Schramm-Loewner Evolution and Path Regularity

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von der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation

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Tag der wissenschaftlichen Aussprache: 26.10.2021

Berlin 2022

Abstract

We investigate fundamental questions regarding (chordal) Loewner chains and the construction of SLE trace. Our problems concern the existence of a continuous trace, its regularity, and dependence on the driving function.

Chapter 1 is the content of [Yua20] (Indiana Univ. Math. J., to appear). We show a deterministic result characterising traces that can arise from Loewner chains driven by general continuous driving functions. By definition, these are the curves that satisfy a local growth property. We show two equivalent characterisations: 1. A continuous curve is a trace if and only if mapping out any initial segment preserves its continuity (which can be seen as an analogue of the domain Markov property of SLE). 2. The (not necessarily simple) traces are exactly the uniform limits of simple traces. Moreover, using methods by [LMR10], we infer that uniform convergence of traces imply uniform convergence of their driving functions.

Chapter 2 is the content of [Yua21b]. We drive the Loewner differential equation with non-constant random parameter, i.e. $d\xi(t) = \sqrt{\kappa_t} dB_t$. We show that in case κ_t is bounded below or above 8, the construction still yields a continuous trace. This is true in both cases either when driving the forward equation or the backward equation by $\sqrt{\kappa_t} dB_t$. In the case of the forward equation, we develop a new argument to show the result, without the need of analysing the time-reversed equation.

Chapter 3 is an earlier version of [Yua21a]. We extend an idea from the second chapter to show refined regularity statements for classical SLE_κ. In particular, we show ψ-variation regularity with $\psi(x) = x^p (\log 1/x)^{-p-\varepsilon}$ and Hölder-type modulus $\varphi(x) = x^\alpha (\log 1/x)^{1+\varepsilon}$ where p and α are the optimal exponents shown in [FT17; JL11] respectively. This sharpens the results of the aforementioned references.

Chapter 4 is the content of [FTY21] (joint with P. K. Friz and H. Tran, Probab. Theory Related Fields). We investigate the stability of SLE_κ in the parameter κ. By driving the Loewner equation with $\sqrt{\kappa}B$, we get a family of SLE_κ traces that we interpret as a random field $\gamma(t, \kappa)$. We improve a result of [JRW14] and show that this random field is jointly Hölder continuous for $\kappa < 8/3$. Moreover, we show that the SLE_κ trace $\gamma(\cdot, \kappa)$ (as a continuous path) is stochastically continuous in κ at all $\kappa \neq 8$. Our proofs rely on a novel variation of the Garsia-Rodemich-Rumsey (GRR) inequality, which is of independent interest.

Zusammenfassung

Wir untersuchen grundlegende Fragestellungen bezüglich (chordal) Löwner chains und der Konstruktion der SLE trace. Wir befassen uns mit Fragen zur Existenz einer stetigen trace, deren Regularität und der Abhängigkeit von der driving function.

Kapitel 1 ist der Inhalt von [Yua20] (Indiana Univ. Math. J., erscheint bald). Wir zeigen ein deterministisches Resultat, die diejenigen traces charakterisiert, die aus Löwner chains mit allgemeinen stetigen driving functions entstehen können. Nach Definition sind das die Kurven, die eine lokale Wachstumseigenschaft aufweisen. Wir zeigen zwei äquivalente Charakterisierungen: 1. Eine stetige Kurve ist genau dann eine trace, wenn das mapping out eines beliebigen Anfangssegments ihre Stetigkeit beibehält. (Dies kann als ein Analogon zur domain Markov property der SLE gesehen werden.) 2. Die stetigen (nicht notwendigerweise einfachen) traces sind genau die gleichmäßigen Limiten von einfachen traces. Weiterhin folgern wir aus den Methoden von [LMR10], dass gleichmäßige Konvergenz von traces die gleichmäßige Konvergenz derer driving functions impliziert.

Kapitel 2 ist der Inhalt von [Yua21b]. Wir betrachten die Löwner-Differentialgleichung mit zeitabhängigem zufälligem Parameter, d.h. $d\xi(t) = \sqrt{\kappa_t} dB_t$. Wir zeigen im Fall, dass κ_t unter oder über 8 beschränkt ist, dass die Konstruktion immer noch eine stetige trace erzeugt. Dies gilt sowohl für die Vorwärts- als auch die Rückwärtsgleichung mit driving function $\sqrt{\kappa_t} dB_t$. Für den Fall der Vorwärtsgleichung entwickeln wir ein neues Argument, das das Resultat beweist, ohne die Rückwärtsgleichung analysieren zu müssen.

Kapitel 3 ist eine frühere Version von [Yua21a]. Wir erweitern eine Idee aus dem zweiten Kapitel und zeigen verfeinerte Regularitätsaussagen für die klassische SLE_κ. Dabei zeigen wir endliche ψ-Variation für $\psi(x) = x^p (\log 1/x)^{-p-\varepsilon}$ und ein Hölder-artiges Stetigkeitsmodul mit $\varphi(x) = x^\alpha (\log 1/x)^{1+\varepsilon}$, wobei p und α die optimalen Exponenten sind, die jeweils in [FT17; JL11] bewiesen worden sind. Unser Resultat verschärft diese beiden Ergebnisse.

Kapitel 4 ist der Inhalt von [FTY21] (gemeinsam mit P. K. Friz und H. Tran, Probab. Theory Related Fields). Wir untersuchen die Stabilität der SLE_κ im Parameter κ. Aus der Löwner-Gleichung mit den driving functions $\sqrt{\kappa} B$ erhalten wir eine Familie von SLE_κ traces, die wir als stochastisches Feld $\gamma(t, \kappa)$ sehen können. Wir verbessern ein Resultat von [JRW14] und zeigen, dass dieses Feld Hölder-stetig gemeinsam in beiden Parametern ist, solange $\kappa < 8/3$. Weiterhin zeigen wir, dass die SLE_κ trace $\gamma(\cdot, \kappa)$ (als stetiger Pfad) stochastisch stetig in κ ist für $\kappa \neq 8$. Wir verwenden im Beweis eine neue Variante der Garsia-Rodemich-Rumsey-Ungleichung, welche auch für sich von Interesse ist.

Acknowledgements

I acknowledge partial support from European Research Council through Consolidator Grant 683164 (PI: Peter Friz). I thank Peter Friz for his supervision and guidance, and more importantly, his encouragement. I had some fears about the academic career, and he encouraged me pursuing it nevertheless. I thank Huy Tran for his tips and collaboration during my Master's thesis and at the beginning of my doctoral phase. I thank Steffen Rohde for many discussions and comments, in particular his visits to Berlin and my visit to the University of Washington. I also thank Dmitry Belyaev for discussions at our Berlin-Oxford meetings.

Moreover, I had many possibilities to learn and broaden my knowledge which include the 49th Saint-Flour Probability Summer School. In my last year I have benefitted from the new IRTG 2544 which enabled many interactions with fellow postgraduate students. The various reading groups were an invaluable opportunity to learn new things beyond my initial horizon.

Of course, there is a lot in my personal life I am grateful for. Without all the people that I have met I would not be in the stage of life that I am currently. Having gone through times of self-doubt, I am grateful for the personal development I went through.

I am lucky having grown up in a family where an academic career is nothing extraordinary, and I did not need to fight for it. I am grateful to my friends who stayed with me through personal and emotional challenges. I am grateful for the personal coaching that I had committed to, and all the changes in my beliefs and my life, some I had not believed to be possible. Most importantly, all the people that I met on the way and shaped me. I apologise that I cannot mention every person individually, but I do hope meeting you in life again and giving you the appreciation that you deserve.

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What evolution?

Many people are interested in studying conformally invariant random objects in two dimensions. This is partly motivated by the physics belief that critical models in the scaling limit are conformally invariant. Describing such objects mathematically has posted quite a few challenges (and still do). For instance, the Ising model is a model for the particle spins in a ferromagnet. At critical temperature, it is believed to converge to some conformal field theory. Despite some recent progresses, these things are still beyond mathematical understanding.

An important milestone was the realisation that certain interfaces in such conformally invariant models can be described mathematically by a family of curves that are nowadays called Schramm-Loewner evolution (originally stochastic Loewner evolution). They were introduced in [Sch00] as candidates for the scaling limits of the loop-erased random walk and uniform spanning tree. This was indeed confirmed in [LSW04], and many other models have been rigorously proved to converge to SLE, cf. [Smi01; SS09; Che+14]. Amongst others, S. Smirnov received a fields medal for his work on the conformal invariance of critical Bernoulli percolation and the Ising model.

SLE has then been studied by many authors. It can be described via a simple equation, and many quantities can be computed explicitly which makes it very attractive. Since it is "the" conformally invariant curve in the plane, it relates to many other objects. By comparing it to planar Brownian motion, G. Lawler, O. Schramm, and W. Werner have proved also many geometric properties of the latter (e.g. the famous Mandelbrot conjecture) ([LSW01a; LSW01b; LSW02]), and ultimately W. Werner was handed a fields medal for his work.

The introduction of SLE has been a key step in further advancing the study of random conformal geometry. SLE was originally introduced as a very specific interface appearing in the models, but from there people have vastly extended the definitions and are now able to describe a great variety of curves that appear in these contexts. This includes SLE with additional force points ([LSW03; SW05; MS16]), conformal loop ensembles ([CN06; She09; SW12]), multiple SLEs ([KL07; Dub07; PW19; BPW21]), etc. It serves also as a key ingredient in studying other objects such as the Gaussian free field and Liouville quantum gravity. Many connection between these objects have been made e.g. in [Dub09; SS13; MS16; She16; DMS20].

Despite the recent huge progress in understanding SLE and related objects, many fundamental questions are still open. In the following, we will concern ourselves with the most basic variant of chordal SLE which is a curve in the upper-half plane from 0 to ∞. One (and so far the only) way of constructing chordal SLE_κ uses (a variant of) the Loewner differential equation driven by a Brownian motion with speed κ. What is different from the usual theory of stochastic processes is that the equation describes not the SLE curve itself, but rather conformal maps of the complement of the curve. As a consequence, we can very well answer questions that are observed "off the curve" (such as the distance to the curve as in [Bef08; LW13]). On the other hand, the behaviour of the curve itself is more tricky. In particular, it is not a semimartingale or a Markov process. (It does satisfy a type of domain Markov property, though.) So even the regularity of SLE requires some arguments beyond classical stochastic analysis.

From the complex analytic point of view, there is an even more fundamental question. The Loewner differential equation can take any continuous function as driving function[1], but not every driving function gives rise to a continuous trace. Moreover, since the Loewner differential equation describes the conformal maps and not the trace (which makes the boundary of the conformal maps), the dependence of the trace on the driving function is not clear. In the more regular world, e.g. [LMR10; RTZ18] for 1/2-Hölder driving functions, we always have a continuous trace which is a quasislit and depends continuously on the driving function. But there are very few results beyond that regularity. In particular, all the proof for Brownian motion are based on probabilistic estimates. This leaves quite many questions open.

- What are the properties of Brownian paths that make the Loewner chain generate a trace?
- Do all multiples of a fixed Brownian path generate a trace? If so, is there a nice (e.g. continuous) dependence of the trace on the Brownian path?
- What is the correct regularity of SLE_κ (in particular SLE_8)?

We will tackle some of the questions in the dissertation. We are still far from a purely deterministic / complex analytic treatment of the questions above. Instead, we are still using probabilistic arguments to answer some of the questions above.

We begin in Chapter 1 with a fundamental question in the deterministic theory of Loewner chains. A basic feature of Loewner chains driven by continuous driving functions is that they produce hulls with the local growth property. One can say that intuitively, the corresponding trace cannot "cross over" itself (but may self-intersect and "bounce off"). In case the trace has finitely many self-intersections, this statement can easily be formalised. But traces may have infinitely many self-intersections and even be space-filling. We will give a topological characterisation of traces that describes what we mean in general by "non-crossing" curves. As a consequence, we answer another fundamental question about the existence of traces, namely that the property of a driving function to generate a trace depends only on its local behaviour. It is easy to see that concatenating

[1] This can be further generalised, cf. [Law05, Chapter 4], [GW08; CR09; APW20]. But continuous driving functions play a special role since they correspond to a local growth property of the Loewner chain.

driving functions that generate traces gives rise to a trace again. We show that the converse is true as well, i.e. when a driving function generates a continuous trace, so does its restriction to any subinterval. Although this seems intuitive at first sight, a rigorous proof is not at all obvious.[2]

There are not many results in the literature about driving functions that are beyond 1/2-Hölder regularity besides Brownian motion (some can be found in [FS17; STW19]). In Chapter 2, we investigate Brownian motion with non-constant speed. We show that when the speed parameter (κ_t) is adapted and bounded either below or above 8, then both the forward and backward Loewner chain generate continuous traces. One interesting thing about the proof in the forward adapted case is that we need an argument to analyse the trace without using the time-reversed equation (as is usually done in the literature). We will use that argument also in Chapter 3.

In Chapter 3, we prove refined regularity results for SLE_κ. The optimal p-variation and Hölder exponents of the SLE_κ trace are known from [JL11; Bef08; FT17], but we do not know whether we have the regularity for the critical exponents. Ideally, we would like to know the precise regularity. We show that adding logarithmic factors to the critical exponents indeed give us regularity. Our arguments build on and extend those from Chapter 2.

Chapter 4 comes from a joint project with Peter Friz and Huy Tran. We show that for almost all Brownian paths B, the driving functions $\sqrt{\kappa}B$, $\kappa < 8/3$, each generate a continuous trace, and the trace depends (Hölder-)continuously on κ in that range. This improves a result by [JRW14] who proved the statement for $\kappa < 8(2-\sqrt{3}) \approx 2.1$. Our two main contributions are the following:
1. We prove a stability result for the conformal maps driven by different κ. This improves the result of [JRW14] and is the main ingredient for pushing the limit from $8(2-\sqrt{3})$ to $8/3$. What is a bit ironic is that although the ultimate goal would be having a purely deterministic treatment, we actually introduce more probability into the game. Our stability result is based on the argument of [JRW14], but instead of writing down an estimate for arbitrary driving functions, our improvement relies on the fact that we use $\sqrt{\kappa}B$ as driving function and use probabilistic estimates to continue from an early point.
2. We prove a new variant of the Garsia-Rodemich-Rumsey inequality (and consequently Kolmogorov continuity theorem) which is then applied with our stability result as an input. This way of utilising the estimates provides a few advantages to the Whitney-type partition arguments used in [RS05; JRW14]: The GRR / Kolmogorov theorem applies to any stochastic process that satisfies a similar estimate on the moments as in our case. Moreover, it provides a more explicit description on the Hölder constants of the process.

The results of the dissertation contribute one step in understanding the path regularity of SLE. We still mainly use probabilistic arguments, and are far from a deterministic theory that build only on pathwise properties of driving functions. But nevertheless we have gained some understanding what kind of results are true in the rough regime. A

[2] After a discussion at a meeting in Berlin without a definite answer, I had posted this question in my Master's thesis with the reward of a cookie.

further project is to find lower bounds for the results in Chapter 3 (as was done in [JL11] on the Hölder scale). Hopefully we will then be able to find the precise regularity of SLE_κ.

It seems also reasonale that the methods used in Chapter 3 may be useful also in the treatment of deterministic Loewner chains such as in [RTZ18; FS17; STW19]. This opens a new approach for proving and improving results related to regularity and stability of Loewner traces.

Moreover, the arguments from Chapters 2 and 3, analysing the trace via the forward Loewner equation may be applied to related problems such as in Chapter 4 (There we still use the backward equation.) It is to be checked whether similar techniques can be applied to improve our result. Another related goal would be to investigate perturbations of the Brownian driving function by other (e.g. smooth) functions.

Background on (chordal) Loewner chains and SLE

We review the basics on (chordal) Loewner chains and SLE. More detailed information can be found in [Law05; Kem17] and the excellent lecture notes [BN16].

We denote the complex plane by $\mathbb{C}$, and the extended complex plane by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The upper half-plane is denoted by $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and the closed upper half-plane by $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$. The unit disc is denoted by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

When one constructs SLE, one does not construct the curve directly but rather its complement $H_t = \mathbb{H} \setminus \gamma[0,t]$. In case of non-simple curves, one takes instead the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$ (see Figure 1). This is a simply connected domain, so one can equivalently describe it by its Riemann map $g_t \colon H_t \to \mathbb{H}$. One way of fixing the normalisation is the hydrodynamic normalisation at ∞, i.e. $g_t(z) = z + O(1/z)$ as $z \to \infty$. It turns out that one can parametrise the SLE curve such that the family of maps (g_t) satisfies (a chordal variant of) Loewner's differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \tag{1}$$

where $\xi(t) = \sqrt{\kappa} B_t$, B a standard Brownian motion, and $\kappa \geq 0$ a parameter. In that case, we have

$$\gamma(t) = \lim_{y \searrow 0} g_t^{-1}(iy + \xi(t))$$

which is called the SLE_κ trace.

There is a general theory that works with arbitrary continuous driving functions $\xi \colon [0,\infty[\to \mathbb{R}$. These turn out to be in one-to-one correspondence with families of compact $\mathbb{H}$-hulls having the local growth property. In the SLE case, the hulls are $\gamma[0,t]$ or, in the case of non-simple curves, its union with the bounded connected components of $\mathbb{H} \setminus \gamma[0,t]$ (which we will denote by $\operatorname{fill}(\gamma[0,t])$).

We call a compact set $K \subseteq \overline{\mathbb{H}}$ a compact $\mathbb{H}$-hull if $\mathbb{H} \setminus K$ is simply connected. We identify two compact $\mathbb{H}$-hulls when $K \cap \mathbb{H} = \tilde{K} \cap \mathbb{H}$. For each such set, there exists a

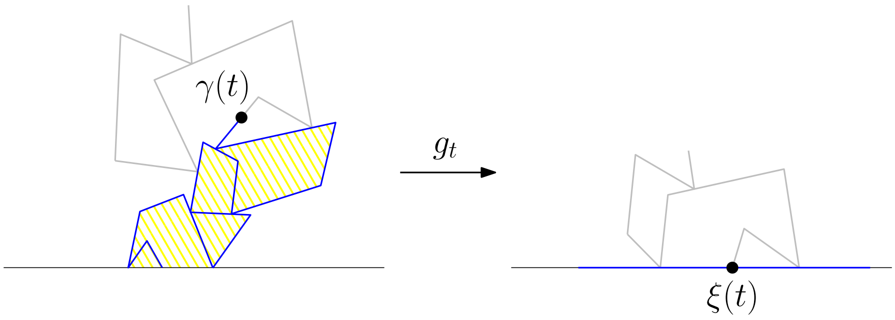

Figure 1: The mapping-out function g_t in case of non-simple curves.

unique Riemann map $g_K\colon \mathbb{H}\setminus K \to \mathbb{H}$ that satisfies the hydrodynamic normalisation at ∞. It has an expansion

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}) \quad \text{as} \quad z \to \infty$$

where $a_K \geq 0$. We call $\operatorname{hcap}(K) = a_K$ the half-plane capacity of K.

Consider a strictly increasing family $(K_t)_{t\geq 0}$ of compact $\mathbb{H}$-hulls. Let us write $g_t = g_{K_t}$ and $K_{s,t} = g_s(K_t \setminus K_s)$ for $s \leq t$. We say that the family $(K_t)_{t\geq 0}$ satisfies the local growth property if for any $T > 0$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that $\operatorname{diam}(K_{t,t+\delta}) < \varepsilon$ for all $t \in [0,T]$. An equivalent condition is that for any $T > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in [0,T]$ there exists some connected subset $C \subseteq \mathbb{H}\setminus K_t$ with $\operatorname{diam} C < \varepsilon$ that separates $K_{t+\delta}\setminus K_t$ from ∞ in the domain $\mathbb{H}\setminus K_t$. One can also show that one can pick C to be a crosscut of length $< \varepsilon$.

For any such family, the function $t \mapsto \operatorname{hcap}(K_t)$ is continuous and strictly increasing. In particular, one can parametrise it by half-plane capacity, i.e. $\operatorname{hcap}(K_t) = 2t$. There is a one-to-one correspondence between such families of hulls and continuous functions $\xi\colon [0,\infty[\to \mathbb{R}$ (called Loewner transform or driving function). The point $\xi(t)$ is characterised as the unique point on $\mathbb{R}$ such that $\xi(t) \in \overline{K_{t,t+h}}$ for all $h > 0$. Conversely, one obtains the family (g_t) from ξ via the Loewner differential equation (1).[3] In particular, K_t is precisely the set of points $z \in \mathbb{H}$ such that the equation is not solvable up to time t, i.e. the denominator hits 0 before time t. This is true also for $z \in \mathbb{R}$ in case $K_t = \overline{K_t \cap \mathbb{H}}$, and g_t extends via Schwarz reflexion to $\mathbb{R}\setminus K_t$.

Recall that we want to describe the SLE curve by $K_t = \operatorname{fill}(\gamma[0,t])$. More generally, we say that the Loewner chain driven by ξ has a continuous trace if there exists a continuous path γ such that $K_t = \operatorname{fill}(\gamma[0,t])$ for every t. An equivalent condition is saying that $\gamma(t) = \lim_{y\searrow 0} g_t^{-1}(iy + \xi(t))$ exists for all t and is continuous in t. This follows from [RS05, Theorem 4.1] and general results about boundary continuity of conformal maps (cf. [Pom92, Section 2.2]). A trace is called simple if it does not intersect itself nor $\mathbb{R}$ (except at $t = 0$).

In case of SLE_κ which is driven by $\sqrt{\kappa}B$ it is shown in [RS05; LSW04] that with probability 1, we have a continuous trace. It is simple when $\kappa \leq 4$, has (infinitely many)

[3] The factor 2 in (1) corresponds to the factor 2 in the half-plane capacity parametrisation.

self-intersections when $\kappa > 4$, and is space-filling when $\kappa \geq 8$. The law of SLE_κ (modulo parametrisation) is invariant under scaling which makes it possible to define SLE_κ on any simply connected domain D from $a \in \partial D$ to $b \in \partial D$ as the conformal image of SLE_κ in $(\mathbb{H}, 0, \infty)$. Moreover, it satisfies the domain Markov property, i.e. conditioned on any initial segment $\gamma[0,t]$, the law of $\gamma[t,\infty]$ in $(D \setminus \gamma[0,t], \gamma(t), b)$ is again SLE_κ. In fact, conformal invariance and the domain Markov property uniquely characterise the law of SLE_κ up to the parameter κ, and this was the initial motivation in [Sch00] to introduce SLE.

The scale-invariance and the domain Markov property of SLE_κ correspond to properties of the driving Brownian motion, namely invariance under Brownian rescaling and the strong Markov property. More precisely, from the Loewner equation (1) one sees that the rescaled driving function $\tilde{\xi}(t) = r\xi(r^{-2}t)$ generates the rescaled conformal maps $\tilde{g}_t(z) = rg_{r^{-2}t}(r^{-1}z)$ and hulls $\tilde{K}_t = rK_{r^{-2}t}$. Furthermore, the concatenation of two driving functions (w.l.o.g. $\xi^2(0) = \xi^1(t_0)$, otherwise there is an additional horizontal shift)

$$\xi(t) = \begin{cases} \xi^1(t) & \text{for } t \leq t_0, \\ \xi^2(t-t_0) & \text{for } t \geq t_0 \end{cases}$$

generates the conformal maps $g_{t_0+s} = g_s^2 \circ g_{t_0}^1$ and hulls $K_{t_0+s} = K_{t_0}^1 \cup (g_{t_0}^1)^{-1}(K_s^2)$. It is easy to see that when ξ^1 and ξ^2 each generate a continuous trace, so does ξ. The converse is much harder to see, and will be proved in Chapter 1.

Chapter 1

Topological characterisations of Loewner traces

Abstract

The (chordal) Loewner differential equation encodes certain curves in the half-plane (aka traces) by continuous real-valued driving functions. Not all curves are traces; the latter can be defined via a geometric condition called the local growth property. In this paper we give two other equivalent conditions that characterise traces: 1. A continuous curve is a trace if and only if mapping out any initial segment preserves its continuity (which can be seen as an analogue of the domain Markov property of SLE). 2. The (not necessarily simple) traces are exactly the uniform limits of simple traces. Moreover, using methods by Lind, Marshall, Rohde (2010), we infer that uniform convergence of traces imply uniform convergence of their driving functions.

1.1 Introduction and main results

Loewner chains provide a way to encode certain curves in a planar domain by real-valued functions called driving functions or Loewner transforms. They had been originally introduced by K. Löwner (1923) as an approach to solve the Bieberbach conjecture, but have recently also been used by O. Schramm (2000) to construct Schramm-Loewner evolution (SLE) which is a random curve driven by a multiple of Brownian motion. The relation between the driving function and the corresponding curve (called trace) is quite involved. In particular, not all curves are traces, but only those that satisfy a geometric condition called the local growth property. (Conversely, not all driving functions do generate a trace either, and there is so far no known characterisation of such driving functions.)

Particularly nice Loewner traces are the so-called simple traces which do neither intersect themselves nor the boundary of the domain. But already SLE produce (for some parameters) examples of non-simple traces. Therefore there is motivation to study the space of (not necessarily simple) Loewner traces. In the following, we will consider chordal Loewner traces in the upper half-plane $\mathbb{H}$. In [TY20] the authors have shown that uniform limits of simple traces provide a (in general not simple) trace again, and they have raised the question whether the converse is true, i.e. whether any trace can be approximated by simple traces. (For SLE$_\kappa$ this has been known from [LSW04; Tra15].) We show in the present paper that this is indeed the case.

Another motivation for studying the space of Loewner traces is characterising the topological support of SLE$_\kappa$ (as a probability measure on the path space). In [TY20] the authors have shown that the support of SLE$_\kappa$ is the closure of the set of simple traces. The result in the present paper implies that this is already the entire space of Loewner traces.

The main result of the present paper is the following characterisation of chordal Loewner traces. See Section 1.2 for definitions of the terminology.

Theorem 1.1.1. *Let $\gamma\colon [0,\infty[\to \overline{\mathbb{H}}$ be a continuous path with $\gamma(0) \in \mathbb{R}$ such that the family of $K_t := \operatorname{fill}(\gamma[0,t])$ is strictly increasing. Then the following are equivalent:*

(i) The family $(K_t)_{t\geq 0}$ satisfies the local growth property.

(ii) For every $t \geq 0$, the path $\gamma_t(s) := g_t(\gamma(s))$, $s \geq t$, is continuous.

(iii) There exists a sequence of simple paths $\gamma^n\colon [0,\infty[\to \overline{\mathbb{H}}$ with $\gamma^n(0) \in \mathbb{R}$ and $\gamma^n(]0,\infty[) \subseteq \mathbb{H}$ such that $\gamma^n \to \gamma$ locally uniformly.

We point out that we identify K_t by their intersection with $\mathbb{H}$ (see Section 1.2), for instance $\gamma \subseteq \mathbb{R}$ are not counted as strictly increasing.

Remark 1.1.2. *To be very precise, a boundary point $z \in \partial K_t$ can belong to several prime ends of $\mathbb{H}\setminus K_t$, so the image $g_t(z)$ would not be unique. Therefore the precise formulation of (ii) is that γ_t can be chosen to be continuous (in case $\gamma(s) \in \partial K_t$ for some $s > t$).*

While all of the above properties seem natural, proving their equivalence requires some work. One should keep in mind that Loewner traces might have infinitely many self-intersections and be space-filling (e.g. SLE$_\kappa$ with $\kappa \geq 8$). This makes none of the equivalences obvious. (More examples of space-filling curves can be found e.g. in [LR12].)

The property (ii) can be seen as a deterministic analogue of the domain Markov property of SLE which O. Schramm defined [Sch00] (i.e. conditioned on an initial segment of the SLE$_\kappa$ trace $\gamma[0,t]$ (in the domain $\mathbb{H}$), the remaining part of the trace $\gamma[t,\infty]$ is again an SLE$_\kappa$ trace in the domain $\mathbb{H}\setminus \operatorname{fill}(\gamma[0,t])$). Analogously, the property (ii) describes that for any t we have that $\gamma[t,\infty]$, mapped from the domain $\mathbb{H}\setminus\operatorname{fill}(\gamma[0,t])$ to $\mathbb{H}$, becomes again a continuous curve.

The property (iii) could remind us of SLE$_\kappa$ which are (for some values of κ) limits of simple curves arising from certain discrete models (e.g. [Smi01; LSW04]). We emphasise that this property is not trivial to show, either. The "obvious" attempt to construct an approximating sequence (γ^n) would be smoothening the driving function of γ, but it is not clear whether the produced traces converge uniformly (they only converge in the Carathéodory sense, see [Law05, Section 4.7]).

Another way of viewing Theorem 1.1.1 is that intuitively Loewner traces are allowed to self-intersect but need to "bounce-off" instead of "crossing over". But especially when the trace is space-filling, it is not obvious what this means precisely. This theorem describes three equivalent ways of phrasing it.

A consequence of the property (ii) is that if we call ξ the driving function of γ, then γ_t is the continuous trace driven by the restriction $\xi_t := \xi\big|_{[t,\infty[}$. To see this, observe that the family of $K_{t,s} := g_t(K_s \setminus K_t)$, $s \geq t$, is the Loewner chain driven by ξ_t. It is then easy to see that for each $s \geq t$, we have $\mathbb{H} \setminus K_{t,s}$ is the unbounded connected component of $\mathbb{H} \setminus \gamma_t[t,s]$.

In particular, the (pathwise) property of a driving function to generate a continuous trace is a local property.

Corollary 1.1.3. *Suppose $\xi \in C([0,\infty[;\mathbb{R})$ generates a trace. Then for any $t \geq 0$, the driving function $\xi_t := \xi\big|_{[t,\infty[} \in C([t,\infty[;\mathbb{R})$ generates a trace, namely γ_t.*

Again, this statement might "feel" obvious to the expert but requires some work to prove. Indeed, D. Zhan has noticed that this statement is not obvious especially for traces with infinitely many self-intersections. The proof would considerably simplify if one only needed to prove that all corresponding hulls are locally connected. But in a discussion with S. Rohde, D. Belyaev noticed that this does not necessarily imply trace continuity, see a counterexample in Figure 1.1.

Remark 1.1.4. *In the formulation of Theorem 1.1.1, there is no need to require the trace to be parametrised by half-plane capacity since the properties do not depend on the parametrisation anyway. But keep in mind that the correspondence between trace and driving function, as in the formulation of Corollary 1.1.3, is defined via half-plane capacity parametrisation (see Section 1.2 for details).*

In case (K_t) in Theorem 1.1.1 is parametrised by half-plane capacity, then we can choose γ^n parametrised by half-plane capacity as well (since reparametrising does not break the convergence, cf. [TY20, Proposition 6.4]).

Another consequence of the property (iii) is the following.

Corollary 1.1.5. *The set of chordal Loewner traces parametrised by half-plane capacity is a closed subset of $C([0,\infty[;\overline{\mathbb{H}})$ (with compact-open topology).*

Remark 1.1.6. *For this statement, some condition on the parametrisation is required, since in general limits of simple traces might fail to be traces (more precisely, the strict monotonicity of the hulls might fail), e.g.*

$$\gamma_n = \begin{cases} i2t & \text{for } t \in [0,1], \\ i2 + (t-1)(1/n - i) & \text{for } t \in [1,2], \\ 1/n + i + (t-2) & \text{for } t \geq 2. \end{cases}$$

Parametrising traces by half-plane capacity prevents such sequences from converging uniformly since the half-plane capacity parametrisation is stable, see e.g. [TY20, Proposition 6.3].

As an application of Theorem 1.1.1, we give in Section 1.5.1 a simple proof that Loewner traces spend zero "capacity time" on the boundary. This statement should be known among experts, but the property (iii) considerably simplifies the proof.

Proposition 1.1.7. *Let $\gamma\colon [0,\infty[\to \overline{\mathbb{H}}$ be a Loewner trace parametrised by half-plane capacity. Then the set $\{t \geq 0 \mid \gamma(t) \in \mathbb{R}\}$ has measure 0.*

Finally, we discuss again the relationship between trace and driving function. As we have commented above, our proof of property (iii) will not involve regularising the driving function of γ. Instead, we are going to construct γ^n in a geometric fashion that does not take the driving function into account. Therefore it is natural to ask what happens to the driving functions during our construction. In fact, we can show that the uniform convergence of traces already implies uniform convergence of their driving functions. Surprisingly, we have not found this explicit statement in the literature. The closest result we have found is [LMR10, Theorem 4.3], and indeed we can use almost the same proof to show our claim. The proof will be given in Section 1.5.2.

Theorem 1.1.8. *Let $\gamma^n \in C([0,\infty[;\overline{\mathbb{H}})$ be a sequence of chordal Loewner traces parametrised by half-plane capacity, with driving functions $\xi^n \in C([0,\infty[;\mathbb{R})$. If $\gamma^n \to \gamma$ locally uniformly, then $\xi^n \to \xi$ locally uniformly, where ξ is the driving function of γ.*

Note that the map from the trace to its driving function is not uniformly continuous, as the example [LMR10, Figure 6] shows. Moreover, the converse of Theorem 1.1.8 is false, i.e. uniform convergence of driving functions does not imply uniform convergence of their traces, as the example [Law05, Example 4.49] shows.

One may ask to what extent the approximating sequence in property (iii) is unique. Since the left/right turns (in the hyperbolic sense) of a trace are dictated by the increments of its driving function, we see that all γ^n will behave similarly in terms of left/right turns. One may also ask for a quantitative description, but we will not investigate it in this paper.

Acknowledgements: I would like to thank Steffen Rohde and Fredrik Viklund for helpful comments on earlier versions of the paper. I also thank the referee for their comments.

1.2 Preliminaries and Outline

We give a brief summary of chordal Loewner chains and traces, and the notation we use in the paper. A compact set $K \subseteq \overline{\mathbb{H}}$ such that $\mathbb{H} \setminus K$ is simply connected is called a compact $\mathbb{H}$-hull. We identify compact $\mathbb{H}$-hulls that have the same intersection with $\mathbb{H}$ (i.e. we distinguish them only by the complementary domains $\mathbb{H} \setminus K$). We call the mapping-out function of K the unique conformal map $g_K\colon \mathbb{H} \setminus K \to \mathbb{H}$ that satisfies the hydrodynamic normalisation $g_K(z) = z + O(\frac{1}{z})$ at ∞. The half-plane capacity of K is $\operatorname{hcap}(K) := \lim_{z\to\infty} z(g_K(z) - z) \in [0,\infty[$. For a compact set $A \subseteq \overline{\mathbb{H}}$, we define $\operatorname{fill}(A) \subseteq \overline{\mathbb{H}}$ to be the union of A with all bounded connected components of $\mathbb{H} \setminus A$. In case A is connected to $\mathbb{R}$, this is the smallest compact $\mathbb{H}$-hull that contains A.

A strictly increasing family $(K_t)_{t\geq 0}$ of compact $\mathbb{H}$-hulls is said to have the local growth property if for any $\varepsilon > 0$ and $T \geq 0$ there exists $\delta > 0$ such that for every $t \in [0,T]$ there exists a crosscut of $\mathbb{H} \setminus K_t$ of length at most ε that separates $K_{t+\delta} \setminus K_t$ from ∞. When we call g_t the mapping-out function of K_t, the local growth property is equivalent to saying that for any $\varepsilon > 0$ and $T \geq 0$ there exists $\delta > 0$ such that $\operatorname{diam} g_t(K_{t+\delta} \setminus K_t) < \varepsilon$ for all $t \in [0,T]$. In particular, the family $(K_{t,s})_{s\geq t}$ with $K_{t,s} := g_t(K_s \setminus K_t)$ again satisfies the local growth property.

For a strictly increasing family $(K_t)_{t\geq 0}$ of compact $\mathbb{H}$-hulls that satisfies the local growth property, there exists a unique continuous real-valued function $\xi\colon [0,\infty[\to \mathbb{R}$ such that $\xi(t) \in \overline{K_{t,s}}$ for all $0 \leq t < s$. This is called the Loewner transform or driving function of $(K_t)_{t\geq 0}$. The correspondence between $(K_t)_{t\geq 0}$ and ξ is one-to-one when we fix the parametrisation of $(K_t)_{t\geq 0}$ in a certain way, e.g. by half-plane capacity, meaning $\operatorname{hcap}(K_t) = 2t$.

A continuous trace is a continuous path $\gamma\colon [0,\infty[\to \overline{\mathbb{H}}$ with $\gamma(0) \in \mathbb{R}$ such that the family $\operatorname{fill}(\gamma[0,t])$ satisfies the local growth property. We say that $\xi \in C([0,\infty[;\mathbb{R})$ generates a continuous trace if there exists such γ that is parametrised by half-plane capacity and has ξ as driving function, which is equivalent to saying that the limit $\gamma(t) = \lim_{y\searrow 0} g_t^{-1}(iy + \xi(t))$ exists for all t and is continuous in t. A trace is called simple if it intersects neither itself nor $\mathbb{R} \setminus \{\gamma(0)\}$.

When we have two traces $\gamma^1\colon [0,t_1] \to \overline{\mathbb{H}}$ and $\gamma^2\colon [t_1,t_2] \to \overline{\mathbb{H}}$, we can glue them to a trace $\gamma(s) = \gamma^1(s)$ on $[0,t_1]$ and $\gamma(s) = g_{t_1}^{-1}(\gamma^2(s) - \gamma^2(t_1) + \xi(t_1))$ on $[t_1,t_2]$, and the driving function of γ is the concatenation of ξ^1 and ξ^2. The converse statement is Corollary 1.1.3 which we will prove in this paper.

1.2.1 Outline

We give a few comments and first steps on the proof of Theorem 1.1.1.

The fact that (iii) implies (i) has been shown in [TY20, Proposition 6.3]. The converse statement, i.e. (i) implies (iii), is proven in Section 1.4. For that part we will also make use of the property (ii) which we will show first (below and in Section 1.3).

The fact that (ii) implies (i) follows almost immediately from [LMR10, Lemma 4.5]. One has to observe that although the lemma is formulated for connected sets S, its proof shows that it suffices when $\overline{g(S)}$ is connected. In particular, when we assume γ_t to be continuous, the lemma can be applied to

$$\operatorname{diam} g_t(\gamma[t,t+\delta]) \leq c\sqrt{\operatorname{diam} \gamma[t,t+\delta]}.$$

With the uniform continuity of γ, the local growth property follows.

For the proof that (i) implies (ii), we gather a few preliminary observations. The continuity of γ tells us an important piece of information about γ_t. Recall the following statement which follows from [Pom92, Theorem 1.7] via a Möbius transformation taking $z \in \partial H$ to ∞.

Lemma 1.2.1. *Let $f\colon \mathbb{D} \to H \subseteq \hat{\mathbb{C}}$ be conformal, and $z \in \partial H$. Then the set $f^{-1}(z) \subseteq \partial\mathbb{D}$ has measure* 0.

Corollary 1.2.2. *Let $s \geq t$. The set of limit points of γ_t at s is a single point or a subset of $\mathbb{R}$ with measure 0.*

Since γ is continuous, all K_t are locally connected, and hence γ_t is right-continuous, and is continuous at times where it is in $\mathbb{H}$. It follows that γ_t consists of a countable number of excursions in $\mathbb{H}$ from $\mathbb{R}$. Together with the previous observation, we conclude the following.

Lemma 1.2.3. *For any $\delta > 0$, there are finitely many excursions of γ_t with diameter greater than δ on finite time intervals.*

Proof. Suppose there are infinitely many excursions of γ_t with diameter greater than δ on some finite time interval $[t, T]$. Since γ_t is bounded, by compactness of the Hausdorff metric (see [Bee93, Theorem 3.2.4]) we can find a sequence of excursions $\tilde{\gamma}_n$ (considered as compact sets in $\overline{\mathbb{H}}$) that converge in the Hausdorff metric to a compact set $A \subseteq \overline{\mathbb{H}}$, and A is connected (see [Bee93, Exercise 3.2.8]). We can choose the sequence such that also the occurring times of $\tilde{\gamma}_n$ converge to some $\bar{s} \in [t, T]$. Then all points in A are limit points of γ_t at $\bar{s}$, and therefore a single point or a subset of $\mathbb{R}$ with measure 0. Since A is connected, it must be a single point, contradicting $\operatorname{diam}(\tilde{\gamma}_n) > \delta$. □

It follows easily that $K_{t,s}$ is locally connected for each $s \geq t$ (see Lemma 1.3.2). Note that this is not enough to show that γ_t is continuous, as the following variation of an example by D. Belyaev in Figure 1.1 shows.

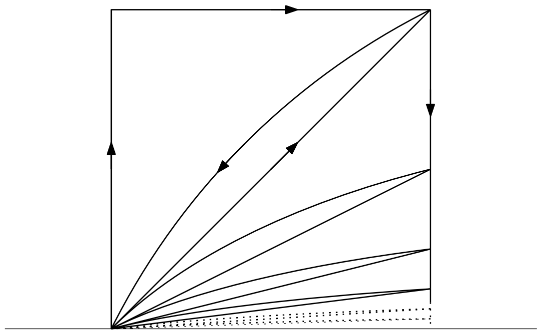

Figure 1.1: All hulls are locally connected, but the "trace" is not continuous. Variation of an example by D. Belyaev [Bel20, p. 212].

Observe that in the above "non-example" there are infinitely many large excursions. We show in Section 1.3 that all counterexamples look like this, and hence do not apply to γ_t. This will establish the continuity of γ_t.

For the convenience of the reader we recall two classical results about the topology of the plane. See [Pom75, Section 1.5] for proofs.

Theorem 1.2.4 (Janiszewski). *Let $A_1, A_2 \subseteq \hat{\mathbb{C}}$ be closed sets such that $A_1 \cap A_2$ is connected. If two points $a, b \in \hat{\mathbb{C}}$ are neither separated by A_1 nor by A_2, then they are not separated by $A_1 \cup A_2$.*

Theorem 1.2.5 (Jordan curve theorem)**.** *If $J \subseteq \hat{\mathbb{C}}$ is a simple loop, then $\hat{\mathbb{C}} \setminus J$ has exactly two components G_0 and G_1, and these satisfy $\partial G_0 = \partial G_1 = J$.*

1.3 Excursions of Loewner traces

In the following, we assume that $\beta\colon [0,\infty[\to \overline{\mathbb{H}}$ has the following properties (we do not a priori assume β to be a continuous function):

- β consists of (a countable number of) excursions in $\mathbb{H}$, i.e. for each $t \ge 0$ if $\beta(t) \in \mathbb{H}$, then there exist $t_1 < t < t_2$ such that β is continuous on $]t_1, t_2[$, has limits $\beta(t_1-), \beta(t_2+) \in \mathbb{R}$, and $\beta(]t_1,t_2[) \subseteq \mathbb{H}$.
- For each $T \ge 0$ and $\delta > 0$ there exist only finitely many excursions of β on the time interval $[0,T]$ with diameter greater than δ.
- For $t \ge 0$, $K_t := \operatorname{fill}(\beta[0,t])$ are compact, strictly increasing, and satisfy the local growth property.

With a slight abuse of notation, an excursion $\tilde\beta$ of β will denote either the path $\tilde\beta \in C([t_1,t_2];\overline{\mathbb{H}})$ or the set $\tilde\beta[t_1,t_2] \subseteq \overline{\mathbb{H}}$ (where $\tilde\beta(t_1)$ and $\tilde\beta(t_2)$ denote the limit points $\beta(t_1-)$, $\beta(t_2+)$). As usual, we write $H_t := \mathbb{H} \setminus K_t$.

Observe that the strict monotonicity of (K_t) implies that the set of times that belong to excursions is dense. Moreover, the local growth property implies that $K_t \cap \mathbb{R}$ is an interval for every t.

Observe also that for $z \in \mathbb{H}$, we have $z \in K_t$ if and only if z lies on or is separated from ∞ by some excursion until time t. This is because only finitely many excursions have diameter larger than $\operatorname{Im} z$.

The main goal of this section is to show the following.

Proposition 1.3.1. *β is continuous in the sense that for every sequence $t_n \to t$ such that $\beta(t_n)$ is on some excursion, the limit $\lim_{n\to\infty} \beta(t_n)$ exists.*
(Equivalently, β can be extended to a continuous function from $[0,\infty[$ to $\overline{\mathbb{H}}$.)

Note that from our assumptions on β, it does not make sense to specify $\beta(t)$ at times t where β is not on any excursion.

Lemma 1.3.2. *For each $t \ge 0$, K_t is locally connected.*

Proof. For $z \in \mathbb{H}$, K_t is clearly locally connected at z since only finitely many excursions intersect z.

For $z \in \mathbb{R}$, let $\delta > 0$. There are only finitely many excursions of diameter at least δ until time t. Call K the union of the fillings of these excursions. Then there exists a connected set $A_1 \subseteq K \cap B(z,\delta)$ that contains $K \cap B(z,r)$ for some $r > 0$. Consider the set

$$A_2 := A_1 \cup \bigcup \{\operatorname{fill}(\tilde\beta) \mid \tilde\beta \text{ is an excursion with } \operatorname{diam}\tilde\beta < \delta \text{ and } \operatorname{dist}(z,\tilde\beta) < r\}$$

which is a connected set contained in $K_t \cap B(z, \delta + r)$. Then $K_t \cap B(z, r) \setminus A_2$ can only consist of connected components of $B(z, r) \setminus A_2$ since all excursions that intersect $B(z, r)$ have been included in A_2. Therefore $A_3 := (K_t \cap B(z, r)) \cup A_2 = A_2 \cup (K_t \cap B(z, r) \setminus A_2)$ is a connected set within $K_t \cap B(z, \delta + r)$ that contains $K_t \cap B(z, r)$. This shows local connectedness at z. □

Lemma 1.3.3. *Let $D \subseteq \hat{\mathbb{C}}$ be a domain with locally connected boundary. Let $z \in \mathbb{C}$ and $0 < r_1 < r_2$. Then only finitely many components of $D \cap B(z, r_1)$ are disconnected in $D \cap B(z, r_2)$.*

Proof. Let $r' := \frac{r_1 + r_2}{2}$. If $D \subseteq B(z, r_2)$, there is nothing to prove. Therefore we can suppose there is some $z_0 \in D \setminus \overline{B(z, r_2)}$. For every $z' \in D \cap B(z, r_1)$ we can find a simple polygonal path $\alpha_{z'}$ in D from z' to z_0. Note that such paths hit any circle only finitely many times. Pick $\alpha_{z'}$ such that it hits $\partial B(z, r')$ as few times as possible.

Suppose that there exist infinitely many $z' \in D \cap B(z, r_1)$ that are disconnected in $D \cap B(z, r_2)$. Let A be an infinite set of such z'. For $z' \in A$ the paths $\alpha_{z'}$ are all disjoint in $B(z, r_2)$. Denote by $w_{z'}$ the first hitting point of $\alpha_{z'}$ with $\partial B(z, r')$. Then $B = \{w_{z'} \mid z' \in A\}$ is an infinite set and hence has a limit point $w_0 \in \partial B(z, r')$.

Clearly $w_0 \in \partial D$ since all points in B are disconnected in $D \cap B(z, r_2)$ by construction. Since ∂D is locally connected, we can find a connected set $C \subseteq \partial D \cap B(w_0, r' - r_1)$ that contains $\partial D \cap B(w_0, 2\delta)$ for some $\delta > 0$. Then each two points in D that are connected in $B(w_0, 2\delta) \setminus C$ are also connected in D. Let $w_{z'} \in B \cap B(w_0, \delta)$. We claim that $\alpha_{z'}$ needs to pass a segment of $\partial B(w_0, \delta) \setminus C$ that intersects $\partial B(z, r')$. This gives us the desired contradiction since there are only two such segments but infinitely many points in $B \cap B(w_0, \delta)$.

Note that $\alpha_{z'}$ needs to enter $B(w_0, \delta)$ through some segment S of $\partial B(w_0, \delta) \setminus C$ before passing $w_{z'}$. We show below that it needs to cross S again. If S does not intersect $\partial B(z, r')$, then $w_{z'}$ is an unnecessary crossing of $\partial B(z, r')$ which contradicts our construction.

Suppose that $\alpha_{z'}$ does not pass S again, which implies that it crosses S an odd number of times. Let $\zeta_1, \zeta_2 \in \partial B(w_0, \delta) \cap C$ be the endpoints of S. We show that ζ_1 and ζ_2 cannot be connected in C which contradicts the connectedness of C. Consider the segment of $\alpha_{z'}$ from when it last enters $B(w_0, r' - r_1)$ until it next leaves $B(w_0, r' - r_1)$ (these times exist since $\alpha_{z'}$ begins inside $B(z, r_1)$ and ends outside $B(z, r_2)$), followed by an arc of $\partial B(w_0, r' - r_1)$. The Jordan curve theorem then implies that any set that connects ζ_1 and ζ_2 in $\hat{\mathbb{C}} \setminus \alpha_{z'}$ needs to intersect $\partial B(w_0, r' - r_1)$. But C cannot do this because $C \subseteq B(w_0, r' - r_1)$. □

Intuitively, the local growth property implies that β might touch but not cross itself again. In particular, it cannot cross any of its past excursions. We make this more precise in the following.

For $h > 0$, we write $\mathcal{S}_h := \{z \in \mathbb{C} \mid \operatorname{Im} z \in]0, h[\}$.

Let $K \subseteq \overline{\mathbb{H}}$ be a compact $\mathbb{H}$-hull and $h > 0$. We say that two points in $\mathcal{S}_h \setminus K$ are *on the same h-side of K* if they are connected in $\mathcal{S}_{h'} \setminus K$ for every $h' > h$. See Figure 1.2 for an illustration of this definition.

Note that if h is smaller than the height of K, then K has at least two h-sides (a left and a right side). If $K_1 \subseteq K_2$, then points on the same h-side of K_2 are also on the same h-side of K_1.

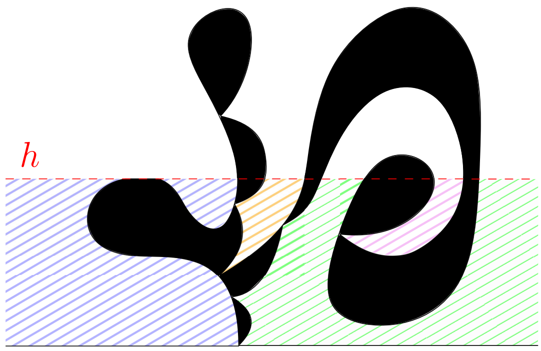

Figure 1.2: A hull with four h-sides.

Lemma 1.3.4. *Let $K \subseteq \overline{\mathbb{H}}$ be a compact $\mathbb{H}$-hull, and $h > 0$. Fix two different h-sides S_1, S_2 of K. Then there exists $\delta > 0$ with the following property:*

If C is a crosscut in $\mathbb{H} \setminus K$ such that there exist $z_1 \in S_1$ and $z_2 \in S_2$ that both are separated from ∞ by C, then $\operatorname{diam} C \geq \delta$.

Proof. Since S_1 and S_2 are different h-sides of K, there exists $h' > h$ such that they are disconnected in $\mathcal{S}_{h'} \setminus K$.

Let $h'' \in]h, h'[$. By definition, all points in S_1 are connected in $\mathcal{S}_{h''} \setminus K$. Pick any $z \in S_1$. Since $\mathbb{H} \setminus K$ is a domain, there exists a path α in $\mathbb{H} \setminus K$ from z to a neighbourhood of ∞. Therefore any crosscut that separates z from ∞ needs to cross α. It follows that any crosscut that separates some point in S_1 from ∞ needs to cross either α or some point connected to S_1 in $\mathcal{S}_{h''} \setminus K$. Let $\delta_1 := \operatorname{dist}(\alpha, \partial K) > 0$. Then any crosscut with diameter smaller than δ_1 that separates some point in S_1 from ∞ needs to contain some point connected to S_1 in $\mathcal{S}_{h''} \setminus K$. Similarly, there is δ_2 such that the analogous statement is true for S_2.

Now let $\delta := \delta_1 \wedge \delta_2 \wedge (h' - h'')$. If C is a crosscut in $\mathbb{H} \setminus K$ with $\operatorname{diam} C < \delta$ and separates points both in S_1 and S_2 from ∞, then C minus its endpoints is a connected set in $\mathcal{S}_{h'} \setminus K$ that contains two points connected to S_1 resp. S_2 in $\mathcal{S}_{h''} \setminus K$. But this is impossible since S_1 and S_2 are separated in $\mathcal{S}_{h'} \setminus K$. □

Corollary 1.3.5. *Let $K \subseteq \overline{\mathbb{H}}$ be a compact $\mathbb{H}$-hull, and $h > 0$. Fix two different h-sides S_1, S_2 of K. Then there exists $\delta > 0$ with the following property:*

If $K' \supseteq K$ is a compact $\mathbb{H}$-hull and C is a crosscut in $\mathbb{H} \setminus K'$ with $\operatorname{diam} C < \delta$ *such that there exist $z_1 \in S_1 \setminus K'$ and $z_2 \in S_2 \setminus K'$ that both are separated from ∞ by C, then C intersects $K' \setminus K$.*

Proof. Choose δ as in Lemma 1.3.4. If C does not intersect $K' \setminus K$, then C is also a crosscut in $\mathbb{H} \setminus K$. We claim that C separates z_1, z_2 from ∞ also in $\mathbb{H} \setminus K$ which is a contradiction to $\operatorname{diam} C < \delta$.

Suppose $C \cup K \cup \hat{\mathbb{R}}$ does not separate z_1 from ∞. Since $K' \cup \hat{\mathbb{R}}$ does not separate z_1 from ∞ either and $(C \cup K \cup \hat{\mathbb{R}}) \cap (K' \cup \hat{\mathbb{R}}) = K \cup \hat{\mathbb{R}}$ is connected (recall that we assumed $C \cap K' \subseteq K$), by Janiszewski's theorem $(C \cup K \cup \hat{\mathbb{R}}) \cup (K' \cup \hat{\mathbb{R}}) = C \cup K' \cup \hat{\mathbb{R}}$ would not separate z_1 from ∞, which contradicts our assumption. The argumentation for z_2 is the same. □

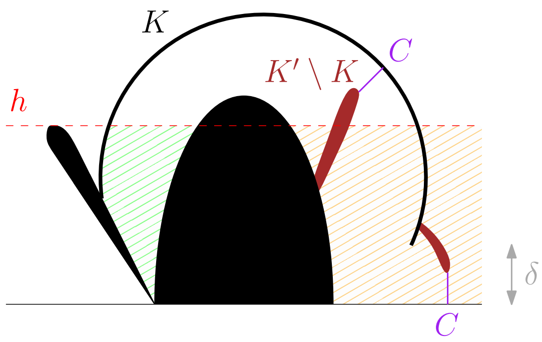

Figure 1.3: The situation in Corollary 1.3.5.

We say that an excursion $\tilde{\beta} \in C([t_1, t_2]; \overline{\mathbb{H}})$ *occurs within a time interval* $[s,t] \subseteq \mathbb{R}$ if $]t_1, t_2[\cap [s,t] \neq \varnothing$.

Let $K \subseteq \overline{\mathbb{H}}$ be a compact $\mathbb{H}$-hull and $h > 0$. We say that $\beta[s,t]$ is *on one h-side of K* if all points of $\beta[s,t] \cap (\mathbb{H} \setminus K)$ lie on the same h-side of K.

Lemma 1.3.6. *Let $t \geq 0$ and $h > 0$. If $\operatorname{Im}\beta(t) < h$, then $\beta[t, t+\varepsilon]$ is on one h-side of K_t for some $\varepsilon > 0$.*

Proof. By compactness we can find a sequence $t_n \searrow t$ such that $\beta(t_n) \in H_t$ converges to some $z \in \overline{\mathbb{H}}$ with $\operatorname{Im} z < h$. By Lemma 1.3.3 only finitely many components of $H_t \cap B(z, (h - \operatorname{Im} z)/2)$ are disconnected in $H_t \cap B(z, h - \operatorname{Im} z)$. Therefore (by the pigeonhole principle) we can pick a subsequence of (t_n) (call it (t_n) again) such that all $\beta(t_n)$ are connected in $H_t \cap B(z, h - \operatorname{Im} z) \subseteq \mathcal{S}_h \setminus K_t$. In particular, they are all on the same h-side of K_t; call that side S_1.

Suppose that there is another sequence $s_n \searrow t$ such that each $\beta(s_n) \in H_t$ is on a different h-side of K_t than S_1. By the same argument as above, we can pick the sequence such that all $\beta(s_n)$ are on the same h-side of K_t; call that side S_2.

By construction $S_1 \neq S_2$. But then Lemma 1.3.4 gives us a contradiction to the local growth property. □

Lemma 1.3.7. *Let $0 \leq s < t$ and $h > 0$. If $\beta[s,t]$ is on one h-side of K_s and $\operatorname{Im}\beta(t) < h$, then $\beta[s, t+\varepsilon]$ is on one h-side of K_s for some $\varepsilon > 0$.*

Proof. If $\beta(t) \in H_s$, then by the continuity of excursions there is nothing to show, so assume $\beta(t) \in K_s \cup \mathbb{R}$. We claim that the set of limit points $\beta(t-)$ is contained in $K_s \cup \mathbb{R}$. In case $\beta(t) \in \mathbb{H}$, this is clear by the continuity of excursions. In case $\beta(t) \in \mathbb{R}$ we have either t as a finishing time of an excursion (in which case the claim is again clear by continuity) or that there are infinitely many excursions finishing shortly before t in which case their diameters have to converge to 0 by the assumption on β which implies the claim.

Call S_1 the h-side of K_s containing $\beta[s,t]$. By Lemma 1.3.6, $\beta[t,t+\varepsilon]$ is on one h-side of K_t and hence also of K_s for some $\varepsilon > 0$; call it S_2. Suppose $S_1 \neq S_2$. Then we can find $h' > h$ such that they are separated in $\mathcal{S}_{h'} \setminus K_s$.

Pick a sequence $t_n \searrow t$ such that $\beta(t_n) \in H_t$. As just observed, we have $\beta(t_n) \in S_2$. Pick any t_N and find a path α in H_t connecting $\beta(t_N)$ to a neighbourhood of ∞. We have seen that the set of limit points $\beta(t-)$ is contained in K_t, so $\delta := \operatorname{dist}(\alpha, \beta(t-)) > 0$. Find $t' < t$ such that $\operatorname{dist}(\beta(t''), \beta(t-)) < \delta/2$ for all $t'' \in [t', t[$.

Since $\beta[s,t]$ is on one h-side of K_s, it follows from Janiszewski's theorem that $\beta[t',t]$ is on one h-side of $K_{t'}$. Recall that we have chosen all $\beta(t_n)$ to be on one different h-side of $K_{t'}$. Applying Corollary 1.3.5 to $K_{t'}$ and by the local growth property there exists some $t'' \in [t', t[$ and some crosscut C in $H_{t''}$ with $\operatorname{diam} C < \delta/2 \wedge (h'-h)$ that separates $\beta(t_n)$ from ∞ for sufficiently large n and intersects $K_{t''} \setminus K_{t'}$.

The choice of δ implies $\operatorname{dist}(C, \alpha) > 0$. Therefore C does not separate $\beta(t_N)$ from ∞. We claim that C does not separate $\beta(t_n)$ from ∞ for any n, producing a contradiction.

We have picked t_n such that all $\beta(t_n)$ are connected in $\mathcal{S}_{h''} \setminus K_t$ for any $h'' > h$. If C separates $\beta(t_n)$ from ∞, C needs to contain some point in the same h-side of K_t as $\beta(t_n)$, and that side is contained in S_2. This means that C needs to contain points from both S_1 and S_2. Since all points of C are less than $h'-h$ away from the set $\beta(t-)$, C contains a connected set in $\mathcal{S}_{h'} \setminus K_s$. But this is impossible since S_1 and S_2 are separated in $\mathcal{S}_{h'} \setminus K_s$. $\square$

Corollary 1.3.8. *Let $0 \le s < t$ and $h > 0$. If all excursions of β that occur within the time interval $[s,t]$ have smaller diameter than h, then $\beta[s,t]$ lies on one h-side of K_s.*

Proof. Let
$$\bar{t} := \sup\{t' \ge s \mid \beta[s,t'] \text{ is on one } h\text{-side of } K_s\}.$$
By Lemma 1.3.6, we have $\bar{t} > s$, and by Lemma 1.3.7, we have $\bar{t} \ge t$. $\square$

Now the proof of Proposition 1.3.1 follows.

Proof of Proposition 1.3.1. First we show left-continuity. Let $t \ge 0$. If some excursion is ongoing or finishes at t, then there is nothing to show. Therefore assume that there are infinitely many excursions of β finishing shortly before t.

Recall that $I := K_t \cap \mathbb{R}$ is an interval. Hence for any $x \in I$, there exists some past excursion $\tilde{\beta}$ such that $\operatorname{fill}(\tilde{\beta})$ has small distance to x. Let $h > 0$ be smaller than the height of $\tilde{\beta}$. From Corollary 1.3.8 and the assumption that only finitely many excursions are larger than h, it follows that when $\varepsilon > 0$ is small enough, $\beta[t-\varepsilon, t]$ will lie on one h-side of $K_{t-\varepsilon}$ and hence also of $\operatorname{fill}(\tilde{\beta})$. Since this holds for all $x \in I$, it implies that $\beta(t-)$ is a Cauchy sequence.

Now let x be any right limit point of β. If $x \neq \beta(t-)$, then as above we can find some past excursion between x and $\beta(t-)$, contradicting Lemma 1.3.7. $\square$

1.4 Proof of (iii) in Theorem 1.1.1

Since this part is about local convergence, we can restrict ourselves to a compact time interval, say $[0, 1]$. Let $\gamma \in C([0, 1]; \overline{\mathbb{H}})$ be a trace. The strategy is to insert a sequence of cut points into γ at a countable dense subset of $[0, 1]$. This will produce a simple trace that approximates γ.

For $\gamma \in C([0, 1]; \overline{\mathbb{H}})$ that satisfies the local growth property, we denote by $\hat{g}_t : \mathbb{H} \setminus \text{fill}(\gamma[0, t]) \to \mathbb{H}$ the conformal map with $g_t(\gamma(t+)) = 0$ and $g_t(z) = z + O(1)$ near ∞, and $\hat{f}_t := \hat{g}_t^{-1}$. In this section, we write $\gamma_t(s) := \hat{g}_t(\gamma(s))$ for $t \le s \le 1$. By the property (ii) of Theorem 1.1.1, this is again a continuous trace (generated by $\xi_t(s) := \xi(s) - \xi(t)$, $s \in [t, 1]$). Note the re-centring here which is a slight change of notation to the previous sections.

We first sketch how we construct a sequence (γ^n) that converge to a simple path γ^∞ such that $\|\gamma - \gamma^\infty\|_\infty < \varepsilon$. To keep the notation a bit simpler, we will care only about γ^∞ being simple and not about boundary hittings. The latter are not a problem since we can remove them via

$$\tilde{\gamma}^\infty := \begin{cases} \gamma^\infty(0) + i2\sqrt{t} & \text{for } t \le \varepsilon, \\ \gamma^\infty(t - \varepsilon) + i2\sqrt{\varepsilon} & \text{for } t \ge \varepsilon. \end{cases}$$

Let (t_n) be a sequence such that $\{t_n \mid n \in \mathbb{N}\}$ is a dense subset of $[0, 1]$. Each γ^n will insert a short simple path into γ which serves as cut points. This path will be inserted in the time interval $[t_n, t_n + h_n]$ for some small $h_n > 0$. As a result, all times $t > t_n$ will shift to $t + h_n$. Therefore it is notationally convenient to introduce another (slight) reparametrisation.

Suppose a summable sequence of $h_n > 0$ have been defined, and write $\bar{h} := \sum_{n\in\mathbb{N}} h_n$. We "stretch" the interval $[0, 1]$ to $[0, 1 + \bar{h}]$ by inserting an additional interval $[t_n, t_n + h_n]$ at time t_n for each n. More precisely, we define $\varphi : [0, 1] \to [0, 1 + \bar{h}]$,

$$\varphi(t) := t + \sum_{m\in\mathbb{N} \text{ s.th. } t_m<t} h_m.$$

Let $s_n := \varphi(t_n)$ and $I_n := [s_n, s_n + h_n] \subseteq [0, 1 + \bar{h}]$. Then

$$\begin{aligned} \varphi^{-1}(s) &:= \sup\{t \in [0, 1] \mid \varphi(t) \le s\} \\ &= \begin{cases} s - \displaystyle\sum_{m\in\mathbb{N} \text{ s.th. } s_m<s} h_m & \text{if } s \notin \bigcup_n I_n, \\ t_n & \text{if } s \in I_n \text{ for some } n. \end{cases} \end{aligned}$$

We will construct $\gamma^n \in C([0, 1 + \bar{h}]; \overline{\mathbb{H}})$ inductively. Let γ^0 be γ but "halted" in the intervals I_n, i.e. $\gamma^0(s) := \gamma(\varphi^{-1}(s))$. Note that the hulls generated by γ^0 are not strictly increasing (they remain constant in the intervals I_n), but this will not worry us because we will construct γ^∞ to be strictly increasing.

For $n \geq 1$, we let (see Figure 1.4)

$$\gamma^n(s) := \begin{cases} \gamma^{n-1}(s) & \text{for } s \leq s_n, \\ \hat{f}^{n-1}_{s_n}(i2\sqrt{s-s_n}) & \text{for } s \in I_n, \\ \hat{f}^{n-1}_{s_n}(i2\sqrt{h_n} + \gamma^{n-1}_{s_n}(s)) & \text{for } s \geq s_n + h_n. \end{cases}$$

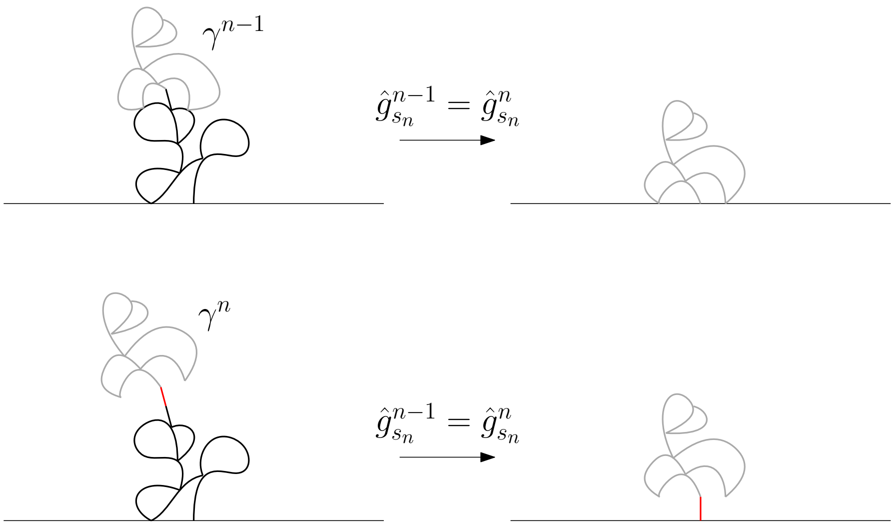

Figure 1.4: The construction of γ^n from γ^{n-1}.

We claim that γ^n satisfies the local growth property again. For $s \leq s_n + h_n$ this is clear. For $s \geq s_n + h_n$ it follows from the local growth property of $\gamma^{n-1}_{s_n}$. (More precisely, for each crosscut C in $\mathbb{H} \setminus \text{fill}(\gamma^{n-1}_{s_n}[s_n, s])$, we can build a crosscut in $\mathbb{H} \setminus \text{fill}(\gamma^n_{s_n}[s_n, s])$ by $\tilde{C} := C + i2\sqrt{h_n}$ and closing $\tilde{C}$ from below in case C terminates on $\mathbb{R}$.)

Note that we have inserted a "cut segment" in the interval I_n which separates $\gamma^n[s_n + h_n, 1+\bar{h}]$ from $\gamma^n[0, s_n]$. We would like to make sure that these two parts remain separated for $m > n$, therefore we introduce the following notation.

For $m \geq n$, we let $d_{n,m} := \text{dist}(\gamma^m[0, s_n], \gamma^m[s_n + h_n, 1 + \bar{h}])$. We will show later that we can pick the sequences (t_n), (h_n) such that the following conditions are satisfied.

- $\|\gamma^n - \gamma^{n-1}\|_\infty < \varepsilon 2^{-n}$.
- $d_{n,m} > d_{n,n}/2 > 0$ for all $m > n$.
- $|\gamma^m(s) - \gamma^m(s')| > \frac{1}{2}|\gamma^n(s) - \gamma^n(s')|$ for $s, s' \in I_n$ and $m > n$.

These conditions will imply that $\gamma^n \to \gamma^\infty$ for some $\gamma^\infty \in C([0, 1 + \bar{h}]; \mathbb{H})$ with $\|\gamma^0 - \gamma^\infty\|_\infty < \varepsilon$. Moreover, we show that γ^∞ is simple. Let $0 \leq s < s'$. We need to show that $\gamma^\infty(s) \neq \gamma^\infty(s')$. There are two cases. In case there exists some n such that $s < s_n < s_n + h_n < s'$, then $|\gamma^m(s) - \gamma^m(s')| \geq d_{n,m} > d_{n,n}/2$ for $m > n$ and hence $|\gamma^\infty(s) - \gamma^\infty(s')| \geq d_{n,n}/2 > 0$. In case no such n exists, by the denseness of the sequence (t_n), we must have $s, s' \in I_n$ for some n. In that case we have $|\gamma^m(s)-\gamma^m(s')| > \frac{1}{2}|\gamma^n(s) - \gamma^n(s')|$ for $m > n$ and hence $|\gamma^\infty(s) - \gamma^\infty(s')| > \frac{1}{2}|\gamma^n(s) - \gamma^n(s')| > 0$.

Now, since γ^0 is just a time-changed version of γ (by at most $\bar{h}$), the uniform continuity of γ implies that $\|\gamma - \gamma^\infty\|_\infty < \varepsilon + \phi(\bar{h})$ for some increasing function ϕ with $\phi(0+) = 0$.

This finishes the proof of (iii) of Theorem 1.1.1 since ε and $\bar{h}$ can be chosen arbitrarily small.

It remains to find suitable sequences (t_n), (h_n) that satisfy our desired conditions.

Lemma 1.4.1. *There exists a countable dense subset $T \subseteq [0,1]$ such that for each $t \in T$ we have $\gamma(t) \notin \gamma([0,t[) \cup \mathbb{R}$.*

Proof. Since the family (K_t) is strictly increasing, there must exist such t in every interval of positive length. The claim follows immediately. □

Lemma 1.4.2. *Let $f \colon \mathbb{H} \to D \subseteq \mathbb{C}$ be a conformal map and $A \subseteq \mathbb{H}$ a bounded set with* $\operatorname{dist}(A, \mathbb{R}) > 0$. *Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z_1+ih)-f(z_2+ih)| \geq (1-\varepsilon)|f(z_1) - f(z_2)|$ for all $z_1, z_2 \in A$ and $h \in [0, \delta]$.*

Proof. Let $d > 0$ be a small number that we specify later. Since f is uniformly continuous on a neighbourhood of A, there certainly exists $\delta > 0$ that work for all $z_1, z_2 \in A$ with $|z_1 - z_2| \geq d^2$.

Suppose now that $|z_1 - z_2| < d^2$. We can assume that $d < \frac{1}{2}\operatorname{dist}(A, \mathbb{R})$. The Koebe distortion theorem and Cauchy integral formula imply that there exists $C > 0$ such that $|f'(w) - f'(z_1)| \leq Cd|f'(z_1)|$ for all $w \in B(z_1, d^2)$. Hence

$$\begin{aligned} |(f(z_1) - f(z_2)) - f'(z_1)(z_1 - z_2)| &\leq \int_{z_1}^{z_2} |f'(w) - f'(z_1)|\,|dw| \\ &\leq Cd|f'(z_1)(z_1 - z_2)| \end{aligned}$$

and consequently

$$|f(z_1) - f(z_2)| \geq (1 - Cd)|f'(z_1)(z_1 - z_2)|.$$

Then, for $h \leq \delta := d^2$,

$$\begin{aligned} |(f(z_1 + ih) - f(z_2 + ih)) - (f(z_1) - f(z_2))| &\leq \int_{z_1}^{z_2} |f'(w + ih) - f'(w)|\,|dw| \\ &\leq \int_{z_1}^{z_2} Cd|f'(w)|\,|dw| \\ &\leq Cd|f'(z_1)(z_1 - z_2)| \\ &\leq \frac{Cd}{1 - Cd}|f(z_1) - f(z_2)| \end{aligned}$$

and consequently

$$|f(z_1 + ih) - f(z_2 + ih)| \geq \left(1 - \frac{Cd}{1 - Cd}\right)|f(z_1) - f(z_2)|.$$

Choosing d small enough such that $\frac{Cd}{1-Cd} \leq \varepsilon$ implies the claim. □

We choose the sequence (t_n) as in Lemma 1.4.1. This implies that $\gamma^0(s_n) \notin \gamma^0([0,s]) \cup \mathbb{R}$ for all $s < s_n$. Inductively, the same is true for all γ^m. Moreover, we see that $\gamma^m(I_n) \cap (\gamma^m([0,s]) \cup \mathbb{R}) = \varnothing$ for all $s < s_n$ and $m \in \mathbb{N}$.

Note we can choose the sequence (h_n) inductively, where the choice of h_n can depend on γ^0,...,γ^{n-1}. This is because although it looks like γ^n depend also on future h_m where $m > n$, they actually do not since we have set γ^n constant on each I_m for $m > n$.

Let $n \in \mathbb{N}$. Since $\hat{f}^{n-1}_{s_n}$ is continuous in $\overline{\mathbb{H}}$, the difference $\|\gamma^n - \gamma^{n-1}\|_\infty$ becomes arbitrarily small when h_n is small. The first condition is then immediately satisfied. For the second condition note that $d_{n,n} > 0$ holds automatically when $h_n > 0$. Then it remains to make sure that $d_{k,n} > d_{k,k}/2$ for all $k < n$. But for each $k < n$, we already have $d_{k,n-1} > d_{k,k}/2$ by induction hypothesis. By continuity of the distance function, this holds also for $d_{k,n}$ when $\|\gamma^n - \gamma^{n-1}\|_\infty$ is small enough.

For the third condition, consider any $k < n$. If $s_k < s_n$, there is nothing to do since $\gamma^n = \gamma^{n-1}$ on $[0, s_n]$. In case $s_k > s_n$, we can apply Lemma 1.4.2 with the map $\hat{f}^{n-1}_{s_n}$ and $A = \gamma^{n-1}_{s_n}(I_k)$ if we know that $\gamma^{n-1}_{s_n}(I_k) \cap \mathbb{R} = \varnothing$. But this is equivalent to $\gamma^{n-1}(I_k) \cap (\gamma^{n-1}([0,s_n]) \cup \mathbb{R}) = \varnothing$ which is true by our construction. Therefore, Lemma 1.4.2 implies that h_n can be chosen small enough such that the third condition is preserved from $n-1$ to n.

1.5 More on trace approximations

In this section we are going to prove Proposition 1.1.7 and Theorem 1.1.8.

1.5.1 Proof of Proposition 1.1.7

We first gather a few general facts.

For a compact set $A \subseteq \overline{\mathbb{H}}$ (not necessarily a hull), we can define $\operatorname{hcap}(A) := \lim_{y\to\infty} y\mathbb{E}[\operatorname{Im} B^{iy}_{\tau_{\mathbb{H}\setminus A}}]$ where B^{iy} denotes Brownian motion started at iy and $\tau_{\mathbb{H}\setminus A}$ denotes the exit time of B^{iy} from $\mathbb{H} \setminus A$.

If $A \subseteq B \subseteq \overline{\mathbb{H}}$ and A is a compact $\mathbb{H}$-hull with mapping-out function g_A, then $\operatorname{hcap}(B) = \operatorname{hcap}(A) + \operatorname{hcap}(g_A(B \setminus A))$. This can be easily shown from [Law05, Proposition 3.41 (3.5)] and the strong Markov property of Brownian motion. In particular, with [Law05, Proposition 3.42] we see that $\operatorname{hcap}(B \setminus A) \geq \operatorname{hcap}(B) - \operatorname{hcap}(A) = \operatorname{hcap}(g_A(B \setminus A))$.

Lemma 1.5.1. *Let $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq \overline{\mathbb{H}}$ be compact $\mathbb{H}$-hulls. Then*

$$\operatorname{hcap}(A_1 \cup (A_3 \setminus A_2) \cup (A_5 \setminus A_4) \cup ...) \geq \operatorname{hcap}(A_1) + \operatorname{hcap}(A_{2,3}) + \operatorname{hcap}(A_{4,5}) + ...$$

where $A_{i,j} := g_{A_i}(A_j \setminus A_i)$.

Proof. By the above observations, we have

$$\begin{aligned} &\operatorname{hcap}(A_1 \cup (A_3 \setminus A_2) \cup (A_5 \setminus A_4) \cup ...) \\ &\quad = \operatorname{hcap}(A_1) + \operatorname{hcap}(g_1(A_3 \setminus A_2) \cup g_1(A_5 \setminus A_4) \cup ...) \\ &\quad \geq \operatorname{hcap}(A_1) + \operatorname{hcap}(A_{2,3} \cup g_2(A_5 \setminus A_4) \cup ...) \end{aligned}$$

and proceed inductively. □

Now we perform the proof of Proposition 1.1.7. It suffices to consider a trace on a compact time interval, say $\gamma\colon [0,1] \to \overline{\mathbb{H}}$. By Theorem 1.1.1 we can find simple traces γ^n such that $\gamma^n \to \gamma$ uniformly. By Remark 1.1.4 we can assume γ^n being parametrised by half-plane capacity.

By the uniform convergence of γ^n, we can find for any $h > 0$ some n such that $\gamma^{-1}(\mathbb{R}) \subseteq (\gamma^n)^{-1}(\mathbb{R}\times[0,h[)$. We would like to show that the latter set has small measure.

The set $(\gamma^n)^{-1}(\mathbb{R}\times[0,h[)$ consists of a countable number of disjoint intervals $]s_i,t_i[$. Since γ^n is simple and parametrised by half-plane capacity, we have $K^n_{t_i}\setminus K^n_{s_i} = \gamma^n(]s_i,t_i])$ and $2|t_i - s_i| = \operatorname{hcap}(K^n_{t_i}) - \operatorname{hcap}(K^n_{s_i}) = \operatorname{hcap}(g^n_{s_i}(K^n_{t_i}\setminus K^n_{s_i}))$.

By Lemma 1.5.1, we have for any $I \in \mathbb{N}$ that

$$\begin{aligned}\sum_{i=1}^{I} \operatorname{hcap}(g^n_{s_i}(K^n_{t_i}\setminus K^n_{s_i})) &\le \operatorname{hcap}\left(\bigcup_{i=1}^{I}(K^n_{t_i}\setminus K^n_{s_i})\right)\\ &= \operatorname{hcap}\left(\bigcup_{i=1}^{I}\gamma^n(]s_i,t_i])\right)\\ &\le ch\end{aligned}$$

where $c < \infty$ depends on $\operatorname{diam}\gamma \approx \operatorname{diam}\gamma^n$. Hence, denoting Lebesgue measure by $|\cdot|$,

$$|\gamma^{-1}(\mathbb{R})| \le |(\gamma^n)^{-1}(\mathbb{R}\times[0,h[)| = \sum_{i\in\mathbb{N}}|t_i - s_i| \le ch.$$

Since $h > 0$ was arbitrary, this implies $|\gamma^{-1}(\mathbb{R})| = 0$.

1.5.2 Proof of Theorem 1.1.8

Since this part is about local convergence, we can restrict ourselves to a compact time interval, say $[0,1]$.

Let $\gamma^n \in C([0,1];\overline{\mathbb{H}})$ be a sequence of chordal Loewner traces, and suppose that $\gamma^n \to \gamma$ uniformly. Note that such a sequence is equicontinuous, and denote their modulus of continuity by ω, i.e. $|\gamma^n(t)-\gamma^n(s)| \le \omega(|t-s|)$ for all n, and the same for γ. As usual, we denote the corresponding hulls by $K_t := \operatorname{fill}(\gamma[0,t])$. Moreover, let $R := \sup_t \operatorname{diam}\gamma_t < \infty$, where $\gamma_t(s) = g_t(\gamma(s))$ as before.

Given $\varepsilon > 0$, we would like to find $\delta > 0$ such that $\|\xi - \xi^n\|$ is small whenever $\|\gamma - \gamma^n\| < \delta$.

Let $h_\varepsilon > 0$ such that $\omega(h_\varepsilon) < \varepsilon$. Let $t \in [0,1]$. We follow the proof of [LMR10, Theorem 4.3] and estimate the difference via

$$\begin{aligned}|\xi(t) - \xi^n(t)| \le |\xi(t) - g_t(\gamma(t+h))| + |g_t(\gamma(t+h)) - g^n_t(\gamma(t+h))|\\ + |g^n_t(\gamma(t+h)) - \xi^n(t)|\end{aligned} \tag{1.1}$$

with a suitable $h \in]0,h_\varepsilon]$ that we will choose below.

By the half-plane capacity parametrisation and [JL11, Lemma 3.4], we have

$$
\begin{aligned}
2h_\varepsilon = \operatorname{hcap}(\gamma_t[t,t+h_\varepsilon]) &\le c \operatorname{diam}(\gamma_t[t,t+h_\varepsilon]) \operatorname{height}(\gamma_t[t,t+h_\varepsilon]) \\
&\le cR \operatorname{height}(\gamma_t[t,t+h_\varepsilon]).
\end{aligned}
$$

Therefore there exists some $h \in \,]0,h_\varepsilon]$ such that $\operatorname{Im}\gamma_t(t+h) \ge \frac{2h_\varepsilon}{cR}$. By [LMR10, Lemma 4.5], it follows that $\operatorname{dist}(\gamma(t+h),K_t) \ge \frac{2h_\varepsilon}{c^2R} \wedge \frac{4h_\varepsilon^2}{c^4R^3} =: d$.

By the uniform continuity of γ, we have $\operatorname{diam}(\gamma[t,t+h]) \le \omega(h) < \varepsilon$, and by [LMR10, Lemma 4.5] it follows that $\operatorname{diam}(\gamma_t[t,t+h]) \le c(\varepsilon \vee R^{1/2}\varepsilon^{1/2})$. In particular, we have

$$
|\xi(t) - g_t(\gamma(t+h))| = |\gamma_t(t) - \gamma_t(t+h)| \le c(\varepsilon \vee R^{1/2}\varepsilon^{1/2})
$$

which bounds the first difference in (1.1).

The third difference in (1.1) can be bounded similarly. When we pick $\delta \le d/2$ so that $\delta < d-\delta < \operatorname{dist}(\gamma(t+h),K_t^n)$, then again by [LMR10, Lemma 4.5]

$$
|g_t^n(\gamma(t+h)) - g_t^n(\gamma^n(t+h))| \le c(\delta \vee R^{1/2}\delta^{1/2})
$$

and

$$
|g_t^n(\gamma^n(t+h)) - \xi^n(t)| \le c(\varepsilon \vee R^{1/2}\varepsilon^{1/2}).
$$

To bound the second difference in (1.1), we use [LMR10, Lemma 4.8]. Let $B := \operatorname{fill}(K_t \cup K_t^n)$.

Pick $\delta \le \frac{d}{2c_0} \wedge \frac{d^2}{4c_0^2R}$, i.e. we have $\|\gamma - \gamma^n\| \le \frac{d}{2c_0} \wedge \frac{d^2}{4c_0^2R}$, where c_0 denotes the constant in [LMR10, Lemma 4.5].

We now estimate the hyperbolic distance from $\gamma(t+h)$ to ∞ in $(\mathbb{C}\setminus B)^*$ where * denotes the reflection through $\mathbb{R}$. By [LMR10, Lemma 4.4], we have $\operatorname{diam} g_t(\partial K_t) \le 4R$. By the choice of δ and [LMR10, Lemma 4.5] it follows that $g_t(\partial B) \subseteq [a, a+4R+d] \times [0, \frac{d}{2}]$ for some $a \in \mathbb{R}$.

Denoting by g_t^* the Schwarz reflection of g_t through $\mathbb{R}$, we have that

$$
\begin{aligned}
\rho_{(\mathbb{C}\setminus B)^*}(\gamma(t+h),\infty) &= \rho_{g_t^*((\mathbb{C}\setminus B)^*)}(\gamma_t(t+h),\infty) \\
&\le \rho_{\mathbb{C}\setminus([a,a+4R+d]\times[-\frac{d}{2},\frac{d}{2}])}(\gamma_t(t+h),\infty).
\end{aligned}
$$

Recalling that $\operatorname{Im}\gamma_t(t+h) \ge d$, an explicit computation (see the lemma below) shows that the hyperbolic distance is at most $\rho \le \sinh^{-1}(\frac{8R+4d}{d/2}) \le \log(17 + \frac{32R}{d})$.

By [LMR10, Lemma 4.8], we then have

$$
\begin{aligned}
&|g_t(\gamma(t+h)) - g_t^n(\gamma(t+h))| \\
&\quad \le |g_t(\gamma(t+h)) - g_B(\gamma(t+h))| + |g_B(\gamma(t+h)) - g_t^n(\gamma(t+h))| \\
&\quad \le 2cR^{1/2}\rho\delta^{1/2} \\
&\quad \le cd\log(17 + \frac{32R}{d}).
\end{aligned}
$$

Since d can be chosen as small as we want, this bounds the second difference in (1.1) and finishes the proof of Theorem 1.1.8.

Lemma 1.5.2. *For $z = x + iy$ with $y > b \geq 0$ we have*

$$\rho_{\hat{\mathbb{C}} \setminus ([-a,a] \times [-b,b])}(z, \infty) \leq \sinh^{-1}\left(\frac{4(a+b)}{y-b}\right).$$

Proof. Let $f \colon \mathbb{H} \to \mathbb{H} \setminus ([-a,a] \times [0,b])$ be the hydrodynamically normalised conformal map. By the Schwarz-Christoffel formula, we have

$$f'(z) = (z - a_1)^{-1/2}(z - a_2)^{1/2}(z - a_3)^{1/2}(z - a_4)^{-1/2}$$

where $a_1, ..., a_4$ are the preimages of the points $-a, -a + ib, a + ib, a$. (The multiplicative constant in the formula is determined by $\lim_{z \to \infty} f'(z) = 1$.)

It follows that $\operatorname{Im} f(z) \leq b + \operatorname{Im} z$ since $|f'(iy)| \leq 1$ and $\operatorname{Im} f'(z) \leq 0$ for $\operatorname{Re} z \geq 0$ and $\operatorname{Im} f'(z) \geq 0$ for $\operatorname{Re} z \leq 0$.

Call g the Schwarz reflection of f^{-1}, so that

$$\rho_{\hat{\mathbb{C}} \setminus ([-a,a] \times [-b,b])}(z, \infty) = \rho_{\hat{\mathbb{C}} \setminus I}(g(z), \infty)$$

where $I = g(\partial([-a,a] \times [-b,b])) \subseteq \mathbb{R}$. By [LMR10, Proposition 4.4], we have $\operatorname{diam} I \leq 4 \operatorname{diam}([-a,a] \times [-b,b]) \leq 8(a+b)$.

By an explicit computation with the map $h \colon \mathbb{D} \to \hat{\mathbb{C}} \setminus I$, $h(z) = c(z + \frac{1}{z})$, we get

$$\rho_{\hat{\mathbb{C}} \setminus I}(g(z), \infty) \leq \sinh^{-1}\left(\frac{2c}{\operatorname{Im} g(z)}\right) \leq \sinh^{-1}\left(\frac{4(a+b)}{(\operatorname{Im} z) - b}\right).$$

$\square$

Chapter 2

SLE with non-constant κ

Abstract

Schramm-Loewner evolution arises from driving the Loewner differential equation with $\sqrt{\kappa}B$ where $\kappa > 0$ is a fixed parameter. In this paper, we drive the Loewner differential equation with non-constant random parameter, i.e. $d\xi(t) = \sqrt{\kappa_t}dB_t$. We show that in case κ_t is bounded below or above 8, the construction still yields a continuous trace. This is true in both cases either when driving the forward equation or the backward equation by $\sqrt{\kappa_t}dB_t$. In the case of the forward equation, we develop a new argument to show the result, without the need of analysing the time-reversed equation.

2.1 Introduction

Schramm-Loewner evolution (SLE) is a family of conformally invariant curves in the plane. They are (either proved or conjectured) to arise in the scaling limits of many physical models that exhibit conformal invariance. One nice feature of SLE (and the reason we study it) is that it can be constructed from a relatively simple differential equation. More precisely, one drives the Loewner differential equation with a Brownian motion with speed κ. This produces conformal maps of domains that are complements of curves, the SLE$_\kappa$ trace. The latter fact is not trivial, and not even true for general driving functions. In case of more regular (e.g. 1/2-Hölder) driving functions this construction is well-understood and we know many of its properties (cf. [LMR10; RTZ18]). For SLE$_\kappa$ which is driven by Brownian motion we still rely on probabilistic techniques. From probabilistic arguments we do know a lot about SLE$_\kappa$ (e.g. [RS05; Bef08; JL11; LW13; FT17; Zha19b]), but we do not understand well how exactly the driving function affects the trace. Such questions have been tackled in [JRW14; FTY21; FS17; STW19], but only partial answers have been obtained so far.

The problem of investigating SLE with non-constant κ has been brought to the surface by [FS17]. We investigate SLE with parameter changing in time, i.e. we drive it by $\xi(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$ where (for some filtration) (κ_t) is an adapted process and B is a standard Brownian motion. All the original proofs for the existence and regularity of SLE trace

[RS05; Law09; JL11] analyse the backward Loewner flow. So do the authors of [FS17], and therefore they consider driving functions that are time reversals of martingales.

Indeed, we will see below that the original proof in [RS05] can be applied to that case without much change.

One interest of this paper is to drive (forward) Loewner chains with ξ. This is the more natural problem when we consider a random curve growing inside a domain that changes its parameter κ_t according to the past. For this model we cannot simply adapt the classical proofs since the time-reversal of a semimartingale can fail to be a semimartingale, so the reverse Loewner flow is not a well-behaved process. We introduce a new argument (Lemma 2.4.2) that allows us to work directly with the forward Loewner flow.

2.1.1 Main Results

For the forward Loewner chain driven by adapted (κ_t), we obtain a continuous trace whenever (κ_t) is bounded below or above 8.

Theorem 2.1.1. *Let $0 = \underline{\kappa} \le \bar{\kappa} < 8$ or $8 < \underline{\kappa} \le \bar{\kappa} < \infty$, and let $\xi(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$ where B is a standard Brownian motion with respect to some filtration and (κ_t) is a measurable adapted process with $\kappa_t \in [\underline{\kappa}, \bar{\kappa}]$ for all t.*

Then the (forward) SLE flow (2.1) driven by ξ almost surely generates a continuous trace.

Our proof is interesting in its own right since it also applies to classical SLE (with constant κ) and gives a new proof of the existence and regularity of the SLE$_\kappa$ trace for $\kappa \neq 8$. In an ongoing work, we follow this idea of proof to obtain refined (variation and Hölder-type) regularity statements for classical SLE$_\kappa$ that include and add logarithmic refinements to the results in [JL11; FT17]. More applications are conceivable, for instance in the context of stability of SLE$_\kappa$ in the driving function (as in [JRW14; FTY21]).

The next result concerns the backward Loewner chain driven by adapted (κ_t) (equivalently the forward Loewner chain driven by its time reversal $\xi(T - \cdot)$). As already mentioned, this case is much more straightforward from the existing arguments of [RS05; Law09; JL11]. As we would expect, we obtain a continuous trace whenever (κ_t) is bounded below or above 8. (The case when (κ_t) is bounded below 2 was already obtained in [FS17].)

Theorem 2.1.2. *Let $(\xi(t))_{t \ge 0}$ be a continuous stochastic process, and suppose that for every $T > 0$, the process $V(t) = \xi(T-t) - \xi(T)$, $t \in [0,T]$, can be written as $V(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$ where B is a standard Brownian motion with respect to some filtration and (κ_t) is a measurable adapted process with $\kappa_t \in [\underline{\kappa}, \bar{\kappa}]$ for all t.*

Suppose that either $0 = \underline{\kappa} \le \bar{\kappa} < 8$ or $8 < \underline{\kappa} \le \bar{\kappa} < \infty$. Then the (forward) SLE flow (2.1) driven by ξ almost surely generates a continuous trace.

Let us mention that it may be more natural to interpret this scenario as a backward SLE flow driven by V. Recall that backward SLE can be seen as a type of conformal welding process, cf. [RZ16; She16].

Corollary 2.1.3. *Let B be a standard Brownian motion with respect to some filtration and (κ_t) a measurable adapted process such that $\kappa_t \in [\underline{\kappa}, \bar{\kappa}]$ for all t. Suppose that either $0 = \underline{\kappa} \le \bar{\kappa} < 8$ or $8 < \underline{\kappa} \le \bar{\kappa} < \infty$. Then the backward SLE flow (2.2) driven by $V(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$, $t \ge 0$, almost surely generates a continuous trace.*

By this we mean that almost surely, for each $t \ge 0$ the domain $h_t(\mathbb{H})$ is the unbounded connected component of the complement of a curve.

This is a direct consequence of Theorem 2.1.2. To see this, recall that for $T > 0$, the backward flow at time T agrees with the forward flow driven by $\xi(t) = V(T-t) - V(T)$. To apply the previous theorem, we observe that $\xi(S-t) - \xi(S) = V(T-S+t) - V(T-S) = \int_{T-S}^{T-S+t} \sqrt{\kappa_s}\, dB_s$ for $0 \le t \le S \le T$.

This paper is structured in a straightforward way. In Section 2.2, we summarise some basic facts on SLE and the notation we use. In Section 2.3, we analyse backward SLE driven by adapted (κ_t) and prove Theorem 2.1.2. In Section 2.4, we analyse forward SLE driven by adapted (κ_t) and prove Theorem 2.1.1.

Acknowledgements: I acknowledge partial support from European Research Council through Consolidator Grant 683164 (PI: Peter Friz). I would like to thank Peter Friz and Steffen Rohde for many discussions and comments.

2.2 Preliminaries

We briefly introduce the forward and backward (chordal) Loewner chains and SLE. More details can be found e.g. in [Law05; Kem17]. Throughout the paper, $\mathbb{H}$ will denote the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and $\overline{\mathbb{H}}$ the closed upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z \ge 0\}$. We will often write $a \lesssim b$ meaning $a \le Cb$ for some constant $C < \infty$ that may depend on the context. We write $a \asymp b$ when $a \lesssim b$ and $b \lesssim a$.

Let $\xi\colon [0,\infty[\to \mathbb{R}$ be a continuous function. Let $(g_t)_{t\ge0}$ be the forward Loewner chain driven by ξ, i.e. the solution of

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \tag{2.1}$$

where $z \in \mathbb{H}$. For given z, this is well defined until the first time $T(z)$ where the denominator hits 0. We obtain a family of conformal maps $g_t\colon H_t \to \mathbb{H}$ where $H_t = \{z \in \mathbb{H} \mid T(z) > t\}$. We write $\hat{f}_t(z) := g_t^{-1}(z + \xi(t))$.

We get another representation by writing $Z_t(z) = X_t(z) + iY_t(z) = g_t(z) - \xi(t)$. Then

$$\begin{aligned} dX_t &= \frac{2X_t}{X_t^2 + Y_t^2}\, dt - d\xi(t), \\ dY_t &= \frac{-2Y_t}{X_t^2 + Y_t^2}\, dt, \end{aligned}$$

and (cf. [RS05])

$$|g_t'(z)| = \exp\left(-2 \int_0^t \frac{X_s^2 - Y_s^2}{(X_s^2 + Y_s^2)^2}\, ds\right).$$

We remark here that every holomorphic function on $\mathbb{H}$ into $\mathbb{H}$ satisfies the following bound which follows from the Schwarz lemma: $|f_t'(z)| \le \frac{\operatorname{Im} f_t(z)}{\operatorname{Im} z} \le \frac{\sqrt{y^2+4t}}{y}$ where $y = \operatorname{Im} z$.

We say that the Loewner chain driven by ξ has a continuous trace if $\gamma(t) = \lim_{y \searrow 0} \hat{f}_t(iy)$ exists and is a continuous function in t. This is equivalent to saying that there exists a continuous $\gamma\colon [0,\infty[\to \overline{\mathbb{H}}$ such that for each $t \ge 0$ the domain H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$.

In case ξ is weakly 1/2-Hölder continuous, a sufficient condition for (g_t) to generate a trace is $|\hat{f}_t'(iy)| \le C y^{-\beta}$ for all t for some $\beta < 1$ (see [JL11]).

The backward Loewner chain is defined by

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - V(t)}, \quad h_0(z) = z. \tag{2.2}$$

Here we suppose again that $V\colon [0,\infty[\to \mathbb{R}$ is a continuous function. This time, $h_t(z)$ is defined for all $z \in \mathbb{H}$ and $t \ge 0$, and each h_t is a conformal map from $\mathbb{H}$ to a subdomain of $\mathbb{H}$.

We have the following relationship between forward and backward Loewner chain. For fixed $t \ge 0$, if we let $V(s) = \xi(t-s) - \xi(t)$, then $\hat{f}_t(z) = h_t(z) + \xi(t)$ and $\hat{f}_t'(z) = h_t'(z)$.

Similarly to the forward case, we can write $Z_t(z) = X_t(z) + iY_t(z) = h_t(z) - V(t)$. Then

$$\begin{aligned} dX_t &= \frac{-2X_t}{X_t^2 + Y_t^2}\, dt - dV(t), \\ dY_t &= \frac{2Y_t}{X_t^2 + Y_t^2}\, dt, \end{aligned}$$

and (cf. [RS05])

$$|h_t'(z)| = \exp\left(2 \int_0^t \frac{X_s^2 - Y_s^2}{(X_s^2 + Y_s^2)^2}\, ds \right). \tag{2.3}$$

SLE_κ is the (forward) Loewner chain driven by $\xi(t) = \sqrt{\kappa} B_t$ with a standard Brownian motion B. Since the time-reversal of B is again a Brownian motion, we see that we can analyse it as well through the backward Loewner chain. Backward SLE_κ can also be seen as an object in its own right, see e.g. [RZ16; She16]. It is known that for any $\kappa \ge 0$, we almost surely have a continuous SLE_κ trace ([RS05; LSW04]).

2.3 (κ_t) adapted to reverse flow

In this section, we prove Theorem 2.1.2. The proof from [RS05] generalises to our setting without considerable extra work.

In the following, we let B be a Brownian motion (with respect to some filtration) and $\kappa_t = \kappa(t,\omega) \ge 0$ a measurable and adapted process. Let $V(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$.

Remark 2.3.1. *The notations in [RS05] differ from the ones in Lawler [Law09; JL11, etc.] and later works. For an easier translation to later works, we use the notation from Lawler. They translate according to the following table. (We have $\nu = 1$ in the notation of [RS05] since we will study the backward SLE flow here.)*

Lawler	*Rohde, Schramm (with $\nu = 1$)*
r	$2b$
$\lambda = r(1+\frac{\kappa}{4}) - \frac{r^2\kappa}{8}$	a
$\zeta = r - \frac{r^2\kappa}{8}$	$\lambda - a$ *(not the same λ as Lawler)*

As already mentioned in Section 2.2, to show existence and regularity of the SLE trace, we would like to study $|h'(z)|$. We follow the idea in [RS05, Theorem 3.2] which we explain briefly now. Due to (2.3), the expectations $\mathbb{E}|h_t'(z)|^\lambda$ can be computed by solving a Feynman-Kac formula. It turns out the formula for $\mathbb{E}\left[|h_t'(z)|^\lambda Y_t^{-\lambda} F(\frac{X_t}{Y_t}, Y_t)\right]$ is easier and can be solved explicitly. Moreover, it is convenient to work in the coordinates $(w, y) = (\frac{x}{y}, y)$. For $F = F(w, y) \in C^2$ we see from Itō's formula that

$$
\begin{aligned}
d\left(|h_t'(z)|^\lambda Y_t^{-\lambda} F(\frac{X_t}{Y_t}, Y_t)\right) &= |h_t'(z)|^\lambda Y_t^{-\lambda-2} \\
&\left[\left(-\frac{4\lambda}{(1+X_t^2/Y_t^2)^2} F + \frac{2Y_t}{1+X_t^2/Y_t^2} F_y - \frac{4X_t/Y_t}{1+X_t^2/Y_t^2} F_w + \frac{\kappa_t}{2} F_{ww}\right) dt \right. \\
&\left. - \sqrt{\kappa_t} Y_t F_w \, dB_t \right]
\end{aligned}
\tag{2.4}
$$

Define the differential operator

$$
\Lambda_\kappa F = \Lambda_\kappa^{(\mathrm{bw})} F := -\frac{4\lambda}{(1+w^2)^2} F + \frac{2y}{1+w^2} F_y - \frac{4w}{1+w^2} F_w + \frac{\kappa}{2} F_{ww}.
$$

In case of constant κ, the equation

$$
\Lambda_\kappa F = 0
$$

is solved by

$$
F(w, y) = (1+w^2)^{r/2} y^{\zeta+\lambda}
$$

where the exponents r, λ, ζ need to be related as in Remark 2.3.1.

In case of non-constant κ, the problem of bounding $\mathbb{E}\left[|h_t'(z)|^\lambda Y_t^{-\lambda} F(\frac{X_t}{Y_t}, Y_t)\right]$ for $\kappa_t \in [\underline{\kappa}, \bar{\kappa}]$ can be interpreted as an optimal stochastic control problem. We would need to solve a Hamilton-Jacobi-Bellman type equation

$$
\sup_{\kappa \in [\underline{\kappa}, \bar{\kappa}]} \Lambda_\kappa F = 0.
$$

Usually one cannot hope for an explicit solution, but it suffices to find supersolutions

$$
\sup_{\kappa \in [\underline{\kappa}, \bar{\kappa}]} \Lambda_\kappa F \le 0.
$$

(Cf. [BS09; CR09] for similar ideas in slightly different settings. We are also reminded of computing superhedging prices under uncertain volatility, cf. [JM10].)

Observe that the function $F(w,y) = (1+w^2)^{r/2} y^{\zeta+\lambda}$ above satisfies

$$F_{ww} = r(1+(r-1)w^2)(1+w^2)^{r/2-2} y^{\zeta+\lambda},$$

i.e. $F_{ww} \ge 0$ if $r \ge 1$. In that case we have $\sup_{\kappa\in[0,\bar\kappa]} \Lambda_\kappa F = \Lambda_{\bar\kappa} F = 0$. In the case $\bar\kappa < 8$ this will suffice. In the case $\underline{\kappa} > 8$ we will need to pick some $r \in \,]0,1[$ (cf. [JL11; FT17] on the choice of r), and we need to modify the exponents in order to get a supersolution.

Lemma 2.3.2. *The function $F(w,y) = (1+w^2)^{r/2} y^{\zeta+\lambda}$ satisfies $\Lambda^{(\mathrm{bw})}_\kappa F \le 0$ on $\mathbb{H}$ if and only if $\lambda - \zeta \ge \frac{r\kappa}{4}$ and $\lambda + \zeta \le 2r + \frac{r\kappa}{4} - \frac{r^2\kappa}{4}$.*

Remark 2.3.3. *In the case $\underline{\kappa} > 8$, one can ask whether there are smarter ways of finding supersolutions that are sharper. Looking at the proofs of [JL11; FT17], the optimal regularity of the SLE trace that can be proved are directly related to the exponents r, λ, ζ (there is a bit more freedom for r though). It is reasonable to believe that the regularity of the trace should be the same as for SLE$_{\underline{\kappa}}$. One possible attempt to prove such a thing would be to find a supersolution to $\sup_{\kappa\in[\underline{\kappa},\bar\kappa]} \Lambda_\kappa F \le 0$ that is asymptotically comparable to $(1+w^2)^{r/2} y^{\zeta+\lambda}$ at least for $y \searrow 0$ (where λ, ζ are chosen according to Remark 2.3.1 with $\kappa = \underline{\kappa}$).*

Under certain conditions on $\underline{\kappa}$, $\bar\kappa - \underline{\kappa}$, and r, a function of the form

$$F(w,y) = y^{\zeta+\lambda}(1+w^2)^{r/2}\exp(g(w))$$

with some bounded g indeed does the trick. More precisely, we can pick g such that g' is of the form

$$g'(w) = \begin{cases} -\alpha_1 w & \text{for } w \le w_0, \\ -\alpha_2 w^{-1-\varepsilon} & \text{for } w \ge w_0. \end{cases}$$

This works whenever $\bar\kappa - \underline{\kappa}$ is sufficiently small (depending on $\underline{\kappa}, r$). Unfortunately, we did not succeed in making this work in general.

Corollary 2.3.4. *Suppose $\kappa_t \in [\underline{\kappa}, \bar\kappa]$ for all t. Let r, λ, ζ be chosen such that $\Lambda_\kappa F \le 0$ for all $\kappa \in [\underline{\kappa}, \bar\kappa]$. Then the process*

$$M_t = |h_t'(z)|^\lambda Y_t^\zeta (1 + X_t^2/Y_t^2)^{r/2}, \quad t \ge 0,$$

is a supermartingale.

Proof. Let $F(w,y) = (1+w^2)^{r/2} y^{\zeta+\lambda}$ as above. Then $M_t = |h_t'(z)|^\lambda Y_t^{-\lambda} F(X_t/Y_t, Y_t)$. By (2.4) and our assumption $\Lambda_{\kappa_t} F \le 0$, we have that (M_t) is a non-negative local supermartingale, and therefore a supermartingale. □

Corollary 2.3.5. *Let $T \ge 0$, and suppose $r \ge 0$, $\lambda \ge 0$, and ζ are chosen according to Corollary 2.3.4. Then there exists a constant $C < \infty$, depending on ζ, λ, T, such that for all $t \in [0,T]$, $x \in \mathbb{R}$, $y \in \,]0,1]$, and $u > 0$ we have*

$$\mathbb{P}(|h_t'(x+iy)| \ge u) \le \begin{cases} C\left(1+(x/y)^2\right)^{r/2} u^{-(\zeta+\lambda)} & \text{if } \zeta > 0, \\ C\left(1+(x/y)^2\right)^{r/2} u^{-\lambda} y^{\zeta} & \text{if } \zeta < 0. \end{cases}$$

Remark 2.3.6. *In the case $\bar\kappa < 8$ and the setting of Remark 2.3.1, the condition is $r \in [1, 2+8/\bar\kappa]$.*

Proof. This is essentially the same as the proof of [RS05, Corollary 3.5]. For the convenience of the reader, we repeat it here with the slight adaptions to our case.

Recall that the Loewner equation implies $Y_t \le \sqrt{y^2+4t}$. Moreover, by the Schwarz lemma we have $|h_t'(z)| \le \frac{\operatorname{Im} h_t(z)}{\operatorname{Im} z} = \frac{Y_t}{y}$, therefore $|h_t'(z)| \ge u$ implies $Y_t \ge yu$. Hence,

$$\begin{aligned}
\mathbb{P}(|h_t'(z)| \ge u) &\le \sum_{m=\log u}^{\frac12 \log(1+4t/y^2)} \mathbb{P}\left(|h_t'(z)| \ge u,\ Y_t \in [ye^{m-1}, ye^m]\right) \\
&\lesssim \sum_{m=\log u}^{\frac12 \log(1+4t/y^2)} u^{-\lambda} y^{-\zeta} e^{-m\zeta}\, \mathbb{E} M_t \\
&\le \sum_{m=\log u}^{\frac12 \log(1+4t/y^2)} u^{-\lambda} e^{-m\zeta} (1+x^2/y^2)^{r/2} \\
&\lesssim (1+x^2/y^2)^{r/2} u^{-\lambda} \begin{cases} u^{-\zeta} & \text{if } \zeta > 0, \\ (1+4t/y^2)^{-\zeta/2} & \text{if } \zeta < 0 \end{cases} \\
&\lesssim (1+x^2/y^2)^{r/2} \begin{cases} u^{-\zeta-\lambda} & \text{if } \zeta > 0, \\ u^{-\lambda} y^{\zeta} & \text{if } \zeta < 0. \end{cases}
\end{aligned}$$

□

The existence of the SLE trace now follows from the proof in [JL11], which we formulate as the following lemma.

Lemma 2.3.7 (see the proof of [JL11, Proposition 4.2 and Theorem 1.1])**.** *Let ξ be a stochastic process that is Hölder continuous for all exponents smaller than $1/2$, and consider the forward SLE flow $(g_t)_{t\ge0}$ driven by ξ. Suppose that there exist constants β, λ, ζ, C with $\beta < 1$, $\lambda\beta > 0$, and $\lambda\beta + \zeta > 2$ such that*

$$\mathbb{P}\left(|(g_t^{-1})'(iy+\xi(t))| \ge y^{-\beta}\right) \le C(1+t/y^2)^{-\zeta/2}\, y^{\lambda\beta}$$

for all $t \ge 0$, $y \in \,]0,1]$.

Then the SLE trace exists almost surely and is α-Hölder continuous for any $\alpha < \frac{1-\beta}{2}$.

Proof of Theorem 2.1.2. In order to apply Lemma 2.3.7, we need to pick $\zeta+\lambda > 2$ in Corollary 2.3.5.

In the case $\bar\kappa < 8$, we pick λ and ζ according to Remark 2.3.1, and $r = \frac12 + \frac{4}{\bar\kappa}$, in which case $\zeta + \lambda > 2$.

In the case $\underline{\kappa} > 8$, we will pick $r \in [0,1]$, in which case the condition becomes $\lambda - \zeta \ge \frac{r\bar\kappa}{4}$ and $\lambda + \zeta \le 2r + \frac{r\underline{\kappa}}{4} - \frac{r^2\underline{\kappa}}{4}$. Picking $\zeta+\lambda > 2$ is now possible if and only if $2r + \frac{r\underline{\kappa}}{4} - \frac{r^2\underline{\kappa}}{4} > 2 \iff r \in \,]\frac{8}{\underline{\kappa}}, 1[$. With any such r, we can then pick $\lambda = r + \frac{r(\bar\kappa+\underline{\kappa})}{8} - \frac{r^2\underline{\kappa}}{8}$ and $\zeta = r - \frac{r(\bar\kappa-\underline{\kappa})}{8} - \frac{r^2\underline{\kappa}}{8}$ which satisfy everything. □

2.4 (κ_t) adapted to forward flow

In this section, we prove Theorem 2.1.1. We will drive the forward Loewner chain by $\xi(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$ where B is a standard Brownian motion with respect to some filtration and $\kappa_t = \kappa(t, \omega)$ is a measurable adapted process. As in the previous section, we would like to find a bound for $|\hat{f}_t'(iy)|$ but this time we do not have the backward Loewner flow at our disposal (since we have no good way of working with the time reversal of ξ). Instead, we use the following idea.

Let $\delta > 0$. We want to find an upper bound for $|\hat{f}_t'(i\delta)| = |g_t'(\hat{f}_t(i\delta))|^{-1}$. Observe that $z = \hat{f}_t(i\delta)$ is the point where we have to start the forward flow in order to reach $Z_t = i\delta$, and this point depends on the behaviour of ξ in the future time interval $[0, t]$. That means we would need to consider all possible points $z \in \mathbb{H}$ that might reach $i\delta$ at time t.

It turns out that (using Koebe's distortion estimates) we can reduce the problem, and we only need to start the flow from a finite number of points. The number of points we need to test already encodes information on $|\hat{f}_t'(i\delta)|$.

In the following, we denote by $B(z, r)$ the open ball about z with radius r, and we denote conformal radius by crad.

Recall Koebe's distortion estimates and a few consequences.

Lemma 2.4.1. *Let $f\colon \mathbb{H} \to \mathbb{C}$ be a univalent function, $g = f^{-1}\colon f(\mathbb{H}) \to \mathbb{H}$, and $z = x + iy \in \mathbb{H}$. Then for every $w \in B(f(z), \frac{1}{8}y|f'(z)|)$ we have*

$$|g(w) - z| < \frac{y}{2} \quad \text{and} \quad \frac{48}{125} \le \frac{|g'(w)|}{|g'(f(z))|} \le \frac{80}{27}.$$

Proof. Note that $y|f'(z)| = \frac{1}{2}\operatorname{crad}(f(z), f(\mathbb{H}))$. In particular, from Koebe's 1/4 theorem we know that $\operatorname{dist}(f(z), \partial f(\mathbb{H})) \ge \frac{1}{4}\operatorname{crad}(f(z), f(\mathbb{H})) = \frac{1}{2}y|f'(z)|$. Another application of Koebe's 1/4 theorem implies $f(B(z, y/2)) \supseteq B(f(z), \frac{1}{8}y|f'(z)|)$.

In particular, every $w \in B(f(z), \frac{1}{8}y|f'(z)|)$ satisfies $|f^{-1}(w) - z| < y/2$. We conclude by Koebe's distortion theorem applied on the domain $B(f(z), \frac{1}{2}y|f'(z)|)$. □

This motivates to start the Loewner flow from the following set of points

$$\begin{aligned} H(a, M, T) = \{x + iy \mid x = \pm aj/8,\ y = a(1 + k/8),\ j, k \in \mathbb{N} \cup \{0\}, \\ |x| \le M,\ y \le \sqrt{1 + 4T}\}. \quad (2.5) \end{aligned}$$

This grid is chosen so that for every $z \in [-M, M] \times [a, \sqrt{1 + 4T}]$ we have $\operatorname{dist}(z, H(a, M, T)) < \frac{a}{8}$.

The following lemma is purely deterministic and holds for any continuous driving function ξ.

Lemma 2.4.2. *Let $\delta \in]0, 1]$, $u > 0$ and suppose $|\hat{f}_t'(i\delta)| \ge u$ for some $t \in [0, T]$. Then there exists $z \in H(u\delta, \|\xi\|_{\infty;[0,T]}, T)$ such that*

$$|Z_t(z) - i\delta| \le \frac{\delta}{2} \quad \text{and} \quad |g_t'(z)| \le \frac{80}{27}\frac{1}{u}$$

where $H(a, M, T)$ is given by (2.5).

Remark 2.4.3. *For later reference, let us note here that the condition $|Z_t(z) - i\delta| \leq \delta/2$ implies in particular*

$$Y_t(z) \in [\delta/2, 3\delta/2] \quad \textit{and} \quad \left|\frac{X_t(z)}{Y_t(z)}\right| \leq 1.$$

Proof. Surely, there is $z_* = \hat{f}_t(i\delta)$ which by definition satisfies everything, but the claim is that we can choose z from the grid $H(u\delta, M, T)$. Indeed, the grid is just defined so that there always exists some $z \in H(u\delta, M, T)$ with $|z - z_*| \leq \frac{1}{8}u\delta$ provided that $z_* \in [-M, M] \times [u\delta, \sqrt{1+4T}]$. By Lemma 2.4.1, such z satisfies the desired properties.

The fact that $z_* \in [-M, M] \times [u\delta, \sqrt{1+4T}]$ just come from the Loewner equation (for the upper bounds) and from the Schwarz lemma (for the lower bound). $\square$

Next, we introduce the parametrisation by imaginary value. For $z \in \mathbb{H}$ and $\delta > 0$, let $\sigma(s) = \sigma(s, z, \delta) = \inf\{t \in \mathbb{R} \mid Y_t \leq \delta e^{-2s}\}$, $s \in \mathbb{R}$. Note that the s-parametrisation is defined such that the flow starts at $s_0 = -\frac{1}{2}\log\frac{y}{\delta}$, while $s = 0$ corresponds to the time t when $Y_t(z) = \delta$. We have the following representations

$$\sigma(s) = \int_{-\frac{1}{2}\log\frac{y}{\delta}}^{s} (X_{\sigma(s')}^2 + Y_{\sigma(s')}^2)\, ds'$$

and

$$\partial_s \log|g_{\sigma(s)}'(z)| = -2\frac{X_{\sigma(s)}^2 - Y_{\sigma(s)}^2}{X_{\sigma(s)}^2 + Y_{\sigma(s)}^2}$$

and consequently $\left|\partial_s \log|g_{\sigma(s)}'(z)|\right| \leq 2$.

Suppose in the following that $\xi(t) = \int_0^t \sqrt{\kappa_s}\, dB_s$ where B is a standard Brownian motion with respect to some filtration and $\kappa_t = \kappa(t, \omega) \geq 0$ is a measurable adapted process.

The moments of $|g_t'(z)|$ can be studied similarly to the case of the backward flow, as was also done in [RS05].

For $F = F(w, y) \in C^2$ we have

$$d\left(|g_t'(z)|^\lambda Y_t^{-\lambda} F(\frac{X_t}{Y_t}, Y_t)\right) = |g_t'(z)|^\lambda Y_t^{-\lambda-2}\big(\Lambda_{\kappa_t} F\, dt - \sqrt{\kappa_t} Y_t F_w\, dB_t\big). \tag{2.6}$$

where

$$\Lambda_\kappa F = \Lambda_\kappa^{(\mathrm{fw})} F := \frac{4\lambda}{(1+w^2)^2} F - \frac{2y}{1+w^2} F_y + \frac{4w}{1+w^2} F_w + \frac{\kappa}{2} F_{ww}.$$

Lemma 2.4.4. *The function $F(w, y) = (1+w^2)^{r/2} y^{\zeta+\lambda}$, satisfies $\Lambda_\kappa^{(\mathrm{fw})} F \leq 0$ on $\mathbb{H}$ if and only if $\lambda - \zeta \leq -\frac{r\kappa}{4}$ and $\lambda + \zeta \geq 2r - \frac{r\kappa}{4} + \frac{r^2\kappa}{4}$.*

Remark 2.4.5. *Here again the regularity of the SLE trace that can be proved are directly related to the exponents λ, ζ (with some restrictions on r). So one may again ask for sharper supersolutions. In contrast to Section 2.3, we had to modify the exponents in F also in the case $\bar{\kappa} < 8$, so optimal regularity of the SLE trace is not clear in that case either. We believe that its regularity should be the same as for SLE$_{\kappa_*}$ where $\kappa_* = \bar{\kappa}$ in the case $\bar{\kappa} < 8$, and $\kappa_* = \underline{\kappa}$ in the case $\underline{\kappa} > 8$.*

Under certain conditions on κ_*, $\bar{\kappa} - \underline{\kappa}$, and r, we can find supersolutions to $\sup_{\kappa\in[\underline{\kappa},\bar{\kappa}]} \Lambda_\kappa F \le 0$ *that are of the form*

$$F(w,y) = y^{\zeta+\lambda}(1+w^2)^{r/2}\exp(g(w))$$

with λ, ζ chosen according to Remark 2.4.7 with $\kappa = \kappa_*$ and a bounded function g. More precisely, we can pick g such that g' is of the form

$$g'(w) = \begin{cases} -\alpha_1 w & \text{for } w \le w_0, \\ -\alpha_2 w^{-1-\varepsilon} & \text{for } w \ge w_0. \end{cases}$$

This works whenever $\bar{\kappa} - \underline{\kappa}$ is sufficiently small (depending on κ_*, r). Again, we did not succeed in making this work in general.

Corollary 2.4.6. *Suppose $\kappa_t \in [\underline{\kappa}, \bar{\kappa}]$ for all t. Let r, λ, ζ be chosen such that $\Lambda_\kappa F \le 0$ for all $\kappa \in [\underline{\kappa}, \bar{\kappa}]$. Then the process*

$$M_t = |g_t'(z)|^\lambda Y_t^\zeta (1 + X_t^2/Y_t^2)^{r/2}, \quad t \ge 0$$

is a supermartingale.

Remark 2.4.7. *In case of constant κ, i.e. $\underline{\kappa} = \bar{\kappa}$, we can take*

$$\lambda = r - \frac{r\kappa}{4} + \frac{r^2\kappa}{8},$$
$$\zeta = r + \frac{r^2\kappa}{8}.$$

In that case, $\Lambda_\kappa F = 0$ and (M_t) is a martingale when stopped before the hull hits a small ball around z.

Proof. Let $F(w,y) = (1+w^2)^{r/2} y^{\zeta+\lambda}$ as above. Then $M_t = |g_t'(z)|^\lambda Y_t^{-\lambda} F(X_t/Y_t, Y_t)$. By (2.6) and our assumption $\Lambda_{\kappa_t} F \le 0$, we have that (M_t) is a non-negative local supermartingale, and therefore a supermartingale. □

Recall that by Lemma 2.4.2, if $|\hat{f}_t'(i\delta)| \ge u$ for some $t \in [0,T]$, then we find $z \in H(u\delta, \|\xi\|_{\infty;[0,T]}, T)$ that satisfies the property stated in the lemma. Note that for such z, we have $\sigma(s,z,\delta) = t$ for some $s \in [-1,1]$. In particular, $|g_{\sigma(s)}'(z)| \lesssim \frac{1}{u}$ and $\left|\frac{X_{\sigma(s)}}{Y_{\sigma(s)}}\right| \le 1$ for some $s \in [-1,1]$.

In case $\lambda < 0$, a lower bound for $|g_t'(z)|$ is equivalent to an upper bound for $|g_t'(z)|^\lambda$. Then

$$\mathbb{P}\left(|g'_{\sigma(s)}(z)| \le \frac{1}{u} \text{ and } |X_{\sigma(s)}| \le Y_{\sigma(s)}\right) \le u^\lambda \mathbb{E}\left[|g'_{\sigma(s)}(z)|^\lambda 1_{|X_{\sigma(s)}| \le Y_{\sigma(s)}}\right]$$

for fixed s. Moreover, since $\left|\partial_s \log |g'_{\sigma(s)}(z)|\right| \le 2$, we have $\dfrac{|g'_{\sigma(s)}(z)|}{|g'_{\sigma(0)}(z)|} \in [e^{-2}, e^2]$ for all $s \in [-1, 1]$.

Let $S = S(z, \delta) = \inf\{s \in [-1, 1] \mid |X_{\sigma(s)}| \le Y_{\sigma(s)}\} \wedge 2$. Together with Corollary 2.4.6, we then have (for any $\lambda \in \mathbb{R}$)

$$\begin{aligned}\mathbb{E}\left[|g'_{\sigma(S)}(z)|^\lambda 1_{S \le 1}\right] &\asymp \delta^{-\zeta} \mathbb{E}\left[M_{\sigma(S)} 1_{S \le 1}\right] \\ &\le \delta^{-\zeta} M_0 \\ &\le \delta^{-\zeta} y^\zeta (1 + x^2/y^2)^{r/2} \end{aligned} \tag{2.7}$$

and consequently (for $\lambda \le 0$)

$$\begin{aligned}&\mathbb{P}\left(|g'_{\sigma(s)}(z)| \le \frac{1}{u} \text{ and } |X_{\sigma(s)}| \le Y_{\sigma(s)} \text{ for some } s \in [-1, 1]\right) \\ &\lesssim u^\lambda \mathbb{E}\left[|g'_{\sigma(S)}(z)|^\lambda 1_{S \le 1}\right] \\ &\lesssim u^\lambda \delta^{-\zeta} y^\zeta (1 + x^2/y^2)^{r/2}. \end{aligned} \tag{2.8}$$

Proposition 2.4.8. *Suppose r, λ, ζ are chosen according to Corollary 2.4.6 and $\lambda \le 0$. Then there exists a constant $C < \infty$, depending on r, ζ, λ, T, M, such that for $\delta, u > 0$ we have*

$$\mathbb{P}(|\hat{f}_t'(i\delta)| \ge u \text{ for some } t \in [0, T],\ \|\xi\|_{[0,T]} \le M) \le \begin{cases} Cu^{\zeta+\lambda} & \text{if } r < -1,\ \zeta + 1 < -1 \\ Cu^{\lambda-2}\delta^{-\zeta-2} & \text{if } r < -1,\ \zeta + 1 > -1, \\ Cu^{\zeta+\lambda-(r+1)}\delta^{-(r+1)} & \text{if } r > -1,\ \zeta - r < -1 \\ Cu^{\lambda-2}\delta^{-\zeta-2} & \text{if } r > -1,\ \zeta - r > -1. \end{cases}$$

Proof. With Lemma 2.4.2, we only need to sum up (2.8) for all points $z \in H(u\delta, M, T)$. To save it for later use, we will perform the calculation in the following Lemma 2.4.9. The result follows. □

Lemma 2.4.9. *Let $r, \zeta \in \mathbb{R}$, $M, T > 0$, $a > 0$. Then there exists $C < \infty$ depending on r, ζ, M, T such that*

$$\sum_{z \in H(a,M,T)} y^\zeta (1 + x^2/y^2)^{r/2} \le \begin{cases} Ca^\zeta & \text{if } r < -1,\ \zeta + 1 < -1, \\ Ca^{-2} & \text{if } r < -1,\ \zeta + 1 > -1, \\ Ca^{\zeta-r-1} & \text{if } r > -1,\ \zeta - r < -1, \\ Ca^{-2} & \text{if } r > -1,\ \zeta - r > -1. \end{cases}$$

Proof. For simplicity, we can write $x_j = aj$, $y_k = ak$ where $j = -Ma^{-1}, ..., Ma^{-1}$ and $k = 1, ..., Ma^{-1}$. (The additional factors do not matter and will be absorbed in the final constant C.)

We have

$$y_k^\zeta (1 + x_j^2/y_k^2)^{r/2} = (ak)^\zeta (1 + j^2/k^2)^{r/2}.$$

We first sum in j.

$$\begin{aligned}\sum_{j \le Ma^{-1}} (1 + j^2/k^2)^{r/2} &\asymp \int_0^{Ma^{-1}} (1 + j^2/k^2)^{r/2}\, dj \\ &= \int_0^{Ma^{-1}/k} (1 + j'^2)^{r/2} k\, dj' \\ &\asymp \begin{cases} k & \text{if } r < -1, \\ a^{-(r+1)} k^{-r} & \text{if } r > -1. \end{cases}\end{aligned}$$

We then sum in k. In case $r < -1$ we have

$$\sum_{k=1}^{Ma^{-1}} (ak)^\zeta k \asymp \begin{cases} a^\zeta & \text{if } \zeta + 1 < -1, \\ a^{-2} & \text{if } \zeta + 1 > -1, \end{cases}$$

and in case $r > -1$ we have

$$\sum_{k=1}^{Ma^{-1}} (ak)^\zeta a^{-(r+1)} k^{-r} \asymp \begin{cases} a^{\zeta - r - 1} & \text{if } \zeta - r < -1, \\ a^{-2} & \text{if } \zeta - r > -1. \end{cases}$$

□

Corollary 2.4.10. *Suppose r, λ, ζ are chosen according to Corollary 2.4.6 and $\lambda \le 0$. Let $\beta > \frac{\zeta+2}{2-\lambda} \vee \frac{r+1}{r+1-\zeta-\lambda} \vee 0$. Then with probability 1 there exists some (random) $y_0 > 0$ such that*

$$|\hat{f}_t'(i\delta)| \le \delta^{-\beta}$$

for all $\delta \in \,]0, y_0]$ and $t \in [0, T]$.

Proof. It suffices to show the claim on the event $\{\|\xi\|_{[0,T]} \le M\}$ for all M. By Proposition 2.4.8

$$\begin{aligned}&\mathbb{P}(|\hat{f}_t'(i\delta)| \ge \delta^{-\beta} \text{ for some } t \in [0, T],\ \|\xi\|_{[0,T]} \le M) \\ &\qquad \le \begin{cases} C\delta^{-\beta(\zeta+\lambda)} & \text{if } r < -1,\ \zeta + 1 < -1 \\ C\delta^{-\beta(\lambda-2)-\zeta-2} & \text{if } r < -1,\ \zeta + 1 > -1, \\ C\delta^{-\beta(\zeta+\lambda-(r+1))-(r+1)} & \text{if } r > -1,\ \zeta - r < -1 \\ C\delta^{-\beta(\lambda-2)-\zeta-2} & \text{if } r > -1,\ \zeta - r > -1. \end{cases}\end{aligned}$$

Our choice of β implies that this probability decays as $\delta \searrow 0$.

For $\delta = 2^{-n}$, $n \to \infty$, the claim then follows from the Borel-Cantelli lemma, and for all other δ from the Koebe distortion theorem. □

Proof of Theorem 2.1.1. If we can pick $\beta < 1$ in the previous corollary, then by [JL11, Corollary 3.12] the trace exists. This is possible if and only if $\frac{\zeta+2}{2-\lambda} < 1 \iff \zeta + \lambda < 0$.

For better readability, we write down the two cases $0 = \underline{\kappa} \leq \bar{\kappa} < 8$ and $8 < \underline{\kappa} \leq \bar{\kappa} < \infty$ separately.

First the case $\bar{\kappa} < 8$. In order to fulfill also the conditions of Corollary 2.4.6, we need to pick r such that $2r - \frac{r\kappa}{4} + \frac{r^2\kappa}{4} < 0$ for all $\kappa \in [0, \bar{\kappa}] \iff r \in \left]1 - \frac{8}{\bar{\kappa}}, 0\right[$. This is a non-empty interval if and only if $\bar{\kappa} < 8$.

Next, we need to fulfill $\lambda - \zeta \leq -\frac{r\kappa}{4}$. Since we allow κ to be as small as 0, and this condition becomes $\lambda - \zeta \leq 0$.

In summary, we need to pick ζ, λ such that $\lambda \leq 0$, $\zeta + \lambda \in \left[2r - \frac{r\bar{\kappa}}{4} + \frac{r^2\bar{\kappa}}{4}, 0\right[$, and $\zeta - \lambda \geq 0$. This can be done by choosing $\zeta = \lambda = r - \frac{r\bar{\kappa}}{8} + \frac{r^2\bar{\kappa}}{8}$.

Now the case $\underline{\kappa} > 8$. Again, we need to pick r such that $2r - \frac{r\kappa}{4} + \frac{r^2\kappa}{4} < 0$ for all $\kappa \in [\underline{\kappa}, \bar{\kappa}] \iff r \in \left]0, 1 - \frac{8}{\underline{\kappa}}\right[$. This is a non-empty interval if and only if $\underline{\kappa} > 8$.

The condition $\lambda - \zeta \leq -\frac{r\kappa}{4}$ for all $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ now becomes $\lambda - \zeta \leq -\frac{r\bar{\kappa}}{4}$.

In summary, we need to pick ζ, λ such that $\lambda \leq 0$, $\zeta + \lambda \in \left[2r - \frac{r\underline{\kappa}}{4} + \frac{r^2\underline{\kappa}}{4}, 0\right[$, and $\zeta - \lambda \geq \frac{r\bar{\kappa}}{4}$. This can be done by choosing $\zeta = r - \frac{r\underline{\kappa}}{8} + \frac{r\bar{\kappa}}{8} + \frac{r^2\underline{\kappa}}{8}$, $\lambda = r - \frac{r\underline{\kappa}}{8} - \frac{r\bar{\kappa}}{8} + \frac{r^2\underline{\kappa}}{8}$. $\square$

Remark 2.4.11. *Of course, the proof also applies to the case of constant κ. In that case, with a bit more work, we can recover the optimal Hölder and p-variation exponents of SLE$_\kappa$ proved in [JL11; FT17], i.e. any $\alpha < (1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}) \wedge \frac{1}{2}$ and $p > (1 + \frac{\kappa}{8}) \wedge 2$.*

In the case of non-constant κ, when $\bar{\kappa} - \underline{\kappa}$ is sufficiently small, it should follow from Remark 2.4.5 that the regularity of the trace is the same as for SLE$_{\kappa_*}$. We believe that this should be true in general, but we are unable to prove it.

Chapter 3

Refined regularity of SLE

Abstract

We prove refined regularity statements for the SLE trace. Previous works by Johansson Viklund and Lawler (2011) and Friz and Tran (2017) have shown that the SLE_κ trace has finite p-variation for any $p > p_*$ and is Hölder continuous with any exponent $\alpha < \alpha_*$ where the exponents p_*, α_* are explicit. We refine their results by showing that the trace has finite ψ-variation for $\psi(x) = x^{p_*}(\log 1/x)^{-p_*-\varepsilon}$ and Hölder-type modulus $\varphi(x) = x^{\alpha_*}(\log 1/x)^{1+\varepsilon}$.

3.1 Introduction

Schramm-Loewner evolution (SLE) is a family of random curves that appear naturally in conformally invariant models. First introduced by O. Schramm to describe the scaling limits of the loop-erased random walk and the uniform spanning tree, they have been shown to appear in the scaling limits of many more models such as the Ising model and Bernoulli percolation. Moreover, they are deeply connected to other conformally invariant objects such as Brownian motion, Gaussian free field, and Liouville quantum gravity.

Regularity of the SLE trace has been studied by many authors, starting from [RS05]. From the works [JL11; Bef08; FT17] we know the optimal p-variation and Hölder exponents which are $p_* = (1+\kappa/8) \wedge 2$ and $\alpha_* = (1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}) \wedge \frac{1}{2}$. More precisely, the trace has finite p-variation for $p > p_*$ and infinite p-variation for $p < p_*$. And (under capacity parametrisation) it is Hölder continuous with exponent $\alpha < \alpha_*$ and not Hölder continuous of exponent $\alpha > \alpha_*$. As for the critical exponents, we do not know but expect that the traces do not have these regularities. This leads to the question of finding the correct modulus. As a comparison, the optimal variation regularity of Brownian motion is shown in [Tay72] to be $\psi(x) = x^2(\log^* \log^*(1/x))^{-1}$, and the optimal modulus of continuity is $\varphi(x) = x^{1/2}(\log^*(1/x))^{1/2}$ (cf. [Lév37]).

Variation regularity seem more natural in the context of SLE since it is parametrisation-independent, and for many applications authors only care about SLE as a curve and not about its parametrisation. It also naturally gives an upper bound on its Hausdorff dimension which turns out to be its true Hausdorff dimension (cf. [Bef08]). The capacity parametrisation, although easier to access via analysis, seem not like the most natural parametrisation of SLE. In fact, the natural parametrisation introduced in [LS11; LZ13] gives much better regularity, as shown in [Zha19b]. Variation regularity ensures that there exists a suitable parametrisation that satisfies a corresponding Hölder-type modulus, but it is not clear whether that agrees with the natural parametrisation of SLE. At least in the Hölder scale, it is shown in [Zha19b; GHM20] that (some variant of) SLE in natural parametrisation is Hölder continuous for any exponent $\alpha < 1/p_*$. So it is plausible that the natural parametrisation may indeed provide the optimal modulus of continuity.

In this paper, we show refined variation and Hölder-type regularities of the SLE$_\kappa$ trace. Moreover, the corresponding variation and Hölder constants have finite moments, see the precise statements in Section 3.3.

Theorem 3.1.1. *Let $\kappa \neq 8$. The SLE$_\kappa$ trace γ on $[0,T]$ almost surely has finite ψ-variation where $\psi(x) = x^p(\log^* 1/x)^{-p-\varepsilon}$, $p = (1+\kappa/8) \wedge 2$.*

Corollary 3.1.2. *Let $\kappa \neq 8$. The SLE$_\kappa$ trace can be parametrised such that*

$$|\tilde\gamma(t) - \tilde\gamma(s)| \le C|t-s|^{1/p}(\log^* \frac{1}{|t-s|})^{1+\varepsilon}$$

where $p = (1+\kappa/8) \wedge 2$.

Theorem 3.1.3. *Let $\kappa \neq 8$, $\kappa \neq 1$. The SLE$_\kappa$ trace γ on $[0,T]$ almost surely satisfies*

$$|\gamma(t) - \gamma(s)| \le C|t-s|^{\alpha}(\log^* \frac{1}{|t-s|})^{1+\varepsilon} \tag{3.1}$$

where $\alpha = (1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}) \wedge \frac{1}{2}$.

In case $\kappa = 1$, we have (3.1) with the exponent 1 being replaced by $\frac{4}{3}$.

Moreover, for $\kappa \neq 8$ and fixed $t_0 > 0$, the SLE$_\kappa$ trace restricted to $[t_0, T]$ satisfies (3.1) with $\alpha = 1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}$.

Remark 3.1.4. *We actually show that the supremum over partitions $0 = t_0 < t_1 < ... < t_r = T$ of*

$$\sup \sum_i \left(\frac{|\gamma(t_i) - \gamma(t_{i-1})|}{|t_i - t_{i-1}|^{\alpha}(\log^* \frac{1}{|t_i - t_{i-1}|})^{1+\varepsilon}} \right)^{\tilde p}$$

is almost surely finite for suitable $\tilde p > 1$.

Moreover, we can interpolate between this statement and Theorem 3.1.1 and have a much wider range of regularity statements.

Existing proofs of SLE regularity in the literature commonly analyse the backward SLE flow. Our proof, in contrast, analyses the forward SLE flow, building on an argument developed in [Yua21b]. This allows us to use certain processes at random stopping times, and ultimately leads to estimates that are uniform in time. One other ingredient is that we parametrise the flow by conformal radius. This parametrisation has been used e.g. in [LW13] to study the distance of points to the trace. Due to [JL11, Lemma 3.5], increments are naturally bounded by certain conformal radii. Arranging the points by conformal radius reduces redundancies in our bounds. Another feature of this parametrisation is that an important quantity, the angle process of the flow, becomes a radial Bessel process which is easy to analyse.

It is reasonable that the approach in this paper can also be used to prove (and maybe improve) results in the case of (more regular) deterministic driving functions, analysed e.g. in [RTZ18; FS17; STW19].

In Section 3.2, we summarise a few preliminaries and basic results. We introduce generalised variation, (chordal) Loewner chains and SLE, and collect a few results on radial Bessel processes that we use later. In Section 3.3.1, we explain the main idea behind our proofs by showing SLE_κ, $\kappa \neq 8$, has a continuous trace. In the remaining part of Section 3.3, we prove our main Theorems 3.1.1 and 3.1.3.

Notation

Throughout the paper, we denote by $\mathbb{H}$ the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and by $\overline{\mathbb{H}}$ the closed upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$. We write $B(z,r)$ for the open ball with radius $r > 0$ around z. We denote conformal radius by crad.

We write $\log^*(x) = (\log x) \vee 1$. We write $a \lesssim b$ meaning $a \leq cb$ for some constant $c < \infty$ that may depend on the context, and $a \asymp b$ meaning $a \lesssim b$ and $b \lesssim a$. Moreover, we write $a \asymp_c b$ to state explicitly the constant c.

Acknowledgements: I acknowledge partial support from European Research Council through Consolidator Grant 683164 (PI: Peter Friz). I thank Peter Friz for discussions on p-variation and ψ-variation.

3.2 Preliminaries

3.2.1 Generalised variation

We summarise the most important facts about ψ-variation. See e.g. [FV10, Section 5.4] for more details.

Let $\psi\colon [0,\infty[\to [0,\infty[$ be a homeomorphism. Let $I \subseteq \mathbb{R}$ be an interval and $x\colon I \to E$ a function with values in a metric space (E,d). Define

$$V^M_{\psi;I}(x) = \sup \sum_i \psi\left(\frac{d(x(t_{i+1}),x(t_i))}{M}\right)$$

where the supremum is taken with respect to all finite subsets $\{t_0 < t_1 < ... < t_r\}$ of I. Usually $V^1_{\psi;I}(x)$ is called the total ψ-variation of x on I.

The ψ-variation constant of x is defined as

$$[x]_{\psi\text{-var};I} = \inf\{M > 0 \mid V^M_{\psi;I}(x) \leq 1\}.$$

In case E is a normed space and ψ is convex, this defines a semi-norm.

Note that in case $\psi(x) = x^p$, this agrees with the notion of p-variation.

We say that ψ satisfies the condition (Δ_c) if[1]

- for any $c > 0$ there exists $\Delta_c > 0$ such that $\psi(cx) \le \Delta_c \psi(x)$ for all x,
- $\lim_{c \searrow 0} \Delta_c = 0$.

If ψ satisfies the condition (Δ_c), we have $[x]_{\psi\text{-var};I} < \infty$ if and only if $V^1_{\psi;I}(x) < \infty$, and in that case $V^M_{\psi;I}(x) < \infty$ for any M.

If x is continuous and $V^1_{\psi;[a,b]}(x) < \infty$, then the function $t \mapsto V^1_{\psi;[a,t]}(x)$ is continuous on $[a, b]$ (cf. [LO73, p. 2.14]). In particular, it can be parametrised by ψ-variation, i.e. $V^1_{\psi;[a,t]}(x) = t - a$. In that parametrisation, it has the following modulus of continuity

$$d(x(t), x(s)) \le \psi^{-1}(t - s).$$

We will later consider the following function ψ. Let $p \ge 1$, $q \ge 0$. Fix any $x_0 \in]0, 1[$. We define

$$\psi(x) = \psi_{p,q}(x) = \begin{cases} x^p (\log \frac{1}{x})^{-q} & \text{for } x \le x_0, \\ \Big(\psi(x_0)^{1/p} + (\psi^{1/p})'(x_0)(x - x_0)\Big)^p & \text{for } x > x_0. \end{cases} \tag{3.2}$$

The advantage of this choice of ψ is that it is convex on all $\mathbb{R}$. Note that $\psi(x) \asymp x^p$ for large x. Moreover, one can easily check that

$$\psi(xy) \lesssim (xy)^p \left(\log^*(1/x)\right)^{-q} \left(\log^*(y)\right)^q \tag{3.3}$$

for $x, y \ge 0$.

3.2.2 Loewner chains: General driving function

We briefly summarise the basics on (chordal) Loewner chains and SLE that we will use in this paper. More details can be found e.g. in [Law05; Kem17]. In this subsection we state the results that hold for any continuous driving function $\xi \colon [0, \infty[\to \mathbb{R}$. In the next subsection we will focus on Brownian driving functions $\xi(t) = \sqrt{\kappa} B_t$.

We consider the forward (chordal) Loewner differential equation

$$g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z. \tag{3.4}$$

The solution $g_t(z)$ exists for $t < T(z)$ where $T(z)$ is the first time where the denominator hits 0. We write

$$\begin{aligned} K_t &= \{z \in \overline{\mathbb{H}} \mid T(z) \le t\}, \\ H_t &= \{z \in \mathbb{H} \mid T(z) > t\}. \end{aligned}$$

[1]This condition is a bit stronger than what is necessary, but will suffice for our purposes.

Then $g_t\colon H_t \to \mathbb{H}$ is a conformal map, the so-called mapping-out function of K_t.

We say that the Loewner chain driven by ξ has a continuous trace if $\gamma(t) = \lim_{y\searrow 0} \hat{f}_t(iy)$ exists and is a continuous function in t. This is equivalent to saying that there exists a continuous $\gamma\colon [0,\infty[\to \overline{\mathbb{H}}$ such that for each $t \geq 0$ the domain H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$. This has been shown for a wide class of driving functions [Lin05; STW19], and a.s. for Brownian motion with speed κ which gives us SLE$_\kappa$ [RS05; LSW04]. (We will give in Section 3.3.1 another proof in the case $\kappa \neq 8$.)

We write $Z_t(z) = X_t(z) + iY_t(z) = g_t(z) - \xi(t)$. Sometimes, to ease the notation, we will leave out the parameter z when there is no confusion. The equation for g_t rewrites to

$$\begin{aligned} dX_t &= \frac{2X_t}{X_t^2+Y_t^2}\,dt - d\xi(t), \\ dY_t &= \frac{-2Y_t}{X_t^2+Y_t^2}\,dt. \end{aligned}$$

Then (cf. [RS05])

$$|g_t'(z)| = \exp\left(-2\int_0^t \frac{X_s^2 - Y_s^2}{(X_s^2+Y_s^2)^2}\,ds\right).$$

Moreover, we write $\hat{f}_t(z) = g_t^{-1}(z+\xi(t))$.

We remark here that we always have $\partial_t Y_t^2 \in [-4, 0[$. Moreover, by the Schwarz lemma we always have $|f_t'(z)| \leq \frac{\operatorname{Im} f_t(z)}{\operatorname{Im} z} \leq \frac{\sqrt{y^2+4t}}{y}$ where $y = \operatorname{Im} z$.

For $z \in \mathbb{H}$ we let $\Upsilon_t(z) = \frac{Y_t(z)}{|g_t'(z)|} = \frac{1}{2}\operatorname{crad}(z, H_t)$ where crad denotes conformal radius. We have

$$\partial_t \Upsilon_t = \frac{-4Y_t^2}{(X_t^2+Y_t^2)^2}\Upsilon_t.$$

The parametrisation by conformal radius is defined via

$$\sigma(s) = \sigma(s,z) = \inf\{t \geq 0 \mid \Upsilon_t(z) = e^{-4s}\}.$$

Notice that the s-parametrisation starts at $s_0(y) := -\frac{1}{4}\log y$, i.e. $\sigma(s_0, z) = 0$. We have the identities

$$d\sigma(s) = Y_{\sigma(s)}^2 \left(1 + \frac{X_{\sigma(s)}^2}{Y_{\sigma(s)}^2}\right)^2 ds, \tag{3.5}$$

$$\partial_s Y_{\sigma(s)}^2 = -4(X_{\sigma(s)}^2 + Y_{\sigma(s)}^2) = -4Y_{\sigma(s)}^2 \left(1 + \frac{X_{\sigma(s)}^2}{Y_{\sigma(s)}^2}\right), \tag{3.6}$$

$$d\frac{X_{\sigma(s)}}{Y_{\sigma(s)}} = 4\frac{X_{\sigma(s)}}{Y_{\sigma(s)}} \left(1 + \frac{X_{\sigma(s)}^2}{Y_{\sigma(s)}^2}\right) ds - \frac{1}{Y_{\sigma(s)}}\,d\xi(\sigma(s)). \tag{3.7}$$

Let $\hat{\theta}(s) = \cot^{-1}\left(\frac{X_{\sigma(s)}}{Y_{\sigma(s)}}\right) \in]0, \pi[$. Writing everything in terms of $\hat{\theta}$ will be convenient in case $\xi(t) = \sqrt{\kappa} B_t$ because $\hat{\theta}$ will be a radial Bessel process (see Section 3.2.3). We then have

$$\begin{aligned} Y_{\sigma(s)} &= Y_0 \exp\left(-2\int_{s_0}^{s} (\sin\hat{\theta}_s)^{-2}\,ds\right) \\ &= Y_0 \exp\left(-2(s-s_0) - 2\int_{s_0}^{s} \cot^2\hat{\theta}_s\,ds\right), \\ |g'_{\sigma(s)}(z)| &= \exp\left(4(s-s_0) - 2\int_{s_0}^{s} (\sin\hat{\theta}_s)^{-2}\,ds\right) \\ &= \exp\left(2(s-s_0) - 2\int_{s_0}^{s} \cot^2\hat{\theta}_s\,ds\right). \end{aligned}$$

We will frequently use the following estimates. Let $s, t \geq 0$ and $y > 0$. From Koebe's distortion theorem, we see that

$$|\hat{f}_t(i2y) - \hat{f}_t(iy)| \leq \int_y^{2y} |\hat{f}'_t(iu)|\,du \asymp y|\hat{f}'_t(iy)| = \Upsilon_t(\hat{f}_t(iy)). \tag{3.8}$$

Moreover, if $|t-s| \asymp y^2$, by [FTY21, Lemma 4.5] (which is a restatement of [JL11, Lemma 3.5 and 3.2]) we have

$$\begin{aligned} |\hat{f}_t(iy) - \hat{f}_s(iy)| &\lesssim |\hat{f}'_s(iy)| \left(\frac{|t-s|}{y} + |\xi(t)-\xi(s)|\left(1 + \frac{|\xi(t)-\xi(s)|^2}{y^2}\right)^l\right) \\ &\asymp \Upsilon_s(\hat{f}_s(iy))) \left(1 + \left(\frac{|\xi(t)-\xi(s)|^2}{|t-s|}\right)^l\right) \end{aligned} \tag{3.9}$$

where $l < \infty$ is a universal constant.

We now explain a proof strategy used in [Yua21b]. For the reader's convenience, we also restate the proofs of the lemmas below.

For $\delta > 0$, we want to find an upper bound for $|\hat{f}'_t(i\delta)| = |g'_t(\hat{f}_t(i\delta))|^{-1}$. Observe that $z = \hat{f}_t(i\delta)$ is the point where we have to start the flow in order to reach $Z_t = i\delta$. But since this point depends on the behaviour of ξ in the future time interval $[0, t]$, we would need to consider all possible points $z \in \mathbb{H}$ that might reach $i\delta$ at time t. Fortunately, using Koebe's distortion and 1/4-theorem, we can reduce the problem to starting the flow from a finite number of points. Then the number of points we need to test already encodes information on $|\hat{f}'_t(i\delta)|$.

Recall Koebe's distortion estimates and a few consequences.

Lemma 3.2.1. *Let $f\colon \mathbb{H} \to \mathbb{C}$ be a univalent function, $g = f^{-1}\colon f(\mathbb{H}) \to \mathbb{H}$, and $z = x + iy \in \mathbb{H}$. Then for every $w \in B(f(z), \frac{1}{8}y|f'(z)|)$ we have*

$$|g(w) - z| < \frac{y}{2} \quad \textit{and} \quad \frac{48}{125} \leq \frac{|g'(w)|}{|g'(f(z))|} \leq \frac{80}{27}.$$

Proof. Note that $y|f'(z)| = \frac{1}{2}\operatorname{crad}(f(z), f(\mathbb{H}))$. In particular, from Koebe's 1/4 theorem we know that $\operatorname{dist}(f(z), \partial f(\mathbb{H})) \geq \frac{1}{4}\operatorname{crad}(f(z), f(\mathbb{H})) = \frac{1}{2}y|f'(z)|$. Another application of Koebe's 1/4 theorem implies $f(B(z, y/2)) \supseteq B(f(z), \frac{1}{8}y|f'(z)|)$.

In particular, every $w \in B(f(z), \frac{1}{8}y|f'(z)|)$ satisfies $|f^{-1}(w) - z| < y/2$. We conclude by Koebe's distortion theorem applied on the domain $B(f(z), \frac{1}{2}y|f'(z)|)$. □

This motivates to start the Loewner flow from the following set of points

$$H(h, M, T) = \{x + iy \mid x = \pm hj/8,\ y = h(1 + k/8),\ j, k \in \mathbb{N} \cup \{0\},$$
$$|x| \leq M,\ y \leq \sqrt{1 + 4T}\}. \quad (3.10)$$

This grid is chosen so that for every $z \in [-M, M] \times [h, \sqrt{1 + 4T}]$ we have $\operatorname{dist}(z, H(h, M, T)) < \frac{h}{8}$.

The following lemma is purely deterministic and holds for any continuous driving function ξ.

Lemma 3.2.2. *Let $\delta \in]0, 1]$, $u > 0$ and suppose $|\hat{f}_t'(i\delta)| \geq u$ for some $t \in [0, T]$. Then there exists $z \in H(u\delta, \|\xi\|_{\infty;[0,T]}, T)$ such that*

$$|Z_t(z) - i\delta| \leq \frac{\delta}{2} \quad \textit{and} \quad |g_t'(z)| \leq \frac{80}{27}\frac{1}{u}$$

where $H(h, M, T)$ is given by (3.10).

Remark 3.2.3. *For later reference, let us note here that the condition $|Z_t(z) - i\delta| \leq \delta/2$ implies in particular*

$$Y_t(z) \in [\delta/2, 3\delta/2] \quad \textit{and} \quad \left|\frac{X_t(z)}{Y_t(z)}\right| \leq 1.$$

Proof. Surely, there is $z_* = \hat{f}_t(i\delta)$ which by definition satisfies everything, but the claim is that we can choose z from the grid $H(u\delta, M, T)$. Indeed, the grid is just defined so that there always exists some $z \in H(u\delta, M, T)$ with $|z - z_*| \leq \frac{1}{8}u\delta$ provided that $z_* \in [-M, M] \times [u\delta, \sqrt{1 + 4T}]$. By Lemma 3.2.1, such z satisfies the desired properties.

The fact that $z_* \in [-M, M] \times [u\delta, \sqrt{1 + 4T}]$ just come from the Loewner equation (for the upper bounds) and from the Schwarz lemma (for the lower bound). □

We will later need to sum up certain expressions on the grid $H(h, M, T)$. We state here the calculation that we will use later.

Lemma 3.2.4. *Let $a, \zeta \in \mathbb{R}$, $M, T > 0$, $h \in]0, 1]$. Then there exists $C < \infty$ depending* *on a, ζ, M, T such that*

$$\sum_{z\in H(h,M,T)} y^\zeta(1+x^2/y^2)^{-a/2} \le C \begin{cases} h^\zeta & \text{if } a>1,\ \zeta+1<-1,\\ h^{-2}\log^*(h^{-1}) & \text{if } a>1,\ \zeta+1=-1,\\ h^{-2} & \text{if } a>1,\ \zeta+1>-1,\\ h^\zeta \log^*(h^{-1}) & \text{if } a=1,\ \zeta+1<-1,\\ h^{-2}\log^*(h^{-1})^2 & \text{if } a=1,\ \zeta+1=-1,\\ h^{-2} & \text{if } a=1,\ \zeta+1>-1,\\ h^{\zeta+a-1} & \text{if } a<1,\ \zeta+a<-1,\\ h^{-2}\log^*(h^{-1}) & \text{if } a<1,\ \zeta+a=-1,\\ h^{-2} & \text{if } a<1,\ \zeta+a>-1. \end{cases}$$

Remark 3.2.5. *The constant C depends on M polynomially.*

Remark 3.2.6. *Suppose we only sum over z with $y = \operatorname{Im} z \ge \varepsilon$. In case $a > 1$, $\zeta+2 < 0$, we then get $\sum \le C\varepsilon^{\zeta+2}h^{-2}$. Analogous statements hold in the other cases.*

Proof. For simplicity, we can write $x_j = hj$, $y_k = hk$ where $j = -Mh^{-1}, ..., Mh^{-1}$ and $k = 1, ..., Mh^{-1}$. (The additional factors do not matter and will be absorbed in the final constant C.)

We have

$$y_k^\zeta(1+x_j^2/y_k^2)^{-a/2} = (hk)^\zeta(1+j^2/k^2)^{-a/2}.$$

We first sum in j.

$$\begin{aligned}\sum_{j\le Mh^{-1}}(1+j^2/k^2)^{-a/2} &\asymp \int_0^{Mh^{-1}} (1+j^2/k^2)^{-a/2}\,dj\\ &= \int_0^{Mh^{-1}/k}(1+j'^2)^{-a/2}k\,dj'\\ &\asymp \begin{cases} k & \text{if } a>1,\\ k\log^*(\frac{M}{hk}) & \text{if } a=1,\\ h^{a-1}k^a & \text{if } a<1.\end{cases}\end{aligned}$$

We then sum in k. In case $a > 1$ we have

$$\sum_{k=1}^{Mh^{-1}}(hk)^\zeta k \asymp \begin{cases} h^\zeta & \text{if } \zeta+1<-1,\\ h^{-2}\log^*(Mh^{-1}) & \text{if } \zeta+1=-1,\\ h^{-2} & \text{if } \zeta+1>-1.\end{cases}$$

In case $a < 1$ we have

$$\sum_{k=1}^{Mh^{-1}}(hk)^\zeta h^{a-1}k^a \asymp \begin{cases} h^{\zeta+a-1} & \text{if } \zeta+a<-1,\\ h^{-2}\log^*(Mh^{-1}) & \text{if } \zeta+a=-1,\\ h^{-2} & \text{if } \zeta+a>-1.\end{cases}$$

In case $a = 1$ we have

$$\sum_{k=1}^{Mh^{-1}} (hk)^{\zeta} k \log^*(\frac{M}{hk}) \asymp \begin{cases} h^{\zeta}\log^*(\frac{M}{h}) & \text{if } \zeta+1<-1, \\ h^{-2}\log^*(\frac{M}{h})^2 & \text{if } \zeta+1=-1, \\ h^{-2} & \text{if } \zeta+1>-1, \end{cases}$$

□

3.2.3 Loewner chains: Brownian driving function

Suppose in the following that $\xi(t) = \sqrt{\kappa}B_t$ where B is a standard Brownian motion and $\kappa \geq 0$. We denote the filtration generated by B by $\mathcal{F} = (\mathcal{F}_t)$.

The equation (3.4) can be seen as a complex version of the (usual) Bessel process. In particular, with initial value $z = x \in \mathbb{R}$, it becomes just a real Bessel process. More precisely, we have

$$dX_t = \frac{2}{X_t}\,dt - \sqrt{\kappa}\,dB_t$$

which is a Bessel process of index $\tilde{\nu} = \frac{2}{\kappa} - \frac{1}{2}$ (equivalently dimension $1 + \frac{4}{\kappa}$) run with speed κ.

Recall that a Bessel process of positive index $\tilde{\nu} > 0$ is transient and satisfies

$$\mathbb{P}(X_t \leq \varepsilon \text{ for some } t \geq 1) \asymp \varepsilon^{2\tilde{\nu}}.$$

The latter can be derived from the following two facts:
1. The transition probability of the Bessel process is (cf. [RY99, p. 446])

$$p_t(0, y) = cy^{2\tilde{\nu}+1}\exp(-y^2/2t).$$

2. The hitting time of the Bessel process satisfies (cf. [RY99, p. 442])

$$\mathbb{P}_x(T_\varepsilon < \infty) = \left(\frac{\varepsilon}{x}\right)^{2\tilde{\nu}} \quad \text{for } x > \varepsilon.$$

From Brownian scaling, it follows that

$$\mathbb{P}(X_t \leq \varepsilon \text{ for some } t \geq t_0) \asymp t_0^{-\tilde{\nu}}\varepsilon^{2\tilde{\nu}}. \tag{3.11}$$

Lemma 3.2.7. *Let $\kappa < 4$. There exists a constant $c < \infty$ such that for any $\varepsilon > 0$ we have with probability larger than $1 - c\varepsilon^{\frac{4}{\kappa}-1}$ that*

$$|X_t(z)| \geq \varepsilon \quad \textit{for all } z \in H_t \cap \{\operatorname{Im} z \leq \varepsilon\},\ t \geq 1.$$

Proof. It suffices to show this for small ε.

We make use of a few known results about SLE. By, [SZ10, (1.4)], the SLE$_\kappa$ trace does not intersect the set $(]-\infty, -\varepsilon^{1/2}] \cup [\varepsilon^{1/2}, \infty[) \times [0, \varepsilon]$ with probability $1 - c\varepsilon^{\frac{4}{\kappa}-1}$. The probability of the trace intersecting the set $[\pm\varepsilon^{1/2}] \times [0, \varepsilon]$ after time 1 will be estimated using [LW13, Lemma 4.5].

Note that the Loewner equation implies $\sup_{t\in[0,t_0]}|\operatorname{Re}\gamma(t)| \le \sup_{t\in[0,t_0]}|\xi(t)|$. Therefore [JL11, Lemma 3.4] implies

$$\begin{aligned}\mathbb{P}(\operatorname{Im}\gamma(t) \ge (\log 1/\varepsilon)^{-1} \text{ for some } t \in [0,1/2]) &\ge \mathbb{P}(\sup_{[0,1/2]}|\xi(t)| \lesssim \log 1/\varepsilon) \\ &\ge 1-\exp(-c(\log 1/\varepsilon)^2).\end{aligned}$$

Suppose that $\tau = \inf\{t \mid \operatorname{Im}\gamma(t) \ge (\log 1/\varepsilon)^{-1}\} \le 1/2$. Restarting the SLE flow at time τ, by (3.11), we then have for fixed $c_1 > 0$

$$\mathbb{P}\begin{pmatrix}\operatorname{Im}\gamma(t) \ge (\log 1/\varepsilon)^{-1} \text{ for some } t \in [0,1/2] \\ |X_t(z)| \ge c_1\varepsilon \text{ for all } z \in \gamma[0,\tau]\cup\mathbb{R},\ t \ge 1\end{pmatrix} \ge 1-c\varepsilon^{2\tilde{\nu}}$$

where $X_t(z) = g_t(z)-\xi(t)$ is understood as the continuous extension of g_t to the boundary.

Suppose that this event happens, and suppose additionally that γ does not re-enter the set $\{\operatorname{Im} z \le \varepsilon\}$ after time τ. In that case, it follows from [Law05, Proposition 3.82] that $|X_t(z)| \ge c_1\varepsilon - c_2\varepsilon$ for all $z \in H_1 \cap \{\operatorname{Im} z \le \varepsilon\}$ where c_2 is a universal constant.

To finish the proof, we need to bound the probability of γ re-entering the set $\{\operatorname{Im} z \le \varepsilon\}$. As remarked above, it remains to bound the probability of re-entering the set $[\pm\varepsilon^{1/2}]\times[0,\varepsilon]$ after time τ. By [LW13, Lemma 4.5], if $\eta \subseteq H_\tau \cap \{\operatorname{Im} z < (\log 1/\varepsilon)^{-1}\}$ is a crosscut of H_τ, the probability of $\gamma[\tau,\infty]$ intersecting η is bounded by $\mathcal{E}_{H_\tau}(\eta, \{\operatorname{Im} z = (\log 1/\varepsilon)^{-1}\})^{8/\kappa-1}$ where $\mathcal{E}$ denotes the excursion measure. The same is true when η is a disjoint union of crosscuts since we are in the case $\kappa < 4 \iff 8/\kappa - 1 > 1$.

From the monotonicity of the excursion measure, comparing to the strip $\{\operatorname{Im} z \in]\varepsilon, (\log 1/\varepsilon)^{-1}[\}$, we see that the sought probability is bounded by $(\varepsilon^{1/2}(\log 1/\varepsilon))^{8/\kappa-1}$. □

For $r \in \mathbb{R}$, let

$$\begin{aligned}\lambda &= r - \frac{r\kappa}{4} + \frac{r^2\kappa}{8}, \\ \zeta &= r + \frac{r^2\kappa}{8}.\end{aligned}$$

Consider the following local martingale (cf. [RS05; LW13])

$$M_t = |g_t'(z)|^\lambda Y_t^\zeta (1+X_t^2/Y_t^2)^{r/2}, \quad t \ge 0. \tag{3.12}$$

It satisfies

$$dM_t = -r\frac{X_t}{X_t^2+Y_t^2}\sqrt{\kappa}\,dB_t$$

and is a martingale when stopped before the hull hits a small ball around z.

Another representation is

$$M_t = G_{H_t}(z;\gamma(t),\infty)$$

where $G = G^{\kappa,r}$ is defined by $G_{\mathbb{H}}(z;0,\infty) = y^{\zeta}(1+x^2/y^2)^{r/2}$ and $G_D(z;w_1,w_2) = |f'(z)|^{\lambda}|f'(w_2)|^{\zeta+\lambda}G_{f(D)}(f(z);f(w_1),f(w_2))$ for any conformal map $f\colon D\to f(D)$, with the convention that if $f(\infty)=\infty$, then $f'(\infty) = \partial_z\big|_{z=0} 1/(f(1/z))$. Moreover, it is also the density of $\mathrm{SLE}_\kappa(\rho)$ with force point at z where $\rho = r\kappa$ (cf. [SW05; Zha19a]).

Now we consider the parametrisation by conformal radius introduced in Section 3.2.2. Recall that we have the representation

$$d\frac{X_{\sigma(s)}}{Y_{\sigma(s)}} = 4\frac{X_{\sigma(s)}}{Y_{\sigma(s)}}\left(1+\frac{X_{\sigma(s)}^2}{Y_{\sigma(s)}^2}\right)ds - \left(1+\frac{X_{\sigma(s)}^2}{Y_{\sigma(s)}^2}\right)\sqrt{\kappa}\,d\hat{B}_s$$

where $d\hat{B}_s = \frac{1}{Y_{\sigma(s)}(1+X_{\sigma(s)}^2/Y_{\sigma(s)}^2)}\,dB_{\sigma(s)}$ is another standard Brownian motion. For the process $\hat{\theta}(s) = \cot^{-1}\left(\frac{X_{\sigma(s)}}{Y_{\sigma(s)}}\right) \in \,]0,\pi[$ we have

$$d\hat{\theta}_s = (\kappa-4)\cot\hat{\theta}_s\,ds + \sqrt{\kappa}\,d\hat{B}_s.$$

This is a (time-changed) radial Bessel process of index $\nu = \frac{1}{2}-\frac{4}{\kappa}$ (i.e. dimension $3-\frac{8}{\kappa}$). In particular, it hits the boundary $\{0,\pi\}$ in finite time if and only if $\kappa<8$. (This reflects the fact that for $\kappa<8$, each point $z\in\mathbb{H}$ is a.s. missed or swallowed in finite time, whereas for $\kappa\geq 8$, each point $z\in\mathbb{H}$ is a.s. hit in finite time.)

We assume absorbing boundary for the process, i.e. we stop the process when $\hat{\theta}_\tau \in \{0,\pi\}$. In case $\kappa\in[0,4]$, this happens only when $\sigma(\tau)=\infty$. In case $\kappa\in\,]4,8[$, this happens when z is swallowed (cf. [Sch01, Lemma 3]).

The martingale (3.12) in this parametrisation is

$$M_{\sigma(s)} = |g'_{\sigma(s)}(z)|^{\lambda}Y_{\sigma(s)}^{\zeta}(1+X_{\sigma(s)}^2/Y_{\sigma(s)}^2)^{r/2} = e^{4s\lambda}Y_{\sigma(s)}^{\zeta+\lambda}(1+X_{\sigma(s)}^2/Y_{\sigma(s)}^2)^{r/2}.$$

In fact, this turns out to be the "change of measure"-martingale for radial Bessel processes, see Section 3.2.4.

3.2.4 Radial Bessel process

Part of the material presented here are contained in [Law19].

Consider a radial Bessel process of index ν (equivalently dimension $\delta = 2+2\nu$)

$$d\theta_t = (\frac{1}{2}+\nu)\cot\theta_t\,dt + dB_t, \quad \theta_0\in\,]0,\pi[.$$

Recall that such process hits $\{0,\pi\}$ in finite time if and only if $\nu<0$.

We introduce the "change of measure"-martingale for the Bessel process: $M_t = (\frac{\sin\theta_t}{\sin\theta_0})^a K_t$ where $K_t = K_t^{(\nu,a)}$ is a compensator that we compute now. We have

$$d(\sin\theta_t)^a = (\sin\theta_t)^a\left(a\cot\theta_t\,dB_t + \left[-\frac{a}{2}+a(\frac{a}{2}+\nu)\cot^2\theta_t\right]dt\right).$$

This leads us to

$$K_t = \exp\left(\int_0^t \left[\frac{a}{2} - a(\frac{a}{2}+\nu)\cot^2\theta_s\right]\, ds\right)$$
$$= \exp\left(a\left[\frac{a}{2}+\frac{1}{2}+\nu\right]t - a(\frac{a}{2}+\nu)\int_0^t (\sin\theta_s)^{-2}\, ds\right).$$

Then

$$M_t = \left(\frac{\sin\theta_t}{\sin\theta_0}\right)^a K_t$$

is a local martingale with

$$dM_t = a\cot\theta_t\, M_t\, dB_t.$$

It is a bounded martingale until time $t\wedge T_\varepsilon$ where $T_\varepsilon := \inf\{t \mid \theta_t \in \{\varepsilon, \pi-\varepsilon\}\}$.

Applying Girsanov's theorem to $M_{t\wedge T_\varepsilon}$ we get a measure $\mathbb{P}^{\nu+a,\varepsilon}$ defined by $d\mathbb{P}^{\nu+a,\varepsilon} = M_{t\wedge T_\varepsilon}\, d\mathbb{P}^\nu$. We can then write

$$d\theta_t = (\frac{1}{2}+\nu+a)\cot\theta_t\, dt + d\hat{B}_t, \quad t \le T_\varepsilon$$

where $\hat{B} = \hat{B}^{(\nu,a)}$ is a standard Brownian motion under the measure $\mathbb{P}^{\nu+a,\varepsilon}$ until time T_ε.

We claim that in case $\nu + a \ge 0$, M_t is a martingale. Indeed, it suffices to show that

$$\mathbb{E}^\nu[M_{t\wedge T_\varepsilon} 1_{T_\varepsilon<t}] = \mathbb{P}^{\nu+a,\varepsilon}(T_\varepsilon < t) \to 0 \quad \text{as } \varepsilon \searrow 0$$

since the optional sampling theorem (applied to $M_{t\wedge T_\varepsilon}$) and the monotone convergence theorem will then imply $\mathbb{E}^\nu M_t = \mathbb{E}^\nu M_0$. But this is true because the index of the Bessel process θ under the law $\mathbb{P}^{\nu+a,\varepsilon}$ is $\nu + a \ge 0$.

In particular, $d\mathbb{P}^{\nu+a} = M_t\, d\mathbb{P}^\nu$ defines a probability measure in case $\nu + a \ge 0$.

In case $\nu + a < 0$, for every t the measure $\mathbb{P}^{\nu+a}$ is still defined on $\mathcal{F}_t\big|_{\{T_0>t\}}$, and $\frac{d\mathbb{P}^{\nu+a}}{d\mathbb{P}^\nu}\big|_{\{T_0>t\}} = M_t$.

Let $q_t(x,y)$ denote the transition density of the process θ under $\mathbb{P}^\nu$. It can be written down explicitly, see [Zha16, Proposition 8.1], and satisfies the following bound. In particular, it converges exponentially fast to its stationary law $f(y) = c_\nu(\sin y)^{1+2\nu}$.

Proposition 3.2.8. *If $\nu \ge 0$, then there exist $c < \infty$ such that for all $t \ge 1$ and $x, y \in \,]0,\pi[$*

$$(1 - ce^{-(1+\nu)t})f(y) \le q_t(x,y) \le (1 + ce^{-(1+\nu)t})f(y)$$

where $f(y) = c_\nu(\sin y)^{1+2\nu}$.

We are interested in the "radial Bessel clock"

$$C_t = \int_0^t (\sin\theta_s)^{-2}\, ds.$$

In case $\nu > 0$, we can pick $a \in \,]-2\nu, 0[$, and the fact that M_t is a martingale implies that C_t has exponential moments of order $\frac{\nu^2}{2}$.

Proposition 3.2.9. *If $\nu > 0$ and $p > 0$, then there exists $c < \infty$ such that*

$$\mathbb{E}^\nu[C_t^p] \leq c(1 + (-\log\sin\theta_0)^p + t^p)$$

for $t \geq 1$.

Proof. We have

$$\log\sin\theta_t = \log\sin\theta_0 + \int_0^t \cot\theta_s\,dB_s - (\frac{1}{2} + \nu)t + \nu\int_0^t (\sin\theta_s)^{-2}\,ds,$$

or equivalently

$$\nu C_t = \log\sin\theta_t - \log\sin\theta_0 - \int_0^t \cot\theta_s\,dB_s + (\frac{1}{2} + \nu)t.$$

Note that

$$\left\langle \int_0^t \cot\theta_s\,dB_s \right\rangle = \int_0^t \cot^2\theta_s\,ds = C_t - t$$

and therefore

$$\begin{aligned} \mathbb{E}C_t^p &\lesssim (-\log\sin\theta_0)^p + \mathbb{E}(-\log\sin\theta_t)^p + t^p + \mathbb{E}(C_t)^{p/2} \\ &\leq (-\log\sin\theta_0)^p + c + t^p + (\mathbb{E}C_t^p)^{1/2} \end{aligned}$$

where we have used Proposition 3.2.8 to bound $\mathbb{E}(-\log\sin\theta_t)^p$ by a constant c independent of $t \geq 1$.

Recall that (in case $\nu > 0$) C_t has exponential moments of small order. Hence $\mathbb{E}C_t^p < \infty$, and solving the quadratic equation yields

$$(\mathbb{E}C_t^p)^{1/2} \lesssim c + \sqrt{(-\log\sin\theta_0)^p + c + t^p}$$

for all t. □

Proposition 3.2.10. *Let $\nu \in \mathbb{R}$ and $\lambda < \frac{\nu^2}{2}$. Let τ be a bounded stopping time and X be a $\mathcal{F}_\tau$-measurable random variable. Then*

$$\mathbb{E}^\nu[X\exp(\lambda C_\tau)\,1_{T_0>\tau}] = (\sin\theta_0)^a\,\mathbb{E}^{\nu+a}\left[X(\sin\theta_\tau)^{-a}\exp\left((\lambda - \frac{a}{2})\tau\right)\right]$$

where $a = -\nu + \sqrt{\nu^2 - 2\lambda}$.
Moreover, suppose that in case $\nu \leq -2$ we also have $\lambda < -2\nu - 2$. Then, for $p > 0$, there exists $c < \infty$ such that

$$\mathbb{E}^\nu[\exp(\lambda C_t)C_t^p\,1_{T_0>t}] \leq c(\sin\theta_0)^a(1 + (-\log\sin\theta_0)^p + t^p)\exp\left((\lambda - \frac{a}{2})t\right)$$

for $t \geq 1$.

Proof. The parameter a is chosen such that $\lambda = -a(\frac{a}{2}+\nu)$ and $\nu + a > 0$. In that case,

$$M_t = \left(\frac{\sin\theta_t}{\sin\theta_0}\right)^a \exp\left((\frac{a}{2}-\lambda)t + \lambda C_t\right)$$

is a $\mathbb{P}^\nu$-martingale, and the law of θ under $\mathbb{P}^{\nu+a}$ is that of a radial Bessel process of index $\nu + a > 0$. Therefore

$$\begin{aligned}\mathbb{E}^\nu[X \exp(\lambda C_\tau)\, 1_{T_0>\tau}] &= \mathbb{E}^{\nu+a}[X \exp(\lambda C_\tau) M_\tau^{-1}\, 1_{T_0>\tau}] \\ &= (\sin\theta_0)^a\, \mathbb{E}^{\nu+a}\left[X(\sin\theta_\tau)^{-a} \exp\left((\lambda-\frac{a}{2})\tau\right)\right].\end{aligned}$$

To get the second claim, we apply this to $X = C_t^p$, then use Hölder's inequality, Proposition 3.2.9, and Proposition 3.2.8 to obtain

$$\begin{aligned}\mathbb{E}^{\nu+a}[C_t^p(\sin\theta_t)^{-a}] &\le \left(\mathbb{E}^{\nu+a}[C_t^{pq'}]\right)^{1/q'} \left(\mathbb{E}^{\nu+a}[(\sin\theta_t)^{-aq}]\right)^{1/q} \\ &\lesssim 1 + (-\log\sin\theta_0)^p + t^p\end{aligned}$$

provided that the second expectation is finite and bounded for some choice of $q > 1$. By Proposition 3.2.8 this is the case whenever $-aq + 1 + 2(\nu + a) > -1$, and we can pick such $q > 1$ if and only if $\nu > -2$ or $\lambda < -2\nu - 2$. □

In the special case $\lambda = 0$, we get the following statement.

Corollary 3.2.11. *Let $\nu < 0$. Let τ be a bounded stopping time and X be a $\mathcal{F}_\tau$-measurable random variable. Then*

$$\mathbb{E}^\nu[X\, 1_{T_0>\tau}] = (\sin\theta_0)^{-2\nu}\, \mathbb{E}^{-\nu}[X(\sin\theta_\tau)^{2\nu} e^{\nu\tau}].$$

Moreover, for $p > 0$, there exists $c < \infty$ such that

$$\mathbb{E}^\nu[C_t^p\, 1_{T_0>t}] \le c(\sin\theta_0)^{-2\nu}(1 + (-\log\sin\theta_0)^p + t^p)e^{\nu t}$$

for $t \ge 1$.

We have a similar statement in case $p < 0$.

Corollary 3.2.12. *Let $\nu \in \mathbb{R}$ and $\lambda \in \,]0, \frac{\nu^2}{2}[$ such that $\nu > -2$ or $\lambda < -2\nu - 2$. Then, for $p > 0$, there exists $c < \infty$ such that*

$$\mathbb{E}^\nu[\exp(\lambda C_t)(1 + C_t)^{-p}\, 1_{T_0>t}] \le c(\sin\theta_0)^a \exp\left((\lambda - \frac{a}{2})t\right) t^{-p}$$

for $t \ge 1$ where $a = -\nu + \sqrt{\nu^2 - 2\lambda}$.

Proof. We split up into the events $\{C_t \le \delta t\}$ and $\{C_t \ge \delta t\}$ where $\delta > 0$ is a suitably chosen number. On the event $\{C_t \ge \delta t\}$ we have

$$\begin{aligned}
&\mathbb{E}^\nu[\exp(\lambda C_t)(1+C_t)^{-p}\, 1_{T_0>t} 1_{C_t\ge\delta t}] \\
&\quad \le (1+\delta t)^{-p}\, \mathbb{E}^\nu[\exp(\lambda C_t)\, 1_{T_0>t}] \\
&\quad \le (1+\delta t)^{-p} (\sin\theta_0)^a \exp\left((\lambda - \frac{a}{2})t\right)\, \mathbb{E}^{\nu+a}\left[(\sin\theta_t)^{-a}\right],
\end{aligned}$$

and the last expectation is bounded as in the proof of Proposition 3.2.10.

For the event $\{C_t \le \delta t\}$ we distinguish the cases $\nu \ge 0$ and $\nu < 0$. In case $\nu \ge 0$ we have $a \le 0$ and therefore (since $\lambda > 0$)

$$\mathbb{E}^\nu[\exp(\lambda C_t)(1+C_t)^{-p}\, 1_{T_0>t} 1_{C_t\le\delta t}] \le \exp(\lambda\delta t) \lesssim (\sin\theta_0)^a \exp\left((\lambda - \frac{a}{2})t\right) t^{-p}.$$

In case $\nu < 0$, applying Corollary 3.2.11, we get

$$\begin{aligned}
\mathbb{E}^\nu[\exp(\lambda C_t)(1+C_t)^{-p}\, 1_{T_0>t} 1_{C_t\le\delta t}] &\le \exp(\lambda\delta t)\, \mathbb{E}^\nu[1_{T_0>t}] \\
&\le \exp(\lambda\delta t)(\sin\theta_0)^{-2\nu} e^{\nu t}\, \mathbb{E}^{-\nu}[(\sin\theta_t)^{2\nu}] \\
&\lesssim (\sin\theta_0)^a \exp\left((\lambda - \frac{a}{2})t\right) t^{-p}
\end{aligned}$$

where we have used $-2\nu \ge a$ and $\nu < \lambda - \frac{a}{2}$. □

3.3 Refined regularity

3.3.1 Warmup: Existence of the trace

In order to illustrate the general idea of our proof, let us give a simple proof showing SLE_κ, $\kappa \neq 8$, generates a continuous trace. The (more technical) proofs that come later are based on the idea that we describe in the following. We remark that the content of this subsection is mainly for illustration, and not required for the rest of the paper (although it greatly helps understanding what comes after).

By [JL11, Corollary 3.12], in order to have a continuous trace, it suffices to show $|\hat{f}_t'(i\delta)| \le \delta^{-\beta}$ for some $\beta < 1$ (uniformly in t and small δ). Due to Koebe's distortion theorem, it suffices to show this for $\delta = e^{-m}$, $m \in \mathbb{N}$.

We restrict to the set $\{\|\xi\|_\infty \le M\}$. Fix $m \in \mathbb{N}$. Suppose that $|\hat{f}_t'(ie^{-m})| \ge e^{\beta m}$ for some t. By Lemma 3.2.2, there exists $z \in H(e^{-(1-\beta)m}, M, T)$ such that $Z_t(z) \approx ie^{-m}$ and $\Upsilon_t(z) \gtrsim e^{-(1-\beta)m}$. Recalling the parametrisation by conformal radius, this means $t \le \sigma(cm)$ where $c = \frac{1-\beta}{4}$. Therefore

$$\begin{aligned}
&\mathbb{P}(|\hat{f}_t'(ie^{-m})| \ge e^{\beta m} \text{ for some } t) \\
&\quad \le \sum_{z\in H(e^{-(1-\beta)m},M,T)} \mathbb{P}\left(Y_{\sigma(s)}(z) \asymp e^{-m} \text{ and } \frac{|X_{\sigma(s)}|}{Y_{\sigma(s)}} \le 1 \text{ for some } s \le cm\right). \qquad (3.13)
\end{aligned}$$

If the sum of the probabilities decays exponentially in m, then by Borel-Cantelli we are done.

For $z = x + iy$, recall from Section 3.2.2 that

$$Y_{\sigma(s)}(z) = y \exp\left(-2 \int_{s_0}^{s} (\sin \hat{\theta}_s)^{-2} \, ds\right)$$

and from Section 3.2.3 that $\hat{\theta}$ is a radial Bessel process of index $\nu = \frac{1}{2} - \frac{4}{\kappa}$ started at $\hat{\theta}_{s_0} = \cot^{-1}(x/y)$ (where $s_0 = -\frac{1}{4} \log y$) and run at speed κs. In particular, we can write

$$Y_{\sigma(s)}(z) = y \exp\left(-\frac{2}{\kappa} C_{\kappa(s-s_0)}\right)$$

where C denotes the radial Bessel clock defined in Section 3.2.4.

For $\kappa \neq 8$, we have $\nu \neq 0$. Therefore the probability on the right-hand side of (3.13) can be estimated by Proposition 3.2.10. Let

$$\tau := \inf \left\{ s \in [s_0, cm] \mid Y_{\sigma(s)}(z) \asymp e^{-m} \text{ and } \hat{\theta}_s \in [\cot^{-1}(\pm 1)] \right\} \wedge (cm+1).$$

Fix some $\lambda \in]0, \frac{\nu^2}{2}[$. We have

$$\begin{aligned}
\mathbb{P}(\tau \leq cm) &\asymp y^{-\lambda} e^{-\lambda m} \mathbb{E}\left[\exp\left(\frac{2\lambda}{\kappa} C_{\kappa(\tau - s_0)}\right) 1_{\tau \leq cm}\right] \\
&\lesssim y^{-\lambda} e^{-\lambda m} (\sin \cot^{-1}(x/y))^a \exp\left(\left(\frac{2\lambda}{\kappa} - \frac{a}{2}\right)^+ \kappa(cm - s_0)\right) \\
&= e^{-\lambda m} \exp\left((2\lambda - \frac{a\kappa}{2})^+ cm\right) y^{-\lambda + (\lambda/2 - a\kappa/8)^+} (1 + |x|/y)^{-a}
\end{aligned}$$

where $a = -\nu + \sqrt{\nu^2 - \frac{4\lambda}{\kappa}}$ and $\eta^+ = \eta \vee 0$ denotes the positive part.

Picking β close to 1 (i.e. c close to 0), we see that after summing in $z \in H(e^{-(1-\beta)m})$ that (according to Lemma 3.2.4) the sum (3.13) is bounded by $e^{-\lambda m + \varepsilon m}$ where we can make $\varepsilon > 0$ as small as we want. This is summable in m, which is exactly what we wanted to show.

3.3.2 Setup of our proofs

We turn to the proofs of the main results of the paper. They follow the same idea as the previous subsection, but require much more care. We begin by discussing the technical setup. Recall the notations Υ_t, σ, and $H(h, M, T)$ introduced in Section 3.2.2.

Let $\delta \in]0,1]$, $t \in [0,T]$, and find $\bar{s} = \bar{s}(t,\delta) \in \mathbb{N}$ such that $\Upsilon_t(\hat{f}_t(i\delta)) \in [e^{-4\bar{s}}, e^{-4(\bar{s}-1)}]$. By Lemma 3.2.2, there exists $w \in H(e^{-4\bar{s}}, \|\xi\|, T)$ such that $\Upsilon_t(w) \in [\frac{27}{160} e^{-4\bar{s}}, \frac{125}{32} e^{-4(\bar{s}-1)}]$ and $|Z_t(w) - i\delta| \leq \delta/2$. For $\delta = 2^{-m}$, call this point $w(t,m)$.

By construction, $t = \sigma(s, w(t,m))$ for some $s = -\frac{1}{4} \log \Upsilon_t(w) \in [\bar{s} - 2, \bar{s} + 1]$.

For $w \in \mathbb{H}$ and $\bar{s} \in \mathbb{N}$, we consider the set

$$P(w,\bar{s}) := \{(m,t) \in \mathbb{N} \times [0,T] \mid |Z_{\sigma(s,w)}(w) - i2^{-m}| \le 2^{-m-1} \text{ and } \sigma(s,w) = t \text{ for some } s \in [\bar{s}-2, \bar{s}+1]\}.$$

Let $P'(w,\bar{s})$ be any subset of $P(w,\bar{s})$ such that the t in the subset have distance at least 2^{-2m} from each other. Let $N(w,\bar{s})$ be the largest possible cardinality of such $P'(w,\bar{s})$.

Let us remark here that if the process $\hat{\theta}$ dies before time $\bar{s}-2$, then $N(w,\bar{s}) = 0$.

To count $N(w,\bar{s})$, we define a sequence of stopping times $S_n = S_n(w,\bar{s})$, $T_n = T_n(w,\bar{s})$ as follows. Fix some $b > 1$ (the exact value does not matter). Let

$$S_0 := \inf\{s \in [\bar{s}-2, \bar{s}+1] \mid \frac{X^2_{\sigma(s)}}{Y^2_{\sigma(s)}} \le 1\},$$

and inductively

$$T_n := \inf\{s \in [S_n, \bar{s}+1] \mid \frac{X^2_{\sigma(s)}}{Y^2_{\sigma(s)}} \ge b\} \wedge (\bar{s}+1),$$

$$S_{n+1} := \inf\{s \in]T_n, \bar{s}+1] \mid \frac{X^2_{\sigma(s)}}{Y^2_{\sigma(s)}} \le 1\}.$$

Let

$$P_n(w,\bar{s}) := \{(m,t) \in \mathbb{N} \times [0,T] \mid |Z_{\sigma(s,w)}(w) - i2^{-m}| \le 2^{-m-1} \text{ and } \sigma(s,w) = t \text{ for some } s \in [S_n, T_n]\},$$

and note that $P(w,\bar{s}) = \bigcup_n P_n(w,\bar{s})$. Similarly to above, let $P'_n(w,\bar{s})$ be any subset of $P_n(w,\bar{s})$ such that the t in the subset have distance at least 2^{-2m} from each other. Let $N_n(w,\bar{s})$ be the largest possible cardinality of such $P'_n(w,\bar{s})$.

Moreover, note that $\frac{X^2_{\sigma(s)}}{Y^2_{\sigma(s)}} \le b$ on any interval $[S_n, T_n]$. Letting

$$p = \mathbb{P}\left(\frac{X^2_{\sigma(s)}}{Y^2_{\sigma(s)}} > b \text{ for some } s \in [0,3] \;\middle|\; \frac{X^2_{\sigma(0)}}{Y^2_{\sigma(0)}} = 1 \right) \in]0,1[,$$

we see that $\mathbb{P}(S_n < \infty) \le p^n$.

Lemma 3.3.1. *There exists $N \in \mathbb{N}$ such that $N_n(w,\bar{s}) \le N$ for any n.*

Proof. If $1 + \frac{X^2_{\sigma(s)}}{Y^2_{\sigma(s)}} \le b$, then by (3.6) we have $dY^2_{\sigma(s)} \ge -4bY^2_{\sigma(s)}\,ds$ and hence (by Grönwall's inequality)

$$Y_{\sigma(s)} \ge e^{-2b(s-\bar{s})} Y_{\sigma(\bar{s})}.$$

Moreover, by (3.5) we have $d\sigma(s) \le b^2 Y_{\sigma(s)}^2\, ds$ and hence

$$\sigma(s) \le \sigma(\bar{s}) + (s-\bar{s}) b^2 Y_{\sigma(\bar{s})}^2.$$

Suppose now that $(m_0, t_0) \in P_n(w, \bar{s})$, i.e. $|Z_{\sigma(s,w)}(w) - i2^{-m_0}| \le 2^{-m_0-1}$ and $\sigma(s, w) = t_0$ for some $s \in [S_n, T_n]$. In particular, $Y_{\sigma(s)} \in [2^{-m_0-1}, 2^{-m_0+1}]$.

If we find another pair $(m,t) \in P_n(w,\bar{s})$, then (by our previous observation) we must have $2^{-m+1} \ge e^{-6b} 2^{-m_0-1}$ and $t \le t_0 + 3b^2 2^{-2m_0+2}$. But there is a fixed maximum number N of such pairs (m,t) where the t also have distance at least 2^{-2m} from each other (and that number N does not depend on m_0 or n). □

We are going to need one more addition to this, the reason of which will become apparent at the end of the proof of Theorems 3.3.7 and 3.3.11.

Lemma 3.3.2. *Let $w \in \mathbb{H}$, $\bar{s} \in \mathbb{N}$. For every n there exists a random variable $M_n(w,\bar{s})$ that is independent of $\mathcal{F}_{\sigma(S_n)}$ and such that for every $(m,t) \in P_n(w,\bar{s})$ and $u > t$ with $|u-t| \in [2^{-2(m+1)}, 2^{-2(m-1)}]$ we have*

$$\frac{|\xi(u)-\xi(t)|}{|u-t|^{1/2}} \le M_n(w,\bar{s}).$$

Moreover, each $M_n(w,\bar{s})$ has the same law and has all exponential moments.

Proof. Let

$$M_n(w,\bar{s}) := \sup_{t,u} \frac{|\xi(u)-\xi(t)|}{|u-t|^{1/2}}$$

where the supremum runs over $t < u$ with $t \in [\sigma(S_n), \sigma(S_n) + 3b^2 Y_{\sigma(S_n)}^2]$ and $|u-t| \in [\frac{1}{16} e^{-12b} Y_{\sigma(S_n)}^2, 16 Y_{\sigma(S_n)}^2]$.

By the strong Markov property and Brownian scaling, we see that each M_n has the same law, and finite exponential moments.

For every $(m,t) \in P_n(w,\bar{s})$, by Lemma 3.3.1, we have $t \in [\sigma(S_n), \sigma(T_n)] \subseteq [\sigma(S_n), \sigma(S_n) + 3b^2 Y_{\sigma(S_n)}^2]$ and $2^{-m} \in [\frac{1}{2} Y_t, 2Y_t] \subseteq [\frac{1}{2} e^{-6b} Y_{\sigma(S_n)}, 2Y_{\sigma(S_n)}]$. If $u > t$ and $|u-t| \in [2^{-2(m+1)}, 2^{-2(m-1)}]$, then also $|u-t| \in [\frac{1}{16} e^{-12b} Y_{\sigma(S_n)}^2, 16 Y_{\sigma(S_n)}^2]$. In particular, the term

$$\frac{|\xi(u)-\xi(t)|}{|u-t|^{1/2}}$$

appears in the supremum defining M_n. □

3.3.3 Generalised variation

In this section, we are going to estimate the ψ-variation of the SLE trace.

We will frequently use the following estimate. Let ψ be a convex function and $p_n \ge 0$ a summable sequence with $p = \sum p_n < \infty$. By Jensen's inequality we have

$$\psi\left(\sum a_n\right) = \psi\left(\sum p \frac{a_n}{p_n} \frac{p_n}{p}\right) \le \sum \psi\left(p \frac{a_n}{p_n}\right) \frac{p_n}{p}.$$

In the following, we will assume that ψ is convex and satisfies the condition (Δ_c) (see Section 3.2.1).

Now let $0 = t_0 < t_1 < ... < t_r = T$ be a partition of $[0,T]$. Recall the notation from Section 3.3.2. For $z \in \mathbb{H}$ and $\bar{s} \in \mathbb{N}$, the following sets of pairs

$$\{(m,t_i) \mid (m,t_i) \in P(z,\bar{s}) \text{ and } |t_i - t_{i-1}| \geq 2^{-2m}\},$$
$$\{(m,t_i) \mid (m,t_i) \in P(z,\bar{s}) \text{ and } |t_{i+1} - t_i| \geq 2^{-2m}\}$$

each form a set $P'(z,\bar{s})$ as described in Section 3.3.2.

Lemma 3.3.3. *Let $M,T > 0$ and $\varepsilon > 0$. There exists $C > 0$ depending on ψ, T, ε such that if $\|\xi\|_{\infty;[0,T]} \leq M$, then*

$$\begin{aligned}
&\sum_i \psi\left(|\gamma(t_i) - \hat{f}_{t_i}(i|t_i - t_{i-1}|^{1/2})|\right) + \psi\left(|\gamma(t_i) - \hat{f}_{t_i}(i|t_{i+1} - t_i|^{1/2})|\right) \\
&\leq C \sum_{s\in\mathbb{N}} \sum_{z \in H(e^{-4s},M,T)} \sum_{n\in\mathbb{N}_0} \\
&\qquad (\log^* Y_{\sigma(S_n,z)}(z)^{-1})^{-1-\varepsilon} \psi\left(e^{-4s}(\log^* Y_{\sigma(S_n,z)}(z)^{-1})^{1+\varepsilon}\right) 1_{S_n(z,s)<\infty}
\end{aligned}$$

for any partition of $[0,T]$.

Remark 3.3.4. *This is almost an upper bound on the ψ-variation of γ. Note that the right-hand side does not depend on the choice of the partition.*

Proof. Pick $m_i \in \mathbb{N}$ with $2^{-m_i} \asymp |t_i - t_{i-1}|^{1/2}$. By (3.8), we have

$$|\gamma(t) - \hat{f}_t(i2^{-m_i})| \lesssim \sum_{m \geq m_i} \Upsilon_t(\hat{f}_t(i2^{-m})).$$

Applying Jensen's inequality as above (and the assumption (Δ_c) for ψ) yields

$$\psi\left(|\gamma(t) - \hat{f}_t(i2^{-m_i})|\right) \lesssim \sum_{m\geq m_i} m^{-1-\varepsilon} \psi\left(\Upsilon_t(\hat{f}_t(i2^{-m}))m^{1+\varepsilon}\right).$$

Applying this to $2^{-m_i} \asymp |t_i - t_{i-1}|^{1/2}$ and summing over t_i, we get

$$\begin{aligned}
&\sum_i \psi\left(|\gamma(t_i) - \hat{f}_{t_i}(i|t_i - t_{i-1}|^{1/2})|\right) \\
&\qquad \lesssim \sum_{m\in\mathbb{N}} \sum_{i \in I_m} m^{-1-\varepsilon} \psi\left(\Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m}))m^{1+\varepsilon}\right) \qquad (3.14)
\end{aligned}$$

where $I_m = \{i \mid |t_i - t_{i-1}| \geq 2^{-2m}\}$.

The same applies with $|t_i - t_{i-1}|$ replaced by $|t_{i+1} - t_i|$, so we can just focus on the former.

We rearrange the sum (3.14) by collecting all terms where $\Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m})) \in [e^{-4s}, e^{-4(s-1)}]$. As we observed in Section 3.3.2, we can find $w = w(t_i, m) \in H(e^{-4s}, \|\xi\|, T)$ such that $\Upsilon_{t_i}(w) \asymp \Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m}))$ and $|Z_{t_i}(w) - i2^{-m}| \le 2^{-m-1}$.

In particular, we have $Y_{t_i}(w) \asymp 2^{-m}$ and $\Upsilon_{t_i}(w) \asymp \Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m})) \asymp e^{-4s}$, and $(m, t_i) \in P_n(w, s)$ for some n. Moreover, by Lemma 3.3.1, for each choice of s,z, and n, we can have at most N pairs of (m, i) with $(m, t_i) \in P_n(z, s)$. Finally, we have shown there also that $Y_{t_i} \asymp Y_{\sigma(S_n)}$. Putting everything together, we get

$$\begin{aligned}
&\sum_i \psi\left(|\gamma(t_i) - \hat{f}_{t_i}(i|t_i - t_{i-1}|^{1/2})|\right) \\
&\quad \lesssim \sum_{s\in\mathbb{N}} \sum_{m\in\mathbb{N}} \sum_{i\in I_m} 1_{\Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m}))\in[e^{-4s}, e^{-4(s-1)}]} \\
&\qquad\qquad (\log^* Y_{t_i}(w(t_i, m))^{-1})^{-1-\varepsilon} \psi\left(e^{-4s}(\log^* Y_{t_i}(w(t_i, m))^{-1})^{1+\varepsilon}\right) \\
&\quad \lesssim \sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s}, M, T)} \sum_{n\in\mathbb{N}_0} \\
&\qquad (\log^* Y_{\sigma(S_n,z)}(z)^{-1})^{-1-\varepsilon} \psi\left(e^{-4s}(\log^* Y_{\sigma(S_n,z)}(z)^{-1})^{1+\varepsilon}\right) 1_{S_n(z,s)<\infty}.
\end{aligned}$$

□

In the following, we consider the function $\psi_{p,q}$ defined in (3.2). By (3.3), we can estimate

$$\psi\left(e^{-4s}(\log^* Y_{\sigma(s)}(z)^{-1})^{1+\varepsilon}\right) \lesssim (...)^p s^{-q} \left(\log^* \log^* Y_{\sigma(s)}(z)^{-1}\right)^q.$$

Recall that by definition $S_n(z, s) \in [s-2, s+1]$ whenever it is finite. Hence, we are reduced to estimate

$$\sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s}, M, T)} \sum_{n\in\mathbb{N}_0} e^{-4ps} s^{-q} \left(\log^* Y_{\sigma(S_n)}^{-1}\right)^{p-1+\varepsilon} \left(\log^* \log^* Y_{\sigma(S_n)}^{-1}\right)^q 1_{S_n<\infty}. \tag{3.15}$$

Recall that $\mathbb{P}(S_n < \infty) \le p^n$ for some $p < 1$. Therefore, when taking expectations, we can (using Hölder's inequality) safely ignore the sum in n at the cost of a multiplicative factor.

Remark 3.3.5. *In the expression* (3.15) *we see again the phase transition of the* p*-variation exponent* $p = (1 + \frac{\kappa}{8}) \wedge 2$*. Recall that* $\hat{\theta}$ *is a radial Bessel process of index* $\nu = \frac{1}{2} - \frac{4}{\kappa}$ *which can hit the boundary in case* $\nu < 0 \iff \kappa < 8$*. Consequently, the process has finite lifetime, and the probability of survival decays like* $e^{\nu t}$*. This allows the summand to be much smaller than* e^{-4ps} *and therefore allows for a choice of* $p < 2$*. In case* $\kappa \ge 8$*, the summand will not be smaller than* e^{-4ps}*, and since for each* s *we have* e^{8s} *summands, we need* $p \ge 2$ *to make the sum converge.*

Recall from Sections 3.2.2 and 3.2.3 (and explained again in Section 3.3.1) that we can write

$$Y_{\sigma(s)}(z) = y \exp\left(-\frac{2}{\kappa} C_{\kappa(s-s_0)}\right)$$

where C is the radial Bessel clock defined in Section 3.2.4, for a radial Bessel process of index $\nu = \frac{1}{2} - \frac{4}{\kappa}$ started at $\hat{\theta}_{s_0} = \cot^{-1}(x/y)$ (where $s_0 = -\frac{1}{4}\log y$). In other words, we have

$$\log Y_{\sigma(s)}(z)^{-1} = \log\frac{1}{y} + \frac{2}{\kappa}C_{\kappa(s-s_0)}.$$

We can now apply the results from Section 3.2.4.

First consider the case $\kappa > 8 \iff \nu > 0$. By Proposition 3.2.9, we have

$$\begin{aligned}\mathbb{E}(\log^* Y_{\sigma(s)}(z)^{-1})^\eta &\lesssim (\log^*(1/y))^\eta + (s-s_0)^\eta + (-\log\sin\cot^{-1}(x/y))^\eta \\ &\lesssim s^\eta + \left(\log^*\frac{1}{y} + \log(1+\frac{|x|}{y})\right)^\eta.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}[\text{ eq. (3.15) }] \\ &\lesssim \sum_{s\in\mathbb{N}}\sum_{z\in H(e^{-4s},M,T)} e^{-4ps}s^{-q}\left(s^{p-1+\varepsilon} + \left(\log^*\frac{1}{y} + \log(1+\frac{|x|}{y})\right)^{p-1+\varepsilon}\right) \\ &\asymp \sum_{s\in\mathbb{N}} s^{p-1-q+\varepsilon}e^{(8-4p)s}\end{aligned}$$

which converges for $p = 2$, $q > 2$.

In case $\kappa < 8 \iff \nu = \frac{1}{2} - \frac{4}{\kappa} < 0$, we apply Corollary 3.2.11. Using also that (by the definition of S_n) $S_n \in [s-2, s+1]$ and $\sin\hat{\theta}_{S_n} \geq \sin\cot^{-1}(1)$ whenever $S_n < \infty$, we get

$$\begin{aligned}&\mathbb{E}[(\log^* Y_{\sigma(S_n)}(z)^{-1})^\eta\, 1_{T_0>S_n}] \\ &\quad\lesssim (\sin\cot^{-1}(x/y))^{-2\nu}\,\mathbb{E}^{-\nu}[(\log^*\frac{1}{y} + C_{\kappa(s+1-s_0)})^\eta e^{\nu\kappa(s-s_0)}] \\ &\quad\lesssim (\sin\cot^{-1}(x/y))^{-2\nu-\varepsilon}(\log^*\frac{1}{y})^\eta(s-s_0)^\eta e^{\nu\kappa(s-s_0)} \\ &\quad\lesssim s^\eta\exp((\frac{\kappa}{2}-4)s)(1+|x|/y)^{1-8/\kappa+\varepsilon}y^{\kappa/8-1-\varepsilon}.\end{aligned}$$

Hence, applying also Lemma 3.2.4, we get

$$\begin{aligned}&\mathbb{E}[\text{ eq. (3.15) }] \\ &\quad\lesssim \sum_{s\in\mathbb{N}}\sum_{z\in H(e^{-4s},M,T)} \\ &\qquad\qquad e^{-4ps}s^{-q}s^{p-1+\varepsilon}\exp((\frac{\kappa}{2}-4)s)(1+|x|/y)^{1-8/\kappa+\varepsilon}y^{\kappa/8-1-\varepsilon} \\ &\quad\asymp \sum_{s\in\mathbb{N}} s^{p-1-q+\varepsilon}e^{(4+\kappa/2-4p)s}\end{aligned}$$

which converges for $p = 1 + \frac{\kappa}{8}$, $q > p = 1 + \frac{\kappa}{8}$.

Remark 3.3.6. *In case $\kappa = 8$, we have $\nu = 0$. The Bessel process of critical dimension barely misses the boundary, and the Bessel clock does not have finite moments. Instead, we have $\mathbb{P}(\log Y_{\sigma(s)}^{-1} > u) \asymp u^{-1/2}$. Unfortunately, this is not enough to estimate the regularity of SLE$_8$.*

Theorem 3.3.7. *Let $\kappa \in]0,8[\cup]8,\infty[$. Let $p = (1+\frac{\kappa}{8}) \wedge 2$, $q > p$, and ψ as in (3.2). Then, restricted to the event $\{\|\xi\|_{[0,T]} \le M\}$, we have*

$$\mathbb{E}\left[V^1_{\psi;[0,T]}(\gamma)\ 1_{\{\|\xi\|_{[0,T]}\le M\}}\right] < \infty.$$

In particular, $\mathbb{E}\left[[\gamma]^{p'}_{\psi\text{-var};[0,T]}\right] < \infty$ for any $p' < p$.

Proof. First note that restricting to the event $\{\|\xi\|_{[0,T]} \le M\}$ is enough since all our previous estimates depend on M polynomially (the dependence comes from Lemma 3.2.4), whereas the probability of $\mathbb{P}(\|\xi\|_{[0,T]} > M)$ decays exponentially in M. The second claim then follows from the fact that $[\gamma]_{\psi\text{-var};[0,T]} \lesssim V^1_{\psi;[0,T]}(\gamma)^{1/p}$.

So we are almost reduced to what we have estimated above.

Let $0 = t_0 < t_1 < ... < t_r = T$ be any partition of $[0,T]$. Write $\delta_i = |t_{i+1}-t_i|^{1/2}$. By our assumption Δ_c on ψ,

$$\psi(|\gamma(t_{i+1})-\gamma(t_i)|) \lesssim \psi\left(|\gamma(t_{i+1}) - \hat{f}_{t_{i+1}}(i\delta_i)|\right) + \psi\left(|\hat{f}_{t_{i+1}}(i\delta_i) - \hat{f}_{t_i}(i\delta_i)|\right) + \psi\left(|\gamma(t_i) - \hat{f}_{t_i}(i\delta_i))|\right).$$

The sums over the first and the third term appear already in Lemma 3.3.3 which we have bounded by (3.15). Recall that the expression (3.15) does not depend on the choice of the partition, and that $\mathbb{E}[$ eq. (3.15) $] < \infty$.

So we are left to estimate the middle term. We show that it is bounded by (3.15) as well.

By (3.9), we have

$$|\hat{f}_{t_{i+1}}(i\delta_i) - \hat{f}_{t_i}(i\delta_i)| \le \Upsilon_{t_i}(\hat{f}_{t_i}(i\delta_i))) \left(1 + \left(\frac{|\xi(t_{i+1})-\xi(t_i)|^2}{|t_{i+1}-t_i|}\right)^l\right).$$

Pick $m_i \in \mathbb{N}$ with $2^{-m_i} \asymp_2 \delta_i$.

As in the proof of Lemma 3.3.3, we can collect the indices where $\Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m_i})) \in [e^{-4s}, e^{-4(s-1)}]$, and as before, we can replace $\hat{f}_{t_i}(i2^{-m_i})$ by $w(t_i,m_i) \in H(e^{-4s}, \|\xi\|, T)$, and we have $Y_{t_i}(w(t_i,m_i)) \asymp 2^{-m_i}$ and $\Upsilon_{t_i}(w(t_i,m_i)) \asymp \Upsilon_{t_i}(\hat{f}_{t_i}(i2^{-m_i})) \asymp e^{-4s}$, and $(m_i,t_i) \in P(w,s)$.

Finally, since $|t_{i+1}-t_i| = \delta_i^2 \asymp_4 2^{-2m_i}$, by Lemma 3.3.2

$$\left(\frac{|\xi(t_{i+1})-\xi(t_i)|^2}{|t_{i+1}-t_i|}\right)^l \le M_n(w,s)^{2l}.$$

Hence,

$$\begin{aligned}
&\sum_i \psi(|\hat f_{t_{i+1}}(i\delta_i) - \hat f_{t_i}(i\delta_i)|)\\
&\quad\le \sum_i \psi\left(\Upsilon_{t_i}(\hat f_{t_i}(i\delta_i))\right)\left(1+\left(\frac{|\xi(t_{i+1})-\xi(t_i)|^2}{|t_{i+1}-t_i|}\right)^l\right)\bigg)\\
&\quad\lesssim \sum_{s\in\mathbb{N}}\sum_{z\in H(e^{-4s},M,T)}\sum_{n\in\mathbb{N}_0} \psi(e^{-4s}(1+M_n(z,s)^{2l}))\, 1_{S_n(z,s)<\infty}\\
&\quad\lesssim \sum_{s\in\mathbb{N}}\sum_{z\in H(e^{-4s},M,T)}\sum_{n\in\mathbb{N}_0} e^{-4ps}s^{-q}(1+M_n(z,s)^{2lp})\, 1_{S_n(z,s)<\infty}
\end{aligned}$$

where we have applied (3.3) in the last step. We see that this sum is also bounded by the expression (3.15) except for a factor $(1+M_n(z,s)^{2lp})$. But when taking the expectation, that factor is irrelevant since it is independent of $\mathcal{F}_{\sigma(S_n)}$ and has exponential moments. □

3.3.4 Hölder-type modulus

We can estimate the Hölder-type modulus of SLE$_\kappa$ in a similar way as in the previous section.

Let $\psi\colon [0,\infty[\, \to [0,\infty[$ be a convex homeomorphism that satisfies the condition (Δ_c). Let $\varphi\colon [0,\infty[\, \to [0,\infty[$ be a non-decreasing function such that for every $c>1$ there exists $\tilde\Delta_c$ such that $\varphi(cx)\le\tilde\Delta_c\varphi(x)$ for all x.

The following is proved in exactly the same way as Lemma 3.3.3.

Lemma 3.3.8. *Let $M,T>0$ and $\varepsilon>0$. There exists $C>0$ depending on $\psi,\varphi,T,\varepsilon$ such that if $\|\xi\|_{\infty;[0,T]}\le M$, then*

$$\begin{aligned}
&\psi\left(\frac{|\gamma(t_1)-\hat f_{t_1}(i|t_1-t_2|^{1/2})|}{\varphi(|t_1-t_2|)}\right)\\
&\le C\sum_{s\in\mathbb{N}}\sum_{z\in H(e^{-4s},M,T)}\sum_{n\in\mathbb{N}_0}\\
&\qquad (\log^* Y_{\sigma(S_n,z)}(z)^{-1})^{-1-\varepsilon}\psi\left(e^{-4s}\frac{(\log^* Y_{\sigma(S_n,z)}(z)^{-1})^{1+\varepsilon}}{\varphi(Y_{\sigma(S_n,z)}(z)^2)}\right)1_{S_n(z,s)<\infty}
\end{aligned}$$

for any $t_1,t_2\in[0,T]$.

Remark 3.3.9. *The proof shows that for any partition of $[0,T]$, the sum*

$$\sum_i \psi\left(\frac{|\gamma(t_i)-\hat f_{t_i}(i|t_i-t_{i-1}|^{1/2})|}{\varphi(|t_i-t_{i-1}|)}\right)+\psi\left(\frac{|\gamma(t_i)-\hat f_{t_i}(i|t_i-t_{i+1}|^{1/2})|}{\varphi(|t_i-t_{i+1}|)}\right)$$

is bounded by the same expression on the right-hand side. In particular, this is a more general version of Lemma 3.3.3.

Now, we again consider $\psi = \psi_{p,q}$ as in (3.2), and $\varphi(x) = x^\alpha(\log^*(\frac{1}{x}))^\beta$. As before, using (3.3), the right-hand side of Lemma 3.3.8 simplifies to

$$\sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s},M,T)} \sum_{n\in\mathbb{N}_0} e^{-4ps} s^{-q}\, Y_{\sigma(S_n)}^{-2p\alpha} \left(\log^* Y_{\sigma(S_n)}^{-1}\right)^{p+q-p\beta-1+\varepsilon} 1_{S_n<\infty}. \tag{3.16}$$

Recall from Sections 3.2.2 and 3.2.3 (and explained again in Section 3.3.1) that we can write

$$Y_{\sigma(s)}(z) = y \exp\left(-\frac{2}{\kappa} C_{\kappa(s-s_0)}\right)$$

where C is the radial Bessel clock defined in Section 3.2.4, for a radial Bessel process of index $\nu = \frac{1}{2} - \frac{4}{\kappa}$ started at $\hat\theta_{s_0} = \cot^{-1}(x/y)$ (where $s_0 = -\frac{1}{4}\log y$).

Let us suppose also that $p+q-p\beta-1 < 0$. (This is not necessary, but makes the calculations a little bit shorter.) It then remains to estimate

$$\mathbb{E}\left[Y_{\sigma(S_n)}^{-\lambda} \left(\log^* Y_{\sigma(S_n)}^{-1}\right)^{-\eta} 1_{S_n<\infty}\right]$$

where $\lambda, \eta > 0$.

Similarly as in Corollary 3.2.12, we can split up into the events $\{Y_{\sigma(S_n)} \le e^{-\delta S_n}\}$ and $\{Y_{\sigma(S_n)} \ge e^{-\delta S_n}\}$ with suitable $\delta > 0$.

Beginning with $\{Y_{\sigma(S_n)} \le e^{-\delta S_n}\}$, we find $\log^* Y_{\sigma(S_n)}^{-1} \gtrsim S_n$. Applying Proposition 3.2.10 and noting that (by the definition of S_n) $S_n \in [s-2, s+1]$ and $\sin\hat\theta_{S_n} \ge \sin\cot^{-1}(1)$ whenever $S_n < \infty$, we get

$$\begin{aligned}
&\mathbb{E}\left[Y_{\sigma(S_n)}^{-\lambda} \left(\log^* Y_{\sigma(S_n)}^{-1}\right)^{-\eta} 1_{S_n<\infty}\, 1_{Y_{\sigma(S_n)}\le e^{-\delta S_n}}\right] \\
&\quad\lesssim s^{-\eta} y^{-\lambda} \mathbb{E}\left[\exp\left(\frac{2\lambda}{\kappa} C_{\kappa(S_n-s_0)}\right) 1_{S_n<\infty}\right] \\
&\quad\lesssim s^{-\eta} y^{-\lambda} (\sin\cot^{-1}(x/y))^a \exp\left(\left(\frac{2\lambda}{\kappa} - \frac{a}{2}\right)\kappa(s-s_0)\right) \\
&\quad\lesssim s^{-\eta} \exp\left((2\lambda - \frac{a\kappa}{2})s\right) y^{-\lambda+\lambda/2-a\kappa/8} (1+|x|/y)^{-a}
\end{aligned}$$

where $a = -\nu + \sqrt{\nu^2 - \frac{4\lambda}{\kappa}} = -\nu + \sqrt{\nu^2 - \frac{8}{\kappa}p\alpha}$.

On the event $\{Y_{\sigma(S_n)} \ge e^{-\delta S_n}\}$ we have

$$\begin{aligned}
&\mathbb{E}\left[Y_{\sigma(S_n)}^{-\lambda} \left(\log^* Y_{\sigma(S_n)}^{-1}\right)^{-\eta} 1_{S_n<\infty}\right] \\
&\quad\lesssim e^{\lambda\delta s}\, \mathbb{E}[1_{S_n<\infty}] \\
&\quad\lesssim s^{-\eta} \exp\left(\left(\frac{2\lambda}{\kappa} - \frac{a}{2}\right)\kappa s\right) (\sin\cot^{-1}(x/y))^a \\
&\quad\lesssim s^{-\eta} \exp\left((2\lambda - \frac{a\kappa}{2})s\right) (1+|x|/y)^{-a}.
\end{aligned}$$

This is true whenever $\frac{8}{\kappa}p\alpha \in \,]0, \nu^2[$, or equivalently $a \in \,]-\nu, (-2\nu) \vee 0[$.

The sum in n is harmless again since we can apply Hölder's inequality in the changed measure (this is important because we do not want to increase the exponent λ, but for η it is no problem). Observe that S_n is defined in terms of $\hat{\theta}_s = \cot^{-1}(X_{\sigma(s)}/Y_{\sigma(s)})$, and the estimate $\mathbb{P}(S_n < \infty) \le p^n$ holds under any measure under which $\hat{\theta}$ is a (time-homogeneous) Markov process. This means that summing over n just gives us an additional multiplicative factor.

Hence, we are left to investigate the convergence of

$$\begin{aligned}
&\mathbb{E}[\text{ eq. (3.16) }] \\
&\quad \lesssim \sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s},M,T)} \\
&\qquad\quad e^{-4ps} s^{-q} s^{p+q-p\beta-1+\varepsilon} \exp\left((4p\alpha - \frac{a\kappa}{2})s\right) y^{(-p\alpha-a\kappa/8)\wedge 0}(1+|x|/y)^{-a} \\
&\quad = \sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s},M,T)} \\
&\qquad\quad \exp\left((-4p+4p\alpha - \frac{a\kappa}{2})s\right) s^{p-p\beta-1+\varepsilon} y^{(-p\alpha-a\kappa/8)\wedge 0}(1+|x|/y)^{-a}.
\end{aligned}$$

Since α, p, a are related by $\frac{4}{\kappa}p\alpha = -a(\frac{a}{2}+\nu) \iff p\alpha = -\frac{a^2\kappa}{8} - \frac{a\kappa}{8} + a$, the expression can be written as

$$\sum_{s\in\mathbb{N}} \exp\left((-4p - \frac{a^2\kappa}{2} - a\kappa + 4a)s\right) s^{p-p\beta-1+\varepsilon} \sum_{z\in H(e^{-4s},M,T)} y^{(a^2\kappa/8-a)\wedge 0}(1+|x|/y)^{-a}.$$

We first sum up in $z \in H(e^{-4s})$. The result is stated in Lemma 3.2.4. There will be three cases relevant to us. The first one will give us the desired result for $\kappa > 1$, the second case will apply to $\kappa \in \,]0,1[$, and the third case to $\kappa = 1$.

Case 1: Suppose either $a \ge 1$, $\frac{a^2\kappa}{8} - a > -2$ or $a \in \,]-1,1]$, $\frac{a^2\kappa}{8} > -1$. In that case the sum over $z \in H(e^{-4s})$ gives us $(e^{-4s})^{-2} = e^{8s}$, and we are left with

$$\sum_{s\in\mathbb{N}} \exp\left((8 - 4p - \frac{a^2\kappa}{2} - a\kappa + 4a)s\right) s^{p-p\beta-1+\varepsilon}.$$

This sum converges when

$$\begin{aligned}
& 8 - 4p - \frac{a^2\kappa}{2} - a\kappa + 4a \le 0 \\
\text{and}\qquad & p - p\beta - 1 < -1.
\end{aligned}$$

Fix $a \in]-\nu, (-2\nu) \vee 0[$. The minimal p to make the sum converge is then $p = 2 - \frac{a^2\kappa}{8} - \frac{a\kappa}{4} + a$, giving us the optimal exponents

$$\alpha = \frac{1}{p}(-\frac{a^2\kappa}{8} - \frac{a\kappa}{8} + a) = \frac{-\frac{a^2\kappa}{8} - \frac{a\kappa}{8} + a}{2 - \frac{a^2\kappa}{8} - \frac{a\kappa}{4} + a}$$

$$\text{and any } \beta > 1$$

provided that we are in Case 1. Optimising over a yields (after a tedious but elementary computation) $a = \frac{16-4\sqrt{\kappa+8}}{\kappa}$ and $\alpha = 1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}$.

One can can check that with this choice of a, we are indeed in Case 1 when $\kappa > 1$. Finally, we indeed have $p > 1$ as assumed.

Case 2: The second relevant case is $a > 1$, $\frac{a^2\kappa}{8} - a < -2$. In that case the sum over $z \in H(e^{-4s})$ gives us $(e^{-4s})^{\frac{a^2\kappa}{8}-a} = e^{(4a-\frac{a^2\kappa}{2})s}$, and we are left with

$$\sum_{s\in\mathbb{N}} \exp\big((-4p - a^2\kappa - a\kappa + 8a)s\big)\, s^{p-p\beta-1+\varepsilon}.$$

This sum converges when

$$-4p - a^2\kappa - a\kappa + 8a \leq 0$$

$$\text{and} \qquad p - p\beta - 1 < -1.$$

Fix $a \in]-\nu, (-2\nu) \vee 0[$. The minimal p to make the sum converge is then $p = -\frac{a^2\kappa}{4} - \frac{a\kappa}{4} + 2a$, giving us the optimal exponents

$$\alpha = \frac{1}{p}(-\frac{a^2\kappa}{8} - \frac{a\kappa}{8} + a) = \frac{1}{2}$$

$$\text{and any } \beta > 1$$

provided that we are in Case 2. Indeed, this holds when $\kappa < 1$, and we also have $p > 1$ as assumed.

Case 3: $\kappa = 1$, $a = 4$. In that case the sum over $z \in H(e^{-4s})$ gives us $(e^{-4s})^{-2}\log^*(e^{4s}) = e^{8s}s$, and we are left with

$$\sum_{s\in\mathbb{N}} \exp\left((12-4p)s\right) s^{p-p\beta+\varepsilon}.$$

The minimal p to make the sum converge is then $p = 3$, giving us the exponents $\alpha = \frac{1}{2}$ and any $\beta > \frac{4}{3}$.

Remark 3.3.10. *In case $\kappa \leq 1$ (i.e. Cases 2 and 3 above), we can get rid of the boundary effect by restricting to the time interval $[t_0, T]$. In that case, by Lemma 3.2.7, it suffices to consider points $z \in H(e^{-4s})$ with $\operatorname{Im} z \geq \varepsilon$. By Remark 3.2.6, the sum over such z is bounded by $\varepsilon^{a^2\kappa/8-a+2}e^{8s}$. From here, we can follow Case 1, with an additional factor of $\varepsilon^{a^2\kappa/8-a+2}$.*

By Lemma 3.2.7, the probability that this ε does not suffice is less than $\varepsilon^{4/\kappa-1}$. Since $\frac{a^2\kappa}{8} - a + 2 + \frac{4}{\kappa} - 1 > 0$, this proves the final assertion in Theorem 3.1.3, together with finite moments of the Hölder constant.

Theorem 3.3.11. *Let $\kappa \in \,]0,8[\, \cup \,]8,\infty[$. Let $\varphi(x) = x^\alpha((\log^*(\frac{1}{x}))^\beta$ with $\alpha = (1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}) \wedge \frac{1}{2}$ and any $\beta > 1$ in case $\kappa \neq 1$, and $\beta > \frac{4}{3}$ in case $\kappa = 1$. Then there exists $\tilde{p} > 1$ such that*

$$\mathbb{E}\left[\sup_{t_1,t_2\in[0,T]} \left(\frac{|\gamma(t_1)-\gamma(t_2)|}{\varphi(|t_1-t_2|)}\right)^{\tilde{p}}\right] < \infty.$$

In particular, there almost surely exists some $C < \infty$ such that

$$|\gamma(t_1)-\gamma(t_2)| \le C\varphi(|t_1-t_2|)$$

for all $t_1, t_2 \in [0,T]$.

Proof. We actually show

$$\mathbb{E}\left[\sup_{t_1,t_2\in[0,T]} \psi\left(\frac{|\gamma(t_1)-\gamma(t_2)|}{\varphi(|t_1-t_2|)}\right)\right] < \infty$$

with $\psi = \psi_{p,q}$ as above. Moreover, we can restrict to the event $\{\|\xi\|_{[0,T]} \le M\}$ since all our previous estimates depend on M polynomially (the dependence comes from Lemma 3.2.4), whereas the probability of $\mathbb{P}(\|\xi\|_{[0,T]} > M)$ decays exponentially in M.

So we are almost reduced to what we have estimated above.

Write $\delta = |t_1-t_2|^{1/2}$. By our assumption Δ_c on ψ,

$$\psi\left(\frac{|\gamma(t_1)-\gamma(t_2)|}{\varphi(|t_1-t_2|)}\right) \lesssim \psi\left(\frac{|\gamma(t_1)-\hat{f}_{t_1}(i\delta)|}{\varphi(|t_1-t_2|)}\right) + \psi\left(\frac{|\hat{f}_{t_1}(i\delta)-\hat{f}_{t_2}(i\delta)|}{\varphi(|t_1-t_2|)}\right) \\ + \psi\left(\frac{|\gamma(t_2)-\hat{f}_{t_2}(i\delta))|}{\varphi(|t_1-t_2|)}\right).$$

The first and the third term appear already in Lemma 3.3.8 which we have bounded by (3.16). Recall that the expression (3.16) does not depend on the choice of t_1, t_2, and that $\mathbb{E}[$ eq. (3.16) $] < \infty$.

So we are left to estimate the middle term. We show that it is bounded by (3.16) as well.

By (3.9), we have

$$|\hat{f}_{t_1}(i\delta)-\hat{f}_{t_2}(i\delta)| \le \Upsilon_{t_1}(\hat{f}_{t_1}(i\delta))) \left(1+\left(\frac{|\xi(t_1)-\xi(t_2)|^2}{|t_1-t_2|}\right)^l\right).$$

Pick $m \in \mathbb{N}$ with $2^{-m} \asymp_2 \delta$.

As in the proof of Lemma 3.3.3, we can find $s \in \mathbb{N}$ such that $\Upsilon_{t_1}(\hat{f}_{t_1}(i2^{-m})) \in [e^{-4s}, e^{-4(s-1)}]$, and as before, we can replace $\hat{f}_{t_1}(i2^{-m})$ by $w(t_1, m) \in H(e^{-4s}, \|\xi\|, T)$, and we have $Y_{t_1}(w(t_1, m)) \asymp 2^{-m}$ and $\Upsilon_{t_1}(w(t_1, m)) \asymp \Upsilon_{t_1}(\hat{f}_{t_1}(i2^{-m})) \asymp e^{-4s}$, and $(m, t_1) \in P(w, s)$.

Finally, since $|t_1 - t_2| = \delta^2 \asymp_4 2^{-2m}$, by Lemma 3.3.2

$$\left(\frac{|\xi(t_1) - \xi(t_2)|^2}{|t_1 - t_2|}\right)^l \le M_n(w, s)^{2l}.$$

Hence,

$$\begin{aligned}
&\psi\left(\frac{|\hat{f}_{t_1}(i\delta) - \hat{f}_{t_2}(i\delta)|}{\varphi(|t_1 - t_2|)}\right) \\
&\quad \le \psi\left(\Upsilon_{t_1}(\hat{f}_{t_1}(i\delta)))\left(1 + \left(\frac{|\xi(t_1) - \xi(t_2)|^2}{|t_1 - t_2|}\right)^l\right)\varphi(|t_1 - t_2|)^{-1}\right) \\
&\quad \lesssim \sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s},M,T)} \sum_{n\in\mathbb{N}_0} \psi(e^{-4s}(1 + M_n(z,s)^{2l})\varphi(Y_{\sigma(S_n)}^2)^{-1})\, 1_{S_n(z,s)<\infty} \\
&\quad \lesssim \sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s},M,T)} \sum_{n\in\mathbb{N}_0} e^{-4ps} s^{-q} \\
&\qquad\qquad \varphi(Y_{\sigma(S_n)}^2)^{-p}\left(\log^* \varphi(Y_{\sigma(S_n)}^2)^{-1}\right)^q (1 + M_n(z,s)^{2lp})\, 1_{S_n(z,s)<\infty} \\
&\quad \lesssim \sum_{s\in\mathbb{N}} \sum_{z\in H(e^{-4s},M,T)} \sum_{n\in\mathbb{N}_0} \\
&\qquad\qquad e^{-4ps} s^{-q} Y_{\sigma(S_n)}^{-2p\alpha}\left(\log^* Y_{\sigma(S_n)}^{-1}\right)^{-p\beta+q} (1 + M_n(z,s)^{2lp})\, 1_{S_n(z,s)<\infty}
\end{aligned}$$

where we have applied (3.3) in the last step. We see that this sum is also bounded by the expression (3.16) except for a factor $(1 + M_n(z, s)^{2lp})$. But when taking the expectation, that factor is irrelevant since it is independent of $\mathcal{F}_{\sigma(S_n)}$ and has exponential moments. $\square$

Chapter 4

Regularity of SLE in (t, κ) and refined GRR estimates

Friz, P.K., Tran, H. & Yuan, Y. Regularity of SLE in (t, κ) and refined GRR estimates. *Probab. Theory Relat. Fields* **180**, 71–112 (2021).

This is an accepted manuscript, available online:
https://doi.org/10.1007/s00440-021-01058-0.

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Abstract

Schramm-Loewner evolution (SLE_κ) is classically studied via Loewner evolution with half-plane capacity parametrization, driven by $\sqrt{\kappa}$ times Brownian motion. This yields a (half-plane) valued random field $\gamma = \gamma(t, \kappa; \omega)$. (Hölder) regularity of in $\gamma(\cdot, \kappa; \omega)$, a.k.a. SLE trace, has been considered by many authors, starting with Rohde-Schramm (2005). Subsequently, Johansson Viklund, Rohde, and Wong (2014) showed a.s. Hölder continuity of this random field for $\kappa < 8(2 - \sqrt{3})$. In this paper, we improve their result to joint Hölder continuity up to $\kappa < 8/3$. Moreover, we show that the SLE_κ trace $\gamma(\cdot, \kappa)$ (as a continuous path) is stochastically continuous in κ at all $\kappa \neq 8$. Our proofs rely on a novel variation of the Garsia-Rodemich-Rumsey (GRR) inequality, which is of independent interest.

4.1 Introduction

Schramm-Loewner evolution (SLE) is a random (non-self-crossing) path connecting two boundary points of a domain. To be more precise, it is a family of such random paths indexed by a parameter $\kappa \geq 0$. It has been first introduced by O. Schramm (2000) to describe several random models from statistical physics. Since then, many authors have intensely studied this random object. Many connections to discrete processes and other geometric objects have been made, and nowadays SLE is one of the key objects in modern probability theory.

The typical way of constructing SLE is via the Loewner differential equation (see Section 4.3) which provides a correspondence between real-valued functions ("driving functions") and certain growing families of sets ("hulls") in a planar domain. For many (in particular more regular) driving functions, the growing families of hulls (or their boundaries) are continuous curves called traces. For Brownian motion, it is a non-trivial fact that for fixed $\kappa \geq 0$, the driving function $\sqrt{\kappa}B$ almost surely generates a continuous trace which we call SLE$_\kappa$ trace (see [RS05; LSW04]).

There has been a series of papers investigating the analytic properties of SLE, such as (Hölder and p-variation) regularity of the trace [RS05; Law09; JL11; FT17]. See also [FS17; STW19] for some recent attempts to understand better the existence of SLE trace.

A natural question is whether the SLE$_\kappa$ trace obtained from this construction varies continuously in the parameter κ. Another natural question is whether with probability 1 the construction produces a continuous trace simultaneously for all $\kappa \geq 0$. These questions have been studied in [JRW14] where the authors showed that with probability 1, the SLE$_\kappa$ trace exists and is continuous in the range $\kappa \in [0, 8(2-\sqrt{3})[$. In our paper we improve their result and extend it to $\kappa \in [0, 8/3[$. (In fact, our result is a bit stronger than the following statement, see Theorem 4.3.2 and Theorem 4.4.1.)

Theorem 4.1.1. *Let B be a standard Brownian motion. Then almost surely the SLE$_\kappa$ trace γ^κ driven by $\sqrt{\kappa}B_t$, $t \in [0,1]$, exists for all $\kappa \in [0, 8/3[$, and the trace (parametrised by half-plane capacity) is continuous in $\kappa \in [0, 8/3[$ with respect to the supremum distance on $[0,1]$.*

Stability of SLE trace was also recently studied in [KS17, Theorem 1.10]. They show the law of $\gamma^{\kappa_n} \in C([0,1], \mathbb{H})$ converges weakly to the law of γ^κ in the topology of uniform convergence, whenever $\kappa_n \to \kappa < 8$. Of course, we get this as a trivial corollary of Theorem 4.1.1 in case of $\kappa < 8/3$. Our Theorem 4.1.2 (proved in Section 4.3.2) strengthens [KS17, Theorem 1.10] in three ways:
(i) we allow for any $\kappa \neq 8$;
(ii) we improve weak convergence to convergence in probability;
(iii) we strengthen convergence in $C([0,1], \mathbb{H})$ with uniform topology to $C^{p\text{-var}}([0,1], \mathbb{H})$ with optimal (cf. [FT17]) p-variation parameter, i.e. any $p > (1+\kappa/8) \wedge 2$. The analogous statement for α-Hölder topologies, $\alpha < \left(1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}\right) \wedge \frac{1}{2}$, is also true.

Here and below we write $\|f\|^p_{p\text{-var};[a,b]} := \sup \sum_{[s,t]\in\pi} |f(t)-f(s)|^p$, with sup taken over all partitions π of $[a,b]$. The following theorem will be proved as Corollary 4.3.12.

Theorem 4.1.2. *Let B be a standard Brownian motion, and γ^κ the SLE$_\kappa$ trace driven by $\sqrt{\kappa}B_t$, $t \in [0,1]$, (and parametrised by half-plane capacity). For any $\kappa > 0$, $\kappa \neq 8$ and any sequence $\kappa_n \to \kappa$ we then have $\|\gamma^\kappa - \gamma^{\kappa_n}\|_{p\text{-var};[0,1]} \to 0$ in probability, for any $p > (1+\kappa/8) \wedge 2$.*

There are two major new ingredients to our proofs. First, we prove in Section 4.5 a refined moment estimate for SLE increments in κ, improving upon [JRW14]. Using standard notation [RS05; Law05], for $\kappa > 0$, we denote by $(g^\kappa_t)_{t\geq 0}$ the forward SLE flow driven by $\sqrt{\kappa}B$, $j=1,2$, and by $\hat{f}_t^\kappa = (g_t^\kappa)^{-1}(\cdot + \sqrt{\kappa}B_t)$ the recentred inverse flow, also defined in Section 4.3 below.

Write $a \lesssim b$ for $a \le Cb$, with suitable constant $C < \infty$. The improved estimate (Proposition 4.3.5) reads

$$\mathbb{E}|\hat{f}_t^\kappa(i\delta) - \hat{f}_t^{\tilde{\kappa}}(i\delta)|^p \lesssim |\sqrt{\kappa} - \sqrt{\tilde{\kappa}}|^p \tag{4.1}$$

for $1 \le p < 1 + \frac{8}{\kappa}$. The interest in this estimate is when p is close to $1 + 8/\kappa$. No such estimate can be extracted from [JRW14], as we explain in some more detail in Remark 4.3.6 below.

Secondly, our way of exploiting moment estimates such as (4.1) is fundamentally different in comparison with the Whitney-type partition technique of "(t, y, κ)"-space [JRW14] (already seen in [RS05] without κ), combined with a Borel-Cantelli argument. Our key tool here is a new higher-dimensional variant of the Garsia-Rodemich-Rumsey (GRR) inequality [GRR71] which is useful in its own right, essentially whenever one deals with random fields with very "different" – in our case t and κ – variables. The GRR inequality has been a useful tool in stochastic analysis to pass from moment bounds for stochastic processes to almost sure estimates of their regularity.

Let us briefly discuss the existing (higher-dimensional) GRR estimates (e.g. [SV79, Exercise 2.4.1], [AI96; FKP06; HL13]) and their shortcomings in our setting. When we try to apply one of these versions to SLE (as a two-parameter random field in (t, κ)), we wish to estimate moments of $|\gamma(t,\kappa) - \gamma(s,\tilde{\kappa})|$, where we denote the SLE$_\kappa$ trace by $\gamma(\cdot, \kappa)$. In [FT17], the estimate

$$\mathbb{E}|\gamma(t,\kappa) - \gamma(s,\kappa)|^\lambda \lesssim |t-s|^{(\lambda+\zeta)/2}$$

with suitable $\lambda > 1$ and ζ has been given. We will show in Proposition 4.3.3 that

$$\mathbb{E}|\gamma(s,\kappa) - \gamma(s,\tilde{\kappa})|^p \lesssim |\kappa - \tilde{\kappa}|^p$$

for suitable $p > 1$. Applying this estimate with $p = \lambda$, we obtain an estimate for $\mathbb{E}|\gamma(t,\kappa) - \gamma(s,\tilde{\kappa})|^\lambda$, and can apply a GRR lemma from [AI96] or [FKP06]. The condition for applying it is $((\lambda+\zeta)/2)^{-1} + p^{-1} = ((\lambda+\zeta)/2)^{-1} + \lambda^{-1} < 1$. But in doing so, we do not use the best estimates available to us. That is, the above estimate typically holds for some $p > \lambda$. On the other hand, we can only estimate the λ-th moment (and no higher ones) of $|\gamma(t,\kappa) - \gamma(s,\kappa)|$. This asks for a version of the GRR lemma that respects distinct exponents in the available estimates, and is applicable when $((\lambda+\zeta)/2)^{-1} + p^{-1} < 1$ with $p > \lambda$ (a weaker condition than above).

We are going to prove the following refined GRR estimates in two dimensions, as required by our application, noting that extension to higher dimension follow the same argument.

Lemma 4.1.3. *Let G be a continuous function (defined on some rectangle) such that,* *for some integers* J_1, J_2,

$$\begin{aligned}|G(x_1,x_2)-G(y_1,y_2)| &\le |G(x_1,x_2)-G(y_1,x_2)| + |G(y_1,x_2)-G(y_1,y_2)|\\ &\le \sum_{j=1}^{J_1} |A_{1j}(x_1,y_1;x_2)| \quad + \sum_{j=1}^{J_1} |A_{2j}(y_1;x_2,y_2)|.\end{aligned}$$

Suppose that for all j,

$$\iiint \frac{|A_{1j}(u_1,v_1;u_2)|^{q_{1j}}}{|u_1-v_1|^{\beta_{1j}}}\, du_1\, dv_1\, du_2 < \infty,$$
$$\iiint \frac{|A_{2j}(v_1;u_2,v_2)|^{q_{2j}}}{|u_2-v_2|^{\beta_{2j}}}\, dv_1\, du_2\, dv_2 < \infty.$$

Then, under suitable conditions on the exponents,

$$|G(x_1,x_2)-G(y_1,y_2)| \lesssim |x_1-y_1|^{\gamma^{(1)}} + |x_2-y_2|^{\gamma^{(2)}}.$$

Observe that the exponents q_{1j}, q_{2j} are allowed to vary, exactly as required for our application to SLE. We also note that the flexibility to have $J_1, J_2 > 1$ is used in the proof of Theorem 4.1.2 but not 4.1.1.

One might ask whether one can further improve Theorem 4.1.1 to all $\kappa \ge 0$. With the methods of this paper, it would require a better moment estimate in the style of (4.1) with larger exponent on the right-hand side. If such an estimate were to hold true with arbitrarily large exponent on the right-hand side (and any suitable exponent on the left-hand side), which is not clear to us, almost sure continuity of the random field in all (t,κ) with $\kappa \ne 8$ would follow.

Acknowledgements: PKF and HT acknowledge funding from European Research Council through Consolidator Grant 683164. All authors would like to thank S. Rohde and A. Shekhar for stimulating discussions. Moreover, we thank the referees for their comments, in particular for pointing out the literature on metric entropy bounds and majorising measures, and for suggesting simplified arguments in the proofs of Lemma 4.2.1 and Theorem 4.2.8.

4.2 A Garsia-Rodemich-Rumsey lemma with mixed exponents

In this section we prove a variant of the Garsia-Rodemich-Rumsey inequality and Kolmogorov's continuity theorem. The classical Kolmogorov's theorem goes by a "chaining" argument (see e.g. [Kun90, Theorem 1.4.1] or [Tal14, Appendix A.2]), but can also be obtained from the GRR inequality (see e.g. [SV79, Corollary 2.1.5]). In the case of proving Hölder continuity of processes, the GRR approach provides more powerful statements (cf. [FV10, Appendix A]). In particular, we obtain bounds on the Hölder constant of the process that are more informative and easier to manipulate, which will be useful in the proof of Theorem 4.4.1. (Although there are drawbacks of the GRR approach when generalising to more refined modulus of continuity, see the discussion in [Tal14, Appendix A.4].)

We discuss some of the extensive literature that deal with the generality of GRR and Kolmogorov's theorem. The reader may skip this discussion and continue straight with the results of this section.

There are some direct generalisations of GRR and Kolmogorov's theorem to higher dimensions, e.g. [SV79, Exercise 2.4.1], [Kun90, Theorem 1.4.1], [AI96; FKP06; HL13]. Moreover, there have been more systematic studies in a general setting under the titles metric entropy bounds and majorising measures. They derive bounds and path continuity of stochastic processes mainly from the structure of certain pseudometrics that the processes induce on the parameter space, such as $d_X(s,t) := (\mathbb{E}|X(s)-X(t)|^2)^{1/2}$. A large amount of the theory is found in the book by Talagrand [Tal14]. These results due to, among others, R. M. Dudley, N. Kôno, X. Fernique, M. Talagrand, and W. Bednorz. Their main purpose is to allow different stuctures of the parameter space and inhomogeneity of the stochastic process (see e.g. [Kôn80; Bed07; Tal14]).

We explain why the existing results do not cover the adaption that we are seeking in this section. The general idea for applying the theory of metric entropy bounds would be considering the metric $d_X(s,t) = (\mathbb{E}|X(s)-X(t)|^q)^{1/q}$ for some $q > 1$.

Let us consider a random process defined on the parameter space $T = [0,1]^2$ that satisfies

$$\begin{aligned} \mathbb{E}|X(s_1,s_2) - X(t_1,s_2)|^{q_1} &\le |s_1 - t_1|^{\alpha_1}, \\ \mathbb{E}|X(t_1,s_2) - X(t_1,t_2)|^{q_2} &\le |s_2 - t_2|^{\alpha_2}, \end{aligned} \tag{4.2}$$

where q_1 and q_2 might be different, say $q_1 < q_2$. By Hölder's inequality,

$$\mathbb{E}|X(t_1,s_2) - X(t_1,t_2)|^{q_1} \le \left(\mathbb{E}|X(t_1,s_2) - X(t_1,t_2)|^{q_2}\right)^{q_1/q_2}. \tag{4.3}$$

Write $t = (t_1,t_2)$, $s = (s_1,s_2)$. We may let

$$(\mathbb{E}|X(s) - X(t)|^q)^{1/q} \le |s_1 - t_1|^{\alpha_1/q_1} + |s_2 - t_2|^{\alpha_2/q_2} =: \|\!|s - t|\!\| =: d(s,t)$$

where we can take $q = q_1$ (but not $q = q_2$ without knowing any bounds on higher moments of $|X(s_1,s_2) - X(t_1,s_2)|$).

We explain now that we have already lost some sharpness when we estimated (4.3) using Hölder's inequality. Indeed, all the results [Kôn80, Theorem 3], [Tal14, (13.141)], [Tal14, Theorem B.2.4], [Bed07, Corollary 1] are based on finding an increasing convex function φ such that

$$\mathbb{E}\varphi\left(\frac{|X(s) - X(t)|}{d(s,t)}\right) \le 1. \tag{4.4}$$

Observe that we can take $\varphi(x) = x^{q_1}$ at best. To apply any of these results, the condition turns out to be $\frac{1}{\alpha_1} + \frac{q_2}{q_1\alpha_2} < 1$. In fact, [Tal14, Theorem 13.5.8] implies that we cannot expect anything better just from the assumption (4.4). More precisely, the theorem states that in general, when we assume only (4.4), in order to deduce any pathwise bounds for the process X, we need to have

$$\int_0^\delta \varphi^{-1}\left(\frac{1}{\mu(B(t,\varepsilon))}\right) d\varepsilon < \infty,$$

with B denoting the ball with respect to the metric d, and μ e.g. the Lebesgue measure. In our setup this turns out to the condition $\frac{1}{\alpha_1} + \frac{q_2}{q_1\alpha_2} < 1$.

We will show in Theorem 4.2.8 that by using the condition (4.2) instead of (4.4), we can relax this condition to $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} < 1$. In case $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} < 1 < \frac{1}{\alpha_1} + \frac{q_2}{q_1\alpha_2}$, this is an improvement. We have not found this possibility in any of the existing references.

We now turn to our version of the Garsia-Rodemich-Rumsey inequality that allows us to make use of different exponents $q_1 \neq q_2$. In addition to the scenario (4.2), we allow also the situation when e.g. $|X(s_1,s_2) - X(t_1,s_2)| \leq A_{11} + A_{12}$ with $\mathbb{E}|A_{1j}|^{q_{1j}} \leq |s_1 - t_1|^{\alpha_{1j}}$ for some q_{1j}, α_{1j}, $j = 1,2$, where possibliy $q_{11} \neq q_{12}$.

Let (E,d) be a metric space. We can assume E to be isometrically embedded in some larger Banach space (by the Kuratowski embedding). To ease the notation, we write $|x-y| = d(x,y)$ both for the distance in E and for the distance in $\mathbb{R}$. For a Borel set A we denote by $|A|$ its Lebesgue measure and $⨍_A f = \frac{1}{|A|}\int_A f$.

In what follows, let I_1 and I_2 be two (either open or closed) non-trivial intervals of $\mathbb{R}$.

Lemma 4.2.1. *Let $G \in C(I_1 \times I_2)$ be a continuous function, with values in a metric space E, such that*

$$|G(x_1,x_2) - G(y_1,y_2)| \leq \sum_{j=1}^{J_1} |A_{1j}(x_1,y_1;x_2)| + \sum_{j=1}^{J_2} |A_{2j}(y_1;x_2,y_2)| \tag{4.5}$$

for all $(x_1,x_2),(y_1,y_2) \in I_1 \times I_2$, where $A_{1j} : I_1 \times I_1 \times I_2 \to \mathbb{R}$, $1 \leq j \leq J_1$, $A_{2j} : I_1 \times I_2 \times I_2 \to \mathbb{R}$, $1 \leq j \leq J_2$, are measurable functions. Suppose that

$$\iiint_{I_1\times I_1\times I_2} \frac{|A_{1j}(u_1,v_1;u_2)|^{q_{1j}}}{|u_1-v_1|^{\beta_{1j}}}\, du_1\, dv_1\, du_2 \leq M_{1j}, \tag{4.6}$$

$$\iiint_{I_1\times I_2\times I_2} \frac{|A_{2j}(v_1;u_2,v_2)|^{q_{2j}}}{|u_2-v_2|^{\beta_{2j}}}\, dv_1\, du_2\, dv_2 \leq M_{2j} \tag{4.7}$$

for all j, where $q_{ij} \geq 1$, $\beta_i := \min_j \beta_{ij} > 2$, $i = 1,2$, and $(\beta_1 - 2)(\beta_2 - 2) - 1 > 0$. Fix any $a, b > 0$. Then

$$\begin{aligned} |G(x_1,x_2) - G(y_1,y_2)| \leq C \sum_j M_{1j}^{1/q_{1j}} \left(|x_1-y_1|^{\gamma_{1j}^{(1)}} + |x_2-y_2|^{\gamma_{1j}^{(2)}}\right) \\ + C \sum_j M_{2j}^{1/q_{2j}} \left(|x_1-y_1|^{\gamma_{2j}^{(1)}} + |x_2-y_2|^{\gamma_{2j}^{(2)}}\right) \end{aligned} \tag{4.8}$$

for all $(x_1, x_2), (y_1, y_2) \in I_1 \times I_2$*, where* $\gamma_{1j}^{(1)} = \dfrac{\beta_{1j} - 2 - b}{q_{1j}}$, $\gamma_{1j}^{(2)} = \dfrac{(\beta_{1j} - 2)a - 1}{q_{1j}}$, $\gamma_{2j}^{(1)} = \dfrac{(\beta_{2j} - 2)b - 1}{q_{2j}}$, $\gamma_{2j}^{(2)} = \dfrac{\beta_{2j} - 2 - a}{q_{2j}}$*, and* $C < \infty$ *is a constant that depends on* $(q_{ij}), (\beta_{ij}), a, b, |I_1|, |I_2|$.

Remark 4.2.2. *The statement is already true when* $q_{ij} > 0$ *(not necessarily* ≥ 1*) and can be shown by an argument similarly as in [SV79, Theorem 2.1.3 and Exercise 2.4.1]. We have decided to stick to* $q_{ij} \geq 1$ *since the proof is simpler here.*

Proof. Note that for any continuous function G and a sequence B_n of sets with $\operatorname{diam}(\{x\} \cup B_n) \to 0$ we have $G(x) = \lim_n \fint_{B_n} G$. (Recall that we can view E as a subspace of some Banach space, so that the integral is well-defined.)

Let $(x_1, x_2), (y_1, y_2) \in I_1 \times I_2$. Using the above observation, we will approximate $G(x_1, x_2)$ and $G(y_1, y_2)$ by well-chosen sequences of sets.

We pick a sequence of rectangles $I_1^n \times I_2^n \subseteq I_1 \times I_2$, $n \geq 0$, with the following properties:

- $(x_1, x_2), (y_1, y_2) \in I_1^0 \times I_2^0$.
- $(x_1, x_2) \in I_1^n \times I_2^n$ for all n.
- $|I_i^n| = R_i^{-n} d_i$, $i = 1, 2$, with parameters
$$R_1, R_2 > 1, \quad d_1, d_2 > 0$$
chosen later.

In order for such a sequence of rectangles to exist, we must have
$$|x_i - y_i| \leq d_i \leq |I_i|, \quad i = 1, 2,$$
since we require $x_i, y_i \in I_i^0 \subseteq I_i$. Conversely, this condition guarantees the existence of such a sequence.

We will bound
$$\left| G(x_1, x_2) - \fint \fint_{I_1^0 \times I_2^0} G \right| \leq \sum_{n \in \mathbb{N}} \left| \fint \fint_{I_1^n \times I_2^n} G - \fint \fint_{I_1^{n-1} \times I_2^{n-1}} G \right|.$$

The same argument applies also to $G(y_1, y_2)$ where we can pick the same initial rectangle $I_1^0 \times I_2^0$. Hence, this will give us a bound on $|G(x_1, x_2) - G(y_1, y_2)|$.

By the assumption (4.5) we have
$$\begin{aligned}
&\left| \fint \fint_{I_1^n \times I_2^n} G - \fint \fint_{I_1^{n-1} \times I_2^{n-1}} G \right| \\
&= \left| \fint \fint_{I_1^n \times I_2^n} \fint \fint_{I_1^{n-1} \times I_2^{n-1}} (G(u_1, u_2) - G(v_1, v_2)) \, du_1 \, du_2 \, dv_1 \, dv_2 \right| \\
&\leq \sum_j \fint_{I_1^n} \fint_{I_1^{n-1}} \fint_{I_2^n} |A_{1j}(u_1, v_1; u_2)| + \sum_j \fint_{I_1^{n-1}} \fint_{I_2^n} \fint_{I_2^{n-1}} |A_{2j}(v_1; u_2, v_2)|.
\end{aligned}$$

Recall that $|I_i^n| = R_i^{-n} d_i$ and that $|u_i - v_i| \leq C R_i^{-n} d_i$ for any $u_i \in I_i^n$, $v_i \in I_i^{n-1}$. This and Hölder's inequality imply

$$\begin{aligned}
&\fint_{I_1^n} \fint_{I_1^{n-1}} \fint_{I_2^n} |A_{1j}(u_1, v_1; u_2)| \\
&\leq C (R_1^{-n} d_1)^{\beta_{1j}/q_{1j}} \fint_{I_1^n} \fint_{I_1^{n-1}} \fint_{I_2^n} \frac{|A_{1j}(u_1, v_1; u_2)|}{|u_1 - v_1|^{\beta_{1j}/q_{1j}}} \\
&\leq C (R_1^{-n} d_1)^{\beta_{1j}/q_{1j}} \left(\fint_{I_1^n} \fint_{I_1^{n-1}} \fint_{I_2^n} \frac{|A_{1j}(u_1, v_1; u_2)|^{q_{1j}}}{|u_1 - v_1|^{\beta_{1j}}} \right)^{1/q_{1j}} \\
&\leq C (R_1^{-n} d_1)^{\beta_{1j}/q_{1j}} \left((R_1^{-n} d_1)^{-2} (R_2^{-n} d_2)^{-1} M_{1j} \right)^{1/q_{1j}} \\
&= C \left((R_1^{-n} d_1)^{\beta_{1j}-2} (R_2^{-n} d_2)^{-1} M_{1j} \right)^{1/q_{1j}} .
\end{aligned}$$

Similarly,

$$\fint_{I_1^{n-1}} \fint_{I_2^n} \fint_{I_2^{n-1}} |A_{2j}(v_1; u_2, v_2)| \leq C \left((R_2^{-n} d_2)^{\beta_{2j}-2} (R_1^{-n} d_1)^{-1} M_{2j} \right)^{1/q_{2j}} .$$

We want to sum the above expressions for all n, which is possible if and only if both $R_1^{\beta_{1j}-2} R_2^{-1} > 1$ and $R_2^{\beta_{2j}-2} R_1^{-1} > 1$. The best pick is $R_2 = R_1^{\frac{\beta_1 - 1}{\beta_2 - 1}}$ (the exact scale of R_1 does not matter), and the condition becomes $(\beta_1 - 2)(\beta_2 - 2) - 1 > 0$ (assuming $\beta_1, \beta_2 > 2$). In that case, we finally get

$$\begin{aligned}
&|G(x_1, x_2) - G(y_1, y_2)| \\
&\qquad \leq C \sum_j \left(d_1^{\beta_{1j}-2} d_2^{-1} M_{1j} \right)^{1/q_{1j}} + C \sum_j \left(d_2^{\beta_{2j}-2} d_1^{-1} M_{2j} \right)^{1/q_{2j}}
\end{aligned} \tag{4.9}$$

It remains to pick $d_1, d_2 > 0$. Let $d_1 := |x_1 - y_1| \vee |x_2 - y_2|^a$, $d_2 := |x_1 - y_1|^b \vee |x_2 - y_2|$, and suppose for the moment that $d_1 \leq |I_1|$, $d_2 \leq |I_2|$. (The conditions $d_1 \geq |x_1 - y_1|$, $d_2 \geq |x_2 - y_2|$ are satisfied by our choice.). In this case the inequality (4.9) becomes

$$\begin{aligned}
&|G(x_1, x_2) - G(y_1, y_2)| \\
&\leq C \sum_j M_{1j}^{1/q_{1j}} \left(|x_1 - y_1|^{\beta_{1j}-2-b} + |x_2 - y_2|^{(\beta_{1j}-2)a-1} \right)^{1/q_{1j}} \\
&\quad + C \sum_j M_{2j}^{1/q_{2j}} \left(|x_1 - y_1|^{(\beta_{2j}-2)b-1} + |x_2 - y_2|^{\beta_{2j}-2-a} \right)^{1/q_{2j}} .
\end{aligned} \tag{4.10}$$

This proves the claim in case $d_1 \leq |I_1|$, $d_2 \leq |I_2|$.

It remains to handle the case when $d_1 > |I_1|$ or $d_2 > |I_2|$. In that case we pick $\hat{d}_1 = d_1 \wedge |I_1|$ and $\hat{d}_2 = d_2 \wedge |I_2|$ instead of d_1 and d_2. The conditions $|x_1 - y_1| \leq \hat{d}_1 \leq |I_1|$

and $|x_2 - y_2| \le \hat{d}_2 \le |I_2|$ are now satisfied, and in (4.9), we instead have

$$\begin{aligned}
\hat{d}_1^{\beta_{1j}-2}\hat{d}_2^{-1} &\le \frac{d_2}{d_2 \wedge |I_2|}\, d_1^{\beta_{1j}-2}d_2^{-1} = \left(\frac{|x_1-y_1|^b}{|I_2|} \vee 1\right) d_1^{\beta_{1j}-2}d_2^{-1}, \\
\hat{d}_1^{-1}\hat{d}_2^{\beta_{2j}-2} &\le \frac{d_1}{d_1 \wedge |I_1|}\, d_1^{-1}d_2^{\beta_{2j}-2} = \left(\frac{|x_2-y_2|^a}{|I_1|} \vee 1\right) d_1^{-1}d_2^{\beta_{2j}-2},
\end{aligned} \tag{4.11}$$

i.e. the same result (4.10) holds with the additional constants $\left(\frac{|x_1-y_1|^b}{|I_2|} \vee 1\right)$ and $\left(\frac{|x_2-y_2|^a}{|I_1|} \vee 1\right)$ (which can be bounded by a constant depending on $a, b, |I_1|, |I_2|$ since $a, b \ge 0$). $\square$

Remark 4.2.3. *The dependence of the multiplicative constant C on $|I_1|$ and $|I_2|$ is specified in (4.11). This can be convenient when we want to apply the lemma to different domains.*

A more accurate version is

$$\begin{aligned}
\hat{d}_1^{\beta_{1j}-2}\hat{d}_2^{-1} &= \left(\frac{d_1 \wedge |I_1|}{d_1}\right)^{\beta_{1j}-2} \frac{d_2}{d_2 \wedge |I_2|}\, d_1^{\beta_{1j}-2}d_2^{-1} \\
&= \left(\frac{|I_1|}{|x_2-y_2|^a} \wedge 1\right)^{\beta_{1j}-2} \left(\frac{|x_1-y_1|^b}{|I_2|} \vee 1\right) d_1^{\beta_{1j}-2}d_2^{-1}, \\
\hat{d}_1^{-1}\hat{d}_2^{\beta_{2j}-2} &= \left(\frac{d_2 \wedge |I_2|}{d_2}\right)^{\beta_{2j}-2} \frac{d_1}{d_1 \wedge |I_1|}\, d_1^{-1}d_2^{\beta_{2j}-2} \\
&= \left(\frac{|I_2|}{|x_1-y_1|^b} \wedge 1\right)^{\beta_{2j}-2} \left(\frac{|x_2-y_2|^a}{|I_1|} \vee 1\right) d_1^{-1}d_2^{\beta_{2j}-2}.
\end{aligned}$$

Remark 4.2.4. *We could have added some more flexibility by allowing the exponents $(q_{ij}), (\beta_{ij})$ to vary with u_1, u_2, but again we will not need it for our result.*

Remark 4.2.5. *We have a free choice of $a, b \ge 0$ which affects the Hölder exponents $\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}$. In general, it is not simple to spell out the optimal choice of a, b and hence the optimal Hölder exponents. Usually we are interested in the overall exponents (i.e. $\min_{i,j} \gamma_{ij}^{(1)}$, $\min_{i,j} \gamma_{ij}^{(2)}$), and we can solve*

$$\begin{aligned}
\min_j \gamma_{1j}^{(1)} &= \min_j \gamma_{2j}^{(1)}, \\
\min_j \gamma_{1j}^{(2)} &= \min_j \gamma_{2j}^{(2)}
\end{aligned}$$

to find the optimal choice for a, b.

For instance, in case $\beta_{1j} = \beta_1$ and $\beta_{2j} = \beta_2$ for all j, the best choice is

$$a = \frac{q_1(\beta_2 - 2) + q_2}{q_2(\beta_1 - 2) + q_1}, \qquad b = \frac{q_2(\beta_1 - 2) + q_1}{q_1(\beta_2 - 2) + q_2},$$

resulting in

$$\gamma^{(1)} = \frac{(\beta_1 - 2)(\beta_2 - 2) - 1}{q_1(\beta_2 - 2) + q_2}, \quad \gamma^{(2)} = \frac{(\beta_1 - 2)(\beta_2 - 2) - 1}{q_2(\beta_1 - 2) + q_1}$$

where $q_i = \max_j q_{ij}$.

In general, we could choose $a = \frac{\beta_2 - 1}{\beta_1 - 1}$, $b = \frac{\beta_1 - 1}{\beta_2 - 1}$, *resulting in*

$$\gamma_{1j}^{(1)} = \frac{(\beta_{1j} - 2)(\beta_2 - 2) - 1 + \beta_{1j} - \beta_1}{q_{1j}(\beta_2 - 1)}, \quad \gamma_{1j}^{(2)} = \frac{(\beta_{1j} - 2)(\beta_2 - 2) - 1 + \beta_{1j} - \beta_1}{q_{1j}(\beta_1 - 1)},$$
$$\gamma_{2j}^{(1)} = \frac{(\beta_1 - 2)(\beta_{2j} - 2) - 1 + \beta_{2j} - \beta_2}{q_{2j}(\beta_2 - 1)}, \quad \gamma_{2j}^{(2)} = \frac{(\beta_1 - 2)(\beta_{2j} - 2) - 1 + \beta_{2j} - \beta_2}{q_{2j}(\beta_1 - 1)}.$$

But this is not necessarily the optimal choice.

Remark 4.2.6. *Notice that the condition to apply the lemma does only depend on* (β_{ij}), *not* (q_{ij}), *but the resulting Hölder-exponents will.*

Remark 4.2.7. *The proof straightforwardly generalises to higher dimensions.*

Using our version of the GRR lemma, we can show another version of the Kolmogorov continuity condition. Here we suppose I_1, I_2 are **bounded** intervals.

Theorem 4.2.8. *Let* X *be a random field on* $I_1 \times I_2$ *taking values in a separable Banach space. Suppose that, for* $(x_1, x_2), (y_1, y_2) \in I_1 \times I_2$, *we have*

$$|X(x_1, x_2) - X(y_1, y_2)| \le \sum_{j=1}^{J_1} |A_{1j}(x_1, y_1; x_2)| + \sum_{j=1}^{J_2} |A_{2j}(y_1; x_2, y_2)| \tag{4.12}$$

with measurable real-valued A_{ij} *that satisfy*

$$\begin{aligned} \mathbb{E}|A_{1j}(x_1, y_1; x_2)|^{q_{1j}} &\le C' \, |x_1 - y_1|^{\alpha_{1j}}, \\ \mathbb{E}|A_{2j}(y_1; x_2, y_2)|^{q_{2j}} &\le C' \, |x_2 - y_2|^{\alpha_{2j}} \end{aligned} \tag{4.13}$$

with a constant $C' < \infty$.

Moreover, suppose $q_{ij} \ge 1$, $\alpha_i = \min_j \alpha_{ij} > 1$, $i = 1, 2$, *and* $\alpha_1^{-1} + \alpha_2^{-1} < 1$.

Then X *has a Hölder-continuous modification* $\hat{X}$. *Moreover, for any*

$$\gamma^{(1)} < \frac{(\alpha_1 - 1)(\alpha_2 - 1) - 1}{q_1(\alpha_2 - 1) + q_2}, \quad \gamma^{(2)} < \frac{(\alpha_1 - 1)(\alpha_2 - 1) - 1}{q_2(\alpha_1 - 1) + q_1},$$

where $q_i = \max_j q_{ij}$, *there is a random variable* C *such that*

$$|\hat{X}(x_1, x_2) - \hat{X}(y_1, y_2)| \le C \left(|x_1 - y_1|^{\gamma^{(1)}} + |x_2 - y_2|^{\gamma^{(2)}} \right)$$

and $\mathbb{E}[C^{q_{min}}] < \infty$ *for* $q_{min} = \min_{i,j} q_{ij}$.

Remark 4.2.9. *In case $\alpha_{1j} = \alpha_1$ and $\alpha_{2j} = \alpha_2$ for all j, the expressions for the Hölder exponents $\gamma^{(1)}, \gamma^{(2)}$ given above are sharp. In the general case, the exponents may be improved, following an optimisation described in Remark 4.2.5.*

Remark 4.2.10. *The constants C' can be replaced by (deterministic) functions that are integrable in (x_1, x_2), without change of the proof. But one would need to formulate the condition more carefully, therefore we decided to not include it.*

We point out that in case $J_1 = J_2 = 1$ and $q_1 = q_2$, this agrees with the two-dimensional version of the (inhomogeneous) Kolmogorov criterion [Kun90, Theorem 1.4.1].

Proof. **Part 1.** Suppose first that X is already continuous. In that case we can directly apply Lemma 4.2.1. The expectation of the integrals (4.6) and (4.7) are finite if $\beta_{ij} < \alpha_{ij}+1$ for all i, j. By choosing β_{ij} as large as possible, the conditions $(\beta_1-2)(\beta_2-2)-1 > 0$ and $\beta_1 > 2$, $\beta_2 > 2$ are satisfied if $\alpha_1^{-1} + \alpha_2^{-1} < 1$ and $\alpha_1 > 1$, $\alpha_2 > 1$.

Since the (random) constants M_{ij} in Lemma 4.2.1 are almost surely finite, X is Hölder continuous as quantified in (4.8), and the Hölder constants $M_{ij}^{1/q_{ij}}$ have q_{ij}-th moments since they are just the integrals (4.6). The formulas for the Hölder exponents follow from the analysis in Remark 4.2.5.

Part 2. Now, suppose X is arbitrary. We need to construct a continuous version of X. It suffices to show that X is uniformly continuous on a dense set $D \subseteq I_1 \times I_2$. Indeed, we can then apply Doob's separability theorem to obtain a separable (and hence continuous) version of X, or alternatively construct $\hat{X}$ by setting $\hat{X} = X$ on D and extend $\hat{X}$ continuously to $I_1 \times I_2$. Then $\hat{X}$ is a modification of X because they agree on a dense set D and are both stochastically continuous (as follows from (4.12) and (4.13)).

We use a standard argument that can be found e.g. in [Tal90, p. 8–9].

We can assume without loss of generality that $X(\bar{x}_1, \bar{x}_2) = 0$ for some $(\bar{x}_1, \bar{x}_2) \in I_1 \times I_2$ (otherwise just consider $Y(x_1, x_2) = X(x_1, x_2) - X(\bar{x}_1, \bar{x}_2)$).

In particular, the conditions (4.12) and (4.13) imply that $X(x_1, x_2)$ is an integrable random variable with values in a separable Banach space for every (x_1, x_2).

Fix any countable dense subset $D \subseteq I_1 \times I_2$. Let

$$\mathcal{G} := \sigma(\{X(x_1, x_2) \mid (x_1, x_2) \in D\}).$$

We can pick an increasing sequence of **finite** σ-algebras $\mathcal{G}_n$ such that $\mathcal{G} = \sigma\left(\bigcup_n \mathcal{G}_n\right)$. By martingale convergence, we have

$$X^{(n)}(x_1, x_2) \to X(x_1, x_2)$$

almost surely for $(x_1, x_2) \in D$ where $X^{(n)}(x_1, x_2) := \mathbb{E}[X(x_1, x_2) \mid \mathcal{G}_n]$.

Moreover, (4.12) implies

$$|X^{(n)}(x_1, x_2) - X^{(n)}(y_1, y_2)| \le \sum_{j=1}^{J_1} |A_{1j}^{(n)}(x_1, y_1; x_2)| + \sum_{j=1}^{J_2} |A_{2j}^{(n)}(y_1; x_2, y_2)|$$

where $|A_{ij}^{(n)}(...)| := \mathbb{E}[|A_{ij}^{(n)}(...)| \mid \mathcal{G}_n]$. By Jensen's inequality and (4.13), we have

$$\mathbb{E}|A_{1j}^{(n)}(x_1, y_1; x_2)|^{q_{1j}} \le \mathbb{E}|A_{1j}(x_1, y_1; x_2)|^{q_{1j}} \le C' \, |x_1 - y_1|^{\alpha_{1j}},$$
$$\mathbb{E}|A_{2j}^{(n)}(y_1; x_2, y_2)|^{q_{2j}} \le \mathbb{E}|A_{2j}(y_1; x_2, y_2)|^{q_{2j}} \le C' \, |x_2 - y_2|^{\alpha_{2j}}.$$

In particular, $X^{(n)}$ is stochastically continuous, and since $\mathcal{G}_n$ is finite, $X^{(n)}$ is almost surely continuous. Applying Lemma 4.2.1 yields

$$\begin{aligned} |X^{(n)}(x_1, x_2) - X^{(n)}(y_1, y_2)| \le C \sum_j (M_{1j}^{(n)})^{1/q_{1j}} \left(|x_1 - y_1|^{\gamma_{1j}^{(1)}} + |x_2 - y_2|^{\gamma_{1j}^{(2)}} \right) \\ + C \sum_j (M_{2j}^{(n)})^{1/q_{2j}} \left(|x_1 - y_1|^{\gamma_{2j}^{(1)}} + |x_2 - y_2|^{\gamma_{2j}^{(2)}} \right) \end{aligned}$$

where $M_{ij}^{(n)}$ are defined as the integrals (4.6) and (4.7) with $A_{ij}^{(n)}$.

It follows that on D we have

$$\begin{aligned} |X(x_1, x_2) - X(y_1, y_2)| \le C \sum_j \tilde{M}_{1j}^{1/q_{1j}} \left(|x_1 - y_1|^{\gamma_{1j}^{(1)}} + |x_2 - y_2|^{\gamma_{1j}^{(2)}} \right) \\ + C \sum_j \tilde{M}_{2j}^{1/q_{2j}} \left(|x_1 - y_1|^{\gamma_{2j}^{(1)}} + |x_2 - y_2|^{\gamma_{2j}^{(2)}} \right) \end{aligned}$$

where $\tilde{M}_{ij} := \liminf_n M_{ij}^{(n)}$. By Fatou's lemma,

$$\mathbb{E}\tilde{M}_{ij} \le \liminf_n \mathbb{E} M_{ij}^{(n)} < \infty,$$

implying that $\tilde{M}_{ij} < \infty$, hence X is uniformly continuous on D. □

One-dimensional variants of Lemma 4.2.1 and Theorem 4.2.8 can also be derived. Having shown the two-dimensional results Lemma 4.2.1 and Theorem 4.2.8, there is no need for an additional proof of their one-dimensional variants, since we can extend any one-parameter function G to a two-parameter function via $\tilde{G}(x_1, x_2) := G(x_1)$. This immediately implies the following results.

Corollary 4.2.11. *Let G be a continuous function on an interval I such that*

$$|G(x) - G(y)| \le \sum_{j=1}^{J} |A_j(x, y)|$$

for all $x, y \in I$, where $A_j : I \times I \to \mathbb{R}$, $j = 1, ..., J$, are measurable functions that satisfy

$$\iint_{I \times I} \frac{|A_j(u, v)|^{q_j}}{|u - v|^{\beta_j}} \, du \, dv \le M_j$$

with some $q_j \geq 1$, $\beta_j > 2$. *Then*

$$|G(x) - G(y)| \leq C \sum_j M_j^{1/q_j} |x-y|^{\gamma_j}$$

for all $x, y \in I$, *where* $\gamma_j = \frac{\beta_j - 2}{q_j}$, *and* $C < \infty$ *is a constant that depends on* $(q_j), (\beta_j)$.

For the sake of completeness we also state the one-dimensional version of Theorem 4.2.8.

Corollary 4.2.12. *Let* X *be a stochastic process on a bounded interval* I *such that*

$$|X(x) - X(y)| \leq \sum_{j=1}^{J} |A_j(x,y)|$$

for all $x, y \in I$, *where* A_j, $j = 1, ..., J$, *are measurable and satisfy*

$$\mathbb{E}|A_j(x,y)|^{q_j} \leq C' |x-y|^{\alpha_j}$$

with $q_j \geq 1$, $\alpha_j > 1$, *and* $C' < \infty$.

Then X *has a continuous modification* $\hat{X}$ *that satisfies, for any* $\gamma < \min_j \frac{\alpha_j - 1}{q_j}$,

$$|\hat{X}(x) - \hat{X}(y)| \leq C_\gamma |x-y|^\gamma$$

with a random variable C_γ *with* $\mathbb{E}[C_\gamma^{q_{min}}] < \infty$ *where* $q_{min} = \min_j q_j$.

4.2.1 Further variations on the GRR theme

We give some additional results that are similar or come as consequence of Lemma 4.2.1. This demonstrates the flexibility and generality that our lemma provides. We do not aim for a complete survey of all implications of the lemma.

We begin by proving the result of Lemma 4.2.1 under slightly weaker assumptions. The assumptions may seem a bit at random, but they will turn out to be what we need in the proof of Theorem 4.4.1.

Lemma 4.2.13. *Consider the same conditions as in Lemma 4.2.1, but instead of* (4.5), *we assume the following weaker condition. Let* $r_j > 1$ *and* $\theta_j > 0$ *such that* $\frac{\beta_{1j} - 2}{q_{1j}} < \theta_j$ *for* $j = 1, ..., J_1$.[1] *Suppose that for some small* $c > 0$, *e.g.* $c \leq |I_1|/4$, *we have*

$$\begin{aligned} &|G(x_1,x_2) - G(y_1,y_2)| \\ &\quad \leq \sum_{j=1}^{J_1} \sum_{k=0}^{\lfloor \log_{r_j}(c/|x_1-y_1|) \rfloor} r_j^{-k\theta_j} |A_{1j}(z_1 + r_j^k(x_1 - z_1), z_1 + r_j^k(y_1 - z_1); x_2)| \\ &\quad\quad + \sum_{j=1}^{J_2} |A_{2j}(y_1; x_2, y_2)| \end{aligned} \tag{4.14}$$

[1] A slightly different result still holds if $\frac{\beta_{1j}-2}{q_{1j}} \geq \theta_j$, as one can see in the proof.

for $(x_1, x_2), (y_1, y_2) \in I_1 \times I_2$ *and* $z_1 \in I_1$ *whenever* $|x_1 - z_1| \vee |y_1 - z_1| \leq 2|x_1 - y_1|$ *and all the points appearing in the sum are also in the domain* I_1.

Then the result of Lemma 4.2.1 still holds, with the constant C *depending also on* $(r_j), (\theta_j)$.

Proof. We proceed similarly as in the proof of Lemma 4.2.1. We pick the sequence I_i^n a bit more carefully. Let $d_i > 0$, $R_i > 1$, $i = 1, 2$, be as in the proof of Lemma 4.2.1, and recall that we can freely pick $R_i \geq 9$. It is not hard to see that we can then pick a sequence of rectangles $I_1^n \times I_2^n$ in such a way that

- $|I_i^n| = \frac{1}{9} R_i^{-n} d_i$,
- $\frac{1}{9} R_i^{-n} d_i \leq \operatorname{dist}(I_i^n, I_i^{n+1}) \leq R_i^{-n} d_i$,
- $\operatorname{dist}(x_i, I_i^n) \to 0$ as $n \to \infty$,

and another analogous sequence of rectangles for (y_1, y_2) that begins with the same $I_1^0 \times I_2^0$.

The proof proceeds in the same way, but instead of the assumption (4.5), we apply (4.14) with some z_1 that we pick now.

Let $n \in \mathbb{N}$. We pick $z_1 := \inf(I_1^n \cup I_1^{n-1})$ if this point is in the left half of I_1, and $z_1 = \sup(I_1^n \cup I_1^{n-1})$ otherwise. From the defining properties of the sequence (I_1^n) it follows that $|u_1 - z_1| \vee |v_1 - z_1| \leq 2|u_1 - v_1|$ for all $u_1 \in I_1^n$, $v_1 \in I_1^{n-1}$. Moreover, all the points $z_1 + r^k(u_1 - z_1)$ and $z_1 + r^k(v_1 - z_1)$, $k \leq \lfloor \log_r(c/|x_1 - y_1|) \rfloor$, are inside I_1 because $|r^k(u_1 - z_1)| \leq \frac{c}{|u_1 - v_1|}|u_1 - z_1| \leq 2c$ and we have chosen z_1 to be more than distance $|I_1|/2 \geq 2c$ away (in the u_1 resp. v_1 direction) from the end of the interval I_1.

We now have to bound

$$\sum_k \fint_{I_1^n} \fint_{I_1^{n-1}} \fint_{I_2^n} r^{-k\theta_j} |A_{1j}(z_1 + r^k(u_1 - z_1), z_1 + r^k(v_1 - z_1); u_2)| \, du_2 \, dv_1 \, du_1$$

With the transformation $\phi_k(u_1) = z_1 + r^k(u_1 - z_1)$ we get

$$\begin{aligned}
&\fint_{I_1^n} \fint_{I_1^{n-1}} \fint_{I_2^n} r^{-k\theta_j} |A_{1j}(z_1 + r^k(u_1 - z_1), z_1 + r^k(v_1 - z_1); u_2)| \\
&= r^{-k\theta_j} \fint_{\phi_k(I_1^n)} \fint_{\phi_k(I_1^{n-1})} \fint_{I_2^n} |A_{1j}(u_1, v_1; u_2)| \\
&\leq C r^{-k\theta_j} (r^k R_1^{-n} d_1)^{\beta_{1j}/q_{1j}} \fint_{\phi_k(I_1^n)} \fint_{\phi_k(I_1^{n-1})} \fint_{I_2^n} \frac{|A_{1j}(u_1, v_1; u_2)|}{|u_1 - v_1|^{\beta_{1j}/q_{1j}}} \\
&\leq C r^{-k\theta_j} (r^k R_1^{-n} d_1)^{\beta_{1j}/q_{1j}} \left(\fint_{\phi_k(I_1^n)} \fint_{\phi_k(I_1^{n-1})} \fint_{I_2^n} \frac{|A_{1j}(u_1, v_1; u_2)|^{q_{1j}}}{|u_1 - v_1|^{\beta_{1j}}} \right)^{1/q_{1j}} \\
&\leq C r^{-k\theta_j} (r^k R_1^{-n} d_1)^{\beta_{1j}/q_{1j}} \left((r^k R_1^{-n} d_1)^{-2} (R_2^{-n} d_2)^{-1} M_{1j} \right)^{1/q_{1j}} \\
&= C r^{k((\beta_{1j}-2)/q_{1j} - \theta_j)} \left((R_1^{-n} d_1)^{\beta_{1j}-2} (R_2^{-n} d_2)^{-1} M_{1j} \right)^{1/q_{1j}}.
\end{aligned}$$

Since we assumed $\frac{\beta_{1j}-2}{q_{1j}} < \theta_j$ this bound sums in k to

$$C\left((R_1^{-n}d_1)^{\beta_{1j}-2}(R_2^{-n}d_2)^{-1}M_{1j}\right)^{1/q_{1j}}$$

which is the same bound as in the proof of Lemma 4.2.1. The rest of the proof is the same as in Lemma 4.2.1. □

The following corollary is only used for Theorem 4.3.8.

Corollary 4.2.14. *Consider the same conditions as in Lemma 4.2.1. For $x_1 \in I_1$, consider $G(x_1, \cdot)$ as an element in the space of continuous functions $C^0(I_2)$. Then the p-variation of $x_1 \mapsto G(x_1, \cdot)$ is at most*

$$C\sum_j M_{1j}^{1/q_{1j}}|I_1|^{\gamma_{1j}^{(1)}} + C\sum_j M_{2j}^{1/q_{2j}}|I_1|^{\gamma_{2j}^{(1)}},$$

where $p = \max_{i,j} \frac{q_{ij}}{1+\gamma_{ij}^{(1)}q_{ij}} = \max_j \frac{q_{1j}}{\beta_{1j}-1-b} \vee \max_j \frac{q_{2j}}{(\beta_{2j}-2)b}$ (with a choice of $b \geq 0$), and C does not depend on $|I_1|$.

Proof. Let $t^0 < t^1 < ... < t^n$ be a partition of I_1. The p-variation of $x_1 \mapsto G(x_1, \cdot) \in C^0(I_2)$ is

$$\sup_{\text{partitions of } I_1}\left(\sum_k \sup_{x_2\in I_2}|G(t^k, x_2) - G(t^{k-1}, x_2)|^p\right)^{1/p}.$$

We estimate the differences using Lemma 4.2.1, applied to $[t^{k-1}, t^k] \times I_2$. Observe that since consider the difference only in the first parameter of G, the constant C in the statement of Lemma 4.2.1 does not depend on the size of $[t^{k-1}, t^k]$, as we explained in Remark 4.2.3. Hence we have

$$\begin{aligned}|G(t^k, x_2) - G(t^{k-1}, x_2)| \leq C\sum_j \left(M_{1j}\big|_{[t^{k-1},t^k]}\right)^{1/q_{1j}} |t^k - t^{k-1}|^{\gamma_{1j}^{(1)}} \\ + C\sum_j \left(M_{2j}\big|_{[t^{k-1},t^k]}\right)^{1/q_{2j}} |t^k - t^{k-1}|^{\gamma_{2j}^{(1)}}\end{aligned}$$

for all $x_2 \in I_2$, where we denote by $M_{1j}\big|_{[s,t]}$ and $M_{2j}\big|_{[s,t]}$ the integrals in (4.6) and (4.7) restricted to $[s, t] \times [s, t] \times I_2$ and $[s, t] \times I_2 \times I_2$, respectively.

Similary to [FV10, Corollary A.3], we can show that

$$\omega(s, t) = C^p\sum_j \left(M_{1j}\big|_{[s,t]}\right)^{p/q_{1j}} |s - t|^{p\gamma_{1j}^{(1)}} + C^p\sum_j \left(M_{2j}\big|_{[s,t]}\right)^{p/q_{2j}} |s - t|^{p\gamma_{2j}^{(1)}}$$

is a control. □

4.3 Continuity of SLE in κ and t

In this section we show the main results Theorems 4.1.1 and 4.1.2. We adopt notations and prerequisite from [JRW14]. For the convenience of the reader, we quickly recall some important notations.

Let $U\colon [0,1] \to \mathbb{R}$ be continuous. The Loewner differential equation is the following initial value ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z \in \mathbb{H}. \tag{4.15}$$

For each $z \in \mathbb{H}$, the ODE has a unique solution up to a time $T_z = \sup\{t > 0 : |g_t(z) - U(t)| > 0\} \in (0, \infty]$. For $t \geq 0$, let $H_t = \{z \in \mathbb{H} : T_z > t\}$. It is known that g_t is a conformal map from H_t onto $\mathbb{H}$. Define $f_t = g_t^{-1}$ and $\hat{f}_t = f_t(\cdot + U(t))$. One says that λ generates a curve γ if

$$\gamma(t) := \lim_{y \to 0+} f_t(iy + U(t)) \tag{4.16}$$

exists and is continuous in $t \in [0,1]$. This is equivalent to saying that there exists a continuous $\overline{\mathbb{H}}$-valued path γ such that for each $t \in [0,1]$, the domain H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$.

It is known ([RS05; LSW04]) that for fixed $\kappa \in [0, \infty)$, the driving function $U = \sqrt{\kappa} B$, where B is a standard Brownian motion, almost surely generates a curve, which we will denote by $\gamma(\cdot, \kappa)$ or γ^κ. But we do not know whether given a Brownian motion B, almost surely all driving functions $\sqrt{\kappa} B$, $\kappa \geq 0$, simultaneously generate a curve. Furthermore, simulations suggest that for a fixed sample of B, the curve γ^κ changes continuously in κ, but only partial proofs have been found so far. We remark that this question is not trivial to answer because in general, the trace does not depend continuously on its driver, as [Law05, Example 4.49] shows.

In [JRW14] the authors show that in the range $\kappa \in [0, 8(2 - \sqrt{3})[\approx [0, 2.1[$, the answer to both of the above questions is positive. Our result Theorem 4.3.2 improves the range to $\kappa \in [0, 8/3[$.

We will often use the following bounds for the moments of $|\hat{f}_t'(iy)|$ that have been shown by F. Johansson-Viklund and G. Lawler in [JL11]. In order to state them, we use the following notation. Let $\kappa \geq 0$. Set

$$\begin{aligned} r_c = r_c(\kappa) &:= \frac{1}{2} + \frac{4}{\kappa}, \\ \lambda(r) = \lambda(\kappa, r) &:= r\left(1 + \frac{\kappa}{4}\right) - \frac{\kappa r^2}{8}, \\ \zeta(r) = \zeta(\kappa, r) &:= r - \frac{\kappa r^2}{8} \end{aligned} \tag{4.17}$$

for $r < r_c(\kappa)$.

With the scaling invariance of SLE, [JL11, Lemma 4.1] implies the following.

Lemma 4.3.1 ([FT17, Lemma 2.1][2]). *Let $\kappa > 0$, $r < r_c(\kappa)$. There exists a constant $C < \infty$ depending only on κ and r such that for all $t, y \in]0,1]$*

$$\mathbb{E}[|\hat{f}_t'(iy)|^{\lambda(r)}] \le Ca(t)y^{\zeta(r)}$$

where $a(t) = a(t,\zeta(r)) = t^{-\zeta(r)/2} \vee 1$.

Moreover, C can be chosen independently of κ and r when κ is bounded away from 0 and ∞, and r is bounded away from $-\infty$ and $r_c(\kappa)$.[3]

Now, for a standard Brownian motion B, and an SLE$_\kappa$ flow driven by $\sqrt{\kappa}B$, we write $\hat{f}_t^\kappa$, γ^κ, etc.

We also use the following notation from [JL11].

$$v(t,\kappa,y) := \int_0^y |(\hat{f}_t^\kappa)'(iu)|\, du.$$

Observe that $v(t,\kappa,\cdot)$ is decreasing in y and

$$|\hat{f}_t^\kappa(iy_1) - \hat{f}_t^\kappa(iy_2)| \le \int_{y_1}^{y_2} |(\hat{f}_t^\kappa)'(iu)|\, du = |v(t,\kappa,y_1) - v(t,\kappa,y_2)|.$$

Therefore $\lim_{y\searrow 0} \hat{f}_t^\kappa(iy)$ exists if $v(t,\kappa,y) < \infty$ for some $y > 0$. For fixed t, κ, this happens almost surely because Lemma 4.3.1 implies

$$\mathbb{E}v(t,\kappa,y) = \int_0^y \mathbb{E}|(\hat{f}_t^\kappa)'(iu)|\, du < \infty.$$

So we can define

$$\gamma(t,\kappa) = \begin{cases} \lim_{y\searrow 0} \hat{f}_t^\kappa(iy) & \text{if the limit exists,} \\ \infty & \text{otherwise,} \end{cases}$$

as a random variable. Note that with this definition we can still estimate

$$|\gamma(t,\kappa) - \hat{f}_t^\kappa(iy)| \le v(t,\kappa,y).$$

4.3.1 Almost sure regularity of SLE in (t,κ)

In this subsection, we prove our first main result.

Theorem 4.3.2. *Let $0 < \kappa_- < \kappa_+ < 8/3$. Let B be a standard Brownian motion. Then almost surely the SLE$_\kappa$ trace γ^κ driven by $\sqrt{\kappa}B$ exists for all $\kappa \in [\kappa_-, \kappa_+]$. Moreover, there exists a random variable C, depending on κ_-, κ_+, such that*

$$|\gamma(t,\kappa) - \gamma(s,\tilde{\kappa})| \le C(|t-s|^\alpha + |\kappa - \tilde{\kappa}|^\eta)$$

for all $t, s \in [0,1]$, $\kappa, \tilde{\kappa} \in [\kappa_-, \kappa_+]$ where $\alpha, \eta > 0$ depend on κ_+. Moreover, C can be chosen to have finite λth moment for some $\lambda > 1$.

[2] Note that in [FT17], λ was called q.
[3] Note that in [JL11], the notation $a = 2/\kappa$ and $q = r_c - r$ is used.

The theorem should be still true near $\kappa \approx 0$ (Without any integrability statement for C, it is shown in [JRW14].), but due to complications in applying Lemma 4.3.1 (cf. [JRW14, Proof of Lemma 3.3]), we decided to omit it.

As in [FT17], we will estimate moments of the increments of γ, using Lemma 4.3.1. We need to be a little careful, though, when applying Lemma 4.3.1, that the exponents do depend on κ. Since we are going to apply that estimate a lot, let us agree on the following.

For every $\kappa > 0$, we will choose some $r_\kappa < r_c(\kappa)$, and we will call $\lambda_\kappa = \lambda(\kappa, r_\kappa)$ and $\zeta_\kappa = \zeta(\kappa, r_\kappa)$ (where r_c, λ, and ζ are defined in (4.17)). (The exact choices of r_κ will be decided later.)

We will use the following moment estimates.

Proposition 4.3.3. *Let $0 < \kappa_- < \kappa_+ < \infty$. Let $t, s \in [0,1]$, $\kappa, \tilde{\kappa} \in [\kappa_-, \kappa_+]$, and $p \in [1, 1 + \frac{8}{\kappa_+}[$. Then (with the above notation) if $\lambda_\kappa \geq 1$, then*

$$\begin{aligned}\mathbb{E}|\gamma(t,\kappa) - \gamma(s,\kappa)|^{\lambda_\kappa} &\leq C(a(t,\zeta_\kappa) + a(s,\zeta_\kappa))\, |t-s|^{(\zeta_\kappa+\lambda_\kappa)/2}, \\ \mathbb{E}|\gamma(s,\kappa) - \gamma(s,\tilde{\kappa})|^p &\leq C|\sqrt{\kappa} - \sqrt{\tilde{\kappa}}|^p,\end{aligned}$$

where $C < \infty$ depends on κ_-, κ_+, p, and the choice of r_κ (see above).

Remark 4.3.4. *Note that $|\sqrt{\kappa} - \sqrt{\tilde{\kappa}}| \leq C|\kappa - \tilde{\kappa}|$ if $\kappa, \tilde{\kappa}$ are bounded away from 0.*

The first estimate is just [FT17, Lemma 3.2].

The second estimate follows from the following result (which we will prove in Section 4.5) and Fatou's lemma.

Proposition 4.3.5. *Let $0 < \kappa_- < \kappa_+ < \infty$ and $\kappa, \tilde{\kappa} \in [\kappa_-, \kappa_+]$. Let $t \in [0,T]$, $\delta \in \,]0,1]$, and $|x| \leq \delta$. Then, for $1 \leq p < 1 + \frac{8}{\kappa_+}$, there exists $C < \infty$, depending on κ_-, κ_+, T, and p, such that*

$$\mathbb{E}|\hat{f}_t^\kappa(x+i\delta) - \hat{f}_t^{\tilde{\kappa}}(x+i\delta)|^p \leq C|\sqrt{\kappa} - \sqrt{\tilde{\kappa}}|^p.$$

If $p > 1 + \frac{8}{\kappa_+}$, then for any $\varepsilon > 0$ there exists $C < \infty$, depending on κ_-, κ_+, T, p, and ε, such that

$$\mathbb{E}|\hat{f}_t^\kappa(x+i\delta) - \hat{f}_t^{\tilde{\kappa}}(x+i\delta)|^p \leq C|\sqrt{\kappa} - \sqrt{\tilde{\kappa}}|^p \delta^{1+\frac{8}{\kappa_+}-p-\varepsilon}.$$

Remark 4.3.6. *Following the proof of [JRW14], in particular using [JRW14, Lemma 2.3] and Lemma 4.3.1, we can show*

$$\mathbb{E}|\hat{f}_t^\kappa(x+i\delta) - \hat{f}_t^{\tilde{\kappa}}(x+i\delta)|^{2\lambda-\varepsilon} \leq C|\sqrt{\kappa} - \sqrt{\tilde{\kappa}}|^{2\lambda-\varepsilon}\delta^{-\lambda+\zeta-\varepsilon}.$$

If we use this estimate instead, we can estimate

$$\begin{aligned}|\gamma(t,\kappa) - \gamma(s,\tilde{\kappa})| &\leq |\gamma(t,\kappa) - \gamma(s,\kappa)| + |\gamma(s,\kappa) - \gamma(s,\tilde{\kappa})| \\ &\leq |\gamma(t,\kappa) - \gamma(s,\kappa)| \\ &\quad + |\gamma(s,\kappa) - \hat{f}_s^\kappa(iy)| + |\hat{f}_s^\kappa(iy) - \hat{f}_s^{\tilde{\kappa}}(iy)| + |\hat{f}_s^{\tilde{\kappa}}(iy) - \gamma(s,\tilde{\kappa})|\end{aligned}$$

with $y = |\Delta\kappa|$. Then, with

$$\begin{aligned}\mathbb{E}|\gamma(t,\kappa)-\gamma(s,\kappa)|^{\lambda} &\le C|t-s|^{(\zeta+\lambda)/2},\\ \mathbb{E}|\gamma(s,\kappa)-\hat{f}_s^{\kappa}(iy)|^{\lambda} &\le Cy^{\zeta+\lambda} = C|\kappa-\tilde{\kappa}|^{\zeta+\lambda},\\ \mathbb{E}|\hat{f}_s^{\kappa}(iy)-\hat{f}_s^{\tilde{\kappa}}(iy)|^{2\lambda-\varepsilon} &\le C|\kappa-\tilde{\kappa}|^{\zeta+\lambda-\varepsilon},\end{aligned}$$

Theorem 4.2.8 applies if $(\frac{\zeta+\lambda}{2})^{-1}+(\zeta+\lambda)^{-1} < 1 \iff \zeta+\lambda > 3$, which happens when $\kappa \in [0, 8(2-\sqrt{3})[\cup]8(2+\sqrt{3}),\infty[$ and with an appropriate choice of r. Hence, we recover the continuity of SLE in the same range as in [JRW14].

Notice that for fixed $\kappa > 0$ the maximal value that $\zeta+\lambda$ can attain is $\frac{\kappa}{4}\left(\frac{1}{2}+\frac{4}{\kappa}\right)^2$ which is (for $\kappa < 8$) less than $p = 1+\frac{8}{\kappa}$ as in our Proposition 4.3.3. In other words, Proposition 4.3.3 is really an improvement to [JRW14].

Below we write $x^+ = x \vee 0$ for $x \in \mathbb{R}$.

Corollary 4.3.7. *Under the same conditions as in Proposition 4.3.5 we have*

$$\mathbb{E}|(\hat{f}_t^{\kappa})'(i\delta)-(\hat{f}_t^{\tilde{\kappa}})'(i\delta)|^p \le C|\sqrt{\kappa}-\sqrt{\tilde{\kappa}}|^p\delta^{-p-(p-1-\frac{8}{\kappa}+\varepsilon)^+}$$

where $C < \infty$ depends on κ_-, κ_+, T, p, and ε.

Proof. For a holomorphic function $f : \mathbb{H} \to \mathbb{H}$, Cauchy Integral Formula tells us that

$$f'(i\delta) = \frac{1}{i2\pi}\int_{\alpha}\frac{f(w)}{(w-i\delta)^2}\,dw$$

where we let α be a circle of radius $\delta/2$ around $i\delta$. Consequently,

$$|(\hat{f}_t^{\kappa})'(i\delta)-(\hat{f}_t^{\tilde{\kappa}})'(i\delta)| \le \frac{1}{2\pi}\int_{\alpha}\frac{|\hat{f}_t^{\kappa}(w)-\hat{f}_t^{\tilde{\kappa}}(w)|}{\delta^2/4}\,|dw|.$$

For all w on the circle α we have $\operatorname{Im} w \in [\delta/2, 3\delta/2]$ and $\operatorname{Re} w \in [-\delta/2, \delta/2]$. Therefore Proposition 4.3.5 implies

$$\mathbb{E}|\hat{f}_t^{\kappa}(w)-\hat{f}_t^{\tilde{\kappa}}(w)|^p \le C|\Delta\sqrt{\kappa}|^p\delta^{-(p-1-\frac{8}{\kappa}+\varepsilon)^+}.$$

By Minkowski's inequality,

$$\mathbb{E}|(\hat{f}_t^{\kappa})'(i\delta)-(\hat{f}_t^{\tilde{\kappa}})'(i\delta)|^p \le \left(\frac{1}{2\pi}\int_{\alpha}\frac{(\mathbb{E}|\hat{f}_t^{\kappa}(w)-\hat{f}_t^{\tilde{\kappa}}(w)|^p)^{1/p}}{\delta^2/4}\,|dw|\right)^p,$$

and the result follows since the length of α is $\pi\delta$. $\square$

With Proposition 4.3.3, we can now apply Theorem 4.2.8 to construct a Hölder continuous version of the map $\gamma = \gamma(t,\kappa)$, whose Hölder constants have some finite moments.

There is just one detail we still have to take into consideration. In order to apply Theorem 4.2.8, we have to use one common exponent λ on the entire range of κ where we want to apply the GRR lemma. Of course, we can choose new values for λ again when we consider a different range of κ.

Alternatively, we could formulate our GRR version to allow exponents to vary with the parameters. But this will not be necessary since we can break our desired interval for κ into subintervals.

Proof of Theorem 4.3.2. Consider the joint SLE$_\kappa$ process in some range $\kappa \in [\kappa_-, \kappa_+]$. We can assume that the interval $[\kappa_-, \kappa_+]$ is so small that $\lambda(\kappa)$ and $\zeta(\kappa)$ are almost constant. Otherwise, break $[\kappa_-, \kappa_+]$ into small subintervals and consider each of them separately.

We perform the proof in three parts. First we construct a continuous version $\tilde{\gamma}$ of γ using Theorem 4.2.8. Then, using Lemma 4.2.1, we show that $\tilde{\gamma}$ is jointly Hölder continuous in both variables. Finally, we show that for each κ, the path $\tilde{\gamma}(\cdot, \kappa)$ is indeed the SLE$_\kappa$ trace generated by $\sqrt{\kappa}B$.

Part 1. For the first part, we would like to apply Theorem 4.2.8. There is just one technical detail we need to account for. In the estimates of Proposition 4.3.3, there is a singularity at time $t = 0$, but we have not formulated Theorem 4.2.8 to allow C' to have a singularity. Therefore, it is easier to apply Theorem 4.2.8 on the domain $[\varepsilon, 1] \times [\kappa_-, \kappa_+]$ with $\varepsilon > 0$. With $\varepsilon \searrow 0$, we obtain a continuous version of γ on the domain $]0, 1] \times [\kappa_-, \kappa_+]$. Due to the local growth property of Loewner chains, we must have $\lim_{t \searrow 0} \gamma(t, \kappa) = 0$ uniformly in κ, so we actually have a continuous version of γ on $[0, 1] \times [\kappa_-, \kappa_+]$. [4]

Now we apply Proposition 4.3.3 on the domain $[\varepsilon, 1] \times [\kappa_-, \kappa_+]$. For this, we pick $\lambda \geq 1$, $r_\kappa < r_c(\kappa)$, and $p \in [1, 1 + \frac{8}{\kappa_+}[$ in such a way that $\lambda_\kappa = \lambda$ for all $\kappa \in [\kappa_-, \kappa_+]$. The condition to apply Theorem 4.2.8 is then $(\frac{\zeta+\lambda}{2})^{-1} + p^{-1} < 1$.

A computation shows that $\zeta + \lambda = \frac{\kappa}{4} r \left(1 + \frac{8}{\kappa} - r\right)$ attains its maximal value $\frac{\kappa}{4}\left(\frac{1}{2} + \frac{4}{\kappa}\right)^2$ at $r = \frac{1}{2} + \frac{4}{\kappa} = r_c$. Note also that $\lambda(r_c) = 1 + \frac{2}{\kappa} + \frac{3}{32}\kappa > 1$. Recall from above that we can pick any $p < 1 + \frac{8}{\kappa}$. Therefore, the condition for the exponents is

$$\frac{2}{\frac{\kappa}{4}\left(\frac{1}{2} + \frac{4}{\kappa}\right)^2} + \frac{1}{1 + \frac{8}{\kappa}} < 1 \iff \kappa < \frac{8}{3}.$$

This completes the first part of the proof and gives us a continuous random field $\tilde{\gamma}$.

Part 2. Now that we have a random continuous function $\tilde{\gamma}$, we can apply Lemma 4.2.1. As in the proof of Theorem 4.2.8, we show that the integrals (4.6) and (4.7) have finite expectation, and therefore are almost surely finite. Denoting $|A_1(t, s; \kappa)| := |\gamma(t, \kappa) - \gamma(s, \kappa)|$, $|A_2(s; \kappa, \tilde{\kappa})| := |\gamma(s, \kappa) - \gamma(s, \tilde{\kappa})|$, and the corresponding integrals by M_1, M_2, we have by Proposition 4.3.3

$$\mathbb{E}M_1 \lesssim \iiint (a(t) + a(s))|t - s|^{(\zeta+\lambda)/2 - \beta_1}\, dt\, ds\, d\kappa,$$
$$\mathbb{E}M_2 \lesssim \iiint |\kappa - \tilde{\kappa}|^{p - \beta_2}\, ds\, d\kappa\, d\tilde{\kappa}.$$

[4] Alternatively, we could also use the same strategy as in the proof of Theorem 4.2.8, and deduce the result directly from Lemma 4.2.1.

Picking $\beta_1 = \frac{\zeta+\lambda}{2} + 1 - \varepsilon$, $\beta_2 = p + 1 - \varepsilon$, the condition for the exponents is again $(\frac{\zeta+\lambda}{2})^{-1} + p^{-1} < 1$. Additionally, we need to account for the singularity at $t = 0$ in the first integrand. This is not a problem if the function $a(t) = t^{-\zeta/2} \vee 1$ is integrable.

To make $a(t) = t^{-\zeta/2} \vee 1$ integrable, we would like to have $\zeta < 2$. [5] Recall that $\zeta = r - \frac{\kappa r^2}{8}$ from (4.17). In case $\kappa > 1$, we always have $\zeta < 2$. In case $\kappa \leq 1$, we have $\zeta < 2$ for $r < \frac{4}{\kappa}(1 - \sqrt{1-\kappa})$, or equivalently $\lambda(r) < 3 - \sqrt{1-\kappa}$. Therefore we can certainly find r such that $\zeta < 2$ and $\zeta + \lambda \approx 2 + (3 - \sqrt{1-\kappa})$, and $p \approx 9 < 1 + \frac{8}{\kappa}$. The condition $(\frac{\zeta+\lambda}{2})^{-1} + p^{-1} < 1$ is still fulfilled.

This proves the statements about the Hölder continuity of $\tilde{\gamma}$.

Part 3. In the final part, we show that for each κ, the path $\tilde{\gamma}(\cdot, \kappa)$ is indeed the SLE$_\kappa$ trace generated by $\sqrt{\kappa}B$.

First, we fix a countable dense subset $\mathcal{K}$ in $[\kappa_-, \kappa_+]$. There exists a set Ω_1 of probability 1 such that for all $\omega \in \Omega_1$, all $\kappa \in \mathcal{K}$, $\gamma(\kappa, t)$ exists and is continuous in t.

Since $\tilde{\gamma}$ is a version of γ, for all t,

$$\mathbb{P}\big(\gamma(t,\kappa) = \tilde{\gamma}(t,\kappa) \text{ for all } \kappa \in \mathcal{K}\big) = 1.$$

Hence, there exists a set Ω_2 with probability 1 such that for all $\omega \in \Omega_2$, we have $\gamma(t,\kappa) = \tilde{\gamma}(t,\kappa)$ for all $\kappa \in \mathcal{K}$ and almost all t. Restricted to $\omega \in \Omega_3 = \Omega_1 \cap \Omega_2$, the previous statement is true for all $\kappa \in \mathcal{K}$ and all t. We claim that on the set Ω_3 of probability 1, the path $t \mapsto \tilde{\gamma}(t,\kappa)$ is indeed the SLE$_\kappa$ trace driven by $\sqrt{\kappa}B$. This can be shown in the same way as [LSW04, Theorem 4.7].

Indeed, fix $t \in [0,1]$ and let $H_t = f_t^\kappa(\mathbb{H})$. We show that H_t is the unbounded connected component of $\mathbb{H} \setminus \tilde{\gamma}([0,t],\kappa)$ [6]. Find a sequence of $\kappa_n \in \mathcal{K}$ with $\kappa_n \to \kappa$ and let $(f_t^{\kappa_n})$ be the corresponding inverse Loewner maps. Since $\sqrt{\kappa_n}B \to \sqrt{\kappa}B$, the Loewner differential equation implies that $f_t^{\kappa_n} \to f_t^\kappa$ uniformly on each compact set of $\mathbb{H}$. By the chordal version of the Carathéodory kernel theorem (see [Pom92, Theorem 1.8]) which can be easily shown with the obvious adaptions, it follows that $H_t^{\kappa_n} \to H_t$ in the sense of kernel convergence. Since $\kappa_n \in \mathcal{K}$, we have $H_t^{\kappa_n} = \mathbb{H} \setminus \gamma([0,t],\kappa_n) = \mathbb{H} \setminus \tilde{\gamma}([0,t],\kappa_n)$. Therefore, the definitions of kernel convergence and the uniform continuity of $\tilde{\gamma}$ imply that H_t is the unbounded connected component of $\mathbb{H} \setminus \tilde{\gamma}([0,t],\kappa)$. $\square$

By Theorem 4.3.2, we now know that with probability one, the SLE$_\kappa$ trace $\gamma = \gamma(t,\kappa)$ is jointly continuous in $[0,1] \times [\kappa_-, \kappa_+]$. Similarly, applying Corollary 4.2.14, we can show the following.

Theorem 4.3.8. *Let $0 < \kappa_- < \kappa_+ < 8/3$. Let γ^κ be the SLE$_\kappa$ trace driven by $\sqrt{\kappa}B$, and assume it is jointly continuous in $(t,\kappa) \in [0,1] \times [\kappa_-, \kappa_+]$. Consider γ^κ as an element of $C^0([0,1])$ (with the metric $\|\cdot\|_\infty$).*

Then for some $0 < p < 1/\eta$ (with η from Theorem 4.3.2), the p-variation of $\kappa \mapsto \gamma^\kappa$, $\kappa \in [\kappa_-, \kappa_+]$, is a.s. finite and bounded by some random variable C, depending on κ_-, κ_+, that has finite λth moment for some $\lambda > 1$.

[5] Alternatively, we can drop this condition if we make statements about the SLE$_\kappa$ process only on $t \in [\varepsilon, 1]$ for some $\varepsilon > 0$.

[6] Actually, there is only one component because it will turn out that $\tilde{\gamma}(\cdot, \kappa)$ is a simple trace.

We know that for fixed $\kappa \leq 4$, the SLE$_\kappa$ trace is almost surely simple. It is natural to expect that there is a common set of probability 1 where all SLE$_\kappa$ traces, $\kappa < 8/3$, are simple. This is indeed true.

Theorem 4.3.9. *Let B be a standard Brownian motion. We have with probability 1 that for all $\kappa < 8/3$ the SLE$_\kappa$ trace driven by $\sqrt{\kappa}B$ is simple.*

Proof. As shown in [RS05, Theorem 6.1], due to the independent stationary increments of Brownian motion, this is equivalent to saying that $K_t^\kappa \cap \mathbb{R} = \{0\}$ for all t and κ, where $K_t^\kappa = \{z \in \overline{\mathbb{H}} \mid T_z^\kappa \leq t\}$ (the upper index denotes the dependence on κ).

Let $(g_t(x))_{t\geq 0}$ satisfy (4.15) with $g_0(x) = x$ and driving function $U(t) = \sqrt{\kappa}B_t$. Then $X_t = \frac{g_t(x)-\sqrt{\kappa}B_t}{\sqrt{\kappa}}$ satisfies

$$dX_t = \frac{2/\kappa}{X_t}\,dt - dB_t,$$

i.e. X is a Bessel process of dimension $1+\frac{4}{\kappa}$. The statement $K_t^\kappa \cap \mathbb{R} = \{0\}$ is equivalent to saying that $X_s \neq 0$ for all $x \neq 0$ and $s \in [0,t]$. This is a well-known property of Bessel processes, and stated in the lemma below. □

Lemma 4.3.10. *Let B be a standard Brownian motion and suppose that we have a family of stochastic processes $X^{\kappa,x}$, $\kappa, x > 0$, that satisfy*

$$X_t^{\kappa,x} = x + B_t + \int_0^t \frac{2/\kappa}{X_s^{\kappa,x}}\,ds, \quad t \in [0, T_{\kappa,x}]$$

where $T_{\kappa,x} = \inf\{t \geq 0 \mid X_t^{\kappa,x} = 0\}$.

Then we have with probability 1 that $T_{\kappa,x} = \infty$ for all $\kappa \leq 4$ and $x > 0$.

Proof. For fixed $\kappa \leq 4$, see e.g. [Law05, Proposition 1.21]. To get the result simultaneously for all κ, use the property that if $\kappa < \tilde{\kappa}$ and $x > 0$, then $X_t^{\kappa,x} > X_t^{\tilde{\kappa},x}$ for all $t > 0$, which follows from Grönwall's inequality. □

4.3.2 Stochastic continuity of SLE$_\kappa$ in κ

In the previous section, we have shown almost sure continuity of SLE$_\kappa$ in κ (in the range $\kappa \in [0, 8/3[$). Weaker forms of continuity are easier to prove, and hold on a larger range of κ. We will show here that stochastic continuity (also continuity in $L^q(\mathbb{P})$ sense for some $q > 1$ depending on κ) for all $\kappa \neq 8$ is an immediate consequence of our estimates. Below we write $\|f\|_{C^\alpha[a,b]} := \sup \frac{|f(t)-f(s)|}{|t-s|^\alpha}$, with sup taken over all $s < t$ in $[a,b]$.

Theorem 4.3.11. *Let $\kappa > 0$, $\kappa \neq 8$. Then there exists $\alpha > 0$, $q > 1$, $r > 0$, and $C < \infty$ (depending on κ) such that if $\tilde{\kappa}$ is sufficiently close to κ (where "sufficiently close" depends on κ), then*

$$\mathbb{E}\left[\|\gamma(\cdot,\kappa) - \gamma(\cdot,\tilde{\kappa})\|_{C^\alpha[0,1]}^q\right] \leq C|\kappa - \tilde{\kappa}|^r.$$

In particular, if $\kappa_n \to \kappa$ exponentially fast, then $\|\gamma(\cdot,\kappa) - \gamma(\cdot,\kappa_n)\|_{C^\alpha[0,1]} \to 0$ almost surely.

Note that without sufficiently fast convergence of $\kappa_n \to \kappa$ it is not clear whether we can pass from L^q-convergence to almost sure convergence.

Proof. Fix $\kappa, \tilde{\kappa} \neq 8$. We apply Corollary 4.2.11 to the function $G : [0,1] \to \mathbb{C}$, $G(t) = \gamma(t,\kappa) - \gamma(t,\tilde{\kappa})$. We have

$$\begin{aligned}|G(t) - G(s)| &\le (|\gamma(t,\kappa) - \gamma(s,\kappa)| + |\gamma(t,\tilde{\kappa}) - \gamma(s,\tilde{\kappa})|)\, 1_{|t-s|\le|\kappa-\tilde{\kappa}|} \\ &\quad + (|\gamma(t,\kappa) - \gamma(t,\tilde{\kappa})| + |\gamma(s,\kappa) - \gamma(s,\tilde{\kappa})|)\, 1_{|t-s|>|\kappa-\tilde{\kappa}|} \\ &=: A_1(t,s) + A_2(t,s)\end{aligned}$$

where by Proposition 4.3.3

$$\begin{aligned}&\mathbb{E}|A_1(t,s)|^\lambda \le C(a^1(t) + a^1(s))\, |t-s|^{(\zeta+\lambda)/2}\, 1_{|t-s|\le|\kappa-\tilde{\kappa}|}, \\ &\mathbb{E}|A_2(t,s)|^p \le C|\kappa-\tilde{\kappa}|^p\, 1_{|t-s|>|\kappa-\tilde{\kappa}|},\end{aligned}$$

for suitable $\lambda \ge 1$, $p \in [1, 1+\frac{8}{\kappa}[$.

It follows that, for $\beta_1, \beta_2 > 0$,

$$\begin{aligned}\mathbb{E} \iint \frac{|A_1(t,s)|^\lambda}{|t-s|^{\beta_1}}\, dt\, ds &\le C \iint_{|t-s|\le|\kappa-\tilde{\kappa}|} (a^1(t) + a^1(s))\, |t-s|^{(\zeta+\lambda)/2-\beta_1}\, dt\, ds \\ &\le C|\kappa-\tilde{\kappa}|^{(\zeta+\lambda)/2-\beta_1+1}, \\ \mathbb{E} \iint \frac{|A_2(t,s)|^p}{|t-s|^{\beta_2}}\, dt\, ds &\le C|\kappa-\tilde{\kappa}|^p \iint_{|t-s|>|\kappa-\tilde{\kappa}|} |t-s|^{-\beta_2}\, dt\, ds \\ &\le C|\kappa-\tilde{\kappa}|^{p-\beta_2+1}\end{aligned}$$

if $\zeta < 2$ and $\beta_1 < \frac{\zeta+\lambda}{2} + 1$.

Recall that if $\kappa \neq 8$ and $\tilde{\kappa}$ is sufficiently close to κ, then the parameters λ, ζ are almost the same for κ and $\tilde{\kappa}$, and (see the proof of Theorem 4.3.2) they can be picked such that $\zeta < 2$ and $\zeta + \lambda > 2$. Hence, we can pick $\beta_1, \beta_2 > 2$ such that $2 < \beta_1 < \frac{\zeta+\lambda}{2} + 1$ and $2 < \beta_2 < 1 + p < 2 + \frac{8}{\kappa}$.

The result follows from Corollary 4.2.11, where we take $\alpha = \frac{\beta_1 - 2}{\lambda} \wedge \frac{\beta_2 - 2}{p}$ and $q = \lambda \wedge p$, which implies

$$\mathbb{E}\left[\|G\|^q_{C^\alpha[0,1]}\right] \le C\mathbb{E}\left[\left(\iint \frac{|A_1(t,s)|^\lambda}{|t-s|^{\beta_1}}\, dt\, ds\right)^{q/\lambda} + \left(\iint \frac{|A_2(t,s)|^p}{|t-s|^{\beta_2}}\, dt\, ds\right)^{q/p}\right].$$

□

Corollary 4.3.12. *For any $\kappa > 0$, $\kappa \neq 8$ and any sequence $\kappa_n \to \kappa$ we then have $\|\gamma^\kappa - \gamma^{\kappa_n}\|_{p\text{-var};[0,1]} \to 0$ in probability, for any $p > (1+\kappa/8) \wedge 2$.*

Proof. Theorem 4.3.11 immediately implies the statement with $\|\cdot\|_\infty$. To upgrade the result to Hölder and p-variation topologies, recall the following general fact which follows from the interpolation inequalities for Hölder and p-variation constants (see e.g. [FV10, Proposition 5.5]):

Suppose X_n, X are continuous stochastic processes such that for every $\varepsilon > 0$ there exists $M > 0$ such that $\mathbb{P}(\|X_n\|_{p\text{-var};[0,T]} > M) < \varepsilon$ for all n. If $X_n \to X$ in probability with respect to the $\|\cdot\|_\infty$ topology, then also with respect to the p'-variation topology for any $p' > p$. The analogous statement holds for Hölder topologies with $\alpha' < \alpha \le 1$.

In order to apply this fact, we can use [FT17, Theorem 5.2 and 6.1] which bound the moments of $\|\gamma\|_{p\text{-var}}$ and $\|\gamma\|_{C^\alpha}$. The values for p and α have also been computed there. □

4.4 Convergence results

Here we prove a stronger version of Theorem 4.3.2, namely uniform convergence (even convergence in Hölder sense) of $\hat{f}_t^\kappa(iy)$ as $y \searrow 0$. For this result, we really use the full power of Lemma 4.2.1 (actually Lemma 4.2.13 as we will explain later). We point out that this is a stronger result than Theorem 4.1.1, and that our previous proofs of Theorems 4.1.1 and 4.1.2 do not rely on this section.

The Hölder continuity in Theorem 4.3.2 induces an (inhomogeneous) Hölder space, with (inhomogeneous) Hölder constant that we denote by

$$\|\gamma\|_{C^{\alpha,\eta}} := \sup_{(t,\kappa)\neq(s,\tilde\kappa)} \frac{|\gamma(t,\kappa)-\gamma(s,\tilde\kappa)|}{|t-s|^\alpha + |\kappa-\tilde\kappa|^\eta}.$$

As before, we write

$$v(t,\kappa,y) = \int_0^y |(\hat{f}_t^\kappa)'(iu)|\,du.$$

Theorem 4.4.1. *Let $\kappa_- > 0$, $\kappa_+ < 8/3$. Then $\|v(\cdot,\cdot,y)\|_{\infty;[0,1]\times[\kappa_-,\kappa_+]} \searrow 0$ almost surely as $y \searrow 0$. In particular, $\hat{f}_t^\kappa(iy)$ converges uniformly in $(t,\kappa) \in [0,1]\times[\kappa_-,\kappa_+]$ as $y \searrow 0$.*

Moreover, both functions converge also almost surely in the same Hölder space $C^{\alpha,\eta}([0,1]\times[\kappa_-,\kappa_+])$ as in Theorem 4.3.2.

Moreover, the (random) Hölder constants of $v(\cdot,\cdot,y)$ and $(t,\kappa) \mapsto |\gamma(t,\kappa) - \hat{f}_t^\kappa(iy)|$ satisfy

$$\mathbb{E}[\|v(\cdot,\cdot,y)\|_{C^{\alpha,\eta}}^\lambda] \le Cy^r \quad \text{and} \quad \mathbb{E}[\|\gamma(\cdot,\cdot) - \hat{f}_\cdot^\cdot(iy)\|_{C^{\alpha,\eta}}^\lambda] \le Cy^r$$

for some $\lambda > 1$, $r > 0$ and $C < \infty$, and all $y \in\,]0,1]$.

As a consequence, we obtain also an improved version of [JRW14, Lemma 3.3].

Corollary 4.4.2. *Let $\kappa_- > 0$, $\kappa_+ < 8/3$. Then there exist $\beta < 1$ and a random variable $c(\omega) < \infty$ such that almost surely*

$$\sup_{(t,\kappa)\in[0,1]\times[\kappa_-,\kappa_+]} |(\hat{f}_t^\kappa)'(iy)| \le c(\omega)y^{-\beta}$$

for all $y \in\,]0,1]$.

Proof. By Koebe's 1/4-Theorem we have $y|(\hat{f}_t^\kappa)'(iy)| \le 4\,\mathrm{dist}(\hat{f}_t^\kappa(iy), \partial H_t^\kappa) \le 4v(t,\kappa,y)$. Theorem 4.4.1 and the Borel-Cantelli lemma imply

$$\|v(\cdot,\cdot,2^{-n})\|_\infty \le 2^{-nr'}$$

for some $r' > 0$ and sufficiently large (depending on ω) n. The result then follows by Koebe's distortion theorem (with $\beta = 1 - r'$). $\square$

The same method as Theorem 4.4.1 can be used to show the existence and Hölder continuity of the SLE_κ trace for fixed $\kappa \neq 8$, avoiding a Borel-Cantelli argument. The best way of formulating this result is the terminology in [FT17].

For $\delta \in \,]0,1[$, $q \in \,]1,\infty[$, define the fractional Sobolev (Slobodeckij) semi-norm of a measurable function $x : [0,1] \to \mathbb{C}$ as

$$\|x\|_{W^{\delta,q}} := \left(\int_0^1 \int_0^1 \frac{|x(t)-x(s)|^q}{|t-s|^{1+\delta q}}\, ds\, dt \right)^{1/q}.$$

As a consequence of the (classical) one-dimensional GRR inequality (see [FV10, Corollary A.2 and A.3]), we have that for all $\delta \in \,]0,1[$, $q \in \,]1,\infty[$ with $\delta - 1/q > 0$, there exists a constant $C < \infty$ such that for all $x \in C[0,1]$ we have

$$\|x\|_{C^\alpha[s,t]} \le C \|x\|_{W^{\delta,q}[s,t]}$$

and

$$\|x\|_{p\text{-var};[s,t]} \le C|t-s|^\alpha \|x\|_{W^{\delta,q}[s,t]},$$

where $p = 1/\delta$ and $\alpha = \delta - 1/q$, and $\|x\|_{C^\alpha[s,t]}$ and $\|x\|_{p\text{-var};[s,t]}$ denote the Hölder and p-variation constants of x, restricted to $[s,t]$.

Fix $\kappa \ge 0$, and as before, let

$$v(t,y) = \int_0^y |\hat{f}_t'(iu)|\, du.$$

Recall the notation (4.17), and let $\lambda = \lambda(r)$, $\zeta = \zeta(r)$ with some $r < r_c(\kappa)$.

The following result is proved similarly to Theorem 4.4.1.

Theorem 4.4.3. *Let $\kappa \neq 8$. Then for some $\alpha > 0$ and some $p < 1/\alpha$ there almost surely exists a continuous $\gamma : [0,1] \to \overline{\mathbb{H}}$ such that the function $t \mapsto \hat{f}_t(iy)$ converges in C^α and p-variation to γ as $y \searrow 0$.*

More precisely, let $\kappa \ge 0$ be arbitrary, $\zeta < 2$ and $\delta \in \left]0, \frac{\lambda+\zeta}{2\lambda}\right[$. Then there exists a random measurable function $\gamma : [0,1] \to \overline{\mathbb{H}}$ such that

$$\mathbb{E}\|v(\cdot,y)\|_{W^{\delta,\lambda}}^\lambda \le C y^{\lambda+\zeta-2\delta\lambda} \quad \text{and} \quad \mathbb{E}\|\gamma - \hat{f}_\cdot(iy)\|_{W^{\delta,\lambda}}^\lambda \le C y^{\lambda+\zeta-2\delta\lambda}$$

for all $y \in \,]0,1]$, where C is a constant that depends on κ, r, and δ. Moreover, a.s. $\|v(\cdot,y)\|_{W^{\delta,\lambda}} \to 0$ and $\|\gamma - \hat{f}_\cdot(iy)\|_{W^{\delta,\lambda}} \to 0$ as $y \searrow 0$.

If additionally $\delta \in \left]\frac{1}{\lambda}, \frac{\lambda+\zeta}{2\lambda}\right[$, then the same is true for $\|\cdot\|_{1/\delta\text{-var}}$ and $\|\cdot\|_{C^\alpha}$ where $\alpha = \delta - 1/\lambda$.

Remark 4.4.4. *The conditions for the exponents are the same as in [FT17]. In particular, the result applies to the (for SLE$_\kappa$) optimal p-variation and Hölder exponents.*

Proof of Theorem 4.4.1. We use the same setting as in the proof of Theorem 4.3.2. For $\kappa \le \kappa_+ < 8/3$, we choose $p \in [1, 1+\frac{8}{\kappa_+}[$, $r_\kappa < r_c(\kappa)$, $\lambda(\kappa, r_\kappa) = \lambda \ge 1$, and the corresponding $\zeta_\kappa = \zeta(\kappa, r_\kappa)$ as in the proof of Theorem 4.3.2. Again, we assume that the interval $[\kappa_-, \kappa_+]$ is small enough so that $\lambda(\kappa)$ and $\zeta(\kappa)$ are almost constant.

Step 1. We would like to show that v and $\hat{f}$ (defined above) are Cauchy sequences in the aforementioned Hölder space as $y \searrow 0$. Therefore we will take differences $|v(\cdot,\cdot,y_1) - v(\cdot,\cdot,y_2)|$ and $|\hat{f}(iy_1) - \hat{f}(iy_2)|$, and estimate their Hölder norms with our GRR lemma. Note that it is not a priori clear that $v(t,\kappa,y)$ is continuous in (t,κ), but $|v(t,\kappa,y_1) - v(t,\kappa,y_2)| = \int_{y_1}^{y_2} |(\hat{f}_t^\kappa)'(iu)|\, du$ certainly is, so the GRR lemma can be applied to this function.

Consider the function

$$G(t,\kappa) := v(t,\kappa,y) - v(t,\kappa,y_1) = \int_{y_1}^{y} |(\hat{f}_t^\kappa)'(iu)|\, du.$$

The strategy will be to show that the condition of Lemma 4.2.1 is satisfied almost surely for G. As in the proof of Kolmogorov's continuity theorem, we do this by showing that the expectation of the integrals (4.6), (4.7) are finite (after defining suitable A_{1j}, A_{2j}) and converge to 0 as $y \searrow 0$. In particular, they are almost surely finite, so Lemma 4.2.1 then implies that G is Hölder continuous, with Hölder constant bounded in terms of the integrals (4.6), (4.7).

We would like to infer that almost surely the functions $v(\cdot,\cdot,y)$, $y > 0$, form a Cauchy sequence in the Hölder space $C^{\alpha,\eta}$. But this is not immediately clear, therefore we will bound the integrals (4.6), (4.7) by expressions that are decreasing in y. We will also define A_{1j}, A_{2j} here.

In order to do so, we estimate

$$\begin{aligned}
&|G(t,\kappa) - G(s,\tilde{\kappa})| \\
&\le \int_0^y \Big| |(\hat{f}_t^\kappa)'(iu)| - |(\hat{f}_s^\kappa)'(iu)| \Big|\, du + \int_0^y \Big| |(\hat{f}_s^\kappa)'(iu)| - |(\hat{f}_s^{\tilde{\kappa}})'(iu)| \Big|\, du \\
&\le \int_0^y |(\hat{f}_t^\kappa)'(iu) - (\hat{f}_s^\kappa)'(iu)|\, du + \int_0^y |(\hat{f}_s^\kappa)'(iu) - (\hat{f}_s^{\tilde{\kappa}})'(iu)|\, du \\
&=: A_{1*}(t,s;\kappa) + A_{2*}(s;\kappa,\tilde{\kappa}),
\end{aligned}$$

Moreover, the function $\hat{G}(t,\kappa) := \hat{f}_t^\kappa(iy) - \hat{f}_t^\kappa(iy_1)$ also satisfies

$$|\hat{G}(t,\kappa) - \hat{G}(s,\tilde{\kappa})| \le A_{1*}(t,s;\kappa) + A_{2*}(s;\kappa,\tilde{\kappa}).$$

Therefore all our considerations for G apply also to $\hat{G}$.

We want to estimate the difference $|(\hat{f}_s^\kappa)'(iu) - (\hat{f}_s^{\tilde\kappa})'(iu)|$ differently for small and large u (relatively to $|\Delta\kappa|$), therefore we we split A_{2*} into

$$\begin{aligned} A_{2*}(s;\kappa,\tilde\kappa) = &\int_0^{y\wedge|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}} |(\hat{f}_s^\kappa)'(iu) - (\hat{f}_s^{\tilde\kappa})'(iu)|\,du \\ &+ \int_{y\wedge|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}}^{y} |(\hat{f}_s^\kappa)'(iu) - (\hat{f}_s^{\tilde\kappa})'(iu)|\,du \\ =: &\, A_{21}(s;\kappa,\tilde\kappa) \\ &+ A_{22}(s;\kappa,\tilde\kappa). \end{aligned}$$

We would like to apply Lemma 4.2.1 with these choices of A_{1*}, A_{21}, A_{22}. We denote the integrals (4.6), (4.7) by

$$\begin{aligned} M_{1*} &:= \iiint \frac{|A_{1*}(t,s;\kappa)|^\lambda}{|t-s|^{\beta_1}}\,ds\,dt\,d\kappa, \\ M_{21} &:= \iiint \frac{|A_{21}(s;\kappa,\tilde\kappa)|^\lambda}{|\kappa-\tilde\kappa|^{\beta_2}}\,ds\,d\kappa\,d\tilde\kappa, \\ M_{22} &:= \iiint \frac{|A_{22}(s;\kappa,\tilde\kappa)|^p}{|\kappa-\tilde\kappa|^{\beta_2}}\,ds\,d\kappa\,d\tilde\kappa. \end{aligned}$$

Suppose that we can show that

$$\mathbb{E}[M_{1*}] \lesssim y^r, \quad \mathbb{E}[M_{2j}] \lesssim y^r$$

for some $r > 0$. This would imply that they are almost surely finite, and that G and $\hat{G}$ are Hölder continuous with $\|G\|_{C^{\alpha,\eta}} \lesssim M_{A*}^{1/\lambda} + M_{21}^{1/\lambda} + M_{22}^{1/p}$ (same for $\hat{G}$).

Notice that now A_{1*}, A_{21}, A_{22}, hence also M_{A*}, M_{21}, M_{22} are decreasing in y. So as we let $y, y_1 \searrow 0$, it would follow that

- $\mathbb{E}[\|G\|_{C^{\alpha,\eta}}^\lambda] \lesssim y^{r'} \to 0$ (same for $\hat{G}$) with a (possibly) different $r' > 0$. In particular, as $y \searrow 0$, the random functions $v(\cdot,\cdot,y)$ and $(t,\kappa) \mapsto \hat{f}_t^\kappa(iy)$ form Cauchy sequences in $L^\lambda(\mathbb{P}; C^{\alpha,\eta})$, and it follows that also $\mathbb{E}[\|v(\cdot,\cdot,y)\|_{C^{\alpha,\eta}}^\lambda] \lesssim y^{r'} \to 0$ and $\mathbb{E}[\|\gamma(\cdot,\cdot) - \hat{f}_\cdot^\cdot(iy)\|_{C^{\alpha,\eta}}^\lambda] \lesssim y^{r'} \to 0$ as $y \searrow 0$.

- By the monotonicity of M_{A*}, M_{21}, M_{22} in y we have that almost surely the functions $v(\cdot,\cdot,y)$ and $(t,\kappa) \mapsto \hat{f}_t^\kappa(iy)$ are Cauchy sequences in the Hölder space $C^{\alpha,\eta}$.

This will show Theorem 4.4.1.

Step 2. We now explain that in fact, our definition of A_{1*} does not always suffice, and we need to define A_{1j} a bit differently in order to get the best estimates. The new definition of A_{1j} will satisfy only the relaxed condition (4.14) (instead of (4.5)).

The reason is that, when $|t-s| \le u^2$, $|\hat{f}_t(iu) - \hat{f}_s(iu)|$ is estimated by an expression like $|\hat{f}_s'(iu)||B_t - B_s|$ which is of the order $O(|t-s|^{1/2})$. The same is true for the difference

$|\hat{f}'_t(iu) - \hat{f}'_s(iu)|$ (see (4.20) below). When we carry out the moment estimate for our choice of A_{1*}, then we will get

$$\mathbb{E}|A_{1*}(t,s;\kappa)|^\lambda = O(|t-s|^{\lambda/2}).$$

But recall from Proposition 4.3.3 that

$$\mathbb{E}|\gamma(t) - \gamma(s)|^\lambda \le C|t-s|^{(\zeta+\lambda)/2},$$

which has allowed us to apply Lemma 4.2.1 with $\beta_1 \approx \frac{\zeta+\lambda}{2} + 1$ in the proof of Theorem 4.3.2. When $\zeta > 0$, this was better than just $\lambda/2$.

To fix this, we need to adjust our choice of A_{1j}. In particular, we should not evaluate $\mathbb{E}|\hat{f}'_t(iu) - \hat{f}'_s(iu)|^\lambda$ when $u \gg |t-s|^{1/2}$ (here "$\gg$" means "much larger"). As observed in [JL11], $|\hat{f}'_s(iu)|$ does not change much in time when $u \gg |t-s|^{1/2}$. More precisely, we have the following results.

Lemma 4.4.5. *Let (g_t) be a chordal Loewner chain driven by U, and $\hat{f}_t(z) = g_t^{-1}(z + U(t))$. Then, if $t, s \ge 0$ and $z = x + iy \in \mathbb{H}$ such that $|t-s| \le C'y^2$, we have*

$$|\hat{f}'_t(z)| \le C|\hat{f}'_s(z)| \left(1 + \frac{|U(t)-U(s)|^2}{y^2}\right)^l, \tag{4.18}$$

$$|\hat{f}_t(z) - \hat{f}_s(z)| \le C|\hat{f}'_s(z)| \left(\frac{|t-s|}{y} + |U(t)-U(s)| \left(1 + \frac{|U(t)-U(s)|^2}{y^2}\right)^l\right), \tag{4.19}$$

$$|\hat{f}'_t(z) - \hat{f}'_s(z)| \le C|\hat{f}'_s(z)| \left(\frac{|t-s|}{y^2} + \frac{|U(t)-U(s)|}{y} \left(1 + \frac{|U(t)-U(s)|^2}{y^2}\right)^l\right), \tag{4.20}$$

where $C < \infty$ depends on $C' < \infty$, and $l < \infty$ is a universal constant.

Proof. The first two inequalities (4.18) and (4.19) follow from [JL11, Lemma 3.5 and 3.2]. The third inequality (4.20) follows from (4.19) by the Cauchy integral formula in the same way as in Corollary 4.3.7. Note that for $z \in \mathbb{H}$ and w on a circle of radius $y/2$ around z, we have $|\hat{f}'_s(w)| \le 12|\hat{f}'_s(z)|$ by the Koebe distortion theorem. □

We now redefine A_{1j}. Let

$$A_{11}(t,s;\kappa) = \int_0^{y \wedge |t-s|^{1/2}} |\hat{f}'_t(iu) - \hat{f}'_s(iu)|\, du,$$

$$A_{12}(t,s;\kappa) = \int_{y\wedge|t-s|^{1/2}}^{y} \frac{|t-s|}{u^2} |\hat{f}'_s(iu)|\, du,$$

$$A_{13}(t,s;\kappa) = \int_{y\wedge|t-s|^{1/2}}^{y\wedge 2|t-s|^{1/2}} u^{-1} |\hat{f}'_s(iu)| \left(1 + \|B\|_{C^{1/2(-)}}\right)^{2l+1} |t-s|^{1/2^{(-)}}\, du,$$

for $s \le t$, where the exponents $1/2^{(-)} < 1/2$ denote some numbers that we can pick arbitrarily close to $1/2$. (Of course, $\hat{f}_t$ still depends on κ, but for convenience we do not write it for now.)

Note that the integrands in A_{12} and A_{13} just make fancy bounds of

$$|\hat{f}'_t(iu) - \hat{f}'_s(iu)|,$$

according to (4.20). But now, in A_{13} we are not integrating up to y any more. Thus, the condition (4.5) is not satisfied any more. But the relaxed condition (4.14) of Lemma 4.2.13 is still satisfied. Indeed, by (4.20),

$$\begin{aligned} A_{1*}(t, s; \kappa) &\le A_{11}(t, s; \kappa) + \int_{y \wedge |t-s|^{1/2}}^{y} |\hat{f}'_t(iu) - \hat{f}'_s(iu)|\, du \\ &\le A_{11}(t, s; \kappa) + A_{12}(t, s; \kappa) \\ &\quad + \int_{y \wedge |t-s|^{1/2}}^{y} u^{-1} |\hat{f}'_s(iu)| \left(1 + \|B\|_{C^{1/2^{(-)}}}\right)^{l+1} |t-s|^{1/2^{(-)}}\, du \end{aligned}$$

where by (4.18)

$$\begin{aligned} &\int_{y \wedge |t-s|^{1/2}}^{y} u^{-1} |\hat{f}'_s(iu)| \left(1 + \|B\|_{C^{1/2^{(-)}}}\right)^{l+1} |t-s|^{1/2^{(-)}}\, du \\ &= \sum_{k=0}^{\lfloor \log_4(y^2/|t-s|) \rfloor} \int_{y \wedge (4^k |t-s|)^{1/2}}^{y \wedge 2(4^k|t-s|)^{1/2}} \cdots \\ &= \sum_{k=0}^{\lfloor \log_4(y^2/|t-s|) \rfloor} 4^{-k(1/2^{(-)})} |A_{13}(t_1 + 4^k(t-t_1), t_1 + 4^k(s-t_1); \kappa)| \end{aligned}$$

whenever $|s - t_1| \le 2|t-s|$ (implying $|s - (t_1 + 4^k(s-t_1))| \le (4^k - 1)2|t-s| \le 2u^2$).

Finally, with this definition of A_{13}, we truly have $\mathbb{E}|A_{13}(t, s; \kappa)|^{\lambda^{(-)}} = O(|t-s|^{(\zeta+\lambda)^{(-)}/2})$ and not just $O(|t-s|^{\lambda/2})$; here $\lambda^{(-)} < \lambda$ is an exponent that can be chosen arbitrarily close to λ.

Proposition 4.4.6. *With the above notation and assumptions, if* $1 < \beta_1 < \frac{\zeta+\lambda}{2} + 1$, $1 < \beta_2 < p+1$, *we have*

$$\begin{aligned} \mathbb{E} \iiint \frac{|A_{1j}(t, s; \kappa)|^{\lambda}}{|t-s|^{\beta_1}}\, ds\, dt\, d\kappa &\le C y^{\zeta+\lambda-2\beta_1+2} \iint a(s, \zeta_\kappa)\, ds\, d\kappa, \quad j = 1, 2, \\ \mathbb{E} \iiint \frac{|A_{13}(t, s; \kappa)|^{\lambda^{(-)}}}{|t-s|^{\beta_1}}\, ds\, dt\, d\kappa &\le C y^{(\zeta+\lambda)^{(-)}-2\beta_1+2} \iint a(s, \zeta_\kappa)^{1^{(-)}}\, ds\, d\kappa, \\ \mathbb{E} \iiint \frac{|A_{21}(s; \kappa, \tilde{\kappa})|^{\lambda}}{|\kappa - \tilde{\kappa}|^{\beta_2}}\, ds\, d\kappa\, d\tilde{\kappa} &\le C y^{(\zeta+\lambda)(p-\beta_2+1)/p} \iint a(s, \zeta_\kappa)\, ds\, d\kappa, \\ \mathbb{E} \iiint \frac{|A_{22}(s; \kappa, \tilde{\kappa})|^{p}}{|\kappa - \tilde{\kappa}|^{\beta_2}}\, ds\, d\kappa\, d\tilde{\kappa} &\le C y^{(\zeta+\lambda)(p-\beta_2+1)/p}, \end{aligned}$$

where C *depends on* κ_-, κ_+, λ, p, β_1, β_2.

Proof. These follow from direct computations making use of Lemma 4.3.1 and Corollary 4.3.7. They can be found in the appendix of the arXiv version of this paper. □

Recall that the condition for Lemma 4.2.1 is $(\beta_1-2)(\beta_2-2)-1>0$. With $\beta_1 < \frac{\lambda+\zeta}{2}+1$, $\beta_2 < p+1$ this is again the condition $(\frac{\zeta+\lambda}{2})^{-1}+p^{-1}<1$, which leads to $\kappa < \frac{8}{3}$. Moreover, we need the additional condition $\frac{\beta_1-2}{\lambda} < 1/2^{(-)}$ for Lemma 4.2.13, which is implied by $\zeta < 2$.

The same analysis of λ and ζ as in the proof of Theorem 4.3.2 applies here. This finishes the proof of Theorem 4.4.1. □

4.5 Proof of Proposition 4.3.5

The proof is based on the methods of [Law09; JRW14].

Let $t \geq 0$ and $U \in C([0,t];\mathbb{R})$. We study the chordal Loewner chain $(g_s)_{s\in[0,t]}$ in $\mathbb{H}$ driven by U, i.e. the solution of (4.15). Let $V(s) = U(t-s)-U(t)$, $s \in [0,t]$, and consider the solution of the reverse flow

$$\partial_s h_s(z) = \frac{-2}{h_s(z)-V(s)}, \quad h_0(z) = z. \tag{4.21}$$

The Loewner equation implies $h_t(z) = g_t^{-1}(z+U(t))-U(t) = \hat{f}_t(z)-U(t)$.

Let $x_s + iy_s = z_s = z_s(z) = h_s(z)-V(s)$. Recall that

$$\partial_s \log |h_s'(z)| = 2\frac{x_s^2-y_s^2}{(x_s^2+y_s^2)^2}$$

and therefore

$$|h_s'(z)| = \exp\left(2\int_0^s \frac{x_\vartheta^2-y_\vartheta^2}{(x_\vartheta^2+y_\vartheta^2)^2}\,d\vartheta\right).$$

For $r\in[0,t]$, denote by $h_{r,s}$ the reverse Loewner flow driven by $V(s)-V(r)$, $s\in[r,t]$. More specifically,

$$\begin{aligned}\partial_s(h_{r,s}(z_r(z))+V(r)) &= \frac{-2}{(h_{r,s}(z_r(z))+V(r))-V(s)},\\ h_{r,r}(z_r(z))+V(r) &= z_r(z)+V(r) = h_r(z),\end{aligned}$$

which implies from (4.21) that

$$\begin{aligned}&h_{r,s}(z_r(z))+V(r) = h_s(z)\\ \text{and}\qquad & z_{r,s}(z_r(z)) = z_s(z) \quad \text{for all } s\in[r,t].\end{aligned}$$

This implies also

$$|h_{r,s}'(z_r(z))| = \exp\left(2\int_r^s \frac{x_\vartheta^2-y_\vartheta^2}{(x_\vartheta^2+y_\vartheta^2)^2}\,d\vartheta\right).$$

The following result is essentially [JRW14, Lemma 2.3], stated in a more refined way.

Lemma 4.5.1. *Let $V^1, V^2 \in C([0,t];\mathbb{R})$, and denote by (h^j_s) the reverse Loewner flow driven by V^j, $j=1,2$, respectively. For $z = x+iy$, denoting $x^j_s + iy^j_s = z^j_s = h^j_s(z) - V^j(s)$, we have*

$$
\begin{aligned}
&|h^1_t(z) - h^2_t(z)| \\
&\qquad \le 2(y^2+4t)^{1/4} \int_0^t |V^1(s) - V^2(s)| \frac{1}{|z^1_s z^2_s|} \frac{1}{(y^1_s y^2_s)^{1/4}} |(h^1_{s,t})'(z^1_s)(h^2_{s,t})'(z^2_s)|^{1/4}\, ds.
\end{aligned}
$$

Proof. The proof of [JRW14, Lemma 2.3] shows that

$$
\begin{aligned}
&|h^1_t(z) - h^2_t(z)| \\
&\qquad \le \int_0^t |V^1(s) - V^2(s)| \frac{2}{|z^1_s z^2_s|} \exp\left(2\int_s^t \frac{x^1_\vartheta x^2_\vartheta - y^1_\vartheta y^2_\vartheta}{((x^1_\vartheta)^2 + (y^1_\vartheta)^2)((x^2_\vartheta)^2 + (y^2_\vartheta)^2)}\, d\vartheta\right) ds.
\end{aligned}
$$

The claim follows by estimating

$$
\begin{aligned}
&2\int_s^t \frac{x^1_\vartheta x^2_\vartheta - y^1_\vartheta y^2_\vartheta}{((x^1_\vartheta)^2 + (y^1_\vartheta)^2)((x^2_\vartheta)^2 + (y^2_\vartheta)^2)}\, d\vartheta \\
&\quad \le 2\int_s^t \frac{x^1_\vartheta x^2_\vartheta}{((x^1_\vartheta)^2 + (y^1_\vartheta)^2)((x^2_\vartheta)^2 + (y^2_\vartheta)^2)}\, d\vartheta \\
&\quad \le \prod_{j=1,2} \left(2\int_s^t \frac{(x^j_\vartheta)^2}{((x^j_\vartheta)^2 + (y^j_\vartheta)^2)^2}\, d\vartheta\right)^{1/2} \\
&\quad = \prod_{j=1,2} \left(\frac{1}{2}\int_s^t \frac{2((x^j_\vartheta)^2 - (y^j_\vartheta)^2)}{((x^j_\vartheta)^2 + (y^j_\vartheta)^2)^2}\, d\vartheta + \frac{1}{2}\int_s^t \frac{2}{(x^j_\vartheta)^2 + (y^j_\vartheta)^2}\, d\vartheta\right)^{1/2} \\
&\quad = \prod_{j=1,2} \left(\frac{1}{2}\log|(h^j_{s,t})'(z^j_s)| + \frac{1}{2}\log\frac{y^j_t}{y^j_s}\right)^{1/2} \\
&\quad \le \sum_{j=1,2} \left(\frac{1}{4}\log|(h^j_{s,t})'(z^j_s)| + \frac{1}{4}\log\frac{y^j_t}{y^j_s}\right)
\end{aligned}
$$

and $y^j_t \le \sqrt{y^2+4t}$. (In the last line we used $\sqrt{ab} \le \frac{a+b}{2}$ for $a, b \ge 0$.) □

4.5.1 Taking moments

Let $\kappa, \tilde\kappa > 0$, and let $V^1 = \sqrt{\kappa}B$, $V^2 = \sqrt{\tilde\kappa}B$, where B is a standard Brownian motion. In the following, C will always denote a finite deterministic constant that might change from line to line.

Lemma 4.5.1 and the Cauchy-Schwarz inequality imply

$$\begin{aligned}
&\mathbb{E}|h_t^1(z) - h_t^2(z)|^p \\
&\quad \le C|\Delta\sqrt{\kappa}|^p\,\mathbb{E}\left|\int_0^t |B_s|\frac{1}{|z_s^1 z_s^2|}\frac{1}{(y_s^1 y_s^2)^{1/4}}|(h_{s,t}^1)'(z_s^1)(h_{s,t}^2)'(z_s^2)|^{1/4}\,ds\right|^p \\
&\quad \le C|\Delta\sqrt{\kappa}|^p\,\mathbb{E}\prod_{j=1,2}\left|\int_0^t |B_s|\frac{1}{|z_s^j|^2}\frac{1}{(y_s^j)^{1/2}}|(h_{s,t}^j)'(z_s^j)|^{1/2}\,ds\right|^{p/2} \\
&\quad \le C|\Delta\sqrt{\kappa}|^p\prod_{j=1,2}\left(\mathbb{E}\left|\int_0^t |B_s|\frac{1}{|z_s^j|^2}\frac{1}{(y_s^j)^{1/2}}|(h_{s,t}^j)'(z_s^j)|^{1/2}\,ds\right|^p\right)^{1/2}.
\end{aligned} \tag{4.22}$$

Now the flows for κ and $\tilde{\kappa}$ can be studied separately. We see that as long as the above integral is bounded, then $\mathbb{E}|\Delta_{\sqrt{\kappa}}h_t^\kappa(z)|^p \lesssim |\Delta\sqrt{\kappa}|^p$. Heuristically, the typical growth of y_s is like $\sqrt{s}$, as was shown in [Law09]. Therefore, we expect the integrand to be bounded by $s^{1/2-1-1/4-\beta/4} = s^{-(3+\beta)/4}$ which is integrable since $\beta = \beta(\kappa) < 1$ for $\kappa \neq 8$.

In order to make the idea precise, we will reparametrise the integral in order to match the setting in [Law09] and apply their results.

4.5.2 Reparametrisation

Let $\kappa > 0$. In [Law09], the flow

$$\partial_s \tilde{h}_s(z) = \frac{-a}{\tilde{h}_s(z) - \tilde{B}_s}, \qquad \tilde{h}_0(z) = z, \tag{4.23}$$

with $a = \dfrac{2}{\kappa}$ is considered. To translate our notation, observe that

$$\partial_s h_{s/\kappa}(z) = \frac{-2/\kappa}{h_{s/\kappa}(z) - \sqrt{\kappa}B_{s/\kappa}}.$$

If we let $\tilde{B}_s = \sqrt{\kappa}B_{s/\kappa}$, then

$$h_{s/\kappa}(z) = \tilde{h}_s(z) \implies h_s(z) = \tilde{h}_{\kappa s}(z).$$

Moreover, if we let $\tilde{z}_s = \tilde{h}_s(z) - \tilde{B}_s$, then $z_s = h_s(z) - \sqrt{\kappa}B_s = \tilde{z}_{\kappa s}$.

Therefore,

$$\begin{aligned}
\int_0^t |B_s|\frac{1}{|z_s|^2}\frac{1}{y_s^{1/2}}|h_{s,t}'(z_s)|^{1/2}\,ds &= \int_0^t \left|\frac{1}{\sqrt{\kappa}}\tilde{B}_{\kappa s}\right|\frac{1}{|\tilde{z}_{\kappa s}|^2}\frac{1}{\tilde{y}_{\kappa s}^{1/2}}|\tilde{h}_{\kappa s,\kappa t}'(\tilde{z}_{\kappa s})|^{1/2}\,ds \\
&= \int_0^{\kappa t} \kappa^{-3/2}|\tilde{B}_s|\frac{1}{|\tilde{z}_s|^2}\frac{1}{\tilde{y}_s^{1/2}}|\tilde{h}_{s,\kappa t}'(\tilde{z}_s)|^{1/2}\,ds.
\end{aligned}$$

For notational simplicity, we will write just t instead of κt and B, h_s, z_s instead of $\tilde{B}, \tilde{h}_s, \tilde{z}_s$.

In the next step, we will let the flow start at $z_0 = i$ instead of $i\delta$. Observe that

$$\partial_s(\delta^{-1}h_{\delta^2 s}(\delta z)) = \frac{-a}{\delta^{-1}h_{\delta^2 s}(\delta z) - \delta^{-1}B_{\delta^2 s}},$$

so we can write $h_s(\delta z) = \delta\tilde{h}_{s/\delta^2}(z)$ where $(\tilde{h}_s)$ is driven by $\delta^{-1}B_{\delta^2 s} =: \tilde{B}_s$. Note that $\tilde{h}'_{s/\delta^2}(z) = h'_s(\delta z)$. As before, we denote $z_s = h_s(\delta z) - B_s$ and $\tilde{z}_s = \tilde{h}_s(z) - \tilde{B}_s$, where $z_s = \delta\tilde{z}_{s/\delta^2}$. Consequently,

$$\begin{aligned}
&\int_0^t |B_s| \frac{1}{|z_s|^2} \frac{1}{y_s^{1/2}} |h'_{s,t}(z_s)|^{1/2}\, ds \\
&= \int_0^t |\delta\tilde{B}_{s/\delta^2}| \frac{1}{\delta^2|\tilde{z}_{s/\delta^2}|^2} \frac{1}{\delta^{1/2}\tilde{y}_{s/\delta^2}^{1/2}} |\tilde{h}'_{s/\delta^2,t/\delta^2}(\tilde{z}_{s/\delta^2})|^{1/2}\, ds \\
&= \delta^{-3/2} \int_0^t |\tilde{B}_{s/\delta^2}| \frac{1}{|\tilde{z}_{s/\delta^2}|^2} \frac{1}{\tilde{y}_{s/\delta^2}^{1/2}} |\tilde{h}'_{s/\delta^2,t/\delta^2}(\tilde{z}_{s/\delta^2})|^{1/2}\, ds \\
&= \delta^{1/2} \int_0^{t/\delta^2} |\tilde{B}_s| \frac{1}{|\tilde{z}_s|^2} \frac{1}{\tilde{y}_s^{1/2}} |\tilde{h}'_{s,t/\delta^2}(\tilde{z}_s)|^{1/2}\, ds.
\end{aligned}$$

Again, for notational simplicity we will stop writing the ˜ from now on.
Now, let $z_0 = i$, and (cf. [Law09])

$$\sigma(s) = \inf\{r \mid y_r = e^{ar}\} = \int_0^s |z_{\sigma(r)}|^2\, dr$$

which is random and strictly increasing in s.

Then

$$\begin{aligned}
&\delta^{1/2} \int_0^{t/\delta^2} |B_s| \frac{1}{|z_s|^2} \frac{1}{y_s^{1/2}} |h'_{s,t/\delta^2}(z_s)|^{1/2}\, ds \\
&\qquad\qquad = \delta^{1/2} \int_0^{\sigma^{-1}(t/\delta^2)} |B_{\sigma(s)}| \frac{1}{y_{\sigma(s)}^{1/2}} |h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{1/2}\, ds.
\end{aligned}$$

This is the integral we will work with.

To sum it up, we have the following.

Proposition 4.5.2. *Let $z \in \mathbb{H}$, and $(h_s(\delta z))_{s\geq 0}$ satisfy (4.21) with $V(s) = \sqrt{\kappa}B_s$ and a standard Brownian motion B, and $(\tilde{h}_s(z))_{s\geq 0}$ satisfy (4.23) with a standard Brownian motion $\tilde{B}$. Let $x_s + iy_s = z_s = h_s(\delta z) - V(s)$, and $\tilde{x}_s + i\tilde{y}_s = \tilde{z}_s = \tilde{h}_s(z) - \tilde{B}_s$. Then, with the notations above,*

$$\int_0^t |B_s| \frac{1}{|z_s|^2} \frac{1}{y_s^{1/2}} |h'_{s,t}(z_s)|^{1/2}\, ds$$

has the same law as

$$\kappa^{-3/2}\delta^{1/2}\int_0^{\sigma^{-1}(\kappa t/\delta^2)} |\tilde{B}_{\sigma(s)}|\frac{1}{\tilde{y}_{\sigma(s)}^{1/2}}|\tilde{h}'_{\sigma(s),\kappa t/\delta^2}(\tilde{z}_{\sigma(s)})|^{1/2}\,ds.$$

(Recall that $\tilde{y}_{\sigma(s)} = e^{as}$.)

4.5.3 Main proof

In the following, we fix $\kappa \in [\kappa_-, \kappa_+]$, $a = \dfrac{2}{\kappa}$, and let $(h_s(x+i))_{s\geq 0}$ satisfy (4.23) with initial point $z_0 = x+i$, $|x| \leq 1$.

Our goal is to estimate

$$\mathbb{E}\left|\delta^{1/2}\int_0^{\sigma^{-1}(t/\delta^2)} |B_{\sigma(s)}|\frac{1}{y_{\sigma(s)}^{1/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{1/2}\,ds\right|^p$$
$$= \mathbb{E}\left|\delta^{1/2}\int_0^{\infty} 1_{\sigma(s)\leq t/\delta^2}|B_{\sigma(s)}|\frac{1}{y_{\sigma(s)}^{1/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{1/2}\,ds\right|^p.$$

With (4.22) and Proposition 4.5.2 this will complete the proof of Proposition 4.3.5.

From the definition of σ it follows that $\sigma(s) \geq \int_0^s e^{2ar}\,dr = \frac{1}{2a}(e^{2as}-1)$, or equivalently, $\sigma^{-1}(t) \leq \frac{1}{2a}\log(1+2at)$. Therefore, $\sigma^{-1}(t/\delta^2) \leq \frac{1}{a}\log\frac{C}{\delta}$ and

$$\mathbb{E}\left|\delta^{1/2}\int_0^{\sigma^{-1}(t/\delta^2)} |B_{\sigma(s)}|\frac{1}{y_{\sigma(s)}^{1/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{1/2}\,ds\right|^p$$
$$\leq \delta^{p/2}\left(\int_0^{\frac{1}{a}\log\frac{C}{\delta}}\left(\mathbb{E}\left[1_{\sigma(s)\leq t/\delta^2}|B_{\sigma(s)}|^p\frac{1}{y_{\sigma(s)}^{p/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2}\right]\right)^{1/p}ds\right)^p \tag{4.24}$$

where we have applied Minkowski's inequality to pull the moment inside the integral.

To proceed, we need to know more about the behaviour of the reverse SLE flow, which also incorporates the behaviour of σ. This has been studied in [Law09]. Their tool was to study the process J_s defined by $\sinh J_s = \frac{x_{\sigma(s)}}{y_{\sigma(s)}} = e^{-as}x_{\sigma(s)}$. By [Law09, Lemma 6.1], this process satisfies

$$dJ_s = -r_c\tanh J_s\,ds + dW_s$$

where $W_s = \int_0^{\sigma(s)}\frac{1}{|z_r|}\,dB_r$ is a standard Brownian motion and r_c is defined in (4.17).

The following results have been originally stated for an equivalent probability measure $\mathbb{P}_*$, depending on a parameter r, such that

$$dJ_s = -q\tanh J_s\,ds + dW_s^*$$

with $q > 0$ and a process W^* that is a Brownian motion under $\mathbb{P}_*$. But setting the parameter $r = 0$, we have $\mathbb{P}_* = \mathbb{P}$, $q = r_c$, and $W^* = W$. Therefore, under the measure $\mathbb{P}$, the results apply with $q = r_c$.

Note also that although the results were originally stated for a reverse SLE flow starting at $z_0 = i$, they can be written for flows starting at $z_0 = x + i$ without change of the proof. One just uses [Law09, Lemma 7.1 (28)] with $\cosh J_0 = \sqrt{1 + x^2}$.

Recall that [Law09; JL11] use the notation $\sinh J_s = \frac{x_{\sigma(s)}}{y_{\sigma(s)}}$ and hence $\cosh^2 J_s = 1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}}$.

Lemma 4.5.3 ([JL11, Lemma 5.6]). *Suppose $z_0 = x+i$. There exists a constant $C < \infty$, depending on κ_-, κ_+, such that for each $s \geq 0$, $u > 0$ there exists an event $E_{u,s}$ with*

$$\mathbb{P}(E^c_{s,u}) \leq C(1 + x^2)^{r_c} u^{-2r_c}$$

on which

$$\sigma(s) \leq u^2 e^{2as} \quad \text{and} \quad 1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} \leq u^2/4.$$

Fix $s \in [0, t]$. Let

$$E_u = \left\{ \sigma(s) \leq u^2 e^{2as} \text{ and } 1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} \leq u^2 \right\}$$

and $A_n = E_{\exp(n)} \setminus E_{\exp(n-1)}$ for $n \geq 1$, and $A_0 = E_1$. Then

$$\mathbb{P}(A_n) \leq \mathbb{P}(E^c_{\exp(n-1)}) \leq C(1 + x^2)^{r_c} e^{-2r_c n}. \tag{4.25}$$

(The constant C may change from line to line.)

Lemma 4.5.4 (see proof of [JL11, Lemma 5.7]). *Suppose $z_0 = x+i$. There exists $C < \infty$, depending on κ_-, and a global constant $\alpha > 0$, such that for all $s \geq 0$, $u > \sqrt{1 + x^2}$, and $k > 2a$ we have*

$$\mathbb{P}\left(\sigma(s) \leq u^2 e^{2as} \text{ and } 1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} \geq u^2 e^k \right) \leq C(1 + x^2)^{r_c} u^{-2r_c} e^{-\alpha(k-2a)^2}.$$

We proceed to estimating

$$\begin{aligned} &\mathbb{E}\left[1_{A_n} 1_{\sigma(s) \leq t/\delta^2} |B_{\sigma(s)}|^p \frac{1}{y^{p/2}_{\sigma(s)}} |h'_{\sigma(s), t/\delta^2}(z_{\sigma(s)})|^{p/2} \right] \\ &\qquad = \mathbb{E}\left[1_{A_n} 1_{\sigma(s) \leq t/\delta^2} |B_{\sigma(s)}|^p \frac{1}{y^{p/2}_{\sigma(s)}} \mathbb{E}\left[|h'_{\sigma(s), t/\delta^2}(z_{\sigma(s)})|^{p/2} \mid \mathcal{F}_{\sigma(s)} \right] \right] \end{aligned} \tag{4.26}$$

where $\mathcal{F}$ is the filtration generated by B.

Note that $y_{\sigma(s)} = e^{as}$ by the definition of σ. Moreover, on the set A_n, the Brownian motion is easy to handle since by Hölder's inequality

$$\begin{aligned}\mathbb{E}[1_{A_n} 1_{\sigma(s)\le t/\delta^2} |B_{\sigma(s)}|^p] &\le \mathbb{E}\left[1_{A_n} 1_{\sigma(s)\le t/\delta^2} \sup_{r\in[0,e^{2n}e^{2as}]} |B_r|^p\right] \\ &\le \mathbb{P}(A_n \cap \{\sigma(s) \le t/\delta^2\})^{1-\varepsilon}\, \mathbb{E}\left[\sup_{r\in[0,e^{2n}e^{2as}]} |B_r|^{p/\varepsilon}\right]^{\varepsilon} \\ &\le C\, \mathbb{P}(A_n \cap \{\sigma(s) \le t/\delta^2\})^{1-\varepsilon}\, e^{np} e^{pas} \end{aligned} \tag{4.27}$$

for any $\varepsilon > 0$.

It remains to handle $\mathbb{E}\left[|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2} \mid \mathcal{F}_{\sigma(s)}\right]$.

The following result is well-known and follows from the Schwarz lemma and mapping the unit disc to the half-plane.

Lemma 4.5.5. *Let $f : \mathbb{H} \to \mathbb{H}$ be a holomorphic function. Then $|f'(z)| \le \frac{\operatorname{Im}(f(z))}{\operatorname{Im}(z)}$ for all $z \in \mathbb{H}$.*

Recall that the Loewner equation implies

$$\operatorname{Im}(h_{\sigma(s),t/\delta^2}(z_{\sigma(s)})) = y_{t/\delta^2} \le \sqrt{1 + 2at/\delta^2} \le C\delta^{-1}.$$

Let $\varepsilon > 0$. By the lemma above, we can estimate

$$\begin{aligned}&\mathbb{E}\left[|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2} \mid \mathcal{F}_{\sigma(s)}\right] \\ &\qquad \le (\delta y_{\sigma(s)})^{-(1-\varepsilon)p/2} \mathbb{E}\left[|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{\varepsilon p/2} \mid \mathcal{F}_{\sigma(s)}\right].\end{aligned} \tag{4.28}$$

From [JL11, Lemma 3.2] it follows that there exists some $l > 0$ such that

$$|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})| \le C \left(1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}}\right)^l |h'_{\sigma(s),t/\delta^2}(iy_{\sigma(s)})|. \tag{4.29}$$

We claim that

$$\mathbb{E}\left[|h'_{\sigma(s),t/\delta^2}(iy_{\sigma(s)})|^{\varepsilon p/2} \mid \mathcal{F}_{\sigma(s)}\right] \le C \tag{4.30}$$

if $\varepsilon > 0$ is sufficiently small.

To see this, first recall that for small $\varepsilon > 0$ we have

$$\mathbb{E}\left[|h'_t(i)|^{\varepsilon}\right] \le C \tag{4.31}$$

uniformly in $t \ge 1$. This follows from [JL11, Theorem 5.4] or, even more elementary, from the proof of [RS05, Theorem 3.2].

Now approximate $\sigma(s)$ by simple stopping times $\tilde{\sigma} \geq \sigma(s)$. A possible choice is $\tilde{\sigma} = \lceil \sigma(s) 2^n \rceil 2^{-n} \wedge t/\delta^2$. It suffices to show

$$\mathbb{E}\left[|h'_{\tilde{\sigma},t/\delta^2}(iy_{\sigma(s)})|^{\varepsilon p/2} \mid \mathcal{F}_{\sigma(s)} \right] \leq C$$

and then apply Fatou's lemma to pass to the limit.

Now that $\tilde{\sigma}$ is simple, we can apply (4.31) on each set $F_r = \{\tilde{\sigma} = r\}$. Using the strong Markov property of Brownian motion and the scaling invariance of SLE, we get

$$\begin{aligned}
\mathbb{E}\left[1_{F_r} |h'_{\tilde{\sigma},t/\delta^2}(ie^{as})|^{\varepsilon p/2} \mid \mathcal{F}_{\sigma(s)} \right] &= 1_{F_r} \mathbb{E}\left[|h'_{r,t/\delta^2}(ie^{as})|^{\varepsilon p/2} \right] \\
&= 1_{F_r} \mathbb{E}\left[|h'_{e^{-2as}(t/\delta^2 - r)}(i)|^{\varepsilon p/2} \right] \\
&\leq 1_{F_r} C
\end{aligned}$$

and the claim follows.

Combining (4.28), (4.29), and (4.30), we have

$$\begin{aligned}
\mathbb{E}\left[|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2} \mid \mathcal{F}_{\sigma(s)} \right] &\leq C\, \delta^{-(1-\varepsilon)p/2}\, y_{\sigma(s)}^{-(1-\varepsilon)p/2} \left(1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} \right)^{l\varepsilon p/2} \\
&\leq C\, \delta^{-(1-\varepsilon)p/2}\, e^{-(1-\varepsilon)pas/2} \left(1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} \right)^{l\varepsilon p/2} \qquad (4.32)
\end{aligned}$$

where on the set A_n we have

$$1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} \leq e^{2n}.$$

Proceeding from (4.26), we get from (4.32) and (4.27)

$$\begin{aligned}
&\mathbb{E}\left[1_{A_n} 1_{\sigma(s) \leq t/\delta^2} |B_{\sigma(s)}|^p \frac{1}{y^{p/2}_{\sigma(s)}} \mathbb{E}\left[|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2} \mid \mathcal{F}_{\sigma(s)} \right] \right] \\
&\leq C\, \mathbb{E}\left[1_{A_n} 1_{\sigma(s) \leq t/\delta^2}\, |B_{\sigma(s)}|^p\, e^{-pas/2}\, \delta^{-(1-\varepsilon)p/2}\, e^{-(1-\varepsilon)pas/2} e^{nl\varepsilon p} \right] \\
&\leq C\, \delta^{-(1-\varepsilon)p/2}\, e^{nl\varepsilon p}\, e^{-pas + \varepsilon pas/2}\, \mathbb{P}(A_n \cap \{\sigma(s) \leq t/\delta^2\})^{1-\varepsilon}\, e^{np} e^{pas} \\
&= C\, \delta^{-(1-\varepsilon)p/2}\, e^{np + nl\varepsilon p}\, e^{\varepsilon pas/2}\, \mathbb{P}(A_n \cap \{\sigma(s) \leq t/\delta^2\})^{1-\varepsilon}. \qquad (4.33)
\end{aligned}$$

We would like to sum this expression in n.

Proposition 4.5.6. *Let $\sigma(s)$ and A_n be defined as above. Then*

$$\sum_{n \in \mathbb{N}} e^{np + nl\varepsilon p}\, \mathbb{P}(A_n \cap \{\sigma(s) \leq t/\delta^2\})^{1-\varepsilon} \leq \begin{cases} C & \text{if } p + l\varepsilon p - 2r_c(1-\varepsilon) < 0 \\ C(e^{-as}\sqrt{t}/\delta)^{p + l\varepsilon p - 2r_c(1-\varepsilon)} & \text{if } p + l\varepsilon p - 2r_c(1-\varepsilon) > 0 \end{cases}$$

where $C < \infty$ depends on κ_-, κ_+, p, and ε.

Proof. We distinguish two cases. If $n \le \log(\sqrt{t}/\delta) - as + 1 + a$, we have (by (4.25))

$$\begin{aligned}
&\sum_{n \le \log(\sqrt{t}/\delta)-as+1+a} e^{np+nl\varepsilon p}\, \mathbb{P}(A_n)^{1-\varepsilon} \\
&\quad \le C \sum_{n \le \log(\sqrt{t}/\delta)-as+1+a} e^{np+nl\varepsilon p} e^{-2nr_c(1-\varepsilon)} \\
&\quad \le \begin{cases} C & \text{if } p + l\varepsilon p - 2r_c(1-\varepsilon) < 0 \\ C(e^{-as}\sqrt{t}/\delta)^{p+l\varepsilon p-2r_c(1-\varepsilon)} & \text{if } p + l\varepsilon p - 2r_c(1-\varepsilon) > 0. \end{cases}
\end{aligned}$$

For $n > \log(\sqrt{t}/\delta) - as + 1 + a$, we have $e^{2(n-1)}e^{2as} > t/\delta^2$ and therefore (by the definition of A_n)

$$\begin{aligned}
A_n \cap \{\sigma(s) \le t/\delta^2\} &\subseteq E^c_{e^{n-1}} \cap \{\sigma(s) \le t/\delta^2\} \\
&\subseteq \left\{ \sigma(s) \le t/\delta^2 \text{ and } 1 + \frac{x^2_{\sigma(s)}}{y^2_{\sigma(s)}} > e^{2(n-1)} \right\},
\end{aligned}$$

so Lemma 4.5.4, applied to $u = e^{-as}\sqrt{t}/\delta$ and $k = 2(n-1) - 2(\log(\sqrt{t}/\delta) - as)$, implies

$$\begin{aligned}
\mathbb{P}(A_n \cap \{\sigma(s) \le t/\delta^2\}) &\le C\,(e^{-as}\sqrt{t}/\delta)^{-2r_c}\, e^{-\alpha(2(n-1)-2(\log(\sqrt{t}/\delta)-as)-2a)^2} \\
&= C\,(e^{-as}\sqrt{t}/\delta)^{-2r_c}\, e^{-2\alpha(n-(\log(\sqrt{t}/\delta)-as+1+a))^2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{n > \log(\sqrt{t}/\delta)-as+1+a} e^{np+nl\varepsilon p}\, \mathbb{P}(A_n \cap \{\sigma(s) \le t/\delta^2\})^{1-\varepsilon} \\
&\quad \le C(e^{-as}\sqrt{t}/\delta)^{p+l\varepsilon p} \sum_{n\in\mathbb{N}} e^{np+nl\varepsilon p}\,(e^{-as}\sqrt{t}/\delta)^{-2r_c(1-\varepsilon)}\, e^{-2\alpha(1-\varepsilon)n^2} \\
&\quad \le C(e^{-as}\sqrt{t}/\delta)^{p+l\varepsilon p-2r_c(1-\varepsilon)}.
\end{aligned}$$

□

Hence, by (4.33) and Proposition 4.5.6,

$$\begin{aligned}
&\mathbb{E}\left[1_{\sigma(s)\le t/\delta^2} |B_{\sigma(s)}|^p \frac{1}{y^{p/2}_{\sigma(s)}} |h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2} \right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[1_{A_n} 1_{\sigma(s)\le t/\delta^2} |B_{\sigma(s)}|^p \frac{1}{y^{p/2}_{\sigma(s)}} |h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2} \right] \\
&\le \begin{cases} C\,\delta^{-(1-\varepsilon)p/2}\, e^{\varepsilon pas/2} & \text{if } p + l\varepsilon p - 2r_c(1-\varepsilon) < 0 \\ C\,\delta^{-(1-\varepsilon)p/2}\,(e^{-as}\sqrt{t}/\delta)^{p+l\varepsilon p-2r_c(1-\varepsilon)}\, e^{\varepsilon pas/2} & \text{if } p + l\varepsilon p - 2r_c(1-\varepsilon) > 0. \end{cases}
\end{aligned} \tag{4.34}$$

Finally, if $p + l\varepsilon p - 2r_c(1-\varepsilon) < 0$, we estimate (4.24) with (4.34), so

$$\begin{aligned}
&\mathbb{E}\left|\delta^{1/2}\int_0^{\sigma^{-1}(t/\delta^2)} |B_{\sigma(s)}|\frac{1}{y_{\sigma(s)}^{1/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{1/2}\,ds\right|^p \\
&\leq \delta^{p/2}\left(\int_0^{\frac{1}{a}\log\frac{C}{\delta}}\left(\mathbb{E}\left[1_{\sigma(s)\leq t/\delta^2}|B_{\sigma(s)}|^p\frac{1}{y_{\sigma(s)}^{p/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{p/2}\right]\right)^{1/p} ds\right)^p \\
&\leq C\delta^{p/2}\left(\int_0^{\frac{1}{a}\log\frac{C}{\delta}}\left(\delta^{-(1-\varepsilon)p/2}\,e^{\varepsilon pas/2}\right)^{1/p} ds\right)^p \\
&= C\delta^{\varepsilon p/2}\left(\int_0^{\frac{1}{a}\log\frac{C}{\delta}} e^{\varepsilon as/2}\,ds\right)^p \\
&\leq C.
\end{aligned}$$

Since $\varepsilon > 0$ can be chosen as small as we want, the condition to apply this is $p < 2r_c = 1 + \frac{8}{\kappa}$.

On the other hand, if $p + l\varepsilon p - 2r_c(1-\varepsilon) > 0$, we have

$$\begin{aligned}
&\mathbb{E}\left|\delta^{1/2}\int_0^{\sigma^{-1}(t/\delta^2)} |B_{\sigma(s)}|\frac{1}{y_{\sigma(s)}^{1/2}}|h'_{\sigma(s),t/\delta^2}(z_{\sigma(s)})|^{1/2}\,ds\right|^p \\
&\leq C\delta^{p/2}\left(\int_0^{\frac{1}{a}\log\frac{C}{\delta}}\left(\delta^{-(1-\varepsilon)p/2}\,(e^{-as}\sqrt{t}/\delta)^{p+l\varepsilon p-2r_c(1-\varepsilon)}\,e^{\varepsilon pas/2}\right)^{1/p} ds\right)^p \\
&\leq C\delta^{\varepsilon p/2-(p+l\varepsilon p-2r_c(1-\varepsilon))}\left(\int_0^{\frac{1}{a}\log\frac{C}{\delta}} e^{as(\varepsilon/2-(1+l\varepsilon-2r_c(1-\varepsilon)/p))}\,ds\right)^p \\
&\leq \begin{cases} C & \text{if } \varepsilon/2-(1+l\varepsilon-2r_c(1-\varepsilon)/p) > 0 \\ C\delta^{\varepsilon p/2-(p+l\varepsilon p-2r_c(1-\varepsilon))} & \text{if } \varepsilon/2-(1+l\varepsilon-2r_c(1-\varepsilon)/p) < 0 \end{cases} \\
&= \begin{cases} C & \text{if } 2r_c(1-\varepsilon)-p(1+\varepsilon(l-1/2)) > 0 \\ C\delta^{2r_c(1-\varepsilon)-p(1+\varepsilon(l-1/2))} & \text{if } 2r_c(1-\varepsilon)-p(1+\varepsilon(l-1/2)) < 0. \end{cases}
\end{aligned}$$

Since $\varepsilon > 0$ can be chosen as small as we want, the condition to apply this is $p > 2r_c = 1 + \frac{8}{\kappa}$, and the exponent can be chosen to be greater than $2r_c - p - \varepsilon'$ for any $\varepsilon' > 0$.

With this estimate for (4.24), the proof of Proposition 4.3.5 is complete.

4.6 Appendix: Proof of Proposition 4.4.6

We begin with estimating the expressions for A_{1j} which involve the time difference, and then estimate the expressions for A_{2j} which involve the κ difference.

The Δt term

For this part, we again suppress writing κ, although all expressions depend on a parameter κ.

The moment estimates are all similar. In A_{11}, we will encounter the expression $\mathbb{E}|\hat{f}'_t(iu) - \hat{f}'_s(iu)|^\lambda$ which we estimate by $\mathbb{E}|\hat{f}'_t(iu) - \hat{f}'_s(iu)|^\lambda \lesssim (a(s) + a(t))u^\zeta$ with Lemma 4.3.1 (which is sufficient since $|t-s| \geq u^2$). Together with Minkowski's inequality, we have

$$\begin{aligned}
\mathbb{E}|A_{11}(t,s;\kappa)|^\lambda &\leq \left(\int_0^{y \wedge |t-s|^{1/2}} \left(\mathbb{E}|\hat{f}'_t(iu) - \hat{f}'_s(iu)|^\lambda \right)^{1/\lambda} du \right)^\lambda \\
&\lesssim \left(\int_0^{y \wedge |t-s|^{1/2}} (a(s) + a(t))^{1/\lambda} u^{\zeta/\lambda} \, du \right)^\lambda \\
&\lesssim (a(s) + a(t)) \left(y \wedge |t-s|^{1/2} \right)^{\zeta+\lambda},
\end{aligned}$$

assuming $\zeta + \lambda > 0$. Consequently,

$$\begin{aligned}
\mathbb{E} \iint \frac{|A_{11}(t,s;\kappa)|^\lambda}{|t-s|^{\beta_1}} \, ds \, dt &\lesssim \iint_{|t-s| \leq y^2} \frac{(a(s) + a(t))|t-s|^{(\zeta+\lambda)/2}}{|t-s|^{\beta_1}} \, ds \, dt \\
&\quad + \iint_{|t-s| > y^2} \frac{(a(s) + a(t)) y^{\zeta+\lambda}}{|t-s|^{\beta_1}} \, ds \, dt \\
&\lesssim y^{\zeta+\lambda-2\beta_1+2} \int a(t) \, dt,
\end{aligned}$$

assuming $1 < \beta_1 < \frac{\zeta+\lambda}{2} + 1$.

The terms A_{12}, A_{13} only appear when $|t-s|^{1/2} < y$.

For A_{12}, we get (again by Minkowski's inequality and Lemma 4.3.1)

$$\begin{aligned}
\mathbb{E}|A_{12}(t,s;\kappa)|^\lambda &\leq \left(\int_{y \wedge |t-s|^{1/2}}^{y} \frac{|t-s|}{u^2} \left(\mathbb{E}|\hat{f}'_s(iu)|^\lambda \right)^{1/\lambda} du \right)^\lambda \\
&\leq |t-s|^\lambda \left(\int_{y \wedge |t-s|^{1/2}}^{y} a(s)^{1/\lambda} u^{\zeta/\lambda} u^{-2} \, du \right)^\lambda \\
&\lesssim a(s)|t-s|^{(\zeta+\lambda)/2}
\end{aligned}$$

using the fact that $\zeta < \lambda$ (see (4.17)).

Finally, for A_{13}, note that $\|B\|_{C^{1/2(-)}}$ has arbitrarily high moments, so that we can apply Hölder's inequality and get

$$\mathbb{E}\left[\left(|\hat{f}'_s(iu)| \left(1 + \|B\|_{C^{1/2(-)}} \right)^{2l+1} \right)^{\lambda(-)} \right] \lesssim \left(\mathbb{E}|\hat{f}'_s(iu)|^\lambda \right)^{1(-)} \lesssim (a(s)u^\zeta)^{1(-)}$$

Consequently, again by Minkowski's inequality,

$$\begin{aligned}\mathbb{E}|A_{13}(t,s;\kappa)|^{\lambda^{(-)}} &\lesssim \left(\int_{y\wedge|t-s|^{1/2}}^{y\wedge 2|t-s|^{1/2}} u^{-1}(a(s)u^{\zeta})^{1/\lambda^{(-)}}|t-s|^{1/2^{(-)}}\,du\right)^{\lambda^{(-)}}\\ &\lesssim a(s)^{1^{(-)}}|t-s|^{(\zeta+\lambda)^{(-)}/2}.\end{aligned}$$

This shows

$$\begin{aligned}\mathbb{E}\iint \frac{|A_{12}(t,s;\kappa)|^{\lambda}}{|t-s|^{\beta_1}}\,ds\,dt &\lesssim y^{\zeta+\lambda-2\beta_1+2}\int a(t)\,dt,\\ \mathbb{E}\iint \frac{|A_{13}(t,s;\kappa)|^{\lambda^{(-)}}}{|t-s|^{\beta_1}}\,ds\,dt &\lesssim y^{(\zeta+\lambda)^{(-)}-2\beta_1+2}\int a(t)^{1^{(-)}}\,dt\end{aligned}$$

if $\beta_1 < \frac{\zeta+\lambda}{2}+1$.

The $\Delta\kappa$ term

A_{21} will again just be estimated using Lemma 4.3.1 on

$$\mathbb{E}|(\hat{f}_s^{\kappa})'(iu)-(\hat{f}_s^{\tilde\kappa})'(iu)|^{\lambda} \lesssim a(s,\zeta_\kappa)u^{\zeta_\kappa}+a(s,\zeta_{\tilde\kappa})u^{\zeta_{\tilde\kappa}}.$$

Then, by Minkowski's inequality,

$$\begin{aligned}\mathbb{E}|A_{21}(s;\kappa,\tilde\kappa)|^{\lambda} &\le \left(\int_0^{y\wedge|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}}\left(\mathbb{E}|(\hat{f}_s^{\kappa})'(iu)-(\hat{f}_s^{\tilde\kappa})'(iu)|^{\lambda}\right)^{1/\lambda}du\right)^{\lambda}\\ &\lesssim \left(\int_0^{y\wedge|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}} a(s,\zeta_\kappa)^{1/\lambda}u^{\zeta/\lambda}\,du\right)^{\lambda}\\ &\lesssim a(s,\zeta_\kappa)(y^{\zeta+\lambda}\wedge|\kappa-\tilde\kappa|^{p}),\end{aligned}$$

assuming $\zeta+\lambda>0$, and consequently

$$\mathbb{E}\iint \frac{|A_{21}(s;\kappa,\tilde\kappa)|^{\lambda}}{|\kappa-\tilde\kappa|^{\beta_2}}\,d\kappa\,d\tilde\kappa \lesssim y^{(\zeta+\lambda)(p-\beta_2+1)/p}\int a(s,\zeta_\kappa)\,d\kappa,$$

assuming $p-\beta_2+1>0$, i.e. $\beta_2<p+1$.

For A_{22} we apply Corollary 4.3.7 when κ, $\tilde\kappa$ are close to each other, i.e. $|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)} \le y$. This gives us

$$\mathbb{E}|(\hat{f}_s^{\kappa})'(iu)-(\hat{f}_s^{\tilde\kappa})'(iu)|^{p} \lesssim |\kappa-\tilde\kappa|^{p}u^{-p}.$$

In this case Minkowski's inequality does not give us quite the optimal estimate (although it is still sufficient), therefore we do something similar. Let $b\in\mathbb{R}$ be a constant that will be chosen later.

By Hölder's inequality,

$$\begin{aligned}\mathbb{E}|A_{22}(s;\kappa,\tilde\kappa)|^p &= \mathbb{E}\left(\int_{|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}}^{y} |(\hat f_s^{\kappa})'(iu)-(\hat f_s^{\tilde\kappa})'(iu)|\, u^b u^{-b}\,du\right)^p \\ &\lesssim \mathbb{E}\left(\int_{|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}}^{y} |(\hat f_s^{\kappa})'(iu)-(\hat f_s^{\tilde\kappa})'(iu)|^p u^{bp}\,du\right) y^{p-1-bp} \\ &\lesssim \left(\int_{|\kappa-\tilde\kappa|^{p/(\zeta+\lambda)}}^{y} |\kappa-\tilde\kappa|^p u^{-p} u^{bp}\,du\right) y^{p-1-bp} \\ &\lesssim |\kappa-\tilde\kappa|^{p+(-p+bp+1)p/(\zeta+\lambda)} y^{p-1-bp},\end{aligned}$$

assuming $p-1-bp>0$.

Then

$$\mathbb{E}\iint \frac{|A_{22}(s;\kappa,\tilde\kappa)|^p}{|\kappa-\tilde\kappa|^{\beta_2}}\,d\kappa\,d\tilde\kappa \lesssim y^{(\zeta+\lambda)(p-\beta_2+1)/p},$$

assuming $p+(-p+bp+1)p/(\zeta+\lambda)-\beta_2+1>0$.

These estimates work if we can choose b such that $bp\in\,](\beta_2-p-1)(\zeta+\lambda)/p+p-1,\ p-1[$, i.e. when $\beta_2<p+1$.

This finishes the estimates of Proposition 4.4.6.

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