Dominic Jost, Kai Nagel

Probabilistic Traffic Flow Breakdown in Stochastic Car Following Models

Jost, D.; Nagel, K. (2005). Probabilistic Traffic Flow Breakdown in Stochastic Car Following Models. Traffic and Granular Flow ’03, 87–103. https://doi.org/10.1007/3-540-28091-X_9

Probabilistic Traffic Flow Breakdown In Stochastic Car Following Models

Dominic Jost[1] and Kai Nagel[2]

[1] Inst. for Computational Science – ETH Zürich
CH-8092 Zürich, Switzerland
e-mail: jost@inf.ethz.ch

[2] Inst. for Land and Sea Transport Systems – TU Berlin
Salzufer 17–19, D-10587 Berlin, Germany
e-mail: nagel@vsp.tu-berlin.de

Summary. There is discussion if traffic displays spontaneous breakdown. This paper presents computational evidence that stochastic car following models can have a control parameter that moves the model between displaying and not displaying spontaneous phase separation for some densities. Those phases can be called "laminar" and "jammed". Models with spontaneous phase separation show three states as a function of density: a first state at low density, where those models are homogeneously laminar; a second state at high density, where they are homogeneously jammed; and a third state at intermediate density, where they consist of a mix between the two phases (phase coexistence). This is the same picture as for a gas-liquid transition when volume of the gas is the control parameter.

Although the gas-liquid analogy to traffic models has been widely discussed, no traffic-related model so far displayed a completely understood *stochastic* version of that transition. Having a stochastic model is important to understand the potentially probabilistic nature of the transition. Most importantly, if indeed models with spontaneous phase separation describe certain aspects correctly, then this leads to an understanding of spontaneous breakdown. Alternatively, if models without spontaneous phase separation describe these aspects better, then there is no spontaneous breakdown (= no breakdown without a reason). Interestingly, even models without spontaneous phase separation can still allow for jam formation on small scales, which may give the impression of having a model with spontaneous phase separation.
Keywords: traffic flow theory, car following models, traffic breakdown, traffic simulation, phase transition, phase separation, critical point

1 Introduction

The capacity of a road is an important quantity. If demand exceeds capacity, queues will form, which represent a cost to the driver and thus to the economic system. In addition, such queues may impact other parts of the system, for example by spilling back into links used by drivers who are on a path that is not overloaded.

This paper discusses freeway capacity. The question concerns the maximum flows that freeways can reach, and if the maximum flows sometimes observed (> 2500 vehicles per hour and lane) are sustainable flows or short-term fluctuations. Let us assume that there is traffic with a fairly high density ρ on a freeway, but vehicles are still able to drive at some fast velocity v. Throughput is $q = \rho\, v$. The question is what will happen if density is further increased: Can q further increase because ρ increases more than v decreases? Will q gradually decrease because ρ increases but v decreases faster? Or is there a possibility that traffic will break down, leading to stop-and-go traffic?

More technically, the question is if there is, for each density ρ, a velocity $V(\rho)$ and corresponding throughput $Q(\rho) = \rho\, V(\rho)$ at which traffic flow is smooth and homogeneous. Or is there a density range where that homogeneous traffic flow is unstable, and traffic has a tendency to reorganize into a stop-and-go pattern, with possibly lower throughput?

It is important to note that this paper's focus is on *homogeneous* situations. This concerns both the geometry of the system, which is assumed to be closed (such as a long ring) and spatially uniform (no bottlenecks, no changes in speed limit, no grades, etc.), and the initial condition, which is assumed to be traffic with the same density everywhere along the ring. Clearly, this is a theoretical construct, but the issue is to sort out theoretical questions. Again, the main question is if in such a situation the initially homogeneous traffic has a tendency to reorganize into a stop-and-go pattern; this is what is meant by (spontaneous) "breakdown" in this paper. This is in contrast to induced phases, such as queues upstream of a bottleneck. Induced phases are important, but they are outside the scope of this paper.

There is in fact a long history of publications about breakdown behavior in freeway traffic, sometimes called "reverse lambda shape of the fundamental diagram" [1, 2], "hysteresis" [3], "capacity drop" [4], "catastrophe theory" [5], and the like. From the modeling side, there have since long been discussions about an analogy to a gas-liquid transition [6, 7], and recent work has established traffic models which display deterministic versions of a liquid-gas-like transition [8, 9].

Yet, measurements by Cassidy [10] indicate that there can be stable homogeneous flow at all densities. Many of the "reverse lambda" observations could also be caused by geometrical constraints, in the following way [11]. A bottleneck downstream of a measurement location can cause the following temporal sequence of measurements: (1) The system starts with low flow at low densities. – (2) Both flow and density keep increasing, along the "free flow" branch of the fundamental diagram. – (3) This flow can be larger than what can flow through the bottleneck. Then, a queue starts forming at the bottleneck, but that does not immediately influence the measurement. – (4) Eventually, the queue will have spilled back to the measurement location. At that point in time, data points will move to a much higher density, while the flow value will drop to the bottleneck capacity. It can take up to 20 minutes for the transition zone (transition from free flow to queue) to traverse a fixed detector location, leading to fundamental diagram data points that lie between the free flow and the queue state [11]. This mechanism generates data that looks similar to data that one would expect from a spontaneous breakdown in a homogeneous system, as explained above. Unfortunately, many of the published data sets do not provide enough information about the geometrical layout and the full spatio-temporal picture of the dynamics in order to resolve this question.

Because measurement locations upstream of bottlenecks generate fundamental diagrams that in the past were used to support the spontaneous breakdown hypothesis, at this point few measurements remain that can truly be used to help with the question. The maybe strongest empirical evidence for spontaneous breakdown is an experiment where a number of vehicles drive in a spatially homogeneous circle for an extended period of time [12]. In that experiment, traffic remains laminar for many minutes, but eventually "breaks down" into a stop-and-go pattern. Other evidence is indirect: Assume homogeneous traffic operating at a certain density, and assume the introduction of a strong disturbance, say by stopping one car for several seconds. If the introduced disturbance heals out over time, then homogeneous traffic at that density is stable; if the disturbance grows over time, then the homogeneous solution is unstable at this density. This implies that stable jams, embedded in laminar traffic, support the spontaneous breakdown hypothesis. There are at least three references (Figs. 2 and 3 in [4]; Fig. 3 in [13]; Fig. 4 in [14]) where the data in fact points to the existence of a stable jam, embedded *both upstream and downstream* in free traffic, and where the outflow from the jam is lower than the inflow. In the 2nd and the 3rd of these references, one can in addition see that the jam is remaining compact. In the 1st of these references, the data to decide this question is not sufficient.

This question is not just academic. The correct use of technical devices such as ramp metering [15] or adaptive speed limits [16] depends on the answer. For example, let us assume that the homogeneous solution is unstable in a certain density range, and that the alternative stop-and-go solution has a lower throughput than homogeneous traffic at the same density. In this case, the task of ramp metering might be to keep the density away from the unstable range. If density approaches this value, on-ramp traffic should be reduced.

If, in addition, breakdown is probabilistic, that is, the homogeneous solution can survive for certain amounts of time, then the question becomes which risk of breakdown one would be willing to accept. Accepting higher flow rates in the ramp metering algorithm might increase *average* throughput, but it might also increase the probability of breakdown. There is discussion to include aspects of stochastic transitions into the Highway Capacity Manual [17].

If, in contrast, the homogeneous solution is stable everywhere, then the potentially positive effects of ramp metering need to be derived from something other than breakdown.

Given this state of affairs, it makes sense to look at modeling. The task is to understand which model solutions are possible at all. This understanding will lead to the predictions of additional features that will go along with one mechanism or the other, and it might be possible to measure them, and so the issue will hopefully be eventually resolved.

It is important to note that this paper looks at the issue of spontaneous jam formation in a spatially homogeneous system, e.g. traffic in a long closed ring. In order to be clear about that, the term "spontaneous phase separation" will be used. This is different from boundary-induced phases, such as queues upstream of bottlenecks. Boundary-induced phases are clearly important in traffic, possibly more important than the issue of spontaneous phase separation. Nevertheless, the issue of spontaneous phase separation needs to be understood before conclusive statements on boundary-induced phases can be made.

This paper starts with Sec. 2 which recalls the general idea of a gas-liquid transition. Sec. 3 describes the simulation setup including the car following model that is used, discusses space-time plots of the resulting dynamics, and investigates transients vs. the steady state. Sec. 4 then establishes how a coexistence state can be numerically detected for a given model. Sec. 5 reports similar results for cellular automata (CA) models. Sec. 6 discusses how these results relate to deterministic models; the paper is concluded by a discussion and a summary.

2 Phases and phase transitions

The analogy between a gas-liquid transition and the laminar-jammed transition of traffic was pointed out many times (e.g. [7, 9]). The description of traffic in the well-known 2-fluid-model [18] assumes the existence of two phases; and all simulation models which use spatial queues (e.g. [19–21]) will display two phases because of the definition of the dynamics. The two phases in models with queues are however much easier to understand than the phases in more realistic models.

In a gas-liquid transition, one observes the following (Fig. 1):

- In the **gas state**, at low densities, particles are spread out throughout the system. Distances between particles vary, but the probability of having two particles close to each other is small.
- In the **liquid state**, at high densities, particles are close to each other. There is no crystalline structure as in solids, but the density is similar. Because of the fact that the particles are so close to each other, it is difficult to compress the fluid any further.
- In between, there is the so-called **coexistence state**, where gas and liquid coexist. In typical experiments in gravity, the liquid will be at the bottom and the gas will be above it. Without gravity, droplets form within the gas

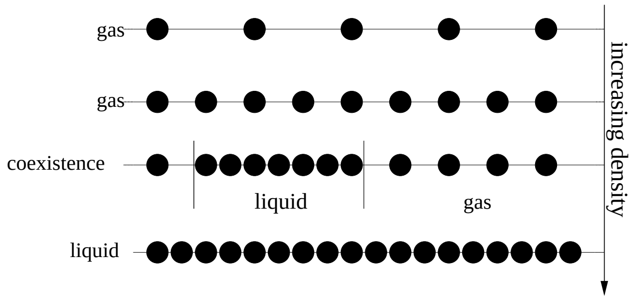

Fig. 1. Schematic representation of the gas-liquid transition in one dimension.

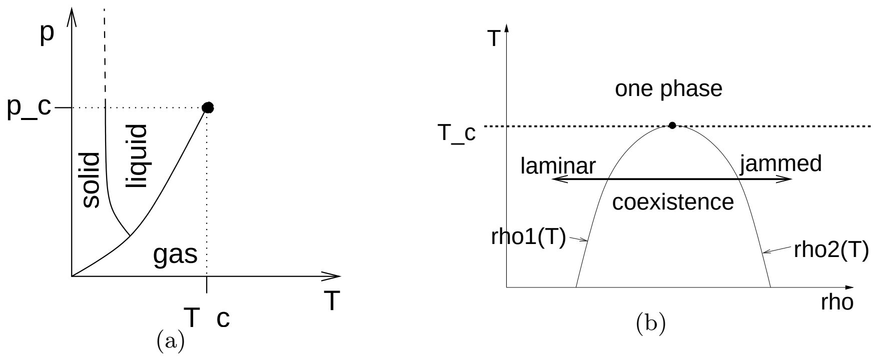

Fig. 2. (a) "Standard" pressure-temperature (PT) diagram with critical point. For $T > T_c$ or $p > p_c$, there is no gas-liquid transition any more. In that case, distinguishing between liquid and gas phases does not really make sense. (b) Phase diagram of the gas-fluid model as a function of the density and the temperature T.

and remain interdispersed. The droplets will slowly merge together into bigger droplets (coagulation). The final state of the system is having one big droplet of liquid, surrounded by gas.
If a system in the coexistence state is compressed, more droplets form and/or existing ones grow, but the density both inside and outside the droplets remains constant. That is, the system reacts by allocating more space to the liquid, but *not* by changing the density either of the gas or the liquid. Let us call those two densities ρ_1 and ρ_2, with $\rho_1 < \rho_2$. Eventually, all the space is used up by the liquid. At this point, the system will be homogeneous again and remain so if density is increased further.

The above picture is probably known to many people, e.g. from high school or undergraduate physics. Still, it is important to be clear about the details. The above description refers to a view where temperature is kept constant and volume is controlled. With regards to pressure, one should recall that pressure does not change in the coexistence state. If one uses pressure instead of volume as the control parameter, then there is no finite range of control parameter values where the system is in coexistence (Fig. 2(a)). For the traffic analogy, it will be important to use volume as the control parameter. The inverse of volume is system-wide density, which is a more common variable for traffic systems. It is, however, important to distinguish system-wide density, which we will denote by $\rho_L = 1/V$, from any local density.

Also note that the above description refers to *two phases*, called "liquid" and "gas", but to *three states*, called "liquid", "gas", and "coexistence". The first two states are homogeneous states, since they contain only one phase and are thus spatially homogeneous. The coexistence state contains both phases together.

The above picture is correct in equilibrium, which essentially means after waiting "long enough" while the system is at a fixed ρ_L. If one compresses the system rather quickly beyond ρ_1, then the system is not able to immediately re-organize into droplets: Some time is necessary to achieve this.

The kinetics of the droplet formation (e.g. [22]) is ruled by a balance between surface tension and vapor pressure. Since surface tension pulls the droplet together, it increases the pressure inside the droplet. This interior pressure pushes water molecules out of the droplet. Vapor pressure outside the droplet is the balancing force – it pushes particles into the droplet.

Surface tension and thus interior pressure depend on the droplet radius – the smaller the droplet, the larger the surface tension and thus the interior pressure. The result is that slightly above ρ_1 large droplets are stable, but small droplets are not. Stable and unstable droplets are separated by a critical radius $r_c(\rho)$: Droplets smaller than r_c in the average shrink and thus in the average eventually dissolve; droplets larger than r_c in the average grow.

When ρ_L comes from a low density, the homogeneous phase can survive for some time even slightly above ρ_1, because small droplets are suppressed, while large droplets are not (yet) there. This super-critical gas is thus *meta-stable*. Only after some waiting time one or more droplets will become, by a fluctuation, large enough to go beyond r_c, at which point these droplets will continue to grow until they have swallowed up enough molecules to reduce the gas density outside the droplets to ρ_1. A direct consequence of meta-stability is *hysteresis*: When coming from low densities, it is possible to have $\rho_L > \rho_1$ and still remain in the gas phase.

The description so far refers to a constant temperature. However, ρ_1 and ρ_2 depend on the temperature (Fig. 2(b)). With increasing temperature the densities approach each other, meaning that the densities inside and outside the droplets become more similar. Eventually, there is a temperature T_c where $\rho_1(T_c) = \rho_2(T_c)$. At this point, the densities inside and outside the droplets become the same, which means that they become indistinguishable. In other words: for $T \geq T_c$ there is no coexistence state any more; the system is homogeneous at every density ρ_L.

Said again differently: Depending on the temperature T, our system will either display spontaneous transitions between gas and coexistence and between coexistence and liquid, *or there will be no spontaneous phase separation at all.* (In that latter case, boundary-induced phase separation is still a possibility.)

We will now move on to describe the supporting evidence for the claim that traffic models can show a similar behavior. As is typical in computational science, our evidence is based on computer simulations. It is backed up by generic knowledge about phase transitions as they are well understood in physics.

3 Simulations

In this paper, we will start by using the model by Krauß [23]. As one will see in Sec. 5, the precise details of the model do not really matter. Nevertheless, they are given for technical completeness. The velocity update of the Krauß model reads as follows:

$$v_{\mathrm{safe}} = \tilde{v}(t) + \frac{\frac{g(t)}{\tau} - \tilde{v}(t)}{\bar{v}(t)/(b\,\tau) + 1} \tag{1}$$

$$v_{\mathrm{des}} = \min\{v(t) + a\,\Delta t, v_{\mathrm{safe}}, v_{\max}\} \tag{2}$$

$$v(t+\Delta t) = \max\{0, v_{\mathrm{des}} - \varepsilon\,a\,\eta\}\ . \tag{3}$$

g is the gap (front-bumper-to-front-bumper distance minus space a vehicle uses in a jam), $\tilde{v}$ is the speed of the car in front, $\bar{v} = (v+\tilde{v})/2$ is the average velocity of the two cars involved, $v_{\max}$ is the maximum velocity, a is the maximum acceleration of the vehicles, b their maximum deceleration for $\varepsilon = 0$, ε is the noise amplitude, and η is a random number in $[0,1]$. The meaning of the terms is as follows:

- Eq. 1: Calculation of a "safe" velocity. This is the maximum velocity that the follower can drive to be sure to avoid a crash [23]. The equation states that the follower tries to have the same velocity as the leader, with a gap proportional to the leader's velocity: $g = \tau\,\tilde{v}$. If the gap is larger than that, v_{safe} is larger than the velocity of the leader; if the gap is smaller than that, then v_{safe} is smaller than the velocity of the leader.
- Eq. 2: The desired velocity is the minimum of: (a) current velocity plus acceleration, (b) safe velocity, (c) maximum velocity (e.g. speed limit).
- Eq. 3: Some randomness is added to the desired velocity.

After the velocities of all vehicles are updated, all vehicles are moved.

The Krauß model has been proven to be free of crashes for numerical time steps Δt smaller than or equal to the reaction time, τ [23]. We will use $\Delta t = \tau = 1$ as has conventionally been used for the Krauß model. We further use $a = 0.2$, $b = 0.6$, $v_{\max} = 3$ for all simulations.

The model as defined above is free of units. A reasonable calibration is: one time unit corresponds to one second, and one space unit correspond to 7.5 meters, which is the space that a vehicle occupies in a jam. The reaction time is then 1 second, and $v_{\max} = 3$ corresponds to 22.5 m/s or 81 km/h. $a = 0.2$ means a maximum acceleration of 1.5 m/s (5.4 km/h) per second. $b = 0.6$ corresponds to a maximum deceleration of 16.2 km/h per second.

All simulations are done in a 1-lane system of length L with periodic boundary conditions (i.e. the road is bent into a ring). Let N be the number of cars on the road. The (global) density is $\rho_L = N/L$.

Before analysing the Krauß model quantitatively, it is instructive to look at space-time plots (Fig. 3). The following refers to the subfigures (i)–(vi) of Fig. 3. They are arranged so that they correspond to Fig. 2(b). The *bottom* row corresponds to a smaller noise amplitude $\varepsilon = 1.0$. One recognizes

(iv) The laminar state: All cars drive at high speed. The available space is shared evenly among the cars. The traffic is homogeneous.
(v) The coexistence state: The slow cars are all together in one big jam. On the rest of the road, the cars drive at high speed. In consequence, the traffic is very inhomogeneous.
(vi) The jammed state: The density is so high that no single car can drive fast. As in (iv), the traffic is homogeneous.

In contrast, the top row (i)–(iii) corresponds to a larger noise amplitude $\varepsilon = 1.8$. Here, many small jams are distributed over the whole system. There is neither a larger area of free flow, nor a major jam. The traffic is homogeneous at all densities. Note that "homogeneous" here means "homogeneous on large scales". In (i) and (ii), there is structure, i.e. small jams and laminar flow, but these are not visible when looking at the plots from a distance. In contrast, the coexistence state, as in (v), will never look homogeneous (see Sec. 4 for a more technical version of this).

For many parameters of the Krauß model, there is a unique equilibrium state, which the system will attain after a finite time t_{relax}, no matter how it was started. Deciding when the equilibrium is reached is not trivial. Our criterion was to look at the number of jams in the system (Fig. 4). The system was once started with equidistant vehicles (maximally homogeneous) and once with all vehicles in a "mega-jam" (maximally inhomogeneous). Initially, the number of jams in the system shows very different behavior in those two simulations. However, eventually that number becomes the same in both simulations, at which point it was assumed that equilibrium was reached. A *jam* here is defined as a sequence of adjacent cars driving with speed less or equal $v_{\max}/2$. This definition of a jam is used nowhere else in this paper; it is only used to decide how long a simulation needs to run until one can assume that it has reached equilibrium.

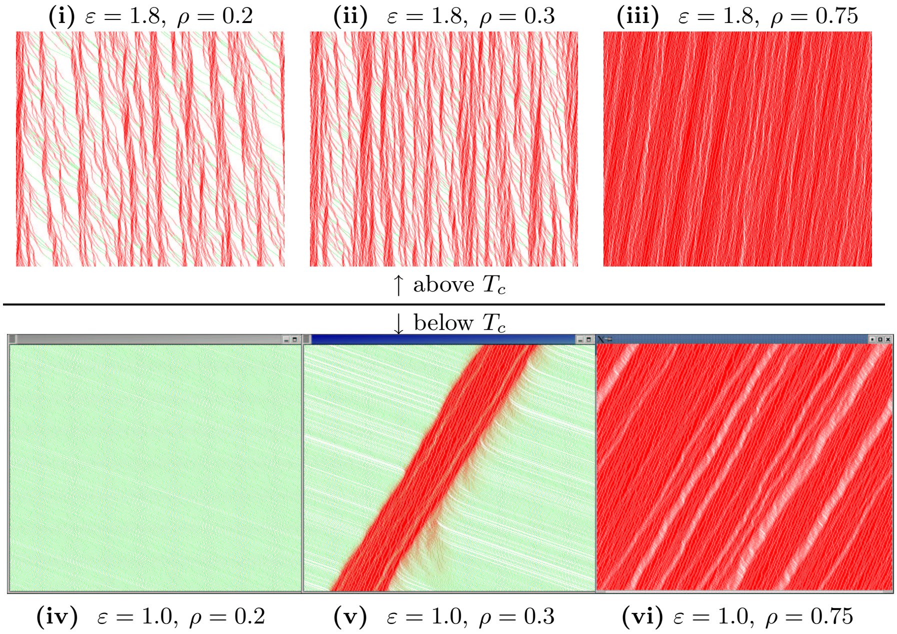

Fig. 3. Space-time plots for different parameters. Space is horizontal; time increases downward; each line is a snapshot; vehicles move from left to right; fast cars are green, slow cars red. $L = 600$ for all plots.

4 Establishment of a phase diagram via a measure of inhomogeneity

One needs to establish a criterion that distinguishes homogeneous from coexistence states. As pointed out before, coexistence states, for example at $\varepsilon = 1.0$ and $\rho = 0.3$ in our model, see Fig. 3(v), are characterized by the coexistence of laminar and jammed traffic. Inside the coexistence regime, the phases coagulate, leading to one large laminar and one large jammed section in the system. When approaching the boundaries of the coexistence regime, this characterization will become less clear-cut, and it may be possible to have more than one jam. Typically, there will be one major jam and many small ones, and for many measurement criteria this will cause enough problems to no longer be able to differentiate between the coexistence and a homogeneous state. This is particularly true for criteria that attempt a binary classification into homogeneous or not. In contrast, our criterion will show a gradual transition.

The criterion is defined as follows: Partition the road into segments of length ℓ (for simplicity let ℓ divide L without remainder). For each segment the local density ρ_ℓ can be computed as the number of cars in that segment divided by ℓ. An interesting value is the variance of the local density (see, e.g.,

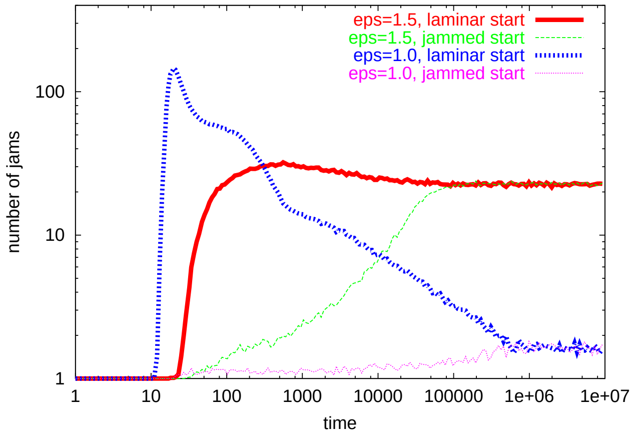

Fig. 4. Time evolution of the number of jams. All four curves are for 1000 cars and $\rho = 0.3$. Each curve is an average over at least 80 realizations, each with a different random seed.

[24]):

$$\mathrm{Var}(\rho_\ell) = \frac{\ell}{L} \sum_{i=1}^{L/\ell} \left(\rho_\ell(i) - \rho_L\right)^2 \quad , \tag{4}$$

where ρ_L is the systemwide average density. Note that since density values always lie within $[0, 1]$, the variance cannot exceed $1/4$.

This value picks up how much each individual measurement segment of length ℓ deviates, in terms of its density, from the average density. Assume a system consisting of jammed and laminar traffic. If there is a jam in one segment, then the segment's density will be much higher than the average density. Conversely, if there is only laminar traffic in a segment, then the segment's density will be much lower than the average density. $\mathrm{Var}(\rho_\ell)$ takes the average over the square of these deviations.

Fig. 5 shows this value as a function of the global density ρ and the noise parameter ε. Each gridpoint is the result of a computer simulation. The simulations run until the average number of jams over the last 100'000 time steps is (almost) equal for a system started with a big jam and a system started with laminar flow (recall Fig. 4). The variance of the local density is averaged over those same 100'000 time steps.

Look at Fig. 5 for fixed noise ε, say $\varepsilon = 1$. One sees that at densities up to $\rho \approx 0.2$, the value of $\mathrm{Var}(\rho_\ell)$ is close to zero, indicating a homogeneous state, which is in this case the laminar state. Similarly, for densities higher than 0.8, $\mathrm{Var}(\rho_\ell)$ is again close to zero, indicating a homogeneous state, which is in this case the jammed state. In between, for $0.2 \le \rho \le 0.8$, the value of $\mathrm{Var}(\rho_\ell)$ is significantly larger than zero, indicating a coexistence state.

Now slowly increase ε. We see that the two critical densities approach each other (see Fig. 5). At $\varepsilon \approx 1.7$, the coexistence phase goes away; for larger ε,

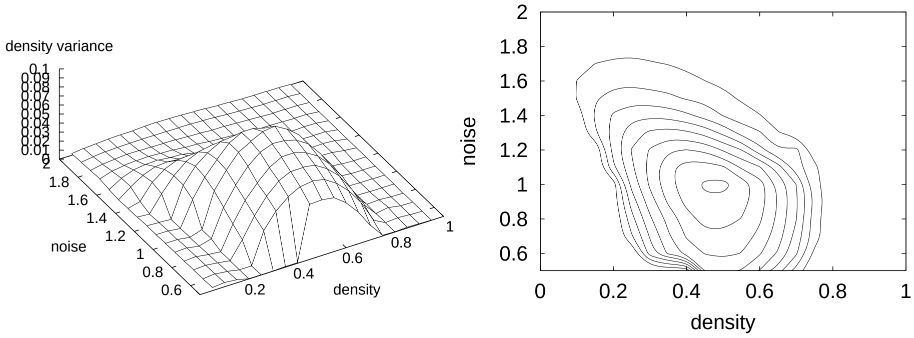

Fig. 5. 3d-plot and contour plot of the density variance for the Krauß model. Both plots show the same data. The outermost isoline is $\mathrm{Var}(\rho_\ell) = 0.01$, the innermost $\mathrm{Var}(\rho_\ell) = 0.09$. $L = 1000$ and $\ell = 62.5$

we do not pick up any inhomogeneity at *any* density (look at the contour plot in order to get information about behavior not visible in the 3d plot). Compare this to the theoretical expectation in Fig. 2(b), where for increasing T the two densities eventually merge and thus the different phases go away. Note that close to but slightly above the critical ε, the system still looks like it possesses different phases (see Fig. 3(i) and locate the corresponding $\varepsilon = 1.8$ and $\rho = 0.2$ in Fig. 5). These structures exist, however, *on small scales only.* This means that for system size $L \to \infty$ and measurement interval $\ell \to \infty$ (but $\ell \ll L$), all intervals of size ℓ will eventually return the same density value. A segment length of $\ell = 62.5$, as used for Fig. 5, is already sufficient in order to not measure any inhomogeneity for the states in Fig. 3(i) and (ii). This will not be the case for coexistence states: In coexistence state, there will always be segments with different densities, unless $\ell \approx L$. This is because droplets will coagulate so that they will eventually show up on all possible length scales ℓ.

Remember again that ε is a model parameter while ρ is a traffic observable. That is, once one has settled for an ε, the model behavior is fixed, and one has decided if one can encounter spontanoues phase separation (= spontaneous jam formation) or not. *If* one can encounter spontaneous phase separation, it will come into existence through changing traffic demand throughout the day – traffic can move from the laminar into the coexistence and potentially into the jammed state and back.

As a side remark, let us note that there is also another regime without spontaneous phase separation for $\varepsilon \to 0$. Albeit potentially interesting, this is outside the scope of this paper.

In summary, one obtains, for the above traffic model and a spatially homogeneous geometry, a phase diagram as in Fig. 2(b), which is the schematic phase diagram for a gas-liquid transition in fluids. Again, the important feature of this phase diagram is that there are three states for low temperatures (small T or small ε): gas/laminar; coexistence; liquid/jammed. For higher temperatures, the coexistence range becomes more and more narrow, while the density of the gas phase and the density of the liquid phase in the coexistence state approach each other. Eventually, these densities become equal, and the coexistence state dies out. The only notable difference is that for our traffic model the phase diagram is bent to the left with increasing ε.

There are other criteria which can be used to understand these types of phase transitions. In particular, one can look at the gap distribution between jams, and one would expect a fractal structure at the critical point, i.e. at $\rho \approx 0.2$ and $\varepsilon \approx 1.7$. This is indeed the case but goes beyond the scope of this paper; see [25] for further information.

5 Cellular automata models

Many of the arguments regarding the nature of a stochastic and possibly critical phase transition [26–30] have been made using so-called cellular automata (CA) models. CA models use coarse spatial, temporal, and state space resolution. For traffic, a standard way is to segment a 1-lane road into cells of length l_c, where l_c is the length a vehicle occupies in the average in a jam, i.e. $l_c = 1/\rho_{jam} \approx 7.5$ m. Cells are either occupied by exactly one car, or are empty. Vehicles move by jumping from one cell to another. As with the Krauß model, the time step for the CA models is best selected similar to the reaction time; a time step of 1 second works well in practice. Taking this time step together with l_c, one finds that a speed of 135 km/h corresponds to five cells per time step; this is often taken as maximum velocity $v_{\max}$.

A possible CA velocity update rule is [31]:

- Deterministic car driving:

$$v_{t+\frac{1}{2}} = \min[g_t, v_t + 1, v_{max}] \ , \tag{5}$$

  where g_t is the gap (number of empty cells ahead) at time t.
- Randomization:

$$v_t = \begin{cases} \max[0, v_{t+\frac{1}{2}} - 1] & \text{with probability } p_{\text{slow}}(v_t) \\ v_{t+\frac{1}{2}} & \text{else} \end{cases} \ , \tag{6}$$

  where $p_{\text{slow}}(v)$ is a velocity-dependent randomization. Often-selected values are

$$p_{\text{slow}}(v) = \begin{cases} p_0 = 0.5 & \text{if } v_t = 0 \\ p_{>0} = 0.01 & \text{if } v_t > 0 \end{cases} \ .$$

  which models that drivers, once stopped, are a bit sloppy in re-starting again. $p_{>0} = p_0 = 0.5$ returns the CA of [32].

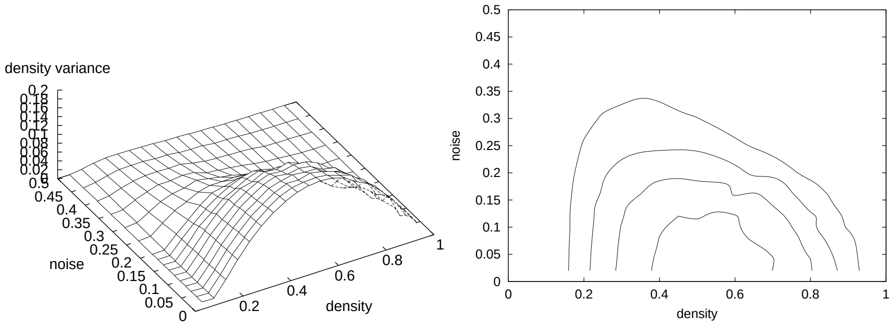

Fig. 6. 3d-plot and contour plot of the density variance for the CA model. Both plots show the same data. Instead of the noise parameter ε, the randomness-parameter $p_{>0}$ is varied from zero to one.

With this family of models, one can again plot the density variance (Fig. 6). Instead of the noise amplitude ε, the parameter $p_{>0}$ is used. $p_{>0} = 0$ means deterministic driving except when accelerating from zero; increasing $p_{>0}$ means increasingly more randomness when moving. In Fig. 6, one finds a behavior similar to Fig. 5: For small $p_{>0}$, the system displays three states (laminar, coexistence, and jammed). For $p_{>0} \to 0.5$, the system becomes eventually a system without spontaneous phase separation.

From this plot, it is impossible to decide exactly at which $p_{>0}$ the transition from a model with to a model without spontaneous phase separation takes place. Nevertheless, this plot makes clear why there was so much discussion about possible fractals for the original model [32] in which $p_{>0} = 0.5$: That model is indeed close to the critical point, and in consequence one should expect fractals up to a certain cut-off length scale. That cut-off length scale should depend on the distance to the critical point; further investigations are necessary to exactly determine the correct value of the critical point.

6 Phase transitions in deterministic models

Only stochastic models can display *spontaneous* transitions between homogeneous and coexistence states. The nature of the transition can however also become clear in deterministic models. We will discuss these similarities first for a deterministic car following model and then for deterministic fluid-dynamical models.

A possible **car-following model** is [33]

$$a(t) = \alpha \cdot (V(g(t)) - v(t)) \ , \quad \text{with} \quad V(g) = v_f \cdot (\tanh(g + l_c) - \tanh(l_c)) \ , \tag{7}$$

where a is the acceleration, g is again the gap, $V(.)$ is a desired velocity, l_c is the space a vehicle occupies in a jam (7.5 m in the previous models), and

v_f is the free speed. For this model, it was shown [9] that the homogeneous solution of the model is linearly unstable for densities where $dV/dg > \alpha/2$. The instability sets in for intermediate densities; for low and high densities *all* models are stable in the homogeneous (laminar or jammed) state. One can thus select the curve $V(g)$ and the parameter α such that the model either has unstable ranges, or not. If all parameters, including the density, are such that the homogeneous solution is not stable, then the system rearranges itself into a pattern of stop-and-go traffic, corresponding to the coexistence state. The density of the laminar and the jammed phase in the coexistence state are independent from the average system density, that is, if in that state system density goes up, it is reflected in the jammed phase using up a larger fraction of space.

Fluid-dynamical theory, of the type

$$\partial_t \rho + \partial_x(\rho\, v) = 0 \tag{8}$$

$$\partial_t v + v\, \partial_x v = \frac{1}{\tau}\, (V(\rho) - v) + \alpha(\rho)\, \partial_x \rho + \nu(\rho)\, \partial_x^2 v \tag{9}$$

can, depending on the choice of parameters including the $V(\rho)$-curve, either display or not display spontaneous phase separation [34]. For example, the homogeneous solution of the model with $\alpha(\rho) = c_0^2/\rho$ and $\nu(\rho) = \nu_0$ is linearly unstable at densities where $|dV/d\rho| > c_0/\rho$ [34]. This is similar to the instability condition for the car following model above; note that $V'(\rho)$ and $V'(g)$ are, albeit related, not the same.

As pointed out before, these models are deterministic. In no situation will these models display *stochastic* transitions.

7 Discussion

As mentioned in the introduction, there is discussion in the literature if traffic shows spontaneous jam formation, or if all jams are caused by geometrical constraints such as bottlenecks. That discussion was in the past hampered by the fact that no clear picture for spontaneous jam formation in stochastic models was available: The introduction of the slow-to-start CA (s2s-CA) models was guided by the observation that the original CA [32] model did not display true meta-stability, but no convincing overall picture emerged. In particular, it was never clarified if or why the origial CA displayed fractal properties, and how these fractal properties change when moving towards s2s-CA models. In contrast, the present paper allows, for people sufficiently versed in the theory, a clear prediction: The original CA should display fractal properties up to a certain cut-off; that cut-off should become larger and eventually diverge with decreasing $p_{>0}$; it should then become smaller again, until eventually one cannot speak of fractals any more.

In addition, better understanding allows to make better predictions for properties besides spontaneous breakdown. For example, one would predict that a traffic queue, when operating at an average density between ρ_1 and ρ_2, would show phase separation; those phases should have the densities ρ_1 and ρ_2, and they should coagulate with increasing distance from the bottleneck. Unfortunately, coagulation is a slow process, and for that reason once more the issue cannot be resolved easily.

8 Summary

This paper shows, via computational evidence, that a specific stochastic car following model can either display or not display spontaneous phase separation, depending on the choice of parameters. The two phases are: "laminar", and "jammed". Models with spontaneous phase separation possess two homogeneous states, which correspond to the phases. They also possess a third state, at intermediate densities, which is a coexistence state. It consists of sections with jammed and sections with laminar traffic.

With respect to cellular automata (CA) models, it turns out that one of the early CA models for traffic [32] is a model without spontaneous phase separation, but close to such a model, which explains the near-fractal structures which have been observed. In contrast, the so-called slow-to-start models [31] display clear phase separation.

Some of these findings can also be understood by looking at deterministic models for traffic, either car-following or fluid-dynamical. However, the stochastic elements of the transition cannot be explained by deterministic models. An important stochastic element is meta-stability, which means that a "super-critical" homogeneous state can survive for long times before it "breaks down" and reorganizes into stop-and-go traffic.

It is important to understand this possibility of stochastic models to be in different regimes if one considers to enter the notion of traffic breakdown probabilities into the Highway Capacity Manual. If traffic is best described by a model without spontaneous phase separation, then there is, in our view, no theoretical justification for (spontaneous) breakdown probabilities. If, however, traffic is best described by a model with spontaneous phase separation, then such breakdown probabilities make sense.

Acknowledgments

F. Wagner, D. Stauffer, and M. Droz have significantly helped in defining computational criteria for the detection of multiple phases. P. Wagner, C. Daganzo, and M. Cassidy have helped indirectly (and maybe inadvertently), via the discussion of a separate but related paper. The paper has also benefited from discussions with P. Nelson, although the consequences of those discussions have not yet completely sunk in. Several weeks of computing time on

192-CPU cluster Xibalba and on 44-CPU cluster Linneus were used for the computational results.

Note added – Gray, Levine, Mukamel and Ziv have made considerable progress with respect to theoretical results in the same area ([35] and references therin). However, none of their results so far completely explains stochastic cases which simultaneously have "slow-to-start" and $v_{max} > 1$.

References

1. M. Koshi, M. Iwasaki, and I. Ohkura. Some findings and an overview of vehicular flow characteristics. In *Proceedings of the 8th International Symposium on Transportation and Traffic Theory*, pages 403–426. Elsevier, Amsterdam, Toronto, 1983.
2. B. S. Kerner. Phase transitions in traffic flow. In D. Helbing, H. J. Hermann, M. Schreckenberg, and D. E. Wolf, editors, *Traffic and Granular Flow '99*, pages 253–284. Springer, Berlin, 1999.
3. J. Treiterer and J.A. Myers. The hysteresis phenomenon in traffic flow. In D.J. Buckley, editor, *Proc. 6th Internat. Sympos. on Transportation and Traffic Theory*, pages 13–38, Sydney, Australia, 1974. Elsevier, New York.
4. B. S. Kerner and H. Rehborn. Experimental features and characteristics of traffic jams. *Phys. Rev. E*, 53(2):R1297–R1300, 1996.
5. J.A. Acha-Daza and F.L. Hall. Graphical comparison of predictions for speed given by catastrophe theory and some classic models. *Transportation Research Record*, 1398:119–124, 1993.
6. I. Prigogine and R. Herman. *Kinetic theory of vehicular traffic*. Elsevier, New York, 1971.
7. H. Reiss, A.D. Hammerich, and E.W. Montroll. Thermodynamic treatment of nonphysical systems: Formalism and an example (single-lane traffic). *Journal of Statistical Physics*, 42(3/4):647–687, 1986.
8. B. S. Kerner and P. Konhäuser. Structure and parameters of clusters in traffic flow. *Phys. Rev. E*, 50(1):54–83, 1994.
9. M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama. Structure stability of congestion in traffic dynamics. *Japan Journal of Industrial and Applied Mathematics*, 11(2):203–223, 1994.
10. M.J. Cassidy. Bivariate relations in nearly stationary highway traffic. *Transportation Research B*, 32B:49–59, 1998.
11. J. C. Muñoz and C. F. Daganzo. Structure of the transition zone behind freeway queues. *Transportation Science*, 37(3):312–329, 2003.
12. Y. Sugiyama, A. Nakayama, M. Fukui, M. Kikuchi, K. Hasebe, K. Nishinari, S.-i. Tadaki, and S. Yukawa. Observations, theories, and experiments for freeway traffic as physics of many-body system. In M. Schreckenberg P. Bovy, S. Hoogendoorn and D.E. Wolf, editors, *Traffic and granular flow '03*. in press.
13. B. S. Kerner and H. Rehborn. Experimental properties of complexity in traffic flow. *Phys. Rev. E*, 53(5):R4275–R4278, 1996.
14. B. S. Kerner and H. Rehborn. Experimental properties of phase transitions in traffic flow. *Phys. Rev. Letters*, 79(20):4030–4033, 1997.

15. B. Persaud, S. Yagar, D. Tsui, and H. Look. Breakdown-related capacity for freeway with ramp metering. *Transportation Research Record*, 1748:110–115, 2001.
16. H. Zackor, R. Kühne, and W. Balz. Untersuchungen des Verkehrsablaufs im Bereich der Leistungsfähigkeit und bei instabilem Fluß. Forschung Straßenbau und Straßenverkehrstechnik 524, Bundesminister für Verkehr, Bonn–Bad Godesberg, 1988.
17. L. Elefteriadou. Personal communication, Nov 2001.
18. R. Herman and I. Prigogine. A two-fluid approach to town traffic. *Science*, 204:148–151, 1979.
19. DYNAMIT www page. See its.mit.edu and dynamictrafficassignment.org, accessed 2003.
20. DYNASMART. See www.dynasmart.com and dynamictrafficassignment.org, accessed 2003.
21. C. Gawron. An iterative algorithm to determine the dynamic user equilibrium in a traffic simulation model. *International Journal of Modern Physics C*, 9(3):393–407, 1998.
22. E.M. Lifschitz and L.P. Pitajewski. Statistische Physik, Teil 1. Lehrbuch der Theoretischen Physik. Akademie-Verlag, 1987.
23. S. Krauß. *Microscopic modeling of traffic flow: Investigation of collision free vehicle dynamics*. PhD thesis, University of Cologne, Germany, 1997. See www.zaik.uni-koeln.de/~paper.
24. D. Helbing and T. Platkowski. Drift- or fluctuation-induced ordering and self-organization in driven many-particle systems. *Europhysics Letters*, 60(2):227–233, 2002.
25. D. Jost. Breakdown and recovery in traffic flow models. Master's thesis, ETH Zurich, Switzerland, 2002. See e-collection.ethbib.ethz.ch.
26. K. Nagel. Life-times of simulated traffic jams. *International Journal of Modern Physics C*, 5(3):567–580, 1994.
27. K. Nagel and M. Paczuski. Emergent traffic jams. *Phys. Rev. E*, 51:2909–2918, 1995.
28. M. Sasvari and J. Kertesz. Cellular automata models of single lane traffic. *Phys. Rev. E*, 56(4):4104–4110, 1997.
29. L. Roters, S. Lübeck, and K.D. Usadel. Critical behavior of a traffic flow model. *Phys. Rev. E*, 59(3):2672–2676, 1999.
30. D. Chowdhury, J. Kertész, K. Nagel, L. Santen, and M. Schadschneider. Comment on: "Critical behavior of a traffic flow model". *Phys. Rev. E*, 61(3):3270–3271, 2000.
31. R. Barlovic, L. Santen, A. Schadschneider, and M. Schreckenberg. Metastable states in CA models for traffic flow. *European Physical Journal B*, 5(3):793–800, 1998.
32. K. Nagel and M. Schreckenberg. A cellular automaton model for freeway traffic. *Journal de Physique I France*, 2:2221–2229, 1992.
33. M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama. Dynamical model of traffic congestion and numerical simulation. *Phys. Rev. E*, 51(2):1035–1042, 1995.
34. R.D. Kühne and R. Beckschulte. Non-linearity stochastics of unstable traffic flow. In C.F. Daganzo, editor, *Proc. 12th Int. Symposium on Theory of Traffic Flow and Transportation*, page 367. Elsevier, Amsterdam, The Netherlands, 1993.

35. E. Levine, D. Mukamel, and G. Ziv. Condensation transition in zero-range processes with diffusion. *cond-mat/042520v1*, 2004.