{\eta} \rangle + |\rho|^2, \quad \tilde{\mu} := \frac{2\sigma_z}{\sigma} - \mu.$$

We know $\langle \eta, \kappa \rangle = 0$, and $\langle \eta, \eta \rangle = 0$ implies $\langle D_z\eta, \eta \rangle = 0 = \langle \xi, \eta \rangle$. Hence

$$\langle \tilde{\kappa}, \tilde{\kappa} \rangle = \frac{4}{\sigma^2}\langle \xi, \xi \rangle.$$

Observe that $\kappa(\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar{z})^{-\frac{1}{2}}$ and $\tilde{\kappa}(\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar{z})^{-\frac{1}{2}}$ are both vector-valued forms. $\sigma \mathrm{d}z\mathrm{d}\bar{z}$ is a $(1,1)$-form. Hence $\langle \kappa, \tilde{\kappa} \rangle \cdot \sigma$ and $\langle \kappa, \kappa \rangle \langle \tilde{\kappa}, \tilde{\kappa} \rangle \cdot \sigma^2$ define separately a $(4,0)$-form and a $(8,0)$-form:

$$\Phi := \langle \kappa, \xi \rangle (\mathrm{d}z)^4, \quad \Psi := \langle \kappa, \kappa \rangle \langle \xi, \xi \rangle (\mathrm{d}z)^8. \tag{4.6}$$

As one might suspect, they are holomorphic forms.

Lemma 4.6. *Φ and Ψ are holomorphic on M_0.*

Proof. First we have

$$\begin{aligned} D_{\bar z}\xi &= D_{\bar z}\big(D_z\eta + \frac{\rho}{2}\kappa\big) = D_z D_{\bar z}\eta - 2\langle\eta,\bar\kappa\rangle\kappa + D_{\bar z}(\frac{\rho}{2}\kappa) \\ &= \frac{\bar\mu}{2}\xi + \big(\rho + \langle\kappa,\bar\kappa\rangle\big)\eta + \frac{1}{2}\big(\underbrace{\rho_{\bar z} - \bar\mu\rho - 4\langle\eta,\bar\kappa\rangle\kappa}_{=0}\big)\kappa \end{aligned} \tag{4.7}$$

by formula (3.12). It follows

$$\langle\kappa,\xi\rangle_{\bar z} = \langle D_{\bar z}\kappa,\xi\rangle + \langle\kappa, D_{\bar z}\xi\rangle = \langle\eta - \frac{\bar\mu}{2}\kappa,\xi\rangle + \langle\kappa,\frac{\bar\mu}{2}\xi + (\cdots)\eta\rangle = 0$$

and $\langle\xi,\xi\rangle_{\bar z} = \bar\mu\langle\xi,\xi\rangle$. Together with $\langle\kappa,\kappa\rangle_{\bar z} = -\bar\mu\langle\kappa,\kappa\rangle$, we find Ψ is also holomorphic, at least on the subset M_0 where μ is defined. □

The problem left to us is: Whether Φ and Ψ have analytic continuations to the whole $M \cong S^2$? Consider the expression of $\langle\xi,\kappa\rangle$. It is easy to find it as a sum of $g_{\bar z}g_z/g$ and other non-singular terms, where $g = \langle\kappa,\kappa\rangle$ as defined before. By Lemma 4.3, we need only to consider the case when $g_{\bar z}g_z$ and g have the same lowest order term $(az + b\bar z)^2$ in the Taylor expansion of each one. If $|a| \neq |b|$, $az + b\bar z$ will have isolated zero at $z = 0$ and the fraction $g_{\bar z}g_z/g$ has continuous limit. But the case $|a| = |b|$ is still troublesome. We face the similar difficulty around the singularities of Ψ. Fortunately, we can establish an technical lemma as follows, whose proof is left to Appendix B.

Lemma 4.7 (Extension Lemma). *Suppose p, q are two analytic functions defined on disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$, where q does not vanish identically, and $f = p/q$ is holomorphic on the open dense subset where $q \neq 0$. Then f is meromorphic on D. Suppose further that the Taylor expansion of q at $z = 0$ contains monomials like $\bar z^l$. Then $z = 0$ is a removable singularity of f.*

Now we can solve the previous problem in affirmative.

Lemma 4.8. *Φ and Ψ extends continously across the zeros of $\langle\kappa,\kappa\rangle$. Thus they are holomorphic forms defined on $M \cong S^2$ and vanish identically.*

Proof. Denote $g = \langle\kappa,\kappa\rangle$ as before. Suppose p is a zero of g and $z(p) = 0$. We compute

$$\begin{aligned} \langle\xi,\kappa\rangle &= \langle D_z\eta + \frac{\rho}{2}\kappa,\kappa\rangle = \langle D_z D_{\bar z}\kappa + \frac{\bar\mu}{2}D_z\kappa + (\frac{1}{2}\bar\mu_z + \frac{\rho}{2})\kappa,\kappa\rangle \\ &= \frac{\bar\mu}{2}\langle D_z\kappa,\kappa\rangle + \bar\mu_z\langle\kappa,\kappa\rangle + \cdots = \frac{3}{4}\frac{g_{\bar z}g_z}{g} + (\text{non-singular part}). \end{aligned}$$

According to Lemma 4.6, this is a quotient of two analytic functions which is holomorphic except around the zeros of g. Lemma 4.3 ensures that the denominator may be chosen to be $\hat g$, which contains monomials like $\bar z^l$ in its Taylor expansion at $z = 0$. Then Lemma 4.7 guarantees the continuous extension to $z = 0$. Hence $\langle \xi, \kappa \rangle$ has only removable singularities and equals zero identically. Keeping this fact in mind, we go on to show

$$\begin{aligned} g \cdot \langle \xi, \xi \rangle &= g \cdot \langle \xi, D_z D_{\bar z}\kappa + \frac{\bar\mu}{2} D_z \kappa \rangle \\ &= g \cdot \langle D_z D_{\bar z}\kappa + \frac{\bar\mu}{2} D_z\kappa + \big(\bar\mu_z - \langle \kappa, \bar\kappa\rangle\big)\kappa, D_z D_{\bar z}\kappa + \frac{\bar\mu}{2} D_z\kappa \rangle \\ &= g\bar\mu \langle D_z D_{\bar z}\kappa, D_z\kappa\rangle + g\frac{\bar\mu^2}{4}\langle D_z\kappa, D_z\kappa\rangle \\ &\quad + g\bar\mu_z \langle \kappa, D_z D_{\bar z}\kappa\rangle + g\big(\bar\mu_z - \langle\kappa,\bar\kappa\rangle\big)\frac{\bar\mu g_z}{4} + (\text{non-singular part}). \end{aligned}$$

Since $g\bar\mu = -g_{\bar z}$, $g\bar\mu^2 = (g_{\bar z}^2)/g$, $\bar\mu g_z = -(g_{\bar z} g_z)/g$, $g\bar\mu_z = (g\bar\mu)_z - \bar\mu g_z = -g_{\bar z z} + (g_{\bar z} g_z)/g$, the previous argument applies to Ψ as well. This finishes the proof. $\square$

Remark 4.9. As a corollary of $\Phi = \Psi = 0$ and the assumption $g \not\equiv 0$, we have

$$\langle \xi, \xi \rangle = \langle \xi, \kappa \rangle = \langle \xi, D_z \eta \rangle = 0 = \langle \xi, \eta \rangle. \tag{4.8}$$

Proposition 4.5 follows from them. In other words, $t = \rho$ is a root of equation

$$\langle D_z\eta + \frac{t}{2}\kappa, D_z\eta + \frac{t}{2}\kappa \rangle = 0.$$

Moreover, $\langle \xi, \kappa \rangle = 0$ implies that $t = \rho$ is a double root. Indeed, another way to prove Proposition 4.5 is to consider the discriminant of the quadratic equation above. One can show it defines a holomorphic $(8,0)$-form on S^2. So this discriminant equals zero and there exists a double root t everywhere. Then one proves $t = \rho$ by showing $\langle \kappa, \kappa \rangle (t - \rho)(\mathrm{d}z)^4$ is a globally defined holomorphic form over S^2. The meromorphicity is easy to verify, and analytic continuation relies on the Extension Lemma. These results also ensure that $(g_{\bar z} g_z)/g$ (as well as $g_{\bar z}^2/g$) extends, not only continuously, but also analytically, to the zeros of g.

Although our previous hope is now in vain, there is still something surprising in this discovery. Why should there be such a strong conclusion about the Hopf differential of $\tilde f$? Can we say something more about them, about these frames $\kappa, \xi, \tilde\kappa$? Could there be some other holomorphic forms? The scalar product between which vectors must vanish? All these questions are worth thinking about.

To extend our exploration, it is natural to consider the higher order differentiations of $\tilde\kappa$ as well as the scalar products between them. Since $\langle \tilde\kappa, \tilde\kappa \rangle = 0$ and the Willmore condition is satisfied, we find

$$0 = \langle \widetilde D_z \tilde\kappa, \tilde\kappa \rangle = \langle \widetilde D_{\bar z}\tilde\kappa, \tilde\kappa\rangle = \langle \widetilde D_{\bar z}\tilde\kappa, \widetilde D_{\bar z}\tilde\kappa\rangle = \langle \widetilde D_z \widetilde D_{\bar z}\tilde\kappa, \widetilde D_{\bar z}\tilde\kappa \rangle = \langle \widetilde D_{\bar z}\widetilde D_z\tilde\kappa, \widetilde D_{\bar z}\tilde\kappa\rangle.$$

Here $\widetilde D$ is the normal connection of $\tilde f$. Next we show

Lemma 4.10. $\langle \widetilde{D}_z \widetilde{D}_{\bar{z}} \tilde{\kappa}, \tilde{\kappa} \rangle = -\langle \widetilde{D}_{\bar{z}} \tilde{\kappa}, \widetilde{D}_z \tilde{\kappa} \rangle = 0$.

Proof. The first equality is obvious. To show they are equal to zero, we need only to prove $\langle \widetilde{D}_z \tilde{\eta}, \tilde{\kappa} \rangle = 0$, where

$$\begin{aligned} \tilde{\eta} := \widetilde{D}_{\bar{z}} \tilde{\kappa} + \frac{\bar{\tilde{\mu}}}{2} \tilde{\kappa} &= -\frac{\tilde{\rho}}{2} \big(\widetilde{Y}_z + \frac{\tilde{\mu}}{2} \widetilde{Y} \big) + \frac{\sigma}{2} \big(Y_z + \frac{\mu}{2} Y \big) \\ &= \frac{1}{\sigma} \Big[\frac{1}{2} (\sigma^2 - |\rho|^2) \big(Y_z + \frac{\mu}{2} Y \big) - \bar{\rho} \eta \Big]. \end{aligned} \tag{4.9}$$

according to equation (3.22). $\tilde{\eta}$ and $\tilde{\kappa}$ are both sections of isotropic subbundle $\mathrm{Span}\{\tilde{\kappa}, \widetilde{D}_{\bar{z}} \tilde{\kappa}\}$. Apply the last structure equation in (1.2) to them, we see $\widetilde{D}_z \tilde{\eta} = \tilde{\eta}_z, \widetilde{D}_z \tilde{\kappa} = \tilde{\kappa}_z$. Also $D_z \eta = \eta_z$ for the same reason. Now it is straightforward to show that $(\partial/\partial z) \tilde{\eta}$ is a linear combination of $Y_z + \frac{\mu}{2} Y, \kappa, \ \eta, D_z \eta$, which is orthogonal to $\tilde{\kappa}$, a linear combination of $Y_z + \frac{\mu}{2} Y, \eta, \ \xi$, by (4.8). $\square$

Remark 4.11. Assume $\tilde{f}$ to be an arbitrary conformally immersed Willmore 2-sphere (without branch points) satisfying $\langle \tilde{\kappa}, \tilde{\kappa} \rangle = 0$. Then $\langle \widetilde{D}_z \tilde{\kappa}, \widetilde{D}_{\bar{z}} \tilde{\kappa} \rangle (\mathrm{d}z)^4$ is a globally defined holomorphic form:

$$\begin{aligned} \langle \widetilde{D}_z \tilde{\kappa}, \widetilde{D}_{\bar{z}} \tilde{\kappa} \rangle_{\bar{z}} &= \langle \widetilde{D}_{\bar{z}} \widetilde{D}_z \tilde{\kappa}, \widetilde{D}_{\bar{z}} \tilde{\kappa} \rangle + \langle \widetilde{D}_z \tilde{\kappa}, \widetilde{D}^2_{\bar{z}} \tilde{\kappa} \rangle \\ &= \langle \widetilde{D}_z \widetilde{D}_{\bar{z}} \tilde{\kappa} - 2 \langle \tilde{\kappa}, \bar{\tilde{\kappa}} \rangle \tilde{\kappa}, \widetilde{D}_{\bar{z}} \tilde{\kappa} \rangle + \langle \widetilde{D}_z \tilde{\kappa}, -\bar{\tilde{s}} \tilde{\kappa} \rangle = 0. \end{aligned}$$

So the same conclusion follows. Combining with previous observation, we have established $\langle \widetilde{D}^i_{\bar{z}} \widetilde{D}^p_z \tilde{\kappa}, \widetilde{D}^j_{\bar{z}} \widetilde{D}^q_z \tilde{\kappa} \rangle = 0, \ \forall \ p + q \leq 1$. This vanishing result has a remarkable generalization in the next chapter.

4.3 Not-vanishing results

Contrasting to above results, under the same assumptions in the last remark, generally we can not determine whether $\langle \widetilde{D}_z \tilde{\kappa}, \widetilde{D}_z \tilde{\kappa} \rangle$ vanishes identically or not . But here we know more. $\tilde{f}$ is a certain adjoint transform of a Willmore 2-sphere, whose Hopf differential takes a special expression. This enables us to obtain a rather surprising *not-vanishing* result.

Proposition 4.12. *Let f be an immersed Willmore 2-sphere satisfying $\langle \kappa, \kappa \rangle \not\equiv 0$, and $\tilde{f}$ is the unique adjoint Willmore surface constructed before. Then*

$$\langle \widetilde{D}_z \tilde{\kappa}, \widetilde{D}_z \tilde{\kappa} \rangle \not\equiv 0.$$

Proof. As we have shown in the proof of Lemma 4.10,

$$\widetilde{D}_z \tilde{\eta} = \tilde{\eta}_z, \quad \widetilde{D}_z \tilde{\kappa} = \tilde{\kappa}_z, \quad D_z \eta = \eta_z, \quad D_z \xi = \xi_z.$$

Differentiate both sides of (4.5), one obtains

$$\sigma \widetilde{D}_z \tilde{\kappa} = (\cdots) \big(Y_z + \frac{\mu}{2} Y \big) + (\cdots) \eta + (\cdots) \xi + 2\zeta,$$

where $\zeta := \xi_z + \lambda\kappa$, $\lambda := \frac{1}{2}\rho_z + \frac{\mu}{4}\rho$. As a consequence of (4.8), $\langle \xi_z, \xi\rangle = \langle \xi_z, \eta\rangle = 0$. Thus

$$\langle \zeta, \xi\rangle = \langle \zeta, \eta\rangle = 0, \tag{4.10}$$

and

$$\langle \widetilde{D}_z\tilde{\kappa}, \widetilde{D}_z\tilde{\kappa}\rangle = \frac{4}{\sigma^2}\langle \zeta, \zeta\rangle.$$

Note that λ satisfies

$$\begin{aligned}
\lambda_{\bar z} &= \frac{1}{2}\rho_{\bar z z} + \frac{\mu}{4}\rho_{\bar z} + \frac{\mu_{\bar z}}{4}\rho \\
&= \frac{1}{2}\Big(\bar\mu\rho + 4\langle \eta, \bar\kappa\rangle\Big)_z + \frac{\mu}{4}\Big(\bar\mu\rho + 4\langle \eta, \bar\kappa\rangle\Big) + \frac{\mu_{\bar z}}{4}\rho \\
&= \bar\mu\Big(\frac{1}{2}\rho_z + \frac{\mu}{4}\rho\Big) + 2\langle D_z\eta, \bar\kappa\rangle + 2\langle \eta, D_z\bar\kappa\rangle + \mu\langle \eta, \bar\kappa\rangle + \Big(\frac{\bar\mu_z}{2} + \frac{\mu_{\bar z}}{4}\Big)\rho \\
&= \bar\mu\lambda + 2\langle \xi - \frac{\rho}{2}\kappa, \bar\kappa\rangle + 2\langle \eta, \bar\eta\rangle + \frac{1}{4}\Big(2\rho + \bar\rho + 6\langle \kappa, \bar\kappa\rangle\Big)\rho \\
&= \bar\mu\lambda + 2\langle \xi, \bar\kappa\rangle + \frac{1}{2}\rho\Big(\rho + \langle \kappa, \bar\kappa\rangle\Big) + \underbrace{\Big(2\langle \eta, \bar\eta\rangle + \frac{1}{4}|\rho|^2\Big)}_{=\frac{1}{4}\sigma^2}
\end{aligned} \tag{4.11}$$

by (3.12). Together with (4.7), there is

$$\begin{aligned}
D_{\bar z}\zeta &= D_{\bar z}(D_z\xi + \lambda\kappa) \\
&= D_z D_{\bar z}\xi - 2\langle \xi, \bar\kappa\rangle\kappa + D_{\bar z}(\lambda\kappa) \\
&= D_z\Big[\frac{\bar\mu}{2}\xi + \Big(\rho + \langle \kappa, \bar\kappa\rangle\Big)\eta\Big] - 2\langle \xi, \bar\kappa\rangle\kappa + D_{\bar z}(\lambda\kappa) \\
&= \frac{\bar\mu}{2}(\zeta - \lambda\kappa) + \frac{\bar\mu_z}{2}\xi + \Big(\rho + \langle \kappa, \bar\kappa\rangle\Big)\Big(\xi - \frac{\rho}{2}\kappa\Big) + \Big(\rho + \langle \kappa, \bar\kappa\rangle\Big)_z\eta \\
&\quad -2\langle \xi, \bar\kappa\rangle\kappa + \lambda\Big(\eta - \frac{\bar\mu}{2}\kappa\Big) + \lambda_{\bar z}\kappa \\
&= \frac{\bar\mu}{2}\zeta + (\cdots)\xi + (\cdots)\eta + \Big[\lambda_{\bar z} - \bar\mu\lambda - 2\langle \xi, \bar\kappa\rangle - \frac{1}{2}\rho\Big(\rho + \langle \kappa, \bar\kappa\rangle\Big)\Big]\kappa \\
&= \frac{\bar\mu}{2}\zeta + (\cdots)\xi + (\cdots)\eta + \frac{\sigma^2}{4}\kappa.
\end{aligned} \tag{4.12}$$

We prove the conclusion by contradiction. Suppose $\langle \zeta, \zeta\rangle \equiv 0$ on an open subset of M_0 where σ and $\langle \kappa, \kappa\rangle$ have no zeros. Then

$$0 = \langle \zeta, \zeta\rangle_{\bar z} = 2\langle D_{\bar z}\zeta, \zeta\rangle = \frac{\sigma^2}{4}\langle \kappa, \zeta\rangle$$

by (4.10)(4.12) and $\langle \zeta, \zeta\rangle = 0$ as assumed. So $\langle \kappa, \zeta\rangle \equiv 0$ on this open subset. Differentiate this equality once more, there will be

$$\begin{aligned}
0 &= \langle D_{\bar z}\kappa, \zeta\rangle + \langle \kappa, D_{\bar z}\zeta\rangle \\
&\qquad = \langle \eta - \frac{\bar\mu}{2}\kappa, \zeta\rangle + \langle \kappa, \frac{\bar\mu}{2}\zeta + (\cdots)\xi + (\cdots)\eta + \frac{\sigma^2}{4}\kappa\rangle = \frac{\sigma^2}{4}\langle \kappa, \kappa\rangle \neq 0.
\end{aligned}$$

This contradiction finishes our proof. □

Now we have a Willmore 2-sphere satisfying $\langle\tilde{\kappa},\tilde{\kappa}\rangle \equiv 0$, $\langle\widetilde{D}_z\tilde{\kappa},\widetilde{D}_z\tilde{\kappa}\rangle \not\equiv 0$. What can we do with it? By intuition one might consider the following quadratic equation about t as analogy to (4.1):

$$\langle\widetilde{D}_{\bar{z}}\widetilde{D}_z\tilde{\kappa} + \frac{t}{2}\widetilde{D}_z\tilde{\kappa}, \widetilde{D}_{\bar{z}}\widetilde{D}_z\tilde{\kappa} + \frac{t}{2}\widetilde{D}_z\tilde{\kappa}\rangle = 0.$$

It is easy to show that the discriminant of this equation again defines a holomorphic form on M. If we assume $\tilde{f}$ is a well-defined map without singularities (branch points), this form must vanish. Hence there is a unique root t, which is found to satisfy the *co-touching condition* that $t_z - \frac{1}{2}t^2 - \tilde{s} = 0$. So we can construct a special adjoint transform of $\tilde{f}$. Motivated by these observations, one might conjecture that there is such a sequence of Willmore 2-spheres, each of them being an adjoint transform of the previous one and satisfies some (non-)isotropic conditions. The exact statement as well as the verification will be presented in the next chapter.

Chapter 5

Willmore 2-spheres: m-isotropic case

In the last chapter, we find the isotropic conditions on the Hopf differential and its derivatives are important to the study of Willmore 2-spheres. So our discussion is naturally divided into two cases. In the first case, the iterated derivative of κ up to some order is not isotropic; in the second, all such derivatives span an isotropic subbundle. We will focus on the first case in this chapter and generalize our results in the special case $\langle\kappa,\kappa\rangle \not\equiv 0$. An orthogonal frame will be constructed, a sequence of Willmore 2-spheres will be obtained. The technical point is still the construction of holomorphic forms and vanishing theorems.

5.1 Isotropic conditions and vanishing theorem

Definition 5.1.

1. An immersed surface $M \to \mathbb{S}^n$ is called *m-isotropic* if it satisfies $\langle D_z^i\kappa, D_z^i\kappa\rangle \equiv 0, \forall\ 0 \le i \le m-1$ for some natural number m. Here $D_z^{i+1}\kappa := D_z D_z^i\kappa, D_z^0\kappa := \kappa$.

2. A surface is *strict m-isotropic* if it is m-isotropic yet $\langle D_z^m\kappa, D_z^m\kappa\rangle \not\equiv 0$.

3. Furthermore, a surface whose $\langle\kappa,\kappa\rangle$ has only isolated zeroes will be called *strict 0-isotropic*. If all such inner products as $\langle D_z^i\kappa, D_z^i\kappa\rangle$ vanish, the surface will be called *totally isotropic*.

Remark 5.2. Our m-isotropic condition corresponds exactly to $(m+1)$-*isotropic* surface defined in [20] according to the following lemma. It is also easy to show that although our definitions are given with respect to a chosen coordinate z, they are independent to such choices, hence well-defined.

Lemma 5.3. *If an immersed surface is m-isotropic, there will be*

$$\langle D_z^p\kappa, D_z^q\kappa\rangle \equiv 0,\ \forall\ p,q\geq 0, p+q\leq 2m-1.$$

Proof. Without loss of generality we always assume $p\geq q$, and prove the conclusion by induction on the difference $r=|p-q|$. It is obviously true for $r=0,1$. Suppose this holds for all $r\leq k$. Then for $p+q\leq 2m-1$ and $p-q=k+1$, we have

$$\langle D_z^{q+k+1}\kappa, D_z^q\kappa\rangle = \langle D_z^{q+k}\kappa, D_z^q\kappa\rangle_z - \langle D_z^{q+k}\kappa, D_z^{q+1}\kappa\rangle = 0.$$

By inductive assumption. This completes the proof. □

In the rest of this section we assume $M\cong S^2$. This global assumption will yields stronger vanishing results. As preparation we set

$$\Pi_r := \mathrm{Span}^{\mathbb{C}}\{D_z^i\kappa, D_{\bar z}D_z^i\kappa : 0\leq i\leq r\}.$$

Note they are independent to the choice of coordinates, hence well-defined. They may be regarded as sub-bundles of the complexified normal bundle degenerating at some points. By $\psi\in\Pi_r$ we mean ψ is a (local) section of Π_r. This should not cause confusion.

Theorem 5.4. *Assume that $f: M\to\mathbb{S}^n$ is a m-isotropic Willmore immersion and $M\cong S^2$. Then*

$$\Omega\begin{bmatrix} i & j\\ p & q\end{bmatrix} := \langle D_{\bar z}^i D_z^p\kappa, D_{\bar z}^j D_z^q\kappa\rangle \equiv 0,\quad \forall\, i+j\leq 2, p+q\leq 2m-1, i,j,p,q\geq 0.$$

Generally, $\langle\psi_p,\psi_q\rangle = 0,\ \forall\ \psi_p\in\Pi_p, \psi_q\in\Pi_q, p+q\leq 2m-1$.

Corollary 5.5. *The conclusions of Theorem 5.4 implies*

1. *$D_{\bar z}(\Pi_i)\subseteq\Pi_i,\ \forall\ 0\leq i\leq 2m;\quad D_z(\Pi_{j-1})\subseteq\Pi_j,\ \forall\ 1\leq j\leq 2m$.*
2. *Let $\{D_1,\dots,D_r\}$ be a series of operators either being D_z or $D_{\bar z}$, and the multiplicity of D_z is i, $i\leq 2m$. Then $(D_1\cdots D_r)\kappa\in\Pi_i$.*
3. *$D_{\bar z}^j D_z^i\kappa - D_z^i D_{\bar z}^j\kappa\in\Pi_{i-1},\ \forall\ 1\leq i\leq 2m,\ j\in\mathbb{N}$.*

Corollary 5.6. *After Theorem 5.4, consider the case $p+q=2m$, we have*

1. $\Omega\begin{bmatrix} i & j\\ m{-}r & m{+}r\end{bmatrix} = (-1)^r\cdot\Omega\begin{bmatrix} i & j\\ m & m\end{bmatrix}$.
2. $\Omega\begin{bmatrix} i & j\\ m{-}r & m{+}r\end{bmatrix}(\mathrm{d}z)^{2m+3}(\mathrm{d}\bar z)^{i+j-1}$ *is independent to the choice of z.*

Proof. To show that Theorem 5.4 implies Corollary 5.5, one need only to use the Ricci equation (1.3c) repeatedly. Here we skip these straightforward yet tedious verification. Corollary 5.6 is also easy to prove. In the following we will take these facts as granted and prove Theorem 5.4, together with these corollaries, by induction on m.

The first case is $m=1$, i.e. $\langle\kappa,\kappa\rangle\equiv 0$, for which we have verified these conclusions in Remark 4.11. Now assume the theorem is true for $m\le l$, and consider the case $m=l+1$. First $\Omega\begin{bmatrix}0&0\\l&l\end{bmatrix}=0$ implies $\Omega\begin{bmatrix}1&0\\l&l\end{bmatrix}=0$. Since Corollary 5.5 is also true for $m\le l$ as assumed, $\forall\ 1\le t\le 2l$ we have

$$\langle D_{\bar z}^2D_z^t\kappa, D_z^{2l-t}\kappa\rangle=\langle D_z^tD_{\bar z}^2\kappa+\Pi_{t-1}\text{-part}, D_z^{2l-t}\kappa\rangle=\langle D_z^t(-\frac{\bar s}{2}\kappa), D_z^{2l-t}\kappa\rangle=0$$

by the inductive assumption and Lemma 5.3. Note this is also true when $t=0$. So we have shown that $\Omega\begin{bmatrix}2&0\\t&2l\text{–}t\end{bmatrix}=0, \forall\ 0\le t\le 2l$. Especially,

$$\Omega\begin{bmatrix}2&0\\l&l\end{bmatrix}=0. \tag{5.1}$$

By differentiation and inductive assumptions we know more:

$$\Omega\begin{bmatrix}1&1\\l&l\end{bmatrix}=\frac{\partial}{\partial\bar z}\Omega\begin{bmatrix}1&0\\l&l\end{bmatrix}-\Omega\begin{bmatrix}2&0\\l&l\end{bmatrix}=0. \tag{5.2}$$

$$\begin{aligned}\Omega\begin{bmatrix}1&1\\l{+}1&l\end{bmatrix} &= \langle D_{\bar z}D_z^{l+1}\kappa, D_{\bar z}D_z^l\kappa\rangle\\ &= \langle D_zD_{\bar z}(D_z^l\kappa)+2\langle D_z^l\kappa,\kappa\rangle\bar\kappa-2\langle D_z^l\kappa,\bar\kappa\rangle\kappa, D_{\bar z}D_z^l\kappa\rangle\\ &= \frac{1}{2}\frac{\partial}{\partial z}\langle D_{\bar z}D_z^l\kappa, D_{\bar z}D_z^l\kappa\rangle=0. \end{aligned}\tag{5.3}$$

$$\begin{aligned}\Omega\begin{bmatrix}1&0\\l\text{–}1&l{+}1\end{bmatrix} &= \langle D_{\bar z}D_z^{l-1}\kappa, D_z^{l+1}\kappa\rangle\\ &= \frac{\partial}{\partial z}\underbrace{\langle D_{\bar z}D_z^{l-1}\kappa, D_z^l\kappa\rangle}_{=0}-\langle D_zD_{\bar z}D_z^{l-1}\kappa, D_z^l\kappa\rangle\\ &= -\langle D_{\bar z}D_z(D_z^{l-1}\kappa)-2\langle D_z^{l-1}\kappa,\kappa\rangle\bar\kappa+2\langle D_z^{l-1}\kappa,\bar\kappa\rangle\kappa, D_z^l\kappa\rangle\\ &= -\Omega\begin{bmatrix}1&0\\l&l\end{bmatrix}=0. \end{aligned}\tag{5.4}$$

$$\begin{aligned}\Omega\begin{bmatrix}2&0\\l&l{+}1\end{bmatrix} &= \langle D_{\bar z}^2D_z^l\kappa, D_z^{l+1}\kappa\rangle=\langle D_z^lD_{\bar z}^2\kappa+\Pi_{l-1}\text{-part}, D_z^{l+1}\kappa\rangle\\ &= \sum_{i=0}^{l}(\cdots)\Omega\begin{bmatrix}0&0\\i&l{+}1\end{bmatrix}+(\cdots)\Omega\begin{bmatrix}1&0\\l\text{–}1&l{+}1\end{bmatrix}\\ &\quad +\langle\Pi_{l-2}\text{-part}, D_z^{l+1}\kappa\rangle\\ &= 0. \end{aligned}\tag{5.5}$$

Next we assert that $\langle D_{\bar z}D_z^l\kappa, D_z^{l+1}\kappa\rangle(\mathrm{d}z)^{2l+4}$ is a globally defined and holomorphic form on $M\cong S^2$, hence

$$\Omega\begin{bmatrix}1 & 0\\ l & l{+}1\end{bmatrix}=0. \tag{5.6}$$

Observe the following facts:

1. $D_{\bar z}D_z^l\kappa(\mathrm{d}z)^{l+\frac{3}{2}}(\mathrm{d}\bar z)^{\frac12}$ is a vector-valued form modulo Π_{l-1} and $D_z^l\kappa$;

2. $D_z^{l+1}\kappa(\mathrm{d}z)^{l+\frac{5}{2}}(\mathrm{d}\bar z)^{\frac12}$ is a vector-valued form modulo $D_z^i\kappa,\ i\le l$.

Since $D_{\bar z}D_z^l\kappa$ is orthogonal to $D_z^i\kappa,\ i\le l$, $D_z^{l+1}\kappa$ is orthogonal to Π_{l-1} and $D_z^l\kappa$ by (5.4) and Lemma 5.3, the scalar product $\langle D_{\bar z}D_z^l\kappa, D_z^{l+1}\kappa\rangle$ defines a differential form. It is holomorphic because

$$\frac{\partial}{\partial\bar z}\Omega\begin{bmatrix}1 & 0\\ l & l{+}1\end{bmatrix}=\Omega\begin{bmatrix}1 & 1\\ l & l{+}1\end{bmatrix}+\Omega\begin{bmatrix}2 & 0\\ l & l{+}1\end{bmatrix}=0$$

by (5.3)(5.5). The vanishing of any holomorphic form over S^2 implies (5.6).

Fix $m=l+1$. Now it is easy to verify by induction on i that

$$\begin{aligned}0=\Omega\begin{bmatrix}1 & 0\\ l{-}i & l{+}i\end{bmatrix}&=\Omega\begin{bmatrix}1 & 0\\ l{+}i & l{-}i\end{bmatrix}=\Omega\begin{bmatrix}1 & 0\\ l{-}i & l{+}i{+}1\end{bmatrix}=\Omega\begin{bmatrix}1 & 0\\ l{+}i & l{-}i{+}1\end{bmatrix}\\ =\Omega\begin{bmatrix}1 & 1\\ l{-}i & l{+}i\end{bmatrix}&=\Omega\begin{bmatrix}1 & 1\\ l{-}i & l{+}i{+}1\end{bmatrix}=\Omega\begin{bmatrix}2 & 0\\ l{-}i & l{+}i{+}1\end{bmatrix}=\Omega\begin{bmatrix}2 & 0\\ l{+}i & l{-}i{+}1\end{bmatrix}.\end{aligned}$$

Here equations (5.1)-(5.6) serve as initial cases when i is small. Now the proof is done. □

Corollary 5.7. *Under the same assumptions of Theorem 5.4, we have*

1. $\Delta_1 := \Big(\langle D_{\bar z}D_z^m\kappa, D_z^m\kappa\rangle^2-\langle D_z^m\kappa, D_z^m\kappa\rangle\langle D_{\bar z}D_z^m\kappa, D_{\bar z}D_z^m\kappa\rangle\Big)(\mathrm{d}z)^{4m+6}$ *is globally defined and holomorphic, thus vanishes identically.*

2. $g:=\langle D_z^m\kappa, D_z^m\kappa\rangle$ *satisfies equation* (4.3), *i.e.* $\frac12 g_{\bar z}^2-gg_{\bar z\bar z}-\bar s g^2=0$. *If* $g\not\equiv 0$, *it does not vanish on an open dense subset* $M_0\subseteq M$, *and* $\bar\mu := -(\ln g)_{\bar z}$ *satisfies equation* (3.7a): $\mu_z-\frac12\mu^2-s=0$. *The conclusion of Lemma 4.3 about the zeros of g holds at here.*

Proof. Δ_1 is globally defined due to Corollary 5.6. To show the holomorphicity, we compute

$$\begin{aligned}&\Big(\langle D_{\bar z}D_z^m\kappa, D_z^m\kappa\rangle^2-\langle D_z^m\kappa, D_z^m\kappa\rangle\langle D_{\bar z}D_z^m\kappa, D_{\bar z}D_z^m\kappa\rangle\Big)_{\bar z}\\ &=2\langle D_{\bar z}D_z^m\kappa, D_z^m\kappa\rangle\langle D_{\bar z}^2D_z^m\kappa, D_z^m\kappa\rangle-2\langle D_z^m\kappa, D_z^m\kappa\rangle\langle D_{\bar z}^2D_z^m\kappa, D_{\bar z}D_z^m\kappa\rangle.\end{aligned}$$

By Corollary 5.5 and the Willmore condition, there holds the following equality modulo Π_{m-1}-part:

$$D_{\bar{z}}^2 D_z^m \kappa \equiv D_z^m D_{\bar{z}}^2 \kappa \equiv -\frac{\bar{s}}{2} D_z^m \kappa.$$

Invoking Theorem 5.4 we obtain

$$\begin{aligned}
&\Big(\langle D_{\bar{z}} D_z^m \kappa, D_z^m \kappa\rangle^2 - \langle D_z^m \kappa, D_z^m \kappa\rangle\langle D_{\bar{z}} D_z^m \kappa, D_{\bar{z}} D_z^m \kappa\rangle\Big)_{\bar{z}} \\
&= 2\,\Omega \begin{bmatrix} 1 & 0 \\ m & m \end{bmatrix} \Big(-\frac{\bar{s}}{2}\Omega \begin{bmatrix} 0 & 0 \\ m & m \end{bmatrix} + \langle \Pi_{m-1}\text{-part}, D_z^m \kappa\rangle\Big) \\
&\quad - 2\,\Omega \begin{bmatrix} 0 & 0 \\ m & m \end{bmatrix} \Big(-\frac{\bar{s}}{2}\Omega \begin{bmatrix} 1 & 0 \\ m & m \end{bmatrix} + \langle \Pi_{m-1}\text{-part}, D_{\bar{z}} D_z^m \kappa\rangle\Big) \\
&= 0.
\end{aligned}$$

Δ_1 is holomorphic, hence vanishes. Since we have found that $\langle D_{\bar{z}}^2 D_z^m \kappa, D_z^m \kappa\rangle = \langle -\frac{\bar{s}}{2} D_z^m \kappa, D_z^m \kappa\rangle$, $\Delta_1 = 0$ implies (4.3). Other conclusions follows easily. Note that whether $g = 0$ or not is a property well-defined due to Corollary 5.6. $\square$

5.2 Orthogonal frames and isotropic subspaces

Let $f : M \to \mathbb{S}^n$ be an immersed Willmore 2-sphere that is strict m-isotropic, i.e. $M \cong S^2$, and

$$\langle \kappa, \kappa\rangle = \cdots = \langle D_z^{m-1}\kappa, D_z^{m-1}\kappa\rangle = 0, \quad \langle D_z^m \kappa, D_z^m \kappa\rangle \not\equiv 0.$$

Obviously $\{\kappa, D_z\kappa, \ldots, D_z^{m-1}\kappa\}$ span an isotropic subbundle of the complexified normal bundle at generic points. We shall construct more orthogonal frame vectors which are also isotropic. This is connected with construction of more holomorphic forms, and based on old and new vanishing results.

For convenience denote $\phi := D_z^m \kappa$, $g := \langle D_z^m \kappa, D_z^m \kappa\rangle$, $\bar{\mu} := -g_{\bar{z}}/g$. On the open dense subset M_0 where $g \neq 0$, $\bar{\mu}$ defines an analytic connection 1-form. Set

$$\phi_1 := D_{\bar{z}}\phi + \frac{\bar{\mu}}{2}\phi = D_{\bar{z}}\phi - \frac{\langle D_{\bar{z}}\phi, \phi\rangle}{\langle \phi, \phi\rangle}\phi.$$

It follows $\langle \phi_1, \phi\rangle = 0$. By Theorem 5.4 and the corollaries it is easy to establish

Lemma 5.8. *ϕ_1 has the following properties:*

i.) $\phi_1 (\mathrm{d}z)^{m+\frac{3}{2}} (\mathrm{d}\bar{z})^{\frac{1}{2}}$ *is a well-defined vector-valued form modulo* Π_{m-1}.

ii.) $D_{\bar{z}}\phi_1 = \dfrac{\bar{\mu}}{2}\phi_1 \pmod{\Pi_{m-1}}$.

iii.) $\langle \phi_1, \phi_1\rangle = \langle \phi_1, \Pi_m\rangle = 0$.

To obtain more orthogonal frames, we differentiate ϕ_1 and take the component orthogonal to ϕ, i.e.

$$\phi_2 := D_z\phi_1 - \frac{\langle D_z\phi_1, \phi\rangle}{\langle \phi, \phi\rangle}\phi.$$

Hence $\langle \phi_2, \phi\rangle = \langle \phi_2, \phi_1\rangle = 0$. There is also $\langle \phi_2, \Pi_{m-1}\rangle = 0$ due to $\langle \phi, \Pi_{m-1}\rangle = 0 = \langle \phi_1, \Pi_m\rangle$. Now we want to show $\langle \phi_2, \phi_2\rangle = 0$. It suffices to prove

Lemma 5.9.

$$\Delta_2 := \Big(\langle D_z\phi_1, \phi\rangle^2 - \langle \phi, \phi\rangle\langle D_z\phi_1, D_z\phi_1\rangle\Big)(\mathrm{d}z)^{4m+8} = \langle \phi, \phi\rangle\langle \phi_2, \phi_2\rangle(\mathrm{d}z)^{4m+8}$$

is globally defined and holomorphic, thus vanishes identically.

Proof. We have established that $\phi_1(\mathrm{d}z)^{m+\frac{3}{2}}(\mathrm{d}\bar{z})^{\frac{1}{2}}$ is well-defined modulo Π_{m-1}. This immediately implies that $\phi_2(\mathrm{d}z)^{m+\frac{5}{2}}(\mathrm{d}\bar{z})^{\frac{1}{2}}$ is a vector-valued form modulo Π_{m-1} and ϕ_1. Thus Δ_2 is independent to the choice of coordinates and well-defined on M_0. On this subset we compute

$$\begin{aligned} D_{\bar{z}}\phi_2 &= D_{\bar{z}}D_z\phi_1 - D_{\bar{z}}\big((\cdots)\phi\big) \\ &= D_z\big(D_{\bar{z}}\phi_1\big) - 2\langle \phi_1, \bar{\kappa}\rangle\kappa - D_{\bar{z}}\big((\cdots)\phi\big) \\ &\equiv \frac{\bar{\mu}}{2}\phi_2 + (\cdots)\phi_1 + \delta_1\phi \quad (\mathrm{mod}\ \Pi_{m-1}). \end{aligned}$$

To determine δ_1, we note $\langle D_{\bar{z}}\phi_2, \phi\rangle = \delta_1\langle \phi, \phi\rangle$. On the other hand,

$$\langle D_{\bar{z}}\phi_2, \phi\rangle = -\langle \phi_2, D_{\bar{z}}\phi\rangle = -\langle \phi_2, \phi_1 - \frac{\bar{\mu}}{2}\phi\rangle = 0.$$

Because $\langle \phi, \phi\rangle \neq 0$, there must be $\delta_1 = 0$ and $D_{\bar{z}}\phi_2 \equiv \frac{\bar{\mu}}{2}\phi_2 + (\cdots)\phi_1$ (mod Π_{m-1}). As a result,

$$\langle \phi_2, \phi_2\rangle_{\bar{z}} = \bar{\mu}\langle \phi_2, \phi_2\rangle,\ \langle \phi, \phi\rangle_{\bar{z}} = -\bar{\mu}\langle \phi, \phi\rangle \quad \Longrightarrow \quad \Delta_2 \text{ is holomorphic on } M_0.$$

Finally we want to show Δ_2 extends continously to the zeros of $g = \langle \phi, \phi\rangle$, hence to the whole M. Suppose $g = 0$ at $z = 0$. Around this point we have

$$\langle D_z\phi_1, \phi\rangle = -\langle \phi_1, D_z\phi\rangle = -\langle D_{\bar{z}}\phi, D_z\phi\rangle - \frac{\bar{\mu}}{2}\langle \phi, \phi\rangle = -\langle D_{\bar{z}}\phi, D_z\phi\rangle + \frac{1}{2}g_{\bar{z}}.$$

So this part is without singularity. The other part is

$$\begin{aligned} \langle \phi, \phi\rangle\langle \phi_2, \phi_2\rangle &= g\bar{\mu}\langle D_z\phi, D_zD_{\bar{z}}\phi\rangle + g\bar{\mu}_z\langle \phi, D_zD_{\bar{z}}\phi\rangle + \frac{1}{4}g\bar{\mu}_z\cdot\bar{\mu}g_z \\ &\quad + g\frac{\bar{\mu}^2}{4}\langle D_z\phi, D_z\phi\rangle + \frac{1}{4}(g\bar{\mu}_z)^2 + g\langle D_zD_{\bar{z}}\phi, D_zD_{\bar{z}}\phi\rangle. \end{aligned}$$

Apply Lemma 4.3 and Lemma 4.7 at here, we find Δ_2 extends continuously. □

The construction above has a natural generalization as stated below.

Theorem 5.10. *There is a series of frame vectors $\{\phi_i, i = 1, 2, \dots\}$ satisfying*

i.) *ϕ_1, ϕ_2 are given as before. Other ones are defined recursively by*

$$\phi_{i+1} := D_z\phi_i - \frac{\langle D_z\phi_i, \phi\rangle}{\langle \phi, \phi\rangle}\phi.$$

Furthermore, $\phi_i(\mathrm{d}z)^{m+i+\frac{1}{2}}(\mathrm{d}\bar{z})^{\frac{1}{2}}$ is a vector-valued differential form modulo $\phi_1, \dots, \phi_{i-1}$ and Π_{m-1}.

ii.) $D_{\bar{z}}\phi_i \equiv \dfrac{\bar{\mu}}{2}\phi_i \pmod{\phi_1, \dots, \phi_{i-1}, \Pi_{m-1}}$.

iii.) *All ϕ_i are orthogonal to Π_m and to each other. Especially, $\langle \phi_i, \phi_i\rangle = 0$.*

Proof. We prove by induction on i. These conclusions are true for $i = 1, 2$. Suppose they hold when $i = 1, \dots, r$. For $i = r + 1$,

$$\phi_{r+1} := D_z\phi_r - \frac{\langle D_z\phi_r, \phi\rangle}{\langle \phi, \phi\rangle}\phi.$$

According to inductive assumptions, $\phi_r(\mathrm{d}z)^{m+r+\frac{1}{2}}(\mathrm{d}\bar{z})^{\frac{1}{2}}$ is well-defined modulo $\phi_1, \dots, \phi_{r-1}$ and Π_{m-1}. In other words, let $\phi'_r, \phi'_{r+1}, \phi'$ be the corresponding frame vectors given with respect to another coordinate w, then there is

$$\phi'_r \equiv \phi_r\Big(\frac{\partial z}{\partial w}\Big)^{m+r+\frac{1}{2}}\Big(\frac{\partial \bar{z}}{\partial \bar{w}}\Big)^{\frac{1}{2}} + \sum_{i=1}^{r-1}(\cdots)\phi_i \pmod{\Pi_{m-1}}.$$

Since $D_z\phi_i = \phi_{i+1} - (\cdots)\phi$, $D_{\bar{z}}\phi = \phi_1 - \frac{\bar{\mu}}{2}\phi$, we have

$$\begin{aligned}
\phi'_{r+1} &= D_w\phi'_r + (\cdots)\phi' \;=\; D_z\phi'_r \cdot \Big(\frac{\partial z}{\partial w}\Big) + (\cdots)\phi' \\
&\equiv D_z\phi_r\Big(\frac{\partial z}{\partial w}\Big)^{m+r+\frac{3}{2}}\Big(\frac{\partial \bar{z}}{\partial \bar{w}}\Big)^{\frac{1}{2}} + (\cdots)\phi_r \\
&\quad + \sum_{i=1}^{r-1}(\cdots)D_z\phi_i + \sum_{i=1}^{r-1}(\cdots)\phi_i + (\cdots)D_{\bar{z}}\phi + (\cdots)\phi \pmod{\Pi_{m-1}} \\
&\equiv \phi_{r+1}\Big(\frac{\partial z}{\partial w}\Big)^{m+r+\frac{3}{2}}\Big(\frac{\partial \bar{z}}{\partial \bar{w}}\Big)^{\frac{1}{2}} + \epsilon_r\phi \pmod{\phi_1, \dots, \phi_r, \Pi_{m-1}}.
\end{aligned}$$

Both of ϕ'_{r+1} and ϕ_{r+1} are orthogonal to the subspace spanned by ϕ and Π_{m-1}, which is the same as $\mathrm{Span}\{\phi', \Pi_{m-1}\}$. So ϵ_r is zero and i.) is proved.

Follow from inductive assumption ii.) when $i = r$,

$$\begin{aligned}
D_{\bar z}\phi_{r+1} &= D_{\bar z}D_z\phi_r - D_{\bar z}\big((\cdots)\phi\big)\\
&= D_z\big(D_{\bar z}\phi_r\big) + 2\underbrace{\langle\phi_r,\kappa\rangle}_{=0}\bar\kappa - 2\langle\phi_r,\bar\kappa\rangle\kappa - D_{\bar z}\big((\cdots)\phi\big)\\
&= D_z\Big(\frac{\bar\mu}{2}\phi_r + \sum_{i=1}^{r-1}\phi_i + \Pi_{m-1}\text{-part}\Big) - 2\langle\phi_r,\bar\kappa\rangle\kappa - D_{\bar z}\big((\cdots)\phi\big)\\
&\equiv \frac{\bar\mu}{2}\phi_{r+1} + \sum_{i=1}^{r}\phi_i + \delta_r\phi \quad (\mathrm{mod}\ \Pi_{m-1}).
\end{aligned}$$

Similarly we can show $\delta_r = 0$ by

$$\delta_r\langle\phi,\phi\rangle = \langle D_{\bar z}\phi_{r+1},\phi\rangle = -\langle\phi_{r+1},D_{\bar z}\phi\rangle = -\langle\phi_{r+1},\phi_1 - \frac{\bar\mu}{2}\phi\rangle = 0.$$

Hence ii.) is also true for $i = r+1$.

By inductive assumption iii.), $\langle\phi_r,\phi_i\rangle = \langle\phi_r,\Pi_m\rangle = 0,\ \forall\ i\le r$. As a result, $\langle D_z\phi_r,\Pi_{m-1}\rangle = 0 = \langle D_z\phi_r,\phi_r\rangle$, and $\langle D_z\phi_r,\phi_i\rangle = -\langle\phi_r,D_z\phi_i\rangle = -\langle\phi_r,\phi_{i+1}-(\cdots)\phi\rangle = 0,\ \forall\ i\le r-1$. So

$$\langle\phi_{r+1},\Pi_m\rangle = \langle\phi_{r+1},\phi_i\rangle,\quad \forall\ i\le r. \tag{5.7}$$

If we can show ϕ_{r+1} is isotropic, too, then the proof is finished. For this purpose, define analogously

$$\begin{aligned}
\Delta_{r+1} &:= \Big(\langle D_z\phi_r,\phi\rangle^2 - \langle\phi,\phi\rangle\langle D_z\phi_r,D_z\phi_r\rangle\Big)(\mathrm{d}z)^{4m+6+2r}\\
&= \langle\phi,\phi\rangle\langle\phi_{r+1},\phi_{r+1}\rangle(\mathrm{d}z)^{4m+6+2r}
\end{aligned}$$

Conclusion i.) guarantees that Δ_{r+1} is independent to the choice of coordinates. Conclusion ii.) implies that on M_0

$$\langle\phi_{r+1},\phi_{r+1}\rangle_{\bar z} = \bar\mu\langle\phi_{r+1},\phi_{r+1}\rangle,\ \langle\phi,\phi\rangle_{\bar z} = -\bar\mu\langle\phi,\phi\rangle \implies \Delta_{r+1}\text{ is holomorphic.}$$

Note we have used (5.7) at here.

The final point is to show that Δ_{r+1} extends to the zeros of $g = \langle\phi,\phi\rangle$ and has no poles. Note that locally the expression of $\phi_1,\ldots,\phi_r$ involves $\bar\mu = -g_{\bar z}/g = -\hat g_{\bar z}/\hat g$ and its derivatives besides other analytic functions, where the decomposition $g = z^l\cdot\hat g$ is given as in Lemma 4.3. Thus $\langle\phi,\phi\rangle\langle\phi_{r+1},\phi_{r+1}\rangle$ is a quotient of two real analytic functions and holomorphic outside the zero set of the denominator. Moreover the denominator may be chosen to be $\hat g^l = (az+b\bar z)^{2l}+\cdots$. By the Extension Lemma, all singularities are removable. Δ_{r+1} vanishes, so does $\langle\phi_{r+1},\phi_{r+1}\rangle$. □

5.3 Derived Willmore surface

Let $f : M \to \mathbb{S}^n$ be a strict m-isotropic Willmore immersion, $m \geq 0, M \cong S^2$. As showed in Corollary 5.7, $\bar{\mu} := -\big(\ln\langle D_z^m\kappa, D_z^m\kappa\rangle\big)_{\bar{z}}$ satisfies the co-touching condition (3.7a), and $\eta := D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa$ must always be isotropic. Thus $\widehat{Y} = \frac{1}{2}|\mu|^2 Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N$ defines an adjoint Willmore surface $\tilde{f}$ on the open dense subset where $g := \langle D_z^m\kappa, D_z^m\kappa\rangle \neq 0$, which we call the *derived Willmore surface.*

In Chapter 4, we have proved that when $m = 0$, the derived Willmore surface must be a strict $(m+1)$-isotropic and branched conformal immersion. This *ascending phenomenon* is very interesting. One would conjecture that the conclusion holds for arbitrary m, and this is indeed true.

Theorem 5.11 (Ascending Theorem). *Let $M \cong S^2$, immersion F : $M \to \mathbb{S}^n$ is strict m-isotropic and Willmore. Suppose F is not congruent to any minimal surface in $\mathbb{R}^n$. Then the derived Willmore surface $\widetilde{F}$ extends to the whole $M \cong S^2$ as a branched conformal immersion , which is a strict $(m+1)$-isotropic Willmore surfaces.*

We will prove this theorem by establishing a series of lemmas. As a preparation, note $\langle\eta,\kappa\rangle = 0$. By (3.23) and (3.22), we have

$$\tilde{\kappa} = \frac{1}{\sigma}\Big[(\rho_z - \tilde{\mu}\rho)\big(Y_z + \frac{\mu}{2}Y\big) + 2D_z\eta - (\mu + 2\tilde{\mu})\eta + \rho\kappa\Big], \tag{5.8}$$

$$\tilde{\eta} = \widetilde{D}_{\bar{z}}\tilde{\kappa} + \frac{\bar{\tilde{\mu}}}{2}\tilde{\kappa} = \frac{1}{\sigma}\Big[4\langle\eta,\bar{\eta}\rangle\big(Y_z + \frac{\mu}{2}Y\big) - \bar{\rho}\eta\Big], \tag{5.9}$$

where $\rho := \bar{\mu}_z - 2\langle\kappa,\bar{\kappa}\rangle$, $\sigma^2 := 8\langle\eta,\bar{\eta}\rangle + |\rho|^2$, $\tilde{\mu} := 2\sigma_z/\sigma - \mu$ as before. Here σ^2 does not vanish identically, and the formulas above are meaningful at least on the open subset where $\sigma^2 \neq 0$. (Otherwise, the corresponding adjoint transform degenerates to a single point, through which all mean curvature spheres of F must pass. Taking a stereographic projection from this point, we find F is Möbius equivalent to a minimal surface in an affine space $\mathbb{R}^n$. This would contradict with our assumption.)

Lemma 5.12. $D_z^r\kappa = (\frac{\partial}{\partial z})^r\kappa, D_z^r(D_{\bar{z}}\kappa) = (\frac{\partial}{\partial z})^r D_{\bar{z}}\kappa, \quad \forall\ r \leq 2m.$

Proof. The last one of structure equations (1.2) implies that $\psi \in \Gamma(V^\perp \otimes \mathbb{C})$ satisfies $D_z\psi = \psi_z$ iff $\langle\psi,\kappa\rangle = \langle\psi, D_{\bar{z}}\kappa\rangle = 0$. The conclusions follow from this criteria, and Theorem 5.4. $\square$

Lemma 5.13. *The derived Willmore surface $\widetilde{F}$ extends continuously to be a branched conformal immersion of S^2 with isolated branch points.*

Proof. The Willmore condition ensures all geometric quantities on M_0 to be analytic. By Corollary 5.7, for $g = \langle D_z^m\kappa, D_z^m\kappa\rangle$ there is again $\frac{1}{2}g_{\bar{z}}^2 - gg_{\bar{z}\bar{z}} - \bar{s}g = 0$, and $\bar{\mu} = -(\ln g)_{\bar{z}}$ satisfies $\mu_z - \frac{1}{2}\mu^2 - s = 0$. Lemma B.1 holds at here. It tells us that μ has no singularities on an open dense subset of M, which we denote by M'.

On M', $\widetilde{F}$ as a map is well defined and analytic. It fails to be immersion exactly at zeros of $\sigma^2 = 8\langle\eta,\bar\eta\rangle + |\rho|^2$. Note η and ρ satisfy an elliptic differential system (3.13)(3.12):

$$D_{\bar z}\eta = \frac{\bar\mu}{2}\eta, \quad \rho_{\bar z} = \bar\mu\rho + 4\langle\eta,\bar\kappa\rangle.$$

Since $\sigma^2 \not\equiv 0$ on an open dense subset, by Lemma B.3 in the appendix, the common zeros of η and ρ are isolated. So are the zeros of σ as well as branch points of $\widetilde{F}$ on M'.

On the other hand, $M \setminus M'$ consists of zeros of g. Taking coordinate z so that $z = 0$is a zero of g, one may express $g(z) = z^j \cdot \hat g$, where j is a non-negative integer and $\hat g$ is a locally defined function with Taylor expansion $(az + b\bar z)^2 + o(|z|^2)$, $b \neq 0$, by Lemma B.1. This time $\bar\mu = -g_{\bar z}/g = -\hat g_{\bar z}/\hat g$ in a suitable neighborhood of $z = 0$. (Note the other type singularity with $\hat g$ locally non-zero yields a regular μ, which we have treated in the last paragraph.) As in Corollary 4.4, with

$$\bar\mu \cdot \hat g_{\bar z} = -\hat g_{\bar z}^2/\hat g = -2\hat g_{\bar z\bar z} - 2\bar s\hat g = -4b^2 + o(|z|) \tag{5.10}$$

being analytic and nonzero locally, we scale the lift $\widehat{Y} = \frac{1}{2}|\mu|^2 Y + \bar\mu Y_z + \mu Y_{\bar z} + N$ by multiplying $|\hat g_{\bar z}|^2$. The resulted local lift is non-zero and analytic. Denote it as Y'. Let's determine when $z = 0$ is a branch point of $\widetilde{F}$. Compute

$$\langle Y'_z, Y'_{\bar z}\rangle = \frac{1}{2}\sigma^2|\hat g_{\bar z}|^4 = \frac{1}{2}(|\rho|^2 + 8\langle\eta,\bar\eta\rangle)|\hat g_{\bar z}|^4 \simeq \frac{1}{2}(|\rho|^2 + 8\langle\eta,\bar\eta\rangle)|\hat g|^2.$$

The last equality holds up to multiplication by a non-zero analytic function $\hat g/\hat g_{\bar z}^2$. Observe that $\hat g\eta$ is an infinitesimal when $z \to 0$, so it suffices to consider $|\hat g\rho|^2$, which is an analytic quantity. It has nonzero limit at $z = 0$ iff $|\hat g^2\rho|^2$ has non-vanishing fourth order term in its Taylor expansion. The latter is equal to

$$|-\hat g\hat g_{\bar z z} + \hat g_z\hat g_{\bar z} - 2\langle\kappa,\bar\kappa\rangle\hat g^2|^2 = 4|ab|^2 \cdot |az + b\bar z|^4 + o(|z|^4).$$

Thus $z = 0$ is a branch point of $\widetilde{F}$ iff $a = 0$, i.e. $g = z^j \cdot \big(b\bar z^2 + o(|z|)\big)$. Obviously, in this case $z = 0$ is an isolated zero of g, hence an isolated branch point of $\widetilde{F}$. □

Lemma 5.14. *$\widetilde{F}$ is $(m+1)$-isotropic.*

Proof. The case $m = 0$ has been shown in the last chapter. In the following suppose $m \geq 1$. We assert that

$$\widetilde{D}_z^r\tilde\kappa = (\frac{\partial}{\partial z})^r\tilde\kappa \ \in \Pi_{r+1} \oplus \{Y_z + \frac{\mu}{2}Y\}, \quad 0 \leq r \leq 2m-2. \tag{5.11}$$

When $r = 0$ this is self-evident. Suppose this assertion holds for integer r. If $r+1$ is still not greater than $2m-2$, by (5.8)(5.9), the inner product between $\widetilde{D}_z^r\tilde{\kappa} \ \in \Pi_{r+1}$ and either of $\tilde{\kappa}$, $\widetilde{D}_{\bar{z}}\tilde{\kappa} \ \in \Pi_1$ must be zero. The conclusion follows similar to the proof of Lemma 5.12. Consequently, $\langle \widetilde{D}_z^r\tilde{\kappa}, \widetilde{D}_z^r\tilde{\kappa}\rangle = 0$ for any $r \le m-2$. In other words, any adjoint surface must be $(m-1)$-isotropic.

Next let's show it is m-isotropic. Since $0 \le m-1 \le 2m-2$, by (5.11) and Lemma 5.12,

$$\widetilde{D}_z^{m-1}\tilde{\kappa} = (\frac{\partial}{\partial z})^{m-1}\tilde{\kappa} \equiv \frac{2}{\sigma}D_z^m\eta \quad (\text{mod } \Pi_{m-1}, Y_z + \frac{\mu}{2}Y). \tag{5.12}$$

Note we have used $(\frac{\partial}{\partial z})^i\eta = D_z^i\eta,\ \forall\ i \le 2m$ (Lemma 5.12). Corollary 5.7 implies

$$\langle D_z^m\eta, D_z^m\kappa\rangle = \langle D_z^m\eta, D_z^m\eta\rangle = 0. \tag{5.13}$$

Hence

$$\begin{aligned}\langle \widetilde{D}_z^{m-1}\tilde{\kappa}, \widetilde{D}_z^{m-1}\tilde{\kappa}\rangle &= \frac{4}{\sigma^2}\langle D_z^m\eta, D_z^m\eta\rangle \\ &= \frac{4}{\sigma^2}\langle D_{\bar{z}}D_z^m\kappa + \frac{\bar{\mu}}{2}D_z^m\kappa, D_{\bar{z}}D_z^m\kappa + \frac{\bar{\mu}}{2}D_z^m\kappa\rangle = 0.\end{aligned}$$

Moreover, (5.12) implies $\widetilde{D}_z^{m-1}\tilde{\kappa}$ is isotropic and orthogonal to $\Pi_m \oplus \{Y_z + \frac{\mu}{2}Y\} \ni \tilde{\kappa}, \widetilde{D}_{\bar{z}}\tilde{\kappa}$ (Keep in mind we have assumed $m \ge 1$). So $\widetilde{D}_z^m\tilde{\kappa} = (\widetilde{D}_z^{m-1}\tilde{\kappa})_z = (\partial^m/\partial z^m)\tilde{\kappa}$. Similarly, $\langle D_z^m\eta, \Pi_0\rangle = 0$ implies $(D_z^m\eta)_z = D_z^{m+1}\eta$ and

$$\widetilde{D}_z^m\tilde{\kappa} \equiv \frac{2}{\sigma}\Big[D_z^m\xi + (\cdots)D_z^m\eta\Big] \quad (\text{mod } \Pi_{m-1}, Y_z + \frac{\mu}{2}Y).$$

where

$$\xi := D_z\eta + \frac{\rho}{2}\kappa.$$

(These conclusions are trivially true when $m = 0$.) For ξ holds the following facts:

1. $\langle D_z^i\xi, \Pi_j\rangle = 0,\ \forall\ i+j \le 2m-1.$
2. $\langle D_z^i\xi, D_z^j\eta\rangle = 0,\ \forall\ i+j \le 2m$. Especially, $D_z^m\xi$ is orthogonal to ϕ_1.
3. $D_{\bar{z}}\xi = \frac{\bar{\mu}}{2}\xi + \big(\rho + \langle\kappa,\bar{\kappa}\rangle\big)\eta.$

The first two properties are corollaries of (5.13). The last one is shown as in (4.7).

Now we can address our problem and find

$$\langle \widetilde{D}_z^m\tilde{\kappa}, \widetilde{D}_z^m\tilde{\kappa}\rangle = \frac{4}{\sigma^2}\langle D_z^m\xi, D_z^m\xi\rangle,$$

To show $\widetilde{D}_z^m\tilde{\kappa}$ is isotropic, similar to Section 4.2, define

$$\Phi := \langle D_z^m\xi, D_z^m\kappa\rangle(\mathrm{d}z)^{2m+4}, \quad \Psi := \langle D_z^m\kappa, D_z^m\kappa\rangle\langle D_z^m\xi, D_z^m\xi\rangle(\mathrm{d}z)^{4m+8}. \tag{5.14}$$

We want to prove that Φ and Ψ are holomorphic forms globally defined over $M \cong S^2$, hence vanish.

It has been shown that F and $\widetilde{F}$ are both m-isotropic. This fact implies that $\langle\widetilde{D}_z^m\tilde{\kappa}, D_z^m\kappa\rangle$ and $\langle\widetilde{D}_z^m\tilde{\kappa}, \widetilde{D}_z^m\tilde{\kappa}\rangle\langle D_z^m\kappa, D_z^m\kappa\rangle$ each determines a differential form. As a result, Φ and Ψ are well-defined on M except at those branch points.

To show that Φ is holomorphic on this subset, differentiate $D_z^m\xi$ and modulo Π_{m-1}-components, by the Ricci equation (1.3c) one obtains

$$\begin{aligned} D_{\bar z}D_z^m\xi &\equiv D_zD_{\bar z}(D_z^{m-1}\xi) + 2\underbrace{\langle D_z^{m-1}\xi,\kappa\rangle}_{=0}\bar\kappa \equiv \cdots \equiv D_z^m D_{\bar z}\xi \\ &\equiv D_z^m\left[\frac{\bar\mu}{2}\xi + (\cdots)\eta\right] \equiv \frac{\bar\mu}{2}D_z^m\xi + (\cdots)D_z^m\eta \qquad (\mathrm{mod}\ \Pi_{m-1}) \end{aligned}$$

We verify directly that

$$\begin{aligned} \langle D_z^m\xi, D_z^m\kappa\rangle_{\bar z} &= \langle D_{\bar z}D_z^m\xi, D_z^m\kappa\rangle + \langle D_z^m\xi, D_{\bar z}D_z^m\kappa\rangle \\ &= \langle\frac{\bar\mu}{2}D_z^m\xi, D_z^m\kappa\rangle + \langle D_z^m\xi, D_z^m\eta - \frac{\bar\mu}{2}D_z^m\kappa\rangle = 0. \end{aligned}$$

The reader is reminded that $D_z^m\eta - \frac{\bar\mu}{2}D_z^m\kappa$ equals $D_{\bar z}D_z^m\kappa$ only up to Π_{m-1}, yet this part is orthogonal to $D_z^m\xi$.

For Ψ, there is $\langle D_z^m\xi, D_z^m\xi\rangle_{\bar z} = 2\langle D_{\bar z}D_z^m\xi, D_z^m\xi\rangle = \bar\mu\langle D_z^m\xi, D_z^m\xi\rangle$. Combining with $g_{\bar z} = -\bar\mu g$ for $g = \langle D_z^m\kappa, D_z^m\kappa\rangle$, Ψ is also shown to be holomorphic outside branch points.

The final step is to show that Φ, Ψ extends continuously, thus analytically, to the branch points. Indeed we need only to consider the zeros of $g = z^r\hat g$ in a local coordinate, where $\hat g(0) \neq 0$ or $\hat g = \bar z^2 + o(|z|)$. Around $z = 0$, it is straightforward to verify

$$\langle D_z^m\xi, \phi\rangle = -\frac{g_{\bar z}g_z}{4g} + \frac{m+2}{2}\frac{g_{\bar z}^2}{g} + (\cdots) = -\frac{\hat g_{\bar z}}{4g}\cdot g_z + \frac{m+2}{2}\frac{\hat g_{\bar z}}{\hat g}\cdot g_{\bar z} + (\cdots)$$

Terms collected in the brackets are non-singular part. The right hand side is able to be expressed as a quotient of two analytic functions whereby the denominator contains $\bar z^l$ in its Taylor expansion at $z = 0$. By the Extension Lemma, Φ has only removable singularities, thus being holomorphic over $M \cong S^2$. So $\langle D_z^m\xi, \phi\rangle = 0$ is proved. As a by-product we finds the locally defined $g_{\bar z}g_z/g$ is always analytic as well as $g_{\bar z}^2/g$. For Ψ, we have

$$\begin{aligned} g\cdot\langle D_z^m\xi, D_z^m\xi\rangle &= g\bar\mu\langle D_z^{m+1}D_{\bar z}\kappa, D_z^{m+1}\kappa\rangle + g\frac{\bar\mu^2}{4}\langle D_z^{m+1}\kappa, D_z^{m+1}\kappa\rangle \\ &\quad + \frac{m+2}{2}g\bar\mu_z\langle D_z^m\kappa, D_z^{m+1}D_{\bar z}\kappa\rangle + \frac{m+2}{4}g\bar\mu_z\bar\mu\cdot\frac{g_z}{2} \\ &\quad + (\text{analytic part}). \end{aligned}$$

Since $g\bar{\mu} = -g,\ g\bar{\mu}^2 = (g_{\bar{z}}^2)/g,\ g\bar{\mu}_z = (g\bar{\mu})_z - \bar{\mu}g_z = -g_{\bar{z}z} + (g_{\bar{z}}g_z)/g$ are all non-singular, we can also extend Ψ analytically to M.

As a globally defined holomorphic form on $M \cong S^2$, Ψ must vanish identically. Yet g is nonzero on an open dense subset. The only possibility left to us is $\langle D_z^m\xi, D_z^m\xi\rangle \equiv 0$. There follows $\langle \widetilde{D}_z^m\tilde{\kappa}, \widetilde{D}_z^m\tilde{\kappa}\rangle \equiv 0$, and we are done. □

Lemma 5.15. $\langle \widetilde{D}_z^{m+1}\tilde{\kappa}, \widetilde{D}_z^{m+1}\tilde{\kappa}\rangle \not\equiv 0$. *In other words, $\widetilde{F}$ is strict $(m+1)$-isotropic.*

Proof. $\widetilde{D}_z^m\tilde{\kappa}$ is orthogonal to itself and $D_z^m\eta$. So it is orthogonal to $\tilde{\kappa}$ and $\tilde{\eta}$ as well. This ensures $\widetilde{D}_z^{m+1}\tilde{\kappa} = (\widetilde{D}_z^m\tilde{\kappa})_z$. Next we determine $\tilde{\kappa}_z$ as in Section 4.3:

$$\sigma\widetilde{D}_z\tilde{\kappa} = (\cdots)\big(Y_z + \frac{\mu}{2}Y\big) + (\cdots)\eta + (\cdots)\xi + 2\zeta,$$

where $\xi := D_z\eta + \frac{\rho}{2}\kappa,\ \zeta := D_z\xi + \lambda\kappa,\ \lambda := \frac{1}{2}\rho_z + \frac{\mu}{4}\rho.$

Before computing $(\partial^m/\partial z^m)\tilde{\kappa}_z$, we remark that $D_z^i\eta$ and $D_z^i\xi$ are orthogonal to Π_0 for all $i \le 2m$ based on previous results. Hence the differential operator $\partial/\partial z$ coincides with D_z when acting on these objects. Especially, $(\partial^m/\partial z^m)\zeta = D_z^m\zeta \equiv D_z^{m+1}\xi \pmod{\Pi_m}$. We point out the following important facts about ζ:

1. $D_z^m\zeta, D_z^m\kappa$ are orthogonal to $D_z^m\xi, D_z^m\eta$.
2. $\langle D_z^i\zeta, \Pi_j\rangle,\ \forall\ i+j \le 2m-1$. Especially, $D_z^m\zeta$ is orthogonal to Π_{m-1}.
3. $D_{\bar{z}}\zeta = \frac{\bar{\mu}}{2}\zeta + (\cdots)\xi + (\cdots)\eta + \frac{\sigma^2}{4}\kappa.$

It is straightforward to verify the first two facts; the last one is the same as (4.12). By induction we further find

$$\begin{aligned} D_{\bar{z}}D_z^m\zeta &\equiv D_zD_{\bar{z}}(D_z^{m-1}\zeta) + 2\underbrace{\langle D_z^{m-1}\zeta, \kappa\rangle}_{=0}\bar{\kappa} \equiv \cdots \equiv D_z^mD_{\bar{z}}\zeta \\ &\equiv D_z^m\Big[\frac{\bar{\mu}}{2}\zeta + (\cdots)\xi + (\cdots)\eta + \frac{\sigma^2}{4}\kappa\Big] \\ &\equiv \frac{\bar{\mu}}{2}D_z^m\zeta + (\cdots)D_z^m\xi + (\cdots)D_z^m\eta + \frac{\sigma^2}{4}D_z^m\kappa \pmod{\Pi_{m-1}} \end{aligned}$$

After these preparations, we can show

$$\widetilde{D}_z^{m+1}\tilde{\kappa} = \Big(\frac{\partial}{\partial z}\Big)^m(\tilde{\kappa}_z) \equiv \frac{2}{\sigma}D_z^m\zeta + (\cdots)D_z^m\xi + (\cdots)D_z^m\eta. \pmod{\Pi_{m-1}}$$

and $\langle \widetilde{D}_z^{m+1}\tilde{\kappa}, \widetilde{D}_z^{m+1}\tilde{\kappa}\rangle = \frac{4}{\sigma^2}\langle D_z^m\zeta, D_z^m\zeta\rangle$. Let's suppose it equals to zero identically on some open subset and see what will happen. If so, It must vanish on an open subset of M without branch points. When restricted to this small subset, we can use all the formulas before without worrying about singularities. Differentiation yields

$$0 = \langle D_{\bar{z}} D_z^m \zeta, D_z^m \zeta \rangle = \langle \frac{\bar{\mu}}{2} D_z^m \zeta + (\sigma^2/4) D_z^m \kappa, D_z^m \zeta \rangle.$$

Since $\langle D_z^m \zeta, D_z^m \zeta \rangle = 0$ and $\sigma^2 \neq 0$ as assumed, we obtain $0 = \langle D_z^m \kappa, D_z^m \zeta \rangle$. Differentiate once more,

$$\begin{aligned}
0 &= \langle D_z^m \kappa, D_z^m \zeta \rangle_{\bar{z}} = \langle D_{\bar{z}} D_z^m \kappa, D_z^m \zeta \rangle + \langle D_z^m \kappa, D_{\bar{z}} D_z^m \zeta \rangle \\
&= \langle D_z^m \eta - \frac{\bar{\mu}}{2} D_z^m \kappa, D_z^m \zeta \rangle + \langle D_z^m \kappa, \frac{\bar{\mu}}{2} D_z^m \zeta + \frac{1}{4} \sigma^2 D_z^m \kappa \rangle \\
&= \frac{1}{4} \sigma^2 \langle D_z^m \kappa, D_z^m \kappa \rangle \not\equiv 0.
\end{aligned}$$

This is a contradiction. Thus we have proved that $\widetilde{F}$ is strict $(m+1)$-isotropic. $\square$

Appendix A

Touch and co-touch: an interpretation by quaternions

In Chapter 2, we have introduced the notion of touch and co-touch between two contact elements or two surfaces. This is inspired by the conception *touch (from left/right)* put forward by Pedit and Pinkall in the context of quaternions $\mathbb{H}$. Here we briefly review the related constructions, and point out the relationship in between.

The skew field $\mathbb{H}$ might be regarded as $\mathbb{R}^4$ with a fixed orientation, whose standard oriented basis is denoted as $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ with multiplication

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \quad \mathrm{ij} = -\mathrm{ji} = \mathrm{k}.$$

For a quaternion $\lambda = a + b\mathrm{i} + \mathrm{cj} + \mathrm{dk}$, the real part is $\operatorname{Re}\lambda = a$ and the imaginary part $\operatorname{Im}\lambda = b\mathrm{i} + \mathrm{cj} + \mathrm{dk}$. The quaternionic conjugation $\bar{\lambda} = \operatorname{Re}\lambda - \operatorname{Im}\lambda$. Then the imaginary quaternions $\operatorname{Im}\mathbb{H} = \{\lambda \mid \operatorname{Re}\lambda = 0\}$ is identified with $\mathbb{R}^3$. Also note the inner product is now expressed as $\langle\alpha, \beta\rangle = \frac{1}{2}(\alpha\bar{\beta} + \beta\bar{\alpha})$, and formula $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$. In this context one makes the following interesting observation:

Lemma A.1 (the Fundamental lemma in [10]; also Lemma 6 in [7]). *For every oriented real subspace $U \subset \mathbb{H}$ of dimension 2 there are unique vectors N and R satisfying $N^2 = R^2 = -1$ with the property that*

$$U = \{x \in \mathbb{H} \mid Nx = -xR\ \} \tag{A.1}$$

and that left multiplication by N rotates vectors in U by $\pi/2$ in positive direction. Then $U^\perp = \{x \in \mathbb{H} \mid Nx = xR\ \}$ is the orthogonal complement of U and left multiplication by N rotates vectors in $U^\perp$ by $\pi/2$ in positive direction.

Conversely, every pair of vectors N and R satisfying $N^2 = R^2 = -1$ defines, via (A.1)*, an oriented 2-plane.*

Remark A.2. By the description

$$S^2 = \{N \in \operatorname{Im}\mathbb{H} \mid |N| = 1\} = \{N \in \mathbb{H} \mid N^2 = -1\},$$

the lemma above indeed gives an algebraic representation of the well-known fact that the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ is $S^2 \times S^2$.

Identify $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$, then $U = \{x \in \mathbb{H} \mid Nx = -xR\}$ is contained in $\mathbb{R}^3$ iff $N = R$. This time N is the Euclidean unit normal vector of U compatible with the given orientation.

Definition A.3. Motivated by the observations above, N and R are called the *left* and *right normal vector* of U respectively, though in general they are *not* orthogonal to U.

For a conformal immersion f from an oriented Riemann surface M into $\mathbb{H}$, there also exist uniquely a pair of N, R such that

$$*\mathrm{d}f = N\mathrm{d}f = -\mathrm{d}fR. \tag{A.2}$$

They are called the *left* and *right normal vectors* of f. Combined together they give the usual Gauss map in 4-space.

Remark A.4. Conversely, the existence of such a pair vectors characterize conformal immersions into $\mathbb{H}$. This formulation of conformality is the starting point of a conformal surface theory by quaternions in [10, Section 2.2] and [7, Section 2.2].

Now we come to the notion of *left (co-)touch* and *right (co-)touch.*

Definition A.5. Let U_i be oriented 2-plane with N_i and R_i as their left and right normal vectors respectively, $i = 1, 2$. Then

1. U_1 and U_2 *touch* each other *from left (right)* if $N_1 = N_2$ ($R_1 = R_2$).
2. U_1 and U_2 *co-touch* each other *from left (right)* if $N_1 = -N_2$ ($R_1 = -R_2$).

Similarly we can define *(co-)touch* of two conformal immersions at their intersection point.

Remark A.6. When the touch is both from left and right, these two immersions are tangent at the intersection point with the same induced orientation (Lemma A.1). So left/right touch may be viewed as a generalization of tangency. They are used to define *Darboux transforms* for general surfaces which are generalization of the classical Darboux transforms of isothermic surfaces ([7, Section 7.1, p.47]).

Remark A.7. In similar terms, it is shown in [10, Section 9.2] that the mean curvature spheres of a Willmore surface touches its *forward (backward) two-step Bäcklund transforms* from left (right) yet with a negative sign. This motivated the author to introduce the dual notion of *co-touch.*

Given two oriented 2-planes in an oriented 4-space, Definition A.5 seems algebraic and depending on the way in which we identify $\mathbb{R}^4 \simeq \mathbb{H}$. Yet by the following two lemmas, we find they are well-defined geometric notions (depending only on different choices of orientations).

Lemma A.8. *Every orientation preserving linear isometry of $\mathbb{H}$ is of the form*

$$x \in \mathbb{H} \quad \mapsto \quad \mu x \lambda \in \mathbb{H},$$

and every orientation reversing linear isometry of $\mathbb{H}$ is of the form

$$x \in \mathbb{H} \quad \mapsto \quad \mu \bar{x} \lambda \in \mathbb{H}.$$

Here $\mu, \lambda \in \mathbb{S}^3$ are unit quaternions uniquely determined up to common multiplication by -1.

Lemma A.9.

(i) Every orientation preserving linear isometry of $\mathbb{H}$ leaves the relationship of left (co-)touch *and* right (co-)touch *invariant.*

(ii) Every orientation reversing linear isometry of $\mathbb{H}$ preserves the property of touch *and* co-touch*, but interchanges between* left *and* right.

(iii) Suppose U_1 touches U_2 from left (right). Then U_1 with opposite orientation co-touches U_2 from left (right). If the orientation of U_2 is also reversed, they turn out to touch each other from left (right) as before.

Proof. Our argument is based on the previous lemma, which is a well-known fact and we omit the proof at here. Now given 2-planes $U_i \subset \mathbb{H}$ with oriented orthonormal basis $\{\alpha_i, \beta_i\},\ i = 1, 2$. Then their left and right normal vectors are $N_i = \beta_i \bar{\alpha}_i$ and $R_i = \bar{\alpha}_i \beta_i$. After an isometry $x \in \mathbb{H} \mapsto \mu x \lambda \in \mathbb{H}$, the new basis of U_i is $\{\mu \alpha_i \lambda, \mu \beta_i \lambda\}$, and the new left and right normal vectors are

$$\begin{aligned} \widetilde{N_i} &= \mu \beta_i \lambda \cdot \overline{(\mu \alpha_i \lambda)} = \mu N_i \bar{\mu}, \\ \widetilde{R_i} &= \overline{(\mu \alpha_i \lambda)} \cdot \mu \beta_i \lambda = \bar{\lambda} R_i \lambda. \end{aligned}$$

Thus the conclusion (i) was proved by Definition A.5. On the other hand, if the isometry is $x \in \mathbb{H} \mapsto \mu \bar{x} \lambda \in \mathbb{H}$, there will be

$$\begin{aligned} \widetilde{N_i} &= \mu \bar{\beta}_i \lambda \cdot \overline{(\mu \bar{\alpha}_i \lambda)} = \mu \bar{R}_i \bar{\mu} = -\mu R_i \bar{\mu}, \\ \widetilde{R_i} &= \overline{(\mu \bar{\alpha}_i \lambda)} \cdot \mu \bar{\beta}_i \lambda = \bar{\lambda} \bar{N}_i \lambda = -\bar{\lambda} N_i \lambda. \end{aligned}$$

So part (ii) follows immediately. Finally, changing the oriented basis of U_i to be $\{\beta_i, \alpha_i\}$, we find

$$\begin{aligned} \widetilde{N_i} &= \beta_i \bar{\alpha}_i = \bar{N}_i = -N_i, \\ \widetilde{R_i} &= \bar{\beta}_i \alpha_i = \bar{R}_i = -R_i. \end{aligned}$$

This completes the proof to (iii). $\square$

From the previous lemma we see compatible or opposite orientations on this pair of 2-planes accounts for the difference between touch and co-touch. On the other hand, the difference between *left* and *right* is due to the orientation induced by the identification $\mathbb{R}^4 = \mathbb{H}$, hence not essential. The next proposition confirms this observation, and unifies two different definitions of touch and co-touch.

Proposition A.10. *Let U and $\widehat{U}$ be a pair of oriented 2-dim subspaces in $\mathbb{R}^4$. They (co-)touch each other as contact elements if, and only if, they (co-)touch each other from left or right. (Whether it is from left or right depends on the orientation induced by the identification $\mathbb{R}^4 \simeq \mathbb{H}$.)*

Proof. Equipped $U, \widehat{U}$ with oriented orthonormal basis $\{\alpha, \beta\}$ and $\{\hat{\alpha}, \hat{\beta}\}$ respectively. Regarding U as a conformally embedded submanifold of $\mathbb{R}^4 \subset \mathbb{S}^4$, we fix a lift $U \subset \mathbb{R}^4 \to \mathbb{R}^{5,1}$ as

$$v \in U \;\mapsto\; \left(\frac{1}{2}(1+|v|^2), \frac{1}{2}(1-|v|^2), v\right),$$

which projects down to $\mathbb{P}(\mathcal{L})$. The image of $0 \in \mathbb{R}^4$ is $Y = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$. The induced contact element at 0, denoted by $\Sigma = \{Y, Y_1, Y_2\}$, is given by

$$Y_1 = (0, 0, \alpha), \quad Y_2 = (0, 0, \beta).$$

We have similar representation of $\widehat{\Sigma} = \{Y, \widehat{Y}_1, \widehat{Y}_2\}$ as the contact element of $\widehat{U}$.

Consider the invariant $\underline{\rho}$ associated with $\Sigma, \widehat{\Sigma}$. By definition (2.7),

$$\underline{\rho} = \frac{1}{2}\langle \alpha - \mathrm{i}\beta, \hat{\alpha} - \mathrm{i}\hat{\beta}\rangle = 0 \iff \begin{cases} \langle \alpha, \hat{\alpha}\rangle &= \langle \beta, \hat{\beta}\rangle, \\ \langle \alpha, \hat{\beta}\rangle &= -\langle \beta, \hat{\alpha}\rangle. \end{cases}$$

We define a complex structure J on $\mathbb{R}^4$ via

$$J\{\alpha, \beta, \hat{\alpha}, \hat{\beta}\} = \{\beta, -\alpha, \hat{\beta}, -\hat{\alpha}\}.$$

In this term the condition of touch holds if, and only if, there is J satisfying the formula above and compatible with the Euclidean metric. By Lemma A.8, it is easy to show that such a complex structure must be of the form $\alpha \mapsto N\alpha$ or $\alpha \mapsto \alpha R$, where $N, R \in \mathbb{H}, N^2 = R^2 = -1$. This implies our conclusion on touch. For co-touch the similar argument applies. $\square$

Remark A.11. For any pair of oriented 2-dim subspaces in $\mathbb{R}^n$, we can always regard them as located in a suitable 4-dim subspace and identify it with $\mathbb{H}$. Since the notion of left/right (co-)touch are independent to such an identification (leaving the choice of orientations aside), it is understandable that we have the similar notion in this general context.

Remark A.12. When $U, \widehat{U} \subset \mathbb{R}^4$ are touching and co-touching at the same time, i.e. $\underline{\theta} = \underline{\rho} = 0$, we conclude easily that they must form an orthogonal direct sum. We also note that they have the same left normal vector and opposite right normal vectors (with orientations suitable chosen). These facts are compatible according to Lemma A.1 and Proposition A.10.

Appendix B

Technical lemmas on singularities

Lemma B.1. *Complex-valued function g is defined in a neighborhood of $z = 0$ and $g(0) = 0$, satisfying*

$$\frac{1}{2}g_{\bar{z}}^2 - gg_{\bar{z}\bar{z}} - \bar{s}g^2 = 0$$

for some analytic function s. Suppose g is analytic, too. Then there is another analytic solution $\hat{g}$ to equation (4.3), whose Taylor expansion at $z = 0$ is either $\hat{g} = (az + b\bar{z})^2 + o(|z|^2)$ with $b \neq 0$, or $\hat{g} = c + o(|z|)$ with $c \neq 0$, so that $g = z^r \cdot \hat{g}$ for a non-negative integer r.

Proof. Expand g at $z = 0$ into power series of z and $\bar{z}$. We claim this expression is either $g = (az + b\bar{z})^2 z^{r-2} + o(|z|^r)$ with $r \geq 2, b \neq 0$, or $g(z) = cz^l + o(|z|^l)$ with $c \neq 0$.

To verify our claim, first note that by (4.3), $g_{\bar{z}}$ also equals 0 at $z = 0$. If g does not vanish at $z = 0$ up to first order terms, there must be $g(z) = cz + o(|z|)$, and the claim is true.

Now suppose g vanish at $z = 0$ up to higher order terms. The expansion is

$$g(z) = \sum_{i=0}^{r} a_i z^{r-i}\bar{z}^i + o(|z|^r),\ r \geq 2.$$

Differentiation yields

$$\begin{aligned} g_{\bar{z}} &= \sum_{i=1}^{r} i \cdot a_i z^{r-i}\bar{z}^{i-1} + o(|z|^{r-1}), \\ g_{\bar{z}\bar{z}} &= \sum_{i=2}^{r} i(i-1)a_i z^{r-i}\bar{z}^{i-2} + o(|z|^{r-2}). \end{aligned}$$

Denote a_t to be the last non-zero element of $\{a_0, a_1, \ldots, a_r\}$. If $t = 0$, our claim is evidently true. If $t = 1$, we have $g = (az + b\bar{z})z^{r-1} + o(|z|^r),\quad b \neq 0.$

There would be

$$0 \equiv \frac{1}{2}g_{\bar{z}}^2 - gg_{\bar{z}\bar{z}} - \bar{s}g^2 = \frac{1}{2}b^2 z^{2r-2} + o(|z|^{2r}).$$

This can not be true when $b \neq 0$. So this possibility is excluded. The final case is $t \geq 2$. At this time

$$\begin{aligned} 0 \quad &\equiv \quad \frac{1}{2}g_{\bar{z}}^2 - gg_{\bar{z}\bar{z}} - \bar{s}g^2 \\ &= \quad \sum_{i=2r-2t+1}^{2r-2}(\cdots)z^i\bar{z}^{2r-i-2} + \big[\frac{1}{2}t^2a_t^2 - t(t-1)a_t^2\big]z^{2r-2t}\bar{z}^{2t-2} + o(|z|^{2t-2}) \end{aligned}$$

implies $\frac{1}{2}t^2 - t(t-1) = 0$, i.e. $t = 2$. Thus we may assume

$$g(z) = (a_0z^2 + a_1z\bar{z} + a_2\bar{z}^2)z^{r-2} + o(|z|^r), r \geq 2.$$

A similar argument by (4.3) yields $\frac{1}{2}a_1^2 - 2a_0a_2 = 0$. It is equivalent to saying that

$$g = (az + b\bar{z})^2 z^{r-2} + o(|z|^r), \quad r \geq 2, \quad b \neq 0. \tag{B.1}$$

This completes the proof of the claim.

Next we prove the stronger conclusion of this lemma. Without loss of generality, assume $g = z^r(az+b\bar{z})^2 + o(|z|^{r+2})$ with $b \neq 0$, or $g = cz^r + o(|z|^r)$ with $c \neq 0$, $r \geq 1$. We assert that in the power series of g at $z = 0$, there is no monomials like $\bar{z}^l$, hence g/z is an analytic function, which obviously satisfies (4.3). By induction we see $\hat{g} := g/z^r$ is an analytic solution to (4.3) with the desired lowest terms.

Suppose our assertion is not true. Then among monomials of the form $\bar{z}^l$ there must be one with the lowest power $t \geq 2$. Hence

$$g = z \cdot g_1 + \bar{z}^t + o(|z|^t),$$

where g_1 is a polynomial whose order is less than or equal to $t - 1$. As a consequence of (4.3),

$$0 \equiv \frac{1}{2}g_{\bar{z}}^2 - gg_{\bar{z}\bar{z}} - \bar{s}g^2 = \frac{1}{2}(t\bar{z}^{t-1})^2 - \bar{z}^t \cdot t(t-1)\bar{z}^{t-2} + z \cdot (\cdots) + o(|z|^{2t-2}).$$

So $t = 2$. It follows $g_1 = A + Bz + C\bar{z}$. Substitute this back into (4.3),

$$0 \equiv \frac{1}{2}g_{\bar{z}}^2 - gg_{\bar{z}\bar{z}} - \bar{s}g^2 = 2Az + o(|z|^2)$$

force $A = 0$. Such a g has second order zero at $z = 0$ with $\bar{z}^2$ term in the Taylor expansion, which is impossible according to our claims at the beginning of this discussion. This contradiction proves our assertion. $\square$

Lemma B.2 (Extension Lemma). *Suppose p, q are two analytic functions defined on disc $D = \{z \in \mathbb{C} : |z| < 1\}$, where q does not vanish identically, and $f = p/q$ is holomorphic on the open dense subset where $q \neq 0$. Then f is meromorphic on D. Suppose further that the Taylor expansion of q at $z = 0$ contains monomials like $\bar{z}^l$. Then $z = 0$ is a removable singularity of f.*

Proof. Suppose $z = 0$ is a zero of q. We want to show that f extends to a meromorphic function around $z = 0$. Since $q \not\equiv 0$, it does not vanish identically on the straight line $\{z : z - e^{i\theta}\bar{z} = 0\}$ for some real constant θ. Without loss of generality, we may assume $\theta = 0$ is such a direction. In other words, $q \not\equiv 0$ on the real axis. Because any homogeneous polynomial $h(z, \bar{z})$ can be decomposed as $(z - \bar{z})h' + cz^n$, the Taylor expansion of p and q at $z = 0$ might be rewritten as

$$p = (z - \bar{z})p_1 + p_2, \quad q = (z - \bar{z})q_1 + q_2,$$

where p_1 and q_1 are power series about z and $\bar{z}$, p_2 and q_2 are power series about z. When restricted to the real axis, p and q are reduced to p_2 and q_2, which must be analytic too, hence are convergent power series. Moreover, $q_2 \neq 0$ in the open interval

$$L_\epsilon = \{z : z - \bar{z} = 0, 0 < |z| < \epsilon\}$$

for sufficiently small $\epsilon > 0$. The same holds for q. So $p/q = f$ is a well defined function on L_ϵ. It must be holomorphic on an open subset containing L_ϵ due to the holomorphicity condition. On the other hand, p_2/q_2 is a meromorphic function around $z = 0$, which coincides with f on a line segment. Thus we see $p_2/q_2 = p/q$ on an open subset. This means $p_2 q - q_2 p \equiv 0$ on an open subset which contains $z = 0$ in its closure. The functions involved are all analytic in a small neighborhood of $z = 0$, hence $p_2 q - q_2 p$ as an analytic function must vanish identically around $z = 0$. So meromorphic function p_2/q_2 is an analytic extension of f in a neighborhood of $z = 0$. The first conclusion is proved.

Next, suppose the Taylor expansion of q at $z = 0$ contains monomials like $\bar{z}^l$. As a meromorphic function, f may be written as p'/q', where $p'(0) \neq 0$ and $q' = z^k$. Consider the identity $q'p = p'q$. When expanding at $z = 0$, every monomial in the left hand side can be divided by z^k, yet there exists monomials on the right hand side looks like $\bar{z}^l$. The only possibility is $k = 0$. Thus $z = 0$ is a removable singularity of f. □

Lemma B.3. *Let $w_\alpha(z)$ be complex-valued functions which satisfy the differential system*

$$\frac{\partial w_\alpha}{\partial \bar{z}} = \sum_\beta a_{\alpha\beta} w_\beta, \qquad 1 \leq \alpha, \beta \leq p,$$

in a neighborhood of $z = 0$*, where* $a_{\alpha\beta}$ *are complex-valued* C^1*-functions. Suppose the* w_α *do not all vanish identically in a neighborhood of* $z = 0$*. Then:*

(1) the common zeros of w_α *are isolated;*

(2) at a common zero of w_α *the ratios* $w_1 : \cdots : w_p$ *tends to a limit.*

Proof. See section 4 in [16]. □

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