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On convergence of an instantaneous control type method for parabolic boundary control

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Abstract

A method of instantaneous control type is considered for a simplified linear-quadratic parabolic boundary control problem with bound constraints on the control. The performance of the method is examined numerically and confirmed by an associated analysis. In particular, the method is shown to converge to a unique fixed point. In the same way, the receding horizon technique is investigated, and a new hybrid method is suggested.

1 Introduction

In this paper, we discuss a suboptimal strategy of instantaneous control type for the following simplified class of parabolic boundary control problems with bound constraints on the control,

$$\text{(P)} \qquad \min J(u) = \frac{1}{2}\int_0^1 (y(x,T) - y_d(x))^2 dx + \frac{\nu}{2}\int_0^T u(t)^2 dt$$

subject to

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= y_0(x) \\ y_x(0,t) &= 0 \\ y_x(1,t) &= \alpha\,(u(t) - y(1,t)), \end{aligned} \tag{1.1}$$

$x \in (0,1)$, $t \in (0,T)$, and subject to the bound constraints

$$u_a \le u(t) \le u_b \tag{1.2}$$

to be fulfilled a.e. on $[0,T]$. The control u is taken from $L^\infty(0,T)$. In this setting, T, ν, α are fixed positive constants, while $u_a < u_b$ are given real numbers. Moreover, y_d and y_0 are given in $L^2(0,1)$.

Problems of this type were frequently discussed in literature. It is easy to show that (P) admits a unique optimal control $\bar{u}$. Necessary and (by convexity) sufficient optimality conditions were derived already years ago, see the references in [14]. Thanks to the low dimension one of the spatial domain $\Omega = (0,1)$, (P) can be easily solved numerically by various methods. It can be fully discretized and hereafter solved as a finite-dimensional quadratic programming problem, [7]. An alternative strategy is to work with different projection methods [11], [12], or to apply active set strategies such as the Bertsekas projection method [3], [7] or primal-dual active set strategies [2]. Today, it is by far not a challenge to solve (P) numerically. On the other hand, its simplicity was the reason that we became interested in the associated numerics again. The problem is a good candidate to study analysis and performance of instantaneous control type methods.

Techniques of this type turned out to be efficient suboptimal strategies to solve very large scale optimal control problems. Their origin is the control of flows described by the instationary Navier-Stokes equations, where it is rather hopeless to try an accurate optimal solution in a reasonable time. We refer to [4], [5], or [8]. In a number of recent papers, the advantages of this method were demonstrated again, see [8], [9], [10]. Encouraged by this success, we applied a method of instantaneous control type to a problem of cooling steel with linear objective functional, nonlinear equation, and constraints on control and state. We were able to drastically reduce the computing time with almost no loss of accuracy in comparison with an exact optimization [15]. Why these methods are so powerful? Are we able to develop a convergence analysis?

It is known that the standard method of instantaneous control may fail for the case of a terminal time functional. This is, what our first computations confirmed:

We applied the following simple suboptimal strategy of instantaneous control type to (P): Split the interval $I = [0, T]$ into n small subintervals $I_j = [t_{j-1}, t_j]$, $t_j = j\,T/n$, and take piecewise constant controls $u(t) = u_j$ on I_j.

The objective of (P) is to approximate y_d as close as possible in the L^2-norm at the final time T. Therefore, it might be natural to first choose a real control value u_1 such that $\|y(\cdot, t_1) - y_d\|$ is minimized on the first subinterval I_1, then - starting from $y(\cdot, t_1)$ - to select u_2 on I_2 such that $\|y(\cdot, t_2) - y_d\|$ is minimal etc. The idea behind reflects part of our experience in daily life. Aiming to reach a target, we try to approach it in each step. However, this simple strategy exhibits a weak performance, which was experienced also by other authors.

In this paper, we first present a mathematical proof, why the standard method of instantaneous control will most likely fail. We show that it converges to a unique fixed point, which might be far from the optimum. While this result explains the weak performance of the standard method, it also serves as the background to develop a much better method. This improved method is the main result of our paper.

It is organized as follows: First, we discuss the convergence analysis. Next we present numerical results of the standard instantaneous control technique described above and explain, why it cannot behave better. Finally we report on other methods, which improve the performance and still are very fast.

2 Analysis of the standard instantaneous control method

Let us first express problem (P) in a shorter setting of functional analysis. We define

$$U_{ad} = \{u \in L^2(0, T) \mid u_a \leq u(t) \leq u_b \ \text{ a.e. on } [0, T]\}.$$

For each $u \in U_{ad}$, there exists a unique weak solution y solving the equations (1.1). This is the *state* associated with u. Let $G = G(x, \xi, t)$ denote the Green's function to (1.1). It is known that y is a weak solution to (1.1) iff

$$y(x,t) = \int_0^1 G(x,\xi,t) y_0(\xi)\, d\xi + \int_0^t G(x,1,t-s)\alpha\, u(s)\, ds. \tag{2.1}$$

As mentioned in the introduction, we split $[0,T]$ into n subintervals of uniform length $\tau = T/n$. Define on $[0,\tau]$ linear and continuous operators $D_\tau : L^2(0,1) \to L^2(0,1)$ and $S_\tau : L^2(0,\tau) \to L^2(0,1)$ by

$$\begin{aligned}(D_\tau\, w)(x) &= \int_0^1 G(x,\xi,\tau)\, w(\xi) d\xi \\ (S_\tau u)(x) &= \int_0^\tau G(x,1,\tau-s)\, \alpha\, u(s) ds.\end{aligned}$$

For the continuity of these operators we refer, for instance, to [14]. Then $(D_\tau\, w)(x) = y(x,\tau)$, where $y(x,t)$ is the unique solution of the initial-boundary value problem

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= w(x) \\ y_x(0,t) &= 0 \\ y_x(1,t) + \alpha\, y(1,t) &= 0, \end{aligned} \tag{2.2}$$

while $(S_\tau u)(x) = z(x,\tau)$, where $z(x,t)$ solves

$$\begin{aligned} z_t(x,t) &= z_{xx}(x,t) \\ z(x,0) &= 0 \\ z_x(0,t) &= 0 \\ z_x(1,t) + \alpha\, z(1,t) &= \alpha\, u(t). \end{aligned} \tag{2.3}$$

Our method of instantaneous control is defined as follows. Let $e_j = e_j(t)$, $j = 1, \ldots, n$, denote the piecewise constant basis functions

$$e_j(t) = \begin{cases} 1 & \text{on } I_j \\ 0 & \text{on } [0,T] \setminus I_j. \end{cases}$$

We will apply controls u having the form

$$u(t) = \sum_{j=1}^n u_j\, e_j(t)$$

with unknown real numbers u_j. Define $y_j(x) = y(x,t_j)$, $j = 0, \ldots, n-1$. Starting at $j = 1$, we subsequently solve the short horizon control problems

$$(\mathrm{P}_j) \qquad \min_u \varphi(y_{j-1}, u) = \frac{1}{2} \| y(\cdot, t_j) - y_d\|^2_{L^2(0,1)} + \frac{\nu}{2} \cdot \tau \cdot u^2$$

subject to

$$
\begin{aligned}
y_t(x,t) &= y_{xx}(x,t) \\
y(x,t_{j-1}) &= y_{j-1}(x) \\
y_x(0,t) &= 0 \\
y_x(1,t) + \alpha\, y(1,t) &= \alpha\, u,
\end{aligned}
\tag{2.4}
$$

$$u_a \le u \le u_b,$$

$t \in [t_{j-1}, t_j]$, where u is a real number.

The heat equation (2.4) is autonomous in time, hence $y(x,t_j) = \tilde{y}(x,\tau)$, where $\tilde{y}(x,\tau)$ solves (2.4) in $(0,\tau)$ subject to $\tilde{y}(x,0) = y_{j-1}(x)$. We can express this fact equivalently by

$$
\begin{aligned}
y(\cdot,t_j) &= \int_0^1 G(\cdot,\xi,\tau)\, y_{j-1}(\xi)\, d\xi + \int_0^\tau \alpha\, G(\cdot,1,\tau-s)\, u\, ds \\
&= D_\tau\, y_{j-1} + u\, S_\tau\, e_1
\end{aligned}
$$

(notice that $u \in \mathbb{R}$). In what follows, we shall indicate suboptimal controls by a bar. Therefore, the optimal solution of (P_j) is denoted by $\bar{u}_j$. Moreover, we introduce for convenience the notation $e := S_\tau\, e_1$. In our paper, $\|\cdot\|$ stands for the norm of $L^2(0,1)$, and $(\cdot,\cdot)$ denotes the associated natural inner product.

(P_j) is equivalent to a very simple one-dimensional quadratic programming problem that can be solved explicitely. In fact, define $f : L^2(0,1) \times \mathbb{R} \to \mathbb{R}$ by

$$f(y,u) := \frac{1}{2}\, \|y_d - D_\tau y - u\, e\|^2 + \frac{\nu}{2}\, \tau\, u^2.$$

Then $\varphi(y_{j-1},u) = f(y_{j-1},u)$ and

$$f(y_{j-1},u) = \frac{1}{2} a\, u^2 + b_j\, u + c_j,$$

where

$$
\begin{aligned}
a &= \|e\|^2 + \nu\, \tau \\
b_j &= (e,\, D_\tau\, y_{j-1} - y_d) \\
c_j &= \frac{1}{2}\, \|D_\tau\, y_{j-1} - y_d\|^2.
\end{aligned}
$$

Therefore, the solution $\bar{u}_j$ of (P_j) is obtained by minimizing $\frac{1}{2} a\, u^2 + b_j\, u$ subject to $u \in [u_a, u_b]$,

$$
\bar{u}_j = \begin{cases} u_a, & \text{if } -\frac{b_j}{a} < u_a \\ -\frac{b_j}{a}, & \text{if } -\frac{b_j}{a} \in [u_a, u_b] \\ u_b, & \text{if } -\frac{b_j}{a} > u_b. \end{cases}
\tag{2.5}
$$

We define the suboptimal control $\bar{u}(t)$ for (P) by $\bar{u}(t) = \bar{u}_j$, $t \in I_j$. It is obvious that $\bar{u}$ can be determined very fast. Since the minimization of (P_j) is done analytically, the only work is to set up (P_j). To do this, the most time consuming step is the computation of $D_\tau\, y_{j-1}$, i.e. the solution of the heat equation (2.2) for $w = y_{j-1}$ on $(0,\tau)\times(0,1)$. Then a, b_j, c_j can be found by numerical integration. The function $e = S_\tau e_1$ has to be computed only once by solving (2.3). Altogether, $n+1$ PDE solves are needed.

Clearly, this suboptimal method stops after n steps. Nevertheless, let us consider infinitely many repetitions of the iteration. Each iteration assigns to an initial function $y = y(x)$ a real optimal control number $\bar u = \bar u(y(\cdot))$ for the time horizon $[0,\tau]$ and a new initial function $\bar y(x)$ by

$$\bar y = D_\tau\, y + \bar u\,(y(\cdot))\, e. \tag{2.6}$$

Definition 1. The mapping $y \longmapsto \bar y$ defined by (2.6) in $L^2(0,1)$ is denoted by Φ, $\Phi(y) := D_\tau\, y + \bar u(y)\, e$.

We shall prove that Φ has a unique fixed point. First, notice that D_τ is a contraction in $L^2(0,1)$. To this aim, define A by

$$\begin{aligned} &D(A) = \{y \in H^2(0,1) \mid y_x = 0 \ \text{ at } \ x = 0,\ y_x + \alpha\, y = 0 \ \text{ at } \ x = 1\},\\ &A\, y = -\, y_{xx}, \end{aligned}$$

$A : L^2(0,1) \supset D(A) \to L^2(0,1)$. It is known that $-A$ is the infinitesimal generator of a contraction semigroup e^{-At} in $L^2(0,1)$. Moreover, it holds $e^{-A\tau}y = D_\tau\, y$. Therefore, D_τ is a contraction. We shall show that Φ is a contraction as well. Preparing this result, we first formulate the necessary optimality conditions for the problem

$$\min_{u\,\in\,[u_a,u_b]} \quad f(y,u), \tag{2.7}$$

where $y \in L^2(0,1)$ is given fixed. Let $\bar u = \bar u(y)$ denote the solution of (2.7). Then

$$\frac{\partial f}{\partial u}\,(y,\bar u)(u-\bar u) \ge 0 \qquad \forall\, u \in\ [u_a,u_b]$$

must hold. Therefore, the variational inequality

$$\{a\,\bar u + (e, D_\tau\, y - y_d)\} \cdot (u - \bar u) \ge 0 \qquad \forall\, u \in\ [u_a,u_b] \tag{2.8}$$

is satisfied.

Lemma 2. *The mapping Φ is a contraction in $L^2(0,1)$.*

Proof. Consider arbitrary y_1, $y_2 \in L^2(0,1)$. First we invoke the variational inequality (2.8) for $\bar u_i = \bar u(y_i)$ and insert the other solution $\bar u_j$ for the free variable u in (2.8),

$$\begin{aligned} &\{a\,\bar u_1 + (e, D_\tau\, y_1 - y_d)\}\,(\bar u_2 - \bar u_1) \ge 0\\ &\{a\,\bar u_2 + (e, D_\tau\, y_2 - y_d)\}\,(\bar u_1 - \bar u_2) \ge 0. \end{aligned}$$

Adding both inequalities,

$$-a\left(\bar{u}_1-\bar{u}_2\right)^2+\left(e, D_\tau\left(y_1-y_2\right)\right)\left(\bar{u}_2-\bar{u}_1\right) \geq 0 \tag{2.9}$$

is found. In particular, this implies

$$\left(e, D_\tau\left(y_1-y_2\right)\right)\left(\bar{u}_1-\bar{u}_2\right) \leq -a\left(\bar{u}_1-\bar{u}_2\right)^2 \leq -\|e\|^2(\bar{u}_1-\bar{u}_2)^2 \tag{2.10}$$

as $a \geq \|e\|^2 > 0$. Now

$$\begin{aligned}
\|\Phi\left(y_1\right)-\Phi\left(y_2\right)\|^2 &= \|D_\tau\left(y_1-y_2\right)+\left(\bar{u}_1-\bar{u}_2\right)e\|^2 \\
&= \|D_\tau\left(y_1-y_2\right)\|^2+2(D_\tau\left(y_1-y_2\right), e)(\bar{u}_1-\bar{u}_2) \\
&\quad +(\bar{u}_1-\bar{u}_2)^2\|e\|^2 \\
&\leq \|D_\tau\|^2\,\|y_1-y_2\|^2-\|e\|^2(\bar{u}_1-\bar{u}_2)^2 \\
&\leq \|D_\tau\|^2\,\|y_1-y_2\|^2
\end{aligned}$$

holds by (2.10). Since D_τ is a contraction, Φ is a contraction as well. □

Let us define the projection of a real number u onto the interval $[u_a, u_b]$ by $Proj_{[u_a,u_b]}\,u$.

Theorem 3. *The mapping Φ has unique fixed point y^* with associated control variable u^*. They satisfy*

$$y^*=u^*y_f, \quad u^*=\bar{u}(y^*), \quad y_f=(I-D_\tau)^{-1}e, \tag{2.11}$$

$$u^*=Proj_{[u_a,u_b]}\left(\frac{(e,y_d)}{(e,y_f)+\nu\tau}\right). \tag{2.12}$$

Proof. Existence and uniqueness of y^* follow from Lemma 2 and the Banach fixed point theorem. By (2.6),

$$y^*=D_\tau\,y^*+\bar{u}\left(y^*\right)e,$$

hence

$$y^*=(I-D_\tau)^{-1}\bar{u}(y^*)e=\bar{u}(y^*)(I-D_\tau)^{-1}e=u^*y_f,$$

and (2.11) follows immediately. Moreover, (2.12) follows from (2.5). □

It should be mentioned, that y_f can be described in two additional alternative ways: on the one hand we know $y_f = D_\tau y_f + e$. In other words, $y_f(x) = y(x,\tau)$, where y is the solution of an initial boundary value problem with initial value $y(x,0) = y_f(x)$. Hence $y(x,0)$ and $y(x,\tau)$ must coincide. We have obtained $y_f(x) = y(x,\tau)$, where y solves the following boundary value problem with periodic boundary conditions with respect to the time variable:

$$\begin{aligned}
y_t(x,t) &= y_{xx}(x,t) \\
y(x,0) &= y(x,\tau) \\
y_x(0,t) &= 0 \\
y_x(1,t)+\alpha\,y(1,t) &= \alpha e_1(t),
\end{aligned} \tag{2.13}$$

This formulation is equivalent to (2.11 iii). It follows that (2.13) is uniquely solvable.
On the other hand, we can express $(I - D_\tau)^{-1}$ by a Neumann's series that converges in view of $\|D_\tau\| \leq c < 1$. So y_f can be approximated by a finite series.
Notice that $y(x,t) \equiv 1$ is a solution of (2.13) for $e_1(t) \equiv 1$.

3 Performance of the standard instantaneous control type method

We tested our instantaneous control method by a known test problem due to Schittkowski [13]. Here, the following data were given:

$T = 1.58$, $y_d(x) = 0.5\,(1 - x^2)$, $y_0(x) = 0$, $\nu = 0.001$, $u_a = -1$, $u_b = 1$, $\alpha = 1$.

To apply the method, the problem must be discretized. The intervals $(0,1)$ and $(0,T)$ were splitted by uniform grids into $n_x = 50$ and $n_t = 100$ subintervals, and the control u was approximated by $n = 100$ basis functions e_j. The heat equation (1.1) was solved by a fully implicit finite difference method.

First, we recall the results for an exact minimization of the discretized problem, i.e. a solution of the associated finite-dimensional quadratic programming problem. They were obtained by the Bertsekas projection method and application of a CG method to the associated unconstrained subproblems, see [1] for details.

The optimal control $u(t)$ and a comparison of the desired temperature profile y_d with the optimal final temperature $\bar{y}(x,T)$ are presented in Figure 3.1. The computed optimal control has 3 switching points separating two interior and two boundary arcs, and the computed optimal value is

$$J(u) = 0.000686.$$

We refer also to associated numerical tests in Eppler and Tröltzsch [6]. Moreover, we should mention here the interesting fact that, up to now, it is an open question, if this reflects the true switching structure of the optimal control for the infinite-dimensional problem. In particular, it is even not yet proved that the number of switching points is finite.

Let us now report on the application of our instantaneous control method to the example, based on the same discretization. The computed suboptimal functions are plotted in the Figures 3.2, and the suboptimal value for the objective was

$$J(\bar{u}) = 0.03645.$$

We observe a striking difference to the optimal data. At the beginning of our computations, we were convinced to have made a simple error in our code.

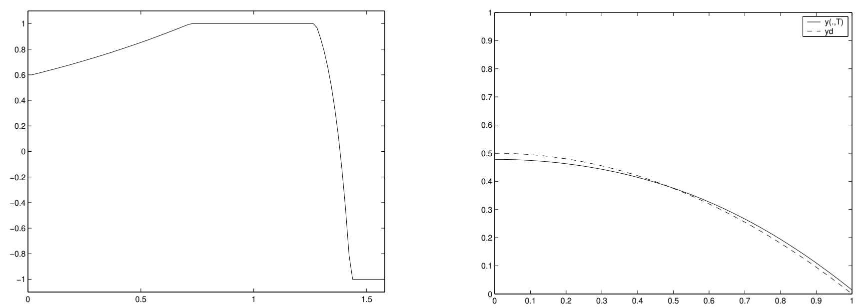

Figure 1: Optimal control u and optimal final state $y(\cdot, T)$

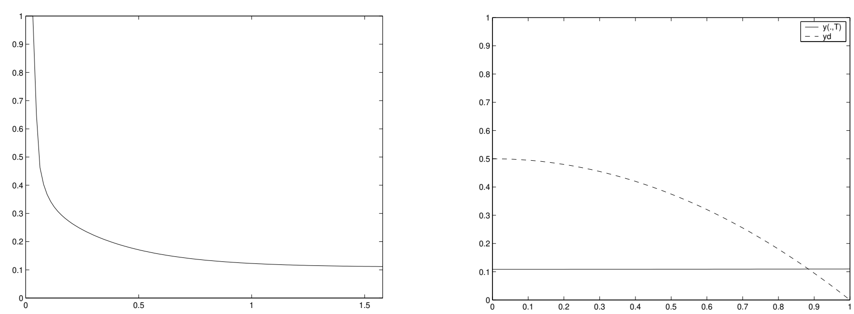

Figure 2: Instantaneous control: Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot, T)$

However, the code was correct and we later understood, why the method exhibits this behaviour.

The first hint came by intuition. In the first part of the time interval $[0, T]$, the optimal control inserts more energy than needed. Finally, where the control is equal to -1, this extra energy is wasted. The optimizer recognizes that this is useful to better fit y_d at the end of the control process. The suboptimal controller does not have this future perspective. It acts too cautiously and supplies only a moderate temperature. It cannot foresee that it might be useful to invest a little bit more of energy in order to reach a better value in the future. To see this, the controller should have more insight into the future.

The final evidence came by the proof of our fixed point theorem 3. The fixed point predicted by (2.12) is $u^* = 0.1094$. This is what Figure 3.2 shows. The right hand figure shows $y(\cdot, T) \approx u^* y_f = 0.1094 \cdot 1$. We recall that $y_f \equiv 1$ is given by (2.13).

4 Improved strategies

4.1 Extended time horizon

As we have seen in the preceding section, a lack of future perspective can be a decisive drawback of the standard instantaneous control method. The time horizon τ is too short. Choosing a larger horizon for the optimization should improve the performance. Therefore, we first proceed as follows: Given an initial state $y(x) := y_0(x)$ we try to approach the desired state $y_d(x)$ on the extended time interval $[0, l\,\tau]$, where $l \in \mathbb{N}$ is given. The ansatz for the control is

$$u(t) = \sum_{i=1}^{l} u_i e_i(t). \tag{4.1}$$

We solve the optimal control problem

$$(\mathrm{P}_l) \qquad \min \quad J(u_1, .., u_l) = \frac{1}{2}\int\limits_0^1 (y(x, l\,\tau) - y_d(x))^2 dx + \frac{\nu}{2}\int\limits_0^{l\,\tau} u(t)^2 dt$$

subject to

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t)\\ y(x,0) &= y_0(x)\\ y_x(0,t) &= 0\\ y_x(1,t) &= \alpha\,(u(t) - y(1,t)) \end{aligned} \tag{4.2}$$

on $[0, l\,\tau]$, subject to the ansatz (4.1) for $u(t)$ and to the constraints

$$u_a \le u_i \le u_b,$$

$i = 1, .., l$. The optimization in (P_l) is performed exactly (of course, for a discretized version of the heat equation). As a result, we obtain control coefficients $\hat{u}_1$,.., $\hat{u}_l$ and the state $y(x, l\,\tau)$ at the end of the extended horizon $[0, l\,\tau]$. Next, we put $\bar{u}_i := \hat{u}_i$, $i = 1, .., l$, and define $y_0(x) := y(x, l\,\tau)$ to be the new initial state for the shifted time horizon $[l\,\tau, 2l\,\tau]$. Then we perform the optimization on this interval. By autonomy in time, this is equivalent to solve problem (P_l) on $(0, l\,\tau)$. Let us denote its solution by $\hat{u}_1$,.., $\hat{u}_l$ again. The suboptimal controls on $[l\,\tau, 2l\,\tau]$ are now $\bar{u}_{l+1} := \hat{u}_1$, ..,$\bar{u}_{2l} = \hat{u}_l$. Next we proceed to $[2l\,\tau, 3l\,\tau]$ etc. If $k\,l\,\tau > T$, then only those controls are used that belong to subintervals contained in $[0, T]$.

This method requires some more effort. Nevertheless, we will prove that a fixed point exists as well. For this reason, we will follow the steps already done above in section 2. First we investigate $S_{l\tau} e_i$. By definition (2.3), $S_{l\tau} e_i =$

$y(\cdot, l\tau)$, where y solves

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= 0 \\ y_x(0,t) &= 0 \\ y_x(1,t) + \alpha\, y(1,t) &= \alpha\, e_i(t). \end{aligned} \tag{4.3}$$

Since $e_i(t) = 0$ for $t < (i-1)\tau$ the initial condition can be replaced by $y(x,(i-1)\tau) = 0$. Furthermore, we get $y(\cdot, i\tau) = S_\tau e_1 = e$ by $e_i(t) = e_1(t-(i-1)\tau)$ and (2.3). Moreover, it holds $e_i(t) = 0$ for $t > i\tau$, hence

$$S_{l\tau} e_i = y(\cdot, l\tau) = D_{(l-i)\tau} y(\cdot, i\tau) = D_{(l-i)\tau} S_\tau e_1 = D_{(l-i)\tau} e = D_\tau^{l-i} e.$$

Putting this together we find

$$S_{l\tau} u = \sum_{i=1}^{l} u_i\, D_\tau^{l-i} e.$$

Let us denote by $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_l) =: \bar{u}(y)$ the solution of (P_l). Now the method assigns to an initial function y a new function $\bar{y}$ by

$$\bar{y} = D_\tau^l y + S_{l\tau} u = D_\tau^l y + \sum_{i=1}^{l} \bar{u}_i D_\tau^{l-i} e =: \Phi_l(y) \tag{4.4}$$

that defines a mapping $\Phi_l : L^2(0,1) \to L^2(0,1)$. We introduce a functional $f : L^2(0,1) \times \mathbb{R}^l \to \mathbb{R}$ by

$$f(y,u) = \frac{1}{2} \| y_d - D_\tau^l y - \sum_{i=1}^{l} u_i D_\tau^{l-i} e \|^2 + \frac{\nu}{2} \tau \sum_{i=1}^{l} u_i^2 = \frac{1}{2} u^T H u + b^T u + c,$$

where

$$\begin{aligned} H &= (h_{ij}) & h_{ij} &= (D_\tau^{l-i} e,\ D_\tau^{l-j} e) + \delta_{ij} \nu \tau \\ b &= (b_i) & b_i &= (D_\tau^{l-i} e,\ D_\tau^l\, y - y_d) \\ c &= \frac{1}{2} \| D_\tau^l\, y - y_d \|^2. \end{aligned}$$

Obviously, the matrix H is symmetric, and $(H - \nu\tau I)$ is positive semidefinite, hence H is positive definite. The necessary condition for $\bar{u}$ to be solution of (P_l) is

$$(H\bar{u} + b)^T (u - \bar{u}) \geq 0 \quad \forall u \in [u_a, u_b]^l. \tag{4.5}$$

Lemma 4. *The mapping Φ_l is a contraction in $L^2(0,1)$.*

Proof. Let $y_1, y_2 \in L^2(0,1)$ be arbitrary functions and $\bar{u}_i = \bar{u}(y_i)$. The variational inequality (4.5) yields

$$\begin{aligned} (H\, \bar{u}_1 + b)^T\, (\bar{u}_2 - \bar{u}_1) &\geq 0 \\ (H\, \bar{u}_2 + b)^T\, (\bar{u}_1 - \bar{u}_2) &\geq 0. \end{aligned}$$

Adding these inequalities

$$-(\bar u_1-\bar u_2)^T H(\bar u_1-\bar u_2)+\sum_{i=1}^{l}\big(D_\tau^{l-i}e,\,D_\tau^{l}(y_1-y_2)\big)\cdot(\bar u_{2i}-\bar u_{1i})\ge 0$$

leads to

$$\sum_{i=1}^{l}\big(D_\tau^{l-i}e,\,D_\tau^{l}(y_1-y_2)\big)\cdot(\bar u_{1i}-\bar u_{2i})\le -(\bar u_1-\bar u_2)^T H(\bar u_1-\bar u_2)\le 0, \tag{4.6}$$

since H is positive definite. Then

$$\begin{aligned}\|\Phi_l\,(y_1)-\Phi_l\,(y_2)\|^2 &= \|D_\tau^l\,(y_1-y_2)+\textstyle\sum_{i=1}^{l}(\bar u_{1i}-\bar u_{2i})D_\tau^{l-i}e\|^2\\ &= \|D_\tau^l\,(y_1-y_2)\|^2+2\textstyle\sum_{i=1}^{l}(\bar u_{1i}-\bar u_{2i})\left(D_\tau^l\,(y_1-y_2),D_\tau^{l-i}e\right)\\ &\quad+(\bar u_1-\bar u_2)^T(H-\nu\tau I)(\bar u_1-\bar u_2)\\ &\le \|D_\tau^l\,(y_1-y_2)\|^2-(\bar u_1-\bar u_2)^T(H+\nu\tau I)(\bar u_1-\bar u_2)\\ &\le \|D_\tau^l\,(y_1-y_2)\|^2\ \le (\|D_\tau\|^2)^l\,\|(y_1-y_2)\|^2\end{aligned}$$

follows immediately from (4.6). This shows that Φ_l is a contraction because D_τ is a contraction as well. □

Theorem 5. *The mapping Φ_l has unique fixed point y^* with associated control variable u^*. They satisfy*

$$y^*=\sum_{i=1}^{l}u_i^*\,(y^*)\;(I-D_\tau^l)^{-1}\,D_\tau^{l-i}e,\quad u^*=\bar u(y^*) \tag{4.7}$$

and

$$(H'u^*+b')^T(u-u^*)\ge 0\quad\forall u\in[u_a,u_a]^l, \tag{4.8}$$

where

$$\begin{aligned}H'&=(h'_{ij})\qquad h'_{ij}=((I-D_\tau^l)^{-1}D_\tau^{l-i}e,\,D_\tau^{l-j}e)+\delta_{ij}\nu\tau\\ b'&=(b'_j)\qquad\ \ b'_j\ =-(D_\tau^{l-i}e,\,y_d)\end{aligned} \tag{4.9}$$

Proof. Existence and uniqueness of y^* follow from Lemma 4 and the Banach fixed point theorem. (4.7) is a consequence of (4.4) and the fact that y^* is the fixed point of Φ_l. (4.8) and (4.9) are the necessary condition (4.5) with respect to $\bar u$. The formulas 4.9 can be obtained as follows:

$$\begin{aligned}(Hu^*+b)_i&=\sum_{j=1}^{l}h_{ij}u_j^*+b_i=\sum_{j=1}^{l}h_{ij}u_j^*+(D_\tau^{l-i}e,D_\tau^l y^*-y_d)\\ &=\sum_{j=1}^{l}\{(D_\tau^{l-i}e,D_\tau^{l-j}e)+\delta_{ij}\nu\tau\}u_j^*+(D_\tau^{l-i}e,D_\tau^l y^*)+b'_i.\end{aligned} \tag{4.10}$$

In view of (4.7) we find

$$(D_\tau^{l-i}e, D_\tau^l y^*) = \sum_{j=1}^{l} (D_\tau^{l-i}e, D_\tau^l (I - D_\tau^l)^{-1} D_\tau^{l-j} e) u_j^*.$$

Inserting this expression in (4.10),

$$\begin{aligned}(Hu^* + b)_i &= \sum_{j=1}^{l} \{(D_\tau^{l-i}e, [I + D_\tau^l (I - D_\tau^l)^{-1}] D_\tau^{l-j} e) + \delta_{ij}\nu\tau\} u_j^* + b_i' \\ &= \sum_{j=1}^{l} h_{ij}' u_j^* + b_i'\end{aligned}$$

is obtained. Here we applied

$$I + D_\tau^l (I - D_\tau^l)^{-1} = ((I - D_\tau^l) + D_\tau^l)(I - D_\tau^l)^{-1} = (I - D_\tau^l)^{-1}.$$

□

This method is not essentially better than the standard instantaneous control method, which is obtained for $l = 1$. For $l = 4$ the following suboptimal controls and final states were computed (all discretization parameters as above):

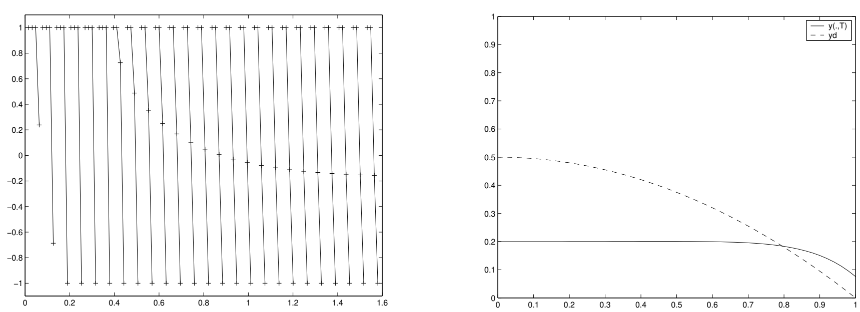

Figure 3: Extended horizon – Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot, T)$

The suboptimal value of the objective is

$$J(\bar{u}) = 0.01878.$$

For $l = 4$, any computed extended horizon control has 4 components. This explains the left hand side picture of Fig. 4.1. Notice that a new horizon begins repeatedly after 4 time steps. The fixed point vector u^*, calculated by (4.7) and (4.8), is $u^* = (1.0, 1.0, -0.1483, -1.0)$. The associated y^* (dashed) and the final state $\bar{y}(\cdot, T)$ (solid line) are presented in the right hand Fig. 4.1. In a sense, on each short horizon the method reflects the long time behaviour shown in the left hand side Fig 3.1.

In (4.7), the functions $y_{f,i} = (I - D_\tau)^{-1} D_\tau^{l-i} e$ were computed by approximating the Neumann series. The expansion was truncated after $\|D_\tau^k(D_\tau^{l-i}e)\|$ was sufficiently small. The matrix H' is symmetric since D_τ is self-adjoint. However, the symmetry can be destroyed due to numerical errors in the discretized form of (4.9). To make H' symmetric, in the numerical approximation its elements were computed by

$$h'_{ij} = ((I - D_\tau^l)^{-\frac{1}{2}} D_\tau^{l-i} e,\, (I - D_\tau^l)^{-\frac{1}{2}} D_\tau^{l-j} e) + \delta_{ij}\nu\tau$$

This can be done since $(D_\tau)^{\frac{1}{2}} = D_{\frac{\tau}{2}}$ and $\|D_{\frac{\tau}{2}}\| < 1$. Therefore, the power series for $(I - D_\tau^l)^{-\frac{1}{2}}$ converges.

4.2 Receding horizon technique

The extension of the time horizon discussed in the last subsection improves the precision of the instantaneous control technique. However, the larger the time horizon is, the more control values are fixed in one step, no matter what happens in the future intervals. This drawback can be avoided by the well known method of receding horizon. Here, we proceed as follows:

Again, the length of the time horizon is taken as $l\,\tau$, i.e. the horizon includes l time steps. Given an initial state $y_0 = y_0(x)$, we try to approach the desired final state y_d on $[0, l\,\tau]$ by controls $u(t)$ having the form (4.1). Again, we solve the optimal control problem (P_l) to obtain control coefficients $\hat{u}_1, \ldots, \hat{u}_l$. By (4.1), they define a control $\hat{u}(t)$ that is optimal for (P_l). In contrast to the preceding subsection, we do not use all of $\hat{u}(t)$ as part of a suboptimal control $\bar{u}(t)$ for (P). We only select the first part of $\hat{u}(t)$ that is defined on $[0, \tau]$. The remaining part, defined on $[\tau, l\,\tau]$, is ignored. Thus we set $\bar{u}_1 := \hat{u}_1$. We compute $y_1(x) = y(x, \tau)$ as the next initial state and repeat the optimization step on the shifted time interval $[\tau, (l+1)\,\tau]$. Thanks to autonomy in time, this is equivalent to solving (P_l) on $[0, l\,\tau]$ subject to $y_0(x) := y_1(x)$. Once again, we adopt only the first part of the associated optimal control, i.e. we put $\bar{u}_2 := \hat{u}_1$, while the other values $\hat{u}_2, \ldots, \hat{u}_l$ are ignored.

In (P), the interval of time is $[0, T]$ and $\tau = T/n$. After $n - l$ steps, the receding horizon method has reached the subinterval $[(n-l)\,\tau, T]$. Then the whole function $\hat{u}(t)$ is taken to define the last part of the suboptimal $\bar{u}(t)$, i.e. we define $\bar{u}_{n-l+1} := \hat{u}_1, \ldots, \bar{u}_n := \hat{u}_l$.

For $l = 1$, this method recovers the instantaneous control technique. If $l = 2$, then it already behaves much better. We have the impression that the method converges to a fixed point as well. The gain of performance is partly connected with the optimal solution on the last interval of time $[(n-l)\,\tau, T]$, which is the most important one, due to the smoothing property of the heat equation.

We tested our example for $l = 4$, $n = n_t = 100$, $n_x = 50$. The results are presented in Fig 4.2. The suboptimal value of the objective is

$$J(\bar{u}) = 0.00605.$$

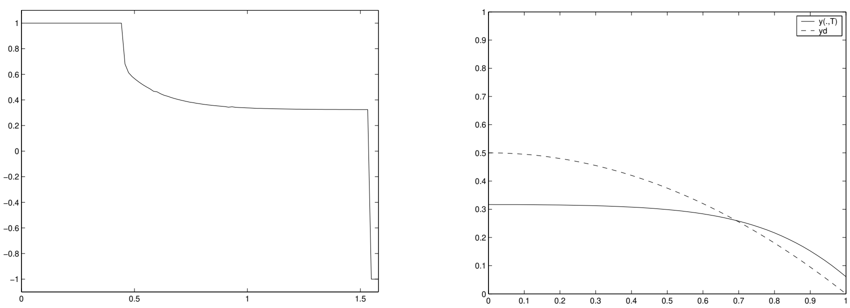

Figure 4: Receding horizon – Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot, T)$

5 Improved instantaneous control method

So far, we have understood the result of Theorem 3 as a negative one explaining the bad performance of the instantaneous control method. However, we can exploit it to set up a new and – as we shall see – much better technique.

We have mentioned why the method of instantaneous control might have difficulties to approach the desired final state y_d. According to Theorem 3, it drives the state function y to the fixed point $y_f = c\,(I - D_\tau)^{-1}\,e$, i.e. to a multiple of $(I - D_\tau)^{-1}\,e$. Another aspect is the cause for the better results of the receding horizon method. These are, to a large extent, connected with the optimal solution on the last time interval $[(n-l)\,\tau, T]$. Now we will combine these two observations and introduce a two-components algorithm. The first step of this method is of instantaneous control type. However, the desired function y_d is changed to $\hat{y}_d = c \cdot y_f$. Then there is a good chance to achieve this goal by the instantaneous method. Proceeding in this way, we find the control u on the time interval $[0, (n-l)\,\tau]$. The remaining control variables will be the result of optimization on $[(n-l)\,\tau, T]$.

How should $\hat{y}_d$ be defined, i.e., how should the unknown constant c be selected? Suppose, we are able to control the initial distribution y_0 into $\hat{y}_d$ exactly after $n - l$ instantaneous control steps. The last step would be the solution of

$$\min \frac{1}{2}\|y_d - c \cdot D_\tau^l y_f - \sum_{i=1}^{l} u_i D_\tau^{l-i} e\|^2 + \frac{\nu}{2}\tau \sum_{i=1}^{l} u_i^2 \qquad (5.11)$$

subject to $u_a \le u_i \le u_b$, $i = 1 \dots l$. The constant c is not fixed yet. Therefore, c can also be included in the optimization of (5.11) subject to the constraints $u_a \le c \le u_b$. Putting this together we get the following algorithm:

1. Determine y_f according to (2.13).
2. Solve (5.11) subject to $u_a \le u_i \le u_b$, $i = 1 \dots l$, $u_a \le c \le u_b$, and get $\hat{c}, \hat{u}_1, \dots, \hat{u}_l$.
3. Compute $u_1, \dots, u_{n-l}$ and $y(\cdot, (n-l)\,\tau)$ by the instantaneous control method described in section 2 with desired state $\hat{y}_d = \hat{c}\, y_f$.
4. Minimize the objective functional

$$\frac{1}{2}\|y_d - D_\tau^l y(\cdot, (n-l)\,\tau) - \sum_{i=1}^{l} u_i D_\tau^{l-i} e\|^2 + \frac{\nu}{2}\tau \sum_{i=1}^{l} u_i^2$$

   subject to $u_a \le u_i \le u_b$, $i = 1 \dots l$, to obtain the solution $\bar{u}_1, \dots, \bar{u}_l$. Set $u_{n-l+1} = \bar{u}_1, \dots, u_n = \bar{u}_l$.

This hybrid technique behaves essentially better than the former ones. With the same test parameters as above we got for $l = 4$

$$J(\bar{u}) = 0.00265.$$

The constant $\hat{c}$ was computed as $\hat{c} = 0.41356$. This can be seen in Fig. 4.3.

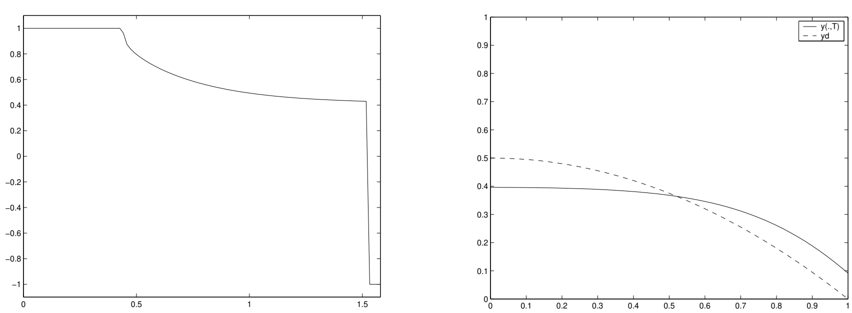

Figure 5: Improved method: Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot, T)$

This strategy has some similarities to the Bellmann optimality principle. First we optimize to find a well approachable state $\hat{y}_d$ at time $(n-l)\tau$, then we optimize on the remaining interval.

Remark 6. The result of step 2 is the state $\hat{y}_d = \hat{c}\; y_f$ that should be approximated by the standard intantaneous control method.

Remark 7. The step 4 may be superfluous. If the state $\hat{y}_d$ is already well approximated by the instantaneous control method in step 3, then we can skip the minimization in step 4. In this case, we take the variables $\hat{u}_1, \ldots, \hat{u}_l$ to define the last control components: $u_{n-l+1} = \hat{u}_1, \ldots, u_n = \hat{u}_l$. This case occurs if the final time T is sufficiently large. Then the convergence theorem 3 gives the convergence of the series $\{\bar{u}_i\}$ towards $u^* \approx \hat{c}$.

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