Traveling Waves in Stochastic Neural Fields

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Abstract

Neural field equations are used to model the spatio-temporal dynamics of the activity in a network of synaptically coupled populations of neurons in the continuum limit. They exhibit traveling wave solutions, modeling the propagation of activity. In this thesis we provide a mathematical framework for the analysis of the influence of noise on these solutions.

The noise influences the dynamics on two scales. First, it causes fluctuations in the wave profile, and second, it causes a random displacement of the wave from its uniformly translating position. In order to analyze the stability of the wave under noise, we study the linear operator that appears in the equation when linearizing around the traveling wave. We show that this nonlocal operator has a spectral gap by proving a functional inequality, expressing that perturbations in directions that are orthogonal to the direction of movement are damped by the neural field dynamics.

By separating the two spatial scales, we obtain a simplified description of the dynamics. Here we use a dynamic phase adaptation of a reference wave to determine the phase shift caused by the noise. We prove an expansion of the stochastic traveling wave describing the effects of the noise to different orders of the noise strength. We prove that, to first order, the shift in the phase is roughly diffusive, and the fluctuations in the wave profile are given by an Ornstein-Uhlenbeck process that is orthogonal to the direction of movement.

The neural field model approximates the network behavior in two ways. The local dynamics in each population are summarized in a mean activity, and the large network is approximated by a continuum. A possible source of noise on this level are deviations from the mean field behavior due to the finite size of the populations, so-called finite-size effects. By describing the evolution of the activity in a finite network of finite populations by a Markov chain and analyzing the fluctuations of this process, we determine a stochastic correction term to the mean field equation. We derive a well-posed L^2-valued stochastic neural field equation with a noise term accounting for finite-size effects on the traveling wave solution, and prove it to be the continuum limit of an associated network of diffusion processes.

Zusammenfassung

Neuronale Feldgleichungen werden benutzt, um die Dynamik der Aktivität in einem Netzwerk synaptisch gekoppelter Populationen von Neuronen in Raum und Zeit zu modellieren. Eine spezielle Art von Lösungen dieser Gleichungen sind wandernde Wellen, welche die Ausbreitung der Aktivität beschreiben. In dieser Dissertation führen wir einen mathematischen Rahmen für die Untersuchung des Einflusses von Rauschen auf diese Art von Lösungen ein.

Das Rauschen beeinflusst die Dynamik auf zwei verschiedenen Skalen. Zum Einen führt es zu Fluktuationen im Profil der Welle. Zum Anderen verursacht es eine Störung der Geschwindigkeit und somit eine Abweichung von der gleichmäßigen Fortbewegung der Welle. Um die Stabilität der Welle unter Einfluss von Rauschen zu untersuchen, analysieren wir Eigenschaften des linearen Operators, der in der Gleichung auftritt, wenn wir um die Wellenlösung linearisieren. Wir zeigen, dass dieser nichtlokale Operator eine Spektrallücke hat, indem wir eine Funktionalungleichung beweisen. Diese drückt aus, dass Störungen in Richtungen, die orthogonal zur Bewegungsrichtung liegen, durch die Dynamik des Feldes ausgeglichen werden.

Indem wir die zwei räumlichen Skalen trennen, erhalten wir eine vereinfachte Darstellung der Dynamik. Hierbei bestimmen wir durch dynamisches Anpassen der Geschwindigkeit eines Referenzprofils die durch das Rauschen verursachte Phasenverschiebung. Wir leiten eine Entwicklung der stochastischen Welle in einem kleinen Parameter ϵ, der die Stärke des Rauschens beschreibt, her, an der man den Einfluss der stochastischen Störung zu beliebiger Ordnung in ϵ ablesen kann. Wir zeigen, dass die Positionsverschiebung der Welle zu erster Ordnung in ϵ in etwa diffusiv ist und dass die Fluktutationen im Wellenprofil durch einen Ornstein-Uhlenbeck Prozess, der orthogonal zur Bewegungsrichtung ist, beschrieben werden.

Das neuronale Feldmodell stellt in zwei Aspekten nur eine näherungsweise Beschreibung des Verhaltens des Netzwerkes dar. Die lokale Dynamik in den einzelnen Populationen wird zusammengefasst in einer mittleren Aktivität und das diskrete Netzwerk wird durch ein stetiges Feld approximiert. Eine möglich Quelle von Rauschen auf diesem Level sind Abweichungen vom Mittelwert, verursacht durch die endliche Größe der Populationen. Indem wir die Evolution der Aktivität in einem endlichen Netzwerk endlicher Populationen durch eine Markovkette beschreiben und deren Fluktuationen analysieren, leiten wir einen stochastischen Korrekturterm her, der die Effekte dieser Abweichungen auf wandernde Wellen beschreibt. Wir erhalten eine wohlgestellte L^2-wertige stochastische neuronale Feldgleichung und zeigen, dass diese Kontinuumsgrenzwert eines zugehörigen Netzwerkes von Diffusionsprozessen ist.

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Contents

Chapter 1

Introduction

1.1 Neural Fields

We start by giving a very brief overview over the neuroscientific background and introduce the models that will be considered in this thesis. Most of the content of this chapter is based on chapter 2 in [39], chapters 10, 11, 12 in [28], chapters 5 and 6 in [33], and chapters 6 and 7 in [10], where also a lot of additional information can be found.

1.1.1 Neurons

A neuron is an electrically excitable cell, the signaling unit of the nervous system. The human brain is estimated to contain on the order of 10^{11} neurons. They are interconnected and communicate by sending electrical signals. A neuron receives input from other neurons or from external stimuli (for example visual, audible, olfactory stimuli) on the dendrites. The input is processed in the soma and a response is sent along the axon and transmitted to other neurons via synapses.

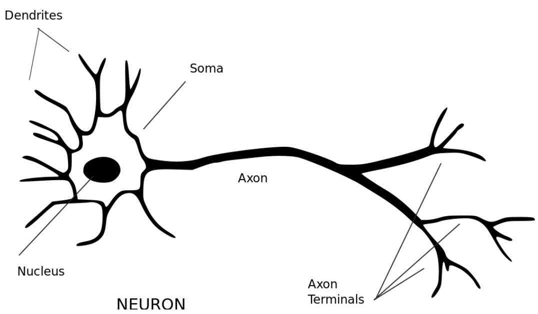

Figure 1.1: A schematic neuron (Notjim, Wikimedia Commons, licensed under creative commons by-sa 3.0; URL https:creativecommons.org/licenses/by-sa/3.0/deed.en)

There is a difference between the electric potential inside and outside the neuron, called the *membrane potential*. This is due to a difference in ion concentrations, which is maintained by means of the selective permeability of the cell membrane. The *resting value* of the membrane potential is the difference between the potential inside and outside the cell when it is at rest. Its value is approximately $-70mV$. The term *excitable* means that the neuron's membrane potential can change rapidly from its resting value to an excited state of significantly higher voltage (*spiking*).

The input a neuron receives from a *presynaptic cell* or due to an external stimulation leads to a depolarization or hyperpolarization of the cell. This causes the activation or inactivation of *voltage-gated ion channels* in the cell membrane, controlling the flux of ions into and out of the cell. If the membrane potential crosses a certain threshold, these ionic mechanisms lead to the generation of an *action potential*. It propagates along the axon and causes the release of neurotransmitter at the synapses. This leads to the activation of receptors in the *post-synaptic cell* and causes a change in the potential, called the *post-synaptic potential*.

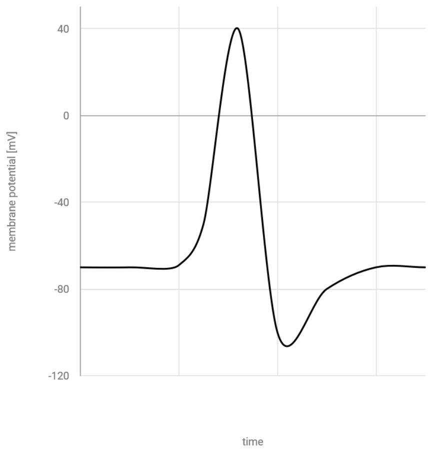

Figure 1.2: Typical form of an action potential

Such an action potential was first recorded by Hodgkin and Huxley in 1939 in a squid giant axon [34]. They were also able to describe the mechanism underlying the generation of the action potential by a system of four coupled differential equations [35]. One equation describes the evolution of the membrane potential, while the other three govern the activation or inactivation of ion channels.

The form and amplitude of the action potential is basically always the same. To convey information, other properties of the spiking network, such as the timing of action potentials, their path in the network, or patterns in the collective signaling of many neurons, must therefore be relevant.

1.1.2 Noise

There are several evidences for 'noise' in neural systems. There is variability in the responses of neurons to external stimuli from trial to trial. Spontaneous activity is observed in the absence of any stimulus. While deterministic models produce regular spike trains, the spike trains recorded in in vivo measurements are very irregular.

Several sources of noise have been identified. A source of *intrinsic noise* is the so-called *channel noise*. While in the Hodgkin-Huxley model the proportion of open ion channels is treated as a deterministic variable, the opening and closing of the channels is really a probabilistic event. The implicit assumption is that their number is so large that the proportion of open channels can be approximated by the probability of a channel being open at a given potential.

A source of *extrinsic noise* is the so-called *synaptic noise*. A neuron receives input from many other neurons, not all of which may be relevant. Input coming from neurons outside the functional unit may rather be considered as background noise.

Other sources of noise are the spontaneous release of neurotransmitter at synapses, causing small changes in the potential of post-synaptic neurons, or transmission failures.

Noise has important effects. While in deterministic models identical neurons receiving the same input all respond in exactly the same way, noise allows for variability. In particular it can lead to sub-threshold firing. The neuron's firing rate as a parameter of the input current becomes a continuous variable and allows to track time-changing stimuli.

1.1.3 Population Models

Networks of neurons of growing size quickly become unwieldy. Even if one uses simpler models than the Hodgkin-Huxley model, the analysis becomes involved from a computational as well as from an analytical perspective. It may therefore be useful to zoom out from the microscopic view and describe populations of neurons in terms of more macroscopic variables. This is the approach used in so-called *population models* or *firing rate models*. Instead of keeping track of single neuron spiking, a *population activity* is identified as an average over a group of neurons.

This corresponds to measurement techniques such as the electroencephalography (EEG). With extracellular electrodes synchronized activity in many neurons is detected, while single spiking activity cannot be identified.

We consider a population of N neurons. We say that a neuron is 'active' if it is in the process of firing an action potential such that its membrane potential is larger than some threshold value κ. If Δ is the width of an action potential, then a neuron is active at time t if it fired a spike in the time interval $(t-\Delta, t]$. We define the population activity at a given time t as the proportion of active neurons,

$$a^N(t) = \frac{\#\text{ neurons that are active at time } t}{N} \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}.$$

We assume that all neurons in the population are identical and receive the same input. If the neurons fire independently from each other, then for a constant input current I,

$$a^N(t) \xrightarrow{N\to\infty} F(I),$$

where $F(I)$ is the probability that a neuron receiving constant stimulation I is active. In the infinite population limit, the population activity is thus related to the input current via the function F, called the *gain function*. Sometimes one also defines F as a function of the potential u, assuming that the potential is proportional to the current as in Ohm's law. F is typically a nonlinear function. It is usually modeled as a sigmoid, for example

$$F(x) = \frac{1}{1+e^{-\gamma(x-\kappa)}}$$

for some $\gamma > 0$ and some threshold $\kappa > 0$, imitating the threshold-like nature of spiking activity.

Sometimes a *firing rate* is considered instead of a probability. We define the population firing rate λ^N as

$$\lambda^{\delta,N}(t) = \frac{\#\text{ spikes in the time interval } (t-\delta,t]}{\delta N}.$$

If $\delta = \Delta$, then $\Delta\lambda^{\delta,N}(t) = a^N(t)$. At constant potential u, $\lim_{\delta\to 0}\lim_{N\to\infty}\lambda^{\delta,N}(t) = \lambda(u)$, where $\lambda(u)$ is the single neuron firing rate. Note that $\lambda \le \frac{1}{\Delta}$. The firing rate is related to the probability $F(u)$ via

$$F(u) \approx \lambda(u)\Delta.$$

If the stimulus varies in time, then the activity may track this stimulus with some delay such that

$$a(t+\tau_a) = F(u(t))$$

for some time constant τ_a. Taylor expansion of the left-hand side gives an approximate description of the (infinite population) activity in terms of the differential equation

$$\tau_a \dot{a}(t) = -a(t) + F(u(t)),$$

to which we refer as the *rate equation*.

Population models describe the evolution of the activity in a network of synaptically coupled populations of neurons. Instead of modeling explicitly the spiking behavior of all the neurons, each population's behavior is summarized in one variable, the activity, and the description of the network dynamics is thus reduced to one equation for each population in the network. In the following subsection, we give a heuristic derivation of a population model.

Heuristic Derivation

Consider a network of P populations, each consisting of N neurons. We assume that each presynaptic spike in population j at time s causes a postsynaptic potential

$$h(t-s) = \frac{1}{N} w_{ij} \frac{1}{\tau_m} e^{-\frac{1}{\tau_m}(t-s)}$$

in population i at time t. Here the (w_{ij}) are weights characterizing the strength of the synaptic connections between populations i and j, and τ_m is the membrane time constant, describing how fast the membrane potential relaxes back to its resting value.

Under the assumption that all inputs add up linearly, the potential in population i at time t is given as

$$u_i^N(t) = \sum_{j=1}^{P} w_{ij} \int_{-\infty}^{t} \frac{1}{\tau_m} e^{-\frac{1}{\tau_m}(t-s)} a_j^N(s) ds.$$

In the infinite population limit we obtain

$$u_i(t) = \sum_{j=1}^{P} w_{ij} \int_{-\infty}^{t} \frac{1}{\tau_m} e^{-\frac{1}{\tau_m}(t-s)} a_j(s) ds, \tag{1.1}$$

where

$$\tau_a \dot{a}_j(t) = -a_j(t) + F(u_j(t)).$$

The behavior of the coupled system (u_i, a_i) depends on the two time constants, τ_m and τ_a. We consider two different regimes in which the model can be reduced to just one of the two variables, u or a.

Case 1: $\tau_m \gg \tau_a \to 0$

In this regime we can assume that the activity reacts to changes in input immediately such that $a_j(t) = F(u_j(t))$. Then (1.1) can be closed in the variables u_i and we obtain

$$u_i(t) = \sum_{j=1}^{P} w_{ij} \int_{-\infty}^{t} \frac{1}{\tau_m} e^{-\frac{1}{\tau_m}(t-s)} F(u_j(s)) ds. \tag{1.2}$$

Differentiation yields the system of ordinary differential equations

$$\frac{d}{dt} u_i(t) = \frac{1}{\tau_m} \Big(-u_i(t) + \sum_{j=1}^{P} w_{ij} F(u_j(t)) \Big), \tag{1.3}$$

which we will call the *voltage-based neural network equation.*

Case 2: $\tau_a \gg \tau_m \to 0$

By (1.1),

$$a_i(t+\tau_a) = F\bigg(\sum_{j=1}^{P} w_{ij} \int_{-\infty}^{t} \frac{1}{\tau_m} e^{-\frac{1}{\tau_m}(t-s)} a_j(s)ds\bigg).$$

Letting $\tau_m \to 0$ we obtain

$$a_i(t+\tau_a) = F\bigg(\sum_{j=1}^{P} w_{ij} a_j(t)\bigg).$$

Using again that $a_i(t+\tau_a) \approx a_i(t)+\tau_a a_i'(t)$, we end up with the system of ordinary differential equations

$$\tau_a \frac{d}{dt} a_i(t) = -a_i(t) + F\bigg(\sum_{j=1}^{P} w_{ij} a_j(t)\bigg), \tag{1.4}$$

which we will call the *activity-based neural network equation.*

1.1.4 The Neural Field Equation

If the number of populations is large, the system of differential equations describing the evolution of the potentials u_i or the activities a_i, respectively, can be further simplified by considering a spatially distributed network and taking the *continuum limit.* We embed the populations into an interval on the real line. Letting the size of the interval and the density of populations go to infinity, we obtain a continuous model

$$\tau_m \frac{\partial}{\partial t} u(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x,y) F(u(y,t)) dy, \tag{1.5}$$

or, in the activity-based setting,

$$\tau_a \frac{\partial}{\partial t} a(x,t) = -a(x,t) + F\bigg(\int_{-\infty}^{\infty} w(x,y) a(y,t) dy\bigg). \tag{1.6}$$

The P-dimensional system is thus reduced to a function-valued evolution equation. The difficulty in the analysis lies in the nonlocal nature of the equation.

It might seem more natural to embed the populations in a domain in $\mathbb{R}^3$. In this thesis we only consider the simplified one-dimensional model. It is argued that this can be seen as an approximation for the dynamics of the activity in cortical columns. A very thin layer of neurons can be treated as two-dimensional. If the neurons are organized according to functionality in vertical columns, then each column may be treated as a population, indexed by its position on the real line (cf. [10], 6.5).

Neural field equations were first introduced by Amari in the voltage-based setting [1], and by Wilson and Cowan in the activity-based setting [61], [62]. Since then they have been used widely to model the propagation of activity in large networks of populations of neurons. While of a relatively simple form, they exhibit a variety of spatiotemporal patterns. In this thesis we focus on the analysis of *traveling wave solutions*, which will be introduced in the next section. Other examples include bump solutions ([40, 45]), traveling pulses [51, 32], or spiral waves ([36, 44]), see also [21].

Neural field models have found application to various phenomena such as wave propagation in brain slices [51], binocular rivalry [14], working memory [45], hallucinations [26], [11], and many others. For details on neural field modeling and many related references we refer to the reviews [25], [9], and the books [28], [10], [22].

1.1.5 Traveling Waves

Assume that the strength of the synaptic connections between two populations depends only on their distance such that $w(x,y) = w(|x-y|)$, and that $\int_{-\infty}^{\infty} w(|x|)dx = 1$. Assume that the gain function F is such that $x \mapsto F(x) - x$ is *bistable*. That is, $x \mapsto F(x) - x$ has exactly three zeroes $0 < a_1 < a < a_2 < 1$, and that $F'(a_i) < 1, i = 1, 2$, and $F'(a) > 1$.

Then the constant functions $\mathbf{a_1}$, $\mathbf{a}$, and $\mathbf{a_2}$ are stationary solutions to the neural field equation (1.5). $\mathbf{a_1}$ and $\mathbf{a_2}$ are stable, while $\mathbf{a}$ is unstable. $\mathbf{a_1}$ represents a stable state of low activity, where basically all the neurons are inactive, and $\mathbf{a_2}$ corresponds to a stable state of high activity, where basically all the neurons are active and the recurrent activity keeps them active.

It was proven in 1992 by Ermentrout and McLeod [27] that, under certain additional assumptions on the parameters, there exists a unique monotone *traveling wave solution* to the neural field equation (1.5) connecting the two stable states. That is, there exists a unique monotone wave profile $\hat{u} : \mathbb{R} \to \mathbb{R}$ and a unique wave speed $c \in \mathbb{R}$ such that

$$u^{TW}(x,t) = \hat{u}(x - ct)$$

is a solution to (1.5), and

$$\lim_{x \to -\infty} \hat{u}(x) = a_1, \qquad \lim_{x \to \infty} \hat{u}(x) = a_2.$$

Note that the traveling wave solution is determined only up to translation: for any $a \in \mathbb{R}$, $(x,t) \mapsto \hat{u}(x - a - ct)$ is also a solution. The traveling wave profile satisfies the differential equation

$$-c\hat{u}_x = -\hat{u} + w * F(\hat{u}). \tag{1.7}$$

The traveling wave solution can be interpreted as activity (or inactivity, depending on the value of c) spreading through the populations. This corresponds to the wave-like propagation of activity that has been observed in brain slice experiments and in in-vivo recordings, see for example [51, 52, 56].

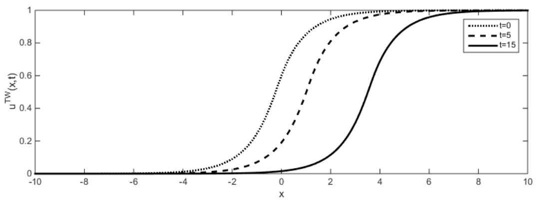

Figure 1.3: Traveling wave at different times t

The wave profile $\hat{u}$ and the wave speed c are usually unknown. In [27] it is proven that the sign of c is the same as the sign of

$$\int_{a_1}^{a_2} x - F(x)dx = \int_{a_1}^{a} x - F(x)dx - \int_{a}^{a_2} F(x) - xdx. \tag{1.8}$$

Whether the activity is spreading or dying out thus depends on whether the mass of $|x-F(x)|$ is concentrated near a_2, in which case the wave is 'pulled up', or near a_1, in which case the wave is 'pulled down'.

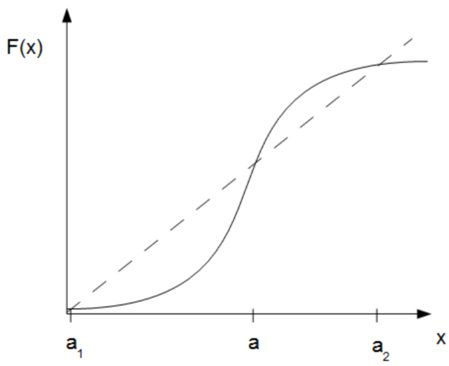

Figure 1.4: A situation where $c > 0$

Example: Heaviside Nonlinearity

Denote by H the Heaviside function,

$$H(x) = \mathbb{1}_{[0,\infty)}(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

In the case where the gain function F is given as

$$F(x) = H(x - \kappa)$$

for some threshold $0 < \kappa < 1$, a traveling front solution to (1.5) can be calculated explicitly.

We fix a translation of the wave profile by setting $\hat{u}(0) = \kappa$. Assume that $c \neq 0$. By (1.7), for $x \geq 0$,

$$\begin{aligned}\hat{u}(x) &= e^{\frac{1}{c}x}\kappa - \frac{1}{c}\int_0^x e^{\frac{1}{c}(x-y)} w * H(\hat{u}(\cdot) - \kappa)(y)dy \\ &= e^{\frac{1}{c}x}\kappa - \frac{1}{c}\int_0^x e^{\frac{1}{c}(x-y)} \int_0^\infty w(y-z)dzdy,\end{aligned}$$

and for $x \leq 0$,

$$\begin{aligned}\hat{u}(x) &= e^{\frac{1}{c}x}\kappa + \frac{1}{c}\int_x^0 e^{\frac{1}{c}(x-y)} w * H(\hat{u}(\cdot) - \kappa)(y)dy \\ &= e^{\frac{1}{c}x}\kappa + \frac{1}{c}\int_x^0 e^{\frac{1}{c}(x-y)} \int_0^\infty w(y-z)dzdy.\end{aligned}$$

Set $W(y) = \int_0^\infty w(y-z)dz$. If $w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$, then

$$W(y) = \begin{cases} 1 - \frac{1}{2}e^{-\frac{y}{\sigma}} & y \geq 0 \\ \frac{1}{2}e^{\frac{y}{\sigma}} & y \leq 0 \end{cases},$$

and

$$\hat{u}(x) = \begin{cases} 1 - \frac{\sigma}{2(\sigma+c)}e^{-\frac{1}{\sigma}x} + e^{\frac{1}{c}x}(\kappa - 1 + \frac{\sigma}{2(\sigma+c)}) & x \geq 0 \\ \left(\kappa - \frac{\sigma}{2(\sigma-c)}\right)e^{\frac{1}{c}x} + \frac{\sigma}{2(\sigma-c)}e^{\frac{1}{\sigma}x} \end{cases}$$

If $c > 0$, then $\hat{u}$ is bounded if and only if $\kappa - 1 + \frac{\sigma}{2(\sigma+c)} = 0$. Similarly, if $c < 0$, then $\hat{u}$ is bounded if and only if $\kappa = \frac{\sigma}{2(\sigma-c)}$. This implies that

$$c = \begin{cases} \sigma\frac{(2\kappa-1)}{2-2\kappa} & c > 0 \\ \sigma\frac{2\kappa-1}{2\kappa} & c \leq 0. \end{cases}$$

In particular $c > 0$ if and only if $\kappa > \frac{1}{2}$ (cf. (1.8)).

1.1.6 Stochastic Neural Field Equations

As outlined above, the communication of neurons is subject to noise. It is therefore interesting to consider stochastic versions of the neural field equation (1.5). While several sources of noise have been identified on the single neuron level (cf. subsection 1.1.2), it is not clear how this translates to the level of populations, in particular because neural field equations have not been rigorously derived from single neuron models.

A usual approach is therefore to add a Gaussian noise term to the equation to account for any source of noise in the system.

Since the neural field equations are (heuristically) derived as mean field limits, the usual sources of noise should have averaged out on this level. However, the equations are derived for the infinite population limit. The actual finite size of the populations causes deviations from the mean-field behavior. A possible way to determine a suitable stochastic correction term is therefore to study the nature of these *finite-size effects*.

1.2 Content of the Thesis

This thesis deals with the modeling and analysis of stochastic neural field equations. The main goal is to provide a mathematical framework allowing for the analysis of the influence of noise on traveling wave solutions. This involves the following questions:

1) How should one model the noise in the neural field?

2) In which sense does the stochastic neural field equation possess a solution?

3) In which way does the continuum model approximate the behavior of the network?

4) How can the influence of noise on spatio-temporal patterns be analyzed?

5) How can the stability of the wave under noise be expressed?

The main contribution to the analysis of stochastic neural fields is the work by P. Bressloff and his coworkers. They have analyzed the influence of noise on diverse phenomena that can be observed in neural fields. We refer to his book [10] and the review [9] for details.

However the methods applied are usually not rigorous. The mathematically rigorous modeling of stochastic neural fields has received more attention recently. This includes the work of M. Riedler and E. Buckwar ([54], questions 1, 3), J. Krüger and W. Stannat ([42], questions 2, 4, 5), J. Touboul ([59], question 3), C. Kuehn and M. Riedler ([43], large deviations), O. Faugeras and J. Inglis ([30], question 2), J. Inglis and J. MacLaurin ([37], questions 4, 5), O. Faugeras, J. Touboul and coworkers ([2, 7, 60, 31] (mean field limits)). Their work will be reviewed in more detail when it becomes relevant in the respective chapters.

Work that does not consider stochastic neural fields but is related in mathematical methods includes the work by W. Stannat [57, 58], where the stochastic stability of traveling wave solutions to reaction-diffusion equations is studied. The approach of obtaining a simplified description of the dynamics in stochastic evolution equations by separating different spatial or temporal scales and approximating to a certain order of the noise strength (cf. Chapter 5) is related to D. Blömker's work on amplitude equations ([6, 5] are examples among many others), .

1.2.1 Structure and Main Results

In the following we outline the structure of this thesis, highlighting in particular the main results. Chapter 5 and sections 4.1 and 4.2 are based on the preprint [46].

1. In Chapter 2 we introduce the mathematical framework in which we will work. We model the noise by a Q-Wiener process and solve the resulting stochastic neural field equation. We introduce the operators that will be subject of our analysis in what follows. In particular we give a possible answer to question 2, using the ideas already presented in [42]. We introduce the notion of the phase of a wave-like function and explain our dynamic approach to determining the phase of the stochastic wave.

2. The question of the stability of the traveling wave solution is related to spectral properties of the linear operator appearing in the neural field equation when linearizing around the traveling wave. In Chapter 3 we show that this nonlocal operator, denoted by $L^{\#}$, has a spectral gap in $L^2(\rho)$ for a certain density ρ by proving the following functional inequality.

   Main Theorem 1. *Under rather general assumptions, there exists $\kappa > 0$ such that for all $v \in H^1(\rho)$,*
$$\langle L^{\#} v, v\rangle_\rho \leq -\kappa\Big(\|v\|_\rho^2 - \langle v, \hat{u}_x\rangle_\rho^2\Big). \tag{1.9}$$

   This inequality expresses that the neural field dynamics is contractive in directions orthogonal to $\hat{u}_x$, the direction of movement of the wave. It will be the basis for the analysis of the stability of the traveling wave under noise and an answer to question 5.

   While it is known that $L^{\#}$ has a spectral gap as an operator on $\mathcal{C}_0$, the space of continuous functions vanishing at infinity (cf. [3]), to our knowledge, the $L^2(\rho)$-spectral gap is a new result, and the problem has not been considered from the perspective of functional inequalities before. We show that this approach allows for a clear and general proof and is in particular amenable to a stochastic setting.

   The result is not restricted to the neural field case, but applies to a general class of nonlocal evolution equations possessing traveling wave solutions.

3. In Chapter 4, we analyze a specific example in detail, namely the case where the strength of the synaptic connections decays exponentially with the distance, that is, where the kernel is given as $w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$ for some $\sigma > 0$. In particular, exploiting the fact that
$$w * h = (I - \sigma^2\Delta)^{-1}h,$$
   we compute explicitly the exponential rates of decay of the derivative of the wave profile $\hat{u}_x$ and of the associated adjoint eigenfunction.

   All the results presented in this thesis apply to this classical example. The assumptions we make should hold in a very general context, but the nonlocal nature of the neural field equation makes the analysis of the properties of the traveling wave solution difficult.

4. Chapter 5 is devoted to the analysis of the effects of the noise on the traveling wave solution and thus an answer to questions 4 and 5. The noise influences the dynamics on two scales. First it causes fluctuations in the wave profile, and second in the phase of the

wave, that is, the wave is shifted randomly from its uniformly translating position. A separation of these two scales allows us to obtain a simplified description of the stochastic dynamics. We derive an expansion of the solution $u(x,t)$ to the stochastic neural field equation in the noise strength ϵ.

Main Theorem 2. *Up to a certain stopping time τ satisfying $P(\tau = T) \xrightarrow{\epsilon \to 0} 1$ for any fixed time horizon T, $u(x,t)$ can be decomposed as*

$$\begin{aligned} u(x,t) &= \hat{u}(x - ct - C(t)) + \epsilon v_0(x,t) + \epsilon^2 v_1(x,t) + \ldots + \epsilon^k v_{k-1}(x,t) + r_k(x,t), \\ C(t) &= \epsilon C_0(t) + \epsilon^2 C_1(t) + \ldots + \epsilon^k C_{k-1}(t), \end{aligned}$$

where the v_k and C_k are independent of ϵ and are given explicitly as the solutions to stochastic evolution/differential equations, and where the rest terms r_k are of higher order in ϵ. The C_{k-1} describe the phase shift caused by the noise to order ϵ^k, and the v_k describe the fluctuations in the wave profile.

An approximate description of u is thus given by

$$u(x,t) \approx \hat{u}(x - ct - \epsilon C_0(t)) + \epsilon v_0(t).$$

We will see that C_0 is roughly diffusive and that v_0 is an Ornstein-Uhlenbeck process orthogonal to the direction of movement $\hat{u}_x$. This also expresses the stability of the traveling wave.

The problem of describing the effects of noise on the traveling wave has already been considered in [15]. Bressloff and Webber decomposed the solution into a shifted wave profile and fluctuations around it, and by carrying out a formal expansion, supported by numerical simulations, found that the phase shift is diffusive to first order of the noise strength.

From a mathematically rigorous perspective, in [37], Inglis and MacLaurin describe the phase of the stochastic wave by a stochastic differential equation and use it to study stability properties. They do not derive an expansion or give an explicit description of the influence of the noise.

To our knowledge, our result, building on work in [42], is the first mathematically rigorous description of the effects of noise to different orders of the noise strength.

5. The neural field equation approximates the network behavior in two ways. First, in the spirit of the law of large numbers, it describes the mean field behavior of the coupled populations as their size tends to infinity. Second, the network is approximated by a continuum. In Chapter 6 we make these two approximation steps explicit. In order to describe deviations from the mean field behavior for finite population sizes, we set up a Markov chain to describe the evolution of the activity in the finite network, extending a

model by Bressloff and Newby [13]. The transition rates are chosen in such a way that we obtain the voltage-based neural network equation in the infinite population limit. We analyze the fluctuations of the Markov chain in order to determine a stochastic correction term describing finite-size effects (question 1). We set up an approximating system of diffusion processes and prove that a stochastic neural field equation is obtained in the continuum limit (question 3).

Main Theorem 3. *The L^2-valued solution to the stochastic neural field equation*

$$du(x,t) = -u(x,t) + w * F(u(t))(x) + \sigma(u(t),t)(x)dW(t),$$

with a noise term accounting for finite-size effects on traveling wave solutions, is obtained as the strong continuum limit of an associated finite-dimensional network.

The model introduced in [13] allows to determine finite-size effects in the activity-based neural field equation. A candidate for a stochastic correction term was rigorously derived in [54] using convergence theorems for infinite-dimensional Markov jump processes, but the question of the well-posedness of the resulting stochastic neural field equation was left open. A detailed comparison of our approach with [13] and [54] will follow in Chapter 6 below.

To our knowledge, a well-posed stochastic neural field equation has not been derived in this context before, and finite-size effects in the voltage-based model have not been described up to now.

1.2.2 Outlook

Stochastic neural field equations constitute a wide area of research. This thesis takes a first step in providing a mathematical framework for their analysis. There are many interesting questions and phenomena related to stochastic neural fields that go beyond what is covered here.

A problem that is directly related is the analysis of the long-time behavior of the traveling wave. It has recently been considered in [37] by J. Inglis and J. MacLaurin. Under assumptions related to the L^2-spectral properties of the system they derive a long-term stability result. However, it is not clear whether or in which cases the assumptions are satisfied. The spectral gap we prove (cf. Chapter 3) allows for a stability analysis and for a description of the influence of the noise up to a fixed time horizon T (cf. Chapter 5), but on larger time scales we lose control over the nonlinear part of the dynamics in the weighted space $L^2(\rho)$. We derive a long-term L^2-stability result for small wave speeds c (cf. section 3.6) but have not been able to generalize it to arbitrary values of c.

A related question is how noise influences the dynamics on larger time scales. Many phenomena related to traveling waves in neural fields, such as stimulus locking or coupling of wave fronts, occur on a larger time-scale than considered in this thesis. A formal analysis of the effects of noise in such situations has recently been carried out in [12].

Another interesting problem is the analysis of the influence of noise on more general patterns, such as bumps or traveling pulses, and in higher-dimensional neural field equations. The methods presented here rely on the monotonicity of the traveling wave. An adaptation to other patterns is therefore not straightforward.

Chapter 2

Mathematical Setting

In this section, we introduce the mathematical setting in which we will work. We model the noise by a Q-Wiener process and solve the stochastic neural field equation. We identify a 'stochastic traveling wave solution' and introduce the operators and measures that will be studied in the following chapters.

2.1 Notation

We denote by L^p the Lebesgue space $L^p(\mathbb{R})$ equipped with the Lebesgue measure and by $\|\cdot\|_p$ the associated norm. We will usually work in L^2 with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. We denote by $H^k = H^k(\mathbb{R}) = W^{k,2}(\mathbb{R})$ the Sobolev space of k times weakly differentiable functions h satisfying $\sum_{i=0}^k \|h^{(i)}\|^2 < \infty$ equipped with the norm and inner product

$$\|h\|_{H^k} = \Big(\sum_{i=0}^k \|h^{(i)}\|^2\Big)^{\frac{1}{2}}, \qquad \langle g,h\rangle_{H^k} = \sum_{i=0}^k \langle g^{(i)}, h^{(i)}\rangle.$$

For a continuous density $\mu : \mathbb{R} \to (0,\infty)$ we denote by $L^p(\mu)$ and $H^k(\mu)$ the Lebesgue and Sobolev spaces with weight μ and associated norms

$$\|h\|_{L^p(\mu)} = \Big(\int h^p(x)\mu(x)dx\Big)^{\frac{1}{p}}, \qquad \|h\|_{H^k(\mu)} = \Big(\sum_{i=0}^k \|h^{(i)}\|^2_{L^2(\mu)}\Big)^{\frac{1}{2}}.$$

The $L^2(\mu)$-norm will also simply be denoted by $\|\cdot\|_\mu$.

We say that a bounded linear operator $A : H \to H$, where H is a separable Hilbert spaces, is of *finite trace*, if for any orthonormal basis (e_k) of H,

$$trA := \sum_k \langle Ae_k, e_k\rangle_H < \infty.$$

If U is another separable Hilbert space, then a bounded linear operator $A : U \to H$ is said to be *Hilbert-Schmidt* if $\sum_k \|Ae_k\|_H^2 < \infty$ for an orthonormal basis (e_k) of U. We denote by $L_2(U,H)$ the space of Hilbert-Schmidt operators from U to H equipped with the norm

$$\|A\|_{L_2(U,H)} := \left(\sum_k \|Ae_k\|_H^2 \right)^{\frac{1}{2}}.$$

In both cases the norm is independent of the choice of basis. We simply write L_2 for the space $L_2(L^2, L^2)$. Given a bounded linear operator $Q : U \to U$, the space $Q(U)$ is a Hilbert space with inner product $\langle Q^{-1}u, Q^{-1}v\rangle_U$. When the reference to Q is clear we write $L_2^0(U,H)$ for the space $L_2(Q(U),H)$ and simply L_2^0 if $U = H = L^2$.

We fix a probability space $(\Omega, \mathcal{F}, P)$. We denote by E the expectation with respect to P and by $L^p(\Omega)$ the space of real-valued random variables X on Ω satisfying $\|X\|_{L^p(\Omega)} := E(|X|^p)^{\frac{1}{p}} < \infty$. We denote by $L^p(\Omega; H)$ the space of random variables on Ω with values in the Hilbert space H for which $E(\|X\|_H^p)^{\frac{1}{p}} < \infty$, and by $L^p(\Omega \times [0,T]; H)$ the space of H-valued stochastic processes satisfying $E\Big(\int_0^T \|X(t)\|_H^p dt\Big)^{\frac{1}{p}} < \infty$.

2.2 A Stochastic Neural Field Equation

As mentioned above, it is not clear what is the nature and consequently also what is the right representation of the noise in neural fields. We therefore start by adding an abstract stochastic correction term to the neural field equation and consider the stochastic evolution equation

$$du(x,t) = \big(- u(x,t) + w * F(u(t))(x)\big)dt + \epsilon\tilde{\sigma}(t, u(t))dW(x,t), \tag{2.1}$$

where W is a Q-Wiener process, $\epsilon > 0$ is the strength of the noise, and $\tilde{\sigma}$ is a diffusion coefficient. Details on the theory of Q-Wiener processes can be found in [23] or [53].

Usually solutions to the neural field equation are not L^2-valued - think of the constant solutions $\mathbf{a_1}$ and $\mathbf{a_2}$, or of the traveling wave, of course. In [30], Faugeras and Inglis proved existence and uniqueness of a (mild) $L^2(\mu)$-valued solution to a stochastic neural field equation for densities $\mu \in L^1 \cap L^\infty$ for which there exists a constant C_μ such that

$$w * \mu \leq C_\mu \mu. \tag{2.2}$$

They show that, under very general assumptions, such a measure can be constructed using the Fourier transform of the kernel w.

Here we follow an approach already used in [57], [58], and [42], that will allow us to work in the unweighted space L^2. Later, in section 2.4, we introduce a weight ρ that is naturally associated to the problem. We will see that typically ρ satisfies (2.2), but grows exponentially so that it is neither bounded nor integrable.

Since we are interested in stochastic perturbations of the traveling wave solution, we formally define

$$v(x,t) = u(x,t) - u^{TW}(x,t)$$

to be the difference between any solution u to the neural field equation and the deterministic traveling wave. Then formally, v satisfies the stochastic evolution equation

$$dv(x,t) = \Big(-v(x,t) + w * \big(F(u^{TW}(t)+v(t)) - F(u^{TW}(t))\big)(x)\Big)dt + \epsilon\sigma(t,v(t))(x)dW_t, \quad (2.3)$$

with $\sigma(t,v) = \tilde{\sigma}(t, v + u^{TW}(t))$. This equation can be expected to possess an L^2-valued solution, and then $u^{TW}(x,t) + v(x,t)$ corresponds to a 'stochastic traveling wave solution'.

In order to give a meaning to (2.1) we start by making precise assumptions on the parameters.

2.2.1 Assumptions on the Parameters

The gain function F

We assume that

(i) $F \geq 0, \lim_{x\to-\infty} F(x) = 0, \lim_{x\to\infty} F(x) = 1$

(ii) $F(x) - x$ has exactly three zeros $0 < a_1 < a < a_2 < 1$

(iii) $F \in \mathcal{C}^k$ for some $k \geq 2$ and the derivatives are bounded

(iv) $F' > 0, F'(a_1) < 1, F'(a_2) < 1, F'(a) > 1$

The kernel w

We assume that

(i) $w \geq 0$ and w is homogeneous: $w(x,y) = w(x-y)$

(ii) w is even and $\int w(x)dx = 1$

(iii) w is absolutely continous and $w_x \in L^1$

Sometimes we will also assume that

$$\left\|\frac{w_x}{w}\right\|_\infty := \sup_{x:w(x)>0}\left|\frac{w_x(x)}{w(x)}\right| < \infty.$$

A classical choice for w is

$$w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$$

for some $\sigma > 0$. We will refer to this example as the *exponential example*. It will be analyzed in detail in Chapter 4.

The traveling wave

Assumption (iv) on F implies that a_1 and a_2 are stable fixed points of the voltage-based neural field equation (1.5), while a is an unstable fixed point. It has been shown in [27] that under these assumptions there exists a unique monotone traveling wave solution to (1.5) connecting the stable fixed points (and in [18], that traveling wave solutions are necessarily monotone). That is, there exists a unique wave profile $\hat{u} : \mathbb{R} \to [0,1]$ with $\hat{u}_x > 0$ and a unique wave speed $c \in \mathbb{R}$ such that $u^{TW}(x,t) := \hat{u}(x-ct)$ is a solution to (1.5), i.e.

$$-c\partial_x u_t^{TW}(x) = \partial_t u_t^{TW}(x) = -u_t^{TW}(x) + w * F(u_t^{TW})(x), \tag{2.4}$$

and

$$\lim_{x\to-\infty} \hat{u}(x) = a_1, \qquad \lim_{x\to\infty} \hat{u}(x) = a_2.$$

As also pointed out in [27] (Thm. 3.4), we can without loss of generality assume that $c \geq 0$ in the following sense: if $(\hat{u}, c)$ is the traveling wave associated with w, F, then $(\tilde{u}, -c)$, where $\tilde{u}(x) = 1 - \hat{u}(-x)$, is the traveling wave associated with $w, \tilde{F}$, where $\tilde{F}(x) = 1 - F(1-x)$ satisfies the same assumptions as F, the fixed points of $\tilde{F}(x) - x$ being $1-a_2, 1-a, 1-a_1$.

Note that $\hat{u}_x \in L^2(\mathbb{R})$, since in the case $c > 0$

$$\begin{aligned}\int \hat{u}_x^2(x)dx &= \int \hat{u}_x(x)\frac{1}{c}\left(\hat{u}(x) - w * F(\hat{u})(x)\right)dx \\ &\leq \frac{1}{c}\left(\|\hat{u}\|_\infty + \|F(\hat{u})\|_\infty \int w(x)dx\right)\int \hat{u}_x(x)dx = \frac{2}{c}a_2(a_2 - a_1),\end{aligned}$$

and in the case $c = 0$,

$$\int \hat{u}_x^2(x)dx = \int \hat{u}_x(x)\; w_x * F(\hat{u})(x)dx \leq \|w_x\|_1 a_2(a_2 - a_1).$$

$(I - c\partial_x)\hat{u} = w * F(\hat{u})$ implies that $\hat{u}$ can be represented as

$$\hat{u}(x) = \int_0^\infty e^{-s} w * F(\hat{u})(x + cs)ds. \tag{2.5}$$

By (2.4), the regularity of F and w determine that of $\hat{u}$. For example, if $F \in \mathcal{C}^k$, then $\hat{u} \in \mathcal{C}^{k+1}$, since

$$-c\hat{u}^{(k+1)}(x) = -\hat{u}^{(k)}(x) + w * ((F(\hat{u}))^{(k)})(x).$$

The noise

Let $Q : L^2 \to L^2$ be a symmetric non-negative bounded linear operator and fix some time horizon T. We consider two different cases.

Assumption (A).

(1) Additive noise: *Q is of finite trace and $\sigma(t,v) \equiv I$ is the identity operator on L^2. For example, $Q^{\frac{1}{2}}$ may be given by a symmetric integral kernel,*

$$Q^{\frac{1}{2}}v(x) = \int q(x,y)v(y)dy,$$

with $q(x,y) \in L^2(\mathbb{R}\times\mathbb{R})$. Then, by Parseval's identity,

$$trQ = \sum_k \|Q^{\frac{1}{2}}e_k\|^2 = \int\int q^2(x,y)dydx < \infty.$$

(2) Multiplicative noise: *$\sigma : [0,T]\times L^2 \to L_2^0(L^2,L^2)$, Lipschitz-continuous in the second variable uniformly in $t \le T$. For example, Q may be given by a symmetric integral kernel $q(x,y)$ satisfying $\sup_{x\in\mathbb{R}}(\|q(x,\cdot)\|_2 + \|q(x,\cdot)\|_1) < \infty$, and $\sigma(t,v)$ may be given by a multiplication operator associated to $\bar{\sigma}(t,v) \in L^2$, where $\bar{\sigma}$ is Lipschitz continuous in v uniformly in $t \le T$. Then $Q^{\frac{1}{2}} : L^2 \to L^2$ since by the Cauchy-Schwarz inequality*

$$\|Q^{\frac{1}{2}}v\|^2 \le \int\int q(x,y)dy \int q(x,y)v^2(y)dydx \le \sup_x \|q(x,\cdot)\|_1^2\|v\|^2,$$

and $\sigma(t,v) \in L_2^0$ since by Parseval's identity

$$\begin{aligned}\|\sigma(t,v)\|_{L_2^0}^2 &= \sum_k \|\bar{\sigma}(t,v)Q^{\frac{1}{2}}e_k\|_2^2 = \int \bar{\sigma}(t,v)^2(x)\|q(x,\cdot)\|^2dx \\ &\le \sup_x \|q(x,\cdot)\|^2\|\bar{\sigma}(t,v)\|^2 < \infty.\end{aligned}$$

The Lipschitz continuity follows similarly since

$$\|\sigma(t,v_1) - \sigma(t,v_2)\|_{L_2^0}^2 \le \sup_x \|q(x,\cdot)\|^2\|\bar{\sigma}(t,v_1) - \bar{\sigma}(t,v_2)\|^2.$$

Sometimes we will ask for some more regularity and assume the following.

Assumption (B).

(1) Additive noise: *$Q^{\frac{1}{2}} \in L_2(L^2,H^1)$ and $\sigma(t,v) \equiv I$. For example, $Q^{\frac{1}{2}}$ may be given by a symmetric integral kernel $q(x,y)$ satisfying $\sup_x \|\partial_x q(x,\cdot)\| < \infty$ and $\int \|q(x,\cdot)\|_{H^1}^2 dx < \infty$. Then $\frac{d}{dx}Q^{\frac{1}{2}}v = \int q_x(x,y)v(y)dy$ for all $v \in L^2$, and for an orthonormal basis (e_k) of L^2, by Parseval's identity,*

$$\begin{aligned}\|Q^{\frac{1}{2}}\|_{L_2(L^2,H^1)}^2 &= \sum_k \left\| \int q(x,y)e_k(y)dy \right\|_{H^1}^2 \\ &= \int \|q(x,\cdot)\|^2 + \|\partial_x q(x,\cdot)\|^2 dx = \int \|q(\cdot,y)\|_{H^1}^2 dy < \infty.\end{aligned}$$

(2) Multiplicative noise: $\sigma : [0,T] \times H^1 \to L_2^0(L^2, H^1)$, *Lipschitz-continuous in v uniformly in t. For example, $Q^{\frac{1}{2}}$ may be given by a homogeneous kernel $q \in H^1$ satisfying $q, q_x \in L^1$ and $\sigma(t,v)$ may be given by a multiplication operator associated to $\bar{\sigma} : [0,T] \times H^1 \to H^1$, Lipschitz continuous in the second variable uniformly in $t \le T$. Then for $v \in L^2$, $Q^{\frac{1}{2}} v \in H^1$ since*

$$\begin{aligned}\|Qv\|_{H^1}^2 &\le \int \int q(x-y)dy \int q(x-y)v^2(y)dy + \int |q_x(x-y)|dy \int |q_x(x-y)|v^2(y)dydx \\ &= (\|q\|_1^2 + \|q_x\|_1^2)\|v\|^2\end{aligned}$$

and $\sigma(t,v) \in L_2^0(L^2, H^1)$ since for an orthonormal basis (e_k) of L^2, by Parseval's identity,

$$\begin{aligned}&\|\sigma(t,v)\|_{L_2^0(L^2,H^1)}^2 \\ &= \sum_k \int \bar{\sigma}^2(t,v)(x)\Big(\Big(\int q(x-y)e_k(y)dy\Big)^2 + \Big(\int q_x(x-y)e_k(y)dy\Big)^2\Big)dx \\ &\quad + \int \Big(\frac{d}{dx}\bar{\sigma}(t,v)(x)\Big)^2 \Big(\int q(x-y)e_k(y)dy\Big)^2 dx \\ &= \|q\|_{H^1}^2 \|\bar{\sigma}(t,v)\|^2 + \|q\|^2 \Big\|\frac{d}{dx}\bar{\sigma}(t,v)\Big\|^2 < \infty.\end{aligned}$$

The covariance operator Q defines a Q-Wiener process W on $(\Omega, \mathcal{F}, \mathbb{P})$. Q describes the correlations of the process W. If $Q^{\frac{1}{2}}$ is given by an integral kernel $q(x,y)$, then formally,

$$\text{"}E(W_t(x)W_s(y)) = E(\langle \delta_x, W_t\rangle \langle \delta_y, W_s\rangle) = s \wedge t \int q(x,z)q(z,y)dz\text{"}.$$

Note that if Q is of finite trace, then the correlations cannot be translation invariant. However, in the multiplicative noise cases it is possible to choose $q(x,y) = \bar{q}(x-y)$.

2.2.2 Solving the SNFE

Proposition 2.2.1. *(i) If Q and σ satisfy assumption (A), then for every initial condition $\eta \in L^2$, there exists a unique strong L^2-valued solution v to (2.3). v admits a continuous modification. For all $p \ge 1$,*

$$E\Big(\sup_{0 \le t \le T} \|v(t)\|^p\Big) < \infty.$$

$u(x,t) := u^{TW}(x,t) + v(x,t)$ is a solution to (2.1).

(ii) If $\eta \in H^1$, Q and σ satisfy assumption (B), and $\big\|\frac{w_x}{w}\big\|_\infty < \infty$, then v takes values in H^1 and for any $p \ge 1$,

$$E\Big(\sup_{0 \le t \le T} \|v(t)\|_{H^1}^p\Big) < \infty.$$

Proof. (i) It is enough to show that the drift $B : [0,T] \times L^2 \to L^2$,

$$B(t,v) = -v + w * (F(u_t^{TW} + v) - F(u_t^{TW})),$$

is Lipschitz continuous in v uniformly in t. Then the claim follows for example from [23], Thm. 7.2 (with $A = 0$). Now for $v_1, v_2 \in L^2$, using Jensen's inequality and the Lipschitz-continuity of F,

$$\begin{aligned}
&\|B(t,v_1) - B(t,v_2)\|^2 \\
&\le 2\|v_1 - v_2\|^2 + 2\int\int w(x-y)\big(F(u^{TW}(y,t) + v_1(y)) - F(u_t^{TW}(y,t) + v_2(y))\big)^2 dydx \\
&\le 2\|v_1 - v_2\|^2 + 2\|F'\|_\infty^2 \int\int w(x-y)(v_1(y) - v_2(y))^2 dydx \\
&= 2(1 + \|F'\|_\infty^2)\|v_1 - v_2\|^2.
\end{aligned}$$

(ii) If $\left\|\frac{w_x}{w}\right\|_\infty < \infty$, then for $v_1, v_2 \in H^1$,

$$\begin{aligned}
&\|B(t,v_1) - B(t,v_2)\|_{H^1} \\
&\le 2\|v_1 - v_2\|_{H^1}^2 + 2\Big(1 + \Big\|\frac{w_x}{w}\Big\|_\infty\Big) \\
&\qquad \int\int w(x-y)\big(F(u^{TW}(y,t) + v_1(y)) - F(u_t^{TW}(y,t) + v_2(y))\big)^2 dydx \\
&\le 2\|v_1 - v_2\|_{H^1}^2 + 2\Big(1 + \Big\|\frac{w_x}{w}\Big\|_\infty\Big)\|F'\|_\infty^2 \|v_1 - v_2\|^2.
\end{aligned}$$

□

2.3 Linearization around the Traveling Wave

2.3.1 The Traveling Wave Operator

The analysis of the influence of the noise on the traveling wave solution is related to properties of the linear operator appearing in the equation when linearizing around the traveling wave. We write

$$dv(x,t) = \Big(-v(x,t) + w * (F'(u^{TW}(t))v)(x) + R(t,v(t))(x)\Big)dt + \epsilon\sigma(t,v(t))(x)dW(x,t),$$

where

$$R(t,v) = w * \Big(F(u^{TW}(t) + v) - F(u^{TW}(t)) - F'(u^{TW}(t))v\Big).$$

We define the *traveling wave operator* as the bounded linear operator on L^2 acting as

$$Lv = -v + w * (F'(\hat{u})v), \tag{2.6}$$

and the family of uniformly bounded time-dependent linear operators

$$L_t v = -v + w * (F'(u^{TW}(t))v). \tag{2.7}$$

Note that

$$L\hat{u}_x = -c\hat{u}_{xx}.$$

In particular, if $c = 0$, then $\hat{u}_x$ is an eigenfunction of L to the eigenvalue 0.

By Taylor's formula, there exists $\xi(y, t)$ such that

$$R(t, v) = \frac{1}{2}\int w(x-y)\Big(F''(u^{TW}(y,t) + \xi(y,t))v^2(y)\Big)dy.$$

Using the Cauchy-Schwarz inequality it follows that

$$\begin{aligned}\|R(t,v)\|^2 &\leq \frac{1}{4}\|F''\|_\infty^2 \int\int w^2(x-y)v^2(y)dy \int v^2(y)dydx \\ &\leq \frac{1}{4}\|F''\|_\infty^2\|w\|_\infty\|v\|^2 \int\int w(x-y)dxv^2(y)dy = \frac{1}{2}\|F''\|_\infty^2\|w\|_\infty\|v\|^4.\end{aligned} \tag{2.8}$$

The rest term $R(t, v)$ is thus of higher order in $\|v\|$. It can therefore be expected that the stability properties of the traveling wave depend only on the linear operator L.

The traveling wave operator L is self-adjoint in $L^2(\mathfrak{m})$, where

$$\mathfrak{m}(dx) = F'(\hat{u}(x))dx. \tag{2.9}$$

Note that since $F'(\hat{u})$ is bounded from above and bounded away from 0, the $L^2(\mathfrak{m})$-norm is equivalent to the L^2-norm.

2.3.2 The Frozen Wave Operator

If $c = 0$, that is, if the traveling wave is actually a standing wave, then $\hat{u}_x$ is an eigenfunction of L to the eigenvalue 0. If $c \neq 0$, then the movement of the wave makes the analysis more difficult. In particular, $\hat{u}_x$ is no longer an eigenfunction of L.

It will therefore sometimes be useful to work in the moving frame picture. That is, we can freeze the wave by moving instead the coordinates. For $h : [0, T] \to H^1$ set $h^\#(x, t) := \Phi_t h(x, t) := h(x + ct, t)$. If the solution v to the stochastic neural field equation (2.3) takes values in H^1, then for $g \in H^1$, by Itô's lemma,

$$\begin{aligned}\langle v^\#(t), g\rangle &= \langle v(t), g(\cdot - ct)\rangle \\ &= \langle v(0), g\rangle - c\int_0^t \langle v(s), g_x(\cdot - cs)\rangle ds \\ &\quad + \int_0^t \langle L_s v(s) + R(s, v(s)), g(\cdot - cs)\rangle ds + \epsilon\int_0^t \langle g(\cdot - cs), \sigma(s, v(s))dW_s\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle v^{\#}(0), g\rangle + c\int_0^t \langle \partial_x v^{\#}(s), g\rangle ds + \int_0^t \langle -v^{\#}(s) + w * (F'(\hat{u})v^{\#}(s)), g\rangle ds \\
&\quad + \int_0^t \langle R^{\#}(v^{\#}(s)), g\rangle ds + \epsilon \int_0^t \langle g, \Phi_s \sigma(s, v^{\#}(\cdot - cs)) dW_s\rangle \\
&= \langle v^{\#}(0), g\rangle + \int_0^t \langle L^{\#} v^{\#}(s), g\rangle ds \\
&\quad + \int_0^t \langle R^{\#}(v^{\#}(s)), g\rangle ds + \epsilon \int_0^t \langle g, \Phi_s \sigma(s, v^{\#}(\cdot - cs)) dW_s\rangle,
\end{aligned}
$$

where $L^{\#}$ is the *frozen wave operator* given by

$$L^{\#} v = -v + c\partial_x v + w * (F'(\hat{u})v), \qquad \mathcal{D}(L^{\#}) = H^1,$$

and $R^{\#}(v) = w * (F(\hat{u}+v) - F(\hat{u}) - F'(\hat{u})v)$. That is, $v^{\#}(t)$ is the weak solution in L^2 to the *frozen wave equation*

$$dv^{\#}(t) = \big(L^{\#} v^{\#}(t) + R^{\#}(v^{\#}(t))\big) dt + \epsilon \Phi_t \sigma(t, v^{\#}(\cdot - ct)) dW_t.$$

Note that $L^{\#}(\hat{u}_x) = 0$.

2.4 The Measure ρ

2.4.1 The Adjoint Eigenfunction

Proposition 2.4.1. *The adjoint of $L^{\#}$ is*

$$L^{\#,*} v = -v - c\partial_x v + F'(\hat{u}) w * v, \qquad \mathcal{D}(L^{\#,*}) = H^1.$$

There exists a unique (up to constant multiples) $0 \not\equiv \psi \in H^1$ such that $L^{\#,*}\psi = 0$, and $\psi > 0$.

Proof. The proof is similar to that of Thm. 4.2 and 4.3 in [27]. There exist $\delta, M > 0$ such that for all x with $|x| \geq M$, $F'(\hat{u}(x)) \leq 1 - \delta$. Consider the operator on L^2

$$w * (F'(\hat{u})v) = Kv + Bv$$

where

$$Kv(x) = \int_{-M}^{M} w(x-y) F'(\hat{u}(y)) v(y) dy$$

and

$$Bv(x) = \left(\int_{-\infty}^{-M} + \int_{M}^{\infty} \right) w(x-y) F'(\hat{u}(y)) v(y) dy.$$

Then $\|B\| \leq 1 - \delta$ and since

$$\int\int \big(w(x-y)\mathbb{1}_{(-M,M)}(y)F'(\hat{u}(y))\big)^2 dydx \leq 2M\|F'\|_\infty^2\|w\|^2 < \infty,$$

K is Hilbert-Schmidt and hence compact. We have $L^\# = A + K$, where

$$A = -I + c\partial_x + B.$$

We show that A has a bounded inverse.

(i) A is injective.

Let $v \in ker(A)$, then

$$0 = \langle Av, v\rangle = -\|v\|^2 + \underbrace{c\langle\partial_x v, v\rangle}_{=0} + \langle Bv, v\rangle \leq -\delta\|v\|^2,$$

which implies that $v \equiv 0$.

(ii) A^{-1} is continuous.

Indeed, for $v \in H^1$ we have as in (i)

$$\delta\|v\|^2 \leq \langle -Av, v\rangle \leq \|Av\|\|v\|,$$

which implies that for all $u \in ran(A)$

$$\|A^{-1}u\| \leq \frac{1}{\delta}\|u\|.$$

(iii) $D(A^*) = H^1$.

$H^1 \subset D(A^*)$ is clear. Let $\phi \in D(A^*)$. Then there exists $\phi^* \in L^2$ such that $\langle\phi, Av\rangle = \langle\phi^*, v\rangle$ for all $v \in H^1$. Then

$$c\langle v_x, \phi\rangle = \langle\phi, Av + v - Bv\rangle = \langle\phi^* + \phi - B^*\phi, v\rangle$$

and it follows that $\phi \in H^1$.

(iv) A^* is injective.

$$\langle A^*\phi, \phi\rangle = -\|\phi\|^2 - c\underbrace{\langle\phi_x, \phi\rangle}_{=0} + \langle B^*\phi, \phi\rangle \leq -\delta\|\phi\|.$$

Thus, if $\phi \in ker(A^*)$, then $\phi \equiv 0$.

(v) A is surjective.

We have $\overline{ran(A)} = ker(A^*)^\perp = L^2$. Let $\phi \in L^2$. Then there exist $\xi_n \in H^1$ such that for $\phi_n := A\xi_n$, $\|\phi_n - \phi\| \xrightarrow{n\to\infty} 0$. Since $\|\xi_m - \xi_n\| \le \|A^{-1}\|\|\phi_n - \phi_m\| \xrightarrow{n,m\to\infty} 0$, $\xi := \lim \xi_n$ exists in L^2. Since for $v \in H^1$

$$\begin{aligned} c\langle \xi, v_x\rangle &= \lim c\langle \xi_n, v_x\rangle = \lim -c\langle (\xi_n)_x, v\rangle \\ &= \lim\langle -A\xi_n - \xi_n + B\xi_n, v\rangle = \langle -\phi - \xi + B\xi, v\rangle, \end{aligned}$$

it follows that $\xi \in H^1$, and since

$$\langle A\xi, v\rangle = \langle \xi, A^*v\rangle = \lim\langle \xi_n, A^*v\rangle = \lim\langle \phi_n, v\rangle = \langle \phi, v\rangle,$$

$\phi = A\xi$.

We have $A^{-1}L^\# = I + A^{-1}K$ and $A^{-1}K$ is compact. Since $A^{-1}L^\#\hat{u}_x = 0$ and since there are no other eigenfunctions with eigenvalue 0 as proven in Thm. 4.2 in [27], there exists a unique $\tilde{\psi} \not\equiv 0$ such that $L^{\#,*}(A^*)^{-1}\tilde{\psi} = 0$, hence $L^{\#,*}\psi = 0$ where $\psi := (A^*)^{-1}\tilde{\psi} \in H^1$.

Since $c\psi_x = -\psi + F'(\hat{u})w * \psi$, we actually have $\psi \in \mathcal{C}^1$. We show that ψ is of one sign. Assume without loss of generality that there exists x such that $\psi(x) > 0$. Set $\psi^+(x) = \psi(x) \vee 0$. Then $\psi^+ \in H^1$ with $\psi_x^+ \equiv \psi_x$ on $\{\psi \ge 0\}$ and $\psi_x^+ \equiv 0$ on $\{\psi < 0\}$. Thus, on $\{\psi \ge 0\}$, $L^{\#,*}\psi^+ = -\psi - c\psi_x + F'(\hat{u})w * \psi^+ \ge L^{\#,*}\psi(x) = 0$, and on $\{\psi < 0\}$, $L^{\#,*}\psi^+ = F'(\hat{u})w * \psi^+ \ge 0$. Since $\hat{u}_x > 0$, $L^{\#,*}\psi^+ \ge 0$, and

$$0 = \langle L^\#\hat{u}_x, \psi^+\rangle = \langle \hat{u}_x, L^{\#,*}\psi^+\rangle,$$

it follows that $L^{\#,*}\psi^+ \equiv 0$ and hence $\psi^+ \equiv \psi$. As in (2.5), since $(I + c\partial_x)\psi = F'(\hat{u})w * \psi$, ψ satisfies

$$\psi(x) = \int_0^\infty e^{-s}F'(\hat{u}(x - cs))w * \psi(x - cs)ds. \tag{2.10}$$

Now the strict positivity of ψ follows as in the proof of Thm. 4.2 in [27]. □

We normalize ψ such that $\langle \hat{u}_x, \psi\rangle = 1$. Set

$$\rho(x) = \frac{\psi(x)}{\hat{u}_x(x)}.$$

Note that for $h \in H^1$,

$$\langle L^\# h, \hat{u}_x\rangle_\rho = \langle L^\# h, \psi\rangle = \langle h, L^{\#,*}\psi\rangle = 0, \tag{2.11}$$

that is, $L^\#(H^1) \subset \hat{u}_x^\perp$, where we denote by $\hat{u}_x^\perp$ the orthogonal complement of $\hat{u}_x$ in $L^2(\rho)$. In $L^2(\rho)$, the direction of movement of the wave $\hat{u}_x$ and the orthogonal directions are thus naturally separated by $L^\#$, which makes it a natural choice of function space to work in.

Note that if $c = 0$, then $\rho(x) = \frac{1}{\int \hat{u}_x^2(x)F'(\hat{u}(x))dx}F'(\hat{u}(x))$ such that the $L^2(\rho)$-norm is equivalent to the L^2-norm. Typically, for $c \neq 0$, the $L^2(\rho)$-norm will not be equivalent to the L^2-norm. We will see this for the exponential example in Proposition 4.2.1 below.

Remark 2.4.2. We cannot expect to have the same control over the rest term R in $L^2(\rho)$ as in L^2 (cf. (2.8)). We will see on the exponential example in section 4.2 that if $c > 0$, then typically there exists $L_\rho > 0$ such that $\rho(y) \leq L_\rho \rho(x)$ for $y \leq x$, and $\lim_{x \to -\infty} \rho(x) = 0$. Now assume that there exists $C > 0$ such that for $v \in L^2(\rho)$,

$$\|w * v^2\|_\rho^2 = \int\int\int w(x-y_1)w(x-y_2)v^2(y_1)v^2(y_2)\rho(x)dy_1dy_2dx \leq C\|v\|_\rho^4.$$

Formally, letting $v^2 \to \delta_y$ we obtain

$$\int w^2(x-y)\rho(x)dx \leq C\rho^2(y).$$

But

$$\frac{\int w^2(x-y)\rho(x)dx}{\rho^2(y)} \geq \frac{\int_y^\infty w^2(x-y)dx\rho(y)}{L_\rho \rho^2(y)} = \frac{\int_0^\infty w^2(x)dx}{L_\rho \rho(y)} \xrightarrow{y \to -\infty} \infty,$$

which is a contradiction.

2.4.2 Solving the SNFE in $H^1(1+\rho)$

To be able to work in $L^2(\rho)$ (or $H^1(\rho)$), we need to verify that the solution to the stochastic neural field equation is an element of this space. We therefore consider the space $H^1(1+\rho) = H^1 \cap H^1(\rho)$ equipped with the norm

$$\|h\|_{H^1(1+\rho)} = \left(\|h\|_{H^1}^2 + \|h\|_{H^1(\rho)}^2\right)^{\frac{1}{2}}.$$

To obtain a solution to (2.3) in $H^1(1+\rho)$ we adapt our assumptions on the noise.

Assumption (C).

(1) Additive noise: $Q^{\frac{1}{2}} \in L_2(L^2, H^1(1+\rho))$. *$Q^{\frac{1}{2}}$ may for example be given by a symmetric integral kernel $q(x,y) \geq 0$ with $q(x,\cdot) \in H^1(1+\rho)$ and $\sup_x \|\partial_x q(x,\cdot)\| < \infty$, and satisfying $\int \|q(x,\cdot)\|_{H^1(1+\rho)}^2 dx < \infty$. Then for any orthonormal basis (e_k) of L^2, using Parseval's identity,*

$$\begin{aligned}\sum_k \|Q^{\frac{1}{2}} e_k\|_{H^1(1+\rho)}^2 &= \sum_k \int \left(\int q(x,y)e_k(y)dy\right)^2 + \left(\int \partial_x q(x,y)e_k(y)dy\right)^2 (1+\rho(x))dx \\ &= \int \left(\|q(x,\cdot)\|^2 + \|\partial_x q(x,\cdot)\|^2\right)(1+\rho(x))dx \\ &= \int \|q(\cdot,y)\|_{H^1(1+\rho)}^2 dy < \infty.\end{aligned}$$

(2) Multiplicative noise: $\sigma : [0,T] \times H^1(1+\rho) \to L_2^0(L^2, H^1(1+\rho))$, *Lipschitz continuous in the second variable uniformly in $t \leq T$. In analogy to (B)(ii), $Q^{\frac{1}{2}}$ may for example be given by a homogeneous kernel $q \in H^1(1+\rho)$ satisfying $q, q_x \in L^1(1+\rho)$, and $\sigma(t,v)$ may be given by a multiplication operator associated to $\bar{\sigma} : [0,T] \times H^1(1+\rho) \to H^1(1+\rho)$, Lipschitz continuous in the second variable uniformly in $t \leq T$.*

We make the following assumptions on ρ.

Assumption 2.4.3.

(i) There exists a constant L_ρ such that for all $x \in \mathbb{R}$ and $y > 0$,

$$\begin{aligned} \rho(x-y) &\leq L_\rho \rho(x) \qquad \text{if } c \geq 0, \\ \rho(x+y) &\leq L_\rho \rho(x) \qquad \text{if } c < 0. \end{aligned} \tag{2.12}$$

(ii) There exists a constant $K_\rho > 0$ such that

$$w * \rho(x) \leq K_\rho \rho(x). \tag{2.13}$$

Condition (i) says that ρ should be roughly increasing. It will be needed only later in Chapter 5. If $\left\| \frac{w_x}{w} \right\|_\infty < \infty$, then we have an a priori bound on the growth of ρ. By (2.10),

$$|\psi_x(x)| \leq \left(\left\| \frac{F''(\hat{u})\hat{u}_x}{F'(\hat{u})} \right\|_\infty + \left\| \frac{w_x}{w} \right\|_\infty \right) \psi(x).$$

Similarly, differentiating in (2.5) yields

$$|\hat{u}_{xx}(x)| \leq \left\| \frac{w_x}{w} \right\|_\infty \hat{u}_x(x),$$

such that

$$|\rho_x| = \left| \frac{\psi_x}{\psi} - \frac{\hat{u}_{xx}}{\hat{u}_x} \right| \rho \leq \left(\left\| \frac{F''(\hat{u})\hat{u}_x}{F'(\hat{u})} \right\|_\infty + 2 \left\| \frac{w_x}{w} \right\|_\infty \right) \rho =: M\rho. \tag{2.14}$$

Condition (ii) says roughly that ρ should neither grow nor decay too quickly relative to w. A condition of this kind has already been introduced in [30] to prove existence of solutions to the stochastic neural field equation in a weighted space (cf. (2.2).

Both conditions are satisfied in the exponential example analyzed in chapter 4.

Proposition 2.4.4. *Assume that σ and Q satisfy assumption (C), ρ satisfies (2.13), and that $\left\| \frac{w_x}{w} \right\|_\infty < \infty$. Then for any initial condition $\eta \in H^1(1+\rho)$ there exists a unique strong $H^1(1+\rho)$-valued solution v to the stochastic evolution equation (2.3). v admits a continuous modification and for all $p \geq 1$,*

$$E\left(\sup_{0 \leq t \leq T} \|v(t)\|^p_{H^1(1+\rho)} \right) < \infty.$$

Proof. As in the proof of Proposition 2.2.1, it is enough to show that $B : [0,T] \times H^1(1+\rho) \to H^1(1+\rho)$, $B(t,v) := -v + w * (F(\hat{u}(\cdot - ct) + v) - F(\hat{u}(\cdot - ct)))$ is Lipschitz continuous in v. This follows from the fact that for $v_1, v_2 \in H^1(1+\rho)$, using (2.13),

$$
\begin{aligned}
&\left\| w * \Big(F(\hat{u}(y-ct) + v_1(y)) - F(\hat{u}(y-ct) + v_2(y)) \Big) \right\|^2_{H^1(1+\rho)} \\
&\leq \Big(1 + \Big\| \frac{w_x}{w} \Big\|^2_\infty \Big) \|F'\|^2_\infty \int \int w(x-y)(1+\rho(x))dx \ (v_1(y) - v_2(y))^2 dy \\
&\leq \Big(1 + \Big\| \frac{w_x}{w} \Big\|^2_\infty \Big) \|F'\|^2_\infty \int (1 + K_\rho \rho(y))(v_1(y) - v_2(y))^2 dy \\
&\leq \Big(1 + \Big\| \frac{w_x}{w} \Big\|^2_\infty \Big) \|F'\|^2_\infty (1 + K_\rho) \|v_1 - v_2\|^2_{1+\rho}.
\end{aligned}
$$
□

2.5 The Stochastic Traveling Wave

The solution $u = u^{TW} + v$ to the stochastic neural field equation (2.1) is a stochastic perturbation of the traveling wave solution u^{TW}. The noise has two effects. First, it causes fluctuations in the wave profile. Second, it causes a random shift in the position, or phase, of the wave. It is one of the main objectives in this thesis to give a mathematically rigorous description of these two effects (cf. Chapter 5). To this end, we need to define what exactly we mean by the 'position' of the wave.

2.5.1 The Phase of the Wave

We loosely define the phase φ of a 'wave-like' function u to be a minimizer of

$$a \mapsto \|u - \hat{u}(\cdot - a)\|, \tag{2.15}$$

the L^2-distance between u and all possible translations of the deterministic wave profile $\hat{u}$.

In order to determine the phase shift caused by the noise, we dynamically adapt the phase of a reference wave to match that of the stochastic solution. The idea is to move along the gradient of (2.15) towards the minimum. This can be seen as a dynamic version of the freezing of traveling waves applied by Lord and Thümmler [47]. If we let a depend on a parameter s and differentiate, we obtain

$$\frac{d}{ds}\|u - \hat{u}(\cdot - a(s))\|^2 = 2\dot{a}(s)\langle u - \hat{u}(\cdot - a(s)), \hat{u}_x(\cdot - a(s))\rangle.$$

If we now choose a such that $\dot{a}(s) = -\langle u - \hat{u}(\cdot - a(s)), \hat{u}_x(\cdot - a(s))\rangle$,
then $\frac{d}{ds}\|u - \hat{u}(\cdot - a(s))\|^2 \leq 0$, which means a should move towards the right phase.

This motivates the following dynamics which were first introduced in [57] and [58] in the context of reaction-diffusion equations and already used in [42] in the neural field setting.

Let $C^m(t)$ be the solution to the pathwise ordinary differential equation

$$\dot{C}^m(t) := -m\langle u(t) - \hat{u}(\cdot - ct - C^m(t)), \hat{u}_x(\cdot - ct - C^m(t))\rangle, \quad C^m(0) = 0, \tag{2.16}$$

which has been proven to exist in [42], Prop. 3.5. Here $m > 0$ is a parameter that determines the rate of relaxation to the right phase.

One might also want to consider the phase of the wave in $L^2(m)$ or $L^2(\rho)$. In a weighted space, it is convenient to move the measure with the wave such that, for example,

$$\|\hat{u}(\cdot - ct - C^m(t))\|_{1+\rho(\cdot - ct - C^m(t))} = \|\hat{u}\|_{1+\rho}$$

for all times t. We define the phase adaptation in $L^2(\mathfrak{m})$ as

$$\dot{C}^m(t) := -m\langle u(t) - \hat{u}(\cdot - ct - C^m(t)), \hat{u}_x(\cdot - ct - C^m(t))\rangle_{\mathfrak{m}(\cdot - ct - C^m(t))}, \tag{2.17}$$

or in $L^2(\rho)$ as

$$\begin{aligned}\dot{C}^m(t) &:= -m\langle u(t) - \hat{u}(\cdot - ct - C^m(t)), \hat{u}_x(\cdot - ct - C^m(t))\rangle_{\rho(\cdot - ct - C^m(t))} \\ &= -m\langle u(t) - \hat{u}(\cdot - ct - C^m(t)), \psi(\cdot - ct - C^m(t))\rangle.\end{aligned} \tag{2.18}$$

Existence of a unique pathwise solution can be proven analogously.

It cannot in general be expected that there exists a unique global minimum of (2.15) as discussed in [37]. Here C^m is designed to follow the local minimum that is closest to the initial phase.

In [37], Inglis and MacLaurin derive an SDE describing the dynamics of this local minimum exactly. This should correspond to the case $m \to \infty$ in our picture (cf. section 5.5). For finite m, our approach gives an approximate description in terms of an ODE. In particular it provides a way of calculating the phase of the stochastic wave from a realization without explicit knowledge of the noise.

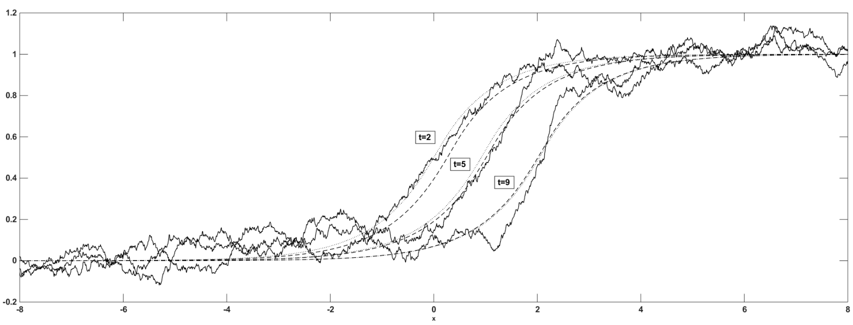

Figure 2.1: A stochastic traveling wave at different times t. Dashed line: $\hat{u}(x - ct)$. Dotted line: $\hat{u}(x - ct - C^m(t))$.

2.5.2 Stability under Noise

When simulating stochastic traveling waves, one can still clearly recognize a traveling front (cf. Fig. 2.1). How can this stability be expressed? We expect that small perturbations of the wave profile will over time be balanced out by the neural field dynamics. However, if we consider two waves with different phases, then we would rather expect the two waves to travel next to each other at a fixed distance.

This intuitive picture can be formalized in terms of the traveling wave operator. As outlined in section 2.3.1,

$$\frac{d}{dt}\|v(t)\|^2 \approx 2\langle L_t v(t), v(t)\rangle.$$

If there exist $\kappa, Z > 0$ such that

$$\langle Lv, v\rangle \leq -\kappa\|v\|^2 + Z\langle v, \hat{u}_x\rangle^2, \tag{2.19}$$

then perturbations that are orthogonal to the direction of movement $\hat{u}_x$ will even out exponentially fast. Perturbations in the direction of movement, on the other hand, correspond to a shift in the phase.

We expect this shift to be compensated by the phase adaptation $\dot{C}^m$ such that the stochastic traveling wave can be represented as a fluctuating wave profile moving at an adapted speed,

$$u(x,t) \approx \hat{u}(x - ct - C^m(t)) + \text{‘fluctuations’}.$$

This question will be considered in detail in chapter 5.

Recall that for the frozen wave operator $L^\#$ acting on $H^1(\rho)$, $L^\#\hat{u}_x = 0$ and $L^\#(H^1(\rho)) \subset \hat{u}_x^\perp$, such that the direction of movement and the orthogonal directions are naturally separated. That suggests to consider the $L^2(\rho)$-version

$$\langle L^\# v, v\rangle_\rho \leq -\kappa\|v\|_\rho^2 + Z\langle v, \hat{u}_x\rangle_\rho^2 \tag{2.20}$$

of (2.19). In the next chapter, we will prove (2.20) for a general class of nonlocal evolution equations possessing traveling wave solutions.

Chapter 3

The Spectral Gap

In this chapter, we prove that the frozen wave operator has a spectral gap in $L^2(\rho)$. The result applies to a general class of nonlocal evolution equations exhibiting traveling wave solutions. In section 3.5 we show that for small wave speeds c, the traveling wave operator satisfies a spectral gap inequality (2.19) in $L^2(m)$. This space has the advantage that the $L^2(\mathfrak{m})$-norm is equivalent to the L^2-norm. Together with the phase adaptation defined in section 2.5.1 it will allow us to derive a stochastic long-time stability result in section 3.6

3.1 Introduction

We consider a nonlocal evolution equation of the form

$$\partial_t u(x,t) = d\partial_{xx} u(x,t) + S(u, w * g(u))(x,t), \tag{3.1}$$

where $x \in \mathbb{R}$, $t \geq 0$, and

- $d \geq 0$
- $g \in C^1(\mathbb{R})$ with $g' > 0$
- $S \in C^1(\mathbb{R} \times \mathbb{R})$ with $\partial_2 S > 0$, and $x \mapsto S(x, g(x))$ is bistable: there exist exactly three zeroes $a_1 < a < a_2$ such that $S(a_i, g(a_i)) = S(a, g(a)) = 0$, $\frac{d}{dx} S(a_i, g(a_i)) < 0$, $i = 1, 2$, and $\frac{d}{dx} S(a, g(a)) > 0$.
- $w \geq 0$ is differentiable almost everywhere and $\int w(x)dx = 1$, $\int |w_x(x)|dx < \infty$, and $w * h$ denotes the convolution $\int w(x-y)h(y)dy$

We assume that there exists a unique monotone traveling wave solution connecting the stable states of the equation, that is, there exists a unique wave profile $\hat{u} \in C^1(\mathbb{R})$ with $\hat{u}_x > 0$ and $\lim_{x\to-\infty} \hat{u}(x) = a_1$, $\lim_{x\to\infty} \hat{u}(x) = a_2$, and a unique wave speed $c \in \mathbb{R}$ such that

$$u^{TW}(x,t) = \hat{u}(x-ct)$$

is a solution to (3.1).

In [19], Chen proved existence, uniqueness, and exponential stability (in L^∞) of a monotone traveling wave solution to (3.1) for a wide class of evolution equations of the form (3.1). Apart from the neural field equation, he collected the following examples.

- *Ising Model.* Here

  $$\partial_t u(t) = \tanh(\beta(w * u(t) + h)) - u(t),$$

  where $\beta > 1$, $0 \le w \in \mathcal{C}^2$ is even and supported on $[-1,1]$, and $0 \le h \le h^*$. Existence, uniqueness and stability of a monotone traveling front was proven in [24] for small h and in [49] in the general case.

- *Convolution Model for Phase Transitions.*

  $$\partial_t u(t) = \lambda w * u(t) - u(t) + f(u(t)),$$

  where $\lambda > 0$, $0 \le w \in \mathcal{C}^1$ is even and f is bistable. Existence and Uniqueness of a monotone traveling front is established in [4].

- *Thalamic Model.*

  $$\partial_t u(t) = -u(t) + h(1-u(t))F(w * (u^p(t)) - \Theta),$$

  where $h, \Theta > 0$, $p \in 1,2,3,4$, $w(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$ for some $\sigma > 0$, and F is a sigmoid function.

- The above example is included as a special case of a class of nonlocal evolution equations of the form

  $$\partial_t u(t) = r(u(t)) + p(u(t))S(w * q(u(t)))$$

  considered in [20], where existence and uniqueness of monotone traveling waves is shown. We refer to [20] for the precise assumptions on the parameters.

So far, in most of the analysis, the stability of the traveling wave and spectral properties of (3.1) were established in $L^\infty(\mathbb{R})$ or $C_0(\mathbb{R})$, the space of functions vanishing at infinity, see also [3]. A more recent exception is [63], where spectral properties in L^p, $1 \le p \le \infty$, are studied.

Here we consider the question of the L^2-stability of the traveling wave solution in the context of functional inequalities. In the frozen wave picture (cf. section 2.3.2), $u^\#(x,t) := u(x+ct,t)$ satisfies

$$\partial_t u^\#(x,t) = c\partial_x u^\#(x,t) + d\partial_{xx} u^\#(x,t) + S(u^\#, w * g(u^\#))(x,t). \tag{3.2}$$

Linearization around the traveling wave yields the frozen wave operator

$$L^{\#}v = \partial_1 S(\hat{u}, w * g(\hat{u}))v + c\partial_x v + d\partial_{xx}v + \partial_2 S(\hat{u}, w * g(\hat{u}))w * (g'(\hat{u})v).$$

Note that $L^{\#}\hat{u}_x = 0$. Recall from section 2.4, that a measure that is naturally associated with $L^{\#}$ is the measure with density $\rho(x) = \frac{\psi(x)}{\hat{u}_x(x)}$, where ψ is the eigenfunction to the eigenvalue 0 of the adjoint operator $L^{\#,*}$. As motivated in section 2.5.2 we are going to prove that $L^{\#}$ satisfies a spectral gap inequality in $L^2(\rho)$,

$$\langle L^{\#}v, v\rangle_\rho \leq -\kappa\big(\|v\|_\rho^2 - \langle v, \hat{u}_x\rangle_\rho^2\big) \tag{3.3}$$

for some $\kappa > 0$.

This way of expressing the stability of the traveling wave is in particular amenable to a stochastic setting. Versions of (3.3) have been assumed in [42] (for the neural field equation) and [37] (in a general setting) to derive stochastic stability results.

3.2 The Setting

Motivated by the traveling wave examples given above we consider the operator

$$L^{\#}v = -fv + c\partial_x v + d\partial_{xx}v + rw * (qv), \qquad D(L^{\#}) = H^2.$$

We make the following assumptions on the parameters.

- $c \in \mathbb{R}$, $d \geq 0$
- $f, r, q \in C(\mathbb{R})$, $r > 0$ and $q > 0$
- f, r, and q are bounded, and q and r are bounded away from 0
- $w \geq 0$ is absolutely continuous, $\int w(x)dx = 1$, and $\int |w_x(x)|dx < \infty$

If $L^{\#}$ is the frozen wave operator associated to (3.1), then

$$f = -\partial_1 S(\hat{u}, w * g(\hat{u})), \quad r = \partial_2 S(\hat{u}, w * g(\hat{u})), \quad q = g'(\hat{u}).$$

We decompose $L^{\#}$ into a local and a nonlocal part,

$$L^{\#}v = Av + Pv,$$

where the local part is given by

$$Av = -fv + c\partial_x v + d\partial_{xx}v,$$

and the nonlocal part is

$$Pv = \int p(x,y)v(y)dy$$

with

$$p(x,y) = r(x)w(x-y)q(y).$$

The adjoint of $L^{\#}$ is

$$L^{\#,*}v = A^*v + P^*v, \qquad D(L^{\#,*}) = H^2,$$

where the local part is

$$A^*v = -fv - c\partial_x v + d\partial_{xx}v,$$

and the nonlocal part is

$$P^*v = \int p^*(x,y)v(y)dy$$

with

$$p^*(x,y) = p(y,x).$$

Assumption. *There exists a unique (up to constant multiples)* $0 \not\equiv \hat{u}_x \in H^2$ *such that* $L^{\#}\hat{u}_x = 0$ *and a unique (up to constant multiples)* $0 \not\equiv \psi \in H^2$ *such that* $L^{\#,*}\psi = 0$*, and* $\hat{u}_x > 0$ *and* $\psi > 0$*.*

Here we denote the eigenfunction of $L^{\#}$ by $\hat{u}_x$ in reference to the traveling wave example. We normalize ψ such that $\langle \hat{u}_x, \psi \rangle = 1$ and introduce the density

$$\rho(x) = \frac{\psi(x)}{\hat{u}_x(x)}.$$

Recall from sections 2.4 and 2.5.2 that ρ is a natural density to consider since

$$\langle L^{\#}v, \hat{u}_x \rangle_\rho = \langle L^{\#}v, \psi \rangle = \langle v, L^{\#,*}\psi \rangle = 0,$$

i.e., $D(L^{\#}) \subset \hat{u}_x^\perp$, the orthogonal complement of $\hat{u}_x$ in $L^2(\rho)$. The direction of movement $\hat{u}_x$ and the orthogonal directions are thus naturally separated.

We assume that (2.13) is satisfied such that $L^{\#} : H^2(\rho) \to L^2(\rho)$.

3.2.1 Reformulation

The following reformulation will be useful. Define the probability measure

$$\nu(x) = \hat{u}_x(x)\psi(x).$$

Expand $v \in H^2(\rho)$ as $v = h\hat{u}_x$ and set

$$L_0^{\#}h = A_0 h + P_0 h, \qquad D(L_0^{\#}) = H^2(\nu)$$

with
$$A_0 h = \frac{A(h\hat{u}_x)}{P\hat{u}_x}, \qquad P_0 h = \frac{P(h\hat{u}_x)}{P\hat{u}_x}.$$
Then
$$\langle L^\# v, v\rangle_\rho = \langle L_0^\# h, h\ P\hat{u}_x\psi\rangle,$$
and $P_0\mathbb{1} \equiv \mathbb{1}$, so that
$$p_0(x,y) = \frac{p(x,y)\hat{u}_x(y)}{P\hat{u}_x(x)} = \frac{w(x-y)q(y)\hat{u}_x(y)}{w*(q\hat{u}_x)(x)}$$
is a Markov kernel. Set
$$\mu(x) = \frac{1}{Z_\mu}P\hat{u}_x(x)\psi(x), \qquad \mu^*(x) = \frac{1}{Z_\mu}\hat{u}_x(x)P^*\psi(x), \tag{3.4}$$
where $Z_\mu = \int P\hat{u}_x(x)\psi(x)dx = \int \hat{u}_x(x)P^*\psi(x)dx$, such that μ and μ^* are probability measures. Then (3.3) is equivalent to
$$E_\mu(L_0^\# h\ h) \le -\frac{\kappa}{Z_\mu}Var_\nu(h).$$

3.3 The Spectral Gap Inequality

Denote by $\mathcal{S}$ the support of w, $\mathcal{S} = \{x \in \mathbb{R} : w(x) > 0\}$. We make the following additional assumption on w.

Assumption 3.3.1.

(i) for all $v \in L^2(\rho)$, $\partial_x w * v = w_x * v$

(ii) $M{:=}\sup_{x\in\mathbb{R}}\int_{x-\mathcal{S}}\left(\frac{w_x(x-y)}{w(x-y)}\right)^2 p_0(x,y)dy < \infty$

Theorem 3.3.2. *Assume that (2.13) and Assumption 3.3.1 are satisfied and that furthermore*

(i) there exist $\delta_i, \tilde{\delta}_i > 0, i = 1, 2,$ such that
$$\delta_1\hat{u}_x \le P\hat{u}_x \le \delta_2\hat{u}_x, \qquad \tilde{\delta}_1\psi \le P^*\psi \le \tilde{\delta}_2\psi.$$
In particular, the ν-, μ-, and μ^*-norms are equivalent.

(ii) there exists $\kappa_0 > 0$ such that for all $h \in H^1(\mu)$,
$$Var_\mu(h) \le \kappa_0 \int h_x^2(x)\mu(dx) \tag{3.5}$$

Then for all $v \in H^2(\rho)$,
$$\langle L^\# v, v\rangle_\rho \le -\kappa\Big(\|v\|_\rho^2 - \langle v, \hat{u}_x\rangle_\rho^2\Big),$$

where

$$\kappa = \frac{\tilde{\delta}_1}{2}\Big(1 - \frac{\kappa_0 M}{1+\kappa_0 M}\Big).$$

Proof. I) Note that

$$E_\mu(L_0^{\#} h) = \frac{1}{Z_\mu}\int L^{\#}(h\hat{u}_x)\psi dx = \frac{1}{Z_\mu}\int h\hat{u}_x L^{\#,*}\psi dx = 0.$$

Therefore

$$\begin{aligned}\frac{1}{Z_\mu}\langle L^{\#} v, v\rangle_\rho &= E_\mu(L_0^{\#} h\ h) = E_\mu\big((L_0^{\#} h - E_\mu(L_0^{\#} h))(h - E_\mu(h))\big)\\ &= Cov_\mu(L_0^{\#} h, h) = Cov_\mu(A_0 h, h) + Cov_\mu(P_0 h, h)\\ &\le Cov_\mu(A_0 h, h) + \frac{1}{2}\big(Var_\mu(P_0 h) + Var_\mu(h)\big).\end{aligned}$$

II) We consider the first summand,

$$Cov_\mu(A_0 h, h) = \langle A_0 h, h\rangle_\mu - E_\mu(A_0 h)E_\mu(h).$$

We have

$$\begin{aligned}Z_\mu\langle A_0 h, h\rangle_\mu &= \int \big(-fh\hat{u}_x + c(h\hat{u}_x)_x + d(h\hat{u}_x)_{xx}\big)h\psi dx\\ &= \int h^2 A\hat{u}_x\psi dx + c\int hh_x\hat{u}_x\psi dx + d\int hh_{xx}\hat{u}_x\psi dx + 2d\int hh_x\hat{u}_{xx}\psi dx.\end{aligned}$$

Note that

$$\int h^2 A\hat{u}_x\psi dx = -\int h^2 P\hat{u}_x\psi dx = -Z_\mu\int h^2 d\mu$$

and

$$\int h^2\hat{u}_x A^*\psi dx = -\int h^2\hat{u}_x P^*\psi dx = -Z_\mu\int h^2 d\mu^*.$$

Using integration by parts,

$$\begin{aligned}c\int hh_x\hat{u}_x\psi dx &= -\frac{c}{2}\int h^2\hat{u}_{xx}\psi dx - \frac{c}{2}\int h^2\hat{u}_x\psi_x dx\\ &= -\frac{1}{2}\int h^2(-f\hat{u}_x + c\hat{u}_{xx})\psi dx + \frac{1}{2}\int h^2(-f\psi - c\psi_x)\hat{u}_x dx,\end{aligned}$$

and

$$\begin{aligned}&d\int hh_{xx}\hat{u}_x\psi dx + 2d\int hh_x\hat{u}_{xx}\psi dx\\ &= -d\int h_x^2\hat{u}_x\psi dx - d\int hh_x\hat{u}_x\psi_x dx + d\int hh_x\hat{u}_{xx}\psi dx\\ &= -d\int h_x^2\hat{u}_x\psi dx + \frac{d}{2}\int h^2\big(\hat{u}_x\psi_{xx} - \hat{u}_{xxx}\psi\big)dx.\end{aligned}$$

Altogether we obtain that

$$\begin{aligned}
Z_\mu \langle A_0 h, h\rangle_\mu &= -Z_\mu \int h^2 d\mu - \frac{1}{2}\int h^2 A\hat{u}_x \psi dx + \frac{1}{2}\int h^2 \hat{u}_x A^* \psi dx - d\int h_x^2 \hat{u}_x \psi dx \\
&= -\frac{Z_\mu}{2}\int h^2 d\mu - \frac{Z_\mu}{2}\int h^2 d\mu^* - d\int h_x^2 \hat{u}_x \psi dx \\
&\leq -\frac{Z_\mu}{2}\int h^2 d\mu - \frac{Z_\mu}{2}\int h^2 d\mu^*.
\end{aligned}$$

Since

$$E_\mu(A_0 h) = \frac{1}{Z_\mu}\int A(h\hat{u}_x)\psi dx = \frac{1}{Z_\mu}\int h\hat{u}_x A^*\psi dx = -E_{\mu^*}(h) \tag{3.6}$$

and

$$E_\mu(h)E_{\mu^*}(h) \leq \frac{1}{2}E_\mu(h)^2 + \frac{1}{2}E_{\mu^*}(h)^2,$$

it follows that

$$Cov_\mu(A_0 h, h) \leq -\frac{1}{2}Var_\mu(h) - \frac{1}{2}Var_{\mu^*}(h).$$

III) *Claim*: $\exists \gamma < 1$ such that

$$Var_\mu(P_0 h) \leq \gamma Var_{\mu^*}(h).$$

Proof of the claim. By assumptions (ii) and 3.3.1(i),

$$Var_\mu(P_0 h) \leq \kappa_0 \int \big(\partial_x P_0 h(x)\big)^2 \mu(dx) = \kappa_0 \int \left(\int \partial_x p_0(x,y)h(y)dy\right)^2 \mu(dx).$$

Since $\int \partial_x p_0(x,y)dy = \partial_x P_0 \mathbb{1}(x) = \partial_x \mathbb{1} = 0$, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
Var_\mu(P_0 h) &\leq \kappa_0 \int \left(\int \partial_x p_0(x,y)\big(h(y) - P_0 h(x)\big)dy\right)^2 \mu(dx) \\
&\leq \kappa_0 \int \int_{x-\mathcal{S}} \frac{\big(\partial_x p_0(x,y)\big)^2}{p_0(x,y)} dy \int p_0(x,y)\big(h(y) - P_0 h(x)\big)^2 dy\mu(dx).
\end{aligned}$$

For $y \in x - \mathcal{S}$ we have

$$\partial_x p_0(x,y) = \left(\frac{w_x(x-y)}{w(x-y)} - \frac{\partial_x w * (q\hat{u}_x)(x)}{w * (q\hat{u}_x)(x)}\right) p_0(x,y)$$

and therefore

$$\begin{aligned}
\int_{x-\mathcal{S}} \frac{(\partial_x p_0(x,y))^2}{p_0(x,y)} dy &= \int_{x-\mathcal{S}} \Big(\frac{w_x(x-y)}{w(x-y)}\Big)^2 p_0(x,y)dy - \left(\frac{\partial_x w * (q\hat{u}_x)(x)}{w * (q\hat{u}_x)(x)}\right)^2 \\
&\leq \int_{x-\mathcal{S}} \Big(\frac{w_x(x-y)}{w(x-y)}\Big)^2 p_0(x,y)dy = M < \infty
\end{aligned}$$

by assumption 3.3.1(ii). It follows that

$$\begin{aligned} Var_\mu(P_0 h) &\le \kappa_0 M \int P_0 h^2(x) - (P_0 h)^2(x)\mu(dx) \\ &= \kappa_0 M\Big(E_\mu(P_0 h^2) - (E_\mu(P_0 h))^2 - Var_\mu(P_0 h)\Big) \end{aligned}$$

and hence

$$(1+\kappa_0 M) Var_\mu(P_0 h) \le \kappa_0 M\Big(E_\mu(P_0 h^2) - (E_\mu(P_0 h))^2\Big).$$

Since for any $g \in L^1(\mu)$, as in (3.6), $E_\mu(P_0 g) = E_{\mu^*}(g)$, it follows that

$$Var_\mu(P_0 h) \le \frac{\kappa_0 M}{1+\kappa_0 M} Var_{\mu^*}(h) =: \gamma Var_{\mu^*}(h).$$

This proves the claim.

IV) From I) , II), and III) it follows that

$$\langle L^{\#} v, v\rangle_\rho \le -\frac{(1-\gamma)Z_\mu}{2} Var_{\mu^*}(h),$$

and since by assumption (i)

$$Z_\mu Var_{\mu^*}(h) = Z_\mu \int \big(h(x) - E_{\mu^*}(h)\big)^2 \mu^*(dx) \ge \tilde{\delta}_1 \int \big(h(x) - E_{\mu^*}(h)\big)^2 \nu(dx) \ge \tilde{\delta}_1 Var_\nu(h),$$

we obtain that

$$\langle L^{\#} v, v\rangle_\rho \le -\kappa\big(\|v\|_\rho^2 - \langle v, \hat{u}_x\rangle_\rho^2\big)$$

with $\kappa = \frac{\tilde{\delta}_1(1-\gamma)}{2}$. □

3.4 Application to the Examples

We show that the assumptions in Theorem 3.3.2 are satisfied under rather general conditions.

Remark 3.4.1. Using a result by Muckenhoupt on Hardy's inequalities with weights (originally obtained by Tomaselli, Talenti, Artola, cf. [48], Thm. 1), assumption (ii) in Theorem 3.3.2 is satisfied if and only if

$$B_1 := \sup_{r>0} \int_r^\infty \mu(x)dx \int_0^r \frac{1}{\mu(x)}dx < \infty$$

and

$$B_2 := \sup_{r>0} \int_{-\infty}^{-r} \mu(x)dx \int_{-r}^0 \frac{1}{\mu(x)}dx < \infty.$$

Then we can bound κ_0 in (3.5) by

$$B_1 \wedge B_2 \leq \kappa_0 \leq 4(B_1 \vee B_2).$$

Theorem 3.4.2. *Assume that $w > 0$ in a neighborhood of 0 and that (2.13) and Assumption 3.3.1 are satisfied. Assume further that there exist $\alpha, \beta, k, l > 0$, such that for all $x \geq 0$, $y \geq 0$*

$$\mu(x+y) \leq ke^{-\alpha y}\mu(x), \qquad \mu(-x-y) \leq le^{-\beta y}\mu(-x), \tag{3.7}$$

and that

$$\left\|\frac{\hat{u}_{xxx}}{\hat{u}_x}\right\|_\infty + \left\|\frac{\hat{u}_{xx}}{\hat{u}_x}\right\|_\infty + \left\|\frac{\psi_{xx}}{\psi}\right\|_\infty + \left\|\frac{\psi_x}{\psi}\right\|_\infty < \infty. \tag{3.8}$$

Then the assumptions of Theorem 3.3.2 are satisfied.

Proof. (I) Since $m := \left\|\frac{\hat{u}_{xx}}{\hat{u}_x}\right\|_\infty < \infty$ by (3.8), $-m\hat{u}_x \leq \hat{u}_{xx} \leq m\hat{u}_x$ and hence for $x, y \geq 0$, $\hat{u}_x(x+y) \geq e^{-my}\hat{u}_x(x)$ and $\hat{u}_x(-x-y) \geq e^{-my}\hat{u}_x(-x)$. It follows that for $x \geq 0$,

$$\begin{aligned} P\hat{u}_x(x) &\geq \min r \min q \int_{-\infty}^0 w(y)\hat{u}_x(x-y)dy \\ &\geq \min r \min q \int_{-\infty}^0 w(y)e^{my}dy \ \hat{u}_x(x), \end{aligned}$$

and analogously for $x \leq 0$. Thus, there exists $\delta_1 > 0$ such that

$$\delta_1\hat{u}_x(x) \leq P\hat{u}_x(x).$$

Using (3.8) it follows that there exists $\delta_2 > 0$ such that

$$P\hat{u}_x = -A\hat{u}_x = f\hat{u}_x - c\hat{u}_{xx} - d\hat{u}_{xxx} \leq \delta_2\hat{u}_x.$$

It can be proven analogously that there exist $\tilde{\delta}_1, \tilde{\delta}_2 > 0$ such that

$$\tilde{\delta}_1\psi \leq P^*\psi \leq \tilde{\delta}_2\psi.$$

Assumption (i) of Theorem 3.3.2 is therefore satisfied.

(II)

$$\begin{aligned} B_1 &:= \sup_{r>0} \int_r^\infty \mu(x)dx \int_0^r \frac{1}{\mu(x)}dx \\ &\leq k^2 \int_r^\infty e^{-\alpha(x-r)}dx\mu(r) \int_0^r e^{-\alpha(r-x)}dx\frac{1}{\mu(r)} \leq \frac{k^2}{\alpha^2}, \end{aligned}$$

and analogously

$$B_2 := \sup_{r>0} \int_{-\infty}^{-r} \mu(x)dx \int_{-r}^0 \frac{1}{\mu(x)}dx \leq \frac{l^2}{\beta^2}.$$

Using Remark 3.4.1, assumption (ii) of Theorem 3.3.2 is satisfied. $\square$

Remark 3.4.3.

1. We will see in Chapter 4 below that in the case of the neural field equation (1.5) with $w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$, $\sigma > 0$, $\hat{u}_x$ and ψ decay exponentially (cf. Thm. 4.1.2), and that ρ grows exponentially at a rate smaller than $\frac{1}{\sigma}$ (cf. Prop. 4.2.1). Since $\left\|\frac{w_x}{w}\right\|_\infty < \infty$, it follows that in this case (2.13) and Assumption 3.3.1, as well as (3.7) and (3.8) are satisfied.

2. In [63] it is shown in a rather general setting that for $q \equiv 1$ and w satisfying
$$\int w(x)e^{\alpha x}dx < \infty$$
for all $\alpha \in \mathbb{R}$, $\hat{u}_x$ decays exponentially and the exact rates are given. Existence and exponential decay of the adjoint eigenfunction are also proven. In particular, (3.7) and (3.8) are satisfied.

3. If $\hat{u}_x$ and ψ decay exponentially, then ρ (or $\frac{1}{\rho}$, depending on whether $c > 0$ or $c < 0$) grows exponentially. Thus, if w has compact support and $\sup_{x\in\mathcal{S}}\left|\frac{w_x(x)}{w(x)}\right| < \infty$, or if w decays faster than exponentially, then (2.13) and assumption 3.3.1 are satisfied.

3.5 The Spectral Gap Inequality in $L^2(\mathfrak{m})$ for Small Wave Speeds

In this section we will assume that $d = 0$ and that $f \geq \min f > 0$. Recall from section 2.3.1 that another measure that is naturally associated with the problem is the symmetrizing measure of the traveling wave operator

$$Lv = -fv + rw * (qv)$$

with density
$$\mathfrak{m}(x) = \frac{q(x)}{r(x)}.$$

Note that the $L^2(\mathfrak{m})$-norm is equivalent to the L^2-norm.

If $c = 0$, then $\psi = \frac{1}{Z}\frac{q}{r}\hat{u}_x$, where $Z = \int \frac{q}{r}\hat{u}_x^2 dx$, and thus $\mathfrak{m} = Z\rho$. In this case, if the assumptions in Theorem 3.3.2 are satisfied, then L has a spectral gap in $L^2(\mathfrak{m})$. We can extend the spectral gap for the case $c = 0$ to small wave speeds c by a perturbation argument.

Theorem 3.5.1. *Assume that Assumption 3.3.1 is satisfied and that furthermore*

(i) there exist $\delta_1 < \delta_2$ such that $\delta_1\hat{u}_x \leq P\hat{u}_x \leq \delta_2\hat{u}_x$

(ii) there exists $\kappa_0^0 > 0$ such that for all $h \in H^1(\mu^0)$,

$$Var_{\mu^0}(h) \leq \kappa_0^0 \int h_x^2(x)\mu^0(dx),$$

where $\mu^0 = \frac{1}{Z_{\mu^0}} \frac{q}{r} P\hat{u}_x \hat{u}_x$ with $Z_{\mu^0} = \int \frac{q}{r} P\hat{u}_x \hat{u}_x dx$.

Then there exists $c^* = c^*(w, f, r, q) > 0$ (see (3.12) for the precise definition) such that if $c = c(w, f, r, q)$ satisfies $|c| \leq c^*$, there exist $\kappa, Z > 0$ such that

$$\langle Lv, v\rangle_{\mathfrak{m}} \leq -\kappa \|v\|_{\mathfrak{m}}^2 + Z\langle v, \hat{u}_x\rangle_{\mathfrak{m}}^2. \tag{3.9}$$

Proof. Set $\varphi^0 = \frac{P\hat{u}_x}{f}$ and $P^0 v = \int p^0(x,y)v(y)dy$ where $p^0(x,y) = p(x,y)\frac{\hat{u}_x(y)}{\varphi^0(y)}$. Then $P^0\varphi^0 = P\hat{u}_x = f\varphi^0$ so that $L^0\varphi^0 = 0$, where

$$L^0 v = -fv + P^0 v, \qquad D(L^0) = L^2.$$

Let $\mathfrak{m}^0$ be the associated symmetrizing measure,

$$\mathfrak{m}^0 = \frac{q\hat{u}_x}{r\varphi^0} = \frac{\hat{u}_x}{\varphi^0}\mathfrak{m}, \tag{3.10}$$

ψ^0 the eigenfunction of $L^{0,*}$,

$$\psi^0 = \frac{1}{Z^0}\frac{q}{r}\hat{u}_x,$$

where

$$Z^0 = \int \frac{q}{r}\hat{u}_x\varphi^0 dx = \int \frac{q}{rf}P\hat{u}_x\hat{u}_x dx,$$

and set

$$\rho^0 = \frac{\psi^0}{\varphi^0} = \frac{1}{Z^0}\mathfrak{m}^0.$$

Note that assumption (i) implies that

$$\frac{\delta_1}{\max|f|} \leq \frac{\varphi^0}{\hat{u}_x} \leq \frac{\delta_2}{\min|f|}.$$

By Theorem 3.3.2 applied to L^0 with eigenfunction φ^0, there exists $\kappa_0 > 0$ such that

$$\langle L^0 v, v\rangle_{\rho^0} \leq -\kappa_0\big(\|v\|_{\rho^0}^2 - \langle v, \varphi^0\rangle_{\rho^0}^2\big). \tag{3.11}$$

Now

$$\begin{aligned}\langle Lv, v\rangle_{\mathfrak{m}} &= \langle -fv + Pv, v\rangle_{\mathfrak{m}} = \Big\langle -fv + P^0\Big(\frac{\varphi^0}{\hat{u}_x}v\Big), v\Big\rangle_{\mathfrak{m}} \\ &= \Big\langle -f\frac{\varphi^0}{\hat{u}_x}v + P^0\Big(\frac{\varphi^0}{\hat{u}_x}v\Big), v\frac{\varphi^0}{\hat{u}_x}\Big\rangle_{\mathfrak{m}^0} - \Big\langle f\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big)v, v\Big\rangle_{\mathfrak{m}}\end{aligned}$$

$$= \Big\langle L^0 \frac{\varphi^0}{\hat{u}_x} v, \frac{\varphi^0}{\hat{u}_x} v \Big\rangle_{\mathfrak{m}^0} - \Big\langle f\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big) v, v \Big\rangle_{\mathfrak{m}}.$$

Applying (3.11) to $v\frac{\varphi^0}{\hat{u}_x}$ and using (3.10) we obtain that

$$\begin{aligned}\langle Lv, v\rangle_{\mathfrak{m}} &\leq -\kappa_0 \Big\| v\frac{\varphi^0}{\hat{u}_x}\Big\|^2_{\mathfrak{m}^0} + \frac{\kappa_0}{Z^0}\Big\langle v\frac{\varphi^0}{\hat{u}_x}, \varphi^0\Big\rangle^2_{\mathfrak{m}^0} - \Big\langle f\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big) v, v\Big\rangle_{\mathfrak{m}} \\ &= -\kappa_0 \int v^2 \frac{\varphi^0}{\hat{u}_x}\mathfrak{m} dx + \frac{\kappa_0}{Z^0}\Big(\int v\varphi^0 \mathfrak{m} dx\Big)^2 - \int f\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big) v^2 \mathfrak{m} dx \\ &= -\kappa_0 \|v\|^2_{\mathfrak{m}} + \int v^2\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big)(\kappa_0 - f)\mathfrak{m} dx + \frac{\kappa_0}{Z^0}\Big(\int v\varphi^0\mathfrak{m} dx\Big)^2.\end{aligned}$$

Now

$$\int v^2\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big)(\kappa_0 - f)\mathfrak{m} dx \leq \Big\|\Big(1 - \frac{\varphi^0}{\hat{u}_x}\Big)(\kappa_0 - f)\Big\|_\infty \|v\|^2_{\mathfrak{m}} = \Big\| c\frac{\hat{u}_{xx}}{\hat{u}_x f}(\kappa_0 - f)\Big\|_\infty \|v\|^2_{\mathfrak{m}}$$

and

$$\begin{aligned}\Big(\int v\varphi^0\mathfrak{m} dx\Big)^2 &= \Big(\int v\Big(\hat{u}_x - c\frac{\hat{u}_{xx}}{f}\Big)\mathfrak{m} dx\Big)^2 \\ &\leq 2\langle v, \hat{u}_x\rangle^2_{\mathfrak{m}} + 2c^2\Big\langle v, \frac{\hat{u}_{xx}}{f}\Big\rangle^2_{\mathfrak{m}} \leq 2\langle v, \hat{u}_x\rangle^2_{\mathfrak{m}} + 2c^2\|v\|^2_{\mathfrak{m}}\Big\|\frac{\hat{u}_{xx}}{f}\Big\|^2_{\mathfrak{m}}.\end{aligned}$$

It follows that

$$\langle Lv, v\rangle_{\mathfrak{m}} \leq -\kappa(c)\|v\|^2_{\mathfrak{m}} + 2\frac{\kappa_0}{Z^0}\langle v, \hat{u}_x\rangle^2_{\mathfrak{m}},$$

where

$$\kappa(c) = \kappa_0\Big(1 - \frac{2c^2}{Z^0}\Big\|\frac{\hat{u}_{xx}}{f}\Big\|^2_{\mathfrak{m}}\Big) - |c|\Big\|\frac{\hat{u}_{xx}}{\hat{u}_x f}(\kappa_0 - f)\Big\|_\infty.$$

Note that $\kappa(c) \xrightarrow{c\to 0} \kappa_0 > 0$. Set

$$c^* = \min\{|c| : \kappa(c) \leq 0\}. \tag{3.12}$$

Then (3.9) is satisfied with $\kappa = \kappa(c)$ if $|c| \leq c^*$. □

Note that c, $\kappa(c)$, c^* are usually unknown variables depending on w, f, q, r. It is a priori not clear that there exists a setting in which Theorem 3.5.1 applies. This can be clarified in the neural field example. Consider the neural field traveling wave operator

$$Lv = -v + w * (F'(\hat{u})v)$$

for some kernel w satisfying $M := \|\frac{w_x}{w}\|_\infty < \infty$ and some gain function F and the corresponding traveling wave $(\hat{u}, c)$. We define an associated standing wave in the following way. Set $\hat{u}^0 = w * F(\hat{u})$ and $F^0(x) = F(\hat{u}((\hat{u}^0)^{-1}(x)))$ (since $\hat{u}^0$ is increasing, $(\hat{u}^0)^{-1}$ is well-defined). Then $\hat{u}^0 = w * F^0(\hat{u}^0)$ is the traveling wave solution to the neural field equation with kernel w and gain function F^0, and $\hat{u}^0_x$ is the eigenfunction to the eigenvalue 0 of L^0, where

$$L^0 v = -v + w * ((F^0)'(\hat{u}^0)v) = -v + w * \Big(F'(\hat{u})\frac{\hat{u}_x}{\hat{u}^0_x}v\Big).$$

(Note that, in the notation of the proof of Theorem 3.5.1, $\hat{u}^0_x = \varphi^0$.) Since $\hat{u}^0 = w * F(\hat{u}) = \hat{u} - c\hat{u}_x$, we have

$$\hat{u}(x) = (I - c\partial_x)^{-1}\hat{u}^0 = \int_0^\infty e^{-s}\hat{u}^0(x+cs)ds. \tag{3.13}$$

In this setting, Theorem 3.5.1 therefore tells us the following. Assume that L^0 satisfies a spectral gap inequality in $L^2(\mathfrak{m}^0)$ with constant κ_0. Set

$$\kappa(c) = \kappa_0\Big(1 - \frac{2c^2}{Z^0}\|\hat{u}_{xx}\|^2_{\mathfrak{m}}\Big) - |c|\Big\|\frac{\hat{u}_{xx}}{\hat{u}_x}(\kappa_0 - 1)\Big\|_\infty.$$

Since $\big\|\frac{w_x}{w}\big\|_\infty = M$ and, as in (2.5),

$$|\hat{u}_{xx}(x)| = \Big|\int_0^\infty e^{-s}w_x * (F'(\hat{u})\hat{u}_x)(x+cs)ds\Big| \le M\hat{u}_x(x),$$

it follows that

$$\frac{\|\hat{u}_{xx}\|^2_{\mathfrak{m}}}{Z^0} = \frac{\int \hat{u}^2_{xx}F'(\hat{u})dx}{\int \hat{u}^0_x\hat{u}_x F'(\hat{u})dx} \le \frac{M^2}{1-|c|M}$$

and

$$\kappa(c) \ge \kappa_0\Big(1 - \frac{2c^2M^2}{1-|c|M}\Big) - |c|M|\kappa_0 - 1|.$$

Then for all c satisfying

$$\kappa_0\Big(1 - \frac{2c^2M^2}{1-|c|M}\Big) - |c|M|\kappa_0 - 1| > 0, \tag{3.14}$$

the traveling wave operator associated to $\hat{u}$ as defined in (3.13) (that is, the operator with kernel w and gain function $F(x) = F^0(\hat{u}^0(\hat{u}^{-1}(x)))$), satisfies a spectral gap inequality in $L^2(\mathfrak{m})$.

It would be desirable to express the smallness condition on c in terms of the parameters of the system, w and F. We will do this in the next chapter for the exponential example (cf. section 4.4).

3.6 $L^2(\mathfrak{m})$-Stability of the Traveling Wave

In this section we will stick to the neural field setting. As outlined in section 2.5, we expect that shifts in the phase of the wave can be accounted for by adapting the speed according to (2.17). We therefore define

$$\tilde{v}(x,t) = u(x,t) - \hat{u}(x - ct - C^m(t)) = v(x,t) + \hat{u}(x-ct) - \hat{u}(x - ct - C^m(t))$$

to be the difference between the solution to the stochastic neural field equation and the traveling wave moving at the adapted speed. Set $\tilde{u}(x,t) = \hat{u}(x - ct - C^m(t))$ and $\mathfrak{m}_t(x) = \mathfrak{m}(x - ct - C^m(t))$. We move the measure with the wave such that $\|\tilde{u}(t)\|_{\mathfrak{m}_t} = \|\hat{u}\|_{\mathfrak{m}}$ for all $t \geq 0$. $\tilde{v}$ satisfies the stochastic evolution equation

$$\begin{aligned} d\tilde{v}(t) = \Big(&- \tilde{v}(t) + w * (F'(\tilde{u}(t))\tilde{v}(t)) + \tilde{R}(t, \tilde{v}(t)) + \dot{C}^m(t)\partial_x \tilde{u}(t)\Big)dt \\ &+ \epsilon\sigma(t, \tilde{v}(t) + \tilde{u}(t) - \hat{u}(\cdot - ct))dW_t, \end{aligned}$$

where

$$\tilde{R}(t,v) = w * \big(F(\tilde{u}(t) + v) - F(\tilde{u}(t)) - F'(\tilde{u}(t))v\big).$$

Note that, analogously to (2.8),

$$\begin{aligned} \|\tilde{R}(t,v)\|_{\mathfrak{m}_t}^2 &\leq \frac{1}{4}\|w\|_\infty \|F''\|_\infty^2 \|v\|^2 \int\int w(x-y)v^2(y)dy\, \mathfrak{m}_t(x)dx \\ &\leq \frac{1}{4}\|w\|_\infty \|F''\|_\infty^2 \frac{\|\mathfrak{m}\|_\infty}{\min|\mathfrak{m}|^2}\|v\|_{\mathfrak{m}_t}^4 =: M_R^2 \|v\|_{\mathfrak{m}_t}^4. \end{aligned} \tag{3.15}$$

Here we assume that assumption (A) on the noise is satisfied and that there exist L_σ, M_σ such that for all $t \geq 0$

$$\|(\sigma(t,v_1) - \sigma(t,v_2)) \circ Q^{\frac{1}{2}}\|_{L_2(L^2, L^2(\mathfrak{m}_t))} \leq L_\sigma \|v_1 - v_2\|_{\mathfrak{m}_t},$$

and

$$M_\sigma := \sup_{t \geq 0} \|\sigma(t,0) \circ Q^{\frac{1}{2}}\|_{L_2(L^2, L^2(\mathfrak{m}_t))} < \infty.$$

Then

$$\begin{aligned} &\|\sigma(t, \tilde{v}(t) + \tilde{u}(t) - \hat{u}(\cdot - ct)) \circ Q^{\frac{1}{2}}\|_{L_2(L^2, L^2(\mathfrak{m}_t))}^2 \\ &\leq 2L_\sigma^2 \|\tilde{v}(t) + \tilde{u}(t) - \hat{u}(\cdot - ct))\|_{\mathfrak{m}_t}^2 + 2M_\sigma^2 \\ &\leq 4L_\sigma^2 \big(\|\tilde{v}(t)\|_{\mathfrak{m}_t}^2 + \|\tilde{u}(t) - \hat{u}(\cdot - ct))\|_{\mathfrak{m}_t}^2\big) + 2M_\sigma^2. \end{aligned}$$

By Taylor's theorem, there exists $\xi(x,t)$ such that

$$\tilde{u}(x,t) - \hat{u}(x - ct) = -C(t)\hat{u}_x(x - ct - \xi(x,t)),$$

and we obtain that

$$\begin{aligned} &\|\sigma(t,\tilde v(t)+\tilde u(t)-\hat u(\cdot - ct))\circ Q^{\frac12}\|^2_{L_2(L^2,L^2(\mathfrak{m}_t))} \\ &\le 4L_\sigma^2\big(1+m^2\|\mathfrak{m}\|_\infty\|\hat u_x\|^2\|\hat u_x\|^2_{\mathfrak{m}}\big)\|\tilde v(t)\|^2_{\mathfrak{m}_t}+2M_\sigma^2 =: \tilde L_\sigma^2\|\tilde v(t)\|^2_{\mathfrak{m}_t}+2M_\sigma^2. \end{aligned} \tag{3.16}$$

If the traveling wave operator L satisfies a spectral gap inequality in $L^2(\mathfrak{m})$, then we can derive a time-uniform bound for $\tilde v(t)$. We obtain the following stability result. Similar results have been proven in the context of reaction-diffusion equations in [57, 58].

Theorem 3.6.1. *Assume that the traveling wave operator L satisfies a spectral gap inequality in $L^2(\mathfrak{m})$,*

$$\langle Lv,v\rangle_{\mathfrak{m}} \le -\kappa\|v\|^2_{\mathfrak{m}} + Z\langle v,\hat u_x\rangle^2_{\mathfrak{m}}.$$

Assume that $\epsilon^2\tilde L_\sigma^2+|c|\|\frac{\partial_x\mathfrak{m}}{\mathfrak{m}}\|_\infty<\kappa$ and $m>Z$. Set $b^*=\frac{\kappa}{2M_R+m\|\hat u_x\|_{\mathfrak{m}}\|\frac{\partial_x\mathfrak{m}}{\mathfrak{m}}\|_\infty}$ and

$$\tau := \inf\{t\ge 0 : \|\tilde v(t)\|_{\mathfrak{m}_t}\ge b^*\}.$$

Then

$$P(\tau<\infty)\le \frac{1}{(b^*)^2}\left(\|\tilde v(0)\|_2^2+\frac{2M_\sigma^2\epsilon^2}{\kappa-\epsilon^2\tilde L_\sigma^2-|c|\|\frac{\partial_x\mathfrak{m}}{\mathfrak{m}}\|}\right).$$

Proof. For $t\le\tau$, by Itô's Lemma,

$$\begin{aligned} d\|\tilde v(t)\|^2_{\mathfrak{m}_t} &= 2\langle -\tilde v(t)+w*(F'(\tilde u(t))\tilde v(t))+\tilde R(t,\tilde v(t)),\tilde v(t)\rangle_{\mathfrak{m}_t}dt \\ &\quad+\Big(2\dot C^m(t)\langle\tilde v(t),\partial_x\tilde u(t)\rangle_{\mathfrak{m}_t}-(c+\dot C^m(t))\int \tilde v^2(t)\partial_x\mathfrak{m}_t dx \\ &\quad+\epsilon^2\|\big(\sigma(t,\tilde v(t)+\tilde u(t)-\hat u(\cdot-ct))\big)\circ Q^{\frac12}\|^2_{L_2(L^2,L^2(\mathfrak{m}_t))}\Big)dt \\ &\quad+2\epsilon\langle\tilde v(t),\sigma(t,\tilde v(t)+\tilde u(t)-\hat u(\cdot-ct))dW_t\rangle_{\mathfrak{m}_t} \end{aligned}$$

Since $m>Z$,

$$\begin{aligned} &\langle -\tilde v(t)+w*(F'(\tilde u(t)))\tilde v(t)),\tilde v(t)\rangle_{\mathfrak{m}_t}+\dot C^m(t)\langle\tilde v(t),\partial_x\tilde u(t)\rangle_{\mathfrak{m}_t} \\ &\le -\kappa\|\tilde v(t)\|^2_{\mathfrak{m}_t}+(Z-m)\langle\tilde v(t),\partial_x\tilde u(t)\rangle^2_{\mathfrak{m}_t}\le-\kappa\|\tilde v(t)\|^2_{\mathfrak{m}_t}. \end{aligned}$$

Set

$$M_t=\int_0^t\langle\tilde v(s),\sigma(s,\tilde v(s)+\tilde u(s)-\hat u(\cdot-cs))dW_s\rangle_{\mathfrak{m}_s}.$$

Using (3.15), (3.16), and

$$\left|(c+\dot C^m(t))\int\tilde v^2(t)\partial_x\mathfrak{m}_t dx\right|\le(|c|+mb^*\|\hat u_x\|_{\mathfrak{m}})\Big\|\frac{\partial_x\mathfrak{m}}{\mathfrak{m}}\Big\|_\infty\|\tilde v(t)\|^2_{\mathfrak{m}_t},$$

we obtain that

$$\begin{aligned} d\|\tilde{v}(t)\|^2_{\mathfrak{m}_t} &\leq \Big(-2\kappa + \Big(2M_R + m\|\hat{u}_x\|_{\mathfrak{m}}\Big\|\frac{\partial_x \mathfrak{m}}{\mathfrak{m}}\Big\|_\infty\Big)b^* + \epsilon^2 \tilde{L}^2_\sigma \\ &\quad + |c|\Big\|\frac{\partial_x \mathfrak{m}}{\mathfrak{m}}\Big\|_\infty\Big)\|\tilde{v}(t)\|^2_{\mathfrak{m}_t} dt + 2\epsilon^2 M^2_\sigma dt + 2\epsilon dM_t \\ &\leq -\Big(\kappa - \epsilon^2 \tilde{L}^2_\sigma - |c|\Big\|\frac{\partial_x \mathfrak{m}}{\mathfrak{m}}\Big\|_\infty\Big)\|\tilde{v}(t)\|^2_{\mathfrak{m}_t} dt \\ &\quad + 2\epsilon^2 M^2_\sigma dt + 2\epsilon dM_t. \end{aligned}$$

Set $\tilde{\kappa} := \kappa - \epsilon^2 \tilde{L}^2_\sigma - |c|\|\frac{\partial_x \mathfrak{m}}{\mathfrak{m}}\|_\infty > 0$. Applying Itô's formula to $e^{\tilde{\kappa}t}\|\tilde{v}(t)\|^2_{\mathfrak{m}_t}$ we get that

$$\begin{aligned} e^{\tilde{\kappa}t}\|\tilde{v}(t)\|^2_{\mathfrak{m}_t} &\leq \|\tilde{v}(0)\|^2_{\mathfrak{m}} + 2\epsilon^2 M^2_\sigma \int_0^t e^{\tilde{\kappa}s} ds + 2\epsilon \int_0^t e^{\tilde{\kappa}s} dM_s \\ &= \|\tilde{v}(0)\|^2_{\mathfrak{m}} + \frac{2\epsilon^2 M^2_\sigma}{\tilde{\kappa}}\left(e^{\tilde{\kappa}t} - 1\right) + 2\epsilon \int_0^t e^{\tilde{\kappa}s} dM_s. \end{aligned}$$

Therefore, since $E\big(\int_0^t e^{\tilde{\kappa}s} dM_s\big) = 0$,

$$\begin{aligned} (b^*)^2 P(\tau < \infty) &= E(\|\tilde{v}(\tau)\|^2_{\mathfrak{m}_\tau}; \tau < \infty) = \lim_{t\to\infty} E(\|\tilde{v}(t \wedge \tau)\|^2_{\mathfrak{m}_{t\wedge\tau}}; \tau < \infty) \\ &\leq \|\tilde{v}(0)\|^2_m + \frac{2\epsilon^2 M^2_\sigma}{\kappa - \epsilon^2 \tilde{L}^2_\sigma - |c|\|\frac{\partial_x \mathfrak{m}}{\mathfrak{m}}\|_\infty}. \end{aligned}$$ □

Remark 3.6.2.

1. The theorem tells us that the difference $\tilde{v}$ between the stochastic solution and the adapted traveling wave stays small uniformly in t on the set $\{\tau = \infty\}$. The probability of this set can be controlled by the initial difference $\|u(0) - \hat{u}\|^2$ and the noise amplitude ϵ. In particular, if $u(0) = \hat{u}$, then

$$P(\tau = \infty) \geq 1 - O(\epsilon^2).$$

Note that this asymptotic result holds for arbitrarily small upper bounds b^*.

2. In section 2.5.1 we defined the phase adaptation C^m in the hope that it would allow us to track the stochastic solution. It is a consequence of the theorem that it indeed does what it is supposed to.

3. We cannot derive an analogous result in $L^2(\rho)$ since we do not have the same control over the rest term $\tilde{R}$, cf. Remark 2.4.2.

4. Note that also in Theorem 3.6.1 $|c|$ is required to be 'small enough' since we assume that $|c|\|\frac{\partial_x \mathfrak{m}}{\mathfrak{m}}\|_\infty < \kappa$. This smallness condition can be made explicit in the exponential example as we will see in the next chapter.

Chapter 4

The Exponential Example

In this chapter, we explicitly analyze the traveling wave solution to the neural field equation for the case where the strength of the synaptic connections decays exponentially with the distance, that is, where the kernel w is given as

$$w(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$$

for some $\sigma > 0$. We exploit the fact that

$$w * h = (I - \sigma^2 \Delta)^{-1} h.$$

We start by analyzing the asymptotic behavior of the derivative of the traveling wave, which is the eigenfunction of the frozen wave operator, and the associated adjoint eigenfunction ψ that has been proven to exist in Proposition 2.4.1. We show that both decay exponentially and compute the rates of decay. This will also give us the asymptotic behavior of the density $\rho = \frac{\psi}{\hat{u}_x}$ in section 4.2. These first two sections are based on the preprint [46]. Together with Theorem 3.4.2, the results imply that the frozen wave operator $L^{\#}$ satisfies the spectral gap inequality in this particular example. Finally, we derive a bound on the wave speed c in section 4.3. In section 4.4 we use this bound to make the smallness assumption on c from sections 3.5 and 3.6 explicit in terms of the parameters of the system, σ and F.

In all of this chapter we assume without loss of generality that $c \geq 0$. Furthermore we make the following assumption.

Assumption. *There exist $z_1 \leq z_2$ such that $F''(x) \geq 0$ for $x \leq z_1$, $F''(x) \leq 0$ for $x \geq z_2$.*

Definition. *We say that F is* convex-concave *if $z_1 = z_2$ in the above assumption.*

4.1 Asymptotic Behavior of $\hat{u}_x$ and ψ

Set $\phi(x) = w * \psi(x)$.

Lemma 4.1.1.

(i) There exist $y_1 < y_2$ such that $\hat{u}_{xx}(x) \geq 0$ for $x < y_1$ and $\hat{u}_{xx}(x) \leq 0$ for $x > y_2$. If F is convex-concave, then so is $\hat{u}_{xx}$ since we can choose $y_1 = y_2$.

(ii) There exist $\tilde{y}_1 < \tilde{y}_2$ such that $\phi_x(x) > 0$ for $x < \tilde{y}_1$ and $\phi_x(x) < 0$ for $x > \tilde{y}_2$.

*(iii) For all $x \leq \bar{y}_1 := \min(\hat{u}^{-1}(z_1), y_1, \tilde{y}_1)$, $\psi_x(x) \geq 0$,
while for all $x \geq \bar{y}_2 := \max(\hat{u}^{-1}(z_2), y_2, \tilde{y}_2)$, $\psi_x(x) \leq 0$.*

Proof. Note first that if $c = 0$, then $\hat{u} = w * F(\hat{u})$ implies that $\sigma^2 \hat{u}_{xx} = \hat{u} - F(\hat{u})$. Set $Z = \int \hat{u}_x^2(x) F'(\hat{u}(x))dx$. We have $\psi_x = \frac{1}{Z} F''(\hat{u})\hat{u}_x^2 + F'(\hat{u})\hat{u}_{xx}$ and $\phi_x = \frac{1}{Z}(w * \psi)_x = \frac{1}{Z}\hat{u}_{xx}$. In this case, (i)-(iii) are therefore satisfied. Now assume that $c > 0$.

(i) We have

$$\begin{aligned}\hat{u}(x) - \sigma^2 \hat{u}_{xx}(x) &= (1 - \sigma^2 \Delta) \int_0^\infty e^{-s} w * F(\hat{u})(x + cs)ds \\ &= \int_0^\infty e^{-s} F(\hat{u}(x + cs))ds \geq F(\hat{u}(x)),\end{aligned}$$

which implies that $\sigma^2 \hat{u}_{xx} \leq \hat{u} - F(\hat{u}) < 0$ for $x > \hat{u}^{-1}(a)$.

Let $b_1 = \min\{x \geq a_1 : F'(x) \geq 1\}$. Assume that there exist $x_1 < x_2 < \hat{u}^{-1}(b_1)$ such that $\hat{u}_{xx}(x_1) = 0, \hat{u}_{xx}(x_2) = 0$ and $\hat{u}_{xx}(x) < 0$ for $x_1 < x < x_2$. Then $\hat{u}_{xxx}(x_1) \leq 0$ and $\hat{u}_{xxx}(x_2) \geq 0$. We have

$$\begin{aligned}\hat{u} - \sigma^2 \hat{u}_{xx} - c\hat{u}_x + c\sigma^2 \hat{u}_{xxx} &= (1 - \sigma^2 \Delta)(\hat{u} - c\hat{u}_x) \\ &= (1 - \sigma^2 \Delta) w * F(\hat{u}) = F(\hat{u})\end{aligned}$$

and therefore

$$\begin{aligned}0 = \sigma^2(\hat{u}_{xx}(x_2) - \hat{u}_{xx}(x_1)) &= \underbrace{c\sigma^2(\hat{u}_{xxx}(x_2) - \hat{u}_{xxx}(x_1))}_{\geq 0} - \underbrace{c(\hat{u}_x(x_2) - \hat{u}_x(x_1))}_{<0} \\ &\quad + \underbrace{\hat{u}(x_2) - F(\hat{u}(x_2)) - (\hat{u}(x_1) - F(\hat{u}(x_1)))}_{= \int_{x_1}^{x_2} (1 - F'(\hat{u}(x)))\hat{u}_x(x)dx > 0} > 0,\end{aligned}$$

which is a contradiction. Thus, since $\hat{u}_x > 0$ implies that $\hat{u}_{xx}(x) > 0$ for arbitrarily small x, the claim follows.

If $z_1 = z$, assume that there exist $\hat{u}^{-1}(b_1) < x_3 < x_4 < \hat{u}^{-1}(a)$ such that $\hat{u}_{xx}(x_3) = \hat{u}_{xx}(x_4) = 0$, $\hat{u}_{xx}(x) > 0$ for $x_3 < x < x_4$. Since $F'(b_1) = 1$ and $F'(a) > 1$, $F'(\hat{u}(x)) \geq 1$ for all $x_3 < x < x_4$. Therefore

$$0 = \sigma^2(\hat{u}_{xx}(x_4) - \hat{u}_{xx}(x_3)) = \underbrace{c\sigma^2(\hat{u}_{xxx}(x_4) - \hat{u}_{xxx}(x_3))}_{<0} - \underbrace{c(\hat{u}_x(x_4) - \hat{u}_x(x_3))}_{>0}$$
$$+ \underbrace{\hat{u}_x(x_4) - F(\hat{u}_x(x_4)) - (\hat{u}_x(x_3) - F(\hat{u}_x(x_3)))}_{=\int_{x_3}^{x_4}(1-F'(\hat{u}(x)))\hat{u}_x(x)dx<0} < 0,$$

which is a contradiction. It follows that we can choose $y_1 = y_2$.

(ii): ϕ satisfies

$$\begin{aligned} F'(\hat{u})\phi &= (1 + c\partial_x)\psi = (1 + c\partial_x)(1 - \sigma^2\Delta)\phi \\ &= \phi + c\phi_x - \sigma^2\phi_{xx} - c\sigma^2\phi_{xxx}. \end{aligned} \tag{4.1}$$

There exist $z_1' < z_2'$ such that $F'(\hat{u}(x)) < 1$ for all $x \le z_1'$ and $x \ge z_2'$. Since $\int_{-\infty}^{x} \phi_x(y)dy = \phi(x) > 0$ for all x, there exist arbitrarily small z such that $\phi_x(z) > 0$. Analogously, since $\int_x^\infty \phi_x(y)dy = -\phi(x) < 0$, there exist arbitrarily large z such that $\phi_x(z) < 0$. We show that there does not exist a non-negative local maximum of ϕ_x on $\{F'(\hat{u}) < 1\}$. Then (ii) follows.

So assume there exists x_0 such that $F'(\hat{u}(x_0)) < 1$ and ϕ_x attains a local maximum at x_0 with $\phi(x_0) \ge 0$. Then, using (4.1),

$$0 < (1 - F'(\hat{u}(x_0)))\phi(x_0) = \underbrace{-c\phi_x(x_0)}_{\le 0} + \underbrace{\sigma^2\phi_{xx}(x_0)}_{=0} + c\sigma^2 \underbrace{\phi_{xxx}(x_0)}_{\le 0} \le 0,$$

which is a contradiction.

(iii): ψ satisfies $\psi + c\psi_x = F'(\hat{u})\phi$. Differentiating we obtain

$$\psi_x(x) + c\psi_{xx}(x) = F''(\hat{u}(x))\hat{u}_x(x)\phi(x) + F'(\hat{u}(x))\phi_x(x) =: g(x).$$

For $x \le \bar{y}_1$, $g(x) > 0$ and for $x \ge \bar{y}_2$, $g(x) < 0$. Thus, ψ does not attain a local maximum on $(-\infty, \bar{y}_1)$, nor a local minimum on $(\bar{y}_2, \infty)$. Since $\psi > 0$ there exist arbitrarily small x such that $\psi_x(x) > 0$ and arbitrarily large x such that $\psi_x(x) < 0$, and the claim follows. □

Set $\delta_1 = 1 - \lim_{x\to-\infty} F'(\hat{u}(x)) = 1 - F'(a_1)$, $\delta_2 = 1 - \lim_{x\to\infty} F'(\hat{u}(x)) = 1 - F'(a_2)$.

Theorem 4.1.2. *Let $\epsilon > 0$. There exist $x_1(\epsilon) < x_2(\epsilon)$ and $\sqrt{\delta_1} < \tilde{\delta}_1(c) < 1, \sqrt{\delta_2} < \tilde{\delta}_2(c) < 1$, such that for all $x \le x_1, y > 0$,*

$$\begin{aligned} \hat{u}_x(x) &\le e^{\frac{\sqrt{\delta_1}+\epsilon}{\sigma}y}\hat{u}_x(x-y) \\ e^{\frac{\tilde{\delta}_1(c)-\epsilon}{\sigma}y}\phi(x-y) \le \phi(x) &\le e^{\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}y}\phi(x-y) \end{aligned}$$

and for all $x \geq x_2, y > 0$,

$$\begin{aligned} e^{-\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}y}\hat{u}_x(x) \leq& \hat{u}_x(x+y) \leq e^{-\frac{\tilde{\delta}_2(c)-\epsilon}{\sigma}y}\hat{u}_x(x) \\ e^{-\frac{\sqrt{\delta_2}+\epsilon}{\sigma}y}\phi(x) \leq& \phi(x+y). \end{aligned}$$

For $i = 1, 2$, $\tilde{\delta}_i(c)$ is the unique positive root of $f_i(x, c) = cx^3 + \sigma x^2 - cx - \delta_i\sigma$, and is increasing in c with $\tilde{\delta}_i(0) = \sqrt{\delta_i}$ and $\lim_{c\to\infty} \tilde{\delta}_i(c) = 1$.

Proof. Let $\bar{y}_1, \bar{y}_2$ be as in Lemma 4.1.1. Note that

$$\begin{aligned} \frac{\sigma^2\hat{u}_{xxx}(x)}{\hat{u}_x(x)} &= 1 - \frac{(I - \sigma^2\Delta)\hat{u}_x(x)}{\hat{u}_x(x)} \\ &= 1 - \frac{(I - \sigma^2\Delta)\int_0^\infty e^{-s}w * (F'(\hat{u})\hat{u}_x)(x + cs)ds}{\hat{u}_x(x)} \\ &= 1 - \frac{\int_0^\infty e^{-s}F'(\hat{u}(x + cs))\hat{u}_x(x + cs)ds}{\hat{u}_x(x)}, \end{aligned}$$

and since $F'(\hat{u})\phi = \psi + c\psi_x = (I + c\partial_x)(I - \sigma^2\Delta)\phi$,

$$\begin{aligned} \frac{\sigma^2\phi_{xx}(x)}{\phi(x)} &= 1 - \frac{(I + c\partial_x)^{-1}(F'(\hat{u})\phi)(x)}{\phi(x)} \\ &= 1 - \frac{\int_0^\infty e^{-s}F'(\hat{u}(x - cs))\phi(x - cs)ds}{\phi(x)}. \end{aligned}$$

So if $c = 0$, then $\frac{\sigma^2\hat{u}_{xxx}(x)}{\hat{u}_x(x)} = \frac{\sigma^2\phi_{xx}(x)}{\phi(x)} = 1 - F'(\hat{u}(x))$, which converges to δ_1 and δ_2 for $x \to -\infty$ and $x \to \infty$, respectively.

Now assume that $c > 0$. For $x \leq \bar{y}_1$, $\hat{u}_{xx}(x) \geq 0$ and thus

$$\begin{aligned} \frac{\sigma^2\hat{u}_{xxx}(x)}{\hat{u}_x(x)} &\leq 1 - \int_0^{\frac{\bar{y}_1 - x}{c}} e^{-s}F'(\hat{u}(x + cs))\frac{\hat{u}_x(x + cs)}{\hat{u}_x(x)}ds \\ &\leq 1 - \int_0^{\frac{\bar{y}_1 - x}{c}} e^{-s}F'(\hat{u}(x + cs))ds \xrightarrow{x\to-\infty} \delta_1. \end{aligned}$$

Thus, there exists $x_1(\epsilon)$ such that for $x \leq x_1$,

$$\hat{u}_{xxx}(x) \leq \frac{\delta_1 + \epsilon}{\sigma^2}\hat{u}_x(x),$$

and since $\frac{d}{dx}(\hat{u}_{xx}^2(x) - \frac{\delta_1+\epsilon}{\sigma^2}\hat{u}_x^2(x)) = 2\hat{u}_{xx}(x)(\hat{u}_{xxx}(x) - \frac{\delta_1+\epsilon}{\sigma^2}\hat{u}_x(x)) \leq 0$ and $\lim_{x\to-\infty}(\hat{u}_{xx}^2(x) - \frac{\delta_1+\epsilon}{\sigma^2}\hat{u}_x^2(x)) = 0$, it follows that $\hat{u}_{xx}(x) \leq \frac{\sqrt{\delta_1+\epsilon}}{\sigma}\hat{u}_x(x)$ and hence for $y > 0$, $\hat{u}_x(x) \leq e^{\frac{\sqrt{\delta_1+\epsilon}}{\sigma}y}\hat{u}_x(x - y)$.

For $x \geq \bar{y}_2$, $\phi_x(x) \leq 0$ and thus

$$\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \leq 1 - \int_0^{\frac{x-\bar{y}_2}{c}} e^{-s}F'(\hat{u}(x-cs))\frac{\phi(x-cs)}{\phi(x)}ds$$
$$\leq 1 - \int_0^{\frac{x-\bar{y}_2}{c}} e^{-s}F'(\hat{u}(x-cs))ds \xrightarrow{x\to\infty} \delta_2$$

and we obtain similarly to the above that there exists x_2 such that for $x \geq x_2$, $y > 0$, $\phi(x+y) \geq e^{-\frac{\sqrt{\delta_2+\epsilon}}{\sigma}y}\phi(x)$.

Next we show that $\tilde{\delta}_1^2(c) := \lim_{x\to-\infty}\frac{\sigma^2\phi_{xx}(x)}{\phi(x)}$ and $\tilde{\delta}_2^2(c) := \lim_{x\to\infty}\frac{\sigma^2\hat{u}_{xxx}(x)}{\hat{u}_x(x)}$ exist. Note that $|\phi_x| \leq \frac{1}{\sigma}\phi$ such that for $x \leq \bar{y}_1$ and $y > 0$, $\phi(x) \leq e^{\frac{1}{\sigma}y}\phi(x-y)$. It follows that for $x \leq \bar{y}_1$,

$$\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \leq 1 - \int_0^{\infty} e^{-s}F'(\hat{u}(x-cs))e^{-\frac{1}{\sigma}cs}ds$$
$$\xrightarrow{x\to-\infty} 1-(1-\delta_1)\frac{\sigma}{\sigma+c} = \frac{c}{\sigma+c} + \delta_1\frac{\sigma}{\sigma+c} =: \delta_1^{(1)}(c),$$

with $\delta_1 < \delta_1^{(1)}(c) < 1$. It follows that there exists x_1 such that for $x \leq x_1$, $y > 0$, $\phi(x) \leq e^{\frac{\sqrt{\delta_1^{(1)}(c)+\epsilon}}{\sigma}y}\phi(x-y)$. Using this improved bound, we obtain that

$$\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \leq 1 - \int_0^{\infty} e^{-s}F'(\hat{u}(x-cs))e^{-\frac{\sqrt{\delta_1^{(1)}(c)+\epsilon}}{\sigma}cs}ds$$
$$\xrightarrow{x\to-\infty} 1-(1-\delta_1)\frac{\sigma}{\sigma+c\sqrt{\delta_1^{(1)}(c)+\epsilon}} =: \delta_1^{(2)}(c,\epsilon).$$

Thus, $\limsup_{x\to-\infty}\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \leq \delta_1^{(2)}(c,\epsilon) \xrightarrow{\epsilon\to0} 1-(1-\delta_1)\frac{\sigma}{\sigma+c\sqrt{\delta_1^{(1)}(c)}} =: \delta_1^{(2)}(c)$ with $\delta_1 < \delta_1^{(2)}(c) < \delta_1^{(1)}(c)$. Iterating this procedure we obtain a decreasing sequence $\delta_1^{(n)}(c) > \delta_1$ satisfying

$$\delta_1^{(n+1)}(c) = 1-(1-\delta_1)\frac{\sigma}{\sigma+c\sqrt{\delta_1^{(n)}(c)}}.$$

Thus, $\tilde{\delta}_1(c) := \lim_{n\to\infty}\sqrt{\delta_1^{(n)}(c)}$ satisfies

$$c\tilde{\delta}_1^3(c) + \sigma\tilde{\delta}_1^2(c) - c\tilde{\delta}_1(c) = \delta_1\sigma$$

and is therefore the unique positive root of $f_1(c,x) = cx^3 + \sigma x^2 - cx - \delta_1\sigma$.

On the other hand, for small enough x,

$$\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \geq 1 - \int_0^{\infty} e^{-s}F'(\hat{u}(x-cs))ds \xrightarrow{x\to-\infty} \delta_1,$$

and hence

$$\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \geq 1 - \int_0^\infty e^{-s}F'(\hat{u}(x-cs))e^{-\frac{\sqrt{\delta_1-\epsilon}}{\sigma}cs}ds$$
$$\xrightarrow{x\to-\infty} 1-(1-\delta_1)\frac{\sigma}{\sigma+c\sqrt{\delta_1-\epsilon}} =: \delta_1'^{(1)}(c,\epsilon).$$

Thus, $\liminf_{x\to-\infty}\frac{\sigma^2\phi_{xx}(x)}{\phi(x)} \geq \delta_1'^{(1)}(c,\epsilon) \xrightarrow{\epsilon\to 0} 1-(1-\delta_1)\frac{\sigma}{\sigma+c\sqrt{\delta_1}} =: \delta_1'^{(1)}(c)$ with $\delta_1 < \delta_1'^{(1)}(c) < 1$. Iteration of the procedure yields an increasing sequence $\delta_1'^{(n)}(c) < 1$ and $\delta_1'(c) := \lim_{n\to\infty}\sqrt{\delta_1'^{(n)}(c)}$ is the unique positive root of $f_1(c,x) = cx^3+\sigma x^2 - cx - \delta_1\sigma$. Hence, $\delta_1'(c) = \tilde{\delta}_1(c)$ and it follows that there exists x_1 such that for $x \leq x_1$, $y > 0$,

$$e^{\frac{\tilde{\delta}_1(c)-\epsilon}{\sigma}y}\phi(x-y) \leq \phi(x) \leq e^{\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}y}\phi(x-y)$$

Analogously, we obtain that there exists x_2 such that for $x \geq x_2$, $y > 0$,

$$e^{-\frac{\tilde{\delta}_2(x)+\epsilon}{\sigma}y}\hat{u}_x(x) \leq \hat{u}_x(x+y) \leq e^{-\frac{\tilde{\delta}_2(c)-\epsilon}{\sigma}y}\hat{u}_x(x),$$

where $\tilde{\delta}_2(c)$ is the unique positive root of $f_2(c,x) = cx^3+\sigma x^2 - cx - \delta_2\sigma$.

Since for $i = 1, 2$,

$$0 = \frac{d}{dc}\big(f_i(c,\tilde{\delta}_i(c))\big) = \frac{\partial}{\partial c}f_i(c,\tilde{\delta}_i(c)) + \frac{d}{dc}\tilde{\delta}_i(c)\frac{\partial}{\partial x}f_i(c,\tilde{\delta}_i(c)),$$

$\frac{\partial}{\partial c}f_i(c,\tilde{\delta}_i(c)) = \tilde{\delta}_i^3(c) - \tilde{\delta}_i(c) < 0$, and $\frac{\partial}{\partial x}f(c,\tilde{\delta}_i(c)) > 0$, it follows that $\tilde{\delta}_i(c)$ is increasing in c with $\tilde{\delta}_i(0) = \sqrt{\delta_i}$ and $\lim_{c\to\infty}\tilde{\delta}_i(c) = 1$. □

4.2 The Asymptotic Behavior of ρ

The asymptotic behavior of $\hat{u}_x$ and ψ determine that of the density ρ.

Proposition 4.2.1. *(i) There exists constants $\tilde{k}_1, \tilde{k}_2$ such that $\tilde{k}_1\phi \leq \psi \leq \tilde{k}_2\phi$.*

(ii) Let $\epsilon > 0$ (small enough) and let $x_1(\epsilon) < x_2(\epsilon)$ be as in Theorem 4.1.2. There exist constants k_1, k_2, k_1', k_2' such that for $x \leq x_1$, $y > 0$,

$$k_1 e^{\frac{\tilde{\delta}_1(c)-\sqrt{\delta_1}-2\epsilon}{\sigma}y}\rho(x-y) \leq \rho(x) \leq k_2 e^{\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}y}\rho(x-y)$$

and for $x \geq x_2$, $y > 0$,

$$k_1' e^{\frac{\tilde{\delta}_2(c)-\sqrt{\delta_2}-2\epsilon}{\sigma}y}\rho(x) \leq \rho(x+y) \leq k_2' e^{\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}y}\rho(x).$$

Proof. (i) We have

$$(1+c\partial_x)\psi_x = F''(\hat{u})\hat{u}_x w * \psi + F'(\hat{u})w_x * \psi$$

and thus

$$\begin{aligned}\psi_x(x) &= \int_0^\infty e^{-s}\big(F''(\hat{u}(x-cs))\hat{u}_x(x-cs)w*\psi(x-cs)\\ &\quad + F'(\hat{u}(x-cs))w_x*\psi(x-cs)\big)ds\\ &\le \Big(\Big\|\frac{F''(\hat{u})\hat{u}_x}{F'(\hat{u})}\Big\|_\infty + \Big\|\frac{w_x}{w}\Big\|_\infty\Big)\psi(x).\end{aligned}$$

It follows that there exists $\tilde{k}_1$ such that

$$\phi = \frac{\psi + c\psi_x}{F'(\hat{u})} \le \frac{1}{\tilde{k}_1}\psi.$$

Let $\bar{y}_1, \bar{y}_2$ be as in Lemma 4.1.1. Fix $\delta > 0$. Then for $x \ge \bar{y}_2 + \delta$,

$$\phi(x) = w*\psi(x) \ge \int_{x-\delta}^{x} w(x-y)\psi(y) \ge \int_0^\delta w(y)dy\psi(x),$$

and for $x \le \bar{y}_1 - \delta$,

$$\phi(x) \ge \int_x^{x+\delta} w(x-y)\psi(y)dy = \int_0^\delta w(y)dy\psi(x).$$

For $x_1 - \delta \le x \le x_2 + \delta$,

$$\phi(x) \ge \frac{\min_{x_1-\delta\le y\le x_2+\delta}\phi(y)}{\max_{x_1-\delta\le y\le x_2+\delta}\psi(y)}\psi(x),$$

and the claim follows.

(ii) For $x \le x_1$, $y > 0$,

$$\rho(x-y) \le \tilde{k}_2\frac{\phi(x-y)}{\hat{u}_x(x-y)} \le \tilde{k}_2 e^{-\frac{\tilde{\delta}_1(c)-\sqrt{\delta_1}-2\epsilon}{\sigma}y}\frac{\phi(x)}{\hat{u}_x(x)} \le \frac{\tilde{k}_2}{\tilde{k}_1}e^{-\frac{\tilde{\delta}_1(c)-\sqrt{\delta_1}-2\epsilon}{\sigma}y}\rho(x)$$

and

$$\rho(x-y) \ge \tilde{k}_1\frac{\phi(x-y)}{\hat{u}_x(x-y)} \ge \tilde{k}_1 e^{-\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}y}\frac{\phi(x)}{\hat{u}_x(x)} \ge \frac{\tilde{k}_1}{\tilde{k}_2}e^{-\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}y}\rho(x).$$

For $x \ge x_2$, $y > 0$,

$$\rho(x+y) \le \tilde{k}_2 e^{\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}y}\frac{\phi(x)}{\hat{u}_x(x)} \le \frac{\tilde{k}_2}{\tilde{k}_1}e^{\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}y}\rho(x),$$

and

$$\rho(x+y) \ge \frac{\tilde{k}_1}{\tilde{k}_2}e^{\frac{\tilde{\delta}_2(c)-\sqrt{\delta_2}-2\epsilon}{\sigma}y}\rho(x).$$

□

We can now verify that ρ satisfies assumption 2.4.3.

Corollary 4.2.2. *There exists a constant L_ρ such that $\rho(x-y) \leq L_\rho \rho(x)$ for all $x \in \mathbb{R}$ and $y \geq 0$.*

Proof. By Proposition 4.2.1, there exist $x_1 < x_2$ and a constant k such that $\rho(x-y) \leq k\rho(x)$ for $x \leq x_1, y > 0$, and $\rho(x-y) \leq k\rho(x)$ for $x - y \geq x_2, y > 0$. If $x - y \leq x_1 \leq x \leq x_2$, then $\rho(x-y) \leq k\rho(x_1) \leq k\frac{\rho(x_1)}{\min_{x_1 \leq z \leq x_2} \rho(z)}\rho(x)$, if $x - y \leq x_1 < x_2 \leq x$, then $\rho(x-y) \leq k\rho(x_1) \leq k^2 \frac{\rho(x_1)}{\rho(x_2)}\rho(x)$, and if $x_1 \leq x - y \leq x_2$, then $\rho(x-y) \leq k\frac{\max_{x_1 \leq z \leq x_2} \rho(z)}{\min_{x_1 \leq z \leq x_2} \rho(z)}\rho(x)$. □

Corollary 4.2.3. *There exists a constant K_ρ such that for all $x \in \mathbb{R}$,*

$$\int w(x-y)\rho(y)dy \leq K_\rho \rho(x).$$

Proof. Fix $\epsilon > 0$ (small enough) and let x_1, x_2 be as in Theorem 4.1.2. We denote by k an arbitrary positive constant that may change from step to step. By Proposition 4.2.1, we have for $x \leq x_1$,

$$\begin{aligned} w * \rho(x) &\leq k \int_{-\infty}^{x} w(x-y)\rho(x)dy + k\int_{x}^{x_1} w(x-y)e^{\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}(y-x)}\rho(x)dy \\ &\quad + \int_{x_1}^{x_2} w(x-y)dy \max_{x_1 \leq y \leq x_2} \rho(y) + k\int_{x_2}^{\infty} w(x-y)e^{\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}(y-x_2)}\rho(x_2)dy \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Clearly, $I_1 \leq k\rho(x)$. Since $\tilde{\delta}_1(c) + \epsilon < 1$, also $I_2 \leq k\rho(x)$. Note that $\rho(x) \geq ke^{-\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}(x_1-x)}\rho(x_1)$. As $\int_{x_1}^{x_2} w(x-y)dy = \frac{1}{2}(e^{-\frac{x_1-x}{\sigma}} - e^{-\frac{x_2-x}{\sigma}})$, it follows that

$$I_3 \leq ke^{-\frac{1-\tilde{\delta}_1(c)-\epsilon}{\sigma}(x_1-x)}\frac{\rho(x)}{\rho(x_1)} \leq k\rho(x).$$

Since $\int_{x_2}^{\infty} w(x-y)e^{\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}(y-x_2)}dy \leq ke^{\frac{x}{\sigma}}$, we have

$$I_4 \leq ke^{\frac{x}{\sigma}}e^{\frac{\tilde{\delta}_1(c)+\epsilon}{\sigma}(x_1-x)}\frac{\rho(x)}{\rho(x_1)} \leq ke^{\frac{x_1}{\sigma}}e^{-\frac{1-\tilde{\delta}_1(c)-\epsilon}{\sigma}(x_1-x)}\frac{\rho(x)}{\rho(x_1)} \leq k\rho(x).$$

For $x_1 \leq x \leq x_2$, we obtain as above that

$$w * \rho(x) \leq k\rho(x_1) + \max_{x_1 \leq y \leq x_2} \rho(y) + k\rho(x_2) \leq k\frac{\rho(x)}{\min_{x_1 \leq y \leq x_2} \rho(y)}.$$

Finally, for $x \geq x_2$,

$$w * \rho(x) \leq k\rho(x_1) + \max_{x_1 \leq y \leq x_2} \rho(y) + k \int_{x_2}^{x} w(x,y)dy\rho(x) + k \int_{x}^{\infty} w(x,y)e^{\frac{\tilde{\delta}_2(c)+\epsilon}{\sigma}(y-x)}\rho(x)dy.$$

Noting that $\tilde{\delta}_2(c) + \epsilon < 1$ and that $\rho(x) \geq k\rho(x_2)$, we see that also in this case $w * \rho(x) \leq k\rho(x)$, which concludes the proof. □

Together with Theorem 4.1.2 it is now easily checked that the conditions of Theorem 3.4.2 are satisfied

Corollary 4.2.4. *If the kernel w is given as $w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$ for some $\sigma > 0$, then the frozen wave operator satisfies a spectral gap inequality in $L^2(\rho)$: there exists $\kappa > 0$ such that for all $v \in H^1(\rho)$,*

$$\langle L^{\#}v, v\rangle_\rho \leq -\kappa(\|v\|_\rho^2 - \langle v, \hat{u}_x\rangle_\rho^2).$$

4.3 Bounds on the Wave Speed

In general, the speed c of the traveling wave is unknown. We have seen in section 1.1.5 that it can be explicitly calculated in the case of a Heaviside nonlinearity. In [27], Thm. 3.1, Ermentrout and McLeod proved that

$$c = \frac{\int_{a_1}^{a_2} x - F(x)dx}{\int \hat{u}_x^2(x)F'(\hat{u}(x))dx}. \tag{4.2}$$

We can use this representation to deduce a lower and an upper bound on c in the exponential example.

Proposition 4.3.1. *Assume that F is convex-concave. The wave speed is bounded in terms of the parameters of the system, σ and F:*

$$\frac{\sigma}{\sqrt{2}} \frac{\int_{a_1}^{a_2} x - F(x)dx}{\sqrt{\int_{a_1}^{a} x - F(x)dx}} \leq c \leq \frac{\sigma}{4} \frac{\int_{a_1}^{a_2} x - F(x)dx}{\int_{a}^{a_2} F(x) - xdx}$$

Proof. By Lemma 4.1.1, there exists a unique x_0 such that $\hat{u}_{xx}(x_0) = 0$ and $\hat{u}_{xx}(x) \geq 0$ for $x \leq x_0$ and $\hat{u}_{xx}(x) \leq 0$ for $x \geq x_0$.

We first prove the upper bound. Since $|\hat{u}_{xx}| \leq \frac{1}{\sigma}\hat{u}_x$, we have that

$$\hat{u}_x(x) \geq e^{-\frac{|x-x_0|}{\sigma}}\hat{u}_x(x_0).$$

This implies that

$$\begin{aligned}\int \hat{u}_x^2(x)F'(\hat{u}(x))dx &\geq \int e^{-\frac{|x-x_0|}{\sigma}} \hat{u}_x(x)\hat{u}_x(x_0)F'(\hat{u}(x))dx \\ &= 2\sigma w * (F'(\hat{u})\hat{u}_x)(x_0)\hat{u}_x(x_0) = 2\sigma(\hat{u}_x(x_0) - c\hat{u}_{xx}(x_0))\hat{u}_x(x_0) \\ &= 2\sigma\hat{u}_x^2(x_0).\end{aligned}$$

Since (2.5) implies that $\sigma^2\hat{u}_{xx} = \hat{u} - \int_0^\infty e^{-s}F(\hat{u}(\cdot + cs))ds$ we obtain

$$\begin{aligned}\hat{u}_x^2(x) &= 2\int_x^\infty -\hat{u}_{xx}(y)\hat{u}_x(y)dy = 2\int_x^\infty -\frac{\hat{u}(y) - \int_0^\infty e^{-s}F(\hat{u}(y+cs)ds}{\sigma^2}\hat{u}_x(y)dy \\ &\geq 2\int_x^\infty -\frac{\hat{u}(y) - F(\hat{u}(y))}{\sigma^2}\hat{u}_x(y)dy = \frac{2}{\sigma^2}\int_{\hat{u}(x)}^{a_2} F(x) - x\ dx.\end{aligned}$$

Therefore

$$\hat{u}_x^2(x_0) = \max \hat{u}_x^2(x) \geq \hat{u}_x^2(\hat{u}^{-1}(a)) \geq \frac{2}{\sigma^2}\int_a^{a_2} F(x) - x\ dx.$$

Using (4.2), we obtain

$$c = \frac{\int_{a_1}^{a_2} x - F(x)dx}{\int \hat{u}_x^2(x)F'(\hat{u}(x))dx} \leq \frac{1}{2\sigma}\frac{\int_{a_1}^{a_2} x - F(x)dx}{\hat{u}_x^2(x_0)} \geq \sigma\frac{\int_{a_1}^{a_2} x - F(x)dx}{\int_a^{a_2} F(x) - xdx}.$$

This yields the upper bound.

We now prove the lower bound. We have

$$\begin{aligned}\hat{u}_x^2(x) &= 2\int_{-\infty}^x \hat{u}_x(y)\hat{u}_{xx}(y)dy = \frac{2}{\sigma^2}\int_{-\infty}^x \left(\hat{u}(y) - \int_0^\infty e^{-s}F(\hat{u}(y+cs))ds\right)\hat{u}_x(y)dy \\ &\leq \frac{2}{\sigma^2}\int_{-\infty}^x (\hat{u}(y) - F(\hat{u}(y)))\hat{u}_x(y)dy = \frac{2}{\sigma^2}\int_{a_1}^{\hat{u}(x)} x - F(x)dx.\end{aligned}$$

Since

$$0 = \hat{u}_{xx}(x_0) = \frac{1}{\sigma^2}\left(\hat{u}(x_0) - \int_0^\infty e^{-s}F(\hat{u}(x_0+cs))ds\right)$$

it follows that

$$\hat{u}(x_0) = \int_0^\infty e^{-s}F(\hat{u}(x_0+cs))ds > F(\hat{u}(x_0))$$

and hence $\hat{u}(x_0) < a$, so that

$$\hat{u}_x^2(x_0) \leq \frac{2}{\sigma^2}\int_{a_1}^{\hat{u}(x_0)} x - F(x)dx \leq \frac{2}{\sigma^2}\int_{a_1}^{a} x - F(x)dx.$$

Thus, using (4.2),

$$c \geq \frac{\int_{a_1}^{a_2} x - F(x)dx}{\hat{u}_x(x_0)\int F'(\hat{u}(x))\hat{u}_x(x)dx} \geq \frac{\sigma}{\sqrt{2}}\frac{\int_{a_1}^{a_2} x - F(x)dx}{\sqrt{\int_{a_1}^{a} x - F(x)dx}}. \qquad \square$$

4.4 The Spectral Gap Inequality in $L^2(\mathfrak{m})$

In Theorem 3.5.1 we proved that the traveling wave operator L satisfies a spectral gap inequality in $L^2(\mathfrak{m})$ if the wave speed c is small enough. In the exponential example we can express this smallness condition in terms of the parameters of the system, σ and F.

Proposition 4.4.1. *Assume that F is convex-concave. For all small enough wave speeds c the traveling wave operator L satisfies a spectral gap inequality in $L^2(\mathfrak{m})$. The smallness condition on c can be expressed explicitly in terms of the parameters of the system, σ and F.*

Proof. I) In view of Theorem 3.5.1, we explicitly calculate κ_0^0 such that for all $h \in H^1(\mu^0)$,

$$Var_{\mu^0}(h) \leq \kappa_0^0 \int h_x^2 d\mu^0,$$

where $\mu^0(x) = \frac{1}{Z_{\mu^0}} w * (F'(\hat{u})\hat{u}_x)(x)\hat{u}_x(x)F'(\hat{u}(x))$ with $Z_{\mu^0} = \int w * (F'(\hat{u})\hat{u}_x)\hat{u}_x F'(\hat{u})dx$.

By Lemma 4.1.1, there exists a unique x_0 such that $\hat{u}_{xx}(x_0) = 0$ and $\hat{u}_{xx}(x) \geq 0$ for $x \leq x_0$, $\hat{u}_{xx}(x) \leq 0$ for $x \geq x_0$. Let $\epsilon_i < \delta_i$, such that $F'(a_i) = 1 - \delta_i < 1 - \epsilon_i$, $i = 1, 2$. Let $b_1 = \min\{x : F'(x) \geq 1 - \epsilon_1\}$ and $b_2 = \max\{x : F'(x) \geq 1 - \epsilon_2\}$. Set $F'_+ = \max_{a_1 \leq y \leq a_2} F'(y)$ and $F'_- = \min_{a_1 \leq y \leq a_2} F'(y)$. Set $\varphi^0 = w * (F'(\hat{u})\hat{u}_x)$. Then $\varphi^0 - \sigma^2\varphi^0_{xx} = F'(\hat{u})\hat{u}_x$ and thus

$$\sigma^2\varphi^0_{xx} = \Big(1 - F'(\hat{u})\frac{\hat{u}_x}{\varphi^0}\Big)\varphi^0.$$

We assume that $c < \epsilon_1\sigma$ such that

$$1 - F'(\hat{u})\frac{\hat{u}_x}{\varphi^0} = 1 - F'(\hat{u})\frac{\hat{u}_x}{\hat{u}_x - c\hat{u}_{xx}} \geq \begin{cases} \epsilon_2, & x \geq b_2 \\ 1 - (1-\epsilon_1)\frac{\sigma}{\sigma - c}, & x \leq b_1 \end{cases}.$$

Then for $x \geq \hat{u}^{-1}(b_2) \vee x_0$

$$\int_x^\infty \mu^0(y)dy \leq \frac{F'_+\hat{u}_x(x)}{Z_{\mu^0}}\int_x^\infty \frac{\sigma^2}{1 - F'(\hat{u})\frac{\hat{u}_x}{\varphi^0}}\varphi^0_{xx}dy.$$

Since $|\varphi^0_x| \leq \frac{1}{\sigma}\varphi^0$ if follows that

$$\int_x^\infty \mu^0(y)dy \leq \frac{F'_+\hat{u}_x(x)}{\epsilon_2 Z_{\mu^0}}(-\sigma^2\varphi^0_x(x)) \leq \frac{F'_+\sigma}{F'_-\epsilon_2}\mu^0(x).$$

Analogously, for $x \le \hat{u}^{-1}(b_1) \wedge x_0$, since $\varphi^0(x) = \hat{u}_x(x) - c\hat{u}_{xx}(x) \ge \frac{\sigma - c}{\sigma}\hat{u}_x$,

$$\int_{-\infty}^{x} \mu^0(y)dy \le \frac{F'_+\sigma}{F'_-\big(1-(1-\epsilon_1)\frac{\sigma}{\sigma-c}\big)}\mu^0(x).$$

If $\hat{u}^{-1}(b_2) > x_0$, then for $x_0 \le x \le \hat{u}^{-1}(b_2)$,

$$\int_x^{\hat{u}^{-1}(b_2)} \mu^0(y)dy \le \frac{1}{Z_{\mu^0}}\frac{\sigma+c}{\sigma}F'_+\int_x^{\hat{u}^{-1}(b_2)} \hat{u}_x^2(y)dy \le \frac{1}{Z_{\mu^0}}\frac{\sigma+c}{\sigma}F'_+\hat{u}_x^2(x)(\hat{u}^{-1}(b_2)-x_0).$$

We have

$$\hat{u}^{-1}(b_2) - x_0 = \int_{\hat{u}(x_0)}^{b_2}\frac{1}{\hat{u}_x(\hat{u}^{-1}(y))}dy \le \frac{1}{\hat{u}_x(\hat{u}^{-1}(b_2))}(b_2 - \hat{u}(x_0))$$

and, since $\hat{u} - c\hat{u}_x = w * F(\hat{u})$ implies that $\sigma^2\varphi^0_x = \hat{u} - F(\hat{u}) - c\hat{u}_x$,

$$\begin{aligned}\frac{1}{\hat{u}_x(\hat{u}^{-1}(b_2))} &\le \frac{\sigma+c}{\sigma}\frac{1}{\varphi^0(\hat{u}^{-1}(b_2))} \le -\frac{\sigma+c}{\sigma^2}\frac{1}{\varphi^0_x(\hat{u}^{-1}(b_2))}\\ &= \frac{\sigma+c}{F(b_2)-b_2+c\hat{u}_x(\hat{u}^{-1}(b_2))} \le \frac{\sigma+c}{F(b_2)-b_2}.\end{aligned}$$

Analogously, if $\hat{u}^{-1}(b_1) < x_0$, then for $\hat{u}^{-1}(b_1) \le x \le x_0$,

$$\int_{\hat{u}^{-1}(b_1)}^{x} \mu^0(y)dy \le \frac{F'_+}{Z_{\mu^0}}\int_{\hat{u}^{-1}(b_1)}^{x} \hat{u}_x^2(y)dy \le \frac{F'_+}{Z_{\mu^0}}\hat{u}_x^2(x)(x_0 - \hat{u}^{-1}(b_1)),$$

$$x_0 - \hat{u}^{-1}(b_1) \le \frac{1}{\hat{u}_x(\hat{u}^{-1}(b_1))}(\hat{u}(x_0) - b_1),$$

$$\begin{aligned}\frac{1}{\hat{u}_x(\hat{u}^{-1}(b_1))} &\le \frac{1}{\varphi^0(\hat{u}^{-1}(b_1))} \le \frac{1}{\sigma\varphi^0_x(\hat{u}^{-1}(b_1))} = \frac{\sigma}{b_1 - F(b_1) - c\hat{u}_x(\hat{u}^{-1}(b_1))}\\ &\le \frac{\sigma}{b_1 - F(b_1) - \frac{c}{\sigma}b_1}.\end{aligned}$$

Thus, for $x_0 \le x \le \hat{u}^{-1}(b_2)$,

$$\begin{aligned}\int_x^{\infty} \mu^0(y)dy \le{}& \frac{\sigma+c}{\sigma}\frac{F'_+}{F'_-}\mu^0(x)(b_2 - \hat{u}(x_0))\frac{\sigma+c}{F(b_2)-b_2}\\ &+ \frac{1}{Z_{\mu^0}}\frac{F'_+\sigma}{F'_-\epsilon_2}\varphi^0(\hat{u}^{-1}(b_2))\hat{u}_x(\hat{u}^{-1}(b_2))F'(b_2)),\end{aligned}$$

and since $\hat{u}_x(\hat{u}^{-1}(b_2)) \le \hat{u}_x(x)$, $F'(b_2) \le F'(\hat{u}(x))$, and

$$\varphi^0(\hat{u}^{-1}(b_2)) \le \frac{\sigma+c}{\sigma}\hat{u}_x(x) \le \frac{\sigma+c}{\sigma}\varphi^0(x),$$

we obtain

$$\int_x^\infty \mu^0(y)dy \le \frac{\sigma+c}{\sigma}\frac{F'_+}{F'_-}\Big(\frac{\sigma+c}{F(b_2)-b_2}(b_2-b_1)+\frac{\sigma}{\epsilon_2}\Big)\mu^0(x) =: \alpha_1(c)\mu^0(x).$$

Similarly we obtain

$$\int_{-\infty}^x \mu^0(y)dy \le \alpha_2(c)\mu^0(x),$$

where the dependence of $\alpha_1(c)$ and $\alpha_2(c)$ on c is explicit.

Set $\alpha(c) = \alpha_1(c) \vee \alpha_2(c)$. Then for $h \in H^1(\mu^0)$

$$\begin{aligned}\int_{-\infty}^{x_0}(h(x)-h(x_0))^2\mu^0(x)dx &= -2\int_{-\infty}^{x_0}(h(x)-h(x_0))h_x(x)\int_{-\infty}^x \mu^0(y)dydx\\ &\le 2\alpha\int_{-\infty}^{x_0}|h(x)-h(x_0)||h_x(x)|\mu^0(x)dx\\ &\le \frac{1}{2}\int_{-\infty}^{x_0}(h(x)-h(x_0))^2\mu^0(x)dx + 2\alpha^2\int_{-\infty}^{x_0}h_x^2(x)\mu^0(x)dx,\end{aligned}$$

and hence

$$\int_{-\infty}^{x_0}(h(x)-h(x_0))^2\mu^0(x)dx \le 4\alpha^2\int_{-\infty}^{x_0}h_x^2(x)\mu^0(x)dx.$$

Similarly,

$$\int_{x_0}^{\infty}(h(x)-h(x_0))^2\mu^0(x)dx \le 4\alpha^2\int_{x_0}^{\infty}h_x^2(x)\mu^0(x)dx.$$

Together we obtain that

$$Var_{\mu^0}(h) \le \int (h(x)-h(x_0))^2\mu^0(x)dx \le 4\alpha^2(c)\int h_x^2(x)\mu^0(x)dx.$$

II) Note that I) together with the fact that

$$\frac{\sigma-c}{\sigma}\hat{u}_x \le w*(F'(\hat{u})\hat{u}_x) \le \frac{\sigma+c}{\sigma}\hat{u}_x$$

and $\left\|\frac{w_x}{w}\right\|_\infty = \frac{1}{\sigma}$ imply that the conditions in Theorem 3.5.1 are satisfied. Therefore, as in (3.14), L satisfies a spectral gap inequality if

$$\kappa_0\Big(1-2\Big(\frac{c}{\sigma}\Big)^2\frac{\sigma}{\sigma-c}\Big)-\frac{c}{\sigma}|\kappa_0-1|>0,$$

where

$$\kappa_0 = \frac{1}{2}\frac{\sigma^2}{\sigma^2+\kappa_0^0} = \frac{1}{2}\frac{\sigma^2}{\sigma^2+4\alpha^2(c)}$$

Here the dependence on c is explicit. Using the bounds on c in terms of σ and F from Proposition 4.3.1, this condition can be translated into a condition on σ and F, which completes the proof. □

Chapter 5

A Multiscale Analysis

In this chapter we analyze the influence of the noise on the traveling wave. We prove an expansion of the solution to the stochastic neural field equation describing the effects to different orders of the noise strength. By separating two scales - fluctuations in the wave profile and shifts in the phase of the wave - we obtain a simplified description of the stochastic traveling wave. The chapter is based on the preprint [46].

5.1 Introduction

In this chapter, it is our main goal to provide a mathematically rigorous analysis of the influence of the noise on the traveling wave dynamics on multiple scales. The term multiscale refers mainly to two different spatial scales: first, shifts in the phase of the wave, that is, displacements of the wave profile from its uniformly translating position, and second, fluctuations in the wave profile. Recall from section 2.5.1 that by tracking the solution u to the stochastic neural field equation (2.1) with a reference wave we obtain an expression for the stochastic phase $C(t)$. We derive an expansion of u in the noise strength ϵ of the form

$$\begin{aligned} u(x,t) &= \hat{u}(x-\varphi_k(t)) + \epsilon v_0(x,t) + \epsilon^2 v_1(x,t) + \ldots + \epsilon^k v_{k-1}(x,t) + r_k(x,t) \\ \varphi_k(t) &= ct + \epsilon C_0(t) + \epsilon^2 C_1(t) + \ldots + \epsilon^k C_{k-1}(t), \end{aligned}$$

where the coefficients v_k and C_k are independent of ϵ and where the rest terms r_k are of higher order in ϵ. The term multiscale may thus also refer to the different orders of the noise strength, and the v_k and C_k describe the influence of the noise on the scale ϵ^{k+1}. Here we have separated the two spatial scales: the C_k describe the effects of the noise on the phase, and the v_k on the wave profile. The expansion is valid up to a stopping time τ which can be shown to be large with high probability converging to 1 as ϵ goes to 0.

An analysis of the properties of the coefficients then allows to describe the effects of the noise. To first order of the noise strength, the phase shift, given by C_0, is roughly diffusive.

Using the spectral properties of the linearized system (cf. Chapter 3) we find that the fluctuations are to first order given by a $L^2(\Omega; L^2(\rho))$-bounded Ornstein-Uhlenbeck process that is orthogonal to the direction of movement, expressing in particular the stability of the traveling wave under the noise.

The question of how noise influences the traveling wave dynamics in neural fields was first considered by Bressloff and Webber in [15]. They identified the two effects and then formally derived a decomposition of the solution. They obtained a stochastic differential equation as an approximate description of the shift of the phase to first order of the noise strength, and found that the noise causes diffusive wandering of the front.

In [42], Krüger and Stannat made first steps towards a mathematically rigorous derivation of a decomposition of the solution and our results can be seen as an extension of their work.

The problem is also considered by Inglis and MacLaurin in [37]. Under assumptions on the spectral properties of the dynamics they analyze the local stability and long-time behavior of the stochastic solution. While we dynamically adapt the speed of the reference wave such that its distance to the stochastic wave becomes minimal, they derive a stochastic differential equation for the phase whose solution realizes the minimum exactly.

Working in the space $L^2(\rho)$ seems natural here for two reasons. First, it will allow us to separate the two spatial scales. The dynamics of the phase and of the fluctuations in the wave profile decouple, such that we can obtain a separate description of the effects of the noise on the two scales (cf. section 5.4). Second, as we have seen in Chapter 3, we can describe the spectral properties of the system in the space $L^2(\rho)$. This will allow us to derive stability properties of the dynamics.

The method of obtaining a description of the stochastic dynamics by separating the dynamics on different spatial (or temporal) scales and approximating to a certain order of ϵ is related to Blömker's work on amplitude equations (see for example [6] or [5]).

5.2 Setting

We consider the stochastic neural field equation

$$dv(t) = \big(L_t v(t) + R(t, v(t))\big)dt + \epsilon dW_t, \qquad v(0) = \epsilon\eta. \tag{5.1}$$

Here (L_t) is the family of operators acting as

$$L_t v = -v + w * (F'(\hat{u}(\cdot - ct))v) \tag{5.2}$$

and

$$R(t, v) = w * \big(F(\hat{u}(\cdot - ct) + v) - F(\hat{u}(\cdot - ct)) - F'(\hat{u}(\cdot - ct))v\big).$$

For the moment we stick to the additive noise case and assume that assumption (C)(1) is satisfied. We will comment on the multiplicative noise case in section 5.4.1 below. We assume that Assumption 2.4.3 is satisfied and that $\left\|\frac{w_x}{w}\right\|_\infty < \infty$. This covers in particular the case of the exponential example.

Recall that by Proposition 2.4.4, for every initial condition $\eta \in H^1(1+\rho)$, there exists a unique strong $H^1(1+\rho)$-valued solution to (5.1) admitting a continuous modification.

The family of linear operators (L_t) satisfies

$$\|L_t h\|^2_{H^1(1+\rho)} \leq 2\Big(1 + \Big(1 + \Big\|\frac{w_x}{w}\Big\|^2_\infty\Big)(1+K_\rho)\|F'\|^2_\infty\Big)\|h\|^2_{H^1(1+\rho)} =: L_*^2 \|h\|^2_{H^1(1+\rho)}.$$

It generates an evolution semigroup $(P_{t,s})_{0\leq s\leq t\leq T}$ with

$$\|P_{t,s}h\|_{H^1(1+\rho)} \leq e^{L_*(t-s)}\|h\|_{H^1(1+\rho)}.$$

v can thus be represented as a mild solution

$$v(t) = \epsilon P_{t,0}\eta + \int_0^t P_{t,s}R(s, v(s))ds + \epsilon \int_0^t P_{t,s}dW_s.$$

Details on evolution semigroups can be found in [50].

Recall from section 2.3.2 that $v^\#(x,t) = v(x+ct, t)$ is the weak solution to the frozen wave equation. We assume that $L^\#$ satisfies a spectral gap inequality: there exists $\kappa > 0$ such that for all $v \in H^1(\rho)$,

$$\langle L^\# v, v\rangle_\rho \leq -\kappa\big(\|v\|^2_\rho - \langle v, \hat{u}_x\rangle^2_\rho\big).$$

Recall from Chapter 3 that this assumption is satisfied under rather general assumptions. In particular it is satisfied in the exponential example analyzed in Chapter 4. Since

$$\langle L^\# v, \hat{u}_x\rangle_\rho = \langle L^\# v, \psi\rangle = \langle v, L^{\#,*}\psi\rangle = 0,$$

$L^\#$ generates a contraction semigroup $(P_t^\#)$ on $L^2(\rho)$ that satisfies

$$\|P_t^\# v\|_\rho \leq e^{-\kappa t}\|v\|_\rho. \tag{5.3}$$

5.3 An SDE for the Wave Speed

As in section 2.5.1, we track the stochastic solution with a reference wave to determine its phase. This is done using the phase adaptation in $L^2(\rho)$,

$$\begin{aligned} c^m(t) := \dot{C}^m(t) &= -m\langle u - \hat{u}(\cdot - ct - C^m(t)), \hat{u}_x(\cdot - ct - C^m(t))\rangle_{\rho(\cdot - ct - C^m(t))} \\ &= -m\langle u - \hat{u}(\cdot - ct - C^m(t)), \psi(\cdot - ct - C^m(t))\rangle. \end{aligned} \tag{5.4}$$

Set $v^m(x,t) = u(x,t) - \hat{u}(x - ct - C^m(t))$ to be the difference between the solution u to the stochastic neural field equation (2.1) and the deterministic wave profile moving at the dynamically adapted speed $c + c^m(t)$. v^m satisfies the stochastic evolution equation

$$\begin{aligned} dv^m(t) = \Big(&- v^m(t) + w * (F'(\hat{u}(\cdot - ct - C^m(t)))v^m(t)) + R^m(t, v^m(t)) \\ &+ c^m(t)\hat{u}_x(\cdot - ct - C^m(t))\Big)dt + \epsilon dW_t, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} R^m(t, v^m(t)) = w * \Big(&F(\hat{u}(\cdot - ct - C^m(t)) + v^m(t)) - F(\hat{u}(\cdot - ct - C^m(t))) \\ &- F'(\hat{u}(\cdot - ct - C^m(t)))v^m(t)\Big). \end{aligned}$$

Set $\varphi^m(t) = ct + C^m(t)$.

Lemma 5.3.1. *The adaptation of the wave speed $c^m(t) = -m\langle v^m(t), \psi(\cdot - \varphi^m(t))\rangle$ solves the SDE*

$$\begin{aligned} dc^m(t) &= \Big(- mc^m(t) + m\langle v^m(t), \psi_x(\cdot - \varphi^m(t))\rangle c^m(t) \\ &\qquad - m\langle R^m(t, v^m(t)), \psi(\cdot - \varphi^m(t))\rangle\Big)dt - \epsilon m\langle \psi(\cdot - \varphi^m(t)), dW_t\rangle, \\ c^m(0) &= -\epsilon m\langle \eta, \psi\rangle. \end{aligned}$$

Proof. By Itô's lemma,

$$\begin{aligned} c^m(t) &= -\epsilon m\langle \eta, \psi\rangle + m\int_0^t (c + c^m(s))\langle v^m(s), \psi_x(\cdot - \varphi^m(s))\rangle ds \\ &\quad - m\int_0^t \langle -v^m(s) + w * \Big(F'(\hat{u}(\cdot - \varphi^m(s)))v^m(s)\Big), \psi(\cdot - \varphi^m(s))\rangle ds \\ &\quad - m\int_0^t \langle R^m(s, v^m(s)), \psi(\cdot - \varphi^m(s))\rangle ds - m\int_0^t c^m(s)ds\langle \hat{u}_x, \psi\rangle \\ &\quad - \epsilon m\int_0^t \langle \psi(\cdot - \varphi^m(s)), dW_s\rangle \\ &= c^m(0) - m\int_0^t \langle v^m(s), L^{\#,*}\psi(\cdot - \varphi^m(s))\rangle ds - m\int_0^t \langle R^m(s, v^m(s)), \psi(\cdot - \varphi^m(s))\rangle ds \\ &\quad + m\int_0^t c^m(s)\langle v^m(s), \psi_x(\cdot - \varphi^m(s))\rangle ds - m\int_0^t c^m(s)ds \\ &\quad - \epsilon m\int_0^t \langle \psi(\cdot - \varphi^m(s)), dW_s\rangle \\ &= c^m(0) - m\int_0^t c^m(s)ds + m\int_0^t c^m(s)\langle v^m(s), \psi_x(\cdot - \varphi^m(s))\rangle ds \\ &\quad - m\int_0^t \langle R^m(s, v^m(s)), \psi(\cdot - \varphi^m(s))\rangle ds - \epsilon m\int_0^t \langle \psi(\cdot - \varphi^m(s)), dW_s\rangle \end{aligned}$$

□

5.4 Expansion with respect to the Noise Strength

We expect $\hat{u}(\cdot - ct - C^m(t))$ to track the stochastic solution, which means v^m should describe the fluctuations in the wave profile. As long as m is finite, this can however only be an approximate description.

We prove an expansion of the solution u to (2.1) that allows to analyze the behavior of the coupled system (v^m, C^m) to arbitrary order of ϵ. In Section 5.5 we will derive the expansion in the limit $m \to \infty$ and analyze properties of the coefficients in the expansion. In particular, we will show that the limiting regime indeed corresponds to immediate relaxation to the right phase, thereby justifying the expansion as a description of the effects of the noise.

Fix a time horizon $T > 0$. Set $\rho_t(x) = \rho(x - ct)$. For $h \in \mathcal{C}([0,T], H^1(1+\rho))$ set

$$\|h\|_T = \sup_{0 \le t \le T} \|h(t)\|_{H^1(1+\rho_t)},$$

and for $f \in C([0,T])$ set $|f|_T = \sup_{0 \le t \le T} |f(t)|$. Here we move the measure with the wave such that for all $t \ge 0$,

$$\|\partial_x u_t^{TW}\|_{H^1(1+\rho_t)} = \|\hat{u}_x(\cdot - ct)\|_{H^1(1+\rho_t)} = \|\hat{u}_x\|_{H^1(1+\rho)}.$$

Note that there exists a constant $K > 0$ such that $\|h\|_\infty \le K\|h\|_{H^1(1+\rho)}$ for all $h \in H^1(1+\rho)$.

We start by formally identifying the highest order terms in $c^m(t)$ using Lemma 5.3.1. Since we expect both C^m and v^m to be of order ϵ (up to the time horizon T) that leads us to define $c_0^m(t)$ to be the unique strong solution to

$$dc_0^m(t) = -mc_0^m(t)dt - m\langle \psi(\cdot - ct), dW_t\rangle, \qquad c_0^m(0) = -m\langle \eta, \psi\rangle. \tag{5.6}$$

Set $C_0^m(t) = \int_0^t c_0^m(s)ds$ and $\varphi_0^m(t) = ct + \epsilon C_0^m(t)$.

Formally identifying the highest order terms in (5.5) we define v_0^m to be the unique strong solution to

$$dv_0^m(t) = \big(L_t v_0^m(t) + c_0^m(t)\hat{u}_x(\cdot - ct)\big)dt + dW_t, \qquad v_0^m(0) = \eta,$$

where L_t is as defined in (5.2).

Remark 5.4.1. Note that, to first order, the dynamics of C^m decouple from those of v^m. This would not be the case if we had defined the phase in the unweighted space L^2. It is by defining the phase adaptation in $L^2(\rho)$ that we can achieve a separate description of the influence of the noise on the two scales.

For $\epsilon > 0$, $0 \le q < 1$, set

$$\tau_{q,\epsilon} = \inf\{0 \le t \le T : \|v(t)\|_{H^1(1+\rho_t)} \ge \epsilon^{1-q}\}, \tag{5.7}$$

where v is the solution to (5.1), and

$$\tau_{q,\epsilon}^m = \inf\{0 \leq t \leq T : \ |C_0^m(t)| \geq \epsilon^{-q}\}.$$

Theorem 5.4.2. *Assume that $F \in \mathcal{C}^3$ so that $\hat{u} \in \mathcal{C}^4$. Let $q < \frac{1}{2}$. Then on $\{\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = T\}$,*

$$u(x,t) = \hat{u}(x - ct - \epsilon C_0^m(t)) + \epsilon v_0^m(t) + \epsilon r_1^m(t)$$

with

$$\|r_1^m\|_T \leq \alpha_1(T)\epsilon^{1-2q}$$

for a constant $\alpha_1(T)$ independent of ϵ and m, and

$$P(\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = T) \xrightarrow{\epsilon \to 0} 1.$$

Proof. Set

$$\tilde{v}_0^m(t) = u(t) - \hat{u}(\cdot - \varphi_0^m(t)) = v(t) + \hat{u}(\cdot - ct) - \hat{u}(\cdot - \varphi_0^m(t)). \tag{5.8}$$

By Taylor's formula, there exist $\xi(x,t), \tilde{\xi}(x,t)$, $|\xi| \leq \epsilon |C_0^m|$, $\tilde{\xi} \leq \epsilon |C_0^m|$, such that

$$\begin{aligned} \hat{u}(x - ct) - \hat{u}(x - \varphi_0^m(t)) &= \hat{u}_x(x - ct + \xi(x,t)), \\ \hat{u}_x(x - ct) - \hat{u}_x(x - \varphi_0^m(t)) &= \hat{u}_{xx}(x - ct + \tilde{\xi}(x,t)), \end{aligned}$$

and therefore

$$\|\tilde{v}_0^m(t)\|_{H^1(1+\rho_t)} \leq \|v(t)\|_{H^1(1+\rho_t)} + \epsilon |C_0^m(t)| \big(\|\hat{u}_x\|_{1+\rho(\cdot - \xi(t))} + \|\hat{u}_{xx}\|_{1+\rho(\cdot - \tilde{\xi}(t))} \big).$$

Using (2.12) and (2.14),

$$\rho(x - \xi) \leq (L_\rho \vee e^{M|\xi|})\rho(x), \tag{5.9}$$

and thus on $\{\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = T\}$,

$$\|\tilde{v}_0^m\|_T \leq \epsilon^{1-q} \big(1 + (L_\rho \vee e^{M\epsilon^{1-q}})^{\frac{1}{2}} \|\hat{u}_x\|_{H^1(1+\rho)} \big). \tag{5.10}$$

r_1^m satisfies the pathwise evolution equation

$$\begin{aligned} dr_1^m(t) = \Big(& L_t r_1^m(t) \\ & + \frac{1}{\epsilon} w * \big(F(\hat{u}(\cdot - \varphi_0^m(t)) + \tilde{v}_0^m(t)) - F(\hat{u}(\cdot - \varphi_0^m(t))) \\ & \qquad - F'(\hat{u}(\cdot - \varphi_0^m(t)))\tilde{v}_0^m(t) \big) \\ & + \frac{1}{\epsilon} w * \big((F'(\hat{u}(\cdot - \varphi_0^m(t))) - F'(\hat{u}(\cdot - ct)))\tilde{v}_0^m(t) \big) \end{aligned}$$

$$
\begin{aligned}
&\quad + c_0^m(t)(\hat{u}_x(\cdot - \varphi_0^m(t)) - \hat{u}_x(\cdot - ct))\Big)dt \\
&=: \big(L_t r_1^m(t) + r_{1,1}^m(t) + r_{1,2}^m(t) + r_{1,3}^m(t)\big)dt.
\end{aligned}
$$

By Taylor's theorem there exist $\xi_{1,1}(x,t)$, $\xi_{1,2}(x,t)$ such that

$$
\begin{aligned}
\epsilon r_{1,1}^m(t) &= \frac{1}{2} w * \big(F''(\hat{u}(\cdot - \varphi_0^m(t)) + \xi_{1,1}(\cdot,t))(\tilde{v}_0^m(t))^2\big), \\
\epsilon r_{1,2}^m(t) &= -\epsilon C_0^m(t) w * \big(F''(\hat{u}(\cdot - ct + \xi_{1,2}(\cdot,t)))\hat{u}_x(\cdot - ct + \xi_{1,2}(\cdot,t))\tilde{v}_0^m(t)\big).
\end{aligned}
$$

We therefore have, using the Cauchy-Schwarz inequality, that

$$
\begin{aligned}
\|r_{1,1}^m(t)\|_{1+\rho_t}^2 &\leq \frac{1}{4\epsilon^2}\|F''\|_\infty^2 \int\int w^2(x-y)(\tilde{v}_0^m)^2(y,t)dy(1+\rho_t(x))dx \int (\tilde{v}_0^m)^2(y,t)dy \\
&\leq \frac{1}{4\epsilon^2}\|F''\|_\infty^2 \|w\|_\infty (1+K_\rho)\|\tilde{v}_0^m(t)\|^2 \|\tilde{v}_0^m(t)\|_{1+\rho_t}^2
\end{aligned}
\tag{5.11}
$$

and

$$
\|r_{1,2}^m(t)\|_{1+\rho_t}^2 \leq |C_0^m(t)|^2 \|F''(\hat{u})\hat{u}_x\|_\infty^2 (1+K_\rho)\|\tilde{v}_0^m(t)\|_{1+\rho_t}^2. \tag{5.12}
$$

Recall that r_1^m can be represented as a mild solution,

$$
r_1^m(t) = \int_0^t P_{t,s}\big(r_{1,1}^m(s) + r_{1,2}^m(s) + r_{1,3}^m(s)\big)ds. \tag{5.13}
$$

Set $R_{1,3}^m(t) = \frac{1}{\epsilon}\big(- \hat{u}(\cdot - \varphi_0^m(t)) + \hat{u}(\cdot - ct) - \epsilon C_0^m(t)\hat{u}_x(\cdot - ct)\big)$. We have $r_{1,3}^m(t) = \big(\frac{d}{dt} + c\partial_x\big)R_{1,3}^m(t)$ and therefore

$$
\begin{aligned}
\int_0^t P_{t,s} r_{1,3}^m(s)ds &= \int_0^t \frac{d}{ds}\big[P_{t,s}R_{1,3}^m(s)\big] + P_{t,s}(L_s + c\partial_x)R_{1,3}^m(s)ds \\
&= R_{1,3}^m(t) + \int_0^t P_{t,s}(L_s + c\partial_x)R_{1,3}^m(s)ds.
\end{aligned}
$$

Recall that $\|P_{t,s}h\|_{H^1(1+\rho_s)} \leq e^{L_*(t-s)}\|h\|_{H^1(1+\rho_s)}$. Using Taylor's theorem, (2.12), and (5.9) it follows that

$$
\begin{aligned}
&\Big\| \int_0^t P_{t,s} r_{1,3}^m(s)ds \Big\|_{H^1(1+\rho_t)} \\
&\leq \frac{\epsilon}{2}|C_0^m(t)|^2 \big(\|\hat{u}_{xx}(\cdot - ct - \xi_{1,3,1}(\cdot,t))\|_{1+\rho_t} + \|\hat{u}_{xxx}(\cdot - ct - \xi_{1,3,2}(\cdot,t)\|_{1+\rho_t}\big) \\
&\quad + L_\rho^{\frac{1}{2}} \int_0^t e^{L_*(t-s)}\frac{\epsilon}{2}|C_0^m(s)|^2 \\
&\quad \Big(L_*\big(\|\hat{u}_{xx}(\cdot - cs - \xi_{1,3,1}(\cdot,s))\|_{1+\rho_s} + \|\hat{u}_{xxx}(\cdot - cs - \xi_{1,3,2}(\cdot,s))\|_{1+\rho_s}\big)
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&\quad + c\Big(\|\hat{u}_{xxx}(\cdot - cs - \xi_{1,3,2}(\cdot,s))\|_{1+\rho_s} + \|\hat{u}_{xxxx}(\cdot - cs - \xi_{1,3,3}(\cdot,s))\|_{1+\rho_s}\Big)\Big)ds \\
&\leq \frac{\epsilon}{2}|C_0^m|_T^2 (L_\rho \vee e^{M\epsilon^{1-q}})^{\frac{1}{2}} \Big((1 + L_\rho^{\frac{1}{2}}(e^{L_* T} - 1))\|\hat{u}_{xx}\|_{H^1(1+\rho)} \\
&\quad + \frac{cL_\rho^{\frac{1}{2}}}{L_*}(e^{L_* T} - 1)\|\hat{u}_{xxx}\|_{H^1(1+\rho)}\Big).
\end{aligned}
$$

Since for $i = 1, 2$

$$
\begin{aligned}
\|P_{t,s} r_{1,i}^m(s)\|_{H^1(1+\rho_t)}^2 &\leq L_\rho e^{2L_*(t-s)} \|r_{1,i}^m(s)\|_{H^1(1+\rho_s)}^2 \\
&\leq L_\rho e^{2L_*(t-s)} \Big(1 + \Big\|\frac{w_x}{w}\Big\|_\infty^2\Big) \|r_{1,i}^m(s)\|_{1+\rho_s}^2,
\end{aligned}
\tag{5.15}
$$

putting (5.11), (5.12), (5.13), (5.14), and (5.15) together, we conclude that there exists a constant $\alpha_1(T)$ independent of m and ϵ such that

$$\|r_1^m\|_T \leq \alpha_1(T)\epsilon^{1-2q}.$$

If $\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = \tau_{q,\epsilon} < T$, then by continuity, almost surely, there exists $t_0 < T$ such that,

$$
\begin{aligned}
\epsilon^{1-q} &= \|v(t_0)\|_{H^1(1+\rho_{t_0})} \\
&= \|-\epsilon C_0^m(t_0)\hat{u}_x(\cdot - ct_0 + \xi(\cdot,t_0)) + \epsilon v_0^m(t_0) + \epsilon r_1^m(t_0)\|_{H^1(1+\rho_{t_0})}
\end{aligned}
$$

and thus

$$\|v_0^m(t_0) - C_0^m(t_0)\hat{u}_x(\cdot - ct_0 + \xi(\cdot,t_0))\|_{H^1(1+\rho_{t_0})} \geq \epsilon^{-q} - \|r_1^m(t_0)\|_{H^1(1+\rho_{t_0})}.$$

We therefore have that

$$
\begin{aligned}
&P(\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = \tau_{q,\epsilon} < T) \\
&\leq P\big(\|v_0^m - C_0^m \hat{u}_x(\cdot - ct + \xi(\cdot,t))\|_T \geq \epsilon^{-q} - \alpha_1(T)\epsilon^{1-2q}\big) \\
&\leq \frac{2\epsilon^{2q}}{(1-\alpha_1(T)\epsilon^{1-q})^2}\big(E(\|v_0^m\|_T^2) + E(|C_0^m|_T^2)(L_\rho \vee e^{M\epsilon^{1-q}})\|\hat{u}_x\|_{H^1(1+\rho)}^2\big) \xrightarrow{\epsilon\to 0} 0.
\end{aligned}
$$

Since

$$P(\tau_{q,\epsilon}^m < T) \leq P(|C_0^m|_T \geq \epsilon^{-q}) \leq \epsilon^{2q} E(|C_0^m|_T^2) \xrightarrow{\epsilon\to 0} 0,$$

it follows that

$$P(\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m < T) \leq P(\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = \tau_{q,\epsilon} < T) + P(\tau_{q,\epsilon}^m < T) \xrightarrow{\epsilon\to 0} 0. \qquad \square$$

Analogously we can obtain an expansion to higher order of ϵ. Formally identifying the highest order terms in $\frac{1}{\epsilon}c^m(t) - c_0^m(t)$ we define $c_1^m(t)$ to be the unique strong solution to

$$\begin{aligned} dc_1^m(t) &= \big(- mc_1^m(t) - \frac{1}{2} m \langle w * (F''(\hat{u}(\cdot - ct))(v_0^m)^2(t)), \psi(\cdot - ct)\rangle \\ &\qquad + mc_0^m(t)\langle v_0^m(t), \psi_x(\cdot - ct)\rangle\big)dt + mC_0^m(t)\langle \psi_x(\cdot - ct), dW_t\rangle, \\ c_1^m(0) &= 0. \end{aligned} \tag{5.16}$$

Set $C_1^m(t) = \int_0^t c_1^m(s)ds$ and $\varphi_1^m(t) = ct + \epsilon C_0^m(t) + \epsilon^2 C_1^m(t)$. Identifying the highest order terms in $\frac{1}{\epsilon}v^m - v_0^m$, set v_1^m to be the unique strong solution to

$$\begin{aligned} dv_1^m(t) &= \Big(L_t v_1^m(t) + w * \Big(F''(\hat{u}(\cdot - ct))\Big(\frac{1}{2}(v_0^m)^2(t) - C_0^m(t)\hat{u}_x(\cdot - ct)v_0^m(t)\Big)\Big) \\ &\qquad - c_0^m(t)C_0^m(t)\hat{u}_{xx}(\cdot - ct) + c_1^m(t)\hat{u}_x(\cdot - ct)\Big)dt, \\ v_1^m(0) &= 0. \end{aligned} \tag{5.17}$$

For arbitrary $k \geq 2$, if $F \in \mathcal{C}^{k+2}$ we can iterate the procedure and define c_{k-1}^m and v_{k-1}^m by successively identifying the highest order terms in $\frac{1}{\epsilon^{k-1}}(c^m - \epsilon c_0^m - \ldots - \epsilon^{k-1}c_{k-2}^m)$ and $\frac{1}{\epsilon^{k-1}}(v^m(t) - \epsilon v_0^m(t) - \ldots - \epsilon^{k-1}v_{k-2}^m(t)$. This way we obtain an expansion of u up to order ϵ^k. Set

$$\varphi_{k-1}^m(t) = ct + \epsilon C_0^m(t) + \ldots + \epsilon^k C_{k-1}^m(t).$$

Theorem 5.4.3. *Assume that $F \in \mathcal{C}^{k+2}$ for some $k \geq 1$. Let $q < \frac{1}{k+1}$.*
Then on $\{\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m = T\}$,

$$u(x,t) = \hat{u}(\cdot - \varphi_{k-1}^m(t)) + \epsilon v_0^m(t) + \ldots + \epsilon^k v_{k-1}^m(t) + \epsilon^k r_k^m(t)$$

with

$$\|r_k^m\|_T \leq \alpha_k(T)\epsilon^{1-(k+1)q}$$

for some constant $\alpha_k(T)$ independent of ϵ and m.

We will prove the expansion for $k = 2$. For larger k the analysis can be carried out analogously, but the formulas become unwieldy.

We start by deriving a useful representation of C_1^m.

Lemma 5.4.4.

$$\begin{aligned} C_1^m(t) = &-\int_0^t (1 - e^{-m(t-s)})\Big\langle w * \Big(F''(\hat{u}(\cdot - cs))\Big(\frac{1}{2}(v_0^m)^2(s) \\ &\qquad - C_0^m(s)\hat{u}_x(\cdot - cs)v_0^m(s)\Big)\Big), \psi(\cdot - cs)\Big\rangle ds \\ &+ \int_0^t me^{-m(t-s)}C_0^m(s)\langle v_0^m(s), \psi_x(\cdot - cs)\rangle ds \\ &- \frac{1}{2}\int_0^t me^{-m(t-s)}(C_0^m)^2(s)ds\langle \psi_x, \hat{u}_x\rangle. \end{aligned}$$

Proof. Integrating (5.16) we obtain

$$C_1^m(t) = \int_0^t (1-e^{-m(t-s)})\Big(-\frac{1}{2}\langle w*(F''(\hat{u}(\cdot-cs))(v_0^m)^2(s)),\psi(\cdot-cs)\rangle + c_0^m(s)\langle v_0^m(s),\psi_x(\cdot-cs)\rangle\Big)ds + \int_0^t (1-e^{-m(t-s)})C_0^m(s)\langle\psi_x(\cdot-cs),dW_s\rangle.$$

By Itô's Lemma,

$$\begin{aligned}&\int_0^t (1-e^{-m(t-s)})c_0^m(s)\langle v_0^m(s),\psi_x(\cdot-cs)\rangle ds\\ &=\int_0^t me^{-m(t-s)}C_0^m(s)\langle v_0^m(s),\psi_x(\cdot-cs)\rangle ds\\ &\quad-\int_0^t (1-e^{-m(t-s)})C_0^m(s)\Big(\langle v_0^m(s),(L_s^*-c\partial_x)\psi_x(\cdot-cs)\rangle + c_0^m(s)\langle\hat{u}_x,\psi_x\rangle\Big)ds\\ &\quad-\int_0^t (1-e^{-m(t-s)})C_0^m(s)\langle\psi_x(\cdot-cs),dW_s\rangle.\end{aligned}$$

Using integration by parts we obtain that

$$-\int_0^t (1-e^{-m(t-s)})c_0^m(s)C_0^m(s)ds = \frac{1}{2}\int_0^t -me^{-m(t-s)}(C_0^m)^2(s)ds,$$

and since $(L_s^*-c\partial_x)\psi_x(\cdot-cs) = -F''(\hat{u}(\cdot-cs))\hat{u}_x(\cdot-cs)w*\psi(\cdot-cs)$, the claim follows. □

Proof of Thm. 5.4.3. Note first that, using Lemma 5.4.4,

$$|C_1^m|_T \le \|\psi\|\int\Big\|w*\Big(F''(\hat{u}(\cdot-cs))\Big(\frac{1}{2}(v_0^m(s))^2 - C_0^m(s)\hat{u}_x(\cdot-cs)v_0^m(s)\Big)\Big)\Big\|ds + |C_0^m|_T\|\psi_x\|\int_0^t me^{-m(t-s)}\|v_0^m(s)\|ds + \frac{1}{2}|C_0^m|_T^2\|\psi_x\|\|\hat{u}_x\|.$$

Since

$$v_0^m(t) = \frac{1}{\epsilon}\big(v(t)+\hat{u}(\cdot-ct)-\hat{u}(\cdot-\varphi_0^m(t))\big) - r_1^m(t) = \frac{1}{\epsilon}\tilde{v}_0^m(t) - r_1^m(t),$$

where $\tilde{v}_0^m$ is defined in (5.8), using Theorem 5.4.2 and (5.10) it follows that there exists a constant $\beta_1(T)$ such that on $\{\tau_{q,\epsilon}\wedge\tau_{q,\epsilon}^m = T\}$

$$\|v_0^m\|_T \le \beta_1(T)\epsilon^{-q}.$$

Therefore there exists a constant $\beta_2(T)$ such that

$$|C_1^m|_T \le \beta_2(T)\epsilon^{-2q}. \tag{5.18}$$

Set $\tilde{v}_1^m(t) = v(t)+\hat{u}(\cdot-ct)-\hat{u}(\cdot-\varphi_1^m(t))$. By Taylor's theorem there exists $\xi(x,t)$ with

$|\xi| \leq |\epsilon C_0^m + \epsilon^2 C_1^m|$ such that

$$\tilde{v}_1^m(t) = v(t) + (\epsilon C_0^m(t) + \epsilon^2 C_1^m(t))\hat{u}_x(\cdot + \xi(\cdot, t))$$

and it follows that there exists a constant $\beta_3(T)$ such that

$$\|\tilde{v}_1^m\|_T \leq \beta_3(T)\epsilon^{1-q}.$$

We have

$$dr_2^m(t) = \big(L_t r_2^m(t) + r_{2,1}^m(t) + r_{2,2}^m(t)\big)dt,$$

where

$$\begin{aligned}
\epsilon^2 r_{2,1}^m(t) &= w * \Big(F(\hat{u}(\cdot - \varphi_1^m(t)) + \tilde{v}_1^m(t)) - F(\hat{u}(\cdot - \varphi_1^m(t))) \\
&\qquad - F'(\hat{u}(\cdot - \varphi_1^m(t)))\tilde{v}_1^m(t) - \frac{1}{2}F''(\hat{u}(\cdot - \varphi_1^m(t)))(\tilde{v}_1^m(t))^2\Big) \\
&\quad + w * \Big(\big(F'(\hat{u}(\cdot - \varphi_1^m(t))) - F'(\hat{u}(\cdot - \varphi_0^m(t)))\big)\tilde{v}_1^m(t)\Big) \\
&\quad + w * \Big(\big(F'(\hat{u}(\cdot - \varphi_0^m(t))) - F'(\hat{u}(\cdot - ct)) \\
&\qquad + \epsilon C_0^m(t)F''(\hat{u}(\cdot - ct))\hat{u}_x(\cdot - ct)\big)\tilde{v}_1^m(t)\Big) \\
&\quad - \epsilon C_0^m(t) w * \Big(F''(\hat{u}(\cdot - ct))\hat{u}_x(\cdot - ct)(\tilde{v}_1^m(t) - \epsilon v_0^m(t))\Big) \\
&\quad + \frac{1}{2} w * \Big(\big(F''(\hat{u}(\cdot - \varphi_1^m(t))) - F''(\hat{u}(\cdot - ct))\big)(\tilde{v}_1^m(t))^2\Big) \\
&\quad + \frac{1}{2} w * \Big(F''(\hat{u}(\cdot - ct))((\tilde{v}_1^m(t))^2 - \epsilon^2(v_0^m(t))^2)\Big) \\
&= \sum_{i=1}^{6} r_{2,1,i}^m(t), \\
\epsilon^2 r_{2,2}^m(t) &= (\epsilon c_0^m(t) + \epsilon^2 c_1^m(t))(\hat{u}_x(\cdot - \varphi_1^m(t)) - \hat{u}_x(\cdot - ct)) \\
&\qquad + \epsilon^2 c_0^m(t) C_0^m(t)\hat{u}_{xx}(\cdot - ct).
\end{aligned}$$

Now

$$\begin{aligned}
\epsilon^4 \|r_{2,1,1}^m(t)\|_{1+\rho_t}^2 &\leq \frac{1}{36}\|F^{(3)}\|_\infty^2(1+K_\rho)\|w\|_\infty \|\tilde{v}_1^m(t)\|_{1+\rho_t}^2 \int (\tilde{v}_1^m)^4(x,t)dx \\
&\leq \frac{1}{36}\|F^{(3)}\|_\infty^2(1+K_\rho)\|w\|_\infty \|\tilde{v}_1^m(t)\|_{1+\rho_t}^2 \|\tilde{v}_1^m(t)\|_\infty^2 \|\tilde{v}_1^m(t)\|^2 \\
&\leq \frac{1}{36}\|F^{(3)}\|_\infty^2(1+K_\rho)\|w\|_\infty K^2 \|\tilde{v}_1^m\|_T^6.
\end{aligned}$$

Concerning $r^m_{2,1,4}$ and $r^m_{2,1,6}$, note that there exists $\tilde{\xi}(x,t)$ such that

$$\begin{aligned}\tilde{v}^m_1(t) &= \tilde{v}^m_0(t) + \hat{u}(\cdot - \varphi^m_0(t)) - \hat{u}(\cdot - \varphi^m_1(t)) \\ &= \epsilon v^m_0(t) + \epsilon r^m_1(t) + \epsilon^2 C^m_1(t)\hat{u}_x(\cdot - \varphi^m_0(t) + \tilde{\xi}(\cdot,t)),\end{aligned}$$

and hence, using (5.18) and Theorem 5.4.2, there exists a constant $\beta_4(T)$ such that

$$\|\tilde{v}^m_1 - \epsilon v^m_0\|_T \leq \beta_4(T)\epsilon^{2-2q}.$$

We have

$$\epsilon^2 r^m_{2,2}(t) = R^m_{2,2}(t) + \int_0^t P_{t,s}(L_s + c\partial_x)R^m_{2,2}(s)ds,$$

where

$$\begin{aligned}R^m_{2,2}(t) = &-\hat{u}(\cdot - \varphi^m_1(t)) + \hat{u}(\cdot - ct) - (\epsilon C^m_0(t) + \epsilon^2 C^m_1(t))\hat{u}_x(\cdot - ct) \\ &+ \epsilon^2 \frac{1}{2}(C^m_0)^2(t)\hat{u}_{xx}(\cdot - ct).\end{aligned}$$

Now all the terms can be estimated as in the proof of Thm. 5.4.2 and we obtain that there exists $\alpha_2(T)$ independent of m and ϵ such that

$$\|r^m_2\|_T \leq \alpha_2(T)\epsilon^{1-3q}. \qquad \square$$

Remark 5.4.5. Note that in the case $k = 1$ (Thm. 5.4.2) we could also define the stopping time $\tau_{q,\epsilon}$ as an exit time of the $L^2(1+\rho_t)$-norm of v and obtain that $\sup_{t\leq T}\|r^m_1(t)\|_{1+\rho_t} \leq \alpha_1(T)\epsilon^{1-2q}$. This is not the case in the proof of Thm. 5.4.3, where we need to control $\|\tilde{v}^m_1(t)\|_\infty$.

5.4.1 Multiplicative Noise

A similar expansion can be proven in the case of a multiplicative diffusion coefficient as in assumption (C)(2). We assume the diffusion coefficient to be of the form $\sigma(t,v) = \sigma_0(t) + \sigma_1(t)v$ for some continuous $\sigma_i : [0,T] \to H^1(1+\rho)$. This covers the case of the diffusion coefficient we will derive in Chapter 6 to describe finite-size effects. (Strictly speaking, the σ_i should depend on $C^m(t)$. For the sake of a clearer presentation, we omit this dependence here. The analysis can be carried out analogously in this case.)

While in the additive case the noise enters directly only in the first order approximation v^m_0, in the multiplicative case, we keep a stochastic integral in the error term r^m_1. v^m_0 satisfies

$$dv^m_0(t) = \big(L_t v^m_0(t) + c^m_0(t)\hat{u}_x(\cdot - ct)\big)dt + \sigma_0(t)dW_t, \qquad v^m_0(t) = \eta,$$

leading to the additional error term $I^m_1(t) = \int_0^t P_{t,s}(\sigma_1(s)v(s))dW_s$ in r^m_1. In order to obtain a pathwise estimate of I_1^m we introduce the stopping time

$$\tau_{q,\epsilon}^{I,1,m} = \inf\{0 \le t \le T : \|I_1^m(t)\|_{H^1(1+\rho_t)} \ge \epsilon^{1-2q}\}$$

and set $\tau_{q,\epsilon}^{1,m} = \tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m \wedge \tau_{q,\epsilon}^{I,1,m}$. Then it follows as above that on $\{\tau_{q,\epsilon}^{1,m} = T\}$, $\|r_1^m\|_T \le \tilde{\alpha}_1(T,\omega)\epsilon^{1-2q}$.

Similarly we can estimate the higher order terms by introducing stopping times

$$\tau_{q,\epsilon}^{I,k,m} = \inf\{0 \le t \le T : \|I_k^m(t)\|_{H^1(1+\rho_t)} \ge \epsilon^{1-(k+1)q}\}$$

and $\tau_{q,\epsilon}^{k,m} = \min\{\tau_{q,\epsilon}, \tau_{q,\epsilon}^m, \tau_{q,\epsilon}^{I,1,m}, \dots, \tau_{q,\epsilon}^{I,k,m}\}$, where for $k > 1$,

$$I_k^m(t) = \int_0^t P_{t,s}\tilde{r}_{k-1}^m(s)ds$$

for some rest term $\tilde{r}_{k-1}^m(s)$ satisfying

$$\|\tilde{r}_{k-1}^m\|_T \le \tilde{\beta}(T)\epsilon^{1-kq}$$

on $\{\tau_{q,\epsilon}^{m,k} = T\}$ for some constant $\tilde{\beta}(T)$. For example,

$$\tilde{r}_1^m(t) = r_1^m(t) + \frac{1}{\epsilon}\big(\hat{u}(\cdot - \varphi_0^m(t)) - \hat{u}(\cdot - ct) - \epsilon C_0^m(t)\hat{u}_x(\cdot - ct)\big).$$

Note that

$$P(\tau_{q,\epsilon}^{1,m} < T) = P(\tau_{q,\epsilon}^{1,m} = \tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m < T) + P(\tau_{q,\epsilon}^{1,m} = \tau_{q,\epsilon}^{I,1,m} < T).$$

The first probability can be estimated as in Theorem 5.4.2. By the stopped maximal inequality for stochastic convolutions (cf. [41]), there exists a constant $K_I(T)$ such that

$$E\Big(\|I_1^m(\cdot \wedge \tau_{q,\epsilon}^{1,m})\|_T^2\Big) \le K_I(T)E\Big(\int_0^{T\wedge\tau_{q,\epsilon}^{1,m}} \|v(s)\|_{H^1(1+\rho_s)}^2 ds\Big) \le K_I(T)\epsilon^{2-2q}T.$$

Thus, the second probability satisfies

$$\begin{aligned} P(\tau_{q,\epsilon}^{1,m} = \tau_{q,\epsilon}^{I,1,m} < T) &\le P\big(\|I^m(\cdot \wedge \tau_{q,\epsilon}^{1,m})\|_T \ge \epsilon^{1-2q}\big) \\ &\le \epsilon^{-2+4q}E\big(\|I^m(\cdot \wedge \tau_{q,\epsilon}^{1,m})\|_T^2\big) \\ &\le K_I(T)T\epsilon^{2q} \xrightarrow{\epsilon\to 0} 0. \end{aligned}$$

$P(\tau_{q,\epsilon}^{k,m} = T)$ can be estimated analogously.

5.5 Immediate Relaxation

We now go over to the limit $m \to \infty$, presumably corresponding to immediate relaxation to the right phase. Since all the estimates in section 5.4 are independent of m, the expansion will translate to the limiting regime once we have computed the limits of the coefficients.

Denote by π_s the projection onto the orthogonal complement of $\hat{u}_x(\cdot - cs)$ in $L^2(\rho_s)$, i.e.,

$$\pi_s h = h - \langle h, \hat{u}_x \rangle_{\rho(\cdot - cs)} \hat{u}_x(\cdot - cs).$$

Note that while $C_0^m(t) = \int_0^t c_0^m(s)ds$ is a process of bounded variation, in the limit $m \to \infty$ we go over to a process of unbounded variation. The convergence is only locally uniform on $(0, T)$ due to the initial jump to the right phase in the limit.

Lemma 5.5.1. *For any $\delta > 0$, for $i = 1, 2$, almost surely*

$$\sup_{\delta \le t \le T} |C_i^m(t) - C_i(t)| \xrightarrow{m \to \infty} 0$$

and

$$\sup_{\delta \le t \le T} \|v_i^m(t) - v_i(t)\|_{H^1(1+\rho_t)} \xrightarrow{m \to \infty} 0,$$

where $C_0(0) = 0$, $v_0(0) = \eta$, and for $t > 0$,

$$\begin{aligned} C_0(t) &= -\langle \eta, \psi \rangle - \int_0^t \langle \psi(\cdot - cs), dW_s \rangle \\ v_0(t) &= P_{t,0} \pi_0 \eta + \int_0^t P_{t,s} \pi_s dW_s, \end{aligned}$$

and where

$$\begin{aligned} C_1(t) &= -\int_0^t \Big\langle w * \Big(F''(\hat{u}(\cdot - cs)) \Big(\frac{1}{2} v_0^2(s) - C_0(s) \hat{u}_x(\cdot - cs) v_0(s) \Big) \Big), \psi(\cdot - cs) \Big\rangle ds \\ &\quad + C_0(t) \langle v_0(t), \psi_x(\cdot - ct) \rangle - \frac{1}{2} C_0^2(t) \langle \hat{u}_x, \psi_x \rangle, \\ v_1(t) &= \int_0^t P_{t,s} w * \Big(F''(\hat{u}(\cdot - cs)) \Big(\frac{1}{2} v_0^2(s) - C_0(s) \hat{u}_x(\cdot - cs) v_0(s) \Big) \Big) ds \\ &\quad + C_1(t) \hat{u}_x(\cdot - ct) - \frac{1}{2} \int_0^t P_{t,s} \hat{u}_{xx}(\cdot - cs) dC_0^2(s) \end{aligned}$$

We postpone the proof to the end of this section.

Theorem 5.5.2. *Let $\tau_{q,\epsilon}$ be as in (5.7) and set $\tau_{q,\epsilon}^\infty = \inf\{0 \le t \le T : |C_0(t)| \ge \epsilon^{-q}\}$. Then on $\{\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^\infty = T\}$, if $F \in \mathcal{C}^3$ and $q < \frac{1}{2}$,*

$$u(x,t) = \hat{u}(x - ct - \epsilon C_0(t)) + \epsilon v_0(x,t) + \epsilon r_1(x,t),$$

and if $F \in \mathcal{C}^4$ and $q < \frac{1}{3}$, then

$$u(x,t) = \hat{u}(x - ct - \epsilon C_0(t) - \epsilon^2 C_1(t)) + \epsilon v_0(x,t) + \epsilon^2 v_1(x,t) + \epsilon^2 r_2(x,t),$$

where for $k = 1, 2$,

$$\|r_k\|_T \leq \alpha_k(T)\epsilon^{1-(k+1)q},$$

with α_k as in Thms. 5.4.2 and 5.4.3, and

$$P(\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^{\infty} = T) \xrightarrow{\epsilon \to 0} 1.$$

Proof. Let $0 < t < \tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^{\infty}$. Integrating (5.6) we obtain that

$$C_0^m(t) = -(1 - e^{-mt})\langle \eta, \psi \rangle - \int_0^t (1 - e^{-m(t-s)})\langle \psi(\cdot - cs), dW_s \rangle. \tag{5.19}$$

By Itô's Lemma,

$$\begin{aligned} C_0^m(t) = -(1 - e^{-mt})\langle \eta, \psi \rangle &- \int_0^t m e^{-m(t-s)} \langle \psi(\cdot - cs), W_s \rangle ds \\ &- c \int_0^t (1 - e^{-m(t-s)})\langle \psi_x(\cdot - cs), W_s \rangle ds. \end{aligned}$$

Therefore, for $0 < \delta < t$,

$$|C_0^m|_\delta \leq |\langle \eta, \psi \rangle| + \big(\|\psi\| + c\delta\|\psi_x\|\big) \sup_{0 \leq t \leq \delta} \|W_t\| \xrightarrow{\delta \to 0} |\langle \eta, \psi \rangle| < \epsilon^{-q}.$$

Since for any $\delta > 0$ $\sup_{\delta \leq s \leq t} |C_0^m(s)| \xrightarrow{m \to \infty} \sup_{\delta \leq s \leq t} |C_0(s)| < \epsilon^{-q}$ it follows that also $t < \tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^m$ for sufficiently large m. Therefore, using Theorem 5.4.2 and Lemma 5.5.1,

$$\begin{aligned} \|\epsilon r_1(t)\|_{H^1(1+\rho_t)} &\leq \|u(t) - \hat{u}(\cdot - ct - \epsilon C_0^m(t)) - \epsilon v_0^m(t)\|_{H^1(1+\rho_t)} \\ &\quad + \|\hat{u}(\cdot - ct - \epsilon C_0^m(t)) - \hat{u}(\cdot - ct - \epsilon C_0(t))\|_{H^1(1+\rho_t)} \\ &\quad + \epsilon\|v_0^m(t) - v_0(t)\|_{H^1(1+\rho_t)} \\ &\leq \alpha_1(T)\epsilon^{2-2q} + \epsilon|C_0^m(t) - C_0(t)|(L_\rho \vee e^{2M\epsilon^{1-q}})^{\frac{1}{2}}\|\hat{u}_x\|_{H^1(1+\rho)} \\ &\quad + \epsilon\|v_0^m(t) - v_0(t)\|_{H^1(1+\rho_t)} \\ &\xrightarrow{m \to \infty} \alpha_1(T)\epsilon^{2-2q}, a.s. \end{aligned}$$

Thus, on $\{\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^{\infty} = T\}$, $\|r_1\|_T \leq \alpha_1(T)\epsilon^{1-2q}$. The proof for the higher order expansion is analogous.

$$P(\tau_{q,\epsilon} \wedge \tau_{q,\epsilon}^{\infty} = T) \xrightarrow{\epsilon \to 0} 1$$

is proven as in Theorem 5.4.2. □

The term $-\langle \eta, \psi \rangle$ in C_0 accounts for the initial phase difference between $u(0)$ and $\hat{u}$. We have

$$Var(C_0(t)) = \int_0^t \langle \psi(\cdot - cs), Q(\psi(\cdot - cs))\rangle ds \approx \langle \psi, Q\psi\rangle t$$

if the correlations are roughly translation invariant (they cannot be translation invariant since Q is of finite trace). This is in accordance with the analysis of Bressloff and Webber in [15].

Note that for $t > 0$,

$$\begin{aligned}
\langle v_0(t), \hat{u}_x(\cdot - ct)\rangle_{\rho_t} &= \langle P_{t,0}\pi_0\eta, \psi(\cdot - ct)\rangle + \Big\langle \int_0^t P_{t,s}\pi_s dW_s, \psi(\cdot - ct)\Big\rangle \\
&= \langle \pi_0\eta, P^*_{t,0}(\psi(\cdot - ct))\rangle + \int_0^t \langle P^*_{t,s}\psi(\cdot - ct), \pi_s dW_s\rangle \\
&= \langle \pi_0\eta, \psi\rangle + \int_0^t \langle \psi(\cdot - cs), \pi_s dW_s\rangle = 0.
\end{aligned} \tag{5.20}$$

In the frozen wave setting, $v_0^{\#}$ is thus orthogonal to $\hat{u}_x$ in $L^2(\rho)$. Recall that the frozen wave operator $L^{\#}$ generates a contraction semigroup on $\hat{u}_x^{\perp}$. For $t > 0$ we can therefore write

$$v_0^{\#}(t) = P_t^{\#}\pi_0\eta + \int_0^t P^{\#}_{t-s}\Phi_s\pi_s dW_s = P_t^{\#}\pi_0\eta + \int_0^t P^{\#}_{t-s}\pi_0\Phi_s dW_s,$$

where $\Phi_s h = h(\cdot + cs)$. Using (5.3) it follows that

$$\|v_0(t)\|_{\rho_t} = \|v_0^{\#}(t)\|_\rho \leq e^{-\kappa t}\|\eta\|_\rho + \left\| \int_0^t P^{\#}_{t-s}\pi_0\Phi_s dW_s \right\|_\rho$$

and hence

$$E(\|v_0(t)\|^2_{\rho_t}) \leq 2e^{-2\kappa t}\|\eta\|^2_\rho + 2\int_0^t \|P^{\#}_{t-s}\pi_0\Phi_s Q^{\frac{1}{2}}\|^2_{L_2(L^2, L^2(\rho))} ds.$$

Let (e_k) be an orthonormal basis of L^2. We have

$$\begin{aligned}
\|P^{\#}_{t-s}\pi_0\Phi_s Q^{\frac{1}{2}}\|^2_{L_2(L^2,L^2(\rho))} &= \sum_k \|P^{\#}_{t-s}\pi_0\Phi_s Q^{\frac{1}{2}} e_k\|^2_\rho \leq e^{-2\kappa(t-s)} \sum_k \|Q^{\frac{1}{2}} e_k\|^2_{\rho_s} \\
&\leq L_\rho e^{-2\kappa(t-s)} \sum_k \|Q^{\frac{1}{2}} e_k\|^2_\rho = L_\rho e^{-2\kappa(t-s)} \|Q^{\frac{1}{2}}\|^2_{L_2(L^2,L^2(\rho))}
\end{aligned}$$

and thus

$$E(\|v_0^{\#}(t)\|^2_\rho) \leq 2e^{-2\kappa t}\|\eta\|^2_\rho + 2L_\rho \frac{1}{2\kappa}(1 - e^{-2\kappa t})\|Q^{\frac{1}{2}}\|^2_{L_2(L^2,L^2(\rho))} \xrightarrow{t\to\infty} \frac{L_\rho \|Q^{\frac{1}{2}}\|^2_{L_2(L^2,L^2(\rho))}}{\kappa}.$$

$v_0^{\#}$ is thus an $L^2(\Omega; L^2(\rho))$-bounded Ornstein-Uhlenbeck process on $\hat{u}_x^{\perp}$.

So far it is not clear that the expansion in Theorem 5.5.2 gives the right description of the influence of the noise on the traveling wave. A different choice of C_0 would yield another expansion and we are left to justify that our particular choice of C_0 provides the right picture.

Set $\varphi_k(t) = ct + \epsilon C_0(t) + \ldots + \epsilon^{k+1} C_k(t)$. The C_k describe the phase shift caused by the noise to order ϵ^{k+1} in the sense of the following proposition.

Proposition 5.5.3. *Under the assumptions of Theorem 5.5.2, for $t < \tau_{q,\epsilon} \wedge \tau^{\infty}_{q,\epsilon}$,*

$$\|u - \hat{u}(\cdot - ct - \epsilon a)\|^2_{\rho_t} = \|u - \hat{u}(\cdot - ct - \epsilon C_0(t))\|^2_{\rho_t} + \epsilon^2 (a - C_0(t))^2 + o(\epsilon^2).$$

$a \mapsto \|u - \hat{u}(\cdot - ct - \epsilon a)\|_{\rho_t}$ is therefore locally minimal to order ϵ at $a = C_0(t)$.
Similarly,

$$\|u - \hat{u}(\cdot - ct - \epsilon C_0(t) - \epsilon^2 a)\|_{\rho_t} = \|u - \hat{u}(\cdot - ct - \epsilon C_0(t) - \epsilon^2 C_1(t))\|^2_{\rho_t} + \epsilon^4 (a - C_1(t))^4 + o(\epsilon^4).$$

$a \mapsto \|u - \hat{u}(\cdot - ct - \epsilon C_0(t) - \epsilon^2 a)\|_{\rho_t(\cdot - \epsilon C_0(t))}$ is therefore locally minimal to order ϵ^2 at $a = C_1(t)$.

Proof. Note that, using (5.20) and Thm. 5.5.2,

$$\begin{aligned}
\frac{1}{2} \frac{d}{da}\Big|_{a=C_0(t)} \|u(t) - \hat{u}(\cdot - ct - \epsilon a)\|^2_{\rho_t} &= \epsilon^2 \langle v_0(t) + r_1(t), \hat{u}_x(\cdot - \varphi_0(t))\rangle_{\rho_t} \\
&= \epsilon^2 \langle v_0(t), \psi(\cdot - ct)\rangle + o(\epsilon^2) = o(\epsilon^2)
\end{aligned}$$

and

$$\frac{1}{2} \frac{d^2}{da^2}\Big|_{a=C_0(t)} \|u(t) - \hat{u}(\cdot - ct - \epsilon a)\|^2_{\rho_t} = \epsilon^2 \langle \hat{u}_x, \psi\rangle + o(\epsilon^2) = \epsilon^2 + o(\epsilon^2).$$

C_0 is thus such that $\|u(t) - \hat{u}(\cdot - ct - \epsilon C_0(t))\|_{\rho_t}$ is locally minimal to order ϵ. Similarly we have that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{da}\Big|_{a=C_1(t)} \|u(t) - \hat{u}(\cdot - ct - \epsilon C_0(t) - \epsilon^2 a)\|^2_{\rho_t(\cdot - \epsilon C_0(t))} \\
&= \epsilon^3 \langle v_0(t) + \epsilon v_1(t) + \epsilon r_2(t), \hat{u}_x(\cdot - \varphi_1(t))\rangle_{\rho_t(\cdot - \epsilon C_0(t))} \\
&= \epsilon^4 \big(- C_0(t) \langle v_0(t), \psi_x(\cdot - ct)\rangle + \langle v_1(t), \psi(\cdot - ct)\rangle \big) + o(\epsilon^4).
\end{aligned}$$

Note that

$$\begin{aligned}
&\langle v_1(t), \psi(\cdot - ct)\rangle \\
&= \int_0^t \langle w * \Big(F''(\hat{u}(\cdot - cs)) \Big(\frac{1}{2} v_0^2(s) - C_0(s) \hat{u}_x(\cdot - cs) v_0(s) \Big) \Big), \psi(\cdot - cs)\rangle ds \\
&\qquad + C_1(t) + \frac{1}{2} \langle \hat{u}_x, \psi_x\rangle C_0^2(t) \\
&= C_0(t) \langle v_0(t), \psi_x(\cdot - ct).
\end{aligned}$$

We thus obtain

$$\frac{1}{2}\frac{d}{da}\bigg|_{a=C_1(t)} \|u(t)-\hat{u}(\cdot-ct-\epsilon C_0(t)-\epsilon^2 a)\|^2_{\rho_t(\cdot-\epsilon C_0(t))} = o(\epsilon^4)$$

and

$$\frac{1}{2}\frac{d^2}{da^2}\bigg|_{a=C_1(t)} \|u-\hat{u}(\cdot-ct-\epsilon C_0(t)-\epsilon^2 a\|^2_{\rho_t(\cdot-\epsilon C_0(t))} = \epsilon^4 + o(\epsilon^4). \qquad \square$$

Together with Proposition 5.5.3 and the properties of $v_0^{\#}$, the expansion expresses the stability of the traveling wave under the noise. With large probability, up to the time horizon T, the stochastic solution can be described as a wave profile moving at an adapted speed, and the variance of the fluctuations in the wave profile stays bounded uniformly in time.

Remark 5.5.4. The spectral gap of $L^{\#}$ expresses linear stability properties. The control over $\|v_0(t)\|_{\rho_t}$ allows us to derive local stability up to the time horizon T, since the rest terms are of smaller order. The main problem in going over to larger time scales is that we lose control of the L^2-norm in estimates such as

$$\|w * v^2\|^2_\rho \le K_\rho \|v\|^2_\rho \|v\|^2.$$

Proof of Lemma 5.5.1. Using (5.19) we obtain that for $t > 0$,

$$C_0^m(t) - C_0(t) = e^{-mt}\langle \eta, \psi\rangle + \int_0^t e^{-m(t-s)}\langle \psi(\cdot - cs), dW_s\rangle =: e^{-mt}\langle \eta,\psi\rangle + S_t.$$

By Itô's Lemma,

$$\begin{aligned} S_t &= \langle \psi(\cdot - ct), W_t\rangle - \int_0^t m e^{-m(t-s)}\langle \psi(\cdot - cs), W_s\rangle ds \\ &\quad + \int_0^t c e^{-m(t-s)} \langle \psi_x(\cdot - cs), W_s\rangle ds \\ &= \int_0^t m e^{-m(t-s)}\big(\langle \psi(\cdot - ct), W_t\rangle - \langle \psi(\cdot - cs), W_s\rangle\big) ds + e^{-mt}\langle \psi(\cdot - ct), W_t\rangle \\ &\quad + \int_0^t c e^{-m(t-s)}\langle \psi_x(\cdot - cs), W_s\rangle ds. \end{aligned}$$

Note that by the Hölder continuity of $t \to \langle \psi(\cdot - ct), W_t\rangle$, for any $0 < \beta < \frac{1}{2}$, $M_\beta(T,\omega) := \sup_{|t-s|\le T} \frac{|\langle \psi(\cdot - ct), W_t\rangle - \langle \psi(\cdot - cs), W_s\rangle|}{|t-s|^\beta} < \infty$ almost surely (cf. [23], Thm. 3.3). We can thus estimate

$$\begin{aligned} &\Big| \int_0^t m e^{-m(t-s)}\big(\langle \psi(\cdot - ct), W_t\rangle - \langle \psi(\cdot - cs), W_s\rangle\big) ds \Big| \\ &\le M_\beta(T,\omega) \int_0^t m e^{-m(t-s)} (t-s)^\beta ds \end{aligned}$$

$$\leq M_\beta(T,\omega)\frac{1}{m^\beta}\int_0^\infty e^{-r}r^\beta dr = M_\beta(T,\omega)\frac{1}{m^\beta}\Gamma(1+\beta),$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the gamma function, and we obtain that

$$|S_t| \leq M_\beta(T,\omega)\frac{1}{m^\beta}\Gamma(1+\beta) + (e^{-mt}\|\psi\| + \frac{c}{m}\|\psi_x\|)\sup_{0\leq s\leq t}\|W(s)\|.$$

Thus,

$$\begin{aligned}
&\sup_{\delta\leq t\leq T}|C_0^m(t) - C_0(t)| \\
&\leq e^{-m\delta}\|\psi\|\big(\|\eta\| + \sup_{0\leq t\leq T}\|W(t)\|\big) \\
&\quad + \frac{c}{m}\|\psi_x\|\sup_{0\leq t\leq T}\|W(t)\| + \frac{1}{m^\beta}M_\beta(T,\omega)\Gamma(1+\beta) \xrightarrow{m\to\infty} 0, a.s.
\end{aligned}$$

Now we consider v_0^m. Since for $s \leq t$, $P_{t,s}\hat{u}_x(\cdot - cs) = \hat{u}_x(\cdot - ct)$, we have for $t > 0$

$$\begin{aligned}
v_0^m(t) &= P_{t,0}\eta + \int_0^t c_0^m(s)P_{t,s}\hat{u}_x(\cdot - cs)ds + \int_0^t P_{t,s}dW_s \\
&= P_{t,0}\pi_0\eta + \langle\eta,\psi\rangle\hat{u}_x(\cdot - ct) + C_0^m(t)\hat{u}_x(\cdot - ct) \\
&\quad + \int_0^t P_{t,s}\pi_s dW_s + \int_0^t \langle\psi(\cdot - cs), dW_s\rangle\hat{u}_x(\cdot - ct) \\
&= v_0(t) + (C_0^m(t) - C_0(t))\hat{u}_x(\cdot - ct),
\end{aligned}$$

and hence

$$\sup_{\delta\leq t\leq T}\|v_0^m(t) - v_0(t)\|_{H^1(1+\rho_t)} \leq \sup_{\delta\leq t\leq T}|C_0^m(t) - C_0(t)|\|\hat{u}_x\|_{H^1(1+\rho)} \xrightarrow{m\to\infty} 0, a.s.$$

The convergence of C_1^m follows from Lemma 5.4.4 and the convergence of C_0^m and v_0^m. Using (5.17) and

$$\begin{aligned}
&- c_0^m(s)C_0^m(s)P_{t,s}\hat{u}_{xx}(\cdot - cs) + c_1^m(s)P_{t,s}\hat{u}_x(\cdot - cs) \\
&= \big(\frac{d}{ds}P_{t,s} + P_{t,s}(L_s + c\partial_x)\big)\Big(-\frac{1}{2}(C_0^m)^2(s)\hat{u}_{xx}(\cdot - cs) + C_1^m(s)\hat{u}_x(\cdot - cs)\Big)
\end{aligned}$$

we obtain

$$\begin{aligned}
v_1^m(t) = &\int_0^t P_{t,s}w * \Big(F''(\hat{u}(\cdot - cs))\Big(\frac{1}{2}(v_0^m)^2(s) - C_0^m(s)\hat{u}_x(\cdot - cs)v_0^m(s)\Big)\Big)ds \\
&- \frac{1}{2}(C_0^m)^2(t)\hat{u}_{xx}(\cdot - ct) + C_1^m(t)\hat{u}_x(\cdot - ct) \\
&- \frac{1}{2}\int_0^t (C_0^m)^2(s)P_{t,s}(L_s + c\partial_x)\hat{u}_{xx}(\cdot - cs)ds.
\end{aligned}$$

Using the convergence of C_0^m, C_1^m, and v_0^m, and the fact that by Itô's Lemma

$$
\begin{aligned}
&-\frac{1}{2}C_0^2(t)\hat{u}_{xx}(\cdot - ct) - \frac{1}{2}\int_0^t C_0^2(s)P_{t,s}(L_s + c\partial_x)\hat{u}_{xx}(\cdot - cs)ds \\
&= -\frac{1}{2}C_0^2(t)P_{t,t}\hat{u}_{xx}(\cdot - ct) + \frac{1}{2}\int_0^t C_0^2(s)\frac{d}{ds}(P_{t,s}\hat{u}_{xx}(\cdot - cs))ds \\
&= -\frac{1}{2}\int_0^t P_{t,s}\hat{u}_{xx}(\cdot - cs)dC_0^2(s),
\end{aligned}
$$

the convergence of v_1^m to v_1 follows. □

Chapter 6

Finite-Size Effects

As outlined in section 1.1.6, it is not clear how noise translates from the single neuron level to the level of populations, and therefore in particular what is the right representation of the noise in stochastic neural field equations. In this chapter, we derive a stochastic correction term accounting for finite-size effects and prove that the solution to the resulting stochastic neural field equation is the continuum limit of an associated system of diffusion processes.

6.1 Introduction

Neural field equations are heuristically derived to model the spatio-temporal dynamics of the activity in a large network of synaptically coupled populations of neurons (cf. section 1.1.3). This involves two approximation steps. First, under the assumption that the number of populations is large, the finite network is approximated by a continuum. Second, the equations describe the mean field behavior of the network. Under the assumption that the number of neurons in each population is large, in the spirit of the law of large numbers, the local dynamics in each population is reduced to a description of the mean activity.

While several sources of noise have been identified on the single neuron level (cf. section 1.1.2), it is not clear how noise translates to the level of populations. Since neural field equations are derived as mean field limits, the usual effects of noise should have averaged out on this level. However, the actual finite size of the populations leads to deviations from the mean field behavior, suggesting finite size effects as an intrinsic source of noise.

In order to derive corrections to the neural field equation accounting for these effects, in [8], Bressloff (following Buice and Cowan [16]) sets up a continuous time Markov chain describing the evolution of the activity in a finite network of populations of finite size N. The rates are chosen such that in the limit as $N \to \infty$ one obtains the usual activity-based network equation. He then carries out a van Kampen system size expansion of the associated master equation in the small parameter $1/N$ to derive deterministic corrections of the neural field equation in the form of coupled differential equations for the moments. To first order, the finite-size effects can be characterized as Gaussian fluctuations around the mean field limit.

The model is considered from a mathematically rigorous perspective by Riedler and Buckwar in [54]. They make use of limit theorems for Hilbert-space valued piecewise deterministic Markov processes recently obtained in [55] as an extension of Kurtz's convergence theorems for jump Markov processes to the infinite-dimensional setting. They derive a law of large numbers and a central limit theorem for the Markov chain, realizing the double limit (number of neurons per population to infinity and continuum limit) at the same time. They formally set up a stochastic neural field equation, but the question of well-posedness is left open.

In [13], Bressloff and Newby extend the original approach of [8] by including synaptic dynamics and consider a Markov chain modeling the activity coupled to a piecewise deterministic process describing the synaptic current (see also section 6.4 in [10] for a summary). In two different regimes, the model covers the case of Gaussian-like fluctuations around the mean-field limit as derived in [8], as well as a situation in which the activity has Poisson statistics as considered in [17].

In this chapter, we consider the question how finite-size effects can be included in the voltage-based neural field equation. To our knowledge, this has not been covered in the literature yet. We take up the approach of describing the dynamics of the activity in a finite-size network by a continuous-time Markov chain and motivate a choice of jump rates that will lead to the voltage-based network equation in the infinite-population limit. The model is not a modification of the one considered in [13] and does not contradict it, but rather extends it to a different regime. We derive a law of large numbers and a central limit theorem for the Markov chain. Instead of realizing the double limit as in [54], we split up the limiting procedure, which in particular allows us to insert further approximation steps. We follow the original approach by Kurtz to determine the limit of the fluctuations of the Markov chain. By linearizing the noise term around the traveling wave solution, we obtain an approximating system of diffusion processes. After introducing correlations between populations lying close together (cf. section 6.5.1) we obtain a well-posed $L^2(\mathbb{R})$-valued stochastic evolution equation, with a noise term approximating finite size effects on traveling waves, which we prove to be the strong continuum limit of the associated network.

The diffusion coefficient we derive satisfies the assumptions made on the noise in the previous chapters. The stability results and the ϵ-expansion therefore also apply to this setting.

6.2 Finite-Size Effects in Population Models

Consider a network of P populations, each consisting of N neurons. Recall from section 1.1.3 that, heuristically, the potential u_i^N in the populations is related to the activities a_j^N

via

$$\dot{u}_i^N(t) = \frac{1}{\tau_m}\Big(- u_i(t) + \sum_{j=1}^{P} w_{ij} a_j^N(t)\Big), \tag{6.1}$$

where the activity $a_j^N(t)$ of population j is defined as the proportion of neurons that are active at time t,

$$a_j^N(t) = \frac{\#\text{ neurons active at time } t \text{ in population } j}{N},$$

and where τ_m is the membrane time constant, determining how fast the membrane potential relaxes back to its resting value. In the infinite population limit, the activity is related to the potential via the nonlinear gain function F,

$$a_j(t + \tau_a) = F(u_j(t)),$$

where τ_a is a possible delay in the reaction of the activity to a change in the potential. Its evolution can therefore approximately be described by the neural rate equation

$$\tau_a \dot{a}_j(t) = -a_j(t) + F(u_j(t)). \tag{6.2}$$

We identified two regimes in which the description of the evolution of the coupled system (a_i, u_i) can be closed in one of the variables:

- the regime $\tau_a \gg \tau_m \to 0$, leading to the activity-based network equation,
- the regime $\tau_m \gg \tau_a \to 0$, leading to the voltage-based network equation.

6.2.1 Models in the Literature so far

In [13], Bressloff and Newby set up a model for the evolution of the activity in a network of finite populations. They define the activity in population j as

$$a_j^{\delta,N}(t) = \frac{\#\text{ spikes in } (t-\delta, t] \text{ in population } j}{\delta N},$$

where δ is a time window of variable size. If δ is chosen as the width of an action potential Δ, then we obtain our original notion of the activity, $\Delta a^{\Delta,N} = a^N$. Here the activity is modeled as a rate rather than a probability. Note that the number of spikes in the time interval $(t-\delta, t]$ is limited by $n_{max} := N \vee \big[\frac{\delta N}{\Delta}\big]$.

They describe the dynamics of $a^{\delta,N}$ by a Markov chain with state space $\{0, \frac{1}{\delta N}, ..., \frac{n_{max}}{\delta N}\}$

and jump rates

$$\begin{aligned} q_a^N(x, x + \frac{1}{\delta N} e_i) &= \frac{1}{\tau_a} \delta N \lambda(u_i^N(t)) \text{ if } x(i) < \tfrac{n_{max}}{\delta N} \\ q_a^N(x, x - \frac{1}{\delta N} e_i) &= \frac{1}{\tau_a} \delta N x_i(t), \end{aligned} \tag{6.3}$$

where e_i denotes the i-th unit vector, where $\lambda(u)$ is the firing rate at potential u, related to the probability $F(u)$ via $\Delta\lambda(u) = F(u)$, and where u^N evolves according to (6.1). The idea is that the activation rate should be proportional to $\lambda(u)$, while the inactivation rate should be proportional to the activity itself. The rates are chosen such that in the limit as N goes to infinity, we obtain the neural rate equation (6.2).

They consider two regimes.

Case 1: $\delta = 1$, $\tau_a \gg \tau_m \to 0$

In the first regime, the size of the time window δ is fixed, say $\delta = 1$. If $\tau_a \gg \tau_m \to 0$, then as in section 1.1.3, $u_i^N(t) = \sum_{j=1}^P w_{ij} a_j^{\delta,N}(t)$. The description of the Markov chain can thus be closed in the variables $a_i^{\delta,N}$, leading to the model already considered in [8]. In the limit $N \to \infty$ one obtains the activity-based network equation

$$\tau_a \dot{a}_i^N(t) = -a_i^N(t) + \lambda\Big(\sum_{j=1}^P w_{ij} a_j^N(t)\Big). \tag{6.4}$$

By formally approximating to order $\frac{1}{N}$ in the associated master equation, they derive a stochastic correction to (6.4), leading to the diffusion approximation

$$\begin{aligned} da_i^{\delta,N}(t) \approx{}& \frac{1}{\tau_a}\Big(- a_i^{\delta,N}(t) + \lambda\Big(w_{ij} \sum_{j=1}^P a_j^{\delta,N}(t)\Big)\Big) dt \\ &+ \frac{1}{\sqrt{\tau_a N}} \Big(a_i^{\delta,N}(t) + \lambda\Big(\sum_{j=1}^P w_{ij} a_j^{\delta,N}(t)\Big)\Big)^{\frac{1}{2}} dB_j(t) \end{aligned}$$

for independent Browning motions B_j.

In [54], Riedler and Buckwar rigorously derive a law of large numbers and a central limit theorem for the sequence of Markov chains as N tends to infinity. Note that the nature of the jump rates is such that the process has to be 'forced' to stay in its natural domain $[0, \frac{n_{max}}{N}]$ by setting the jump rate to 0 at the boundary. As they point out, this discontinuous behavior is difficult to deal with mathematically. They therefore have to slightly modify the model and allow the activity to be larger than $\frac{n_{max}}{N}$. They embed the Markov chain into $L^2(D)$ for a bounded domain $D \subset \mathbb{R}^d$ and derive the LLN in $L^2(D)$ and the CLT in the Sobolev space $H^{-\alpha}(D)$ for some $\alpha > d$.

Case 2: $\delta = \frac{1}{N}, \tau_m \gg \tau_a$

In the second regime, the size of the time window δ goes to 0 as N goes to infinity such that $\delta N = 1$. In this case,

$$a_i^{\delta,N}(t) = \frac{\#\text{ spikes in } (t-\delta, t] \text{ in pop. } i}{\delta N} \approx \frac{\lambda(u_i^N(t))\delta N}{\delta N} = \lambda(u_i^N(t)).$$

They show that at fixed voltage u, the stationary distribution of the activity $a^{\delta,N}$ evolving according to (6.3) is approximately Poisson with rate $\lambda(u)$. This corresponds to the regime considered in [17].

In the limit $N \to \infty$, $a_j(t) = \lambda(u_j(t))$ and the system reduces to the voltage-based network equation

$$\tau_m \dot{u}_j(t) = -u_j(t) + \sum_{j=1}^{P} w_{ij}\lambda(u_j(t)).$$

Case 3: $\delta = \Delta$**,** $\tau_m \gg \tau_a \to 0$

The third regime has not been considered explicitly in [13]. It is the one which is relevant for us.

We go back to our original definition of the activity and fix the time window δ to be the length of an action potential Δ. We assume that the potential evolves slowly, $\tau_m \gg 0$. Speeding up time, we define

$$\tilde{u}_i^N(t) = u_i^N(t\tau_m).$$

Then

$$\begin{aligned}
\tilde{u}_i^N(t) &= \sum_{j=1}^{P} w_{ij} \int_{-\infty}^{t\tau_m} \frac{1}{\tau_m} e^{-\frac{1}{\tau_m}(t\tau_m - s)} a_j^N(s)ds \\
&= \sum_{j=1}^{P} w_{ij} \int_{-\infty}^{t} e^{-(t-s)} a_j^N(s\tau_m)ds
\end{aligned}$$

For some large n,

$$\tilde{u}_i^N(t) \approx \sum_{j=1}^{P} w_{ij} \sum_{k=-\infty}^{[tn]-1} e^{-(t-\frac{k}{n})} \int_{\frac{k}{n}}^{\frac{k+1}{n}} a_j^N(s\tau_m)ds.$$

The potentials $\tilde{u}_i^N$ therefore only depend on the time-averaged activities given for $\frac{k}{n} \le t < \frac{k+1}{n}$ as

$$\tilde{a}_i^N(t) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} a_i^N(s\tau_m)ds.$$

We have

$$\tilde{u}_i^N(t) \approx \sum_{j=1}^P w_{ij} \sum_{k=-\infty}^{[tn]-1} \frac{1}{n} e^{-(t-\frac{k}{n})} \tilde{a}_j^N(\frac{k}{n}) \approx \sum_{j=1}^P w_{ij} \int_{-\infty}^t e^{-(t-s)} \tilde{a}_j^N(s) ds. \tag{6.5}$$

If $\tau_a \ll \tau_m$, the activity relaxes to its stationary distribution quickly on this time scale. At fixed voltage u, under the stationary distribution $\nu(u)$,

$$\tilde{a}_i^N(\frac{k}{n}) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} a_i^N(s\tau_m) ds \approx E_{\nu(u)}(a_i^N(\frac{k}{n}\tau_m)) = F(u),$$

with equality if $N \to \infty$. If u is time-varying, then differentiation of (6.5) yields

$$\begin{aligned} \frac{d}{dt}\tilde{a}_i(t) &= \frac{d}{dt} F(\tilde{u}_i(t)) \\ &= F'(\tilde{u}_i(t))\Big(- \tilde{u}_i(t) + \sum_{j=1}^P w_{ij} F(\tilde{u}_j(t))\Big) \\ &= F'(F^{-1}(\tilde{a}_i(t)))\Big(- F^{-1}(\tilde{a}_i(t)) + \sum_{j=1}^P w_{ij} \tilde{a}_j(t)\Big). \end{aligned} \tag{6.6}$$

If $N < \infty$, then the finite size of the populations causes deviations from (6.6). In order to determine these finite-size effects, in the next section we will set up a Markov chain $X^{P,N}$ to describe the evolution of the time-averaged activity $\tilde{a}^N$.

6.3 A Markov Chain Model for the Activity

We describe the evolution of the time-averaged activity by a Markov chain $X^{P,N}$ with state space $E^{P,N} = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}^P$. We define the jump rates as

$$\begin{aligned} q^{P,N}(x, x + \frac{1}{N} e_i) &= N F'(F^{-1}(x_i))\Big(- F^{-1}(x_i) + \sum_{j=1}^P w_{ij} x_j \Big)_+ \\ q^{P,N}(x, x - \frac{1}{N} e_i) &= N F'(F^{-1}(x_i))\Big(- F^{-1}(x_i) + \sum_{j=1}^P w_{ij} x_j \Big)_- \end{aligned} \tag{6.7}$$

where for $x \in \mathbb{R}$, $x_+ := x \vee 0$, $x_- := -x \vee 0$, and where e_i denotes the i-th unit vector.

The idea behind this choice is the following: the time-averaged activity tends to jump up (down) if the potential in the population, which is approximately given by $F^{-1}(x_i)$, is lower (higher) than the input from the other populations, which is given by $\sum_{j=1}^P w_{ij} x_j$. The probability that the activity jumps down (up) when the potential is lower (higher) than the input is assumed to be negligible. The jump rates are proportional to the difference between the two quantities, scaled by the factor $F'(F^{-1}(x_i))$. They are therefore higher in the sensitive regime where $F' \gg 1$, that is, where small changes in the potential have large effects on the activity. If $a_i^N = F(u_i^N)$ in all populations i, then the system is in balance.

Note that the state space is naturally bounded since $\lim_{x\uparrow 1} F^{-1}(x) = \infty$ and $\lim_{x\downarrow 0} F^{-1}(x) = -\infty$, such that $(-F^{-1}(x_k) + \sum_{l=1}^P w_{kl}x_l)_+ = 0$ for x with $x_k = 1 - \frac{1}{N}$, $x_l \leq 1 - \frac{1}{N}$, when $N \geq N_0$ is large enough, and similarly at 0.

We will see in Proposition 6.3.1 below that the Markov chain converges to the solution of (6.6) as the size of the populations N goes to infinity.

In [8] a different choice of jump rates was suggested in analogy to (6.3):

$$\begin{aligned}\tilde{q}(x, x + \frac{1}{N}e_i) &= NF'(F^{-1}(x_i)) \sum_{j=1}^P w_{ij}x_j \\ \tilde{q}(x, x - \frac{1}{N}e_i) &= NF'(F^{-1}(x_i))F^{-1}(x_i).\end{aligned}$$

Also this choice leads to (6.6) in the limit. In this picture, the jump rates are high in regions where the activity is high. Since, as explained above, one should think of the Markov chain as governing a slowly varying time-averaged activity, (6.7) seems like a more natural choice.

The generator of $Q^{P,N}$ of $X^{P,N}$ is given for bounded measurable $f : E^{P,N} \to \mathbb{R}$ by

$$\begin{aligned}Q^{P,N}f(x) = N\sum_{k=1}^P F'(F^{-1}(x_k))\bigg(&\Big(-F^{-1}(x_k) + \sum_{j=1}^P w_{kj}x_j\Big)_+ (f(x + \tfrac{1}{N}e_k) - f(x)) \\ &+ \Big(-F^{-1}(x_k) + \sum_{j=1}^P w_{kj}x_j\Big)_- (f(x - \tfrac{1}{N}e_k) - f(x))\bigg).\end{aligned}$$

Let N_0 be such that the jump rates out of the interval $\left[\frac{1}{N_0}, 1 - \frac{1}{N_0}\right]$ are 0.

Proposition 6.3.1. *Let X^P be the (deterministic) Feller process on $[1/N_0, 1-1/N_0]^P$ with generator*

$$L^P f(x) = \sum_{k=1}^P F'(F^{-1}(x_k))\Big(-F^{-1}(x_k) + \sum_{j=1}^P w_{kj}x_j\Big)\partial_k f(x).$$

If $X^{P,N}(0) \xrightarrow[N\to\infty]{d} X^P(0)$, then $X^{P,N} \xrightarrow[N\to\infty]{d} X^P$ on the space of càdlàg functions $D([0,\infty), [1/N_0, 1 - 1/N_0]^P)$ with the Skorohod topology (where $\xrightarrow{d}$ denotes convergence in distribution).

Proof. By a standard theorem on the convergence of Feller processes (cf. [38], Thm. 19.25) it is enough to prove that for $f \in \mathcal{C}^\infty([1/N_0, 1 - 1/N_0]^P)$ there exist bounded measurable f_N such that $\|f_N - f\|_\infty \xrightarrow{N\to\infty} 0$ and $\|Q^{P,N}f_N - L^P f\|_\infty \xrightarrow{N\to\infty} 0$.

Let thus $f \in \mathcal{C}^\infty([1/N_0, 1 - 1/N_0]^P)$ and set $f_N(x) = f\big(\big(\frac{[x_1 N]}{N}, ..., \frac{[x_P N]}{N}\big)\big)$. Then it is easy to see that

$$Q^{P,N}f_N(x) \xrightarrow{N\to\infty} L^P f(x)$$

uniformly in x. $\square$

6.4 Diffusion Approximation

We are now going to approximate $X^{P,N}$ by a diffusion process. To this end, we follow the standard approach due to Kurtz and derive a central limit theorem for the fluctuations of $X^{P,N}$. This will give us a candidate for a stochastic correction term to (6.6).

6.4.1 A Central Limit Theorem

We write

$$X_k^{P,N}(t) = X_k^{P,N}(0) + \int_0^t Q^{P,N}\pi_k(X^{P,N}(s))ds + M_k^{P,N}(t)$$

where $\pi_k : (0,1)^P \to (0,1), x \mapsto x_k$, is the projection onto the k-th coordinate, and

$$M_k^{P,N}(t) := X_k^{P,N}(t) - X_k^{P,N}(0) - \int_0^t Q^{P,N}\pi_k(X^{P,N}(s))ds$$

is a martingale describing the fluctuations of the process.

We start by determining the limit of these fluctuations.

Proposition 6.4.1.

$$\left(\sqrt{N}M_k^{P,N}\right) \xrightarrow[N\to\infty]{d} \left(\int_0^t \left(F'(F^{-1}(X_k^P(s)))\Big| - F^{-1}(X_k^P(s)) + \sum_{j=1}^P w_{kj}X_j^P(s)\Big|\right)^{\frac{1}{2}} dB_k(s)\right)$$

on $\mathcal{D}([0,\infty),\mathbb{R}^P)$, where B is a P-dimensional standard Brownian motion, and X^P is the Feller process from Proposition 6.3.1 .

Proof. The bracket process of $M_k^{P,N}$ is given in terms of the carré du champ operator as

$$\begin{aligned}
\langle M_k^{P,N}\rangle_t &= \int_0^t Q^{P,N}\pi_k^2(X^{P,N}(s)) - 2Q^{P,N}\pi_k(X^{P,N}(s))X_k^{P,N}(s)ds \\
&= \int_0^t \sum_{y\in E^{P,N}} q^{P,N}(X^{P,N}(s),y)(y_k - X_k^{P,N}(s))^2 ds \\
&= \frac{1}{N}\int_0^t F'(F^{-1}(X_k^{P,N}(s)))\Bigg(\Big(-F^{-1}(X_k^{P,N}(s)) + \sum_{j=1}^P w_{kj}X_j^{P,N}(s)\Big)_+ \\
&\qquad + \Big(-F^{-1}(X_k^{P,N}(s)) + \sum_{j=1}^P w_{kj}X_j^{P,N}(s)\Big)_-\Bigg)ds \\
&= \frac{1}{N}\int_0^t F'(F^{-1}(X_k^{P,N}(s)))\Big| - F^{-1}(X_k^{P,N}(s)) + \sum_{j=1}^P w_{kj}X_j^{P,N}(s)\Big|ds.
\end{aligned}$$

Thus,

$$\big\langle \sqrt{N} M_k^{P,N} \big\rangle_t \xrightarrow{N\to\infty} \int_0^t F'(F^{-1}(X_k^P(s)))\Big| - F^{-1}(X_k^P(s)) + \sum_{j=1}^P w_{kj} X_j^P(s)\Big| ds$$

in probability. For $k \neq l$,

$$\big\langle M_k^{P,N}, M_l^{P,N} \big\rangle_t = \int_0^t \sum_y q^{P,N}(X^{P,N}(s), y)(y_k - X_k^{P,N}(s))(y_l - X_l^{P,N}(s)) ds = 0,$$

since for y with $q^{P,N}(X^{P,N}(s), y) > 0$ at least one of $y_k - X_k^{P,N}(s)$ and $y_l - X_l^{P,N}(s)$ is always 0. Now

$$E\left(\sup_t \sqrt{N} \| M^{P,N}(t) - M^{P,N}(t-) \|_2 \right) \leq \frac{1}{\sqrt{N}} \xrightarrow{N\to\infty} 0,$$

and the statement follows by the martingale central limit theorem, see for example Theorem 1.4, Chapter 7 in [29] . □

This suggests to approximate $X^{P,N}$ by the system of coupled diffusion processes

$$\begin{aligned} da_k^{P,N}(t) &= F'(F^{-1}(a_k^{P,N}(t)))\Big(- F^{-1}(a_k^{P,N}(t)) + \sum_{j=1}^P w_{kj} a_j^{P,N}(t) \Big) dt \\ &\quad + \frac{1}{\sqrt{N}} \Big(F'(F^{-1}(a_k^{P,N}(t)))\Big| - F^{-1}(a_k^{P,N}(t)) + \sum_{j=1}^P w_{kj} a_j^{P,N}(t) \Big| \Big)^{\frac{1}{2}} dB_k(t), \end{aligned}$$

$1 \leq k \leq P$.

Using Itô's formula, we formally obtain an approximation for $u_k^{P,N} := F^{-1}(a_k^{P,N})$,

$$\begin{aligned} du_k^{P,N}(t) = \Big(&- u_k^{P,N}(t) + \sum_{j=1}^P w_{kj} F(u_j^{P,N}(t)) \\ &- \frac{1}{2N} \frac{F''(u_k^{P,N}(t))}{F'(u_k^{P,N}(t))^2} \Big| - u_k^{P,N}(t) + \sum_{j=1}^P w_{kj} F(u_j^{P,N}(t)) \Big| \Big) dt \\ &+ \frac{1}{\sqrt{N}} \left(\frac{\Big| - u_k^{P,N}(t) + \sum_{j=1}^P w_{kj} F(u_j^{P,N}(t)) \Big|}{F'(u_k^{P,N}(t))} \right)^{\frac{1}{2}} dB_k(t). \end{aligned} \tag{6.8}$$

Since the square root function is not Lipschitz continuous near 0, we cannot apply standard existence theorems to obtain a solution to (6.8) with the full multiplicative noise term. Instead we will linearize around a deterministic solution to the neural field equation and approximate to a certain order of $\frac{1}{\sqrt{N}}$.

6.4.2 Fluctuations around the Traveling Wave

Let $\bar{u}$ be a solution to the neural field equation (1.5). To determine the finite-size effects on $\bar{u}$, we consider a spatially extended network, that is, we look at populations distributed over an interval $[-L, L] \subset \mathbb{R}$ and use the stochastic integral derived above to describe the local fluctuations on this interval.

Let $m \in \mathbb{N}$ be the density of populations on $[-L, L]$ and consider $P = 2mL$ populations located at $\frac{k}{m}, k \in \{-mL, -mL+1, ..., mL-1\}$. We choose the weights w_{kl} as a discretization of the integral kernel $w : \mathbb{R} \to [0, \infty)$,

$$w_{kl}^m = \int_{\frac{l}{m}}^{\frac{l+1}{m}} w(\tfrac{k}{m} - y)dy, \quad -mL \leq k, l \leq mL - 1. \tag{6.9}$$

Since we think of the network as describing only a section of the actual domain $\mathbb{R}$, we add to each population an input $F(\bar{u}_t(-L))$ and $F(\bar{u}_t(L))$, respectively, at the boundaries with corresponding weights

$$\begin{aligned} w_k^{m,+} &= \int_L^{\infty} w(\tfrac{k}{m} - y)dy, \\ w_k^{m,-} &= \int_{-\infty}^{-L} w(\tfrac{k}{m} - y)dy. \end{aligned} \tag{6.10}$$

Fix a population size $N \in \mathbb{N}$. Set $\bar{u}_k(t) = \bar{u}(\frac{k}{m}, t)$ and for $u \in \mathbb{R}^P$,

$$\hat{b}_k^m(t, u) = -u_k(t) + \sum_{l=-mL}^{mL-1} w_{kl}^m F(u_l(t)) + w_k^{m,+} F(\bar{u}(L, t)) + w_k^{m,-} F(\bar{u}(-L, t)).$$

We write

$$u_k = \bar{u}_k + v_k \tag{6.11}$$

and assume that v_k is of order $1/\sqrt{N}$. Linearizing (6.8) around $(\bar{u}_k)$ we obtain the approximation

$$du_k(t) = \hat{b}_k^m(t, u)dt + \frac{1}{\sqrt{NF'(\bar{u}_k(t))}} |\hat{b}_k^m(t, \bar{u})|^{\frac{1}{2}} dB_k(t)$$

to order $1/\sqrt{N}$.

Note that $\hat{b}_k^m(t, \bar{u}) \approx \partial_t \bar{u}_k(t) = 0$ for a stationary solution $\bar{u}$, with equality if $\bar{u}$ is constant. The finite-size effects are hence of smaller order. Since the square root function is not differentiable at 0 we cannot expand further.

However, the situation is different if we linearize around a moving pattern. We consider the traveling wave solution $u_t^{TW}(x) = \hat{u}(x - ct)$ to (1.5) and we assume without loss of generality that $c > 0$. Then $\hat{b}_k^m(t, \bar{u}) \approx \partial_t \bar{u}_k(t) = -c\partial_x u_t^{TW} < 0$. This monotonicity property allows us to approximate to order $1/N$ in (6.8). Indeed, note that since $\hat{u}$ and F

are increasing,

$$
\begin{aligned}
&-\hat{b}_k^m(t,u_t^{TW})\\
&= u_t^{TW}(\tfrac{k}{m}) - \sum_l w_{kl}^m F(u_t^{TW}(\tfrac{l}{m})) - w_k^{m,+}F(u_t^{TW}(L)) - w_k^{m,-}F(u_t^{TW}(-L))\\
&\geq u_t^{TW}(\tfrac{k}{m}) - \int_{-L}^{\infty} w(\tfrac{k}{m}-y)F(u_t^{TW})(y)dy - \int_{-\infty}^{-L} w(\tfrac{k}{m}-y)(F(u_t^{TW}(-L))dy\\
&= c\hat{u}_x(\tfrac{k}{m}-ct) - \int_{-\infty}^{-L} w(\tfrac{k}{m}-y)(F(u_t^{TW}(-L)) - F(u_t^{TW}(y)))dy\\
&\geq c\hat{u}_x(\tfrac{k}{m}-ct) - \big(F(u_t^{TW}(-L)) - F(a_1)\big)\\
&\xrightarrow{L\to\infty} c\hat{u}_x(\tfrac{k}{m}-ct) > 0.
\end{aligned}
\tag{6.12}
$$

So for L large enough, $-\hat{b}_k^m(t,u_t^{TW}) > 0$ and we have, using Taylor's formula and (6.11),

$$
\begin{aligned}
\left(\frac{|\hat{b}_k^m(t,u)|}{F'(u_k(t)))}\right)^{\frac12} = {}& \left(\frac{-\hat{b}_k^m(t,u_t^{TW})}{F'(u_t^{TW}(\frac{k}{m}))}\right)^{\frac12} + \frac{1}{2\sqrt{-\hat{b}_k^m(t,u_t^{TW})F'(u_t^{TW}(\frac{k}{m}))}}\\
&\left(\frac{F''(u_t^{TW}(\frac{k}{m}))}{F'(u_t^{TW}(\frac{k}{m}))}\hat{b}_k^m(t,u_t^{TW})v_k(t)\right.\\
&\left. + v_k(t) - \sum_l w_{kl}F'(u_t^{TW}(\tfrac{l}{m}))v_l(t)\right) + O\Big(\frac{1}{N}\Big).
\end{aligned}
$$

As a possible diffusion approximation in the case of traveling wave solutions we therefore obtain the system of stochastic differential equations

$$
\begin{aligned}
du_k(t) = {}& \Big(\hat{b}_k^m(t,u) + \frac{1}{2N}\frac{F''(u_t^{TW}(\frac{k}{m}))}{F'(u_t^{TW}(\frac{k}{m}))^2}\hat{b}_k^m(t,u_t^{TW})\Big)dt\\
&+ \frac{1}{\sqrt{N}}\left[\left(\frac{-\hat{b}_k^m(t,u_t^{TW})}{F'(u_t^{TW}(\frac{k}{m}))}\right)^{\frac12} + \frac{1}{2\sqrt{-\hat{b}_k^m(t,u_t^{TW})F'(u_t^{TW}(\frac{k}{m}))}}\right.\\
&\left(\frac{F''(u_t^{TW}(\frac{k}{m}))}{F'(u_t^{TW}(\frac{k}{m}))}\hat{b}_k^m(t,u_t^{TW})v_k(t)\right.\\
&\left.\left. + v_k(t) - \sum_l w_{kl}F'(u_t^{TW}(\tfrac{l}{m}))v_l(t)\right)\right]dB_k(t),
\end{aligned}
\tag{6.13}
$$

for which there exists a unique solution as we will see in the next section.

6.5 The Continuum Limit

In this section we take the continuum limit of the network of diffusions (6.13), that is, we let the size of the domain and the density of populations go to infinity in order to obtain a stochastic neural field equation with a noise term describing the fluctuations around the deterministic traveling wave solution due to finite size effects.

We thus have to deal with functions that 'look almost like the wave' and choose to work in the space $\mathcal{S} := \{u : \mathbb{R} \to \mathbb{R} : \; u - \hat{u} \in L^2\}$. Note that since for $u_1, u_2 \in \mathcal{S}$, $\|u_1 - u_2\| < \infty$, the L^2-norm induces a topology on $\mathcal{S}$.

6.5.1 A Word on Correlations

Recall the definition of the Markov chain introduced in section 6.3. Note that as long as we allow only single jumps in the evolution, meaning that there will not be any jumps in the activity in two populations at the same time, the martingales associated with any two populations will be uncorrelated, yielding independent driving Brownian motions in the diffusion limit (cf. Proposition 6.4.1).

This only makes sense for populations that are clearly distinguishable. In order to determine the fluctuations around traveling wave solutions, we consider spatially extended networks of populations. The population located at $x \in \mathbb{R}$ is to be understood as the ensemble of all neurons in the ϵ-neighborhood $(x - \epsilon, x + \epsilon)$ of x for some $\epsilon > 0$. If we consider two populations located at $x, y \in \mathbb{R}$ with $|x - y| < 2\epsilon$, then they will overlap. Consequently, simultaneous jumps will occur, leading to correlations between the driving Brownian motions.

Thus the Markov chain model (and the associated diffusion approximation) is only appropriate as long as the distance between the individual populations is large enough. When we go over to the continuum limit and increase the density of populations, we therefore adapt the model by introducing correlations between the driving Brownian motions of populations lying close together.

6.5.2 The Stochastic Neural Field Equation

We start by defining the limiting object. For $u \in \mathcal{S}$ and $t \in [0, T]$ set

$$b(t, u)(x) = -u(x) + \int_{-\infty}^{\infty} w(x - y)F(u(y))dy = -u(x) + w * F(u)(x).$$

As in assumption (A)(ii), let $\mathcal{W}^Q$ be a (cylindrical) Q-Wiener process on L^2 with covariance operator $\sqrt{Q}$ given as $\sqrt{Q}h(x) = \int_{-\infty}^{\infty} q(x, y)h(y)dy$ for some symmetric kernel $q(x, y)$ with $q(x, \cdot) \in L^2 \cap L^1$ for all $x \in \mathbb{R}$ and $\sup_{x \in \mathbb{R}}(\|q(x, \cdot)\| + \|q(x, \cdot)\|_1) < \infty$. We assume that the diffusion coefficient is given as the multiplication operator associated with $\sigma : [0, T] \times \mathcal{S} \to L^2(\mathbb{R})$, which we also denote by σ, where σ is Lipschitz continuous with respect to the second variable uniformly in $t \leq T$, that is, we assume that there exists $L_\sigma > 0$ such that for all $u_1, u_2 \in \mathcal{S}$ and $t \in [0, T]$,

$$\|\sigma(t, u_1) - \sigma(t, u_2)\| \leq L_\sigma \|u_1 - u_2\|. \tag{6.14}$$

Recall that the correlations are described by $q * q(x,y) := \int q(x,z)q(z,y)dz$. We could for example take

$$q(x,y) = q(x-y) = \frac{1}{2\epsilon}\mathbb{1}_{(-\epsilon,\epsilon)}(x-y) \tag{6.15}$$

for some small $\epsilon > 0$ (cf. section 6.5.1).

As in assumption (A)(ii), $\sigma \in L_2^0$. Note that for uncorrelated noise (i.e. Q=E), this is not the case. Therefore, in [54] Riedler and Buckwar derive the central limit theorem in the Sobolev space $H^{-\alpha}$. Splitting up the limiting procedures, $N \to \infty$ and continuum limit, allows us to incorporate correlations and finally to work in the more natural function space L^2.

Proposition 6.5.1. *For any initial condition $u^0 \in \mathcal{S}$, the stochastic evolution equation*

$$\begin{aligned} du_t(x) &= \Big(- u_t + w * F(u_t) + \frac{1}{2N}\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\partial_t u_t^{TW}\Big)\, dt + \sigma(t,u_t)d\mathcal{W}_t^Q(x) \\ u_0 &= u^0, \end{aligned} \tag{6.16}$$

has a unique strong $\mathcal{S}$-valued solution. u has a continuous modification. For any $p \geq 2$,

$$E\Big(\sup_{t\in[0,T]} \|u_t - u_t^{TW}\|^p\Big) < \infty.$$

The proof is exactly analogous to the proof of Proposition 2.2.1.

6.5.3 Embedding of the Diffusion Processes

As a next step we embed the systems of coupled diffusion processes (6.13) into $L^2(\mathbb{R})$. Let $m \in \mathbb{N}$ be the population density and $L^m \in \mathbb{N}$ the length of the domain with $L^m \uparrow \infty$ as $m \to \infty$. For $k \in \{-mL^m, -mL^m+1, ..., mL^m-1\}$ set $I_k^m = [\frac{k}{m}, \frac{k+1}{m})$ and $J_k^m = (\frac{k}{m} - \frac{1}{4m}, \frac{k}{m} + \frac{1}{4m})$, and let

$$W_k^m(t) = 2m\langle \mathcal{W}_t^Q, \mathbb{1}_{J_k^m}\rangle$$

be the average of $\mathcal{W}_t^Q$ on the interval J_k^m. Then the W_k^m are one-dimensional Brownian motions with covariances

$$\begin{aligned} E(W_k^m W_l^m) &= 4m^2\langle \sqrt{Q}\mathbb{1}_{J_k^m}, \sqrt{Q}\mathbb{1}_{J_l^m}\rangle \\ &= 4m^2 \int_{-\infty}^{\infty} \int_{\frac{k}{m}-\frac{1}{4m}}^{\frac{k}{m}+\frac{1}{4m}} q(x,y)dy \int_{\frac{l}{m}-\frac{1}{4m}}^{\frac{l}{m}+\frac{1}{4m}} q(x,z)dzdx \\ &= 4m^2 \int_{J_k^m}\int_{J_l^m} q * q(y,z)dydz. \end{aligned}$$

Note that while $m < \frac{1}{4\epsilon}$, the Brownian motions are independent.

For $m \in \mathbb{N}$ let $\hat{\sigma}^m : [0,T] \times \mathbb{R}^P \to \mathbb{R}^P$ and assume that there exists $L_{\hat{\sigma}^m} > 0$ such that for any $t \in [0,T]$ and $u_1, u_2 \in \mathbb{R}^P$,

$$\|\hat{\sigma}^m(t,u_1) - \hat{\sigma}^m(t,u_2)\|_2 \leq L_{\hat{\sigma}^m} \|u_1 - u_2\|_2.$$

Consider the system of coupled stochastic differential equations

$$\begin{aligned} du_k^m(t) = \hat{b}_k^m(t,(u_k^m)) &+ \frac{1}{2N} \frac{F''(u_t^{TW}(\frac{k}{m}))}{F'(u_t^{TW}(\frac{k}{m}))^2} \hat{b}_k^m(t,(u_t^{TW}(\tfrac{k}{m}))_k) \\ &+ w_k^{m,+} F(u_t^{TW}(L^m)) + w_k^{m,-} F(u_t^{TW}(-L^m))\Big) dt \\ &+ \hat{\sigma}_k^m(t,u^m(t)) dW_k^m(t), \qquad -mL^m \leq k \leq mL^m - 1, \end{aligned}$$

with weights as in (6.9) and (6.10).

We identify $u = (u_k)_{-mL^m \leq k \leq mL^m - 1} \in \mathbb{R}^P$ with its piecewise constant interpolation as an element of L^2 via the embedding

$$\iota^m(u) = \sum_{k=-mL^m}^{mL^m-1} u_k \mathbb{1}_{I_k^m}.$$

For $u \in \mathcal{C}(\mathbb{R})$ set

$$\pi^m(u) = \sum_{k=-mL^m}^{mL^m-1} u(\tfrac{k}{m}) \mathbb{1}_{I_k^m}.$$

Then $u_t^m := \iota^m((u_k^m(t))_k)$ satisfies

$$\begin{aligned} du_t^m(x) = b^m(t,u_t^m)(x)dt &+ \frac{1}{2N} \pi^m\Big(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big) b^m(t,\pi^m(u_t^{TW}))dt \\ &+ \sigma^m(t,u_t^m) \circ \Phi^m d\mathcal{W}_t^Q(x), \end{aligned} \tag{6.17}$$

where $b^m : [0,T] \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\Phi^m : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are given as

$$\begin{aligned} b^m(t,u) = -u_t + \sum_k \Bigg[&\int_{-L^m}^{L^m} w(\tfrac{k}{m} - y) F(u_t)(y) dy \\ &+ w_k^{m,+} F(u_t^{TW}(L^m)) + w_k^{m,-} F(u_t^{TW}(-L^m)) \Bigg] \mathbb{1}_{I_k^m}, \\ \Phi^m(u) = 2m &\sum_{k=-mL^m}^{mL^m-1} \langle u, \mathbb{1}_{J_k^m} \rangle \mathbb{1}_{I_k^m}, \end{aligned}$$

and where $\sigma^m : [0,T] \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is such that for $u \in \mathbb{R}^P$, $\sigma^m(t,\iota^m(u)) = \hat{\sigma}_k^m(t,u)$ on

I_k^m. We assume joint continuity and Lipschitz continuity in the second variable uniformly in m and $t \le T$, that is, there exists $L_\sigma > 0$ such that for $u_1, u_2 \in L^2(\mathbb{R})$ and $t \le T$,

$$\|\sigma^m(t,u_1) - \sigma^m(t,u_2)\| \le L_\sigma \|u_1 - u_2\|.$$

Proposition 6.5.2. *For any initial condition $u^0 \in L^2(\mathbb{R})$ there exists a unique strong L^2-valued solution u^m to (6.17). u^m admits a continuous modification. For any $p \ge 2$,*

$$E\Big(\sup_{t\le T}\|u_t^m\|^p\Big) < \infty.$$

Proof. Again we check that the drift and diffusion coefficients are Lipschitz continuous. Note that

$$\begin{aligned}
\sum_k \frac{1}{m} w(\tfrac{k}{m}, y) &= \int_{-L^m}^{L^m} w(x-y)dx + \sum_k \int_{\frac{k}{m}}^{\frac{k+1}{m}} w(\tfrac{k}{m}-y) - w(x-y)dx \\
&\le \|w\|_1 + \sum_k \int_{\frac{k}{m}}^{\frac{k+1}{m}} \int_{\frac{k}{m}}^{\frac{k+1}{m}} |w_x(z-y)|dzdx \\
&= 1 + \sum_k \frac{1}{m} \int_{\frac{k}{m}}^{\frac{k+1}{m}} |w_x(z-y)|dz \le 1 + \frac{1}{m}\|w_x\|_1.
\end{aligned} \tag{6.18}$$

Therefore, for $u_1, u_2 \in L^2(\mathbb{R})$,

$$\begin{aligned}
\|b^m(t,u_1) - b^m(t,u_2)\|_2^2 &\le 2\|u_1-u_2\|^2 + 2\|F'\|_\infty^2 \int_{-L^m}^{L^m} \sum_k \frac{1}{m} w(\tfrac{k}{m}-y)\left(u_1(y)-u_2(y)\right)^2 dy \\
&\le 2\|u_1-u_2\|^2 + 2\|F'\|_\infty^2 \Big(1 + \frac{1}{m}\|w_x\|_1\Big)\|u_1-u_2\|^2
\end{aligned}$$

and for an orthonormal basis (e_k) of $L^2(\mathbb{R})$ we obtain, using Parseval's identity,

$$\begin{aligned}
&\|\left(\sigma^m(t,u_1) - \sigma^m(t,u_2)\right) \circ \Phi^m\|_{L_2^0}^2 \\
&= \sum_k \Big\| \left(\sigma^m(t,u_1) - \sigma^m(t,u_2)\right) \sum_l 2m\langle e_k, \sqrt{Q}\mathbb{1}_{J_l^m}\rangle \mathbb{1}_{I_l^m} \Big\|^2 \\
&= \int \left(\sigma^m(t,u_1) - \sigma^m(t,u_2)\right)^2(x) \sum_l 4m^2 \int \Big(\int_{J_l^m} q(z,y)dy\Big)^2 dz\ \mathbb{1}_{I_l^m}(x)dx \\
&\le \int \left(\sigma^m(t,u_1) - \sigma^m(t,u_2)\right)^2(x) \sum_l 2m \int\int_{J_l^m} q(z,y)^2 dydz\ \mathbb{1}_{I_l^m}(x)dx \\
&= \int \left(\sigma^m(t,u_1) - \sigma^m(t,u_2)\right)^2(x) \sum_l 2m \int_{J_l^m} \|q(y,\cdot)\|^2 dy\ \mathbb{1}_{I_l^m}(x)dx \\
&\le \sup_x \|q(x,\cdot)\|^2 L_\sigma^2 \|u_1-u_2\|^2
\end{aligned}$$

□

6.5.4 Convergence

We are now able to state the main convergence result. We will need the following assumption on the kernel w: there exists $C_w > 0$ such that for $x \geq 0$,

$$\int_x^\infty w(y)dy \leq C_w w(x). \tag{6.19}$$

That assumption is satisfied for classical choices of w such as $w(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$ or $w(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$.

Theorem 6.5.3. *Fix $T > 0$. Let u and u^m be the solutions to (6.16) and (6.17), respectively. Assume that*

(i) $\sup_k \sup_{x\in I_k^m} \|2m\sqrt{Q}\mathbb{1}_{J_k^m} - q(x,\cdot)\| \xrightarrow{m\to\infty} 0,$

(ii) for any $u : [0,T] \to \mathcal{S}$ with $\sup_{t\leq T}\|u_t - \hat{u}\| < \infty$,

$$\sup_{t\leq T} \|\sigma^m(t, u_t\mathbb{1}_{(-L^m,L^m)}) - \sigma(t,u_t)\| \xrightarrow{m\to\infty} 0.$$

Then for any initial conditions $u_0^m \in L^2(\mathbb{R})$, $u_0 \in \mathcal{S}$ such that $\|u_0^m - u_0\|_{L^2((-L^m,L^m))} \xrightarrow{m\to\infty} 0$, and for all $p \geq 2$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|u_t^m - u_t\|^p_{L^2((-L^m,L^m))}\Big) \xrightarrow{m\to\infty} 0.$$

We postpone the proof to section 6.6.

Remark 6.5.4. Let $\epsilon > 0$. The kernel $q(x,y) = \frac{1}{2\epsilon}\mathbb{1}_{(x-\epsilon,x+\epsilon)}(y)$ satisfies assumption (i) of the theorem. Indeed, note that for x, z with $|x-z| \leq \frac{1}{m}$,
$|\{y : \mathbb{1}_{(x-\epsilon,x+\epsilon)}(y) \neq \mathbb{1}_{(z-\epsilon,z+\epsilon)}(y)\}| \leq |z-x| \leq \frac{1}{m}$. Therefore we obtain that for all k and for any $x \in I_k^m$,

$$\begin{aligned}
&\|2m\sqrt{Q}\mathbb{1}_{J_k^m} - q(x,\cdot)\|_2^2 = 4m^2 \int_{-\infty}^{\infty} \Big(\int_{J_k^m} q(z,y) - q(x,y)dz\Big)^2 dy \\
&\leq 2m \int_{-\infty}^{\infty} \int_{J_k^m} (q(z,y) - q(x,y))^2\, dz\; dy \\
&\leq 2\int_{J_k^m} \frac{1}{4\epsilon^2}dz \leq \frac{1}{4\epsilon^2 m} \xrightarrow{m\to\infty} 0.
\end{aligned}$$

We show that we can apply the theorem in the case of the fluctuations described in section 6.4.2. In order to ensure that the diffusion coefficients are in $L^2(\mathbb{R})$, we cut off the noise outside a compact set $\{\partial_x u_t^{TW} \geq \delta\}, \delta > 0$. Note that the neglected region moves with the wave such that we always retain the fluctuations in the relevant regime away from the fixed points.

Theorem 6.5.5. *Assume that the wave speed is strictly positive, $c > 0$. Fix $\delta > 0$. The diffusion coefficients as derived in Section 6.4.2,*

$$\begin{aligned}
\sigma(t,u) &= \frac{1}{\sqrt{N}}\Big(\alpha(t) + \beta(t)(u - u_t^{TW}) \\
&\qquad - \gamma(t) w * (F'(u_t^{TW})(u - u_t^{TW}))\Big)\mathbb{1}_{\{\partial_x u_t^{TW} \geq \delta\}}, \\
\sigma^m(t,u) &= \frac{1}{\sqrt{N}}\Big(\alpha^m(t) + \beta^m(t)(u - \pi^m(u_t^{TW})) \\
&\qquad - \gamma^m(t)\pi^m(w * (\pi^m(F'(u_t^{TW}))(u - \pi^m(u_t^{TW}))))\Big)\mathbb{1}_{\{\partial_x u_t^{TW} \geq \delta\}},
\end{aligned}$$

where

$$\begin{aligned}
\alpha(t) &= \sqrt{\frac{|-u_t^{TW} + w * F(u_t^{TW})|}{F'(u_t^{TW})}} = \sqrt{\frac{c\partial_x u_t^{TW}}{F'(u_t^{TW})}}, \\
\beta(t) &= \frac{1}{2\sqrt{c\partial_x u_t^{TW} F'(u_t^{TW})}}\left(-\frac{F''(u_t^{TW}(x))}{F'(u_t^{TW}(x))}c\partial_x u_t^{TW} + 1\right), \\
\gamma(t) &= \frac{1}{2\sqrt{c\partial_x u_t^{TW} F'(u_t^{TW})}}, \\
\alpha^m(t) &= \sqrt{\frac{-b^m(t, \pi^m(u_t^{TW}))}{\pi^m(F'(u_t^{TW}))}}\mathbb{1}_{[-L^m, L^m)}, \\
\beta^m(t) &= \frac{1}{2\sqrt{-b^m(t, \pi^m(u_t^{TW}))\pi^m(F'(u_t^{TW}))}} \\
&\qquad \times \left(-\frac{\pi^m(F''(u_t^{TW}))}{\pi^m(F'(u_t^{TW}))}(-b^m(t, \pi^m(u_t^{TW}))) + 1\right)\mathbb{1}_{[-L^m, L^m)}, \\
\gamma^m(t) &= \frac{1}{2\sqrt{-b^m(t, \pi^m(u_t^{TW}))\pi^m(F'(u_t^{TW}))}}\mathbb{1}_{[-L^m, L^m)},
\end{aligned}$$

are jointly continuous and Lipschitz continuous in the second variable with Lipschitz constant uniform in m and $t \leq T$, and satisfy condition (ii) of Theorem 6.5.3.

The proof is carried out in section 6.6.

6.6 Proofs of the Theorems

6.6.1 Proof of Theorem 6.5.3

Set $v_t = u_t - u_t^{TW}$ and $v_t^m = u_t^m - \pi^m(u_t^{TW})$. Note that

$$\begin{aligned}\int_{-L^m}^{L^m} (\pi^m(u_t^{TW})(x) - u_t^{TW}(x))^2 dx &\leq \sum_k \int_{\frac{k}{m}}^{\frac{k+1}{m}} \Big(\int_{\frac{k}{m}}^{\frac{k+1}{m}} \partial_x u_t^{TW}(z)dz\Big)^2 dx \\ &\leq \frac{1}{m^2}\sum_k \int_{\frac{k}{m}}^{\frac{k+1}{m}} \big(\partial_x u_t^{TW}(z)\big)^2 dz \\ &\leq \frac{1}{m^2}\|\hat{u}_x\|^2.\end{aligned} \tag{6.20}$$

We show that

$$E\Big(\sup_{t\leq T}\|v_t - v_t^m\|_2^p\Big) \xrightarrow{m\to\infty} 0,$$

then it follows that

$$\begin{aligned}&E\Big(\sup_{t\leq T}\|u_t - u_t^m\|^p_{L^2((-L^m,L^m))}\Big) \\ &\leq const \times \Big[E\Big(\sup_{t\leq T}\|u_t^{TW} - \pi^m(u_t^{TW})\|^p_{L^2((-L^m,L^m))}\Big) + E\Big(\sup_{t\leq T}\|v_t - v_t^m\|_2^p\Big)\Big] \xrightarrow{m\to\infty} 0.\end{aligned}$$

By Itô's formula,

$$\begin{aligned}&\frac{1}{2}d\|v_t^m - v_t\|_2^2 \\ &= \langle b^m(t, v_t^m + \pi^m(u_t^{TW})) - b(t, v_t + u_t^{TW}) - \pi^m(\partial_t u_t^{TW}) + \partial_t u_t^{TW}, v_t^m - v_t\rangle dt \\ &\quad + \frac{1}{2N}\Big\langle \pi^m\Big(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big) b^m(t,\pi^m(u_t^{TW})) - \frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\partial_t u_t^{TW}, v_t^m - v_t\Big\rangle dt \\ &\quad + \frac{1}{2}\|\sigma^m(t, v_t^m + \pi^m(u_t^{TW}))\circ\Phi^m - \sigma(t, v_t + u_t^{TW})\|^2_{L_2^0} \\ &\quad + \langle v_t^m - v_t, \big(\sigma^m(t, v_t^m + \pi^m(u_t^{TW}))\circ\Phi^m - \sigma(t, v_t + u_t^{TW})\big)\, d\mathcal{W}_t^Q\rangle.\end{aligned}$$

In order to finally use Gronwall's Lemma, we estimate the terms one by one.

The Drift

We start by regrouping the terms in a suitable way. We have

$$\begin{aligned}&\|v_t^m - v_t + b^m(t, v_t^m + \pi^m(u_t^{TW})) - b(t, v_t + u_t^{TW}) - \pi^m(\partial_t u_t^{TW}) + \partial_t u_t^{TW}\|^2 \\ &= \int_{-\infty}^{\infty}\Big[\sum_k \Big(\int_{-L^m}^{L^m} w(\tfrac{k}{m} - y)F(v_t^m(y) + \pi^m(u_t^{TW})(y))dy\end{aligned}$$

$$
\begin{aligned}
&+\int_{L^m}^{\infty} w(\tfrac{k}{m}-y)F(u_t^{TW}(L^m))dy+\int_{-\infty}^{-L^m} w(\tfrac{k}{m}-y)F(u_t^{TW}(-L^m))dy\Big)\mathbb{1}_{I_k^m}(x)\\
&-\int_{-\infty}^{\infty} w(x-y)F(v_t(y)+u_t^{TW}(y))dy-\pi^m(w*F(u_t^{TW}))+w*F(u_t^{TW})(x)\Big]^2dx\\
\le\ & 6\int_{-\infty}^{\infty}\Big[\sum_k\int_{-L^m}^{L^m} w(\tfrac{k}{m}-y)\Big(F(v_t^m(y)+\pi^m(u_t^{TW})(y))-F(v_t(y)+u_t^{TW}(y))\Big)dy\mathbb{1}_{I_k^m}(x)\Big]^2\\
&+\Big[\sum_k\int_{-L^m}^{L^m}\big(w(\tfrac{k}{m}-y)-w(x-y)\big)\big(F(v_t(y)+u_t^{TW}(y))-F(u_t^{TW}(y))\big)dy\mathbb{1}_{I_k^m}(x)\Big]^2\\
&+\Big[\sum_k\int_{L^m}^{\infty} w(\tfrac{k}{m}-y)\big(F(u_t^{TW}(L^m))-F(u_t^{TW}(y))\big)dy\mathbb{1}_{I_k^m}(x)\Big]^2\\
&+\Big[\sum_k\int_{-\infty}^{-L^m} w(\tfrac{k}{m}-y)\big(F(u_t^{TW}(-L^m))-F(u_t^{TW}(y))\big)dy\mathbb{1}_{I_k^m}(x)\Big]^2\\
&+\Big[\Big(\int_{L^m}^{\infty}+\int_{-\infty}^{-L^m}\Big)w(x-y)\big(F(v_t(y)+u_t^{TW}(y))-F(u_t^{TW}(y))\big)dy\Big]^2\\
&+\Big[\int_{-L^m}^{L^m} w(x-y)\big(F(v_t(y)+u_t^{TW}(y))-F(u_t^{TW}(y))\big)dy\mathbb{1}_{(-\infty,-L^m)\cup[L^m,\infty)}(x)\Big]^2dx\\
=:\ & 6(S_1+S_2+S_3+S_4+S_5+S_6).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
S_1 &\le \|F'\|_\infty^2\int_{-L^m}^{L^m}\sum_k\frac{1}{m}w(\tfrac{k}{m}-y)\big(v_t^m(y)+\pi^m(u_t^{TW})(y)-v_t(y)-u_t^{TW}(y)\big)^2dy\\
&\overset{(6.18)}{\le} 2\Big(1+\frac{1}{m}\|w_x\|_1\Big)\|F'\|_\infty^2\Big(\|v_t^m-v_t\|_2^2+\int_{-L^m}^{L^m}\big(\pi^m(u_t^{TW})(y)-u_t^{TW}(y)\big)^2dy\Big).
\end{aligned}
$$

With (6.20) it follows that

$$
S_1\le 2\Big(1+\frac{1}{m}\|w_x\|_1\Big)\|F'\|_\infty^2\Big(\|v_t^m-v_t\|_2^2+\frac{1}{m^2}\|\hat{u}_x\|_2^2\Big).
$$

Another application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
S_2 &= \sum_k\int_{\frac{k}{m}}^{\frac{k+1}{m}}\Big(\int_{-L^m}^{L^m}\big(w(\tfrac{k}{m}-y)-w(x-y)\big)\\
&\qquad\big(F(v_t(y)+u_t^{TW}(y))-F(u_t^{TW}(y))\big)dy\Big)^2dx\\
&\le\sum_k\frac{1}{m}\Big(\int_{-L^m}^{L^m}\int_{\frac{k}{m}}^{\frac{k+1}{m}}|w_x(z-y)|dz\,\big(F(v_t(y)+u_t^{TW}(y))-F(u_t^{TW}(y))\big)dy\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_k \frac{1}{m} \int_{-L^m}^{L^m} \int_{\frac{k}{m}}^{\frac{k+1}{m}} |w_x(z-y)| dz dy \\
&\qquad \times \int_{-L^m}^{L^m} \int_{\frac{k}{m}}^{\frac{k+1}{m}} |w_x(z-y)| dz \left(F(v_t(y) + u_t^{TW}(y)) - F(u_t^{TW}(y))\right)^2 dy \\
&\leq \frac{1}{m^2} \|w_x\|_1^2 \|F'\|_\infty^2 \|v_t\|_2^2
\end{aligned}
$$

Using integration by parts, (6.18), and assumption (6.19), we obtain

$$
\begin{aligned}
S_3 = \sum_k \frac{1}{m} \Bigg(& \underbrace{\left[-\int_y^\infty w(z - \tfrac{k}{m}) dz \left(F(u_t^{TW}(L^m)) - F(u_t^{TW}(y))\right) \right]_{y=L^m}^\infty}_{=0} \\
& - \int_{L^m}^\infty \int_y^\infty w(z - \tfrac{k}{m}) dz \; F'(u_t^{TW}(y)) \partial_x u_t^{TW}(y) dy \Bigg)^2 \\
\leq C_w^2 \sum_k \frac{1}{m} & \left(\int_{L^m}^\infty w(y - \tfrac{k}{m}) F'(u_t^{TW}(y)) \partial_x u_t^{TW}(y) dy \right)^2 \\
\leq C_w^2 \Big(1 + & \frac{1}{m} \|w_x\|_1 \Big) \|F'\|_\infty^2 \int_{L^m}^\infty (\partial_x u_t^{TW}(y))^2 dy.
\end{aligned}
$$

Analogously,

$$
S_4 \leq C_w^2 \Big(1 + \frac{1}{m} \|w_x\|_1 \Big) \|F'\|_\infty^2 \int_{-\infty}^{-L^m} (\partial_x u_t^{TW}(y))^2 dy.
$$

Last we observe that

$$
S_5 \leq \|F'\|_\infty^2 \left(\int_{L^m}^\infty + \int_{-\infty}^{-L^m} \right) v_t^2(y) dy.
$$

and

$$
\begin{aligned}
S_6 &\leq \|F'\|_\infty^2 \left(\int_{L^m}^\infty + \int_{-\infty}^{-L^m} \right) \left(\int_{-L^m}^{L^m} w(x-y) |v_t(y)| dy \right)^2 dx \\
&\leq \|F'\|_\infty^2 \left(\int_{L^m}^\infty + \int_{-\infty}^{-L^m} \right) (w * |v_t|(x))^2 dx.
\end{aligned}
$$

Finally we consider

$$
\begin{aligned}
S_7 &:= \left\| \frac{F''(u_t^{TW})}{F'(u_t^{TW})^2} \partial_t u_t^{TW} - \pi^m \Big(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2} \Big) b^m(t, \pi^m(u_t^{TW})) \right\|^2 \\
&\leq 4 \Bigg[\left(\int_{-\infty}^{-L^m} + \int_{L^m}^\infty \right) \left(\frac{F''(u_t^{TW})(y)}{F'(u_t^{TW})^2(y)} \partial_t u_t^{TW}(y) \right)^2 dy \\
&\qquad + \left\| \left(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2} - \pi^m \left(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2} \right) \right) \partial_t u_t^{TW} \mathbb{1}_{(-L^m, L^m)} \right\|^2 \\
&\qquad + \left\| \pi^m \Big(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2} \Big) (\partial_t u_t^{TW} - \pi^m(\partial_t u_t^{TW})) \mathbb{1}_{(-L^m, L^m)} \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\|\pi^m\Big(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big)\Big(\pi^m(w*F(u_t^{TW}))-\pi^m(w*\pi^m(F(u_t^{TW})))\\
&\qquad-\sum_k\big(w_k^{m,+}F(u_t^{TW}(L^m))+w_k^{m,-}F(u_t^{TW})(-L^m)\big)\mathbb{1}_{I_k^m}\Big)\Big\|^2\Big]\\
&=4(S_{7,1}+S_{7,2}+S_{7,3}+S_{7,4}).
\end{aligned}
$$

We have

$$
S_{7,2}\le\Big\|\frac{F^{(3)}(\hat u)\hat u_x}{(F'(\hat u))^2}-\frac{2(F''(\hat u)^2\hat u_x}{(F'(\hat u))^3}\Big\|_\infty^2\frac{1}{m^2}\|\partial_t u_t^{TW}\|^2
$$

and, as in (6.20),

$$
\begin{aligned}
S_{7,3}&\le\Big\|\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big\|_\infty^2\|(\partial_t u_t^{TW}-\pi^m(\partial_t u_t^{TW}))\mathbb{1}_{(-L^m,l^m)}\|^2\\
&\le\frac{1}{m^2}c^2\Big\|\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big\|_\infty^2\|\hat u_{xx}\|^2.
\end{aligned}
$$

The last summand satisfies, using (6.18) and (6.20),

$$
\begin{aligned}
S_{7,4}\le{}&3\Big\|\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big\|_\infty^2\\
&\Big[\sum_k\frac{1}{m}\Big(\int_{-L^m}^{L^m}w(\tfrac{k}{m}-y)(F(u_t^{TW}(y))-\pi^m(F(u_t^{TW}))(y))^2dy\Big)^2+S_3+S_4\Big]\\
\le{}&3\Big\|\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big\|_\infty^2(1+\frac{1}{m}\|w_x\|_1)\|F'\|_\infty^2\frac{1}{m^2}\|\hat u_x\|^2+S_3+S_4).
\end{aligned}
$$

The Itô Correction

$$
\begin{aligned}
&\|\sigma^m(t,v_t^m+\pi^m(u_t^{TW}))\circ\Phi^m-\sigma(t,v_t+u_t^{TW})\|_{L_2^0}^2\\
&\le3\Big(\|\big(\sigma^m(t,v_t^m+\pi^m(u_t^{TW}))-\sigma^m(t,(v_t+u_t^{TW})\mathbb{1}_{(-L^m,L^m)})\big)\circ\Phi^m\|_{L_2^0}^2\\
&\quad+\|\big(\sigma^m(t,(v_t+u_t^{TW})\mathbb{1}_{(-L^m,L^m)})-\sigma(t,v_t+u_t^{TW})\big)\circ\Phi^m\|_{L_2^0}^2\\
&\quad+\|\sigma(t,v_t+u_t^{TW})\circ\Phi^m-\sigma(t,v_t+u_t^{TW})\|_{L_2^0}^2\Big)\\
&=:3(S_8+S_9+S_{10}).
\end{aligned}
$$

Let (e_k) be an orthonormal basis of $L^2(\mathbb{R})$. Note that by Parseval's identity

$$
\begin{aligned}
&\sum_k\Big(\Phi^m(\sqrt{Q}e_k)\Big)^2\\
&=\sum_k\Big(\sum_l 2m\langle\sqrt{Q}e_k,\mathbb{1}_{J_l^m}\rangle\mathbb{1}_{I_l^m}\Big)^2=\sum_l\sum_k 4m^2\langle\sqrt{Q}\mathbb{1}_{J_l^m},e_k\rangle^2\mathbb{1}_{I_l^m}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_l 4m^2 \|\sqrt{Q}\mathbb{1}_{J_l^m}\|^2 \mathbb{1}_{I_l^m} = \sum_l 4m^2 \int \left(\int_{J_l^m} q(x,y)dy \right)^2 dx \ \mathbb{1}_{I_l^m} \\
&\leq \sum_l 2m \int \int_{J_l^m} q^2(x,y)dydx \ \mathbb{1}_{I_l^m} \leq \sup_x \|q(x,\cdot)\|^2 \mathbb{1}_{[-L^m,L^m)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_8 &= \sum_k \int_{-\infty}^{\infty} \Big(\sigma^m(t, v_t^m + \pi^m(u_t^{TW})) \\
&\qquad - \sigma^m(t, (v_t + u_t^{TW})\mathbb{1}_{(-L^m,L^m)}) \Big)^2 (x) \left(\Phi^m(\sqrt{Q}e_k) \right)^2 (x)dx \\
&\leq \sup_x \|q(x,\cdot)\|^2 \ L_\sigma^2 \|v_t^m + \pi^m(u_t^{TW}) - (v_t + u_t^{TW})\mathbb{1}_{(-L^m,L^m)}\|^2 \\
&\overset{(6.20)}{\leq} 2 \sup_x \|q(x,\cdot)\|^2 \ L_\sigma^2 \Big(\|v_t^m - v_t\|^2 + \frac{1}{m^2} \|\hat{u}_x\|^2 \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
S_9 &= \| \left(\sigma^m(t, (v_t + u_t^{TW})\mathbb{1}_{(-L^m,L^m)}) - \sigma(t, v_t + u_t^{TW}) \right) \circ \Phi^m \|_{L_2^0}^2 \\
&\leq \sup_x \|q(x,\cdot)\|^2 \|\sigma^m(t, (v_t + u_t^{TW})\mathbb{1}_{(-L^m,L^m)}) - \sigma(t, v_t + u_t^{TW})\|^2.
\end{aligned}
$$

Using Parseval's identity again we get

$$
\begin{aligned}
S_{10} &= \int_{-\infty}^{\infty} \sigma(t, v_t + u_t^{TW})^2(x) \sum_k \left(\sum_l 2m \langle \sqrt{Q}e_k, \mathbb{1}_{J_l^m} \rangle \mathbb{1}_{I_l^m}(x) - \sqrt{Q}e_k(x) \right)^2 dx \\
&= \sum_l \int_{\frac{l}{m}}^{\frac{l+1}{m}} \sigma(t, v_t + u_t^{TW})^2(x) \sum_k \left(2m \int_{J_l^m} \int_{-\infty}^{\infty} (q(z,y) - q(x,y))e_k(y)dydz \right)^2 dx \\
&\quad + \left(\int_{-\infty}^{-L^m} + \int_{L^m}^{\infty} \right) \sigma(t, v_t + u_t^{TW})^2(x) \sum_k (\sqrt{Q}e_k(x))^2 dx \\
&\leq \sum_l \int_{\frac{l}{m}}^{\frac{l+1}{m}} \sigma(t, v_t + u_t^{TW})^2(x) \left\| 2m \int_{J_l^m} (q(z,\cdot) - q(x,\cdot))dz \right\|^2 dx \\
&\quad + \left(\int_{-\infty}^{-L^m} + \int_{L^m}^{\infty} \right) \sigma(t, v_t + u_t^{TW})^2(x) \sum_k \left(\int_{-\infty}^{\infty} q(x,y)e_k(y)dy \right)^2 dx \\
&\leq \|\sigma(t, v_t + u_t^{TW})\|^2 \sup_l \sup_{x \in I_l^m} \left\| 2m \int_{J_l^m} q(z,\cdot)dz - q(x,\cdot) \right\|^2 \\
&\quad + \sup_x \|q(x,\cdot)\|^2 \left(\int_{-\infty}^{-L^m} + \int_{L^m}^{\infty} \right) \sigma(t, v_t + u_t^{TW})^2(x)dx.
\end{aligned}
$$

Application of Gronwall's Lemma

We use $K, K_1, K_2, \tilde{K}$, etc. to denote suitable constants that may differ from step to step. Summarizing the previous steps and using Young's inequality we arrive at

$$
\begin{aligned}
&\frac{1}{2} d\|v_t^m - v_t\|^2 \\
&\leq \Big[- \|v_t^m - v_t\|^2 + \frac{1}{2}\|v_t^m - v_t\|^2 \\
&\quad + \frac{1}{2}\Big\| v_t^m - v_t + b^m(t, v_t^m + \pi^m(u_t^{TW})) - b(t, v_t + u_t^{TW}) - \pi^m(\partial_t u_t^{TW}) + \partial_t u_t^{TW} \Big\|_2^2 \\
&\quad + \frac{1}{4N}\|v_t^m - v_t\|^2 + \frac{1}{4N}\Big\| \frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\partial_t u_t^{TW} - \pi^m\Big(\frac{F''(u_t^{TW})}{F'(u_t^{TW})^2}\Big) b^m(t, \pi^m(u_t^{TW}))\Big\|^2 \Big] dt \\
&\quad + \frac{1}{2}\|\sigma^m(t, v_t^m + \pi^m(u_t^{TW})) \circ \Phi^m - \sigma(t, v_t + u_t^{TW})\|_{L_2^0}^2 dt \\
&\quad + \langle v_t^m - v_t, \big(\sigma^m(t, v_t^m + \pi^m(u_t^{TW})) \circ \Phi^m - \sigma(t, v_t + u_t^{TW})\big)\, d\mathcal{W}_t^Q\rangle \\
&\leq K_1\|v_t^m - v_t\|^2 dt + K_2 R(t, v_t, m) dt + dM_t,
\end{aligned}
$$

where

$$
\begin{aligned}
R(t, v_t, m) = &\frac{1}{m^2}\left(\|v_t\|^2 + \|\hat{u}_x\|^2 + \|\hat{u}_{xx}\|^2\right) + \left(\int_{L^m}^{\infty} + \int_{-\infty}^{-L^m}\right)\Big(v_t^2(x) + (w * |v_t|)^2(x) \\
&+ \sigma(t, v_t + u_t^{TW})^2(x) + (\partial_x u_t^{TW}(x))^2\Big) dx \\
&+ \|\sigma(t, v_t + u_t^{TW})\|^2 \sup_k \sup_{x \in I_k^m} \|2m\sqrt{Q}\mathbb{1}_{J_k^m} - q(x, \cdot)\|^2 \\
&+ \|\sigma^m(t, (v_t + u_t^{TW})\mathbb{1}_{(-L^m, L^m)}) - \sigma(t, v_t + u_t^{TW})\|^2
\end{aligned}
$$

and

$$
M_t = \int_0^t \langle v_s^m - v_s, \big(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m - \sigma(s, v_s + u_s^{TW})\big)\, d\mathcal{W}_s^Q\rangle
$$

is a martingale with quadratic variation process

$$
\begin{aligned}
[M]_t = &\int_0^t \sum_k \langle v_s^m - v_s, \Big(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m \\
&- \sigma(s, v_s + u_s^{TW})\Big) \circ \sqrt{Q} e_k\rangle^2 ds \\
\leq &\int_0^t \|v_s^m - v_s\|^2 \Big\| \Big(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m \\
&- \sigma(s, v_s + u_s^{TW})\Big)\Big\|_{L_2^0}^2 ds.
\end{aligned}
\tag{6.21}
$$

Applying Itô's formula to the real-valued stochastic process $\|v_t^m - v_t\|^2$ we obtain for $p \geq 2$

$$\begin{aligned}
&d\|v_t^m - v_t\|^p \\
&= \frac{p}{2}\|v_t^m - v_t\|^{p-2} d\|v_t^m - v_t\|^2 + \frac{p(p-2)}{8}\|v_t^m - v_t\|^{p-4} d\left[\|v_t^m - v_t\|^2\right]_t \\
&\overset{(6.21)}{\leq} K_1 p\|v_t^m - v_t\|^p dt + K_2 p\|v_t^m - v_t\|^{p-2} R(t, v_t, m)dt + p\|v_t^m - v_t\|^{p-2} dM_t \\
&\quad + \frac{p(p-2)}{2}\|v_t^m - v_t\|^{p-2}\|\left(\sigma^m(t, v_t^m + \pi^m(u_t^{TW})) \circ \Phi^m - \sigma(t, v_t + u_t^{TW})\right)\|_{L_2^0}^2 dt.
\end{aligned}$$

Estimating the last term as above and using Young's inequality we obtain

$$\begin{aligned}
&d\|v_t^m - v_t\|^p \\
&\leq \tilde{K_1}\|v_t^m - v_t\|^p dt + \tilde{K_2}\|v_t^m - v_t\|^{p-2} R(t, v_t, m)dt + p\|v_t^m - v_t\|^{p-2} dM_t \\
&\leq \left(\tilde{K_1} + \tilde{K_2}\tfrac{p-2}{p}\right)\|v_t^m - v_t\|^p dt + \tilde{K_2}\tfrac{2}{p} R(t, v_t, m)^{\frac{p}{2}} dt + p\|v_t^m - v_t\|^{p-2} dM_t.
\end{aligned}$$

Integrating, maximizing over $t \leq T$, and taking expectations we get

$$\begin{aligned}
&E\left(\sup_{t\leq T}\|v_t^m - v_t\|^p\right) \\
&\leq \|v_0^m - v_0\|^p + \left(\tilde{K_1} + \tilde{K_2}\frac{p-2}{p}\right) E\left(\sup_{t\leq T}\int_0^t \|v_s^m - v_s\|^p ds\right) \\
&\quad + \tilde{K_2}\frac{2}{p} E\left(\sup_{t\leq T}\int_0^t R(s, v_s, m)^{\frac{p}{2}} ds\right) + pE\left(\sup_{t\leq T}\int_0^t \|v_s^m - v_s\|^{p-2} dM_s\right) \qquad (6.22) \\
&\leq \|v_0^m - v_0\|^p + \left(\tilde{K_1} + \tilde{K_2}\frac{p-2}{p}\right)\int_0^T E\left(\sup_{s\leq t}\|v_s^m - v_s\|^p\right) dt \\
&\quad + \tilde{K_2}\frac{2}{p} E\left(\int_0^T R(s, v_s, m)^{\frac{p}{2}} ds\right) + pE\left(\sup_{t\leq T}\int_0^t \|v_s^m - v_s\|^{p-2} dM_s\right).
\end{aligned}$$

We estimate the last term using the Burkholder-Davis-Gundy inequality, (6.21), and Young's inequality:

$$\begin{aligned}
&pE\left(\sup_{t\leq T}\int_0^t \|v_s^m - v_s\|^{p-2} dM_s\right) \\
&\leq KE\left(\int_0^T \|v_s^m - v_s\|_2^{2p-2}\|\left(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m - \sigma(s, v_s + u_s^{TW})\right)\|_{L_2^0}^2 ds\right)^{\frac{1}{2}} \\
&\leq KE\Bigg(\sup_{t\leq T}\|v_t^m - v_t\|^{p-1} \\
&\qquad \left(\int_0^T \|\left(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m - \sigma(s, v_s + u_s^{TW})\right)\|_{L_2^0}^2 ds\right)^{\frac{1}{2}}\Bigg) \\
&\leq \frac{1}{2}E\left(\sup_{t\leq T}\|v_t^m - v_t\|^p\right)
\end{aligned}$$

$$+ \tilde{K}_3 E \left(\int_0^T \| \left(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m - \sigma(s, v_s + u_s^{TW}) \right) \|_{L_2^0}^2 ds \right)^{\frac{p}{2}}.$$

Bringing the first summand to the left-hand side of (6.22) this implies that

$$\begin{aligned} & E\left(\sup_{t\le T} \|v_t^m - v_t\|^p\right) \\ & \le 2\|v_0^m - v_0\|^p + 2\left(\tilde{K}_1 + \tilde{K}_2 \frac{p-2}{p}\right) \int_0^T \ E\left(\sup_{s\le t} \|v_s^m - v_s\|^p\right) dt \\ & \quad + \tilde{K}_2 \frac{4}{p} E\left(\int_0^T R(s, v_s, m)^{\frac{p}{2}} ds\right) \\ & \quad + 2\tilde{K}_3 E\left(\int_0^T \| \left(\sigma^m(t, v_t^m + \pi^m(u_t^{TW})) \circ \Phi^m - \sigma(t, v_t + u_t^{TW})\right) \|_{L_2^0}^2 ds\right)^{\frac{p}{2}}. \end{aligned}$$

We estimate the last term as before and obtain

$$\begin{aligned} & E\left(\int_0^T \| \left(\sigma^m(s, v_s^m + \pi^m(u_s^{TW})) \circ \Phi^m - \sigma(s, v_s + u_s^{TW})\right) \|_{L_2^0}^2 ds\right)^{\frac{p}{2}} \\ & \le KE\left(\int_0^T \|v_s^m - v_s\|^2 + R(s, v_s, m) ds\right)^{\frac{p}{2}} \\ & \le \tilde{K}E\left(\int_0^T \sup_{s\le t} \|v_s^m - v_s\|^p dt + \int_0^T R(s, v_s, m)^{\frac{p}{2}} ds\right). \end{aligned}$$

Altogether we arrive at

$$\begin{aligned} E\left(\sup_{t\le T} \|v_t^m - v_t\|^p\right) \le 2\|v_0^m - v_0\|_2^p + \hat{K}_1 E \int_0^T R(t, v_t, m)^{\frac{p}{2}} dt \\ + \hat{K}_2 \int_0^T E\left(\sup_{s\le t} \|v_s^m - v_s\|^p\right) dt. \end{aligned}$$

An application of Gronwall's Lemma yields

$$E\left(\sup_{t\le T} \|v_t^m - v_t\|^p\right) \le K\left(\|v_0^m - v_0\|^p + E\left(\sup_{t\le T} R(t, v_t, m)^{\frac{p}{2}}\right)\right).$$

The sequence of continuous functions $f^m : [0,T] \to \mathbb{R}$,

$$f^m(t) = \left(\left(\int_{L^m}^{\infty} + \int_{-\infty}^{-L^m} \right) v_t^2(x) + (w * |v_t|)^2(x) + \sigma(t, v_t + u_t^{TW})^2(x) + (\partial_x u_t^{TW}(x))^2 dx \right)^{\frac{p}{2}}$$

is decreasing and converges pointwise to 0 since all the integrands are in $L^2(\mathbb{R})$. By Dini's Theorem the convergence is uniform. This together with the facts that

$\|\sigma(t,v_t)\|_2^2 \leq K(1+\|v_t\|^2)$ and $E\left(\sup_{t\leq T}\|v_t\|^2\right) < \infty$ by Proposition 6.5.1, assumptions (i) and (ii), and dominated convergence implies that

$$
\begin{aligned}
&E\left(\sup_{t\leq T} R(t,v_t,m)^{\frac{p}{2}}\right)\\
&\leq K\Bigg(\frac{1}{m^p}\left(E\left(\sup_{t\leq T}\|v_t\|^p\right)+\|\hat{u}_x\|^p+\|\hat{u}_{xx}\|^p\right)+E\left(\sup_{t\leq T} f^m(t)\right)\\
&\quad + E\left(\sup_{t\leq T}\|\sigma^m(t,(v_t+u_t^{TW})\mathbb{1}_{(-L^m,L^m)})\circ\Phi^m-\sigma(t,v_t+u_t^{TW})\|^p\right)\\
&\quad + E\left(\sup_{t\leq T}\|\sigma(t,v_t+u_t^{TW})\|^p\right)\sup_k\sup_{x\in I_k^m}\|2m\sqrt{Q}\mathbb{1}_{J_k^m}-q(x,\cdot)\|^p\Bigg) \xrightarrow{m\to\infty} 0
\end{aligned}
$$

and hence

$$
E\left(\sup_{t\leq T}\|v_t^m-v_t\|^p\right)\xrightarrow{m\to\infty} 0.
$$

6.6.2 Proof of Theorem 6.5.5

The joint continuity follows essentially from the continuity of $t\mapsto u_t^{TW}$. Note that because of (6.12) there exists $0<\delta'<\delta$ such that $-b^m(t,\pi^m(u_t^{TW}))\geq c\delta'$ on $\{\partial_x u_t^{TW}\geq\delta\}$. To see the Lipschitz continuity, observe that for $u_1,u_2\in\mathcal{S}$,

$$
\begin{aligned}
&\|\sigma(t,u_1)-\sigma(t,u_2)\|_2^2\\
&\leq \frac{2}{N}\|\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\Bigg(\|\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\beta(t)\|_\infty^2\|u_1-u_2\|_2^2\\
&\qquad + \|\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\gamma(t)\|_\infty^2\int (w*(F'(u_t^{TW})(u_1-u_2)))^2(x)dx\Bigg)\\
&\leq \frac{1}{2Nc\delta\min_{[0,1]}F'}\left(\left(\|\hat{u}\|_\infty\max_{[0,1]}\left|\frac{F''}{F'}\right|+1\right)^2+\|F'\|_\infty^2\right)\|u_1-u_2\|_2^2,
\end{aligned}
$$

and similarly for $u_1,u_2\in L^2(\mathbb{R})$,

$$
\begin{aligned}
&\|\sigma^m(t,u_1)-\sigma^m(t,u_2)\|_2^2\\
&\leq \frac{1}{2Nc\delta'\min_{[0,1]}F'}\left(\left(\|\hat{u}\|_\infty\max_{[0,1]}\left|\frac{F''}{F'}\right|+1\right)^2+(1+\frac{1}{m}\|w_x\|_1)\|F'\|_\infty^2\right)\|u_1-u_2\|_2^2.
\end{aligned}
$$

We now show that the convergence property (ii) in Theorem 6.5.3 holds. Note first that for $u\in\mathcal{C}^1$ with $\|u_x\|_\infty<\infty$,

$$
\begin{aligned}
&\|(u-\pi^m(u))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty\\
&\leq \sup_x\left|\sum_k\int_{\frac{k}{m}}^x u_x(y)dy\mathbb{1}_{I_k^m}(x)\right|+\sup_{x\notin[-L^m,L^m)}|u(x)\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}|
\end{aligned}
$$

$$\leq \frac{1}{m}\|u_x\|_\infty + \sup_{x\notin[-L^m,L^m]} |u(x)\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}| \xrightarrow{m\to\infty} 0.$$

Thus,

$$\begin{aligned}
&\|(c\partial_x u_t^{TW} - b^m(t,\pi^m(u_t^{TW})))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty \\
&\leq \|u_t^{TW} - \pi^m(u_t^{TW})\|_\infty \\
&\quad + \sup_k \int_{-L^m}^{L^m} w(\tfrac{k}{m},y)|\pi^m(F(u_t^{TW}))(y) - F(u_t^{TW}(y))|dy \\
&\quad + \sup_k \int_{L^m}^{\infty} w(\tfrac{k}{m},y)|F(u_t^{TW}(L^m)) - F(u_t^{TW}(y))|dy \\
&\quad + \sup_k \int_{-\infty}^{-L^m} w(\tfrac{k}{m},y)|F(u_t^{TW}(L^m)) - F(u_t^{TW}(y))|dy \\
&\quad + \|(w*F(u_t^{TW}) - \pi^m(w*F(u_t^{TW})))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty \\
&\leq \|u_t^{TW} - \pi^m(u_t^{TW})\|_\infty + \|F'\hat{u}_x\|_\infty\frac{1}{m} + |a_2 - F((u_t^{TW}(L^m))| + |a_1 - F(u_t^{TW}(-L^m))| \\
&\quad + \|(w*F(u_t^{TW}) - \pi^m(w*F(u_t^{TW})))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty \xrightarrow{m\to\infty} 0.
\end{aligned}$$

The convergence is uniform in $t \leq T$ in both cases.
Using this one can show that

$$\begin{aligned}
&\sup_{t\leq T}\|(\alpha(t)-\alpha^m(t))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty + \sup_{t\leq T}\|(\beta(t)-\beta^m(t))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty \\
&\qquad + \sup_{t\leq T}\|(\gamma(t)-\gamma^m(t))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty \xrightarrow{m\to\infty} 0.
\end{aligned}$$

It follows that for $u : [0,T] \to \mathcal{S}$ with $\sup_{t\leq T}\|u_t - \hat{u}\| < \infty$,

$$\begin{aligned}
&\sup_{t\leq T}\|\sigma(t,u_t) - \sigma^m(t,u_t\mathbb{1}_{(-L^m,L^m)})\|^2 \\
&\leq \frac{5}{N}\Bigg(\sup_{t\leq T}\|(\alpha(t)-\alpha^m(t))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty^2 |\{\hat{u}_x\geq\delta\}| \\
&\quad + \sup_{t\leq T}\|(\beta(t)-\beta^m(t))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty^2 \sup_{t\leq T}\|u_t\|^2 \\
&\quad + \sup_{t\leq T}\|(\gamma(t)-\gamma^m(t))\mathbb{1}_{\{\partial_x u_t^{TW}\geq\delta\}}\|_\infty^2 \sup_{t\leq T}\|w*(F'(u_t^{TW})(u_t-u_t^{TW}))\|^2 \\
&\quad + \frac{1}{2c\delta'\min_{[0,1]}F'}\Bigg(\sup_{t\leq T}\Big\|\big(w*(F'(u_t^{TW})(u_t-u_t^{TW})) - \pi^m(w*(F'(u_t^{TW})(u_t-u_t^{TW})))\big)\Big\|^2 \\
&\quad + \sup_{t\leq T}\Big\|\pi^m(w*(F'(u_t^{TW})(u_t-u_t^{TW}))) - \pi^m(w*(\pi^m(F'(u_t^{TW}))(u_t-\pi^m(u_t^{TW}))))\Big\|^2\Bigg)\Bigg).
\end{aligned}$$

We have already shown that all but the last two summands converge to 0 as $m \to \infty$. We can show as in the proof of Theorem 6.5.3 that

$$\begin{aligned}
&\Big\| w * (F'(u_t^{TW})(u_t - u_t^{TW})) - \pi^m(w * (F'(u_t^{TW})(u_t - u_t^{TW})))\Big\|^2 \\
&\leq \frac{1}{m^2}\|w_x\|_1^2 \|F'\|_\infty^2 \|u - u_t^{TW}\|^2 + \left(\int_{-\infty}^{-L^m} + \int_{L^m}^\infty\right)(w * (F'(u_t^{TW})(u_t - u_t^{TW}))^2(x)dx,
\end{aligned}$$

and using (6.18) we obtain that

$$\begin{aligned}
&\|\pi^m(w * (F'(u_t^{TW})(u_t - u_t^{TW}))) - \pi^m(w * (\pi^m(F'(u_t^{TW}))(u_t - \pi^m(u_t^{TW}))))\|^2 \\
&\leq 3\Bigg(\sum_k \frac{1}{m}\Bigg(\sum_l \int_{\frac{l}{m}}^{\frac{l+1}{m}} w(\tfrac{k}{m} - y)\left(F'(u_t^{TW}(y)) - F'(u_t^{TW}(\tfrac{l}{m}))\right)(u_t(y) - u_t^{TW}(y))dy\Bigg)^2 \\
&\quad + \sum_k \frac{1}{m}\Bigg(\sum_l \int_{\frac{l}{m}}^{\frac{l+1}{m}} w(\tfrac{k}{m} - y)F'(u_t^{TW}(\tfrac{l}{m}))(u_t^{TW}(y) - \pi^m(u_t^{TW})(y))dy\Bigg)^2 \\
&\quad + \sum_k \frac{1}{m}\Bigg(\Bigg(\int_{-\infty}^{-L^m} + \int_{L^m}^\infty\Bigg)w(\tfrac{k}{m} - y)F'(u_t^{TW}(y))(u_t(y) - u_t^{TW}(y))dy\Bigg)^2\Bigg) \\
&\leq 3\Bigg(\frac{1}{m^2}(1 + \frac{1}{m}\|w_x\|_1)\|F''\hat{u}_x\|_\infty^2\|u_t - u_t^{TW}\|^2 \\
&\quad + (1 + \frac{1}{m}\|w_x\|_1)\|F'\|_\infty^2\|u_t^{TW} - \pi^m(u_t^{TW})\|_{L^2((-L^m,L^m))}^2 \\
&\quad + \|F'\|_\infty^2(1 + \frac{1}{m}\|w_x\|_1)\Bigg(\int_{-\infty}^{-L^m} + \int_{L^m}^\infty\Bigg)(u_t - u_t^{TW})^2(y)dy\Bigg).
\end{aligned}$$

Now $\sup_{t\leq T}\|u_t^{TW} - \pi^m(u_t^{TW})\|_{L^2((-L^m,L^m))}^2 \xrightarrow{m\to\infty} 0$ by (6.20). The sequence of functions

$$f^m(t) = \left(\int_{-\infty}^{-L^m} + \int_{L^m}^\infty\right)(u_t - u_t^{TW})^2(y) + (w * (F'(u_t^{TW})(u_t - u_t^{TW}))^2(y)\ dy$$

is decreasing in m and converges pointwise to 0 since $u_t - u_t^{TW}$ and $w * (F'(u_t^{TW})(u_t - u_t^{TW}))$ are in $L^2(\mathbb{R})$. The convergence is uniform by Dini's Theorem. It follows that

$$\sup_{t\leq T}\|\sigma(t, u_t) - \sigma^m(t, u_t\mathbb{1}_{(-L^m,L^m)})\|^2 \xrightarrow{m\to\infty} 0.$$

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