Capacity Computation and Duality for Wireless Communication Systems

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Abstract

The first part of the thesis deals with the problem of admission control and air interface selection in heterogeneous communication networks, i.e., systems in which several access technologies are used jointly. For this, statistical information about user traffic and channel conditions is taken into account for the optimization of admission and assignment policies, employing a semi-Markov decision process formulation. We propose an efficient approximation algorithm for the optimization of such policies.

The remaining chapters of the thesis take a more fundamental perspective. We first study the problem of computing the boundary of the capacity region of the discrete memoryless multiple access channel. We demonstrate that a result on the optimality conditions for this problem reported in the literature is incorrect. We then investigate optimality conditions of the capacity computation problem for the discrete memoryless two-user multiple access channel with binary input and binary output alphabets. For a subclass of these channels, earlier results in the literature are generalized. Moreover, we investigate a relaxation approach for the capacity computation problem. For a subclass of channels, we derive conditions under which it is possible to construct an optimal solution for the actual (non-relaxed) problem from the solution to a convex (relaxed) problem.

Next, building upon some results for the multiple access channel capacity computation problem, we investigate duality relations between the multiple access and the broadcast channel both for discrete memoryless and for fading channels without channel knowledge at the transmitter, respectively. We introduce the notion of weak duality and derive a sufficient condition and a necessary condition for a discrete memoryless broadcast channel to be weakly dual to a certain class of discrete memoryless multiple access channels. Concerning fading channels, we show that duality holds only under certain conditions. However, we demonstrate that, even in absence of duality, the dual multiple access channel can be used in order to approximatively solve optimization problems for the broadcast channel.

In the last part of the thesis, we turn to a more general network structure, a basic cellular channel. We analyze it using the linear deterministic approximation model, which constitutes an approximative description of the underlying physical channel. We derive the capacity and optimal coding schemes under various interference scenarios.

Zusammenfassung

Das erste Kapitel der Arbeit behandelt die Optimierung der Zugriffssteuerung in heterogenen Systemen, bei denen mehrere verschiedene Zugriffstechnologien zur Verfügung stehen. Der hier verfolgte Ansatz besteht darin, statistische Kenntnisse über das Nutzerverhalten und die Kanäle in die Optimierung mit einzubeziehen. Hierzu wird das System als Semi-Markov Decison Process modelliert und, basierend auf einer geeigneten Zustandsraumaggregation, ein effizienter approximativer Optimierungsalgorithmus entwickelt.

Die übrigen Kapitel der Arbeit widmen sich eher grundlegenden und informationstheoretischen Fragestellungen. Zunächst wird das Problem der expliziten Berechnung der Kapazitätsregion des diskreten gedächtnislosen Vielfachzugriffskanals behandelt. Es wird gezeigt, dass ein in der Literatur beschriebenes Ergebnis bezüglich der Optimalitätsbedingungen nicht gültig ist. Weiterhin werden Optimalitätsbedingungen für den Kanal mit zwei Sendern und binären Eingangs- und Ausgangsalphabeten untersucht und bekannte Resultate verallgemeinert. Darüber hinaus wird ein Optimierungsansatz betrachtet, der mittels Relaxierung auf ein konvexes Problem führt.

Einen weiteren Schwerpunkt der Arbeit bildet die Untersuchung von Dualitätsbeziehungen zwischen dem Vielfachzugriffskanal und dem Broadcastkanal sowohl für diskrete gedächtnislose Kanäle als auch für Schwundkanäle ohne Kanalkenntnis am Sender. Hierbei wird zunächst der Begriff der schwachen Dualität definiert und ein hinreichendes und ein notwendiges Kriterium dafür angegeben, dass ein Broadcastkanal schwach dual zu einer bestimmten Klasse von diskreten gedächtnislosen Vielfachzugriffskanälen ist. In Bezug auf Schwundkanäle wird nachgewiesen, dass Dualität nur unter speziellen Bedingungen vorliegt. Es wird jedoch gezeigt, dass auch bei nicht vorhandener (perfekter) Dualität der duale Vielfachzugriffskanal für die (approximative) Optimierung für den Broadcastkanal genutzt werden kann.

Der letzte Teil der Arbeit befasst sich mit einem allgemeineren Kanalmodell, welches ein einfaches zellulares System beschreibt. Dieses System wird mittels des linearen deterministischen Modells untersucht, bei dem der physikalische Kanal näherungsweise modelliert wird. Für dieses Kanalmodell werden Kapazitätsresultate für unterschiedliche Interferenzszenarien hergeleitet und die entsprechenden effizienten Kodierungsstrategien entwickelt.

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1 Introduction

1.1 Background

Wireless communication technology has undergone a rapid growth in the recent years, and this trend is expected to further accelerate in the future. Besides classical services mainly designed for voice and data communication, new applications of wireless technology are emerging. Among them are wireless sensor networks, which can be used for a multitude of applications such as environmental monitoring, object tracing, supervision of manufacturing processes or intelligent monitoring of buildings. Other fields of application that are envisioned for the future are telemedical applications, machine to machine (M2M) communication, automated highways or communication among nano-sized devices.

From an engineering viewpoint, the wireless channel is rather difficult to handle and poses severe challenges concerning the design of reliable and efficient systems. These difficulties arise especially from the specific properties of the communication medium, that is, the propagation of electromagnetic waves in a constantly changing environment and include *fading* (small-scale and large-scale) and *interference* in systems where multiple communicating entities share the same resources. *Information theory* provides tools for a fundamental study of the trade-offs and potentials involved in (wireless) communication systems. Since Shannon's seminal publication [Sha48], which introduced the central concepts of the field, the recent decades have seen quite substantial progress in information theory, especially concerning the case of several users participating in the communication, and many concepts and techniques have found their way to implementation in practical systems. However, there are still many fundamental questions that are completely unsolved or not understood to a satisfactory degree. For an example, consider two basic structures of multi-user communication, namely the *broadcast channel* (BC) and the *multiple access channel* (MAC). In a broadcast channel, there is one transmitter and several receivers; the transmitter intends to convey a message to each of the receivers (where the messages can differ from each other). The multiple access scenario is the reverse setup, where multiple transmitters each have a message for the central receiver (cf. Figure 1.1). Note that the broadcast channel and the multiple access channel model the *downlink* and the *uplink* in a conventional cellular system: the downlink refers to the transmission of

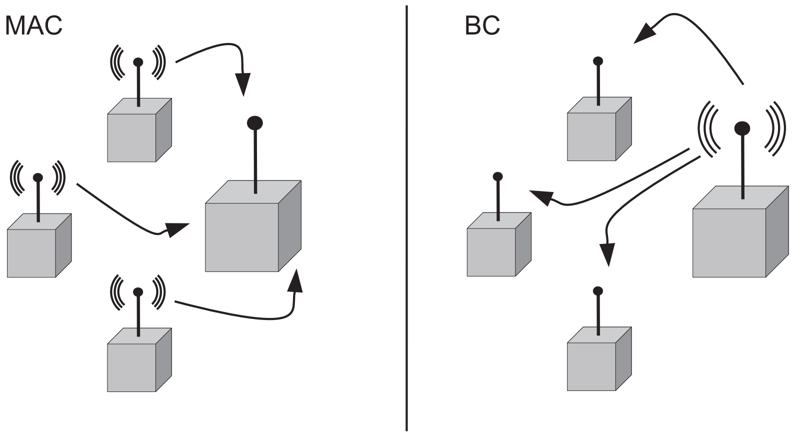

Figure 1.1: Multiple access channel (left) and broadcast channel (right).

a central base station to the users in the cell the base station serves, whereas the uplink means communication of the cell users to the base station. A rather generic model for a communication channel is the *discrete memoryless channel*, which has a finite set of input and output symbols and which is specified by the pairwise probabilities of switching from an input symbol to an output symbol for the transmission of a single symbol. Results for this channel can often be translated to the more specific *Gaussian* channels, which have continuous (typically complex) input and output alphabets and are disturbed by additive white Gaussian noise (AWGN). While the capacity region (i.e., the set of all user rates achievable under reliable communication) is known for the Gaussian MAC and BC and for the discrete memoryless MAC, it is still unknown for the discrete memoryless BC. Despite the similarities of the multiple access and the broadcast structure, there seems to be a tendency of the BC to be more difficult to handle in terms of an information-theoretic analysis. Several problems that are comparatively easy to solve for the MAC are substantially harder to solve in the broadcast scenario. This has motivated researchers to investigate the connection between the two channel classes more closely, unveiling surprising and very useful relationships, known as *duality*, between them. While duality has been shown for a number of channel classes, it is an interesting open question to what extent duality holds for the quite generic class of discrete memoryless channels. Investigating this issue, a natural problem that arises is to find appropriate characterizations of the boundaries of the MAC or the BC. The capacity region for the discrete memoryless MAC is known—however, the boundary of the region is given implicitly as the solution of an optimization problem. Solving this problem is, for itself, an interesting task.

Considering the enormous difficulties encountered in the analysis of multi-user systems, approximative solutions to information-theoretic problems have drawn increasing attention in the recent past. One of the successful strategies to obtain such results involves an approximate description of the physical (Gaussian)[1] channel. Here, the channel operates deterministically on bit vectors. Basically, the effect of the channel is to erase a certain number of bits while keeping the remaining bits unchanged. The *linear deterministic model* obtained in this way is considerably easier to analyze. However, some important properties of the wireless medium are retained in the model. It is frequently the case that efficient communication strategies devised for the linear deterministic model can be translated to (almost optimal) strategies for the physical channel.

Another challenging aspect of wireless communication is the increasingly heterogeneous structure of systems: when new wireless technologies are deployed, system operators generally are interested to keep existing infrastructure in operation as long as possible. Moreover, it is nowadays possible to support multiple wireless technologies in a single mobile device, allowing a flexible choice of the access technology to be employed. This flexibility also provides operators with new degrees of freedom for the optimization of system performance with respect to different performance measures such as throughput, fairness or system reliability. Analytical studies in this context naturally take a higher-level perspective than the information-theoretic investigations described above. Yet, information-theoretic results for the systems in question are valuable in order to assess the data rates that are achievable in the different systems when a certain amount of resources (such as power or bandwidth) are available.

1.2 Outline and Main Results of the Thesis

This thesis deals with different, but interconnected aspects of the issues mentioned above that arise in the context of wireless communication systems and their information-theoretic evaluation. The first chapter takes a more higher-level view and deals with the question of how resource usage can be optimized in heterogeneous network environments, taking statistical information about user behavior into account. For the remaining chapters, the perspective switches to be more fundamental and information-theoretic. First, we study the problem of explicitly computing the boundary of the capacity region of a discrete memoryless MAC, which is still a largely unsolved problem. Building upon these results, we investigate duality relations for discrete memoryless channels and for Gaussian fading channels without channel state information at the transmitter. Finally, we derive approx-

[1]By the "physical channel", we generally mean the corresponding Gaussian channel model. Of course, this is also an approximative model of the "real-world" channel in the sense that it is obtained by making certain assumptions and simplifications.

imative capacity results for cellular channels using the linear deterministic approximation model. More precisely, the main topics and results of the thesis are outlined below.

Chapter 2 deals with the problem of admission control and air interface selection in heterogeneous network environments. The statistics of random user arrivals, channel conditions and service durations are considered for the optimization of heterogeneous access management strategies with respect to minimizing the expected mean cost for blocking events. Based on state aggregation in a semi-Markov decision process formulation, an efficient approximation algorithm for policy optimization is proposed. Though this solution is suboptimal, it still offers considerable performance gains in comparison to simpler heuristic strategies, which is demonstrated by simulations in a heterogeneous GSM/UMTS scenario. Furthermore, structural properties of optimal user assignment policies are studied, proving certain monotonicity properties for a specific type of systems.

The work presented in Chapter 2 has been published in [2], [11] and [12].

Chapter 3 is devoted to the problem of explicitly computing the boundary of the capacity region of the discrete memoryless MAC. The following topics form the core of this chapter:

- In the literature, it has been claimed that the Karush-Kuhn-Tucker conditions provide a necessary and sufficient condition for sum-rate optimality for a certain sub-class of discrete memoryless MACs. We first show that this claim does not hold, even for two-user channels with binary input and binary output alphabets. As a consequence, the capacity computation problem for the discrete MAC remains an interesting and mostly unsolved problem.
- We study optimality conditions of the capacity computation problem for the discrete memoryless two-user MAC with binary input and binary output alphabets. For a subclass of these channels and weight vectors, previous results known from the literature are generalized. It is shown that, depending on an ordering property of the channel matrix, the optimal solution is located on the boundary, or the objective function has at most one stationary point in the interior of the domain. For this, the problem is reduced to a pseudoconcave one-dimensional optimization and the single-user problem.
- We investigate properties of the optimality conditions for a relaxation (user cooperation) approach for the MAC capacity computation problem. We give conditions under which a solution to the relaxed problem has the same value as the actual optimal solution and show that these conditions can in some cases be applied to construct solutions for a restricted class of discrete multiple-access channel using convex optimization.

Parts of the work discussed in Chapter 3 have been published in [1], [9] and [10].

In **Chapter 4**, we turn to the duality relations between the MAC and the BC. The motivation for the work reported in this chapter stems from the question to what extent the existing duality relations that have been found for certain channel classes, such as Gaussian channels, can be extended to other relevant channel models. Two important classes are investigated in this work with respect to duality, namely the discrete memoryless channels and fading channels with imperfect channel state information. The chapter consists of two main parts, each dealing with one of these two channel classes:

- Motivated by the existing duality relations, we define a new notion of duality, namely *weak BC-to-MAC duality*. Based on some results for the MAC capacity computation problem from Chapter 3, we derive both a sufficient condition and a necessary condition for a discrete memoryless BC to be weakly dual to a certain class of discrete memoryless MAC channels. Applying these conditions, we show that the binary symmetric BC is weakly dual to this class of MACs for a large set of parameter choices.
- In the second part, we study duality relations for the fading BC under additive white Gaussian noise and a strict ordering of the fading distributions without channel state information at the transmitter and the natural corresponding dual fading MAC. We prove that if the fading distribution for the weaker user is non-deterministic, the achievable rate region using superposition coding with Gaussian signaling and successive decoding, which is conjectured to be the capacity region, differs from the dual MAC region. Duality holds only in the case of one-sided fading, where the fading distribution for the weaker user is deterministic. Despite this lack of duality, we propose to use the dual MAC in order to approximatively solve a non-convex problem for the BC. Specifically, we consider the problem of weighted sum-rate optimization for the BC and give, under some assumptions, upper bounds on the error incurred by using this procedure.

Parts of the work presented in Chapter 4 have been published in [7] and [8].

The concluding chapter of the thesis deals with the linear deterministic approximation model for Gaussian channels, focusing on cellular-type channels. Effective coding and interference mitigation schemes for cellular channels are still an active area of research. Approximative models such as the linear deterministic approach might help to gain more insight into these problems, which motivated the work presented in this chapter. Here, we study a basic model for the uplink of a cellular system in the linear deterministic setting: there are two users transmitting to a receiver, mutually interfering with a third transmitter that communicates with a second receiver. In short, the model consists of a two-user MAC (forming cell 1) mutually interfering with a point-to-point link (cell 2).

We first consider the case of symmetric weak interference, where the interference links from cell 1 user to the cell 2 base station are identical and the sum of the interference gains are less than or equal to the smallest direct link. We derive an efficient coding scheme and prove its optimality, i.e., characterize the capacity region for the system. This coding scheme is a form of interference alignment which exploits the channel gain difference of the two-user cell. We then relax the restriction on the interference strength and derive outer bounds on the achievable sum rate. Again, we develop a coding scheme achieving the outer bounds, thereby obtaining the sum capacity. In this case, interference alignment is employed as well; however, it is generally required to code across the bit levels forming the channel input. Establishing the connection to the Gaussian counterpart of the channel, we use the results to derive lower bounds on the generalized degrees of freedom of the Gaussian channel for rational link ratio parameters. This measure represents a high SNR description of the system, where the channel gains are kept in constant relation in the dB scale, which allows a more detailed asymptotic description of the system as opposed to the degrees of freedom characterization (in which only the transmit power is scaled).

We conclude the thesis by returning to one of the central themes of the thesis, namely duality. A general network duality result for linear coding implies achievable rates for the dual (downlink) cellular setup, i.e., a BC interfering with a point-to-point link. We conjecture that these rates cannot be exceeded, which would imply that duality holds between the uplink and the downlink scenario in the linear deterministic model. The proof of this conjecture is left for future studies.

The work in Chapter 5 has (partially) been published in [4], [5] and [6].

1.3 Notation

Unless stated otherwise, we use the following notation and conventions throughout the thesis: $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ denotes the non-negative reals, and $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ denotes the positive reals. The set of natural numbers (including 0) is denoted by $\mathbb{N}$, and $\mathbb{F}_p$ is the finite field (Galois field) of size p.

Concerning notation for certain basic functions, we denote the natural logarithm function (to the base e) by ln, whereas the logarithm function to base 2 is written as log. We use the standard expressions I for mutual information and H for entropy. For a scalar $p \in [0, 1]$, the function $H(p)$ denotes the binary entropy [CT06]. Unless stated otherwise, we express all entropy and mutual information quantities in nats. exp denotes the exponential function, i.e., $\exp(x) = e^x$. The symbol $\mathbb{1}$ denotes the indicator function, that is $\mathbb{1}[A] = 1$ if statement A is true and $\mathbb{1}[A] = 0$ otherwise. $\lfloor x \rfloor$ denotes the maximum integer lower or equal to x, while $\lceil x \rceil$ is the minimum integer greater or equal to x. Moreover, div and mod denote integer division and the modulo operation, respectively, where we use the convention $x \operatorname{div} 0 = 0$. For $x \in \mathbb{R}$, the positive part is denoted by $(x)^+ = \max(x, 0)$. By $D(p||q)$, we will denote the Kullback-Leibler divergence between two binary probability functions defined by $p, q \in [0, 1]$, that is,

$$D(p||q) = p \ln \frac{p}{q} + (1-p) \ln \frac{1-p}{1-q}, \tag{1.1}$$

where we use the standard convention that $0 \ln \frac{0}{q} = 0$ for $q \in \mathbb{R}_+$ and $p \ln \frac{p}{0} = \infty$ for $p \in \mathbb{R}_{++}$. The symbol $\oplus$ denotes addition in the finite field $\mathbb{F}_2$, i.e., addition modulo-2. For a function f, we write $f(x) \nearrow$ ($\searrow$) to indicate that f is monotonically increasing (decreasing). The notation $f(x) \nearrow^y$ ($\searrow_y$) also specifies the maximum (minimum) value y of f. Generally, derivatives on the boundary of closed intervals are to be understood as one-sided derivatives. As usual, ∇f denotes the gradient of a function f.

In general, matrices and vectors are denoted by bold letters and the entries referred to with the corresponding plain letters, such as for the matrix $\mathbf{A} = (A_{ij})$. Where appropriate, the entries of a matrix $\mathbf{A}$ may be referred to by $A(i, j)$ instead of $A_{i,j}$ (similarly for vectors). $\mathbf{1}$ denotes the matrix (vector) with all entries equal to 1, $\mathbf{0}$ is the zero matrix (or vector). As usual, the unit matrix is denoted by $\mathbf{I}$ and $\mathbf{A}^T$ denotes matrix transposition. Occasionally, the dimension $m \times n$ is also specified by $\mathbf{0}_{m \times n}$, $\mathbf{1}_{m \times n}$ and by $\mathbf{I}_n$ for the $n \times n$ unit matrix. For two vectors $\mathbf{x}, \mathbf{y}$, we denote component-wise inequalities as $\mathbf{x} < \mathbf{q}, \mathbf{x} \leq \mathbf{q}$ etc. For two matrices $\mathbf{A}$ and $\mathbf{B}$, we denote by $[\mathbf{A}; \mathbf{B}]$ the matrix that is obtained by stacking $\mathbf{A}$ over $\mathbf{B}$. Moreover, for a sequence of matrices $\mathbf{A}_1, \ldots, \mathbf{A}_n$, we write $(\mathbf{A}_k)_{k=1}^n$ for the stacked matrix, i.e., $(\mathbf{A}_k)_{k=1}^n = [(\mathbf{A}_k)_{k=1}^{n-1}; \mathbf{A}_n]$. Similarly, $[\mathbf{A}|\mathbf{B}]$ and $[\mathbf{A}\ \ \mathbf{B}]$ stand for placing $\mathbf{A}$ next to $\mathbf{B}$. For a matrix $\mathbf{A} \in \mathbb{F}_2^{n \times m}$, for $1 \leq i \leq j \leq n$, the sub-matrix $\mathbf{A}[i : j] \in \mathbb{F}_2^{j-i+1 \times m}$ is given by taking only the rows i to j of the matrix $\mathbf{A}$. Moreover, we write $|\mathbf{A}|$ for the number of ones in $\mathbf{A}$. The rank of the matrix is denoted by $\operatorname{rank}(\mathbf{A})$, $\det(\mathbf{A})$ is its determinant and $\langle \mathbf{A} \rangle$ its image. The dimension of a vector space V is expressed as $\dim(V)$.

Finally, $\mathbf{Pr}[\mathcal{E}]$ denotes the probability of an event $\mathcal{E}$, $\mathbf{Pr}[\mathcal{E}|\mathcal{F}]$ is conditional probability (conditioning on the event $\mathcal{F}$), $\mathbf{E}$ is the expectation operator, and $\mathbf{Co}$ denotes the operation of forming the closure of the convex hull of a set.

2 Heterogeneous Access Management

Today's communication infrastructure is becoming more and more heterogeneous in terms of different access technologies and the services that are provided to users. New radio access technologies emerge frequently, and system operators are interested in a continued utilization of systems that have already been deployed. Moreover, advances in integrated circuit design and software-defined radio nowadays allow to support multiple access technologies (AT, also referred to as *air interfaces*) in a single wireless user terminal. The number of different types of services is also increasing: While voice services made up the major part of the system load until recently, the traffic generated by data services (such as WWW, video conferencing, gaming etc.) is increasing dramatically.

In order to efficiently manage the resources provided by the different ATs, it is envisioned to employ a central entity that controls the assignment of user requests (each possibly requesting a different type of service) to the available ATs. This is generally called *heterogeneous access management (HAM)* and involves admission control and resource sharing policies among the different access technologies. HAM also endows operators with new degrees of freedom for the optimization of system performance with respect to throughput, fairness or system reliability. HAM concepts have been studied in the literature in various contexts and assumptions and for different objectives. Among them are load-balancing [PRSA07], service-based [FZ05] and cost-based [BW07], [BWKS09] strategies and strategies based on Markov or Markov decision process formulations [YK07], [SN-MRW08], [GPRSA08], [HF08] and fuzzy neural methodologies [GAPRS08].

In this chapter, we present a generic framework for the optimization of admission control and access selection in heterogeneous networks. We study the cost-based approach, introduced in [BW07], in a dynamic setting where users randomly arrive over time according to a Poisson process and require a random, exponentially distributed service time. The cost-based approach allows to bundle many relevant system parameters such as different service requirements, characteristics of the ATs, user positions and channel gains into one cost value per user and AT. The formulation is general enough to be employed for various types of systems for which resource costs required to support a user in an AT are either orthogonal among the users or, for interference-limited systems, depend only on the sum of interference but not on the composition among the different interference sources [BW07], [BWKS09]. We remark that for analytical and algorithmic tractability, our model requires the assumption of Poisson arrivals and exponential holding times, which is used widely in the literature [YK07], [SNMRW08], [GPRSA08], [HF08]. Moreover, if these assumptions are not satisfied perfectly for a practical system to be optimized, we can at least expect to get reasonable suboptimal solutions.

The system is analyzed using a semi-Markov decision approach under a system reliability (expected mean blocking cost) optimization criterion, which is a common call-level objective for admission control policy design [GB06]. We remark that a similar Markov model for the case of a single available resource (AT) has been studied in [AJK01], [RT89b], [RT89a] in the context of the *stochastic Knapsack problem*. Even though the theory of Markov decision processes provides a powerful tool for control optimization under uncertainty, a typical problem that occurs is the *curse of dimensionality*, which refers to the immense size of the state space for problems of realistic size, which renders standard algorithms such as Policy Iteration or Value Iteration infeasible. The main contribution of the work presented in this chapter is the proposal of a suboptimal solution to the HAM control problem based on state aggregation, thereby significantly reducing the state space size (and hence, computational requirements) and allowing the numerical solution of scenarios of realistic size. In a sample heterogeneous GSM-EDGE/UMTS scenario (which is also used in [BWKS09] for performance evaluation in a similar fashion), we demonstrate that our algorithm, even though suboptimal, shows remarkable performance gains in comparison to straightforward heuristic control policies. Secondly, structural properties of optimal HAM control policies are analyzed in this chapter. We demonstrate certain non-monotonicity properties of optimal policies. For the situation of two ATs and two user classes and a specific ordering of the system parameters, we prove that the heuristic "Greedy" user assignment is optimal except for unforced (not necessarily required) blocking events and that the optimal policy exhibits a certain monotonicity property. These results are obtained employing the technique of event-based Dynamic Programming [Koo06].

2.1 System Model, Problem Formulation and Heuristic Policies

2.1.1 System Model

The system model considered here is as follows: The system offers S ATs, each with load capacity C. Depending on physical layer capabilities, channel conditions and user requirements, each user belongs to one of K different *user classes*. Users arrive to the system over time according to a Poisson arrival process with mean arrival rate λ. An arriving user belongs to a specific class k with probability p_k. Note that due to the probabilistic splitting property of Poisson processes, we can equivalently assume a Poisson arrival process for each class k with rates $\lambda_k = \lambda p_k$. Each time a user arrives to the system (and requests to be served), the HAM control policy specifies if the user should be offered service or not. If the user is served, the controller also decides which AT the user will be assigned to. A user of class k that is assigned to the AT s will demand an amount c_{sk} of resources in this AT for the duration of service. We will refer to these values as *cost values*. The resources taken by an assigned user will be released after an exponentially distributed service time with class-dependent mean time $1/\mu_k$. If a user of class k is blocked, a *penalty* b_k is incurred. We assume a prespecified quantization of the available resources: We assume that all cost values in each AT s are integer multiples of a basic resource unit Δ_s, so that $c_{sk} = \tilde{c}_{sk}\Delta_s$ where $1 \leq \tilde{c}_{sk} \in \mathbb{N}$. This quantization may arise due to different reasons. First of all, it is often the case that distributable resources such as time slots or power levels are inherently quantized. The quantization might also be the result of an approximate system modeling.

We describe the state of the system at time t by a matrix $\mathbf{X}(t) \in \mathcal{X}$, where $X(t)_{sk}$ denotes the number of users of class k in the AT s and the state space $\mathcal{X}$ (the set of possible states of the system) is given by

$$\mathcal{X} := \left\{ \mathbf{M} \in \mathbb{N}^{S \times K} : \forall 1 \leq s \leq S \ \sum_{k=1}^{K} c_{sk} M_{sk} \leq C \right\}. \tag{2.1}$$

The problem we consider is to construct HAM policies that minimize the expected mean cost for blocking events. More precisely, our goal is to find an optimal *HAM policy* $\pi : \mathcal{X} \times \mathbb{R} \times \{1, \dots, K\} \to \{0, 1, \dots, S\}$. Here, $\pi(\mathbf{X}, t, k)$ specifies the AT that a class k user arriving at time t to the system in state $\mathbf{X}$ is assigned to (0 stands for blocking). Let $t_1, \dots, t_{N(t)}$ be the (random) epochs of user arrivals up to time t and the class of the arrival at t_i be denoted by $k(t_i)$. The policy π is considered optimal [for initial state $\mathbf{X}(t_1)$] if it minimizes the expected blocking cost per unit time (expected mean cost)

$$J_\pi := \limsup_{t \to \infty} \frac{1}{t} \mathbf{E} \left[\sum_{i=1}^{N(t)} \mathbb{1} \left[\pi(\mathbf{X}(t_i), t_i, k(t_i)) = 0 \right] b_{k(t_i)} \right], \tag{2.2}$$

where $N(t)$ specifies the number of arrivals up to time t. A policy is called *stationary* if it is independent of time t, and we write $\pi(\mathbf{X}, t, k) = \pi(\mathbf{X}, k)$ in this case. We remark that from the properties of the semi-Markov decision process formulation given subsequently, it follows that an optimal stationary policy always exists. For this, the cost J_π is also independent of the initial state and the limit (not only the limit superior) in (2.2) exists.

2.1.2 Heuristic Policies

A simple heuristic policy is given by the *Load-based policy* π^l which assigns each user to the AT with minimum total load [BW07]. More precisely, it is defined as follows. Let $\overline{\mathcal{X}_k} \subseteq \mathcal{X}$ be the set of states that are totally loaded for class k users, i.e., that do not allow the admission of class k user to any of the ATs and $\mathbf{E}_{ij}$ be the $S \times K$ matrix that contains a one at position (i, j) and zeros elsewhere. The Load-based policy performs HAM control as follows:

$$\pi^{\mathrm{l}}(\mathbf{X}, k) := \begin{cases} 0, & \mathbf{X} \in \overline{\mathcal{X}_k} \\ \underset{\{s:\mathbf{X}+\mathbf{E}_{sk} \in \mathcal{X}\}}{\operatorname{argmin}} \sum_k c_{sk} X_{sk}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Here, ties (equal load situations) are broken arbitrarily (in our examples, we use the AT with smallest index number). Another simple heuristic for HAM control is given by assigning each arriving user to the AT in which it causes the minimum resource usage: The *Greedy policy* π^g is defined as the stationary policy

$$\pi^{\mathrm{g}}(\mathbf{X}, k) := \begin{cases} 0, & \mathbf{X} \in \overline{\mathcal{X}_k} \\ \underset{\{s:\mathbf{X}+\mathbf{E}_{sk} \in \mathcal{X}\}}{\operatorname{argmin}} c_{sk}, & \text{otherwise.} \end{cases} \tag{2.4}$$

If there are several s that minimize the cost value, the policy chooses among them the one corresponding to the AT having minimum current total load; ties among equally loaded ATs are broken arbitrarily (again, in our examples, we use the AT with smallest index number).

2.2 Semi-Markov Decision Process Formulation and Optimal HAM Policies

In this section, we present a semi-Markov decision process (SMDP) [Tij86], [Ber07] formulation of the problem. We define $\mathcal{X}$ as the state space of the SMDP. Following the approach chosen in [RT89a] for the stochastic Knapsack problem, we choose the action space $\mathcal{A} = \{0, 1, \ldots, S\}^K$. The interpretation of the actions is as follows: Assume that in state $\mathbf{X}$ the action $\mathbf{a} = (a_1, \ldots, a_K)$ is chosen and that a class k user arrives while the system is in state $\mathbf{X}$. Then the user will be blocked if $a_k = 0$ (and the system remains in state $\mathbf{X}$ and a penalty b_k is incurred); otherwise, the user will be assigned to AT a_k and the system state changes. Consequently, for a state $\mathbf{X} \in \mathcal{X}$, the set of admissible actions is defined by

$$A_{\mathbf{X}} := \begin{cases} \left\{\mathbf{a} \in \mathcal{A} \setminus \{\mathbf{0}\} : \mathbf{X} + \mathbf{E}_{a_j j} \in \mathcal{X} \text{ if } a_j > 0\right\}, & \mathbf{X} = \mathbf{0} \\ \left\{\mathbf{a} \in \mathcal{A} : \mathbf{X} + \mathbf{E}_{a_j j} \in \mathcal{X} \text{ if } a_j > 0\right\}, & \mathbf{X} \neq \mathbf{0}. \end{cases} \tag{2.5}$$

The transition rate from a state $\mathbf{X} \in \mathcal{X}$ to the state $\mathbf{Y} \neq \mathbf{X} \in \mathcal{X}$ under action $\mathbf{a}$ is given by

$$\lambda_{\mathbf{XY}}(\mathbf{a}) = \begin{cases} \lambda_k, & \mathbf{Y} = \mathbf{X} + \mathbf{E}_{sk} \text{ and } a_k = s \\ \mu_k X_{sk}, & \mathbf{Y} = \mathbf{X} - \mathbf{E}_{sk} \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

The sojourn times (i.e., the expected amount of time spent in a state before moving to a different state) and transition probabilities are given by

$$\tau_{\mathbf{X}}(\mathbf{a}) = \frac{1}{\sum_{\mathbf{Y} \neq \mathbf{X}} \lambda_{\mathbf{XY}}(\mathbf{a})} \tag{2.7}$$

and

$$p_{\mathbf{XY}}(\mathbf{a}) = \tau_{\mathbf{X}}(\mathbf{a}) \lambda_{\mathbf{XY}}(\mathbf{a}), \tag{2.8}$$

respectively. The expected cost in state $\mathbf{X}$ until transition to the next state using action $\mathbf{a} \in A_{\mathbf{X}}$ is calculated as

$$c_{\mathbf{X}}(\mathbf{a}) = \tau_{\mathbf{X}}(\mathbf{a}) \sum_{k=1}^{K} \lambda_k \mathbb{1}[a_k = 0] b_k. \tag{2.9}$$

Note that the state $\mathbf{0}$ is reachable from any other state, so that the SMDP satisfies the unichain property [Tij86]. This implies the existence of a stationary policy $\gamma : \mathcal{X} \to \mathcal{A}, \gamma(\mathbf{X}) \in A_{\mathbf{X}}$ minimizing the expected cost per unit time (which is independent of the initial state). Using $\pi(\mathbf{X}, k) = \gamma(\mathbf{X})_k$ then yields the policy π with the expected mean cost J_π given in (2.2).

The optimal policy γ can be found using standard algorithms such as Value Iteration or Policy Iteration. Value Iteration is an iterative procedure that updates in the nth step a *value function* V_n that maps each element of the state space to a real value. The procedure stops when a specified stopping criterion is satisfied. For the implementation of Value Iteration, a transformation to an equivalent discrete-time model is carried out using uniformization [Tij86], [Ber07]: For the uniformization constant, we choose

$$\tau = \frac{1}{2} \left(\sum_{s=1}^{S} \sum_{k=1}^{K} \mu_k \left\lfloor \frac{C}{c_{sk}} \right\rfloor + \sum_{k=1}^{K} \lambda_k \right)^{-1}. \tag{2.10}$$

Uniformization results in the following data transformation of cost values and transition

Table 2.1: Experimental parameters.

No.	S	K	C	(λ_k)	(μ_k)	(c_{sk})	(b_k)
I	2	2	2	(1 1)	(1 1)	$\begin{pmatrix} 0.5 & 0.6 \\ 0.4 & 0.5 \end{pmatrix}$	(1 1)
II	2	3	1.8	(3 1.5 2)	(1 2 1.5)	$\begin{pmatrix} 0.5 & 0.6 & 0.4 \\ 0.3 & 0.7 & 0.4 \end{pmatrix}$	(1 6 2)

probabilities: $\tilde{c}_{\mathbf{X}}(\mathbf{a}) \coloneqq c_{\mathbf{X}}(\mathbf{a})/\tau_{\mathbf{X}}(\mathbf{a})$,

$$\tilde{p}_{\mathbf{XY}}(\mathbf{a}) = \begin{cases} \frac{\tau}{\tau_{\mathbf{X}}(\mathbf{a})} p_{\mathbf{XY}}(\mathbf{a}), & \mathbf{X} \neq \mathbf{Y} \\ 1 - \frac{\tau}{\tau_{\mathbf{X}}(\mathbf{a})}, & \mathbf{X} = \mathbf{Y}. \end{cases} \tag{2.11}$$

Then, the Value Iteration step for the value function $V_n(\mathbf{X})$ is given by

$$V_{n+1}(\mathbf{X}) = \min_{\mathbf{a} \in A_{\mathbf{X}}} \left[\tilde{c}_{\mathbf{X}}(\mathbf{a}) + \sum_{\mathbf{Y} \in \mathcal{X}} \tilde{p}_{\mathbf{XY}}(\mathbf{a}) V_n(\mathbf{Y}) \right]. \tag{2.12}$$

Similarly as in [RT89a] for the stochastic Knapsack problem, the Value Iteration step (2.12) allows the following simplification: As

$$\begin{aligned} \sum_{\mathbf{Y} \in \mathcal{X}} \tilde{p}_{\mathbf{XY}}(\mathbf{a}) V_n(\mathbf{Y}) - \frac{\tau}{\tau_{\mathbf{X}}(\mathbf{a})} V_n(\mathbf{X}) &= \tau \sum_{\mathbf{Y} \neq \mathbf{X}} \lambda_{\mathbf{XY}}(\mathbf{a}) V_n(\mathbf{Y}) - \frac{\tau}{\tau_{\mathbf{X}}(\mathbf{a})} \sum_{\mathbf{Y} \neq \mathbf{X}} p_{\mathbf{XY}}(\mathbf{a}) V_n(\mathbf{X}) \\ &= \tau \sum_{\mathbf{Y} \neq \mathbf{X}} \lambda_{\mathbf{XY}}(\mathbf{a}) \left(V_n(\mathbf{Y}) - V_n(\mathbf{X}) \right), \end{aligned} \tag{2.13}$$

the Value Iteration step decomposes to

$$V_{n+1}(\mathbf{X}) = V_n(\mathbf{X}) + \sum_k \lambda_k M_n(\mathbf{X}, k) + \tau \sum_{s,k} \mu_k X_{sk} \left(V_n(\mathbf{X} - \mathbf{E}_{sk}) - V_n(\mathbf{X}) \right), \tag{2.14}$$

where

$$M_n(\mathbf{X}, k) = \min \left\{ \min_{s \in \mathcal{S}_{\mathbf{X},k}} \tau \left(V_n(\mathbf{X} + \mathbf{E}_{sk}) - V_n(\mathbf{X}) \right), b_k \right\} \tag{2.15}$$

and $\mathcal{S}_{\mathbf{X},k} \coloneqq \{ s \in \{1, \ldots, S\} : \mathbf{X} + \mathbf{E}_{sk} \in \mathcal{X} \}$.

2.2.1 Examples for Optimal Policies

In the following, we discuss some general properties of optimal policies by means of a simple example. While the Greedy and the Load-Based assignment decisions take only

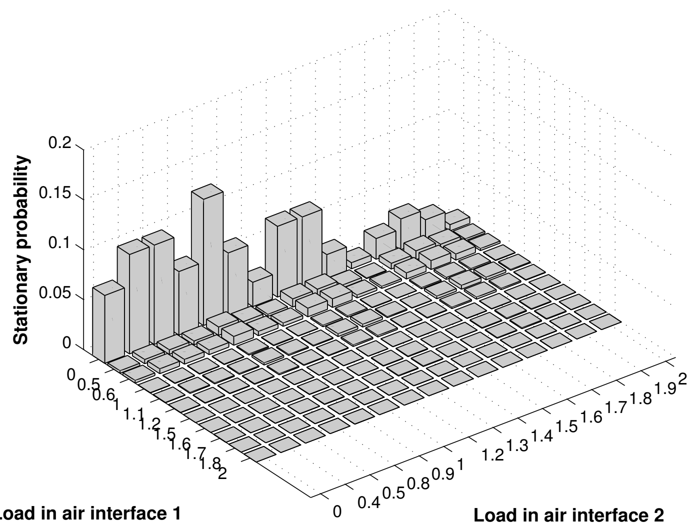

Figure 2.1: Stationary distribution for different load states for the Greedy policy.

the current system state into account, the optimal solution also considers the randomness of future developments of the system. Most importantly, the number of user arrivals in a certain time interval is random, and the optimal policy takes the probabilities for these random variations into account.

This can be illustrated by two effects that typically occur for optimal policies. First of all, it is generally not optimal to assign an arriving user to the AT in which the user causes minimum resource cost (as the Greedy policy does). We call such a control decision a *non-Greedy user assignment*. This is due to the fact that the optimal policy generally tends to balance usage of resources across the different ATs. Thereby, it is possible to accommodate more users (or more users with a higher blocking cost) in the "efficient" AT in the case of a large number of users arriving in a short period of time. Clearly, load balancing is also the intention behind the Load-based policy, which however is often quite wasteful in terms of resource usage. Optimal strategies therefore can be thought of as seeking for the best possible trade-off between load-balancing and efficient resource usage. We illustrate this by means of a simple example. Consider the system I given in Table 2.1. For this system, the user class characteristics are identical except for the cost values, and for both users AT 2 offers a slightly more resource efficient service. Looking at the optimal policy (obtained by Value Iteration as given by (2.14)), it can be seen that Greedy assignments are not optimal in all states. In fact, the expected cost for the Greedy policy is about $6.3 \cdot 10^{-3}$, whereas the optimal policy reduces this cost by about 37% down to

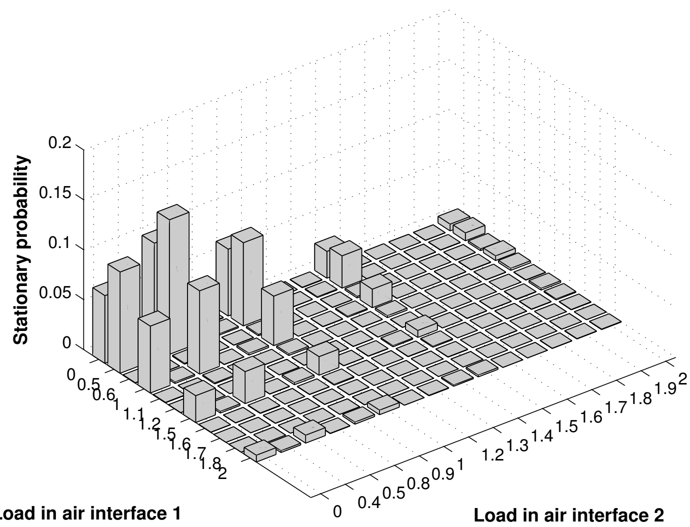

Figure 2.2: Stationary distribution for different load states for the optimal policy.

$3.9 \cdot 10^{-3}$. This gain can be explained, as discussed above, by looking at the stationary probability for different load states, displayed in Figure 2.1 for the Greedy policy and in Figure 2.2 for the optimal policy. Here, one can see that the optimal policy achieves a more balanced resource usage for the different systems. Moreover, optimal policies might deny service to users requesting service even if there are enough resources in one of the ATs to accommodate the user, which we call *unforced blocking*. By doing so, optimal policies reserve resources for potentially arriving users that have lower resource requirements or a higher blocking penalty. Structural properties of optimal policies with respect to unforced blocking are discussed in more detail in Section 2.5.

2.3 The Aggregated SMDP

The SMDP formulation and Value Iteration algorithm described in the last section in principle allow to find optimal HAM policies. However, for problem instances encountered in real-world scenarios, the size of the state space renders these algorithms computationally infeasible. To encounter this problem, we define an aggregated [Ber07] SMDP model which considerably reduces the computational complexity but still allows substantial performance gains in comparison to the Greedy and the Load-based strategy. From now on, we consider the case $S = 2$. For technical reasons to become clear later, we introduce an additional artificial "dummy" user class to the system, for which $c_{s,K+1} = \Delta_s$ and

$p_{K+1} = \epsilon$ for some small $\epsilon > 0$. We adjust the occurrence probability of each original user class k to $p_k - \epsilon/K$. For notational convenience, we will in the following still denote the number of user classes by K, assuming that the dummy class is already among these K classes.

2.3.1 State Aggregation

We form state clusters in the original SMDP model and specify an aggregated SMDP model on the cluster set. The states are clustered according to the total load they require in the system: For each load pair (l_1, l_2) for which $l_1 = k_1\Delta_1, l_2 = k_2\Delta_2, k_1, k_2 \in \mathbb{N}, 0 \leq k_1 \leq \lfloor C/\Delta_1 \rfloor, 0 \leq k_2 \leq \lfloor C/\Delta_2 \rfloor$, we define the corresponding state cluster as

$$C(l_1, l_2) \coloneqq \left\{ \mathbf{X} \in \mathcal{X} : \sum_k c_{sk} X_{sk} = l_s, s = 1, 2 \right\}. \tag{2.16}$$

As state space for the aggregated SMDP model, we use

$$\hat{\mathcal{X}} \coloneqq \{0, \Delta_1, \ldots, \Delta_1 \lfloor C/\Delta_1 \rfloor\} \times \{0, \Delta_2, \ldots, \Delta_2 \lfloor C/\Delta_2 \rfloor\} \times \{0, 1, \ldots, K\}. \tag{2.17}$$

Here, the state $(l_1, l_2, 0)$ represents the system state with load (l_1, l_2) *just after* a user departure, and (l_1, l_2, k), $k > 0$ represents the system state with load (l_1, l_2) *just prior* to a class k user arrival. The action set is defined as $\mathcal{A} = \{0, 1, 2\}$. Again, 0 represents the blocking action and $1, 2$ denote the admission to AT 1 and 2 respectively. The set of admissible actions in the state $\mathbf{x} = (l_1, l_2, k), k > 0$ is then given by

$$\mathcal{A}_{\mathbf{x}} \coloneqq \{0\} \cup \{s \in \{1, 2\} : l_s + c_{sk} \leq C\}. \tag{2.18}$$

For the states $(l_1, l_2, 0)$, the only admissible action is 1; it merely plays the role of a dummy action and has no operational meaning. A stationary policy is given by the mapping $\pi : \hat{\mathcal{X}} \to \mathcal{A}$ such that $\pi(\mathbf{x}) \in \mathcal{A}_{\mathbf{x}}$ for all $\mathbf{x} \in \hat{\mathcal{X}}$ and such that $\pi(0, 0, k) > 0$ for at least one $k, 1 \leq k \leq K$ (this is just a technical requirement in order to ensure the finiteness of the sojourn time in the zero state). For the probability in state (l_1, l_2, k) for moving to the state (l_1', l_2', k') using policy π, we use the notation $p_\pi(l_1, l_2, k, l_1', l_2', k')$. Due to the state aggregation, information about the actual state composition and hence over the actual departure statistics is lost. We encapsulate some information about the departure statistics in the aggregated states as follows. For the load l, we define the set $D_s(l) \coloneqq \{k : l - c_{sk} \geq 0\}$ of classes that are possible constituents of a system state with load l in AT s. If class k was the only user class present in AT s with load l_s, then there would be l_s/c_{sk} users in the system. To approximate the departure rate of class k

users (where $k \in D_s(l_s)$) in AT s, we weight this number by the occurrence probability fraction $p_k / \sum_{j \in D_s(l_s)} p_j$ of this class among the possible constituents, intending to reflect the "typical" proportional share of class k. By finally weighting by the class departure rate μ_k, we define

$$\mu_{sk}(l_1, l_2) := 1\!\!1[k \in D_s(l_s)] \frac{\mu_k p_k l_s}{c_{sk} \sum_{j \in D_s(l_s)} p_j} \tag{2.19}$$

as the exponential departure rate in AT s of a class k user in the states $(l_1, l_2, q), q \geq 0$. Note that $\mu_{sk}(l_1, l_2)$ only depends on the state component l_s, so that we also use the notation $\mu_{sk}(l_s)$ in the following. By $\tau_\pi(l_1, l_2, k)$, we denote the sojourn time in the state (l_1, l_2, k) when using policy π. It is easily verified to be given by

$$\tau_\pi(l_1, l_2, k) = \frac{1}{\sum_{s,k} \mu_{sk}(l_s) + \sum_k 1\!\!1[\pi(l_1, l_2, k) \neq 0] \lambda_k}. \tag{2.20}$$

We further write

$$l_s^\pi(l_1, l_2, k) = l_s + 1\!\!1[\pi(l_1, l_2, k) = s] c_{sk} \tag{2.21}$$

and $\overline{1} = 2, \overline{2} = 1$. Then, the transition probabilities are evaluated as

$$\begin{aligned}
p_\pi(l_1, l_2, 0, l_1', l_2', k') &= \lambda_{k'} \tau_\pi(l_1, l_2, 0) 1\!\!1[l_1' = l_1, l_2' = l_2], \\
p_\pi(l_1, l_2, 0, l_1', l_2', 0) &= \mu_{sk}(l_s) \tau_\pi(l_1, l_2, 0) 1\!\!1[l_s' = l_s - c_{sk} \text{ and } l_{\overline{s}} = l_{\overline{s}}' \text{ for } s = 1 \text{ or } 2, \text{ some } k], \\
p_\pi(l_1, l_2, k, l_1', l_2', k') &= \lambda_{k'} \tau_\pi(l_1, l_2, k) 1\!\!1[l_s' = l_s^\pi(l_1, l_2, k) \text{ for } s = 1 \text{ and } 2], \\
p_\pi(l_1, l_2, k, l_1', l_2', 0) &= \mu_{sl}(l_s) \tau_\pi(l_1, l_2, k) 1\!\!1[l_{\overline{s}}' = l_{\overline{s}}^\pi(l_1, l_2, k) \text{ and } l_s' = l_s^\pi(l_1, l_2, k) - c_{sl} \\
&\qquad \text{for } s = 1 \text{ or } 2 \text{ and some } l].
\end{aligned} \tag{2.22}$$

The expected cost $c_\pi(l_1, l_2, k)$ until transition to the next *different* state is zero for states just after a user departure, i.e., $c_\pi(l_1, l_2, 0) = 0$. Since (l_1, l_2, k) for $k > 0$ denotes the state just prior to a class k arrival, it is also zero for $k > 0$ if class k users are admitted. Otherwise, the expected cost is composed of the blocking cost b_k of the current arrival and b_k times the number $\lambda_k \tau_\pi(l_1, l_2, k)$ of expected class k arrivals during the following transition period. We can thus write the expected cost in state $(l_1, l_2, k), k \geq 0$ as

$$c_\pi(l_1, l_2, k) = 1\!\!1[\pi(l_1, l_2, k) = 0](b_k + \lambda_k \tau_\pi(l_1, l_2, k) b_k). \tag{2.23}$$

(note that for $k = 0$, the only admissible action is defined to be 1). For the above SMDP model, we find the optimal (stationary) policy with respect to minimizing the expected mean cost. Owing to the introduction of the unit cost dummy user class, the model satisfies the unichain property [Tij86] since the zero state $(0, 0, 0)$ is reachable from any other state: It is assured that from each state $(l_1, l_2, k) \in \hat{\mathcal{X}}$, there is a "downward chain" to the zero state in the sense that there exists $n \geq 1$ such that the probability of moving in n steps from (l_1, l_2, k) to $(0, 0, 0)$ is strictly positive (which would generally not be the case without the dummy user class). From the solution π^* of the aggregated model, we define the HAM policy $\tilde{\pi}$ as $\tilde{\pi}(\mathbf{X}, k) := \pi^*(l_1, l_2, k)$ for $\mathbf{X} \in C(l_1, l_2)$.

2.3.2 Implementation of Policy Iteration

The Policy Iteration algorithm starts with an arbitrary stationary policy. We use the Greedy policy as the initial policy [which is defined for the aggregated model in the obvious manner similar to (2.4)]. For the value determination step (which evaluates the performance of the current iteration policy) in iteration step n, we have to solve the following system of linear equations in $\mathbf{v}_n \in \mathbb{R}^{|\hat{\mathcal{X}}|-1}, \gamma \in \mathbb{R}$:

$$\left[-\boldsymbol{\tau}_{\pi_n} | \overline{\mathbf{P}_{\pi_n} - \mathbf{I}}\right] \begin{pmatrix} \gamma \\ \mathbf{v}_n \end{pmatrix} = -\mathbf{c}_{\pi_n}. \tag{2.24}$$

Here, the notation for a policy π is as follows: By $\boldsymbol{\tau}_\pi$, we denote the column vector of state sojourn times $\tau_\pi(x), x \in \hat{\mathcal{X}}$. The transition probability matrix is denoted as $\mathbf{P}_\pi = (p_\pi(x, y))_{x,y \in \hat{\mathcal{X}}}$, and $\mathbf{c}_{\pi_n}$ denotes the column vector of cost values $c_\pi(x), x \in \hat{\mathcal{X}}$. By $\overline{\mathbf{A}}$, we denote the matrix that is obtained from the matrix $\mathbf{A}$ by removing the first column, and $|$ separates the first column from the others. From the structure of the transition probabilities, we see that in the matrix $\left[-\boldsymbol{\tau}_{\pi_n} | \overline{\mathbf{P}_{\pi_n} - \mathbf{I}}\right]$, most of the entries are zero. In fact, there are only $O(|\hat{\mathcal{X}}|) = O\left((K+1)C^2/(\Delta_1 \Delta_2)\right)$ non-zero entries. Hence, the system is sparse and can efficiently be solved using iterative methods for sparse and large linear systems such as BI-CGSTAB (biconjugate gradients stabilized method) [vdV92], which we used in our implementation and typically shows fast convergence in our applications. Note that by the same sparsity argument, we see that the policy improvement step can be implemented using only $O(|\hat{\mathcal{X}}|)$ steps. We also remark that the overall Policy Iteration procedure in our applications typically requires only a few (about 5) iterations until the optimal policy is found.

2.4 Example Application: GSM/UMTS system

In this section, we present an example for the application of the algorithm described in the previous section in a heterogeneous GSM-EDGE/UMTS environment. In the following, we will refer to the GSM-EDGE system simply as GSM. We consider a single cell in a multi-cell downlink scenario in which each cell is equipped with a UMTS and a GSM base

Table 2.2: System parameters.

Transmit power P_{gsm}	15W
Transmit power P_{umts}	20W
Time slots GSM T_{gsm}	21
Path loss constant GSM κ_{gsm}	12.8
Path loss constant UMTS κ_{umts}	15.3
Path loss exponent GSM γ_{gsm}	3.8
Path loss exponent UMTS γ_{umts}	3.76
Orthogonality factor ρ	0.4
Noise power η_{gsm}, η_{umts}	-100 dBm
Inter-cell interference GSM I_{gsm}	-105 dBm
C_b	$1.4 \cdot 10^9$
D_b	10^{-3}
Cell radius r_{cell}	1000m

station (BS) at the cell center and we assume that each mobile is capable of establishing a connection to both of them, but exclusively to one at the same time.

2.4.1 System Model

We assume that there are two main types of users: The first group of users [Quality-of-Service (QoS) class] is subject to quality-of-service guarantees: For each of these users, a minimum data rate out of the set of possible rate requirements $\{R_{QoS}(1), \ldots, R_{QoS}(Q)\}$ is specified. The probability for a user to have rate requirement $R_{QoS}(q)$ is denoted by $p_{QoS}(q)$. The service time of the qth QoS class is exponentially distributed with mean $1/\mu_{QoS}(q)$. Furthermore, for each QoS class a blocking cost $b_{QoS}(q)$ is specified. The users of the second group are best-effort users that share the resources that are unused by the users in the QoS group and are not subject to optimization here.

For simplicity, we assume the cell area to be of circular shape of radius r_{cell}. QoS users randomly arrive to a location inside the cell area according to a Poisson process with rate λ and remain at this position for the duration of their service. We assume that the probability for a user to be located at a specific location is uniform over the cell area. The propagation model includes only path loss; the channel gain of a user located at distance d from the base station follows the path loss model $g_s(d) = d^{-\gamma_s} 10^{-\frac{\kappa_s}{10}}$, where $s \in \{gsm, umts\}$ and γ_s, κ_s are constants.

We model the system characteristics of the two ATs as follows (the modeling is mostly along the lines of [BWKS09]):

Table 2.3: QoS class specifications.

Parameter	QoS class I	QoS class II
Rate requirement $R_{QoS}(q)$	36 kbps	12.2 kbps
Occurrence probability $p_{QoS}(q)$	0.3	0.7
Service rate $\mu_{QoS}(q)$	0.5/min	1/min
Blocking cost $b_{QoS}(q)$	2	1

GSM system characteristics. We assume a fixed BS transmission power P_{gsm}. The signal to interference and noise ratio (SINR) of a GSM user depends on the distance d of the user from the base station. For a user at distance d, it is given by

$$\beta(d) = \frac{g_{gsm}(d)P_{gsm}}{\eta_{gsm} + I_{gsm}}. \tag{2.25}$$

In (2.25), η_{gsm} is the noise power and I_{gsm} the inter-cell interference power, which is assumed to be constant. The achievable rate at a certain SINR per time slot is defined by a SINR-rate mapping $f(\beta)$, given in Figure 2.3. This mapping curve is obtained from the MRRM Simulator specification [KSB+08] and is based on link level simulations from [Agi99].

Each user i is assigned an amount of bandwidth (time slots or frequency slots) s_i; the rate that is achieved at SINR β_i is then given by $s_i f(\beta_i)$. The total amount of bandwidth is limited by T_{gsm}, implying that it must hold that

$$\sum_{\text{user } i \text{ in the system}} s_i \leq T_{gsm}. \tag{2.26}$$

UMTS system characteristics. For the interference-limited UMTS AT, we again assume a fixed BS transmission power P_{umts} and use the rate approximation from [BWKS09], which leads to a linear relationship between the data rate and the assigned resources and which offers a good approximation for fixed transmit powers and low link SINRs. More precisely, for a UMTS connection at BS distance d which is assigned the power P, the achievable data rate is approximated by $R_{umts}(d) \approx P g_{umts}(d) \frac{C_b D_b}{I(r)}$ where C_b, D_b are constants parameterizing the system characteristics such as bandwidth, modulation and bit error rate and

$$I(d) = \rho g_{umts}(d) P_{umts} + \sum_{n \in \mathcal{N}} g(n) P_{umts} + \eta_{umts}. \tag{2.27}$$

Here, ρ is the orthogonality factor which accounts for reduced inter-cell interference, η_{umts}

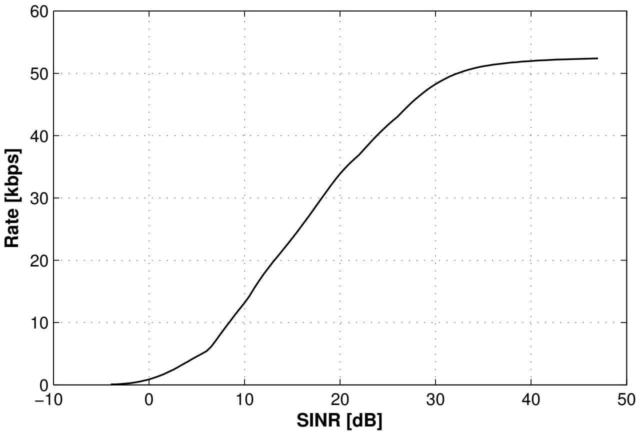

Figure 2.3: SINR-rate mapping for GSM.

is the noise power, $\mathcal{N}$ denotes a set of neighboring base stations and $g(n)$ the channel gain between the cell center and the neighboring base station n. In our implementation, we chose for $\mathcal{N}$ the base stations of two cell rings surrounding the cell under consideration.

Resulting cost and arrival model. We assume a quantization of the normalized resource values by $\Delta_{gsm}, \Delta_{umts}$. For the normalization, we let the AT capacities both equal one, that is, $C = 1$. The normalized and quantized resource amounts required to support a QoS class q at BS distance d are then given by

$$\begin{aligned} c_{gms}(d,q) &= \Delta_{gsm} \left\lceil \frac{R_{QoS}(q)}{f(\beta(d))T_{gsm}\Delta_{gsm}} \right\rceil, \\ c_{umts}(d,q) &= \Delta_{umts} \left\lceil \frac{I(d)R_{QoS}(q)}{g_{umts}(d)C_b D_b P_{umts}\Delta_{umts}} \right\rceil. \end{aligned} \tag{2.28}$$

The areas for which the quantized and normalized cost values are constant for both ATs for QoS class q users form $C(q)$ concentric circular areas (rings) around the cell center. Since arrivals are geometrically uniform, the probability for a user to be located in a specific ring with area A is given by $p(A) = \frac{A}{\pi r_{cell}^2}$. Altogether, we have $K = \sum_{q=1}^{Q} C(q)$ different user classes, where the arrival rate for a rate $R_{QoS}(q)$ user in ring with area A is given by $\lambda p_{QoS}(q)p(A)$.

2.4.2 Simulation Results

We implemented the algorithm described in the last section and applied it to the scenario described above using MATLAB. The system parameters that were used are given in Table

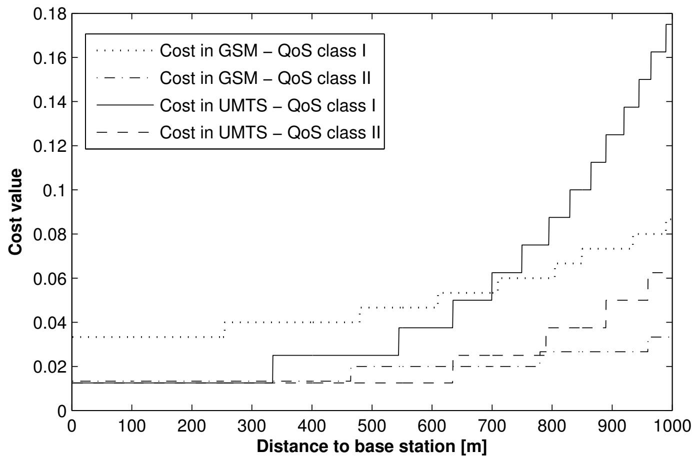

Figure 2.4: Normalized and quantized cost values for the two QoS classes in different ATs.

2.2. As an example, we consider a scenario of two QoS user classes, denoted as I and II and having the specifications given in Table 2.3. As (normalized) resource granularities, we assume $\Delta_{gsm} = 1/150, \Delta_{umts} = 1/80$. The resulting normalized cost values for the two QoS classes are illustrated in Figure 2.4. This cost model gives rise to 21 constant cost rings for QoS class I and 7 rings for QoS class II, so that there are $K = 29$ different user classes (including the dummy user class). The resulting number of states is 366 930, which is a reasonable number from the viewpoint of computational requirements. We remark that computing the actual number of states in the non-aggregated state space $\mathcal{X}$ is difficult. However, it is easy to see that the size of $\mathcal{X}$ is extremely large in comparison to the size of the aggregated state space: Counting only the number of possible states with up to three positive entries in the state matrix, one obtains a count of 636 807 614 states. Clearly, the actual size is much larger than this number. The system was simulated using the (non-aggregated) SMDP model from Section 2.2 for an equivalent of about $1.5{\cdot}10^5$ minutes and for different mean arrival rates λ, comparing the expected mean blocking cost results for the aggregation, the Greedy and the Load-based policy. The computation time required for the optimization for each arrival parameter λ was roughly 60 minutes, which is reasonable considering the complexity of the problem that is dealt with here. Figure 2.5 shows the simulation results in terms of the improvement in percentage that the aggregation policy shows in comparison to the other two strategies. We see that the performance improvement in comparison to the Load-based policy is big over the whole range of mean arrival rates. It is also substantial with respect to the Greedy policy, especially for lower arrival rates.

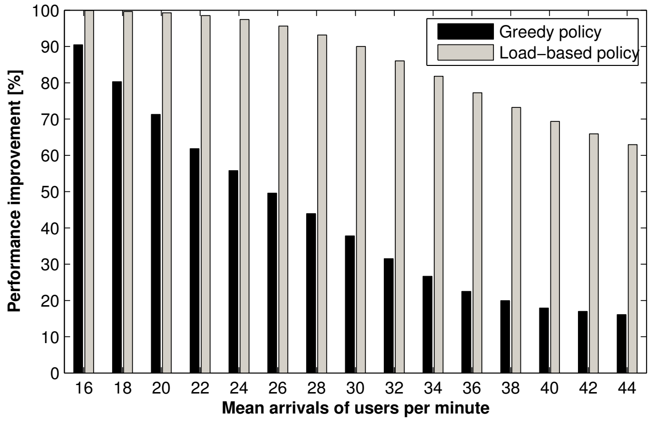

Figure 2.5: Performance improvement of the aggregation policy with respect to the Greedy and the Load-based policy.

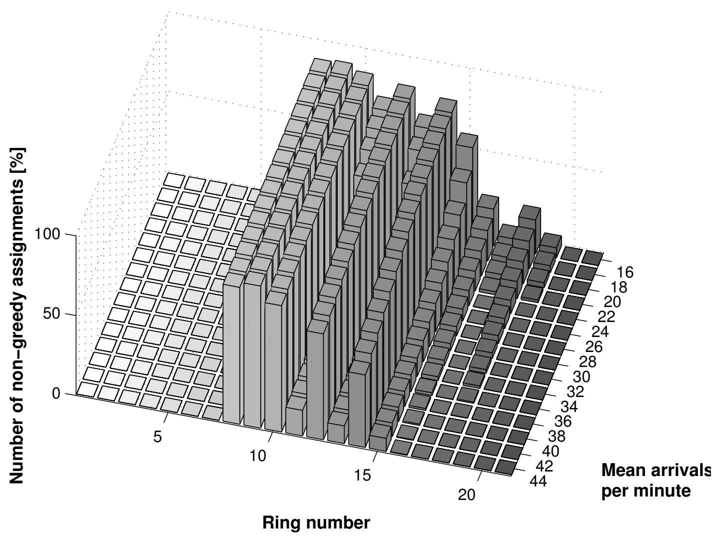

Figure 2.6: Fraction of HAM assignments for QoS class I in the aggregation policy that differ from the Greedy policy (excluding unforced blocking actions).

In the following, we illustrate some properties of the aggregation policies that were obtained. Figure 2.6 displays for each arrival rate λ and each ring of QoS class I the fraction of non-Greedy assignments, i.e., the fraction of states $(l_1, l_2, k), k > 0$ in which the aggregation policy differs from the Greedy policy (on the aggregated state space), not counting the cases in which blocking occurs but is not enforced due to capacity limits. Just as for optimal policies (as discussed in Section 2.2), these non-Greedy assignments increase the system performance by occasionally assigning users of QoS class I to an AT with higher cost, thereby balancing the load across the two ATs and reserving capacity for a potentially arriving burst of QoS class II users, which have a higher occurrence probability. We see that such assignments take place mainly between the rings 8 and 15, where roughly every second class is redirected almost completely to the more expensive AT, in this case the UMTS system. Figure 2.7 shows, again for QoS class I, the fraction of states in which blocking occurs in the aggregation policy, but is not necessary due to the capacity constraint (unforced blocking). We see that unforced blocking is rather rare, but happens more frequently with increasing arrival rates and increasing cost values. Since non-Greedy assignments for QoS class I are directed to the UMTS system, one intuitively expects them to be less frequent if the load in the UMTS system is high and more frequent if the GSM load is high. In fact, this can be seen in Figure 2.8, which displays for the case $\lambda = 28$ users/minute the number of non-Greedy assignments for QoS class I in the aggregation policy for each possible system load pair.

Finally, we remark that the Greedy and the Load-based policy are oblivious of differences in the blocking cost values of the user classes. However, the performance improvement using the aggregation algorithm for systems with identical blocking cost is typically in the same order of magnitude as in examples with differing blocking cost values.

2.5 Structural Results for Optimal Policies

In this section, we discuss some structural properties of optimal policies. Particularly, we are interested in certain monotonicity properties. In general, optimal policies are of complex and often non-intuitive structure. For example, this can be demonstrated by looking at monotonicity properties with respect to user blocking: For user classes k, l and an AT s, we say that a stationary policy π is (k, s, l)-*monotone* if for all $\mathbf{X} \in \mathcal{X}$ the following holds:

$$\pi(\mathbf{X}, k) = 0 \Rightarrow \pi(\mathbf{X} + \mathbf{E}_{sl}, k) = 0 \text{ if } \mathbf{X} + \mathbf{E}_{sl} \in \mathcal{X}. \tag{2.29}$$

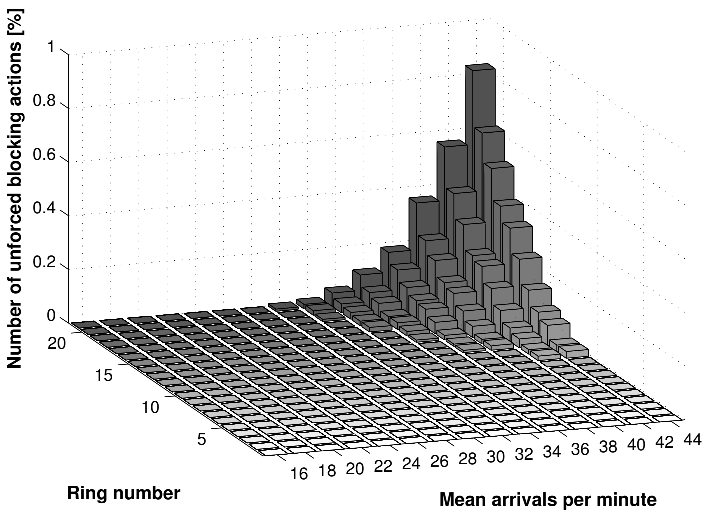

Figure 2.7: Fraction of unforced blocking actions for QoS class I in the aggregation policy.

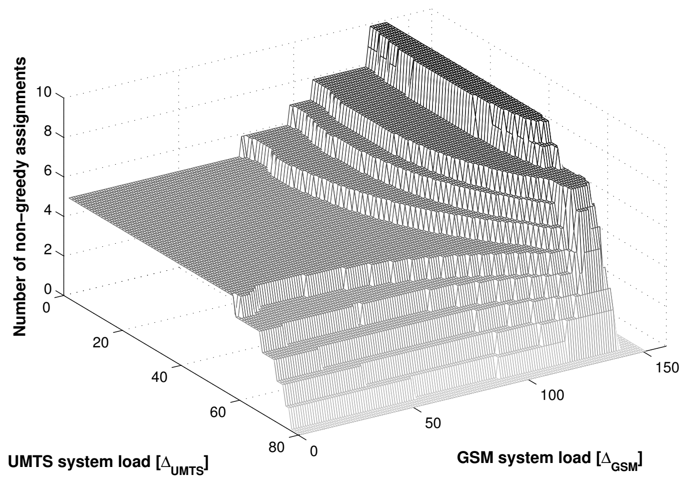

Figure 2.8: Number of non-Greedy assignments in the aggregation policy for QoS class I, $\lambda = 28$ users/minute and different load states.

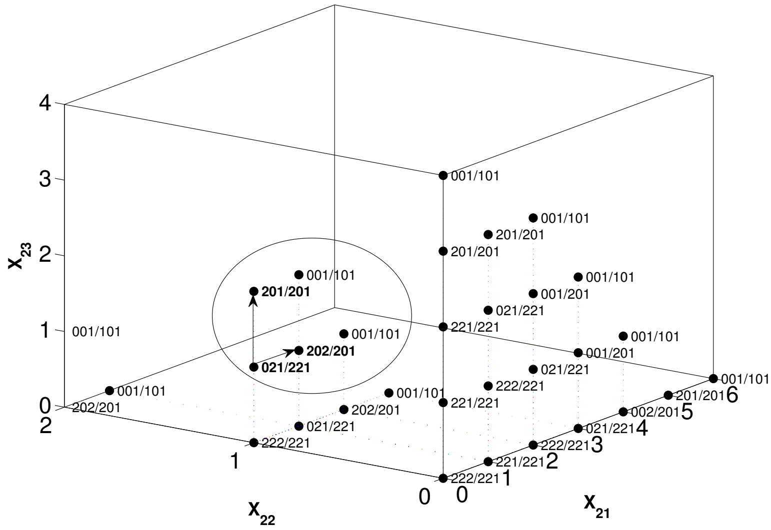

Figure 2.9: Optimal actions and Greedy actions (displayed as $\pi(\mathbf{X},1)\pi(\mathbf{X},2)\pi(\mathbf{X},3)$ / $\pi^g(\mathbf{X},1)\pi^g(\mathbf{X},2)\pi^g(\mathbf{X},3)$ for each state $\mathbf{X}$) in AT 2 for Example II for $X_{11} = 1, X_{12} = 0, X_{13} = 2$.

In other words, if a (k,s,l)-monotone policy blocks a class k user in some state $\mathbf{X}$, then it also blocks this user class in the system state that is reached by assigning a class l user to AT s. Generally, monotonicity is violated for optimal policies. As an example, consider the optimal policy for the system II given in Table 2.1. It is (k,s,l)-monotone except for $k=1, s=2, l=1$ and $k=1, s=2, l=3$. Figure 2.9 shows the projection of the state space to AT 2 where the state of AT 1 is fixed to $X_{11}=1, X_{12}=0, X_{13}=2$ and for each state the optimal policy π and the Greedy policy π^g. The state $X_{21}=0, X_{22}=1, X_{23}=1$ violates both $(1,2,1)$-monotonicity and $(1,2,3)$-monotonicity. Even though monotonicity is generally violated, there are some cases where such a property can be proved. We demonstrate this in the remainder of this section.

A powerful technique for the investigation of structural properties of MDP models is provided by the method of *event-based Dynamic Programming* introduced in [AK95] (see also [AJK01], [Koo06] and [Koo96] for more details on this approach). Here, the value function evolution is described by the application of a *Dynamic Programming (DP) operator*, which is a linear combination of *event operators*, allowing to prove properties of the value function by proving these properties for the event operators separately. This technique has been applied for the stochastic Knapsack problem in [AJK01] to obtain structural policy properties, showing submodularity of the value function and a resulting monotonicity property of the optimal policy for the case of two user classes. In this section, we extend some of these results to the situation of multiple ATs. Here, we prove monotonicity and supermodularity of the value function in certain layers of the state space for the case of two user classes and ATs and a specific ordering of the cost values, also resulting in a monotonicity property of the optimal HAM policy and "almost-optimality" of the Greedy policy. We remark that since we are dealing with a cost-based model rather than a reward-based model as in [AJK01], we obtain supermodularity instead of submodularity.

In order to apply event-based Dynamic Programming, we define the following event operators on the set of functions $f : \mathbb{N}^{S \times K} \to \mathbb{R}$: For each class k, the *HAM operator* $T_A^{(k)}$ is defined as

$$T_A^{(k)} f(\mathbf{X}) := \min \left\{ \min_{s \in \{1,\dots,s\}} \{ f(\mathbf{X} + \mathbf{E}_{sk}), f(\mathbf{X}) + b_k \} \right\}. \tag{2.30}$$

For each class k, each AT s and each l with $1 \leq l \leq \lfloor C/c_{sk} \rfloor$, we define the *departure operator* $T_D^{(k,s,l)}$ by

$$T_D^{(s,k,l)} f(\mathbf{X}) := \begin{cases} f(\mathbf{X} - \mathbf{E}_{sk}), & X_{sk} \geq l \\ f(\mathbf{X}), & \text{otherwise.} \end{cases} \tag{2.31}$$

For the modeling of capacity constraints, we further introduce the boundary function $Q(\mathbf{X})$, which is defined as $Q(\mathbf{X}) = 0, \mathbf{X} \in \mathcal{X}$ and $Q(\mathbf{X}) = \infty$ if $\mathbf{X} \notin \mathcal{X}$. The event-based *DP operator* is then given by the following linear combination of control and departure operators:

$$Tf(\mathbf{X}) := Q(\mathbf{X}) + \sum_{k=1}^{K} \lambda_k T_A^{(k)} f(\mathbf{X}) + \sum_{s=1}^{S} \sum_{k=1}^{K} \mu_k \sum_{l=1}^{\lfloor C/c_{s,k} \rfloor} T_D^{(s,k,l)} f(\mathbf{X}). \tag{2.32}$$

The (event-based) *value function* v_n is then recursively defined using the DP operator by $v_0(\mathbf{X}) := Q(\mathbf{X}),\ v_{n+1}(\mathbf{X}) := Tv_n(\mathbf{X})$. We now show some structural properties of the event-based value function. Following notation in [Koo06], for an operator O and properties $\mathcal{P}_1, \mathcal{P}_2$ of the value function v_n, we write $O : \mathcal{P}_1 \to \mathcal{P}_2$ if the following holds:

$$\forall n \in \mathbb{N}:\ v_n \text{ satisfies property } \mathcal{P}_1 \Rightarrow Ov_n \text{ satisfies property } \mathcal{P}_2. \tag{2.33}$$

In the following, we prove several properties of the event-based value function. The basic proof strategy is induction over n. For the induction step, it is shown for each operator O independently that $O : \mathcal{P} \to \mathcal{P}$ for the property $\mathcal{P}$ to be proved, and then extending the result to the value function by using the fact that the properties $\mathcal{P}$ are closed under linear combinations [Koo06]. In the following, we will again denote $\overline{1} = 2, \overline{2} = 1$. We start with the following general monotonicity property:

Lemma 1. *(Monotonicity) For all $n \geq 0$, $\mathbf{X} \in \mathbb{N}^{S\times K}$ and all $i, j \in \{1,2\}$, it holds that $v_n(\mathbf{X} + \mathbf{E}_{ij}) \geq v_n(\mathbf{X})$.*

Proof. The proof is similar to the monotonicity proof in [AJK01] (Lemma II.1). □

We now consider the more specific case of a system with $S = K = 2$ and where there is the specific ordering of the cost values $c_{11} \leq c_{12}, c_{11} \leq c_{21}, c_{22} \leq c_{12}, c_{22} \leq c_{21}$ and $\mu_1 = \mu_2$. We refer to such systems as *ordered binary systems.* For these systems, we have the following lemmata:

Lemma 2. *(Ordering monotonicity) For an ordered binary system, for all $n \geq 0$ and all $\mathbf{X} \in \mathbb{N}^{S\times K}$ and $j \in \{1,2\}$, it holds that $v_n(\mathbf{X} + \mathbf{E}_{j\bar{j}}) \geq v_n(\mathbf{X} + \mathbf{E}_{jj})$ Moreover, if $\mathbf{X} + \mathbf{E}_{jj} \in \mathcal{X}$ then $v_n(\mathbf{X} + \mathbf{E}_{\bar{j}j}) \geq v_n(\mathbf{X} + \mathbf{E}_{jj})$.*

Proof. Due to the ordering of the cost values, the inequalities are certainly true for $n = 0$. Let $\mathcal{M}_1, \mathcal{M}_2$ denote the first and the second monotonicity property, respectively and $\mathcal{M}$ denote $\mathcal{M}_1$ and $\mathcal{M}_2$ together. For $T_A^{(k)} : \mathcal{M} \to \mathcal{M}$, it suffices to prove $T_A^{(k)} : \mathcal{M}_1 \to \mathcal{M}_1$ and $T_A^{(k)} : \mathcal{M} \to \mathcal{M}_2$, which are easily established using the induction assumption. $T_D^{(s,k,l)}$: $\mathcal{M}_2 \to \mathcal{M}_2$ follows immediately except for the cases $(\mathbf{X} + \mathbf{E}_{\bar{j}j})_{sk} = l, (\mathbf{X} + \mathbf{E}_{jj})_{sk} < l$ and $(\mathbf{X} + \mathbf{E}_{\bar{j}j})_{sk} < l, (\mathbf{X} + \mathbf{E}_{jj})_{sk} = l$, in which case the statement is proved by combining these two cases together as in the monotonicity proof in [AJK01] (Lemma II.2). It is here where we need that $\mu_1 = \mu_2$. Similarly, one can prove $T_D^{(s,k,l)} : \mathcal{M}_1 \to \mathcal{M}_1$. □

Lemma 3. *(Partial supermodularity) For an ordered binary system, for all $n \geq 0$ and all $\mathbf{X} \in \mathbb{N}^{S\times K}$, it holds that*

$$v_n(\mathbf{X}) + v_n(\mathbf{X} + \mathbf{E}_{11} + \mathbf{E}_{22}) \geq v_n(\mathbf{X} + \mathbf{E}_{11}) + v_n(\mathbf{X} + \mathbf{E}_{22}), \tag{2.34}$$

i.e., the event-based value function is supermodular in each X_{11}-X_{22} layer of the state space. Moreover, for $s \in \{1,2\}$ such that $\mathbf{X} + \mathbf{E}_{\bar{s}\bar{s}} \notin \mathcal{X}$, it holds that

$$v_n(\mathbf{X}) + v_n(\mathbf{X} + \mathbf{E}_{s1} + \mathbf{E}_{s2}) \geq v_n(\mathbf{X} + \mathbf{E}_{s1}) + v_n(\mathbf{X} + \mathbf{E}_{s2}). \tag{2.35}$$

Proof. We denote the claimed supermodularity properties (2.34) and (2.35) by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and $\mathcal{S}_1$ and $\mathcal{S}_2$ together by $\mathcal{S}$. Clearly, $\mathcal{S}$ holds for $n = 0$. For the induction step, we prove that $T_D^{(s,k,l)} : \mathcal{S} \to \mathcal{S}$ and $T_A^{(k)} : \mathcal{S} \to \mathcal{S}$. It is easily verified that $T_D^{(s,k,l)} : \mathcal{S} \to \mathcal{S}$. To show $T_A^{(k)} : \mathcal{S} \to \mathcal{S}$, it obviously suffices to prove that $T_A^{(k)} : \mathcal{S}_2 \to \mathcal{S}_2$ and $T_A^{(k)} : \mathcal{S} \to \mathcal{S}_1$.

We start with $T_A^{(k)} : \mathcal{S}_2 \to \mathcal{S}_2$. For this, we let $s \in \{1,2\}$ and $\mathbf{X}$ be such that $\mathbf{X}+\mathbf{E}_{\overline{s}\overline{s}} \notin \mathcal{X}$. We denote by $p^{(k)}$ the minimizing action of $T_A^{(k)}$ in state $\mathbf{X}$ and by $q^{(k)}$ the minimizing action of $T_A^{(k)}$ in state $\mathbf{X}+\mathbf{E}_{s1}+\mathbf{E}_{s2}$ (where 0 denotes blocking). Due to $\mathbf{X}+\mathbf{E}_{\overline{s}\overline{s}} \notin \mathcal{X}$ and the cost value ordering, $p^{(k)} \neq \overline{s}$ and $q^{(k)} \neq \overline{s}$. In the case $p^{(k)} = q^{(k)}$, the statement immediately follows from (2.35). For $p^{(k)} = 0$ and $q^{(k)} = s$, we have

$$\begin{aligned} T_A^{(k)} v_n(\mathbf{X}) + T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{s1}+\mathbf{E}_{s2}) &= v_n(\mathbf{X}) + b_k + v_n(\mathbf{X}+\mathbf{E}_{s1}+\mathbf{E}_{s2}+\mathbf{E}_{sk}) \\ &= v_n(\mathbf{X}) + b_k + v_n(\mathbf{X}+2\mathbf{E}_{sk}+\mathbf{E}_{s\overline{k}}) \\ &\geq v_n(\mathbf{X}+\mathbf{E}_{s\overline{k}}) + v_n(\mathbf{X}+2\mathbf{E}_{sk}) \\ &\geq T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{s\overline{k}}) + T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{sk}) \end{aligned} \tag{2.36}$$

by applying (2.35) twice. For $p^{(k)} = s$ and $q^{(k)} = 0$, we get

$$\begin{aligned} T_A^{(k)} v_n(\mathbf{X}) + T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{s1}+\mathbf{E}_{s2}) &= v_n(\mathbf{X}+\mathbf{E}_{sk}) + v_n(\mathbf{X}+\mathbf{E}_{s1}+\mathbf{E}_{s2}) + b_k \\ &\geq T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{s\overline{k}}) + T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{sk}). \end{aligned} \tag{2.37}$$

Next, we prove $T_A^{(k)} : \mathcal{S} \to \mathcal{S}_1$. For this, let $\mathbf{X} \in \mathbb{N}^{S \times K}$. In the following, we denote by $p^{(k)}$ the minimizing action of $T_A^{(k)}$ in state $\mathbf{X}$ and by $q^{(k)}$ the minimizing action of $T_A^{(k)}$ in state $\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}$ (where 0 denotes blocking). Again, the case $p^{(k)} = q^{(k)}$ easily follows from (2.34). Lemma 2 implies that the situation $p^{(k)} = \overline{k}, q^{(k)} = k$ is impossible. Now consider the case $p^{(k)} = k, q^{(k)} = \overline{k}$. Here,

$$\begin{aligned} T_A^{(k)} v_n(\mathbf{X}) + T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) &= v_n(\mathbf{X}+\mathbf{E}_{kk}) + v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}+\mathbf{E}_{\overline{k}k}) \\ &= v_n(\mathbf{X}+\mathbf{E}_{kk}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}\overline{k}}+\mathbf{E}_{\overline{k}k}) \\ &\geq v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}k}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}\overline{k}}) \\ &\geq T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{11}) + T_A^{(k)} v_n(\mathbf{X}+\mathbf{E}_{22}), \end{aligned} \tag{2.38}$$

where the second-last inequality follows from $\mathbf{X}+2\mathbf{E}_{kk} \notin \mathcal{X}$ (this follows from Lemma 2 since $q^{(k)} = \overline{k}$) and applying inequality (2.35). Next, we study the case $p^{(k)} = 0, q^{(k)} = \overline{k}$. Here again, it holds that $\mathbf{X}+2\mathbf{E}_{kk} \notin \mathcal{X}$, so that

$$v_n(\mathbf{X}+\mathbf{E}_{kk}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}\overline{k}}+\mathbf{E}_{\overline{k}k}) \geq v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}k}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}\overline{k}}). \tag{2.39}$$

Adding this inequality to inequality (2.34), we obtain

$$v_n(\mathbf{X}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}\overline{k}}+\mathbf{E}_{\overline{k}k}) \geq v_n(\mathbf{X}+\mathbf{E}_{\overline{k}\overline{k}}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}k}). \tag{2.40}$$

Then

$$\begin{aligned}T_A^{(k)}v_n(\mathbf{X}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) &= v_n(\mathbf{X}) + b_k + v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}+\mathbf{E}_{\overline{k}k}) \\ &\geq v_n(\mathbf{X}+\mathbf{E}_{\overline{kk}}) + v_n(\mathbf{X}+\mathbf{E}_{kk}+\mathbf{E}_{\overline{k}k}) + b_k \\ &\geq T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{22}). \quad (2.41)\end{aligned}$$

For the case $p^{(k)} = 0, q^{(k)} = k$, we have

$$\begin{aligned}T_A^{(k)}v_n(\mathbf{X}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) &= v_n(\mathbf{X}) + b_k + v_n(\mathbf{X}+2\mathbf{E}_{kk}+\mathbf{E}_{\overline{kk}}) \\ &\geq v_n(\mathbf{X}+\mathbf{E}_{\overline{kk}}) + b_k + v_n(\mathbf{X}+2\mathbf{E}_{kk}) \\ &\geq T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{22}). \quad (2.42)\end{aligned}$$

In the case $p^{(k)} = k, q^{(k)} = 0$, we get

$$\begin{aligned}T_A^{(k)}v_n(\mathbf{X}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) &= v_n(\mathbf{X}+\mathbf{E}_{kk}) + v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) + b_k \\ &\geq T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{22}).\end{aligned} \quad (2.43)$$

It only remains the situation $p^{(k)} = \overline{k}, q^{(k)} = 0$. Here,

$$\begin{aligned}T_A^{(k)}v_n(\mathbf{X}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) &= v_n(\mathbf{X}+\mathbf{E}_{\overline{k}k}) + v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) + b_k \\ &= \infty \\ &\geq T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{11}) + T_A^{(k)}v_n(\mathbf{X}+\mathbf{E}_{22}), \quad (2.44)\end{aligned}$$

since $p^{(k)} = \overline{k}$ implies $\mathbf{X}+\mathbf{E}_{kk} \notin \mathcal{X}$ by Lemma 2. This completes the proof. □

From the above lemmata, we conclude the following properties of optimal policies for ordered binary systems:

Theorem 1. *For an ordered binary system, the optimal policy π is $(2,1,1)$-monotone. Moreover, if no unforced blocking occurs for user class k in state $\mathbf{X} \in \mathcal{X}$, the optimal HAM assignment in state $\mathbf{X}$ for class k users is according to the Greedy policy: $\pi(\mathbf{X},k) = \pi^g(\mathbf{X},k)$.*

Proof. For the first statement, let $n \in \mathbb{N}$ and $\mathbf{X} \in \mathbb{N}^{S\times K}$ such that $\mathbf{X}+\mathbf{E}_{22} \in \mathcal{X}$ and $v_n(\mathbf{X}) + b_2 \leq v_n(\mathbf{X}+\mathbf{E}_{22})$. Using equation (2.34) in Lemma 3 then implies

$$v_n(\mathbf{X}+\mathbf{E}_{11}) - v_n(\mathbf{X}+\mathbf{E}_{11}+\mathbf{E}_{22}) \leq v_n(\mathbf{X}) - v_n(\mathbf{X}+\mathbf{E}_{22}) \leq -b_2 \quad (2.45)$$

For the case $\mathbf{X} + \mathbf{E}_{22} \notin \mathcal{X}$, we similarly use (2.34) with $s = 1$ in Lemma 3. Since these inequalities hold for any n (and hence also in the limit), the statement for the optimal policy follows. Likewise, the inequalities in Lemma 2 carry over to the limit, so that the second statement is a direct consequence of this lemma. □

2.6 Summary and Conclusions

In this chapter, we studied the cost-based model for HAM, that is, access management (admission control and access selection) in heterogeneous network environments, thereby considering information about user traffic (random arrivals and departures) for the minimization of the expected mean cost for blocking events. Two heuristic strategies for HAM are given by the Greedy policy and the Load-based policy. The Greedy policy "greedily" assigns users to the AT which is most efficient in terms of resource requirements, whereas the intention behind the Load-based policy is to balance resource usage over the ATs. The problem was modeled as a semi-Markov decision model, allowing to find optimal policies using standard algorithms such as Value Iteration. For complexity reasons, this is only possible for relatively small system dimensions. The model sizes that arise when dealing with models of real-world systems preclude this approach. In order to tackle this problem, we presented a state aggregation procedure for the computation of HAM policies which results in a drastic state space (and hence, complexity) reduction.

The algorithm was applied in a heterogeneous GSM-EDGE/UMTS scenario, obtaining significant performance gains in comparison to the straightforward Greedy and Load-based policies. This indicates that the aggregated model, though of greatly reduced state space size, still carries enough information about the problem structure to be able to find good control policies. Therefore, the techniques proposed in this chapter represent a promising approach for the optimization of admission control and access selection in heterogeneous network environments. Moreover, we also investigated structural properties of optimal admission control policies. It was shown that generally, one cannot expect monotonicity properties with respect to blocking of users. However, for the case of ordered binary systems (systems with two ATs and two user classes with identical service rates and satisfying a specific cost value ordering), we proved such a monotonicity property and also showed that for these systems, the Greedy policy is "almost" optimal.

3 Capacity Computation for the Multiple Access Channel

In the previous chapter, the problem of user assignment for heterogeneous networks has been studied. For this, a simple orthogonal model has been used to establish the connection between the achievable data rates and the amount of resources spent. The capacity regions (or achievable rate regions) provided by information theory predict that generally, much higher rates can be attained in multi-user systems such as the multiple access channel by employing more sophisticated transmission techniques.

For some multiuser channel models, the capacity region can be characterized in terms of mutual information expressions. However, even for channels where such a single-letter representation is available, evaluation of the capacity region is often a hard problem since computation of the capacity region boundary is generally a difficult and non-convex optimization problem. For the single-user discrete memoryless channel, computation of capacity is a convex problem, and several numerical methods that allow to calculate the capacity within arbitrary precision have been developed, e.g. the Blahut-Arimoto algorithm [Bla72], [Ari72]. For the discrete memoryless multiple access channel, no algorithms for the computation of the capacity region boundary are known. A fundamental step in this direction has been taken in [Wat96], where a numerical method for calculating the sum-rate capacity (also called total capacity) of the two-user MAC with binary output has been developed. This was achieved by showing that the calculation of the sum capacity can be reduced to the calculation of the sum capacity for the two-user MAC with binary input and binary output and by giving necessary and sufficient conditions for sum-rate optimality by a partial modification of the Karush-Kuhn-Tucker (KKT) conditions. The work in [CPFV07] considers the computation of not only the sum capacity, but of the whole capacity region of the two-user discrete MAC, i.e., the problem of maximizing weighted sum-rate over the capacity region. The problem of sum-rate optimization as in [Wat96] is obtained as a special case by choosing equal weights in the weighted sum-rate problem. In [CPFV07] it is shown for the two-user case that the only non-convexity in the problem stems from the requirement of the input probability distributions to be independent, that is, from the constraint for the probability matrix specifying the joint probability input distribution to be of rank one. An approximate solution to the problem is proposed by removing this independence constraint (i.e., relaxation of the rank-one constraint), obtaining an outer bound region to the actual capacity region. By projecting the obtained probability distribution to independent distributions by calculating the marginals, one obtains an inner bound region.

In this chapter, we discuss three different aspects of the MAC capacity computation problem. First of all, in a recent work [WK09] (the main results were already reported in [WK02] and [WK04]), the results in [Wat96] have been extended to the $(n_1, \dots, n_K; m)$-MAC, which is the K-user discrete MAC with m users, each with an alphabet of size n_k and output alphabet of size m. Such a channel is defined as *elementary* if the size of each input alphabet is less than or equal to the size of the channel output alphabet, i.e., $n_k \leq m$ for all $k = 1, \dots, K$. For any $(n_1, \dots, n_K; m)$-MAC, one can construct an elementary sub-MAC by restricting the admissible input distributions. In [WK09], it is shown that for any $(n_1, \dots, n_K; m)$-MAC, there exists such an elementary sub-MAC that achieves the same total capacity as the $(n_1, \dots, n_K; m)$-MAC. The main part of the paper is concerned with the investigation of the (KKT) conditions for the sum-rate optimization for the elementary MAC. It is claimed that the KKT-conditions provide a necessary and sufficient condition for optimality for the elementary MAC. Here, we demonstrate that this claim is not true. It is also noteworthy that the work in [RG04] uses the sufficiency result for the KKT conditions to generalize the Blahut-Arimoto algorithm for the computation of the sum capacity of the $(n_1, \dots, n_K; m)$-MAC. From this, we conclude that the capacity computation problem for the discrete multiple-access channel remains an interesting unsolved problem.

Secondly, we investigate properties of the optimality conditions for the capacity computation problem for the case of two users and binary input and output alphabets. We prove that for a class of $(2, 2; 2)$-MACs, the weighted sum objective function has at most one stationary point in the interior of the domain. Beside the fact that this is an interesting structural property which gives valuable insight into the general problem, it can also be employed for numerical solutions of the problem. Since the maximum of the objective function on the boundary can be found by solving the single-user problem, it suffices to search for stationary points in the interior of the domain: as there is at most one stationary point in the interior, methods such as gradient descent can return a suboptimal solution only if the global optimum is located on the boundary, which is then found by the boundary search. What is more, we prove the statement by showing that the problem in the interior can be reduced to a pseudoconcave one- dimensional problem, resulting in an efficient optimization procedure for a specified tolerance of deviation from the optimal point for one of the input parameters. We remark that there is numerical evidence for the conjecture that also for the general $(2, 2; 2)$-MAC, there is at most one stationary point in the interior of the domain, which we unfortunately could not prove. In addition, we prove comprehensiveness of the set of rates achievable using successive interference cancellation in a fixed decoding order.

Finally, we study properties of the optimality conditions for the relaxation (cooperation) approach suggested in [CPFV07]. We derive conditions under which a solution to the relaxed problem has the same value as the actual optimal solution and show how these conditions can in some cases be applied to construct solutions for a restricted class of $(2, 2; 2)$-MACs using convex optimization. Opposed to this, we demonstrate by examples that even for these channels, the marginal approach suggested in [CPFV07] generally offers only suboptimal solutions. We remark that in addition, these results offer a convenient characterization of the capacity region for this channel subclass which can be used for further studies such as the investigation of duality relations between the discrete multiple access and broadcast channel (see Chapter 4).

3.1 Problem Formulation

The communication model under study in this chapter is the *discrete memoryless K-user multiple-access channel*. In this channel model, there are K transmitters, each conveying a message to a single receiver. The channel has, for each user $k \in \{1, \ldots, K\}$, an input alphabet $\mathcal{X}_k = \{1, \ldots, n_k\}$ of size n_k and an output alphabet $\mathcal{Y} = \{1, \ldots, m\}$ of size m. The channel is given by a channel matrix

$$\mathbf{Q} = (Q(y|x_1, \ldots, x_K))_{1 \leq y \leq m, 1 \leq x_k \leq n_k} \tag{3.1}$$

of size $m \times (n_1 \cdots n_K)$ which specifies the channel transition probabilities. More precisely, denoting the input random variable for user k as X_k and the output random variable as Y, we have $\mathbf{Pr}[Y = y|X_1 = x_1, \ldots, X_K = x_K] = Q(y|x_1, \ldots, x_K)$. Hence, $\mathbf{Q}$ is required to satisfy $\sum_{y=1}^{m} Q(y|x_1, \ldots, x_K) = 1$ for all $x_k \in \mathcal{X}_k$. Moreover, the channel is memoryless, meaning that for input sequences $\mathbf{x}_k \in \mathcal{X}_k^n$ and an output sequence $\mathbf{y} \in \mathcal{Y}^n$

$$\mathbf{Pr}[Y^n = \mathbf{y}|X_1^n = \mathbf{x}_1, \ldots, X_K^n = \mathbf{x}_K] = \prod_{i=1}^{n} Q(y(i)|x_1(i), \ldots, x_K(i)), \tag{3.2}$$

where the input random variable sequences are denoted as X_k^n and the output random variable sequence as Y^n. Note that here, and in the next section, we address the kth component of a vector $\mathbf{x}$ by $x(k)$. We denote the channel defined above as $(n_1, \ldots, n_K; m)$-MAC in the following. In [WK09], the $(n_1, \ldots, n_K; m)$-MAC is called *elementary* if $n_k \leq m$

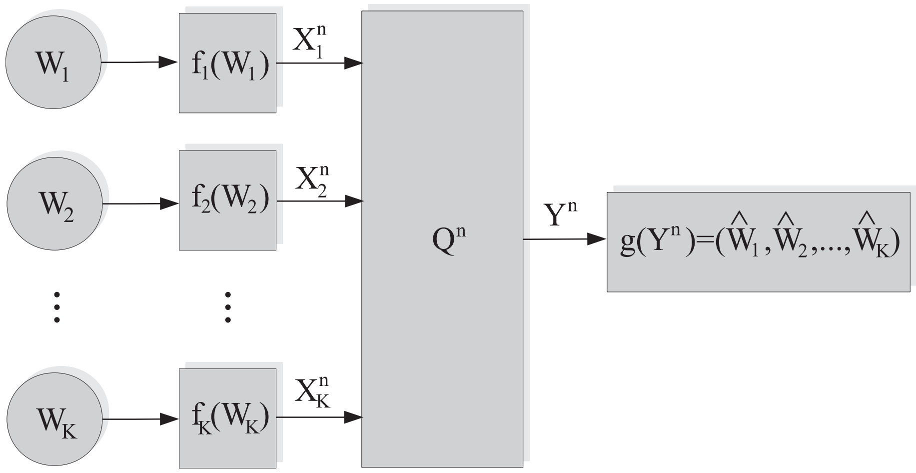

Figure 3.1: The $(n_1, \dots, n_K; m)$-MAC with encoders f_k and the decoder g. Here, Q^n denotes the extension of Q to n time slots according to (3.2).

holds for all users $k \in \{1, \dots, K\}$, that is if each input alphabet size is less or equal than the cardinality of the output alphabet.

We now give some basic information-theoretic notions for the discrete memoryless MAC (see Figure 3.1 for an illustration). A $[(|\mathcal{W}_1|, \dots, |\mathcal{W}_K|), n]$-*code* is given by the message sets $\mathcal{W}_k = \{1, \dots, |\mathcal{W}_k|\}$ and *encoding functions*

$$f_k : \mathcal{W}_k \to \mathcal{X}_k^n \tag{3.3}$$

for the users $k \in \{1, \dots, K\}$, i.e., transmitter k uses the codeword $f_k(W_k)$ of length n to encode the message W_k, which is assumed to be drawn uniformly out of the message set $\mathcal{W}_k$. Finally, the receiver uses the *decoding function*

$$g : \mathcal{Y}^n \to \mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_K \tag{3.4}$$

in order to produce an estimate $g(Y^n) = (\widehat{W}_1, \widehat{W}_2, \dots, \widehat{W}_K)$ of the transmitted messages.

The figures of merit associated with the code are the *transmission rate* (in nats/symbol transmission), given by $R_k = \frac{\ln |\mathcal{W}_k|}{n}$ for user k, and the *average probability of error*[1]

$$P_{\text{err}}^{(n)} = \frac{1}{\prod_k |\mathcal{W}_k|} \sum_{(w_1,\dots,w_K) \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_K} \mathbf{Pr}\left[g(Y^n) \neq (w_1, \dots, w_K) | (w_1, \dots, w_K) \text{ sent}\right]. \tag{3.5}$$

[1]Note that unlike for the single-link discrete memoryless channel, the capacity region is generally smaller (and still unknown) under the *maximal* error probability criterion [Due78].

From this, the notion of the capacity region of the channel is defined as follows:

- A rate vector $(R_1, \ldots, R_K)$ is said to be *achievable* if there exists a sequence of $[(\lceil e^{nR_1} \rceil, \ldots, \lceil e^{nR_K} \rceil), n]$-codes with $P_{\text{err}}^{(n)} \to 0$ for $n \to \infty$.
- The *capacity region* $\mathcal{C}_{(n_1,\ldots,n_K;m)}$ of the $(n_1, \ldots, n_K; m)$-MAC is defined as the closure of the set of achievable rate vectors.

The capacity region $\mathcal{C}_{(n_1,\ldots,n_K;m)}$ can be expressed as follows: for each user k, we define the set $\mathcal{P}_k := \left\{\mathbf{p} \in \mathbb{R}_+^{n_k} : \sum_{i=1}^{n_k} p(i) = 1\right\}$ of probability vectors and set $\mathcal{P} := \mathcal{P}_1 \times \cdots \times \mathcal{P}_K$. We remark that we use the Cartesian product instead of the Kronecker product as in [WK09] to form the space of admissible input probability distributions. However, this amounts just to a difference in notation. For a subset $S \subseteq \{1, \ldots, K\}$ we let $\overline{S}$ denote the complement of S and set $X(S) := \{X_i : i \in S\}$. Moreover, for a vector $\mathbf{r} \in \mathbb{R}^K$ and $S \subseteq \{1, \ldots, K\}$, we let $r(S) := \sum_{s \in S} r(s)$. For each input probability distribution (IPD) $(\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathcal{P}$, define the set

$$\mathcal{R}(\mathbf{p}_1, \ldots, \mathbf{p}_K) := \left\{\mathbf{r} \in \mathbb{R}_+^K : r(S) \leq I(X(S); Y | X(\overline{S})) \, \text{for all} \, S \subseteq \{1, \ldots, K\}\right\}. \tag{3.6}$$

Then, the capacity region $\mathcal{C}_{(n_1,\ldots,n_K;m)}$ of the $(n_1, \ldots, n_K; m)$-MAC is given by [Ahl71], [Lia72], [CT06]

$$\mathcal{C}_{(n_1,\ldots,n_K;m)} = \mathbf{Co} \left(\bigcup_{(\mathbf{p}_1,\ldots,\mathbf{p}_K) \in \mathcal{P}} \mathcal{R}(\mathbf{p}_1, \ldots, \mathbf{p}_K) \right). \tag{3.7}$$

We note that each set $\mathcal{R}(\mathbf{p}_1, \ldots, \mathbf{p}_K)$ forms a convex polytope with $K!$ corner points, corresponding to rate points achievable by the $K!$ possible *successive interference cancellation* orders: for each user permutation (ordering) π and IPD $(\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathcal{P}$, such a corner point is given by $\mathbf{c}^\pi(\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathbb{R}_+^K$, where

$$c_k^\pi(\mathbf{p}_1, \ldots, \mathbf{p}_K) = I(Y; X_{\pi(k)} | X(\{\pi(l) : l < k\})) \tag{3.8}$$

Here, π specifies the order in which the decoding is performed in order to achieve the corresponding rate point: user $\pi(1)$ is decoded first, revealing the message $W_{\pi(1)}$ (assuming perfect reconstruction). Then, user $\pi(2)$ is decoded under using the knowledge of $W_{\pi(1)}$, giving $W_{\pi(2)}$, and so on, i.e., user $\pi(k)$ decodes using the messages $W_{\pi(1)}, \ldots, W_{\pi(k-1)}$.

For a weight vector $\mathbf{w} \in \mathbb{R}_+^n$, the weighted sum-rate optimization problem is formulated as

$$\begin{aligned} \max_{\substack{(\mathbf{p}_1,\ldots,\mathbf{p}_K) \in \mathcal{P} \\ \mathbf{r} \in \mathbb{R}_+^K}} \quad & \mathbf{w}^T \mathbf{r} \\ \text{subject to} \quad & \forall S \subseteq \{1, \ldots, K\} : r(S) \leq I(X(S); Y | X(\overline{S})). \end{aligned}$$

The special case of maximizing sum-rate as studied in [WK09] arises for the case $\mathbf{w} = \mathbf{1}$. Writing $I(\mathbf{p}_1, \ldots, \mathbf{p}_K)$ instead of $I(X_1, \ldots, X_K; Y)$ for the mutual information for an IPD $(\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathcal{P}$, it is easy to see that the problem of maximizing sum-rate can be written as

$$C^* = \max_{(\mathbf{p}_1,\ldots,\mathbf{p}_K)\in\mathcal{P}} I(\mathbf{p}_1, \ldots, \mathbf{p}_K). \tag{3.9}$$

In [WK09], the quantity C^* is also called the *total capacity* of the channel. The mutual information $I(\mathbf{p}_1, \ldots, \mathbf{p}_K)$ is given by

$$I(\mathbf{p}_1, \ldots, \mathbf{p}_K) = \sum_{j,i_1,\ldots i_K} p_1(i_1) \cdots p_K(i_K) Q(j|i_1, \ldots, i_K) \ln \frac{Q(j|i_1, \ldots, i_K)}{q(j)}, \tag{3.10}$$

where

$$q(j) = \sum_{i_1,\ldots i_K} p_1(i_1) \cdots p_K(i_K) Q(j|i_1, \ldots, i_K) \tag{3.11}$$

is the output probability of the symbol j. Defining the function

$$\begin{aligned} J(\mathbf{p}_1, \ldots, \mathbf{p}_K; k, i_k) &:= \frac{\partial I(\mathbf{p}_1, \ldots, \mathbf{p}_K)}{\partial p_k(i_k)} + 1 \\ &= \sum_{j,i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n} p_1(i_1) \cdots p_{k-1}(i_{k-1}) p_{k+1}(i_{k+1}) \cdots p_K(i_K) \\ &\qquad \times Q(j|i_1, \ldots, i_K) \ln \frac{Q(j|i_1, \ldots, i_K)}{q(j)} \end{aligned} \tag{3.12}$$

for $k \in \{1, \ldots, K\}$,$i_k \in \{1, \ldots, n_k\}$, it is easy to see that the KKT conditions for problem (3.9) can be formulated as [WK09]

$$J(\mathbf{p}_1, \ldots, \mathbf{p}_K; k, i_k) \begin{cases} = I(\mathbf{p}_1, \ldots, \mathbf{p}_K), & p_k(i_k) > 0 \\ \leq I(\mathbf{p}_1, \ldots, \mathbf{p}_K), & p_k(i_k) = 0 \end{cases} \tag{3.13}$$
$$\text{for all } k \in \{1, \ldots, K\}, i_k \in \{1, \ldots, n_k\}.$$

These conditions provide a necessary condition for an IPD $(\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathcal{P}$ to be optimal. In [WK09], Theorem 2, it is claimed that the KKT conditions above provide a necessary *and sufficient* condition for optimality for the elementary $(n_1, \ldots, n_K; m)$-MAC. In the following section, we demonstrate that this is not true.

We remark that there are two important features of the $(n_1, \ldots, n_K; m)$-MAC that add to the difficulty of capacity calculation:

- Unlike for the situation of additive white Gaussian noise (AWGN MAC), the capacity region of the $(n_1, \ldots, n_K; m)$-MAC is generally not a polyhedron. This is because in general, there is no input distribution that simultaneously optimizes the mutual information bounds in (3.6) (also see Chapter 4).

- In general, the union over all polyhedron regions for different input distributions is not convex, so the convex hull operation is actually needed. Of course, this is not the case for the AWGN channel.

3.1.1 The Elementary MAC Decomposition

We now briefly sketch how the sum capacity for the $(n_1, \ldots, n_K; m)$-MAC can be reduced to finding the capacity of elementary channels, as is shown in [WK09] and [Wat96]. For this, an *elementary sub-MAC* of a given $(n_1, \ldots, n_K; m)$-MAC with channel matrix $\mathbf{Q}$ is defined to be the MAC that has the same channel matrix $\mathbf{Q}$, but under restricted input: for each user k with $n_k > m$, a set Λ_k of m indices is chosen and each input probability distribution for user k is restricted to satisfy $p_k(l) = 0$ for $l \notin \Lambda_k$. We remark that equivalently, an elementary MAC could be defined by retaining m input symbols for each user k with $n_k > m$, and removing the appropriate columns in the transition matrix. The set of all elementary sub-MACs is denoted as $\Phi_K^{(m)}$. Writing the sum capacity of an elementary sub-MAC $\mathcal{E} \in \Phi_K^{(m)}$ as $C(\mathcal{E})$, the following decomposition of the sum capacity problem is proven in [WK09]:

$$C^* = \max_{(\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathcal{P}} I(\mathbf{p}_1, \ldots, \mathbf{p}_K) = \max_{\mathcal{E} \in \Phi_K^{(m)}} C(\mathcal{E}). \tag{3.14}$$

Note that this involves an optimization over a number of (at most)

$$\left| \Phi_K^{(m)} \right| = \prod_{k: n_k > m} \binom{n_k}{m} \tag{3.15}$$

different elementary channels, which can be very large.

3.1.2 The Blahut-Arimoto Algorithm

We briefly turn to the problem of capacity computation for the single-user case, i.e., $K = 1$. Besides other algorithms using conventional Lagrangian techniques [Mur53] or that are based on convex optimization [MO67], a well-known and standard algorithm for this task is the *Blahut-Arimoto algorithm*, discovered independently by Arimoto and Blahut [Bla72], [Ari72] and which will subsequently be described in some detail. For an IPD $\mathbf{p} \in \mathcal{P}$, the mutual information is given by

$$I(\mathbf{p}) = \sum_{j=1}^{m} \sum_{i=1}^{n} Q(j|i) p(i) \ln \frac{Q(j|i)}{\sum_{k=1}^{n} Q(j|k) p(k)}, \tag{3.16}$$

and the capacity of the channel is $C = \max_{\mathbf{p}} I(\mathbf{p})$. Introducing additional variables by the matrix $\Phi = (\phi(i,j)) \in [0,1]^{n\times m}$, define the function

$$\Lambda(\mathbf{p},\Phi) := \sum_{j=1}^{m}\sum_{i=1}^{n} Q(j|i)p(i)\ln\frac{\phi(i,j)}{p(i)}. \tag{3.17}$$

Then, for fixed $\mathbf{p} \in \mathcal{P}$, the mutual information $I(\mathbf{p})$ can be expressed as the maximum of Λ over all matrices $\Phi \in [0,1]^{n\times m}$:

$$I(\mathbf{p}) = \max_{\Phi} \Lambda(\mathbf{p},\Phi) = \Lambda(\mathbf{p},\Phi^*_{\mathbf{p}}), \tag{3.18}$$

where the entries of the maximizing matrix $\Phi^*_{\mathbf{p}}$ are

$$\phi^*_{\mathbf{p}}(i,j) = \frac{Q(j|i)p(i)}{\sum_{k=1}^{n} Q(j|k)p(k)}. \tag{3.19}$$

Hence, the capacity C can be written as

$$C = \max_{\mathbf{p}} \max_{\Phi} \Lambda(\mathbf{p},\Phi). \tag{3.20}$$

For fixed Φ, the maximizing input probability distribution can also be found analytically:

$$\max_{\mathbf{p}} \Lambda(\mathbf{p},\Phi) = \Lambda(\mathbf{p}^*_{\Phi},\Phi), \tag{3.21}$$

with

$$p^*_{\Phi}(i) = \frac{\exp\left(\sum_{k=1}^{m} Q(k|i)\ln\phi(i,k)\right)}{\sum_{l=1}^{n}\exp\left(\sum_{k=1}^{m} Q(k|l)\ln\phi(i,l)\right)}. \tag{3.22}$$

The Blahut-Arimoto algorithm works iteratively by optimizing over one variable (using the expressions (3.19) and (3.22), respectively) while keeping the other one fixed: starting from an arbitrary $\mathbf{p}_1$ (recommended to be chosen uniformly), the algorithm produces the sequence

$$\mathbf{p}_1 \to \Phi^*_{\mathbf{p}_1} = \Phi_1 \to \mathbf{p}^*_{\Phi_1} = \mathbf{p}_2 \to \Phi^*_{\mathbf{p}_2} = \Phi_2 \to \mathbf{p}^*_{\Phi_2} = \mathbf{p}_3 \to \cdots \tag{3.23}$$

This sequence converges to the optimal $\mathbf{p}^*$: we write $V(t) = \Lambda(\mathbf{p}_{t+1},\Phi_t)$ for the value of the objective function after t iterations. In [Ari72], it is shown that this sequence converges monotonically from below to the capacity: $\lim_{t\to\infty} V(t) = C$. It is noteworthy that an important ingredient of the convergence proof is that the KKT conditions (3.13) are necessary and sufficient for optimality in the single user case. Moreover, the convergence speed is also addressed: the approximation error is inversely proportional to the number of iterations, and if $\mathbf{p}_1$ is chosen as the uniform distribution, then the error is bounded by

$$C - V(t) \leq \frac{\ln n - H(\mathbf{p}^*)}{t}, \tag{3.24}$$

where H is binary entropy (in nats). What is more, if $\mathbf{p}^*$ is unique and strictly positive, it is even shown that the error decreases exponentially with the number of iterations t.

The work in [RG04] generalizes the Blahut-Arimoto algorithm described above for the computation of the total capacity of the MAC. Here, a sequence of probability distributions is constructed in a similar way as in (3.23), and it is shown that it converges to a point satisfying the KKT conditions (3.13). Clearly, this implies that a capacity-achieving distribution is found if the conditions (3.13) are sufficient for optimality, as is claimed in [WK09]. However, this is not the case, as will be shown in the next section. Other generalizations of the Blahut-Arimoto algorithm include the capacity computation for channels with side information known non-causally at the transmitter [DYW04]. Finally, we remark that a simple iterative procedure that alternatingly optimizes over one of the two input probabilities is suggested in [Wat96] for the computation of the sum capacity of the $(2, 2; 2)$-MAC.

3.2 Non-sufficiency of the KKT Conditions for the Elementary MAC

In this section, we show the non-sufficiency of the KKT conditions for the elementary MAC. For this, it even suffices to consider the $(2, 2; 2)$-MAC, which is elementary. For the purpose of illustration, we now consider I as a function on the domain $[0, 1]^2$ and define $I(p_1, p_2) := I(\mathbf{p}_1, \mathbf{p}_2)$ where $\mathbf{p}_1 = (p_1, 1 - p_1)^T, \mathbf{p}_2 = (p_2, 1 - p_2)^T$. The sum-rate capacity problem then reads as

$$C^* = \max_{(p_1,p_2)\in[0,1]^2} I(p_1, p_2). \tag{3.25}$$

It is easy to see that if $(p_1, p_2) \in [0, 1]^2$ satisfies the KKT conditions for problem (3.25), then $\mathbf{p}_1 = (p_1, 1-p_1)^T, \mathbf{p}_2 = (p_2, 1-p_2)^T$ also satisfy the KKT conditions for problem (3.9). Particularly, each stationary point of I in the interior of the domain $[0, 1]^2$ corresponds to a KKT point for problem (3.9). We demonstrate non-sufficiency of the KKT conditions by means of two examples. The first example (Example 1) is the $(2, 2; 2)$-MAC with channel matrix specified by

$$\begin{array}{cc} & \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 2/3 & 1/4 & 0.001 & 5/8 \\ 1/3 & 3/4 & 0.999 & 3/8 \end{pmatrix} \end{array}$$

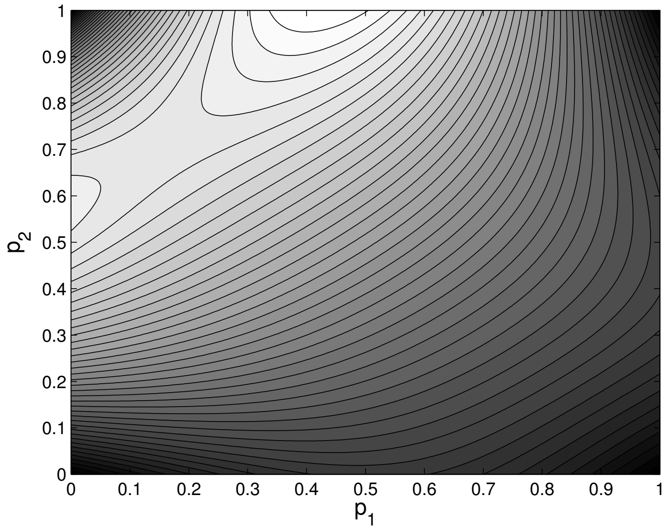

Figure 3.2: Equal level lines of mutual information I for Example 1.

and the second one (Example 2) given by the channel matrix

$$\begin{array}{cc} & \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.4 & 0.68 & 0.7 & 0.5 \\ 0.6 & 0.32 & 0.3 & 0.5 \end{pmatrix} \end{array}$$

Figure 3.2 and Figure 3.3 display the equal level lines of the mutual information $I(p_1, p_2)$ over the range $p_1 \in [0,1], p_2 \in [0,1]$ for the two sample channels. One can see that in both cases, there is a saddle point in the interior of the domain. We denote these saddle points by (p_1^1, p_2^1) for Example 1 and by (p_1^2, p_2^2) for Example 2. These points are stationary points of I, and $\mathbf{p}_1^1 = (p_1^1, 1-p_1^1)^T, \mathbf{p}_2^1 = (p_2^1, 1-p_2^1)^T$ and $\mathbf{p}_1^2 = (p_1^1, 1-p_1^1)^T, \mathbf{p}_2^1 = (p_2^2, 1-p_2^2)^T$ satisfy the KKT conditions (3.13). However, these points are not optimal. We remark that $\mathbf{p}_1^1$ and $\mathbf{p}_1^2$ can be found numerically and be verified to satisfy (3.13).

In what follows, we give an explanation for the occurrence of such non-optimal KKT points and indicate where the proof in [WK09] fails to be valid. The details of the definitions and notations are not given here; the reader is referred to [WK09].

Proposition 3 in [WK09] states that, for any $(n_1, \ldots, n_K; m)$-MAC (elementary or not), every solution to the KKT conditions is a local maximum. The examples presented above show that is not true. The basic idea behind the proof of Proposition 3 is the fact that the sum-rate optimization problem (3.9) can, for each user ordering, be equivalently formulated as an optimization over the region traced out over all possible input distributions by the polyhedron corner point corresponding to this user ordering: for each of the possible

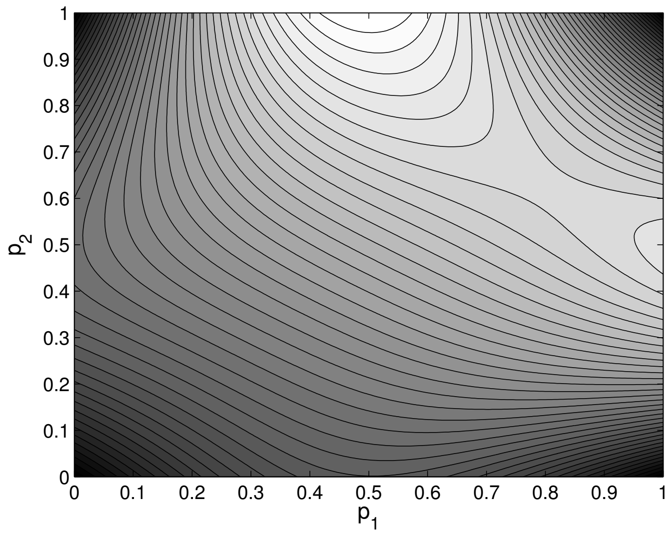

Figure 3.3: Equal level lines of mutual information I for Example 2.

user orderings $\pi_i, i = 1, \ldots, K!$, one obtains the set of "achievable rates"

$$\mathcal{G}_i = \bigcup_{(\mathbf{p}_1,\ldots,\mathbf{p}_K)\in\mathcal{P}} \{\mathbf{c}^{\pi_i}(\mathbf{p}_1, \ldots, \mathbf{p}_K)\}. \tag{3.26}$$

Then

$$C^* = \max_{(\mathbf{p}_1,\ldots,\mathbf{p}_K)\in\mathcal{P}} \mathbf{1}^T \mathbf{c}^{\pi_i}(\mathbf{p}_1, \ldots, \mathbf{p}_K). \tag{3.27}$$

Roughly speaking, the idea then is to show that an IPD satisfying the KKT conditions corresponds to a point on the boundary of each $\mathcal{G}_i$. By looking at projections to different two-dimensional "cross-sections" and local properties around the optimal boundary point, it is aimed to show that any IPD satisfying the KKT conditions is a local maximum. However, there are two problems with this approach:

(1) The proof implicitly assumes convexity of the cross-sections.

(2) It is not ensured rigorously that each point satisfying the KKT conditions actually corresponds to a boundary point on the regions $\mathcal{G}_i$.

We emphasize that the non-convexity of the cross-sections is, for itself, enough to make the proof of Proposition 3 invalid and will be demonstrated by means of examples in the following. The second point is mentioned here more for the sake of completeness. We remark that we could not find an example where an IPD satisfying the KKT conditions does not correspond to a point on the boundary of each $\mathcal{G}_i$. However, for a rigorous proof,

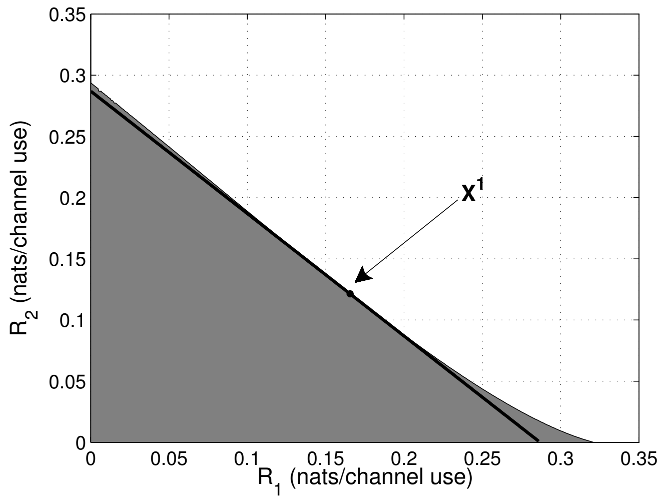

Figure 3.4: The non-convex region $\mathcal{G}_1$ for Example 1.

there is one step missing, as will be discussed in more detail later.

We detail on point (1) for the $(2,2;2)$-MAC in the following. Consider, as in the proof of Proposition 3, a solution $(\mathbf{p}_1^*, \mathbf{p}_2^*) \in \mathcal{P}$ to the KKT equations with $\mathbf{p}_1^* > \mathbf{0}$, $\mathbf{p}_2^* > \mathbf{0}$. Choosing $\mathbf{p}_1'' = \mathbf{p}_2'' = (1 \;\; 0)^T$ and $\mathbf{p}_1' = \mathbf{p}_2' = (0 \;\; 1)^T$, it is easy to see that the set $\mathcal{G}_1$ of achievable rates is given by

$$\mathcal{G}_1 = \bigcup_{(\mathbf{p}_1,\mathbf{p}_2)\in\mathcal{P}} \left\{ (I(Y;X_1|X_2), I(Y;X_2))^T \right\} = \bigcup_{(\mathbf{p}_1,\mathbf{p}_2)\in\mathcal{P}} \left\{ \mathbf{c}^\tau(\mathbf{p}_1,\mathbf{p}_2) \right\}, \tag{3.28}$$

where $\tau(1) = 2, \tau(2) = 1$.

Also, since there are only two users, the cross-section $\mathcal{G}_1(R_1, R_2)$ defined by equation (15) in [WK09] is given by $\mathcal{G}_1(R_1, R_2) = \mathcal{G}_1$. Now, the argumentation that $(\mathbf{p}_1^*, \mathbf{p}_2^*)$ must be a local maximum implicitly assumes that $\mathcal{G}_1$ is a convex set. This assumption however, cannot be made in general. Again, we demonstrate this by an example. Figure 3.4 shows a plot of the non-convex region $\mathcal{G}_1$ for Example 1; Figure 3.5 shows the same for Example 2. These figures also show the points $\mathbf{x}^1 = \mathbf{c}^\tau(\mathbf{p}_1^1, \mathbf{p}_2^1)$ and $\mathbf{x}^2 = \mathbf{c}^\tau(\mathbf{p}_1^2, \mathbf{p}_2^2)$, respectively, which are located on the boundary of $\mathcal{G}_1$ where the tangent line of slope -1 touches the region $\mathcal{G}_1$. As a consequence, the claim in Proposition 3 does not hold. Also, Proposition 4 (the connectedness property), which bases on Proposition 3, is wrong. This can also be easily seen from the contour level plots in Figure 3.2 and Figure 3.3. Similarly and also as a consequence, Lemma 1 and the main result, Theorem 2, are invalid.

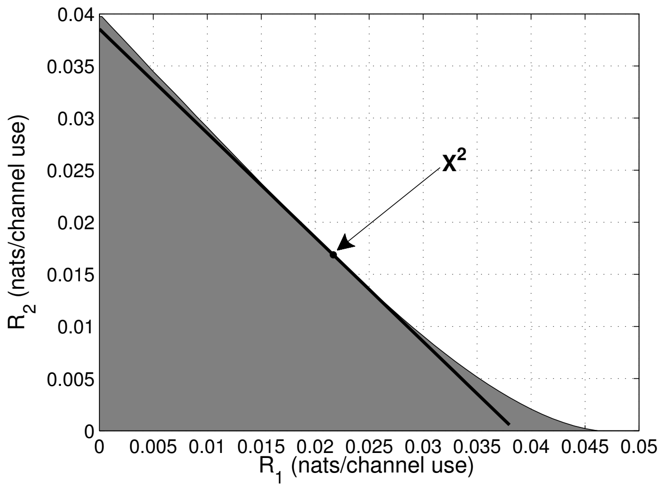

Figure 3.5: The non-convex region $\mathcal{G}_1$ for Example 2.

Concerning point 2), the "boundary equations" state the KKT conditions for a input probability distribution to correspond to a point on the boundary of $\mathcal{G}_i$. Proposition 2 states that any IPD $(\mathbf{p}_1^*, \ldots, \mathbf{p}_K^*)$ that satisfies the KKT conditions (3.13) and for which $\mathbf{p}_i^* > \mathbf{0}$ also satisfies the boundary equations. In the proof of Proposition 3, the point corresponding to such an IPD in $\mathcal{G}_1$ is denoted by $\mathbf{R}(\theta_1^*, \ldots, \theta_K^*) = \mathbf{c}^\tau(\mathbf{p}_1^*, \ldots, \mathbf{p}_K^*)$. The proof of Proposition 3 relies on the fact that $\mathbf{R}(\theta_1^*, \ldots, \theta_K^*)$ is located on the boundary of $\mathcal{G}_1$, and hence on the boundary of the cross-sections $\mathcal{G}_1(R_1, R_k)$. However, in order to be able to make this assumption, one would have to show that the boundary equations provide a *sufficient* condition for an IPD to result in a point $\mathbf{c}^\tau(\mathbf{p}_1^*, \ldots, \mathbf{p}_K^*)$ on the boundary of $\mathcal{G}_1$.

As a final remark, we consider the examples above with respect to the work in [Wat96]. Here, two classes of $(2,2;2)$-MACs are distinguished: *case A* and *case B* channels. For case B channels, the KKT conditions as given above are proved to be sufficient for optimality. For case A channels, the conditions have to be slightly modified to be sufficient; essentially the modification consists in requiring the optimal point to be located on a certain boundary of the domain. The definition of the two different cases in [Wat96] is quite cumbersome, we remark here that one can derive a simpler condition to distinguish between the two cases: Define $P_{\min} = \min_{(i,j)\in\{1,2\}^2} Q(1|i,j)$, $P_{\max} = \max_{(i,j)\in\{1,2\}^2} Q(1|i,j)$ and for real numbers a, b, we let $\langle a, b\rangle = [a, b]$ if $a \leq b$ and $\langle a, b\rangle = [b, a]$ if $a > b$. Then a $(2,2;2)$-MAC is of case B if

$$[P_{\min}, P_{\max}] = \langle Q(1|1,1), Q(1|2,2)\rangle \text{ or } [P_{\min}, P_{\max}] = \langle Q(1|1,2), Q(1|2,1)\rangle \tag{3.29}$$

and of case A if the condition (3.29) is not satisfied.

Now both the channels given by Example 1 and Example 2 are of case A. Also, they satisfy the condition of $(\mathbf{v}(1,1), \mathbf{v}(2,1))$ being in the strictly outer position [Wat96] (which simply means that $P_{\min} = Q(1|1,1), P_{\max} = Q(1|2,1)$). According to [Wat96], Proposition 2, a necessary and sufficient condition for an optimal IPD $(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}$ is given by

$$\mathbf{p}_2(1) = 1, \tag{3.30}$$

$$\mathbf{p}_2(2) = 0, \tag{3.31}$$

$$J(\mathbf{p}_1, \mathbf{p}_2; 1, i_1) = I(\mathbf{p}_1, \mathbf{p}_2) \text{ for } i_1 \in \{1,2\}, \tag{3.32}$$

$$J(\mathbf{p}_1, \mathbf{p}_2; 2, 1) = I(\mathbf{p}_1, \mathbf{p}_2), \tag{3.33}$$

$$J(\mathbf{p}_1, \mathbf{p}_2; 2, 2) < I(\mathbf{p}_1, \mathbf{p}_2). \tag{3.34}$$

One can verify that for the two examples, these conditions actually provide necessary and sufficient conditions for optimality. As a result, our examples are in accordance with and without contradiction to the work in [Wat96].

3.3 Optimality Conditions for the $(2,2;2)$-MAC

In the previous section, we have demonstrated that the KKT conditions generally do not provide a sufficient condition for optimality of the discrete memoryless MAC. This is even the case for the setup of two users and binary alphabets, i.e., the $(2,2;2)$-MAC. This suggests that even the restricted $(2,2;2)$-MAC captures the difficulties involved in the MAC capacity computation problem. In this section, we exclusively study this channel with respect to computation of the entire capacity region, that is, weighted sum-rate optimization for arbitrary weight vectors. More precisely, the problem we consider now is the maximization of the weighted sum-rate over the capacity region $\mathcal{C}_{(2,\,2;\,2)}$ of the $(2,2;2)$-MAC for a given weight vector $\mathbf{w} = (w_1, w_2)^T > \mathbf{0}$:

$$\max_{\mathbf{r} \in \mathcal{C}_{(2,\,2;\,2)}} \mathbf{w}^T \mathbf{r}. \tag{3.35}$$

Each polyhedron region $\mathcal{R}(\mathbf{p}_1, \mathbf{p}_2)$ [cf. (3.6)] is specified by the corner points

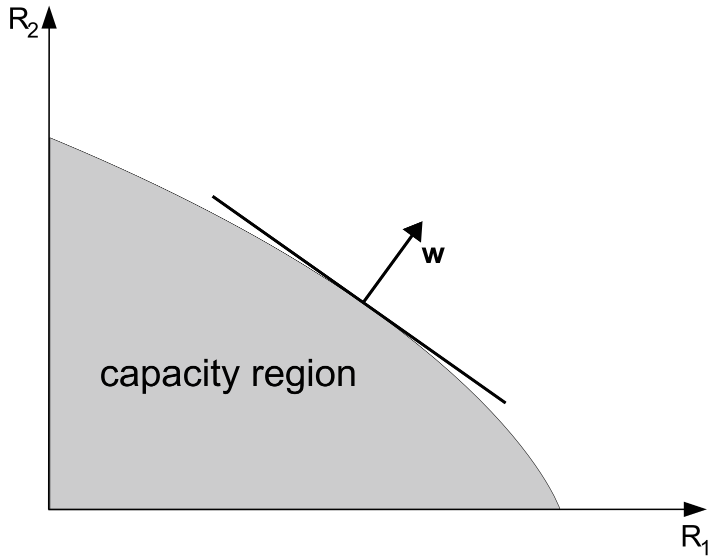

Figure 3.6: Illustration of the capacity computation problem: weighted sum-rate optimization for the weight vector $\mathbf{w}$.

$$C_1(\mathbf{p}_1, \mathbf{p}_2) := \mathbf{c}^{\pi_1}(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}_+^2, \tag{3.36}$$

$$C_2(\mathbf{p}_1, \mathbf{p}_2) := \mathbf{c}^{\pi_2}(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}_+^2, \tag{3.37}$$

where π_1 corresponds to decoding user 1 first, i.e., $\pi_1(1) = 1, \pi_1(2) = 2$ and π_2 results from decoding user 2 first, that is $\pi_2(1) = 2, \pi_2(2) = 1$ (cf. Section 3.1). It is easily verified that the weighted sum-rate optimization problem (3.35) can be stated in terms of optimization over the region defined by the C_1, C_2 points as follows: for $w_1 \leq w_2$, it holds that

$$\max_{\mathbf{r} \in \mathcal{C}_{(2,\,2;\,2)}} \mathbf{w}^T \mathbf{r} = \max_{(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}} \mathbf{w}^T C_1(\mathbf{p}_1, \mathbf{p}_2) \tag{3.38}$$

and for $w_1 > w_2$, the optimization can similarly be performed by optimizing over the C_2 points.

For the channel transition matrix, we write

$$\mathbf{Q} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{pmatrix} a & b & c & d \\ 1-a & 1-b & 1-c & 1-d \end{pmatrix} \end{array}$$

in the following, that is $a := Q(1|1, 1), b := Q(1|1, 2), c := Q(1|2, 1)$ and $d := Q(1|2, 2)$. Moreover, we let $\Delta_1 := a - b, \Delta_2 := c - d$ and recall that $D(p||q)$ denotes the Kullback-Leibler divergence between two binary probability functions defined by $p, q \in [0, 1]$. We assume without loss of generality (w.l.o.g.) that $0 < w_1 \leq w_2$: for the case $w_1 > w_2$, we can use the fact that $I(Y;X_2)$ and $I(Y;X_2|X_1)$ are obtained from $I(Y;X_1)$ and $I(Y;X_1|X_2)$ by interchanging the roles of $\mathbf{p}_1$ and $\mathbf{p}_2$ and the roles of b and c. For $(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}$, we define

$$\Psi(\mathbf{p}_1, \mathbf{p}_2) \coloneqq \mathbf{w}^T C_1(\mathbf{p}_1, \mathbf{p}_2). \tag{3.39}$$

For channels with $a = b$ and $c = d$, it is $I(Y;X_2|X_1) = 0$ for all $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}$; we exclude this degenerate case from investigation. Similarly, for channels with $a = c$ and $b = d$, $I(Y;X_1) = 0$ for all $(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}$, and we also omit this case. In what follows, we are thus concerned with the optimization problem

$$\max_{(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}} \Psi(\mathbf{p}_1, \mathbf{p}_2). \tag{3.40}$$

Obviously, for $w_1 = w_2$ the problem (3.40) reduces to the sum capacity problem studied in the previous section.

3.3.1 Reduction to a One-dimensional Problem

As already stated in the previous section, it is helpful to consider Ψ as a function on the domain $[0,1]^2$ instead, i.e., we are concerned with the maximization of $\Psi(p_1, p_2)$ where the input probability distribution is specified by $p_1 = \mathbf{Pr}[X_1 = 1]$ and $p_2 = \mathbf{Pr}[X_2 = 1]$. Our derivation is based on the following observations: First of all, the boundary points of the capacity region on the two rate axis are $(e_1, 0)^T$ and $(0, e_2)^T$, where

$$e_1 = \max_{i \in \{0,1\}} \max_{p_1 \in [0,1]} I(X_1, X_2; Y)_{p_2 = i}, \tag{3.41}$$

and e_2 is given similarly by fixing the value of p_1 to 0 and 1. Observe that e_1 and e_2 can be found by solving the single-user capacity maximization problem (for example, using the Arimoto-Blahut algorithm as in Section 3.1.2).

Furthermore, $I(X_2;Y|X_1)$ is linear in p_1, and for (all but at most one, namely the one that satisfies $h_2(p_2) = 0$, see below) fixed values of p_2, $I(X_1;Y)$ is strictly concave in p_1. Hence, the first component in the stationarity equation

$$\nabla\Psi(p_1, p_2) \stackrel{!}{=} \mathbf{0},\ p_1, p_2 \in (0,1) \tag{3.42}$$

has a unique solution in p_1 for fixed p_2 in the case of strict concavity. What is more, we can find an explicit expression for this solution by simplifying the partial derivative of $I(X_1;Y)$ with respect to p_1 such that p_1 occurs only once in the expression: the partial derivatives of mutual information at $p_1, p_2 \in (0,1)$ with respect to p_1 are given by

$$\frac{\partial I(Y;X_1)(p_1,p_2)}{\partial p_1} = h_1(p_2) + h_2(p_2)\ln\left(\frac{1}{h_3(p_2)+p_1h_2(p_2)} - 1\right) \tag{3.43}$$

and

$$\begin{aligned}\frac{\partial I(Y;X_2|X_1)(p_1,p_2)}{\partial p_1} &= -p_2H(a) + (p_2-1)H(b) + p_2H(c) - (p_2-1)H(d) \\ &\quad + H(b+p_2(a-b)) - H(d+p_2(c-d)).\end{aligned} \tag{3.44}$$

Here, we used the notation

$$h_1(p_2) := H(d+p_2(c-d)) - H(b+p_2(a-b)), \tag{3.45}$$

$$h_2(p_2) := -b + d + p_2(-a+b+c-d), \tag{3.46}$$

$$h_3(p_2) := 1 - d + p_2(d-c). \tag{3.47}$$

We will also write $h_4(p_2) := \frac{\partial I(Y;X_2|X_1)(p_1,p_2)}{\partial p_1}$. Let $P_2 := \{p \in (0,1) : h_2(p) \neq 0\}$ and $\overline{P}_2 := \{p \in P_2 : f(p) \in (0,1)\}$. Note that we have excluded the case $a = c$ and $b = d$, so that there is at most one $p \in (0,1)$ for which $h_2(p) = 0$. For fixed $p \in P_2$, the explicit solution for p_1 in the first component of (3.42) is given by $f(p)$, where $f : P_2 \to \mathbb{R}$ is defined by

$$f(p) := \frac{1}{\left(e^{h(p)}+1\right)h_2(p)} - \frac{h_3(p)}{h_2(p)}, \tag{3.48}$$

with

$$h(p) := \frac{-\frac{w_2}{w_1}h_4(p) - h_1(p)}{h_2(p)}. \tag{3.49}$$

For $p_2 \in (0,1) \setminus P_2$, it is easy to show that $I(X_1;Y) = 0$ for all p_1. Define $\phi : \overline{P}_2 \to \mathbb{R}$ by $\phi(p) := \Psi(f(p), p)$. Collectively considering the above facts, we obtain the following lemma:

Lemma 4. *If $\nabla\Psi(p_1,p_2) = \mathbf{0}$ for $(p_1,p_2) \in (0,1) \times P_2$, then $p_2 \in \overline{P}_2$, $\phi'(p_2) = 0$ and there is no $\tilde{p}_1 \in (0,1)$ such that $\tilde{p}_1 \neq p_1$ and $\nabla\Psi(\tilde{p}_1,p_2) = \mathbf{0}$. Moreover,*

$$\max_{p_1,p_2\in[0,1]^2} \Psi(p_1,p_2) = \max\left\{\max_{p\in\overline{P}_2}\phi(p), w_1e_1, w_2e_2\right\}. \tag{3.50}$$

3.3.2 The 3-parameter $(2,2;2)$-MAC

The channels we focus on in the following are $(2,2;2)$-MACs with the restriction $a = Q(1|1,1) = Q(1|1,2) = b$ on the channel transition probabilities. We call such a channel a *3-parameter* $(2,2;2)$*-MAC*. The information-theoretic interpretation of such channels is as follows: conditioned on the event that user 1 transmits the symbol 1, the channel that user 2 sees is the single-user antisymmetric binary channel, which has zero capacity. In other words, whenever user 1 transmits 1, the symbol of user 2 cannot be distinguished at the receiver. In fact, it is easily verified that $I(Y;X_2|X_1=1)=0$. If we give the same property to user 1 for user 2 transmitting 1, i.e., $a=b=c$, then $I(Y;X_1|X_2=1)=0$ and all the points on the boundary of the capacity region can be achieved by "time-sharing between the extremal points on the rate axis". Moreover, we have $e_1=e_2$, so that the capacity region is an isosceles triangle. Furthermore, for $a=b$, we can exchange the values c and d without changing the capacity region. In the following, we thus assume w.l.o.g. that $a \neq c$, $a \neq d$ and $c > d$. We also restrict to weight vectors $\mathbf{w} = (w_1, w_2)^T$ with $0 < w_1 \leq w_2$. Unlike in the previous section, this actually is a restriction here: the case $w_1 > w_2$ cannot be treated by exchanging the roles of c and b and p_1 and p_2, since this would result in a $(2,2;2)$-MAC that is not of 3-parameter type.

Now consider the extension of ϕ to P_2, i.e., $\hat{\phi} : P_2 \to \mathbb{R}$ defined by $\hat{\phi}(p) := \Psi(f(p), p)$ and which is easily verified to be well-defined. We will show that Ψ can have at most one stationary point in the interior. We prove this by showing that $\hat{\phi}$ is pseudoconcave on $(0,1)$. Recall that a twice differentiable function $c : D \subseteq \mathbb{R} \to \mathbb{R}$ is called *pseudoconcave* if $c'(p) = 0 \Rightarrow c''(p) < 0$. In this case, each local maximum of c is also a global maximum, and c has at most one stationary point. More precisely, we prove

Proposition 1. *The function $\hat{\phi}$ has the following properties:*

- *For $a \in (d,c)$: $\hat{\phi}'(p) \neq 0$ for all $p \in P_2$.*
- *For $a \notin (d,c)$: $\hat{\phi}$ is pseudoconcave on $P_2 = (0,1)$.*

Proof. We first prove the following properties of h:

- For $a \in (d,c) : h'(p) \neq 0$ for all $p \in P_2$.
- For $a \in [0,d) : h'(p) = 0 \Rightarrow h''(p) < 0$.
- For $a \in (c,1] : h'(p) = 0 \Rightarrow h''(p) > 0$.

The first derivative of h can be written as

$$h'(p) = \frac{\Delta_2 \frac{w_1 - w_2}{w_1} D(a||d + p\Delta_2) + \frac{w_2}{w_1}\delta(a,c,d)}{h_2(p)^2}, \tag{3.51}$$

where

$$\delta(a,c,d) := (c-d)(H(d) - H(a)) - (H(c) - H(d))(d-a). \tag{3.52}$$

For $a \in (0,1)$,

$$\frac{\partial^2}{\partial a^2}\delta(a,c,d) = (c-d)\left(\frac{1}{1-a} + \frac{1}{a}\right) > 0, \tag{3.53}$$

implying that δ is strictly convex in a. Now $\delta(c,c,d) = \delta(d,c,d) = 0$, so that $\delta(a,c,d) > 0$ for $a \notin (d,c)$ and $\delta(a,c,d) < 0$ for $a \in (d,c)$. A similar argument shows $\delta(0,c,d) > 0$ and $\delta(1,c,d) > 0$. In addition, $D(a||d + p\Delta_2) \geq 0$ by the non-negativity of Kullback-Leibler divergence. This implies that $h'(p) < 0$ for $a \in (d,c)$. Now consider the situation $a \notin (d,c)$. For $a \in [0,d)$, we have $h_2(p) > 0$ for all $p \in (0,1)$ and for $a \in (c,1]$, $h_2(p) < 0$ for all $p \in (0,1)$, so that $P_2 = (0,1)$ for $a \notin (d,c)$. Since

$$h''(p) = \frac{\frac{w_2 - w_1}{w_1} h_1''(p) - 2\Delta_2 h'(p)}{h_2(p)}, \tag{3.54}$$

the claimed property of h follows for $w_1 < w_2$ from the strict concavity of h_1. If $w_1 = w_2$, then $h'(p) > 0$ from (3.51), so that the statement also holds in this case.

We now show that $f(p) < 1$ for all $p \in (0,1)$. For this, we first prove that $f(p) \neq 1$ for all $p \in (0,1)$. This follows easily for $a \in \{0,1\}$, so that we let $a \in (0,1)$. We also assume $c,d \in (0,1)$; the situations $c \in \{0,1\}$ or $d \in \{0,1\}$ can be treated similarly. It suffices to prove that

$$v(p) := \left.\frac{\partial I(Y;X_1)(p_1,p)}{\partial p_1}\right|_{p_1=1} < 0 \tag{3.55}$$

and $h_4(p) < 0$ for all $p \in (0,1)$. To see this, we first note that it can be shown that

$$v'(p) = \ln\left(\frac{1}{1-a} - 1\right) h_2'(p) + h_1'(p) \neq 0 \tag{3.56}$$

for all $p \in (0,1)$. Moreover, we get $v(0) = -D(d||a) < 0$ and $v(1) = -D(c||a) < 0$, which together with (3.56) imply (3.55). For the second statement, observe that

$$h_4''(p) = \frac{\Delta_2^2}{(p\Delta_2 + d)(1 - (p\Delta_2 + d))} > 0, \tag{3.57}$$

so that $h_4(p)$ is strictly convex in p. $h_4(p) < 0$ then follows from the fact that $h_4(0) = h_4(1) = 0$. Secondly, also using non-negativity of Kullback-Leibler divergence, one can prove $\lim_{p \to 0+} f(p) < 1$, which together with $f(p) \neq 1$ and the continuity of f implies that $f(p) < 1$ for all $p \in (0,1)$.

To conclude the proof of the proposition, one can find the following factorized representation for $\hat{\phi}'$:

$$\hat{\phi}'(p) = w_1(1 - f(p))h_2(p)h'(p). \tag{3.58}$$

With the shown properties of h and f, the statement follows directly from (3.58) for the case $a \in (d,c)$. For the other case, we have that

$$\hat{\phi}''(p) = w_1 h'(p)\left(-f'(p)h_2(p) + \Delta_2(1-f(p))\right) + w_1(1-f(p))h_2(p)h''(p), \tag{3.59}$$

implying that with the properties of h and f follows that

$$\hat{\phi}'(p^*) = 0 \Rightarrow h'(p^*) = 0 \Rightarrow \hat{\phi}''(p^*) < 0, \tag{3.60}$$

which concludes the proof. □

It can be verified that for $h_2(p_2) = 0$, it holds that $\Psi(p_1, p_2) \neq \mathbf{0}$ for all $p_1 \in (0,1)$. As a consequence, Proposition 1 and Lemma 4 imply that in the case $a \in (d,c)$ the function Ψ has no stationary point in the interior; the optimum input probability distribution is located on the boundary and it suffices to solve the single-user problems. Speaking in terms of the KKT conditions (3.13), this means that each point satisfying these equations is located on the boundary, as in the case of the sum-rate problem [Wat96]. For the case $a \notin (d,c)$, pseudoconcavity of $\hat{\phi}$ implies that there is at most one stationary point (or, equivalently, at most one KKT point) in the interior of the domain. We summarize this in the following theorem:

Theorem 2. *For the 3-parameter $(2,2;2)$-MAC with $a \neq c$, $a \neq d$, $c > d$ and $w_1 \leq w_2$, it holds that:*

- *For $a \in (d,c)$, the optimal input distribution is located on the boundary of $[0,1]^2$, and there is no stationary point of Ψ in the interior of $[0,1]^2$.*
- *For $a \notin (d,c)$, there is at most one stationary point of Ψ in the interior of $[0,1]^2$.*

By Lemma 4, the problem of finding the maximizing input distribution can be reduced to the single-user problem and the optimization of ϕ (for which it suffices to optimize $\hat{\phi}$, as described in the following). Since $\hat{\phi}$ is pseudoconcave, it can efficiently be optimized using a simple standard bisection algorithm: for a given tolerance ε, we start with the interval $[\epsilon, 1-\epsilon]$ and determine if one of the intervals $[\epsilon, 1/2]$, $[1/2, 1-\epsilon]$ contains the optimal point by checking the sign of h' (i.e., the sign of $\hat{\phi}'$ by (3.58)) at the interval boundaries. If this is not the case, we assume the optimal point to be on the boundary. Otherwise, we continue bisecting the interval that contains the stationary point until the interval length is smaller than ε, and find a solution p_ϵ within ϵ deviation tolerance from the optimal using only $O(\log(1/\epsilon))$ evaluations of h', which is much more efficient than a brute-force search. By the definition of ϕ, $(p_1^*, p_2^*) = (f(p_\epsilon), p_\epsilon)$ is used as optimization output if $f(p_\epsilon) \in (0,1)$. If $f(p_\epsilon) \notin (0,1)$, we assume that the stationary point of $\hat{\phi}$ is outside of $\overline{P}_2$, and we also assume the optimal point for Ψ to be on the boundary. Note that in principle, we could find the optimal p_1 for this choice of $p_2 = p_\varepsilon$ by solving the single-user problem for fixed $p_2 = p_\varepsilon$. However, it is clear that for sufficiently small ϵ, we will have $f(p_\epsilon) \in (0,1)$ if Ψ has a stationary point in the interior. Note that we can only give a deviation tolerance for p_2^*. However, $p_1^* = f(p_\epsilon)$ is still a reasonable solution since it is optimal "conditioned" on the choice of $p_2^* = p_\epsilon$. We remark that a gradient descent algorithm employed for Ψ also typically shows fast convergence to the optimal input distribution.

3.3.3 An Explicit Solution

We finally consider the 3-parameter $(2,2;2)$-MAC with $a = b = 0, 0 < d < c$ and $w_1 < w_2$. We refer to this channel as the 2-*MAC*. For this channel, it is possible to give an explicit solution for the optimization of ϕ, as described in what follows.

It can be shown that $\overline{P}_2 = (0,1)$, so that $\hat{\phi} \equiv \phi$. Moreover, we can find the zero of (3.58) by solving $h'(p) = 0$ for p, resulting in

$$p = p^*(c,d,\mathbf{w}) := \frac{1 - d - \exp\left(-\frac{w_2\delta(0,c,d)}{\Delta_2(w_2-w_1)}\right)}{\Delta_2}, \tag{3.61}$$

where $\delta(0,c,d) = (c-d)H(d) - d(H(c) - H(d))$. Hence, the maximum weighted sum-rate is given by

$$\max_{\mathbf{r}\in\mathcal{C}_{(2,\,2;\,2)}} \mathbf{w}^T\mathbf{r} = \begin{cases} \phi(p^*(c,d,\mathbf{w})), & \text{if } p^*(c,d,\mathbf{w}) \in (0,1) \\ \max\{w_1e_1, w_2e_2\}, & \text{otherwise.} \end{cases} \tag{3.62}$$

We summarize this in the following corollary:

Corollary 1. *For the* $(2,2;2)$*-MAC with* $a = b = 0, 0 < d < c$ *and* $w_1 < w_2$*, the optimal input distribution* p_1^*, p_2^* *is given by* $p_1^* = f(p_2^*)$ *and*

$$p_2^* \begin{cases} = p^*(c,d,\mathbf{w}) = \frac{1-d-\exp\left(-\frac{w_2\delta(0,c,d)}{\Delta_2(w_2-w_1)}\right)}{\Delta_2}, & \textit{if } p^*(c,d,\mathbf{w}) \in (0,1) \\ \in \{0,1\}, & \textit{otherwise.} \end{cases} \tag{3.63}$$

3.3.4 Relation to Prior Work

We now briefly discuss how the above results relate to the work in [Wat96], which deals with optimality conditions for the $(2,2;2)$-MAC with respect to the sum rate. Recall from Section 3.2 that according to the results in [Wat96], the set of $(2,2;2)$-MACs can be divided into two subsets, which are referred to as case A and case B channels, respectively. For case B channels, the KKT conditions are sufficient for optimality, whereas for case A channels, the KKT conditions have to be slightly modified to be sufficient and require the optimal point to be located on a certain boundary of the domain. In Section 3.2, we have

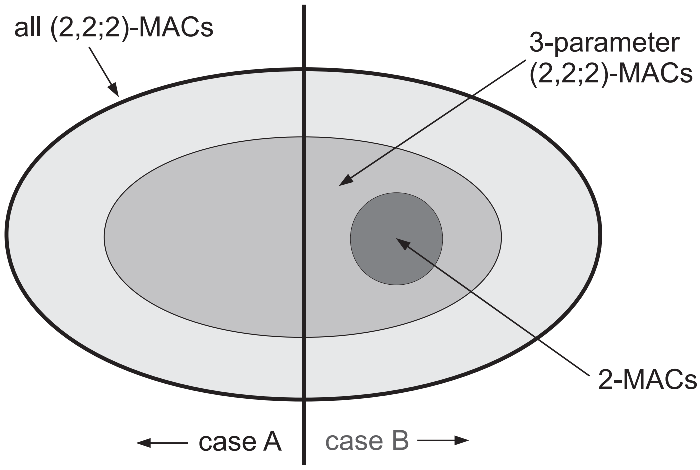

Figure 3.7: Illustration of the different classes of $(2, 2; 2)$-MACs.

also given a simple condition for a channel to be of class A (B), and it is easy to see that for the 3-parameter $(2, 2; 2)$-MAC with $a \neq c$, $a \neq d$, $c > d$, the case $a \in (d, c)$ corresponds to a case A channel, and $a \notin (d, c)$ to a case B channel. Hence, Theorem 2 can be regarded as a generalization (for a subclass of $(2, 2; 2)$-MACs) of the results in [Wat96] concerning sum rate optimality conditions for the $(2, 2; 2)$-MAC to the case of more general weight vectors. Note that the set of 2-MACs, for which an explicit solution has been given above, is a subset of class B channels. The subdivisions of the set of $(2, 2; 2)$-MACs into the different classes is illustrated by a Venn diagram in Figure 3.7.

3.4 Comprehensiveness of the Achievable Point Set

In the previous sections, we have derived structural properties of the capacity computation problem for the $(2, 2; 2)$-MAC with respect to the optimality conditions. In this brief section, we look at a second interesting structural property. We have seen that the capacity computation problem can be formulated as an optimization over the region $\mathcal{G}_1$ or $\mathcal{G}_2$ formed by the SIC points C_1 or C_2 [cf. equations (3.26) and (3.36)]. Generally, the sets $\mathcal{G}_1$ and $\mathcal{G}_2$ are not convex. However, we will prove that $\mathcal{G}_1$ and $\mathcal{G}_2$ are comprehensive: a set $A \subseteq \mathbb{R}_+^2$ is called *comprehensive* if for each $\mathbf{x} \in A$ and $\mathbf{y} \in \mathbb{R}_+^2$ it holds that $\mathbf{y} \leq \mathbf{x} \Rightarrow \mathbf{y} \in A$. This comprehensiveness property of $\mathcal{G}_1$ and $\mathcal{G}_2$ might be useful for further investigations or optimization approaches for the problem.

Due to symmetry, it suffices to show comprehensiveness of $\mathcal{G}_1$ only, and for simplicity we write $G := \mathcal{G}_1$. Furthermore, we use the notation from Section 3.3 and again consider C_1 and the weighted sum rate objective function Ψ as specified on the domain $[0, 1]^2$, i.e., as $C_1(p_1, p_2)$, and $\Psi(p_1, p_2)$, where the input probability distribution is specified by $p_1 = \mathbf{Pr}[X_1 = 1]$ and $p_2 = \mathbf{Pr}[X_2 = 1]$, respectively. Recall that we define the set of achievable points by $G = \{C_1(p_1, p_2) : p_1, p_2 \in [0, 1]\}$, [cf. equations (3.26)] and refer to the rate points in G as *achievable.* Note that for weight vectors $\mathbf{w}$ with $w_1 \leq w_2$, we have the following two formulations of the problem in the probability and the rate domain, respectively:

$$\max_{(p_1,p_2)\in[0,1]^2} \Psi(p_1, p_2) = \max_{\mathbf{x}\in G} \mathbf{w}^T\mathbf{x}. \tag{3.64}$$

We then have the following property of the set of achievable rates:

Theorem 3. *G is a comprehensive set.*

Proof. A set $A \subseteq \mathbb{R}^2_+$ is called *left-comprehensive* if for each $\mathbf{x} \in A$ and $y_1 \geq 0$ it holds that $y_1 \leq x_1 \Rightarrow \mathbf{y} = (y_1, x_2)^T \in A$. The *left-comprehensive hull* of a set $A \subseteq \mathbb{R}^2_+$ is defined as $\operatorname{lcomp}(A) \coloneqq \{\mathbf{x} \in \mathbb{R}^2_+ : x_1 \leq a_1, x_2 = a_2 \,\text{for some}\, \mathbf{a} \in A\}$. In the following, we will write $\phi_1(p_1, p_2) = I(X_1; Y)_{p1,p2}$ and $\phi_2(p_1, p_2) = I(X_2; Y|X_1)_{p1,p2}$, so that $C_1(p_1, p_2) = (\phi_1(p_1, p_2), \phi_2(p_1, p_2))^T$. We first show that G is left-comprehensive. Consider the achievable point $(R_1, R_2)^T = C_1(p_1, p_2)$. For fixed $p \in [0, 1]$, the solution in $\tilde{p}_1$ of the equation $\phi_2(\tilde{p}_1, p) = R_2$ is given by

$$\tilde{p}_1 = s(p) \coloneqq \frac{R_2 + u(p, a, b, c, d)}{h(p)} \tag{3.65}$$

where $u(p, a, b, c, d) \coloneqq -H(cp + d(1 - p)) + pH(c) + (1 - p)H(d)$ and $h(p) \coloneqq \frac{\partial \phi_2(p_1,p)}{\partial p}$. Define $\hat{s}(p) \coloneqq \phi_2(p_1, p_2) + u(p, a, b, c, d)$ and $\tilde{s}(p) \coloneqq \phi_2(p_1, p_2) + u(p, a, b, c, d) - h(p)$. One can easily establish the following properties of these functions:

- $\hat{s}, \tilde{s}$ are convex functions.
- $\hat{s}(0) = \hat{s}(1) = \tilde{s}(0) = \tilde{s}(1) = \phi_2(p_1, p_2)$.
- $\hat{s}'(0) < 0, \hat{s}'(1) > 0, \tilde{s}'(0) < 0, \tilde{s}'(1) > 0$.

It follows that $\hat{s}'(\hat{p}) = 0$ and $\tilde{s}'(\tilde{p}) = 0$ each have a solution in $[0, 1]$ (at which the functions $\hat{s}$, respectively $\tilde{s}$ attain their minimum), and these solutions can, after some algebra, be shown to be given by

$$\hat{p} = \frac{e^{\xi(c,d)}}{(1 + e^{\xi(c,d)})(c - d)} - \frac{d}{c - d} \text{ and } \tilde{p} = \frac{e^{\xi(a,b)}}{(1 + e^{\xi(a,b)})(a - b)} - \frac{b}{a - b}, \tag{3.66}$$

where

$$\xi(x, y) \coloneqq \frac{H(y) - H(x)}{x - y}. \tag{3.67}$$

Hence, the minimum of $\tilde{s}$ is given by $\tilde{\gamma} \coloneqq \tilde{\gamma}(p_1, p_2) \coloneqq \tilde{s}(\tilde{p})$. Now $\tilde{\gamma}(p_1, p_2)$ is linear in p_1 and increasing for $h(p_2) \geq 0$, so that $\tilde{\gamma} \leq \tilde{\gamma}(1, p_2)$ in this case. It holds the identity $\tilde{\gamma}(1, p_2) =$

$-u(p_2,a,b,a,b)+u(\tilde{p},a,b,a,b)$. From the properties of $\tilde{s}$, it follows that $-u(p_2,a,b,a,b)+u(\tilde{p},a,b,a,b) \leq 0$ for all $p_2 \in [0,1]$. This implies that in the case $h(p_2) \geq 0$, $\tilde{s}$ has at least one zero $\tilde{p}_2 \in [0,1]$, so that $s(\tilde{p}_2) = 1$ and $C_1(s(\tilde{p}_2),\tilde{p}_2) = C_1(1,\tilde{p}_2) = (0,R_2)$. A similar line of argument shows that if $h(p_2) < 0$, there exists a zero $\hat{p}_2 \in [0,1]$ of $\hat{s}$, which is also a zero of s, i.e., $C_1(s(\hat{p}_2),\hat{p}_2) = C_1(0,\hat{p}_2) = (0,R_2)$. This shows the existence of $p^* := \underset{p\in[0,1]:s(p)\in\{0,1\}}{\operatorname{argmin}} |p-p_2|$. We assume w.l.o.g. that $p^* < p_2$. We now prove that any point $(R_1-\Delta,R_2) \in G$ for all $\Delta \in [0,R_1)$. For this, we fix $\Delta \in [0,R_1)$ and define $q_\Delta(p) := \phi_1(s(p),p) - \phi_1(p1,p2) + \Delta$. To prove that $(R_1-\Delta,R_2) \in G$, it suffices to show that q_Δ has a zero $p_\Delta \in [0,1]$ such that $s(p_\Delta) \in [0,1]$. Clearly, it holds that $s(p_2) = p_1$, $s(p) = 0 \Rightarrow q_{\phi_1(p_1,p_2)}(p) = 0$, $s(p) = 1 \Rightarrow q_{\phi_1(p_1,p_2)}(p) = 0$ and $q_\Delta(p_2) = \Delta$. Now $s(p^*) \in \{0,1\}, s(p_2) = p_1 \in [0,1]$ so that by continuity of s and the choice of p^*, there is no value p between p^* and p_2 such that $s(p) \notin [0,1]$. In other words, $s([p^*,p_2]) \subseteq [0,1]$. Furthermore, $q_{\phi_1(p_1,p_2)}(p^*) = 0$ and $q_{\phi_1(p_1,p_2)}(p_2) = \phi_1(p_1,p_2)$. To conclude the proof of left-comprehensiveness, we use the following fact which easily follows from the intermediate value theorem for continuous functions: let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function that has a zero in an interval $[l,r] \subseteq D$ and let there be $d \in [l,r]$ such that $f(d) \geq C$ for some $C > 0$. Then for any $\epsilon \in [0,C]$, there exists $z \in [l,r]$ such that $f(z)-\epsilon = 0$. From this and the relation $q_\Delta(p) = q_{\phi_1(p_1,p_2)}(p) + \Delta - \phi_1(p_1,p_2)$, it follows that there exists $p_\Delta \in [p^*,p_2]$ such that $q_\Delta(p_\Delta) = 0$ and $s(p_\Delta) \in [0,1]$. The following properties of ϕ_1,ϕ_2 can easily be verified: $\phi_1(p_1,p_2)$ is convex in p_2, $\phi_2(p_1,p_2)$ is concave in p_2 and $\phi_2(p_1,0) = \phi_2(p_1,1) = 0$.

For each $p_1 \in [0,1]$, define $G^{p_1} := C_1(p_1,[0,1])$ as the trace of C_1 points obtained by varying p_2, so that $G = \bigcup_{p_1\in[0,1]} G^{p_1}$. Using the properties of ϕ_1 and ϕ_2 given above, one can prove that for each $p_1 \in [0,1]$, the left-comprehensive hull $\mathrm{lcomp}(G^{p_1})$ of G^{p_1} is comprehensive. By left-comprehensiveness of G, it holds that $\mathrm{lcomp}(G^{p_1}) \subseteq G$, so that G can be written as the union $G = \bigcup_{p_1\in[0,1]} \mathrm{lcomp}(G^{p_1})$ of comprehensive sets. Since the union operation preserves comprehensiveness, G is comprehensive. □

3.5 Solutions by User Cooperation

The work presented in this section is based on the relaxation approach for the MAC capacity computation problem proposed in [CPFV07] and is, to some extent, parallel to further work presented in a recent journal article [CPFV10]. The channel considered here is the $(N_1,N_2;M)$-MAC, i.e., the discrete memoryless two-user MAC with alphabet sizes N_1,N_2 (for the inputs) and M (for the output) and which is specified by a channel transition probability matrix $\mathbf{Q} = (Q(j|n_1,n_2)) \in \mathbb{R}_+^{M\times N_1N_2}$.

The main result of [CPFV07] consists in showing that the only non-convexity of the weighted sum rate optimization problem for this channel is due to the requirement of the input probability distributions to be independent, that is, the constraint for the probability matrix specifying the joint probability input distribution to be of rank one. For this, the following functions for matrices $\mathbf{P} = (P_{ij}) \in \mathbb{R}^{N_1 \times N_2}$ and vectors $\mathbf{p}_1 = (p_{1i}) \in \mathbb{R}^{N_1}, \mathbf{p}_2 = (p_{2i}) \in \mathbb{R}^{N_2}$ are defined:

$$f_1(\mathbf{P}, \mathbf{p}_2) \coloneqq \sum_{y,i,j} P_{ij} Q(y|i,j) \ln \frac{Q(y|i,j) p_{2j}}{\sum_k P_{kj} Q(y|k,j)}, \tag{3.68}$$

$$f_2(\mathbf{P}, \mathbf{p}_1) \coloneqq \sum_{y,i,j} P_{ij} Q(y|i,j) \ln \frac{Q(y|i,j) p_{1i}}{\sum_k P_{ik} Q(y|i,k)}, \tag{3.69}$$

$$f_{12}(\mathbf{P}) \coloneqq \sum_{y,i,j} P_{ij} Q(y|i,j) \ln \frac{Q(y|i,j)}{\sum_{k,l} P_{kl} Q(y|k,l)}. \tag{3.70}$$

Here, $\mathbf{P}$ models the joint input probability distribution (not necessarily independent for the two users), $\mathbf{p}_1$ and $\mathbf{p}_2$ specify the marginal distributions for each user and f_1, f_2, f_{12} are obtained by a slight modification of the mutual information expressions (allowing dependent distributions). The capacity computation problem can then, for a weight vector $\mathbf{w} = (\theta, 1-\theta)^T \geq \mathbf{0}$, be equivalently formulated as (note that we consider minimization instead of maximization here)

$$\begin{aligned}
\min_{R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2} \quad & -(\theta R_1 + (1-\theta) R_2) && (3.71)\\
\text{subj. to} \quad & R_1 \geq 0, R_2 \geq 0 && \\
& R_1 \leq f_1(\mathbf{P}, \mathbf{p}_2) && (3.72)\\
& R_2 \leq f_2(\mathbf{P}, \mathbf{p}_1) && (3.73)\\
& R_1 + R_2 \leq f_{12}(\mathbf{P}) && (3.74)\\
& \mathbf{P}\mathbf{1} = \mathbf{p}_1, \mathbf{P}^T\mathbf{1} = \mathbf{p}_2, \mathbf{P} \geq \mathbf{0}, \mathbf{1}^T\mathbf{P}\mathbf{1} = 1 && \\
& \operatorname{rank}(\mathbf{P}) = 1. && (3.75)
\end{aligned}$$

In what follows, we refer to this problem as the *extended capacity computation problem (ECCP)*. It is shown in [CPFV07] that f_1, f_2, f_{12} are concave functions, so that non-convexity in the above problem is only caused by the rank one constraint (3.75) which forces the joint distribution specified by $\mathbf{P}$ to be independent. By removing this constraint, and hence allowing cooperation of the users, one obtains a convex problem, which is efficiently solvable using standard convex optimization techniques. Subsequently, we refer to this problem as the *relaxed problem*. The objective function value f_{relaxed} at a solution $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ for the relaxed problem gives a lower bound to the objective value f_{optimal} of the actual (non-relaxed) problem. The matrix $\mathbf{P}$ obtained by solving the relaxed problem is generally not of rank one. In [CPFV07], it has been proposed to use the product distribution $\mathbf{P'} = \mathbf{p}_1^T \mathbf{p}_2$ (which is of rank one and obtained by forming the independent distribution specified by the marginals of $\mathbf{P}$) as a heuristic solution, called the *marginal approach* here: one fixes $\mathbf{P'}, \mathbf{p}_1, \mathbf{p}_2$ in the problem above and solves the convex problem in R_1, R_2, resulting in an objective value function f_{marginal}, which is an upper bound to the actual optimum value. We remark that it is easy to see that this solution is, for $\theta \leq \frac{1}{2}$, given by $R_1 = f_{12}(\mathbf{P'}) - f_2(\mathbf{P'}, \mathbf{p}_1)$, $R_2 = f_2(\mathbf{P'}, \mathbf{p}_1)$. From this, it also follows in this case that $f_{\text{marginal}} = -\Psi(\mathbf{p}_1, \mathbf{p}_2)$. For $\theta > \frac{1}{2}$, one similarly obtains $R_1 = f_1(\mathbf{P'}, \mathbf{p}_2)$, $R_2 = f_{12}(\mathbf{P'}) - f_1(\mathbf{P'}, \mathbf{p}_2)$.

In the following, we first give, for the general case, a sufficient condition for the situation that $f_{\text{relaxed}} = f_{\text{optimal}}$. We then apply this condition to a specific class of channels, showing that for this channel class, the capacity computation problem can in some cases be solved by first solving the relaxed problem and then calculating a rank one solution with the same objective value function. Additionally, we show by examples that in these cases the approach by taking the marginals generally yields suboptimal solutions, i.e., $f_{\text{marginal}} > f_{\text{relaxed}} = f_{\text{optimal}}$.

Subsequently, we study optimality conditions for the relaxed problem. This is a convex problem (as shown in [CPFV07]), and the KKT conditions provide necessary and sufficient conditions for optimality. Introducing Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}_+^5, \boldsymbol{\alpha} \in \mathbb{R}^{N_1}, \boldsymbol{\beta} \in \mathbb{R}^{N_2}, \boldsymbol{\mu} \in \mathbb{R}_+^{N_1 \times N_2}, \tau \in \mathbb{R}$, the Lagrangian function [BV04] for the relaxed problem is given by

$$\begin{aligned}\mathcal{L}(\cdots) = &-(\theta R_1 + (1-\theta) R_2) - \lambda_1 R_1 - \lambda_2 R_2 + \lambda_3 (R_1 - f_1(\mathbf{P}, \mathbf{p}_2)) \\ &+ \lambda_4 (R_2 - f_2(\mathbf{P}, \mathbf{p}_1)) + \lambda_5 (R_1 + R_2 - f_{12}(\mathbf{P})) \\ &+ \boldsymbol{\alpha}^T (\mathbf{P1} - \mathbf{p}_1) + \boldsymbol{\beta}^T (\mathbf{P}^T \mathbf{1} - \mathbf{p}_2) - \sum_{i,j} \mu_{i,j} P_{i,j} + \tau (\mathbf{1}^T \mathbf{P1} - 1).\end{aligned} \tag{3.76}$$

We define

$$A_{ij} := \frac{\partial}{\partial P_{ij}} f_1(\mathbf{P}, \mathbf{p}_2) = \sum_y Q(y|i,j) \ln \frac{Q(y|i,j) p_{2j}}{\sum_k P_{kj} Q(y|k,j)} - 1 \tag{3.77}$$

$$B_{ij} := \frac{\partial}{\partial P_{ij}} f_2(\mathbf{P}, \mathbf{p}_1) = \sum_y Q(y|i,j) \ln \frac{Q(y|i,j) p_{1i}}{\sum_k P_{ik} Q(y|i,k)} - 1, \tag{3.78}$$

$$C_{ij} := \frac{\partial}{\partial P_{ij}} f_{12}(\mathbf{P}) = \sum_y Q(y|i,j) \ln \frac{Q(y|i,j)}{\sum_{k,l} P_{kl} Q(y|k,l)} - 1. \tag{3.79}$$

Moreover, we write $D_i := \frac{\partial}{\partial p_{1i}} f_1(\mathbf{P}, \mathbf{p}_2) = \frac{\sum_j P_{ij}}{p_{1i}}$ and $E_i := \frac{\partial}{\partial p_{2i}} f_2(\mathbf{P}, \mathbf{p}_1) = \frac{\sum_j P_{ji}}{p_{2i}}$. Then, the stationarity conditions of the Lagrangian at the KKT point are given by the set of equations

$$-\lambda_1 + \lambda_3 + \lambda_5 = \theta \tag{3.80}$$

$$-\lambda_2 + \lambda_4 + \lambda_5 = 1 - \theta \tag{3.81}$$

$$\forall i : -\lambda_4 D_i - \alpha_i = 0 \tag{3.82}$$

$$\forall i : -\lambda_3 E_i - \beta_i = 0 \tag{3.83}$$

$$\forall i, j : -\lambda_3 A_{ij} - \lambda_4 B_{ij} - \lambda_5 C_{ij} + \alpha_i + \beta_j - \mu_{ij} + \tau = 0 \tag{3.84}$$

Subsequently, we restrict to the case $\theta \leq \frac{1}{2}$, the other case can, by symmetry, be treated similarly. Consider an optimal solution $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ with $R_1 > 0, R_2 > 0$. From complementary slackness, we have $\lambda_1 = \lambda_2 = 0$. Furthermore, it is clear that at least one of the constraints (3.72)-(3.74) must be active for this solution. First consider the case that (3.72) is not active. Then $\lambda_3 = 0$, implying $\lambda_4 = 1 - 2\theta > 0, \lambda_5 = \theta > 0$, so that (3.73) and (3.74) are active. If (3.73) is not active ($\lambda_4 = 0$), then $\lambda_5 = 1 - \theta$ and $\lambda_3 = 2\theta - 1 < 0$, so that (3.73) must always be active. For inactive (3.74), it follows that (3.72) and (3.73) must be active. Summarizing, there are three cases possible:

1. (3.72) active, (3.73) active, (3.74) active
2. (3.72) inactive, (3.73) active, (3.74) active
3. (3.72) active, (3.73) active, (3.74) inactive

Now consider the case 2) where (3.72) is inactive and $\mathbf{P} > \mathbf{0}$. Clearly, at the optimum, $D_i = E_i = 1$ and by complementary slackness, $\boldsymbol{\mu} = \mathbf{0}$, simplifying the system (3.80)-(3.84) to

$$\forall i, j : (1 - 2\theta) B_{ij} + \theta C_{ij} = 2\theta - 1 + \tau \tag{3.85}$$

Now, for each $\mathbf{P} \in \mathbb{R}_+^{N_1 \times N_2}$ and channel transition matrix $\mathbf{Q}$, we define the sets of matrices

$$\begin{aligned} \mathcal{S}(\mathbf{Q}, \mathbf{P}) := \Big\{ & \tilde{\mathbf{P}} \in \mathbb{R}^{N_1 \times N_2} : \tilde{\mathbf{P}} \geq \mathbf{0}, \tilde{\mathbf{P}}\mathbf{1} = \mathbf{P}\mathbf{1}, \mathbf{1}^T \tilde{\mathbf{P}} \mathbf{1} = 1, \\ & \forall y : \sum_{k,l} (P_{kl} - \tilde{P}_{kl}) Q(y|k,l) = 0, \\ & \forall i, y : \sum_{k} (P_{ik} - \tilde{P}_{ik}) Q(y|i,k) = 0 \Big\} \end{aligned} \tag{3.86}$$

and

$$\mathcal{S}^1(\mathbf{Q}, \mathbf{P}) := \left\{ \tilde{\mathbf{P}} \in \mathcal{S}(\mathbf{Q}, \mathbf{P}) : \operatorname{rank}(\tilde{\mathbf{P}}) = 1 \right\}. \tag{3.87}$$

The set $\mathcal{S}(\mathbf{Q}, \mathbf{P})$ consists of the positive solutions of a linear system defined by the channel matrix $\mathbf{Q}$ and the solution matrix $\mathbf{P}$. One obtains the following sufficient conditions for $f_{\text{relaxed}} = f_{\text{optimal}}$:

Proposition 2. *Let $\theta \le \frac{1}{2}$ and $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ be a solution to the relaxed problem such that $f_1(\mathbf{P}, \mathbf{p}_2) < R_1$ [i.e., (3.72) is inactive], $\mathbf{P} > \mathbf{0}$ and $R_1, R_2 > 0$. If $\overline{\mathbf{P}} \in \mathcal{S}^1(\mathbf{Q}, \mathbf{P})$, then $(\overline{R}_1, \overline{R}_2, \overline{\mathbf{P}}, \overline{\mathbf{P}}\mathbf{1}, \overline{\mathbf{P}}^T\mathbf{1})$ is a solution to the non-relaxed problem, where $\overline{R}_1 = f_{12}(\overline{\mathbf{P}}) - f_2(\overline{\mathbf{P}}, \overline{\mathbf{P}}\mathbf{1}), \overline{R}_2 = f_2(\overline{\mathbf{P}}, \overline{\mathbf{P}}\mathbf{1})$.*

Proof. The expressions B_{ij} and C_{ij} are kept invariant by the matrices in $\mathcal{S}(\mathbf{Q}, \mathbf{P})$. Hence, choosing the same Lagrange multipliers as for $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$, by inspection of the stationarity condition (3.85) and observing that the complementary slackness conditions hold, it follows that $(\overline{R}_1, \overline{R}_2, \overline{\mathbf{P}}, \overline{\mathbf{P}}\mathbf{1}, \overline{\mathbf{P}}^T\mathbf{1})$ also satisfies the KKT conditions. Finally, note that $\overline{R}_1 \ge 0$ from the properties of mutual information. □

3.5.1 Relaxation Solutions for the 3-parameter $(2, 2; 2)$-MAC

For some channels, it is possible to explicitly construct matrices in $\mathcal{S}^1(\mathbf{Q}, \mathbf{P})$. For example, this is the case for the 3-parameter $(2, 2; 2)$-MAC as defined in Section 3.3.2, i.e., the $(2, 2; 2)$-MAC with the restriction $Q(1|1, 1) = Q(1|1, 2)$ on the channel matrix $\mathbf{Q}$. For a matrix $\mathbf{P} \in \mathbb{R}_+^{2\times 2}$, we define

$$\hat{\mathbf{P}} := \begin{cases} \mathbf{P} + \frac{\det(\mathbf{P})}{P_{21}+P_{22}} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, & \text{if } P_{21} + P_{22} > 0 \\ \mathbf{P}, & \text{if } P_{21} + P_{22} = 0. \end{cases} \tag{3.88}$$

For the 3-parameter $(2, 2; 2)$-MAC, it holds that for each $\mathbf{P} \in \mathbb{R}_{++}^{2\times 2}, \mathbf{P} > \mathbf{0}$, the set $\mathcal{S}^1(\mathbf{Q}, \mathbf{P})$ contains the matrix $\hat{\mathbf{P}}$: it is easily verified that $\hat{\mathbf{P}} \in \mathcal{S}(\mathbf{Q}, \mathbf{P})$, and one quickly checks that $\det(\hat{\mathbf{P}}) = 0$, implying $\operatorname{rank}(\hat{\mathbf{P}}) = 1$ since $\mathbf{P} \ne \mathbf{0}$. From this, we get the following sufficient conditions for the constructability of a rank one solution out of the relaxed solution:

Corollary 2. *For the 3-parameter $(2, 2; 2)$-MAC and $\theta \le 1/2$, it holds that if the solution $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ to the relaxed problem satisfies $f_1(\mathbf{P}, \mathbf{p}_2) < R_1$, $\mathbf{P} > \mathbf{0}$ and $R_1, R_2 > 0$, then $(\hat{R}_1, \hat{R}_2, \hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}, \hat{\mathbf{P}}^T\mathbf{1})$ is a solution to the ECCP (3.71), where $\hat{R}_1 = f_{12}(\hat{\mathbf{P}}) - f_2(\hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}), \hat{R}_2 = f_2(\hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1})$ and yields the same objective value, i.e., $f_{optimal} = f_{relaxed}$.*

Applying this corollary, the following solution approach for the 3-parameter $(2, 2; 2)$-MAC for $\theta \le 1/2$ can be applied:

1. Solve the relaxed problem, obtaining a solution $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$.

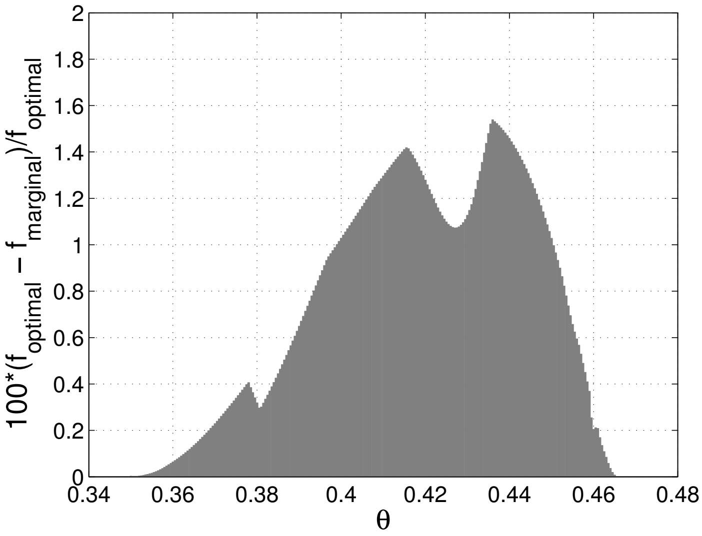

Figure 3.8: Relative performance loss using the marginal approach in comparison to the optimal solution for example I.

2. Check if $\operatorname{rank}(\mathbf{P}) = 1$ (if yes, then a solution to the non-relaxed problem has been found).
3. If $f_1(\mathbf{P}, \mathbf{p}_2) < R_1$, $\mathbf{P} > \mathbf{0}$ and $R_1, R_2 > 0$, then a solution to the non-relaxed problem is $(\hat{R}_1, \hat{R}_2, \hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}, \hat{\mathbf{P}}^T\mathbf{1})$ with $\hat{R}_1 = f_{12}(\hat{\mathbf{P}}) - f_2(\hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}), \hat{R}_2 = f_2(\hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1})$, i.e., optimal input probability distributions are given by $\mathbf{p}_1 = \hat{\mathbf{P}}\mathbf{1}, \mathbf{p}_2 = \hat{\mathbf{P}}^T\mathbf{1}$.

Opposed to this, the approach of obtaining a rank one solution via the marginals (i.e., using $\mathbf{P}' = \mathbf{p}_1^T\mathbf{p}_2$ as described above) is generally suboptimal even for the 3-parameter $(2, 2; 2)$-MAC. We demonstrate this by two examples in the following. The first example (example I) is given by the 3-parameter $(2, 2; 2)$-MAC with channel parameters $Q(1|11) = Q(1|12) = 0.7, Q(1|21) = 0.6, Q(1|22) = 0.1$. For the second example (example II), we use the parameters $Q(1|11) = Q(1|12) = 0.4, Q(1|21) = 0.1, Q(1|22) = 0.36$. Here, we choose a discretization of weight values θ in the interval $(0, \frac{1}{2})$ of step size 10^{-4}. For each such value θ, we first solve the relaxed (convex) problem using the MATLAB function `fmincon`, resulting in a solution $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ with objective function value f_{relaxed}. The marginal approach results in the objective function value $f_{\text{marginal}} = -\Psi(\mathbf{p}_1, \mathbf{p}_2)$ and rate pairs $\left(R_1^{\text{marginal}}, R_2^{\text{marginal}}\right)$. If the relaxed solution satisfies the conditions in Theorem 4, then $f_{\text{optimal}} = f_{\text{relaxed}}$ and the optimal input distributions are computed as $\mathbf{p}_1^* = \hat{\mathbf{P}}\mathbf{1}, \mathbf{p}_2^* = \hat{\mathbf{P}}^T\mathbf{1}$. The corresponding optimal rates are given by $R_1^{\text{optimal}} = \hat{R}_1, R_2^{\text{optimal}} = \hat{R}_2$. For the two examples and the different θ values, we compare the objective function

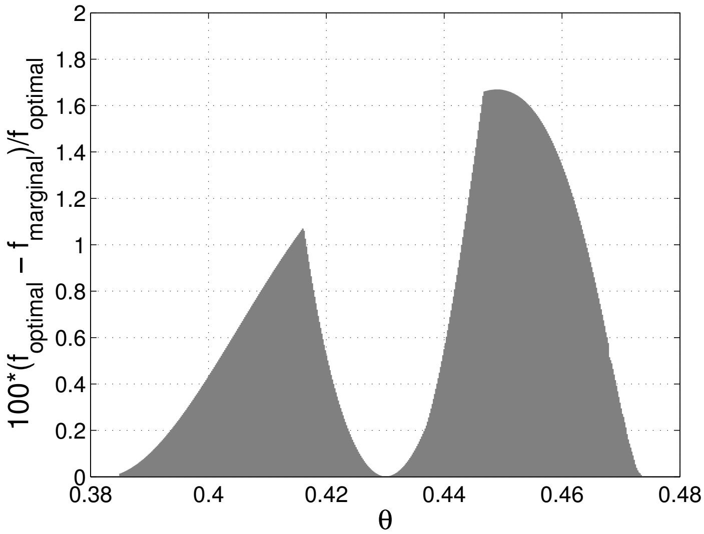

Figure 3.9: Relative performance loss using the marginal approach in comparison to the optimal solution for example II.

values f_{optimal} and f_{marginal}, the rate regions given by the points $\left(R_1^{\text{marginal}}, R_2^{\text{marginal}}\right)$ and $\left(R_1^{\text{optimal}}, R_2^{\text{optimal}}\right)$, respectively, and the input probability distributions obtained.

Figure 3.8 and Figure 3.9 show the relative performance loss (in percentage with respect to the optimal solution) of the marginal approach in comparison to the optimal solution for example I and example II, respectively. In each of the figures, it is displayed for a range of θ values (which suffices to trace out the boundary of the capacity regions). We remark that for all values in this range, the conditions in Theorem 4 hold. One sees that there is a small, but notable performance loss for the marginal approach as compared to the optimal solution. Figure 3.10 and Figure 3.11 display a section of the rate regions obtained, illustrating that the boundary obtained via the marginal approach is strictly contained inside the actual capacity boundary. Finally, Figure 3.12 displays the input probability distributions $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_1^*, \mathbf{p}_2^*$ for both examples. For each example, the curves show the pairs of the first components of the vectors $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_1^*, \mathbf{p}_2^*$, respectively. The figure demonstrates that the input probability distributions obtained by the marginal approach generally differ considerably from the optimal ones.

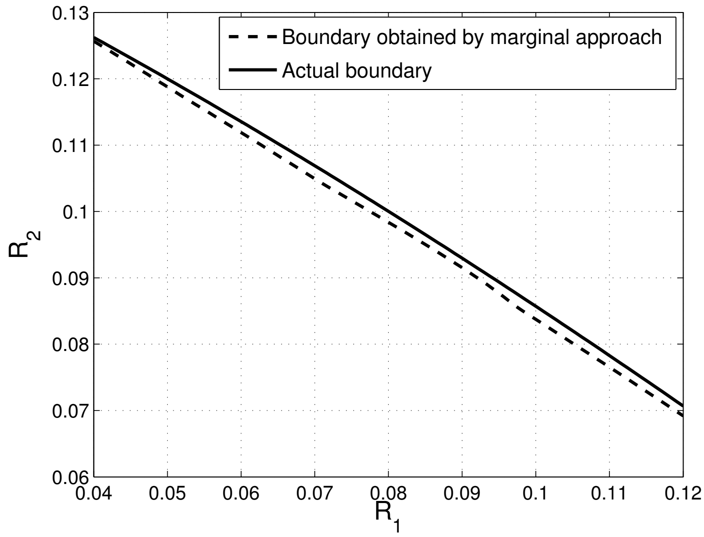

Figure 3.10: Section of the actual capacity boundary and the boundary obtained by the marginal approach for example I.

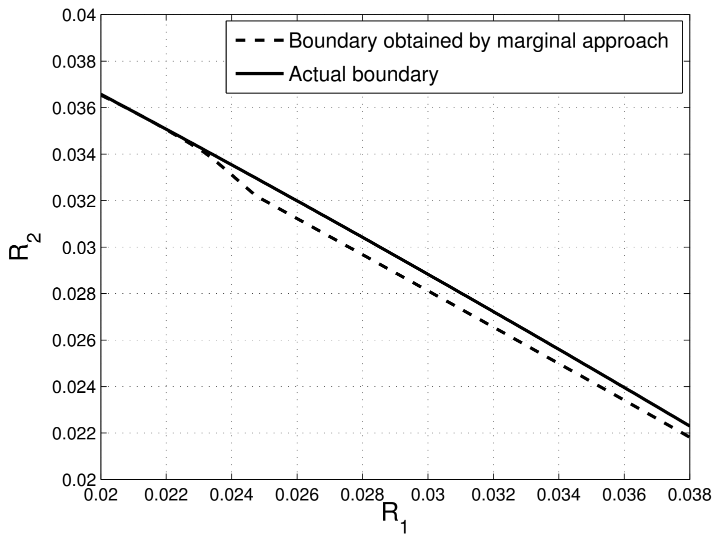

Figure 3.11: Section of the actual capacity boundary and the boundary obtained by the marginal approach for example II.

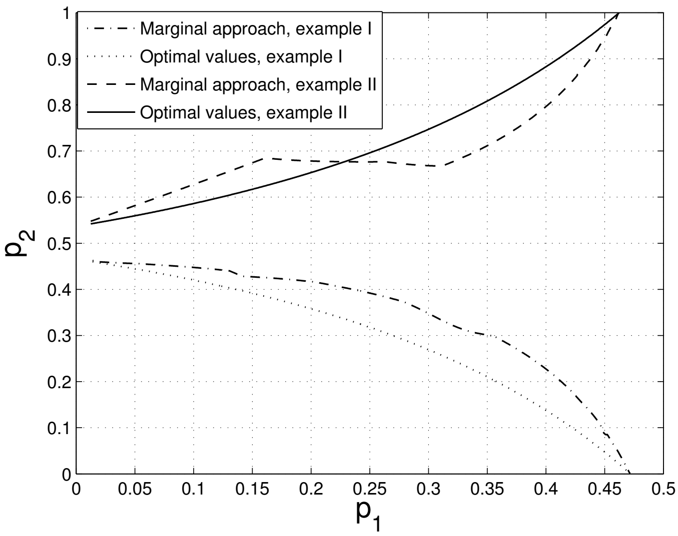

Figure 3.12: Comparison of the optimal input probability distribution and the distribution obtained by the marginal approach for example I and example II.

3.5.2 Sufficient Condition for the Constructability of an Optimal ECCP Solution for the 3-parameter $(2,2;2)$-MAC

We now give a sufficient condition under which for a matrix $\mathbf{P} \in \mathbb{R}_+^{2\times 2}$, a solution to the ECCP can be constructed. This condition will be useful for the (weak) duality investigations for discrete memoryless channels in the following chapter. For this, we define

$$\Gamma_{ij}(\mathbf{P}) := (1-2\theta)B_{ij} + \theta C_{ij}. \tag{3.89}$$

Then we have the following lemma:

Lemma 5. *For the three-parameter $(2,2;2)$-MAC and $\theta \leq 1/2$, it holds that if $\mathbf{P} \in \mathbb{R}_+^{2\times 2}$ satisfies $\mathbf{1}^T\mathbf{P}\mathbf{1} = 1$ and*

$$\forall i,j,k,l : \Gamma_{ij}(\mathbf{P}) = \Gamma_{kl}(\mathbf{P}) \tag{3.90}$$

then $(R_1, R_2, \hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}, \hat{\mathbf{P}}^T\mathbf{1})$ is a solution to the ECCP (3.71), where $R_1 = f_{12}(\mathbf{P}) - f_2(\mathbf{P}, \mathbf{P}\mathbf{1})$, $R_2 = f_2(\mathbf{P}, \mathbf{P}\mathbf{1})$ and $\hat{\mathbf{P}}$ is given in (3.88). Furthermore, the optimal value is, for any $i,j \in \{1,2\}$, given by

$$f(\mathbf{P}) := -(\theta R_1 + (1-\theta)R_2) = -(\Gamma_{ij}(\mathbf{P}) + 1 - \theta). \tag{3.91}$$

Proof. It is easily verified that for all $i,j \in \{1,2\}$, we have $\Gamma_{ij}(\mathbf{P}) = \Gamma_{ij}(\hat{\mathbf{P}})$, $f_2(\hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}) =$

$f_2(\mathbf{P}, \mathbf{P1})$ and $f_{12}(\hat{\mathbf{P}}) = f_{12}(\mathbf{P})$. Moreover, $\text{rank}(\hat{\mathbf{P}}) = 1$. Using $\frac{\partial}{\partial p_{1i}} f_1(\hat{\mathbf{P}}, \hat{\mathbf{P}}\mathbf{1}) = 1$ and choosing Lagrange multipliers $\lambda_1 = \lambda_2 = \lambda_3 = 0, \lambda_4 = 1 - 2\theta, \lambda_5 = \theta, \boldsymbol{\mu} = \mathbf{0}$ and $\mathbf{p}_1 = \hat{\mathbf{P}}\mathbf{1}, \mathbf{p}_2 = \hat{\mathbf{P}}^T\mathbf{1}$, the stationarity conditions (3.80)-(3.84) reduce to

$$\forall i, j : \Gamma_{ij}(\hat{\mathbf{P}}) = 2\theta - 1 + \tau, \tag{3.92}$$

which is, by (3.90), true for the appropriate choice of τ. Furthermore, $R_1, R_2 \geq 0$ and the complementary slackness conditions are satisfied by the choice of R_1, R_2 and the Lagrange multipliers. Now, for any $\mathbf{P}, \mathbf{p}_1, \mathbf{p}_2$, we have the following useful relations:

$$\sum_{i,j} P_{ij} \frac{\partial}{\partial P_{ij}} f_2(\mathbf{P}, \mathbf{p}_1) = f_2(\mathbf{P}, \mathbf{p}_2) - 1, \tag{3.93}$$

$$\sum_{i,j} P_{ij} \frac{\partial}{\partial P_{ij}} f_{12}(\mathbf{P}) = f_{12}(\mathbf{P}) - 1. \tag{3.94}$$

By weighting equation (3.92) by the $\hat{P}_{ij}$ values and summing, one obtains that

$$\sum_{i,j} \hat{P}_{ij} \Gamma_{ij}(\hat{\mathbf{P}}) = -f(\mathbf{P}) - (1 - 2\theta) - \theta = 2\theta - 1 + \tau, \tag{3.95}$$

so that $f(\mathbf{P}) = -(\tau + \theta)$. □

Note that Corollary 2 can also be derived from Lemma 5 instead of from the more general condition given by Proposition 2.

As a final remark, we note that the extension of the relaxation approach to the case of an arbitrary number of users is quite straightforward, as is demonstrated in [CPFV10]. This work also describes a randomization approach for as an alternative to the marginal approach. Here, a rank one solution is constructed randomly under guidance of the solution to the relaxed problem. What is more, this work also gives conditions under which the inner bound and / or the outer bound obtained by the relaxation and marginal approach are tight.

3.6 Summary and Conclusions

In this chapter, we discussed the problem of computing the capacity region of the discrete memoryless MAC. In the literature, it has been claimed [WK09] that if the MAC is elementary, i.e., if the size of each input alphabet is less than or equal to the size of the channel output alphabet, then the KKT conditions provide necessary and sufficient conditions for an input probability distribution to be optimal. We have demonstrated that this claim is invalid, even for the case of the $(2,2;2)$-MAC, that is, the two-user MAC with binary alphabets. We used two example channels to serve as counter-examples to the sufficiency claim in [WK09] and outlined where the proof given in [WK09] fails to hold. However, the examples are not in contradiction to earlier work on the problem in [Wat96]. Our first conclusion is that the capacity computation problem and even the problem of sum-rate optimization for the discrete multiple-access channel remains open and an interesting subject for further research.

In the light of the facts mentioned above, we further focused on weighted sum rate optimization for the $(2,2;2)$-MAC. The problem was first reduced to a one-dimensional problem and the single user problem and then studied for the 3-parameter $(2,2;2)$-MAC, for which $Q(1|1,1) = Q(1|1,2)$. For weights satisfying $w_1 \leq w_2$, we showed that, depending on an ordering property of the channel transition probability matrix, either the maximum is attained on the boundary, or there is at most one stationary point in the interior of the optimization domain. These results represent a generalization of the results in [Wat96] to more general weight vectors. The proof was obtained by showing that the reduction to the one-dimensional problem leads to a pseudoconcave formulation, which can also be used numerically for the capacity optimization. For a further restricted class of $(2,2;2)$-MAC channels, an explicit solution for the one-dimensional problem could be given. As a second structural property, we showed that the region G formed by the successive interference cancellation points C_1 or C_2 is a comprehensive set, which might be a useful property for further investigations of the problem.

For future work, it is an interesting question whether it is possible to find a general classification of discrete memoryless MACs, stating for which channels the KKT conditions are necessary and sufficient and for which they are not.

In the last part of the chapter, we studied the relaxation (user cooperation) approach suggested in [CPFV07]. This method identifies a rank one constraint as the only non-convexity of the problem, allowing to find suboptimal solutions via convex optimization by removing this constraint. We derived conditions under which a solution to the (convex) relaxed problem has the same value as the actual optimal solution. For the class of 3-parameter $(2,2;2)$-MAC, we applied these results to show that, in some cases, it is possible to obtain the optimal solution by first solving the convex problem and then computing an optimal solution for the actual (non-relaxed) problem. By means of two examples, we demonstrated that even for these channels, the marginal approach suggested in [CPFV07] offers only suboptimal solutions. These results also provide a useful characterization of the capacity region for the class of 3-parameter $(2,2;2)$-MACs, which will be used for duality investigations for the discrete memoryless MAC and BC in the following chapter.

4 MAC-BC Duality for Discrete Memoryless and Fading Channels

The (discrete memoryless) multiple access channel has been the subject of the previous chapter. We now also consider communication in the reverse direction, that is, the *broadcast channel* (BC), for which there is one transmitter intending to convey a message to each of several receivers. Despite the fact that the MAC and the BC are structurally similar, the BC is generally more difficult to handle in terms of an information-theoretic analysis. Most notably, while the discrete memoryless MAC is understood quite well information-theoretically, this is not true at all for the discrete memoryless BC—the capacity region is (except for some special cases) still unknown even for the case of two users. The capacity region of the *Gaussian* BC is known, and has recently also been found for the MIMO case [WSS06], i.e., the case of multiple antennas at the transmitter and the receivers. However, several problems that are comparatively easy to solve for the MAC are substantially harder to solve in the BC. This has motivated researchers to study the connection between the MAC and the BC more closely, resulting in the uncovering of very useful relationships between them, known as *MAC-BC duality* or *uplink-downlink duality*. Speaking in general terms, duality means that it is possible to express a certain performance criterion for the BC (such as data rate or capacity) in terms of a dual MAC and vice versa. There are many examples where a problem for the BC can be "translated" to the dual MAC setup, solved there efficiently and "translated" back to a solution for the original BC problem. For example, an efficient calculation of optimal transmit covariance matrices for the Gaussian MIMO BC can be performed by employing this procedure [VJG03]. While duality is known for a variety of different types of channel models (see below), there is still interest in finding dualities for other classes of channels. From a more general viewpoint, MAC-BC duality is also believed to contribute to a better understanding of network information theory. For example, duality might guide a way to a better comprehension of the capacity region of the discrete BC, which is still unknown. Considering this, it is of interest to what extent there is a duality between the discrete memoryless MAC and BC. A kind of such a relation has been found for a class of deterministic channels in [JVG03], but the general problem remains open.

In this chapter, we study two different channel classes with respect to the existence (or non-existence) of duality between the MAC and the BC. At first, we turn to the discrete memoryless channels and define weak BC-to-MAC duality, which is a slightly more general condition than the existing duality definitions. In short, weak BC-to-MAC duality means that for each $\mathbf{x}$ that is a solution to a maximum weighted sum-rate problem for a weight vector $\mathbf{w}$ over the BC capacity region, there exists a MAC such that $\mathbf{x}$ is also a solution to the $\mathbf{w}$-weighted sum-rate problem over the MAC capacity region. While our definition of weak duality increases the degrees of freedom for the shape of the dual MAC channels as opposed to duality in the sense of the Gaussian case, it is still a useful definition since a necessary condition for weak duality is also necessary for duality in the "stronger" Gaussian sense. What is more, establishing weak duality can be used as a step to showing duality in the stronger sense by additionally proving that all "dual" MACs are contained in the BC region. Using the results from the previous chapter, we derive a sufficient condition as well as a necessary condition for a BC to be weakly dual to a class of discrete memoryless MACs, namely the 2-MAC. Note that this channel has been studied in Section 3.3.3. Furthermore, we apply these results for the binary symmetric BC and prove that under some conditions, this channel is weakly dual to the 2-MAC.

Secondly, duality relations are explored for *fading* channels for which no channel state information is provided to the transmitter. Concerning duality of fading channels, some results have been reported in the literature. As already pointed out in [JVG04], in the case where the fading states are known to the transmitter (and the receiver), the transmitter can adapt its transmission to the current fading state, and the duality for the non-fading case directly extends to channels with fading. Lack of duality for channel state information at the transmitter only has been shown in [Jor06] for the case of linear transmitters and receivers without time-sharing. In this chapter, we study duality relationships for the (fast) fading BC under additive white Gaussian noise (AWGN) and a strict ordering of the (discrete) fading distributions where the instantaneous fading states are perfectly known to the receiver, but unknown to the transmitter. An achievable rate region is obtained by using superposition coding and successive decoding (SCSD) and Gaussian signaling. This region is conjectured to be the capacity region of the channel [TS03]. We consider the natural corresponding dual fading MAC, which is simply obtained by reversing the roles of the transmitters and receivers and letting the sum of the individual power constraints equal the BC power constraint. We show that if the fading distribution for the weaker user is non-deterministic, the achievable region using SCSD and Gaussian signaling is different from the dual MAC region. Opposed to this, in the case of a deterministic fading distribution for the weaker user, i.e., the situation of one-sided fading, the achievable region equals the achievable MAC region using successive interference cancellation, where

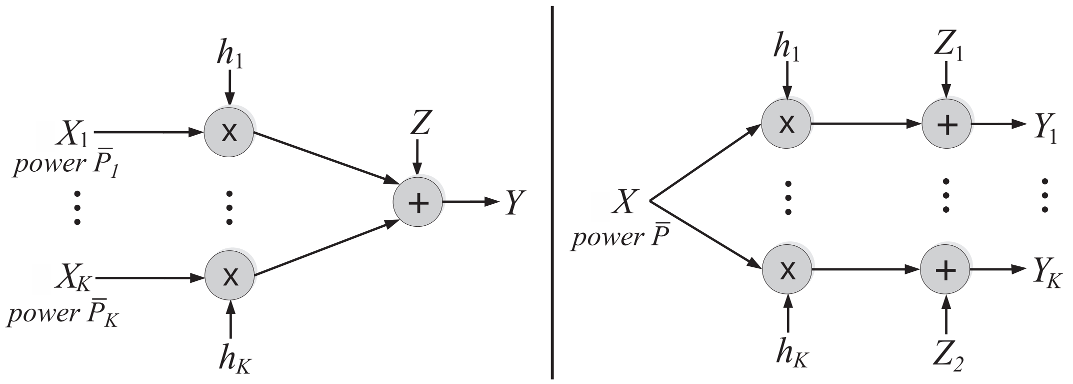

Figure 4.1: Gaussian multiple access channel (left) and Gaussian broadcast channel (right).

the stronger user is decoded first. Hence, duality holds in this case and the well-known way for solving BC problems via the dual MAC can be applied. If there is no duality, such an approach appears not to be applicable. However, we suggest that for some problems, it might be possible to obtain good solutions for a BC problem by solving the problem over the corresponding dual MAC and then heuristically constructing a solution for the BC based on the outcome of the MAC optimization. As an example, we consider the problem of weighted sum-rate optimization for the fading BC, which is non-convex. We demonstrate that for the channels at hand, such a procedure can successfully be applied for this problem. Moreover, under a certain assumption, we derive upper bounds depending only on the channel statistics for the error induced by this approach.

The duality concept can best be illustrated by the real-valued Gaussian SISO (single antenna) MAC and BC, which is the topic of the following section. We remark here that the extension of the duality property to channels with complex-valued symbols is straightforward.

4.1 Duality for the Gaussian MAC and BC

The (real) Gaussian MAC with K transmitters has received signal

$$Y = \sum_{k=1}^{K} \sqrt{h_k} X_k + Z, \tag{4.1}$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is additive white Gaussian noise, h_k are the channel gains and the transmit signals X_k are subject to individual average power constraints given by the vector $\overline{\mathbf{P}} = (\overline{P}_1, \ldots, \overline{P}_K)^T$ (cf. Figure 4.1). We collect the channel gains in the vector

$\mathbf{h} = (h_1, \ldots, h_K)^T$ and write the capacity region as $\mathcal{C}_{\text{MAC}}(\overline{\mathbf{P}}, \mathbf{h})$. It is given by

$$\mathcal{C}_{\text{MAC}}(\overline{\mathbf{P}}, \mathbf{h}) = \left\{ \mathbf{R} \in \mathbb{R}_+^K : \forall M \subseteq \{1, \ldots, K\} \sum_{k \in M} R_k \le \frac{1}{2} \ln \left(1 + \frac{\sum_{k \in M} h_k \overline{P}_k}{\sigma^2} \right) \right\}. \tag{4.2}$$

This region is a polytope in K dimensions and the rate points contained in it can be achieved by the technique of *successive interference cancellation* (in combination with time sharing), and each corner point of the polytope corresponds to one of the possible decoding orders (cf. Section 3.1 or [Gol05]). Alternatively, the rates in $\mathcal{C}_{\text{MAC}}(\overline{\mathbf{P}}, \mathbf{h})$ can be achieved by the technique of *rate splitting* [RU96].

The (real-valued) Gaussian BC with K receivers takes input symbols $X \in \mathbb{R}$, and the received signal $Y_k \in \mathbb{R}$ at the kth receiver is given by

$$Y_k = \sqrt{h_k} X + Z_k, \tag{4.3}$$

where $Z_k \sim \mathcal{N}(0, \sigma^2)$ represents additive white Gaussian noise, h_k are the channel gains and the transmit signal is subject to an average power constraint of $\overline{P}$.

Under a power allocation specified by the vector $\mathbf{P} = (P_1, \ldots, P_K)^T$, the rates

$$\mathcal{R}_{\text{BC}}(\mathbf{P}, \mathbf{h}) = \left\{ \mathbf{R} \in \mathbb{R}_+^K : R_k \le \frac{1}{2} \ln \left(1 + \frac{h_k P_k}{\sigma^2 + h_k \sum_{j=1}^K P_j \mathbb{1}[h_j > h_k]} \right) \right\} \tag{4.4}$$

are achievable using *superposition coding and successive decoding* [BC74], [Ber74] or, as an alternative, by *dirty paper coding* [Cos83], [YC01], [WSS06], [Gol05].

Then, the capacity region $\mathcal{C}_{\text{BC}}(\overline{P}, \mathbf{h})$ is obtained by varying over all admissible power allocations:

$$\mathcal{C}_{\text{BC}}(\overline{P}, \mathbf{h}) = \bigcup_{\mathbf{P} \in \mathbb{R}_+^K : \sum_{k=1}^K P_k = \overline{P}} \mathcal{R}_{\text{BC}}(\mathbf{P}, \mathbf{h}). \tag{4.5}$$

As already mentioned in Chapter 1, these channels are commonly used as models for cellular systems: the MAC corresponds to the *uplink* in a cell (transmission of the mobile users to the base station), and the BC models the *downlink* (where the base station transmits to the mobile users). For the Gaussian channels, we can see that the major differences between the MAC and the BC are as follows: in the uplink, there is only one additive noise term, but K power constraints. Opposed to this, for the downlink, there are K noise terms to be considered, but only one power constraint. Moreover, in the downlink, both the signal and interference for a specific user pass through the same channel, whereas in the uplink the user signal and the interfering signals are subject to different channels. Despite the structural similarities between the two channels, the capacity regions differ

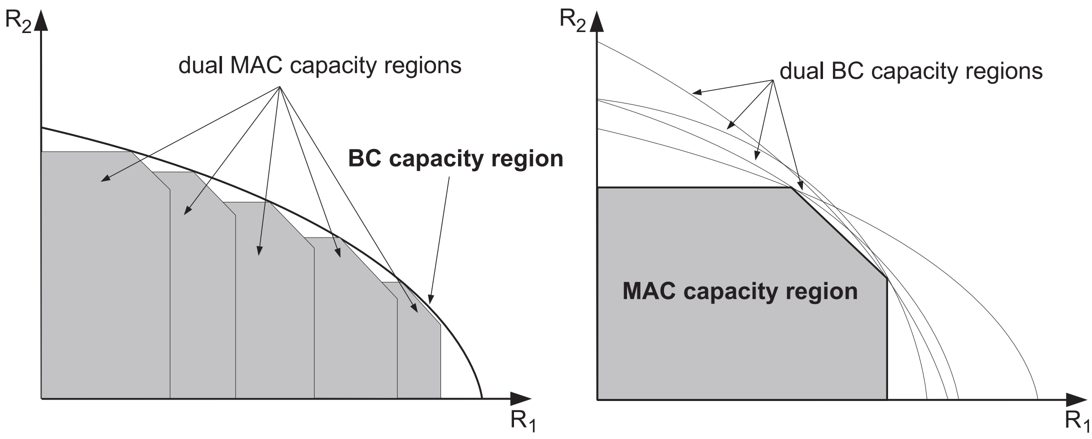

Figure 4.2: Illustration of MAC-BC duality: BC-to-MAC duality (left) and MAC-to-BC duality (right).

quite substantially in their shapes. While the MAC region is a polytope in K dimensions, the BC region has a smooth boundary (cf. Figure 4.2). Yet, the capacity regions of the Gaussian MAC and BC exhibit a close connection, as has been demonstrated in [JVG04]: the capacity region of the Gaussian K-user BC with power constraint $\overline{P}$ can be expressed as a union of capacity regions of the Gaussian K-user MAC, where the union is taken over all channels having sum power constraint $\overline{P}$. On the other hand, the capacity region of a Gaussian BC can be written as the intersection of a set of Gaussian MAC regions. This relation is known as *duality* between the MAC and the BC. More precisely, the Gaussian BC capacity region can be expressed in terms of a set of Gaussian MAC channels:

$$\mathcal{C}_{\mathrm{BC}}(\overline{P}, \mathbf{h}) = \bigcup_{\overline{\mathbf{P}}: \sum_i \overline{P}_i = \overline{P}} \mathcal{C}_{\mathrm{MAC}}(\overline{\mathbf{P}}, \mathbf{h}). \tag{4.6}$$

This relation is also referred to as *BC-to-MAC duality*. Conversely, the Gaussian MAC capacity region can be written as an intersection of Gaussian BC capacity regions as

$$\mathcal{C}_{\mathrm{MAC}}(\overline{\mathbf{P}}, \mathbf{h}) = \bigcap_{\boldsymbol{\alpha} > \mathbf{0}} \mathcal{C}_{\mathrm{BC}}\left(\sum_{i=1}^{K} \frac{P_i}{\alpha_i}, (\alpha_1 h_1, \ldots, \alpha_K h_K) \right), \tag{4.7}$$

which is called *MAC-to-BC duality*. These duality relations are illustrated in Figure 4.2.

4.2 Other Dualities

Channel duality has been investigated for more general network structures and different types of channel models. A couple of these extensions and variations are listed below. Note that some of these duality results are only mentioned here and will be treated in more detail later on.

- First of all, the duality results given above easily extend to the fading Gaussian MAC and BC where channel state information is available at the transmitter. This has already been reported in [JVG04] with respect to **ergodic capacity** and **outage capacity**. In the case where no channel state information is provided to the transmitter, duality holds only under specific conditions, as will be discussed in detail in Section 4.5.
- The extension to the **MIMO case of the Gaussian MAC and BC**, i.e., with multiple antennas at the transmitters and receivers, yields duality relations analogously to the expressions (4.6) and (4.7), where the capacity regions are replaced by the corresponding regions for the MIMO channels [VJG03], [Yu06]. Note that the capacity region for the BC equals the dirty paper coding region, as shown in [WSS06].
- Another line of extension of duality results is concerned with **relaying structures**. Here, the MAC and BC are augmented by one or more *relays* in between the transmitters and receivers in order to help the communication. The known duality results for relaying structures mostly consider the relaying technique of *amplify-and-forward* (AF) where each relay transmits a scaled version of its received signal to the destination(s). Known dualities include results for two-hop AF relaying with multiple antennas [GJ10] and for multi-hop AF MIMO relaying with single antenna source and destination nodes [JGH07]. Note that the work in [RK11] establishes SINR duality (see below) for a more general case.
- Some work concerning duality of **discrete memoryless channels** is presented for the case of deterministic channels in [JVG03]. This topic will also be discussed in detail subsequently (see Section 4.4).
- A few duality results are known for the **linear deterministic model** [ADT11], which was designed to represent an approximative model for Gaussian channels. Under **linear coding**, a duality results holds for general networks: any rate region that is achievable by linear encoding strategies at the network nodes is also achievable in the corresponding dual network (that is obtained by reversing link directions and transposing channel gain matrices) [RPV09]. Note that this applies not only to the linear deterministic model, but also to models with more general finite-field

channel gain matrices. Beyond that, there are also duality results with respect to the capacity regions of linear deterministic networks. These include the duality between the **many-to-one and the one-to-many interference channel** [BPT10]. For the two-user **interference channel with cooperative links**, duality holds between the receiver cooperation and the transmitter cooperation channel both for in-band [PV08] and out-of-band cooperation [WT11a], [WT11b]. The linear deterministic model is treated in detail in Chapter 5, including an elaboration on the corresponding duality results.

The duality relations described above all refer to duality with respect to the capacity region of a system and its corresponding dual. While this notion of duality takes center stage in this thesis, it is important to note that duality has also been investigated for other performance criteria. Specifically, two other types of MAC-BC dualities are known as *SINR duality* and *MSE duality*.

SINR duality refers to the equivalence of the uplink and downlink regions of the achievable signal-to-interference-and-noise ratios (SINR). Early work using the SINR duality of uplink and downlink can be found in the context of power control [ZF94] and beamforming [RFLT98] and has been extended in later works [BS02a], [BS02b], [VT03], [SB04]. Recent work in [RK11] establishes SINR duality for the MAC and BC with an arbitrary number of AF relaying hops and an arbitrary number of antennas.

MSE duality refers to the performance criterion of *mean square error* (MSE) between the transmitted symbol and the received symbol, which is also preserved when changing from downlink to the uplink or vice versa. These dualities have been shown for linear filtering using the SINR duality in [SS05b], [SSB07], later in [KTA06] and for (non-linear) dirty paper coding in [SS05a] and [MHJU06]. Moreover, MSE duality has also been shown directly in [HJU09] and also for the case where the transmitter has partial or even only statistical knowledge of the channel [JVU10]. Moreover, a so-called *filter-based duality* with respect to data rate using MMSE receivers is proposed in [HJ08], and a rate duality under some conditions for MIMO channels under linear precoding is given in [NCH11].

4.3 Weak Duality

In this section, we introduce a generalization of BC-to-MAC duality. The definition is based on the standard notion of duality: let a BC $\mathcal{B}$ with capacity region $\mathcal{C}_{\mathcal{B}}$ be given. Generally, for BC-to-MAC duality, in the light of (4.6), one would be interested in expressing the BC capacity in terms of MAC capacity regions, i.e., in relationships of the

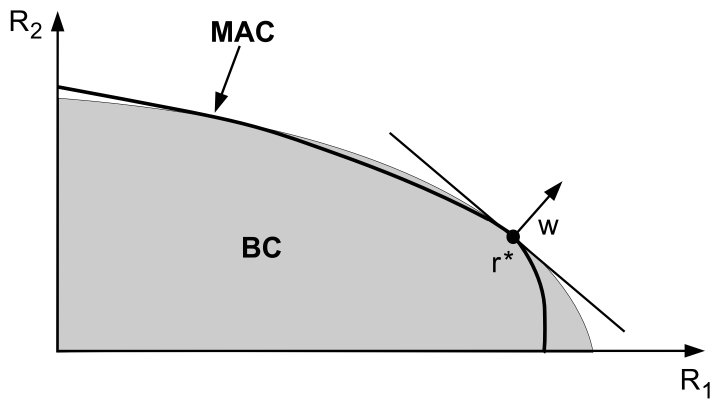

Figure 4.3: Illustration of the definition of weak duality: for the weight vector $\mathbf{w}$, there exists a MAC region (displayed in red) such that there is a common solution for the weighted sum rate problem for both the BC and the MAC (here r^*); weak duality requires this property to hold for all weight vectors.

form

$$\mathcal{C}_{\mathcal{B}} = \bigcup_{\mathcal{D} \in \mathcal{M}} \mathcal{D} \tag{4.8}$$

for a set of MAC channels with corresponding set of capacity regions $\mathcal{M}$. We now define a slightly more general duality condition: we say that there is a *weak BC-to-MAC duality* between the capacity region $\mathcal{C}_{\mathcal{B}}$ and the set $\mathcal{M}$ of MAC capacity regions if the following condition is satisfied:

$$\forall\, \mathbf{w} \geq \mathbf{0} : \; \mathbf{x} = \operatorname*{argmax}_{\mathbf{r} \in \mathcal{C}_{\mathcal{B}}} \mathbf{w}^T \mathbf{r} \Rightarrow \exists\, \mathcal{D} \in \mathcal{M} : \; \mathbf{x} = \operatorname*{argmax}_{\mathbf{r} \in \mathcal{D}} \mathbf{w}^T \mathbf{r}. \tag{4.9}$$

In this case, we also write

$$\mathcal{C}_{\mathcal{B}} \cong \mathcal{M} \tag{4.10}$$

to indicate the weak duality relation. Note that occasionally, we will refer also to the weak duality of a broadcast channel and set of MAC channels, meaning weak duality with respect to the corresponding capacity region.

For a schematic illustration of the definition, see Figure 4.3. Clearly, the definition of weak duality increases the degrees of freedom for the shape of the dual MAC channels as opposed to duality in the sense of (4.8). However, weak duality relations are still worth investigating for the following reasons:

- Any necessary condition for weak duality is also necessary for (4.8).
- If $\mathcal{C}_{\mathcal{B}} \cong \mathcal{M}$ and additionally $\mathcal{D} \subseteq \mathcal{C}_{\mathcal{B}}$ holds for all $\mathcal{D} \in \mathcal{M}$, then we have (4.8). In

other words, establishing weak duality can be used as a step to showing duality in the form of (4.8).

In the following section, we derive weak duality conditions for the BC region with respect to a subclass of discrete memoryless MACs, namely the 2-MAC.

4.4 Duality Relations for Discrete Memoryless Channels

The discrete memoryless MAC has been the subject of the previous chapter; the discrete memoryless broadcast channel is defined in a similar manner: in this channel model, there is one transmitter communicating an individual message to each of K receivers. Before turning to duality results, for the sake of completeness, we briefly recall the definition of the capacity region for this channel. The channel has an input alphabet $\mathcal{X} = \{1, \dots, n\}$ and output alphabets $\mathcal{Y}_k = \{1, \dots, m_k\}$ for $k = 1, \dots, K$ and is specified by a channel matrix

$$\mathbf{Q} = (Q(y_1, \dots, y_K|x))_{1 \le y_k \le m_k, 1 \le x \le n} \tag{4.11}$$

containing the channel transition probabilities and satisfying $\sum_{y_1,\dots,y_K} Q(y_1, \dots, y_K|x) = 1$ for all $x \in \mathcal{X}$. Moreover, the channel is memoryless, meaning that for an input sequence $\mathbf{x} \in \mathcal{X}^n$ and output sequences $\mathbf{y}_k \in \mathcal{Y}_k^n$

$$\mathbf{Pr}[Y_1^n = \mathbf{y}_1, \dots, Y_K^n = \mathbf{y}_K|X^n = \mathbf{x}] = \prod_{i=1}^{n} Q(y_1(i), \dots, y_K(i)|x(i)), \tag{4.12}$$

where the input random variable sequence is denoted as X^n and the output random variable sequences as Y_k^n. We now define some important information-theoretic notions for the channel (see Figure 4.4 for an illustration). A $[(|\mathcal{W}_1|, \dots, |\mathcal{W}_K|), n]$-*code* is given by the message sets $\mathcal{W}_k = \{1, \dots, |\mathcal{W}_k|\}$ and an *encoding function*

$$f : \mathcal{W}_1 \times \cdots \times \mathcal{W}_K \to \mathcal{X}^n. \tag{4.13}$$

Each receiver k uses the *decoding function* g_k

$$g_k : \mathcal{Y}_k^n \to \mathcal{W}_k \tag{4.14}$$

to get an estimate $g_k(Y_k^n) = \widehat{W}_k$ of the transmitted message intended for her. The transmission rates are

$$R_k = \frac{\ln |\mathcal{W}_k|}{n}, \tag{4.15}$$

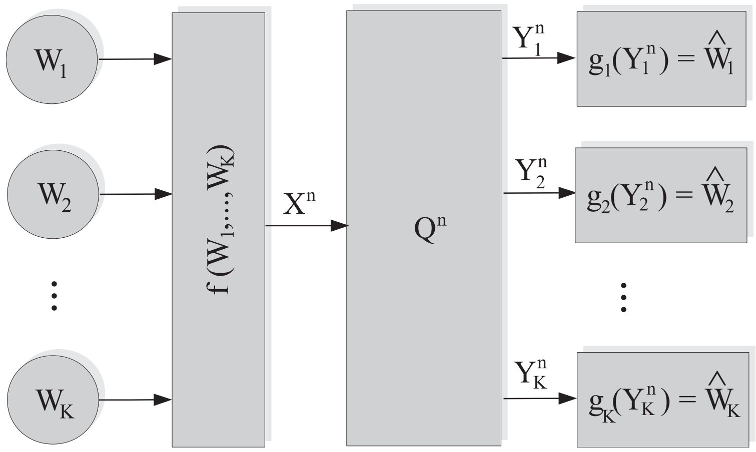

Figure 4.4: The discrete memoryless BC with encoder f and the decoders g_k. Here, Q^n denotes the extension of Q to n time slots according to (4.12).

and the *average probability of error* is given by

$$P_{\text{err}}^{(n)} = \frac{1}{\prod_k |\mathcal{W}_k|} \sum_{(w_1,\ldots,w_K)\in\mathcal{W}_1\times\cdots\times\mathcal{W}_K} \mathbf{Pr}\left[g_k(Y_k^n) \neq w_k \text{ for some } k|(w_1,\ldots,w_K) \text{ sent}\right]. \tag{4.16}$$

Again, the notion of the capacity region of the channel is defined as follows:

- A rate vector $(R_1,\ldots,R_K)$ is said to be *achievable* if there exists a sequence of $[(\lceil e^{nR_1}\rceil,\ldots,\lceil e^{nR_K}\rceil), n]$-codes with $P_{\text{err}}^{(n)} \to 0$ for $n\to\infty$.
- The *capacity region* is defined as the closure of the set of achievable rate vectors.

Unlike for the discrete memoryless MAC, the capacity region of the discrete memoryless BC is, in general, unknown. However, it is known for special cases such as deterministic channels [Pin78], [Mar79], degraded channels [Ber73], [Gal74], less noisy channels [KM77] or more capable channels [Gam79].

4.4.1 Duality for Deterministic Discrete Memoryless Channels

Concerning duality relations for the discrete memoryless BC and MAC, there are very little results in the literature at the time of writing of this thesis. In fact, these results primarily concern *deterministic* channels [JVG03]. For these channels, the channel transition matrices $\mathbf{Q}$ contain only zeros or ones, specifying a deterministic mapping of the inputs to the outputs. In what follows, we briefly sketch the duality relations for such channels. The deterministic BC with input alphabet $\mathcal{X}$ and output alphabets $\mathcal{Y}_1,\mathcal{Y}_2$ is specified by a function

$$f : x\in\mathcal{X} \mapsto (y_1,y_2)\in\mathcal{Y}_1\times\mathcal{Y}_2. \tag{4.17}$$

Similarly, the deterministic MAC with input alphabets $\mathcal{X}_1, \mathcal{X}_2$ and output alphabet $\mathcal{Y}$ is given by a function

$$g : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \mapsto y \in \mathcal{Y}. \tag{4.18}$$

We denote the corresponding capacity regions by $\mathcal{C}_{\mathrm{BC}}(f)$ and $\mathcal{C}_{\mathrm{MAC}}(g)$, respectively. Furthermore, we assume the alphabet sizes of the channels to be given by

$$|\mathcal{X}| = |\mathcal{Y}| = M \tag{4.19}$$

and

$$|\mathcal{Y}_1| = |\mathcal{Y}_2| = |\mathcal{X}_1| = |\mathcal{X}_2| = N. \tag{4.20}$$

In this case, a duality relationship can be proved in the case that $M = aN$ for an integer a with $1 \leq a \leq N$: The capacity regions satisfy

$$\mathbf{Co}\left(\bigcup_f \mathcal{C}_{\mathrm{BC}}(f)\right) = \mathbf{Co}\left(\bigcup_g \mathcal{C}_{\mathrm{MAC}}(g)\right). \tag{4.21}$$

4.4.2 Conditions for Weak 2-MAC Duality

In this section, we derive conditions under which a capacity region $\mathcal{C}$ is weakly dual to a set of 2-MAC channels. Recall that this channel is defined as the $(2,2;2)$-MAC with the additional restriction $Q(1|1,1) = Q(1|1,2) = 0$ on the channel transition probabilities (cf. Section 3.3.3). This channel is fully specified by two parameters $c = Q(1|2,1)$ and $d = Q(1|2,2)$, where w.l.o.g. we may assume $d \geq c$. In the following, we denote the capacity region of this channel by $\mathcal{C}_{2-\mathrm{MAC}}(c,d)$. We let

$$V(\mathcal{C},\theta) := \max_{\mathbf{r}\in\mathcal{C}} \ (\theta, 1-\theta)^T \mathbf{r}, \tag{4.22}$$

$$\mathbf{b}(\mathcal{C},\theta) := \operatorname*{argmax}_{\mathbf{r}\in\mathcal{C}} \ (\theta, 1-\theta)^T \mathbf{r} \tag{4.23}$$

for weight vectors $\mathbf{w} = (\theta, 1-\theta)^T$; if there are several maximizers in (4.23), we choose any one of them. Furthermore, we write

$$\theta_{\min}(\mathcal{C}) = \max\left\{\theta \in [0,1] : \exists\ \mathbf{b}(\mathcal{C},\theta) \text{ that satisfies (4.23) with } b_1(\mathcal{C},\theta) = 0\right\}, \tag{4.24}$$
$$\theta_{\max}(\mathcal{C}) = \min\left\{\theta \in [0,1] : \exists\ \mathbf{b}(\mathcal{C},\theta) \text{ that satisfies (4.23) with } b_2(\mathcal{C},\theta) = 0\right\}.$$

for the weights corresponding to the slopes of the region on the two rates axis, i.e., the minimum and maximum values of θ required to describe the region, respectively.

Then, we have a sufficient condition and a necessary condition for weak duality as given in the following theorem:

Theorem 4. *Let $\mathcal{C}$ be a (BC) capacity region for which $0 < \theta_{\min}(\mathcal{C}) \leq \theta_{\max}(\mathcal{C}) \leq \frac{1}{2}$. If $\mathcal{C}$ satisfies*

$$\frac{V(\mathcal{C},\theta)}{\theta} \leq \ln 2 \tag{4.25}$$

for all $\theta \in [\theta_{\min}(\mathcal{C}), \theta_{\max}(\mathcal{C})]$, then there exists a parameter set $\mathcal{P} \subseteq [0,1]^2$ for the 2-MAC such that

$$\mathcal{C} \cong \{\mathcal{C}_{\text{2-MAC}}(c,d) : (c,d) \in \mathcal{P}\}\,. \tag{4.26}$$

Conversely, if there exists a parameter set $\mathcal{P} \subseteq [0,1]^2$ for the 2-MAC such that (4.26) holds with $c \neq d$ for all $(c,d) \in \mathcal{P}$, then the (BC) capacity region $\mathcal{C}$ satisfies

$$\exp\left(-\frac{V(\mathcal{C},\theta)}{\theta}\right) + \exp\left(-\frac{V(\mathcal{C},\theta)}{1-\theta}\right) \geq 1 \tag{4.27}$$

for all $\theta \in (\theta_{\min}(\mathcal{C}), \theta_{\max}(\mathcal{C}))$.

Proof. We use the notation and some of the results from Section 3.5 in the following. For the first part of the theorem, assume that (4.25) holds for all $\theta \in [\theta_{\min}(\mathcal{C}), \theta_{\max}(\mathcal{C})]$. Using Lemma 5, it suffices to show the existence of $c, d \in [0,1]$ and $\mathbf{P} \in \mathbb{R}_+^{2\times 2}$ such that

(a) $\mathbf{1}^T\mathbf{P}\mathbf{1} = 1$ and condition (3.90) holds
(b) $-f(\mathbf{P}) = \Gamma_{ij}(\mathbf{P}) + 1 - \theta = V(\mathcal{C},\theta)$ for some $i,j \in \{1,2\}$
(c) $f_2(\mathbf{P},\mathbf{P}\mathbf{1}) = b_2(\mathcal{C},\theta)$.

In what follows, for a probability distribution matrix $\mathbf{P}$ and channel parameters c, d, we let $\gamma := cP_{21} + dP_{22}, \epsilon := P_{21} + P_{22}$. Now

$$\Gamma_{11}(\mathbf{P}) = \Gamma_{12}(\mathbf{P}) = \theta - 1 + \theta \ln\left(\frac{1}{1-\gamma}\right) \tag{4.28}$$

and $\Gamma_{21}(\mathbf{P})$ and $\Gamma_{22}(\mathbf{P})$ can be written as

$$\Gamma_{21}(\mathbf{P}) = T_\theta(c,\gamma,\epsilon) + \theta - 1, \tag{4.29}$$

$$\Gamma_{22}(\mathbf{P}) = T_\theta(d,\gamma,\epsilon) + \theta - 1, \tag{4.30}$$

where for $x \in [0,1]$ and $\epsilon \in [\gamma, 1]$

$$T_\theta(x,\gamma,\epsilon) := \theta D(x||\gamma) + (1-2\theta) D\left(x \,\middle|\middle|\, \frac{\gamma}{\epsilon}\right). \tag{4.31}$$

Moreover,

$$f_2(\mathbf{P},\mathbf{P}\mathbf{1}) = P_{21} D\left(c \,\middle|\middle|\, \frac{\gamma}{\epsilon}\right) + P_{22} D\left(d \,\middle|\middle|\, \frac{\gamma}{\epsilon}\right). \tag{4.32}$$

Using (4.28), in order to satisfy (b), we let $\gamma = 1 - e^{-V(\mathcal{C},\theta)/\theta}$ in the following. For $x \in [0,1]$, define $\hat{T}_\theta(x,\epsilon) \coloneqq T_\theta\left(x,\gamma,\epsilon\right)$. Then, it is easy to see that (a)-(c) are equivalent to the existence of a solution $\epsilon \in [\gamma,1], c, d, P_{21}, P_{22} \in [0,1]$ to the following system of equations:

$$\gamma = cP_{21} + dP_{22} \tag{4.33}$$

$$\epsilon = P_{21} + P_{22} \tag{4.34}$$

$$\hat{T}_\theta(c,\epsilon) = \hat{T}_\theta(d,\epsilon) \tag{4.35}$$

$$\hat{T}_\theta(c,\epsilon) = V(\mathcal{C},\theta) \tag{4.36}$$

$$f_2(\mathbf{P},\mathbf{P1}) = b_2(\mathcal{C},\theta). \tag{4.37}$$

As a positive linear combination of Kullback-Leibler divergences, $\hat{T}_\theta(x,\epsilon)$ is convex in x. Define $\hat{T}_{\mathrm{l},\theta}(\epsilon) \coloneqq \hat{T}_\theta(0,\epsilon)$ and $\hat{T}_{\mathrm{r},\theta}(\epsilon) \coloneqq \hat{T}_\theta(1,\epsilon)$. Let $\tilde{x}$ be such that $\frac{\partial}{\partial x}\hat{T}_\theta(\tilde{x},\epsilon) = 0$, which can be found explicitly and shown to be contained in $[0,1]$. Then $\hat{T}_{\min,\theta}(\epsilon) = \hat{T}_\theta(\tilde{x},\epsilon)$ is the minimum value of $\hat{T}_\theta$ over $x \in [0,1]$. Hence, if $\hat{T}_{\mathrm{l},\theta}(\epsilon) \geq V(\mathcal{C},\theta), \hat{T}_{\mathrm{r},\theta}(\epsilon) \geq V(\mathcal{C},\theta)$ and $\hat{T}_{\min,\theta}(\epsilon) \leq V(\mathcal{C},\theta)$ for some $\epsilon \in [\gamma,1]$, subsequently referred to as *condition A*, convexity of $\hat{T}_\theta(x,\epsilon)$ implies that there are $c, d \in [0,1]$ such that (4.35) and (4.36) are satisfied. First assume $d > c$. Then, from (4.33) and (4.34), we have

$$P_{21} = \frac{\gamma - d\epsilon}{c-d} \text{ and } P_{22} = \frac{\gamma - c\epsilon}{d-c}, \tag{4.38}$$

so that $P_{21} \geq 0$ and $P_{22} \geq 0$ if $c \leq \gamma/\epsilon \leq d$. For $c = d$, we get $c = d = \gamma/\epsilon$. Define $\hat{T}_{\mathrm{b},\theta}(\epsilon) \coloneqq \hat{T}_\theta(\gamma/\epsilon,\epsilon)$. Then, if for some $\epsilon \in [\gamma,1]$, $\hat{T}_{\mathrm{b},\theta}(\epsilon) \leq V(\mathcal{C},\theta)$ (denoted by *condition B* in the following) and also condition A holds, it follows again by convexity of $\hat{T}_\theta$ that the system (4.33)-(4.36) for this choice of ϵ has a (unique) solution in $c, d, P_{21}, P_{22} \in [0,1], d \geq c$. We define $\mathcal{F}_\theta$ as the set of $\epsilon \in [\gamma,1]$ for which both condition A and condition B are satisfied. Furthermore, for $\epsilon \in \mathcal{F}_\theta$, we define $f_2(\epsilon) = f_2(\mathbf{P},\mathbf{P1})$ where $c, d, P_{21}, P_{22}, d \geq c$ form the unique solution to the system (4.33)-(4.36).

Now, after some algebra, the following properties of $\hat{T}_{\mathrm{l},\theta}, \hat{T}_{\min,\theta}, \hat{T}_{\mathrm{l},\theta}$ and $\hat{T}_{\mathrm{b},\theta}$ can be verified: $\hat{T}_{\mathrm{l},\theta}(\epsilon) \geq V(\mathcal{C},\theta)$, $\hat{T}_{\mathrm{r},\theta}(\epsilon) \nearrow$, $\hat{T}_{\min,\theta}(\epsilon) \searrow_0$ and $\hat{T}_{\mathrm{b},\theta}(\epsilon) \searrow_0$. From this, we deduce that either $\mathcal{F}_\theta = \emptyset$ or $\mathcal{F}_\theta = [\epsilon^*,1]$ is an interval. Now one can show that $\hat{T}_{\min,\theta}(\gamma) = \hat{T}_{\mathrm{b},\theta}(\gamma) = -\theta\ln(\gamma)$. From (4.25), we have that $\hat{T}_{\min,\theta}(\gamma) = \hat{T}_{\mathrm{b},\theta}(\gamma) = -\theta\ln(1-e^{-V(\mathcal{C},\theta)/\theta}) \geq V(\mathcal{C},\theta)$, so that there exists $\epsilon_{\min} \in [\gamma,1]$ such that $\hat{T}_{\min,\theta}(\epsilon_{\min}) = V(\mathcal{C},\theta)$, and for which clearly $\hat{T}_{\mathrm{r},\theta}(\epsilon_{\min}) \geq V(\mathcal{C},\theta)$. In addition, there is $\epsilon_{\mathrm{b}} \in [\gamma,1]$ such that $\hat{T}_{\mathrm{b},\theta}(\epsilon_{\mathrm{b}}) = V(\mathcal{C},\theta)$. We conclude that $\mathcal{F}_\theta = [\epsilon^*,1]$ where $\epsilon^* = \max\{\epsilon_{\min},\epsilon_{\mathrm{b}}\}$. By continuity, the image $f_2(\mathcal{F}_\theta)$ of $\mathcal{F}_\theta$ under f_2 forms an interval as well, and $\langle f_2(\epsilon^*), f_2(1)\rangle \subseteq f_2(\mathcal{F}_\theta)$, where $\langle a,b\rangle = [a,b]$ for $a \leq b$ and $\langle a,b\rangle = [b,a]$ for $b \leq a$. If $\epsilon^* = \epsilon_{\mathrm{b}}$, then for the solution values for $\epsilon = \epsilon_{\mathrm{b}}$, we have $d = \gamma/\epsilon_{\mathrm{b}}$, so that $P_{21} = 0$, implying

$$f_2(\epsilon_{\mathrm{b}}) = P_{21} D\left(c \,\middle\|\, \frac{\gamma}{\epsilon_{\mathrm{b}}}\right) + P_{22} D\left(\frac{\gamma}{\epsilon_{\mathrm{b}}} \,\middle\|\, \frac{\gamma}{\epsilon_{\mathrm{b}}}\right) = 0. \tag{4.39}$$

For the case $\epsilon^* = \epsilon_{\min}$, we similarly have $c = d = \gamma/\epsilon_{\min}$, resulting in $f_2(\epsilon_{\min}) = 0$. Finally, for $\epsilon = 1$, we have for the solution values that

$$D(c\|\gamma) = D(d\|\gamma) = \frac{V(\mathcal{C},\theta)}{1-\theta}, \tag{4.40}$$

implying

$$\begin{aligned} f_2(1) &= (1-P_{22})D(c\|\gamma) + P_{22}D(d\|\gamma) = \frac{V(\mathcal{C},\theta)}{1-\theta} \\ &= b_2(\mathcal{C},\theta) + \frac{\theta}{1-\theta} b_1(\mathcal{C},\theta) \geq b_2(\mathcal{C},\theta). \end{aligned} \tag{4.41}$$

Consequently, $\left[0, b_2(\mathcal{C},\theta) + \frac{\theta}{1-\theta} b_1(\mathcal{C},\theta)\right] \subseteq f_2(\mathcal{F}_\theta)$, which implies the existence of a solution for the system (4.33)-(4.37).

For the converse part of the theorem, assume that (4.26) holds and that for some $\theta \in (\theta_{\min}(\mathcal{C})), \theta_{\max}(\mathcal{C}))$, condition (4.27) is *not* satisfied. Then, for this θ, there is a solution $(R_1, R_2, \mathbf{P}, \mathbf{p}_1, \mathbf{p}_2)$ to the ECCP with $R_1 > 0, R_2 > 0, \mathbf{P} > \mathbf{0}$. In this case, since $c \neq d$ and from the properties of mutual information, $f_1(\mathbf{P}, \mathbf{p}_2) + f_2(\mathbf{P}, \mathbf{p}_1) > f_{12}(\mathbf{P})$, it is easy to see from (3.80)-(3.84), complementary slackness and weak duality that (a)-(c) must hold. This in turn implies the existence of solution to the system (4.33)-(4.37). Now $\hat{T}_{\mathrm{r},\theta}(1) = (\theta-1)\ln(\gamma)$, so that $\exp\left(-\frac{V(\mathcal{C},\theta)}{\theta}\right) + \exp\left(-\frac{V(\mathcal{C},\theta)}{1-\theta}\right) < 1$ implies $\hat{T}_{\mathrm{r},\theta}(1) < V(\mathcal{C},\theta)$, i.e., $\mathcal{F}_\theta = \emptyset$. Hence, (4.33)-(4.37) has no solution, which is a contradiction and concludes the proof of the theorem. □

4.4.3 Duality 2-MAC / Binary Symmetric BC

In this section, we exemplarily apply Theorem 4 in order to derive (weak) duality conditions for the 2-MAC with respect to the *Binary Symmetric Broadcast Channel (BSBC)* [CT06]. This binary memoryless broadcast channel is specified by the channel matrix $\mathbf{Q}$ with

$$Q(x_1, x_2|y) = (s\mathbb{1}[x_1 \neq y] + (1-s)\mathbb{1}[x_1 = y])(t\mathbb{1}[x_2 \neq y] + (1-t)\mathbb{1}[x_2 = y]), \tag{4.42}$$

where $x_1, x_2, y \in \{1, 2\}$, $0 < s < t < \frac{1}{2}$ (cf. Figure 4.5). Note that since both the 2-MAC and the BSBC offer two degrees of freedom in the specification of the transition

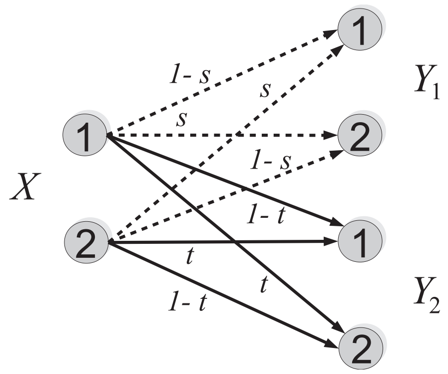

Figure 4.5: Binary Symmetric Broadcast Channel.

probabilities, the pair 2-MAC/BSBC is a natural candidate for duality investigations.

For the BSBC, the boundary $\partial\mathcal{C}_{\text{BSBC}}(s,t)$ of the capacity region $\mathcal{C}_{\text{BSBC}}(s,t)$ can easily be characterized in terms of a parametric representation [CT06]: $\partial\mathcal{C}_{\text{BSBC}}(s,t) = \{\mathbf{r}(\beta) : \beta \in [0,1/2]\}$, where

$$\mathbf{r}(\beta) = \begin{pmatrix} H(\beta * s) - H(s) \\ \ln 2 - H(\beta * t) \end{pmatrix}, \tag{4.43}$$

with the "convolution" $x * y = x(1-y) + (1-x)y$. The sum-rate at the capacity boundary point corresponding to $\beta \in [0,1/2]$ is then given by

$$V(\mathcal{C}_{\text{BSBC}}, \beta) := \theta(\beta) r_1(\beta) + (1-\theta(\beta)) r_2(\beta), \tag{4.44}$$

where $\theta(\beta) = \left(1 - \frac{r_1'(\beta)}{r_2'(\beta)}\right)^{-1}$. We obtain

Proposition 3. *Let s_0 be the solution to $\frac{1}{2}(1-2s)\operatorname{arctanh}(1-2s) = \frac{\ln 2}{2}$ ($s_0 \approx 0.131988$). For any $s_0 \leq s < t < \frac{1}{2}$, there exists a parameter set $\mathcal{P} \subseteq [0,1]^2$ for the 2-MAC such that*

$$\mathcal{C}_{\text{BSBC}}(s,t) \cong \{\mathcal{C}_{\text{2-MAC}}(c,d) : (c,d) \in \mathcal{P}\}. \tag{4.45}$$

Proof. First of all, $0 < \theta_{\min}(\mathcal{C}_{\text{BSBC}}) \leq \theta_{\max}(\mathcal{C}_{\text{BSBC}}) \leq \frac{1}{2}$. By Theorem 4, it suffices to show that for $s_0 \leq s < t < \frac{1}{2}$,

$$\frac{V(\mathcal{C}_{\text{BSBC}}, \beta)}{\theta(\beta)} = r_1(\beta) + \frac{1-\theta(\beta)}{\theta(\beta)} r_2(\beta) \leq \ln 2 \tag{4.46}$$

for all $\beta \in [0,1/2]$. For this, we prove that, for channel parameters as above, both of the summands are less than or equal to $\ln/2$. Clearly, $r_1(\beta)$ is maximal for $\beta = 1/2$, and

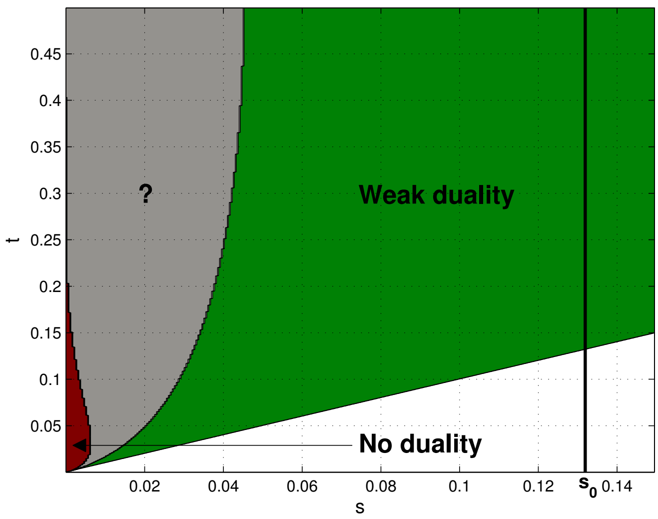

Figure 4.6: Numerical evaluation of the weak duality conditions given in Theorem 4 for the BSBC.

it is quickly verified that $r_1(\frac{1}{2}) = \ln 2 - H(s) \leq \ln 2/2$ for $s > s_0$. Define $h(\beta, s, t) := \frac{1-\theta(\beta)}{\theta(\beta)} r_2(\beta)$. After some algebra, one verifies that for each fixed $s, \beta \in [0, 1/2]$, $h(\beta, s, t) \nearrow$ in t, and that the maximum value of h is given by

$$h(\beta, s, 1/2) = \frac{1}{2}(2\beta - 1)(2s - 1)\text{arctanh}[(2\beta - 1)(2s - 1)]. \tag{4.47}$$

Now $h(\beta, s, 1/2) \searrow$ in β and $h(0, s, 1/2) = \frac{1}{2}(1-2s)\text{arctanh}(1-2s) \searrow$ in s, which concludes the proof. $\square$

Figure 4.6 illustrates the results of evaluating the conditions in Theorem 4 numerically for the BSBC over a range of channel parameters s, t. Three areas are to be distinguished:

1. The parameter range in which the sufficient condition (4.25) is satisfied represents channels for which weak duality to the 2-MAC holds. It is displayed in green in Figure 4.6. Note that for parameters s, t with $s > s_0$, weak duality always holds according to Proposition 3.
2. There is a small range (shown in red) of choices of s, t where weak duality does not hold (i.e., the necessary condition (4.27) is not satisfied).
3. Finally, for some parameters s, t, condition (4.27) is satisfied, but (4.25) does not hold, so that no statement about weak duality can be made using the conditions in Theorem 4. The corresponding region is indicated by the "?".

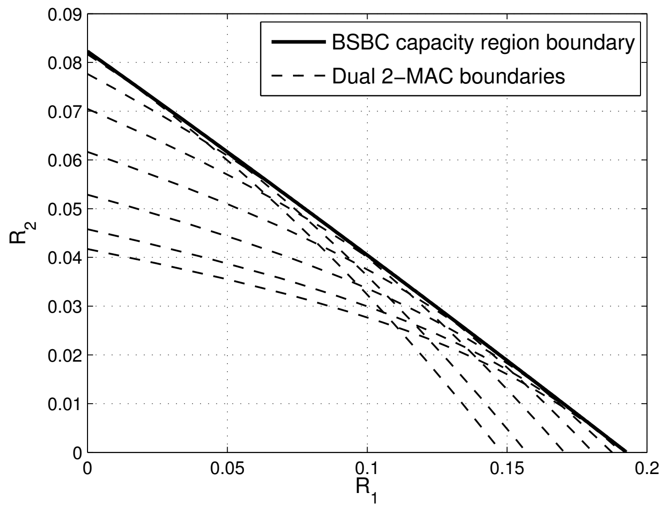

Figure 4.7: Capacity region of the BSBC with parameters $s = 0.2, t = 0.3$ and some of the corresponding dual 2-MAC capacity regions.

Figure 4.7 shows the capacity region of the BSBC for $s = 0.2 > s_0, t = 0.3$ and some of the dual 2-MAC channels found by solving the system (4.33)-(4.37) numerically. We remark that for this example, all the MAC capacity regions are strictly contained in the BC region, so that actually BC-to-MAC duality in the sense of (4.8) holds. Looking at experimental results, this property seems to hold generally: it appears that for the BSBC in the case of weak duality to the 2-MAC, the corresponding dual channels are always contained in the capacity region of the BSBC. However, this remains to be proven and is left open as a conjecture.

4.5 Duality for Fading Channels Without Transmitter Channel State Information

In this section, we investigate duality relations for Gaussian (fast) fading channels. Here, two different scenarios are of specific interest: in the case where *channel state information (CSI)* is available to the transmitter, it is possible to adapt to the current fading state, and the duality for the non-fading case extends to channels with fading [JVG04]. Opposed to this, in the case that the transmitter has no access to CSI, the situation is more involved. The situation of no CSI at the transmitter has already been investigated in [Jor06] with respect to the existence of MAC-BC duality. This work is restricted to linear transmitters and receivers and also precludes time-sharing. Under these assumptions, it is shown that duality does not hold. In the work presented in this section, we study a more general case and allow the usage of dirty paper coding, superposition coding and successive interference cancellation, i.e., non-linear transmitters and receivers, respectively.

4.5.1 Fading BC System Model

The channel we study here is the scalar two-user BC with additive white Gaussian noise and discrete fading where CSI is available only to the receiver, but not to the transmitter. For this channel, the received signals Y_1 at receiver 1 and Y_2 at receiver 2 are given by

$$\begin{aligned} Y_1 &= GX + Z_1, \\ Y_2 &= HX + Z_2. \end{aligned} \tag{4.48}$$

Here, $Z_i \sim \mathcal{N}(0,1), i = 1,2$ is additive white Gaussian noise, $X \in \mathbb{R}$ is the transmit signal, subject to an average power constraint $\mathbf{E}[X^2] \leq P$, where $P > 0$. G and H are the fading coefficients, which for each symbol duration are randomly chosen out of a set $\mathcal{G} = \{g_1, \ldots, g_{N_1}\}$ and $\mathcal{H} = \{h_1, \ldots, h_{N_2}\}$, respectively. We remark here that all the results in this section easily carry over to the case of continuous-type fading distributions. The probability mass functions on $\mathcal{G}$ and $\mathcal{H}$ are denoted as p_1 and p_2, respectively, that is, $\mathbf{Pr}[G = g_k] = p_1(k)$ and $\mathbf{Pr}[H = h_l] = p_2(l)$ for $1 \leq k \leq N_1, 1 \leq l \leq N_2$. Without loss of generality, we assume that $p_1(k) > 0$ and $p_2(l) > 0$ for $1 \leq k \leq N_1, 1 \leq l \leq N_2$. We further assume G and H to be independent and the instantaneous fading states (i.e., realizations of G and H) unknown to the transmitter, but perfectly known to the receivers. However, the transmitter is aware of the fading distributions specified by p_1, p_2. We refer to this channel as the *CSIR-AWGN-BC*. A graphical illustration of the system model is given in Figure 4.8. For this chapter, we make the following additional ordering assumption for the fading distributions:

$$\max_{1 \leq j \leq N_2} h_j \leq \min_{1 \leq i \leq N_1} g_i, \tag{4.49}$$

in which case we speak of an *ordered CSIR-AWGN-BC*.

Unfortunately, very little is known in terms of capacity of the CSIR-AWGN-BC; see [TS03], [TY09] for more details on this channel and the following capacity and achievability discussion. For the AWGN BC (without fading), the capacity region is known and all rate points in the capacity region are achievable by superposition coding and successive decoding (SCSD) or, alternatively, by dirty paper coding (DPC) (cf. Section 4.1). If the CSIR-AWGN-BC is degraded, SCSD is always possible, and the capacity region is known in terms of mutual information expressions, also achieved by SCSD. However, this representation involves an optimization over an auxiliary and the input distribution,

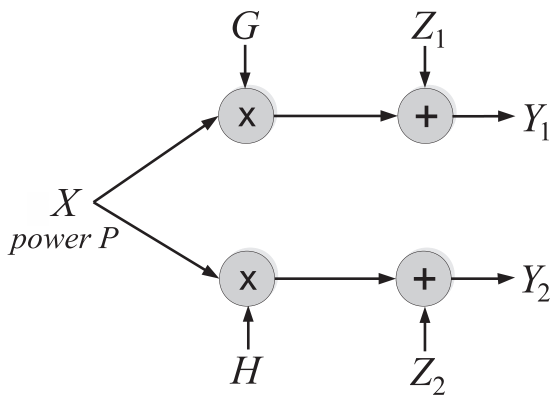

Figure 4.8: System model for the fading broadcast channel.

which are not known how to be chosen optimally. Yet, an achievable rate region is given by choosing these distributions jointly Gaussian. We remark that for non-degraded channels, performing SCSD is generally not possible, but there are cases for which the channel is not degraded, but SCSD is still possible. In such situations, an achievable rate region is also given by evaluating the degraded region for Gaussian signaling. For the ordered CSIR-AWGN-BC, assumption (4.49) ensures that the channel is degraded with user 1 being the "stronger" user, and the achievable rate region using SCSD and Gaussian input is then evaluated as [TS03] $\tilde{\mathcal{C}}_{\mathrm{BC}} = \mathbf{Co}\left(\mathcal{R}_{\mathrm{BC}}\right)$, where

$$\mathcal{R}_{\mathrm{BC}} = \bigcup_{\beta\in[0,1]} \left\{(R_1, R_2)^T : R_1 \leq r_1^{\mathrm{B}}(\beta), R_2 \leq r_2^{\mathrm{B}}(\beta)\right\} \tag{4.50}$$

with

$$r_1^{\mathrm{B}}(\beta) = \mathbf{E}\left[\frac{1}{2}\ln\left(1+\beta P G^2\right)\right] \tag{4.51}$$

and

$$r_2^{\mathrm{B}}(\beta) = \mathbf{E}\left[\frac{1}{2}\ln\left(\frac{1+PH^2}{1+\beta PH^2}\right)\right]. \tag{4.52}$$

Here, $\beta \in [0,1]$ determines the *power split* among the superimposed signals intended for each user. We remark that it is conjectured in [TS03] that Gaussian input is optimal for the CSIR-AWGN-BC. In our case, this reduces to the conjecture that $\tilde{\mathcal{C}}_{\mathrm{BC}}$ is the capacity region for the ordered CSIR-AWGN-BC. We also note that in the next section, it is shown that $\mathcal{R}_{\mathrm{BC}}$ is a convex set, so that the convex hull operation is actually not necessary. However, this property is mainly due to the assumption (4.49) and might not hold if (4.49) is not satisfied.

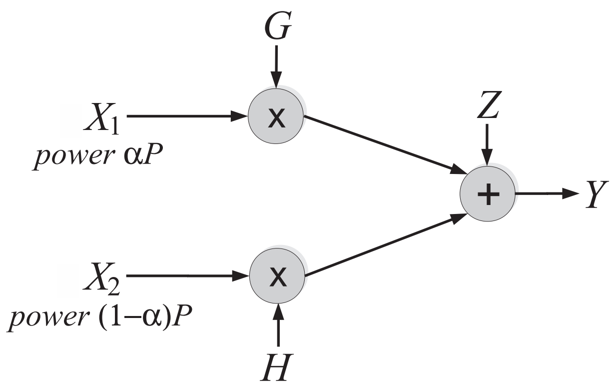

Figure 4.9: System model for the dual fading multiple access channel.

We now consider the problem of weighted sum-rate optimization over the region $\tilde{\mathcal{C}}_{\mathrm{BC}}$. For this, let $\theta \leq \frac{1}{2}$ and define

$$f_\theta^{\mathrm{BC}}(\beta) := \theta r_1^{\mathrm{B}}(\beta) + (1-\theta) r_2^{\mathrm{B}}(\beta). \tag{4.53}$$

The problem of maximizing weighted sum-rate then reads as

$$\max_{\beta \in [0,1]} f_\theta^{\mathrm{BC}}(\beta). \tag{4.54}$$

Now $f_\theta^{\mathrm{BC}}(\beta)$ generally is not concave in β, so that (4.54) is a non-convex problem, just like for the AWGN BC. For the AWGN BC, it is well-known that problem (4.54) can be solved via its dual MAC. The question that arises at this point, then, is whether for the fading case, a similar solution by duality is possible.

4.5.2 The Dual MAC and Lack of Duality

A natural extension of the definition of the dual MACs for channels without fading to the CSIR-AWGN-BC is as follows: one defines the set of dual MACs as the MACs that have identical noise and fading statistics as the BC and individual power constraints P_1, P_2 for the two users such that $P_1 + P_2 = P$, i.e., the sum of the two individual power constraints equals the BC power constraint. More precisely, for each $\alpha \in [0,1]$ we define the *dual α-MAC* as the fading MAC channel with received signal

$$Y = GX_1 + HX_2 + Z, \tag{4.55}$$

where $Z \sim \mathcal{N}(0,1)$ is additive white Gaussian noise, $X_1, X_2 \in \mathbb{R}$ are the transmit signals of user 1 and 2, subject to average power constraints $\mathbf{E}[X_1^2] \leq P_1 = \alpha P$ and $\mathbf{E}[X_2^2] \leq P_2 = (1-\alpha)P$, respectively. The fading random variables G and H follow the same distribution as for the BC, i.e., are independent and specified by p_1 and p_2 and their realizations are assumed to be perfectly known to the receiver, but unknown to the transmitters. The system model is graphically depicted in Figure 4.9.

Using Gaussian inputs and successive interference cancellation (SIC), an achievable rate region is given by $\tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)$, constituting a pentagon which is specified by the two rate points achievable by SIC by decoding user 1 first (decoding order $1 \to 2$) and decoding user 2 first (decoding order $2 \to 1$), respectively. Taking the union over all dual α-MACs, we define

$$\tilde{\mathcal{C}}_{\mathrm{MAC}} \coloneqq \mathbf{Co}\left(\bigcup_{\alpha\in[0,1]} \tilde{\mathcal{C}}_{\mathrm{MAC}}(\alpha)\right). \tag{4.56}$$

It is helpful to express $\tilde{\mathcal{C}}_{\mathrm{MAC}}$ in terms of the two SIC points as follows: tracing the SIC point corresponding to the decoding order $1 \to 2$ for varying α, we obtain the set

$$\tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2} \coloneqq \bigcup_{\alpha\in[0,1]} \left\{(R_1,R_2)^T : R_1 \leq r_1^{\mathrm{M}}(\alpha), R_2 \leq r_2^{\mathrm{M}}(\alpha)\right\}, \tag{4.57}$$

where

$$r_1^{\mathrm{M}}(\alpha) = \mathbf{E}\left[\frac{1}{2}\ln\left(\frac{1+\alpha P G^2 + (1-\alpha)PH^2}{1+(1-\alpha)PH^2}\right)\right] \tag{4.58}$$

and

$$r_2^{\mathrm{M}}(\alpha) = \mathbf{E}\left[\frac{1}{2}\ln\left(1+(1-\alpha)PH^2\right)\right]. \tag{4.59}$$

Similarly, one defines $\tilde{\mathcal{C}}_{\mathrm{MAC}}^{2\to 1}$ by tracing over the SIC rate point corresponding to the decoding order $2 \to 1$. Clearly, we then have $\tilde{\mathcal{C}}_{\mathrm{MAC}} = \mathbf{Co}\left(\tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2} \cup \tilde{\mathcal{C}}_{\mathrm{MAC}}^{2\to 1}\right)$. Now $\tilde{\mathcal{C}}_{\mathrm{BC}}$ and $\tilde{\mathcal{C}}_{\mathrm{MAC}}$ are related as follows:

Theorem 5. *If the fading distribution for H is non-deterministic, i.e., $N_2 > 1$, then $\tilde{\mathcal{C}}_{BC} \neq \tilde{\mathcal{C}}_{MAC}$. If H is deterministic, then $\tilde{\mathcal{C}}_{BC} = \tilde{\mathcal{C}}_{MAC}^{1\to 2}$.*

Proof. We first show that $\mathcal{R}_{\mathrm{BC}}$ given in (4.50) is a convex set, i.e., that $\tilde{\mathcal{C}}_{\mathrm{BC}} = \mathcal{R}_{\mathrm{BC}}$. The boundary of the region $\mathcal{R}_{\mathrm{BC}}$ is given by the image of the parametric curve $\psi_{\mathrm{B}} : [0,1] \to \mathbb{R}^2, \beta \mapsto (r_1^{\mathrm{B}}(\beta), r_2^{\mathrm{B}}(\beta))^T$. Since r_1^{B} is strictly increasing and r_2^{B} is strictly decreasing, ψ_{B} is an injective parametric curve. Moreover, it is differentiable, and the tangent slope of the curve in the point $\psi_{\mathrm{B}}(\beta)$ is given by $t_B(\beta) := \frac{d}{d\beta} r_2^{\mathrm{B}}(\beta) / \frac{d}{d\beta} r_1^{\mathrm{B}}(\beta)$ as

$$t_B(\beta) = -\mathbf{E}\left[\frac{PH^2}{1+\beta PH^2}\right]\left(\mathbf{E}\left[\frac{PG^2}{1+\beta PG^2}\right]\right)^{-1}. \tag{4.60}$$

Now

$$\begin{aligned}\frac{d}{d\beta} t_B(\beta) = \Bigg\{&\mathbf{E}\left[\frac{PG^2}{1+\beta PG^2}\right]\mathbf{E}\left[\frac{P^2H^4}{(1+\beta PH^2)^2}\right] \\ &-\mathbf{E}\left[\frac{PH^2}{1+\beta PH^2}\right]\mathbf{E}\left[\frac{P^2G^4}{(1+\beta PG^2)^2}\right]\Bigg\}\left(\mathbf{E}\left[\frac{PG^2}{1+\beta PG^2}\right]\right)^{-2}.\end{aligned} \tag{4.61}$$

Using condition (4.49), the independence of G and H and the fact that $x/(1+\beta x)$ is increasing in x, it is easy to see that $\frac{d}{d\beta} t_B(\beta) \leq 0$, which implies that ψ_{B} is convex, i.e., that $\mathcal{R}_{\mathrm{BC}}$ is a convex set.

First consider the case that H is non-deterministic, i.e., $N_2 > 1$. We show that $\tilde{\mathcal{C}}_{\mathrm{BC}} \neq \tilde{\mathcal{C}}_{\mathrm{MAC}}$. For this, we consider the boundary of $\tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2}$ as the image of the injective and differentiable parametric curve $\psi_{\mathrm{M}} : [0,1] \to \mathbb{R}^2, \alpha \mapsto (r_1^{\mathrm{M}}(\alpha), r_2^{\mathrm{M}}(\alpha))^T$, which has tangent slope $t_M(\alpha) := \frac{d}{d\alpha} r_2^{\mathrm{M}}(\alpha) / \frac{d}{d\alpha} r_1^{\mathrm{M}}(\alpha)$. Clearly, it suffices to show that there exists a point $\mathbf{x} \in \tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2}$ for which $\mathbf{x} \notin \tilde{\mathcal{C}}_{\mathrm{BC}}$. Since $\psi_{\mathrm{M}}(1) = \psi_{\mathrm{B}}(1)$ and $\tilde{\mathcal{C}}_{\mathrm{BC}}$ is a convex set, the existence of such a point $\mathbf{x}$ follows by comparing the tangent slopes at the end of the curves (on the rate axis for user 1) as $t_M(1) < t_B(1)$. To see this, we have

$$t_M(1) = -\mathbf{E}\left[PH^2\right]\left(\mathbf{E}\left[\frac{PG^2(1+PH^2)}{1+PG^2}\right]\right)^{-1}, \tag{4.62}$$

$$t_B(1) = -\mathbf{E}\left[\frac{PH^2}{1+PH^2}\right]\left(\mathbf{E}\left[\frac{PG^2}{1+PG^2}\right]\right)^{-1}. \tag{4.63}$$

Since $x/(1+x)$ is strictly concave on the interval $(0,\infty]$ and H is non-deterministic, it follows from Jensen's inequality that $\mathbf{E}\left[\frac{PH^2}{1+PH^2}\right] < \frac{\mathbf{E}[PH^2]}{1+\mathbf{E}[PH^2]}$, which yields

$$\begin{aligned}-t_B(1) &= \mathbf{E}\left[\frac{PH^2}{1+PH^2}\right]\left(\mathbf{E}\left[\frac{PG^2}{1+PG^2}\right]\right)^{-1} \\ &< \frac{\mathbf{E}\left[PH^2\right]}{\mathbf{E}\left[1+PH^2\right]}\left(\mathbf{E}\left[\frac{PG^2}{1+PG^2}\right]\right)^{-1} = -t_M(1),\end{aligned} \tag{4.64}$$

where the last equality follows from independence of G^2 and H^2. Note that similarly, one can show that $t_M(0) > t_B(0)$. Finally, we turn to the case that H is deterministic. Here, there is a one-to-one mapping between the images of ψ_{M} and ψ_{B}, induced by the bijective parameter mapping

$$b(\alpha) := \frac{\alpha}{1 + h_1^2(1-\alpha)P}. \tag{4.65}$$

For each $\alpha \in [0,1]$, we have $\psi_{\mathrm{M}}(\alpha) = \psi_{\mathrm{B}}(b(\alpha))$. Similarly, for each $\beta \in [0,1]$, it is $\psi_{\mathrm{M}}(b^{-1}(\beta)) = \psi_{\mathrm{B}}(\beta)$. □

Two remarks concerning Theorem 5 and its proof are in order. First, there is strong numerical evidence that the stronger condition $\tilde{\mathcal{C}}_{\mathrm{BC}} \subsetneq \tilde{\mathcal{C}}_{\mathrm{MAC}}$ holds in the case of non-deterministic H, but has yet to be proven. In fact, the conditions $t_M(1) < t_B(1)$ and $t_M(0) > t_B(0)$ support this conjecture. Second, the MAC capacity region can only be larger than the achievable region $\tilde{\mathcal{C}}_{\mathrm{MAC}}$, and there is, for non-deterministic H, always a point in $\tilde{\mathcal{C}}_{\mathrm{MAC}}$ that is not contained in $\tilde{\mathcal{C}}_{\mathrm{BC}}$. This means that if $\tilde{\mathcal{C}}_{\mathrm{BC}}$ is, as conjectured, actually the capacity region of the ordered CSIR-AWGN-BC, then Theorem 5 implies that there is no duality for non-deterministic H even with respect to the capacity regions.

4.5.3 Approximative Solutions via the Dual MAC

In the previous section, we have demonstrated that if the fading random variable H is deterministic, we have BC-MAC duality in the sense that $\tilde{\mathcal{C}}_{\mathrm{BC}} = \tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2}$. In this case, the non-convex weighted sum-rate problem for the BC can be solved via the dual MAC as follows: define

$$f_\theta^{\mathrm{MAC}}(\alpha) := \theta r_1^{\mathrm{M}}(\alpha) + (1-\theta) r_2^{\mathrm{M}}(\alpha), \tag{4.66}$$

which is easily verified to be a concave function. Then, the weighted-sum rate optimization problem over the region $\tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2}$

$$\alpha^* = \operatorname*{argmax}_{\alpha \in [0,1]} f_\theta^{\mathrm{MAC}}(\alpha) \tag{4.67}$$

is a convex problem, and since $\tilde{\mathcal{C}}_{\mathrm{BC}} = \tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2}$, the solution α^* immediately gives the optimal argument value β^* for the optimization over $\tilde{\mathcal{C}}_{\mathrm{BC}}$ as

$$\beta^* = \frac{\alpha^*}{1 + h_1^2(1-\alpha^*)P} \tag{4.68}$$

[cf. equation (4.65)].

Considering the case that H is non-deterministic, this approach cannot be applied directly since in this situation, there is no perfect duality between the BC region and its corresponding dual MAC region. However, the weighted sum-rate optimization problem over $\tilde{\mathcal{C}}_{\mathrm{MAC}}^{1\to 2}$ is still convex, and it is reasonable to surmise that it might still be possible to obtain suboptimal, but good solutions for the BC problem by solving the problem over the dual MAC region and then heuristically constructing a solution for the BC problem based on the outcome of the MAC optimization. To be precise, we suggest and analyze the following line of approach:

1. Solve $\alpha^* = \underset{\alpha\in[0,1]}{\operatorname{argmax}}\, f_\theta^{\mathrm{MAC}}(\alpha)$.
2. Compute a power split $\tilde{\beta}$ as an approximative solution for problem (4.54) by $\tilde{\beta} = \pi(\alpha^*)$, where $\pi : [0,1] \to [0,1]$.

We refer to π as the *power split mapping.* There are different conceivable choices for the power split mapping π. A specific choice for π is motivated by (4.65): for each $\gamma > 0$,

$$\pi_\gamma(\alpha) := \frac{\alpha}{1+\gamma(1-\alpha)P} \tag{4.69}$$

is a possible power split mapping. This form of π can experimentally found to produce good results in many cases for an appropriate choice of γ. For example, taking γ as the mean channel gain for user two, i.e., $\gamma = \mathbf{E}\left[H^2\right]$, is generally a good choice.

Under the assumption that $\tilde{\mathcal{C}}_{\mathrm{BC}} \subseteq \tilde{\mathcal{C}}_{\mathrm{MAC}}$, the error $\Delta^\pi(\theta)$ incurred by this procedure can be bounded as

$$\Delta^\pi(\theta) = f_\theta^{\mathrm{BC}}(\beta^*) - f_\theta^{\mathrm{BC}}(\pi(\alpha^*)) \le f_\theta^{\mathrm{MAC}}(\alpha^*) - f_\theta^{\mathrm{BC}}(\pi(\alpha^*)) =: \Delta_g^\pi(\theta), \tag{4.70}$$

where β^* is the optimizer of (4.54). $\Delta_g^\pi(\theta)$ is an "a posteriori" bound that can be evaluated only after the MAC optimization problem has been solved; it then gives a guarantee on the maximum deviation from the optimum. One might also be interested in an "a priori" error bound that does not depend on α, but only on the channel parameters. Such a bound can, for instance, be obtained as

$$\Delta^\pi(\theta) \le \max_{\alpha\in[0,1]} \left(f_\theta^{\mathrm{MAC}}(\alpha) - f_\theta^{\mathrm{BC}}(\pi(\alpha))\right) \le \theta\Delta_1^\pi + (1-\theta)\Delta_2^\pi, \tag{4.71}$$

where for $i = 1,2$, we let $\Delta_i^\pi := \max_{\alpha\in[0,1]} \left[r_i^{\mathrm{M}}(\alpha) - r_i^{\mathrm{B}}(\pi(\alpha))\right]$.

Theorem 6. *$\Delta_1^{\pi_\gamma}$ is bounded by*

$$\Delta_1^{\pi_\gamma} \le \frac{1}{2} \sum_{j:h_j^2\le\gamma} p_2(j) \ln\frac{\gamma}{h_j^2}. \tag{4.72}$$

For $\gamma \le \mathbf{E}[H^2]$, it also holds that

$$\Delta_2^{\pi_\gamma} \le -\frac{(\sqrt{1+\gamma P}-1)^2}{2\gamma^2}\mathbf{E}\left[\frac{H^2(\gamma-H^2)}{1+PH^2}\right]. \tag{4.73}$$

Proof. We have

$$r_1^{\mathrm{M}}(\alpha) - r_1^{\mathrm{B}}(\pi_\gamma(\alpha)) = \mathbf{E}_{G,H}\left[s_1(\alpha, G^2, H^2, \gamma, P)\right] \tag{4.74}$$

with

$$s_1(\alpha, x, y, \gamma, P) = \frac{1}{2}\ln\frac{(1+(1-\alpha)P\gamma)(1+(1-\alpha)Py+\alpha Px)}{(1+(1-\alpha)Py)(1+(1-\alpha)P\gamma+\alpha Px)}. \tag{4.75}$$

Now for $y \leq \gamma$, $s_1(\alpha, x, y, \gamma, P)$ is increasing in P and

$$\lim_{P\to\infty} s_1(\alpha, x, y, \gamma, P) = \frac{1}{2}\ln\left[\frac{\gamma(\alpha(x-y)+y)}{(\gamma-\alpha\gamma+\alpha x)y}\right], \tag{4.76}$$

which, in turn, is increasing in α. This shows that $s_1(\alpha, x, y, \gamma, P) \leq \frac{1}{2}\ln\frac{\gamma}{y}$ for $y \leq \gamma$. Now it is easy to see that $s_1(\alpha, x, y, \gamma, P) \geq 0$ iff $y \leq \gamma$, from which (4.72) follows.

For the second bound, assume $\gamma \leq \mathbf{E}[H^2]$. Without loss of generality, we also assume that H is non-deterministic. Here, we have

$$r_2^{\mathrm{M}}(\alpha) - r_2^{\mathrm{B}}(\pi_\gamma(\alpha)) = \mathbf{E}_H\left[s_2(\alpha, H^2, \gamma, P)\right], \tag{4.77}$$

where

$$s_2(\alpha, y, \gamma, P) = \frac{1}{2}\ln\frac{(1+(1-\alpha)Py)(1+P((1-\alpha)\gamma+\alpha y))}{(1+(1-\alpha)\gamma P)(1+Py)}. \tag{4.78}$$

Now, using $\ln x \leq x-1$, one can bound $\mathbf{E}_H\left[s_2(\alpha, H^2, \gamma, P)\right] \leq \phi(\alpha, \gamma, P)$, where

$$\phi(\alpha, \gamma, P) = t(\alpha, \gamma, P)\mathbf{E}\left[\frac{H^2(\gamma - H^2)}{1+PH^2}\right] \tag{4.79}$$

with

$$t(\alpha, \gamma, P) = \frac{-\alpha(\alpha-1)P^2}{2((\alpha-1)\gamma P)-1)}. \tag{4.80}$$

Applying Jensen's inequality, we get

$$\mathbf{E}\left[\frac{H^2(\gamma - H^2)}{1+PH^2}\right] \leq \frac{\mathbf{E}\left[H^2\right](\gamma - \mathbf{E}\left[H^2\right])}{1+P\mathbf{E}\left[H^2\right]} \leq 0, \tag{4.81}$$

where the last inequality follows from $\gamma \leq \mathbf{E}[H^2]$. Now it holds that $\frac{\partial^2}{\partial\alpha^2}t(\alpha, \gamma, P) > 0$, which together with (4.81) implies that $\phi(\alpha, \gamma, P)$ is concave in α. After some algebra, one finds that $\frac{\partial}{\partial\alpha}\phi(\alpha, \gamma, P) = 0$ has exactly one solution $\hat{\alpha}$ in $[0, 1]$, which is given by $\hat{\alpha} = \frac{1+\gamma P-\sqrt{1+\gamma P}}{\gamma P}$ and for which, by concavity, $\phi(\alpha, \gamma, P)$ takes on its maximum value for $\alpha \in [0, 1]$. Substituting $\hat{\alpha}$ into $\phi(\alpha, \gamma, P)$, one obtains (4.73). □

Note that $\Delta_1^{\pi_\gamma} \to 0$ and $\Delta_2^{\pi_\gamma} \to 0$ for H becoming "increasingly deterministic", which means that the duality becomes "better" when the randomness for H is reduced.

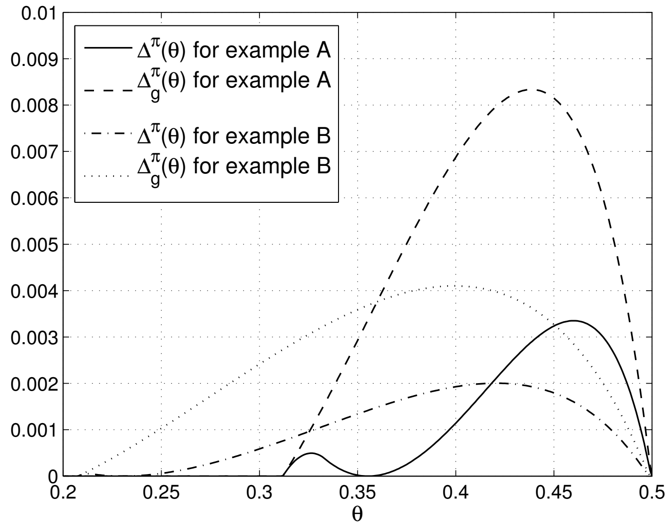

Figure 4.10: Actual errors $\Delta^{\pi}(\theta)$ and “a posteriori” error bounds $\Delta_g^{\pi}(\theta)$.

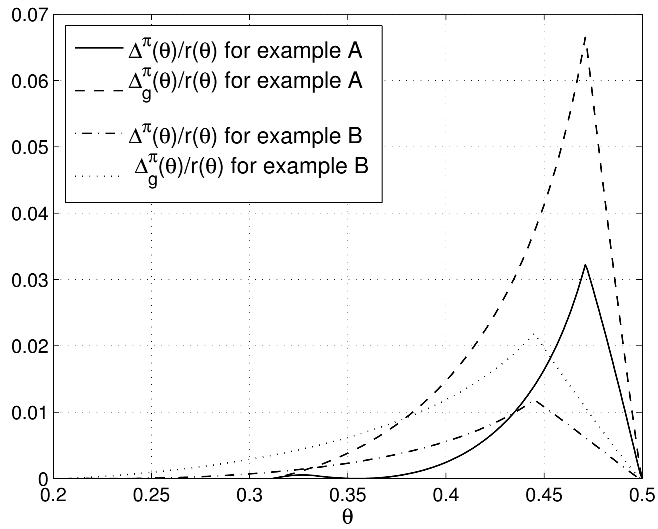

Figure 4.11: Relative actual errors $\Delta^{\pi}(\theta)/r(\theta)$ and relative “a posteriori” error bounds $\Delta_g^{\pi}(\theta)/r(\theta)$.

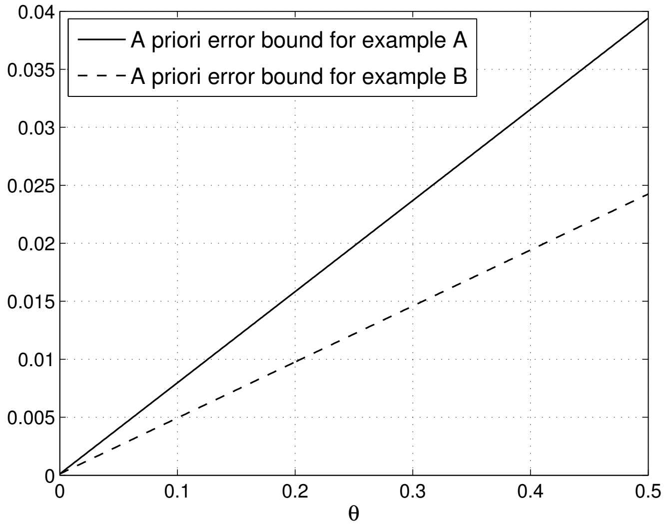

Figure 4.12: “A priori” error bounds using Theorem 6 for examples A and B.

We discuss two examples in the following. We consider the case $N_1 = N_2 = 50, P = 10^3$ and uniform fading: p_1 and p_2 are chosen such that the fading gains G^2 and H^2 are distributed uniformly in the interval $[0.6, 1]$ and $[0.2, 0.6]$ (example A) and $[0.2, 0.9]$ and $[0.1, 0.2]$ (example B), respectively. We compare, for different values of θ out of a fine discretization of $[0, \frac{1}{2}]$, the approximative approach using the power split mapping $\pi = \pi_\gamma$ for $\gamma = \mathbf{E}\left[H^2\right]$ to the solutions obtained by a simple brute force optimization, which is, in principle, possible for the channels at hand but is likely to be impractical for more general channel models.

Figure 4.10 depicts, for the two examples, the actual error $\Delta^\pi(\theta)$ and the “a posteriori” bound $\Delta_g^\pi(\theta)$ for $\theta \in [0.2, 0.5]$ (they are zero outside this interval). For each θ, we determine the range $r(\theta) := \max\limits_{\beta\in[0,1]} f_\theta^{\mathrm{BC}}(\beta) - \min\limits_{\beta\in[0,1]} f_\theta^{\mathrm{BC}}(\beta)$, which is the maximum possible error. Figure 4.11 displays the actual errors and error bounds relative to $r(\theta)$, i.e., $\Delta^\pi(\theta)/r(\theta)$ and $\Delta_g^\pi(\theta)/r(\theta)$, respectively. Finally, Figure 4.12 shows the error bounds obtained using Theorem 6 and (4.71). Clearly, these figures illustrate that the approximative solutions are close to the optimum.

4.6 Summary and Conclusions

Duality between the MAC and the BC is an interesting concept in information theory and refers to the possibility of expressing a certain performance criterion (such as capacity, SINR or MSE) for the BC in terms of a dual MAC and conversely. These relations are not only interesting for themselves, both also from an algorithmic perspective in that they often allow to efficiently solve difficult optimization problems for the BC via a detour over the dual MAC.

In this chapter we investigated the existence of duality relations between the MAC and the BC for two specific classes of channels, namely

- discrete memoryless channels and
- fading channels without channel state information at the transmitter.

Motivated by the existing duality relations, we defined the concept of weak BC-to-MAC duality: A BC is weakly dual to a set of MACs if every weighted sum-rate maximizer over the BC region is also a maximizer for one of the capacity regions of the MACs. This notion of duality is more general than the existing duality definitions as it imposes less restrictions on the shape of the dual MAC channel capacity regions. Yet, it is still a useful definition, since any necessary condition for weak duality is also necessary for "strong duality"—non-existence of weak duality implies non-existence of "strong duality". In addition, establishing weak duality can be used as a step to showing "strong duality": if weak duality can be shown to hold, it suffices to prove that the capacity regions of the weakly dual channels are all contained in the BC capacity region.

Based on a boundary characterization for the 2-MAC, we gave a sufficient condition and a necessary condition for a BC to be weakly dual to the class of 2-MACs. These conditions were applied to the binary symmetric BC, showing that this channel is weakly dual to the class of 2-MACs for a large set of parameter choices. Concerning future work in this area, it would be interesting to investigate to what extent the (weak) duality found here can be carried over to more general classes of channels.

For fading channels without CSI at the transmitter, there generally is a lack of duality: we have demonstrated that the achievable SCSD region (conjectured to constitute the capacity region) is different from the SIC region for the corresponding dual MAC if the fading distribution for the weaker user is non-deterministic. Duality holds only in the case of one-sided fading, where the fading distribution for the weaker user is deterministic. However, we demonstrated by means of the problem of weighted sum-rate optimization that the principle of solving problems for the BC by the construction of the dual MAC is promising even in the case where no perfect duality is present. This raises the question for future work whether similar principles can applied to more general channel models and or different optimization problems.

5 Capacity Results for Approximative Channel Models

In recent years, approximate characterizations of capacity regions of multi-user systems have gained more and more attention, with a well-known example being the characterization of the capacity region of the Gaussian interference channel to within one bit/s/Hz in [ETW08]. One of the tools that arose in the context of capacity approximations and which has been shown to be useful in many cases is the *linear deterministic model (LDM)*, also known in the literature as the *ADT model* or *shift linear deterministic model*, introduced in [ADT07], [ADT11]. In the LDM, the effect of the channel is to erase a certain number of ingoing bits, while superposition of signals is given by the modulo addition. Even though this model is deterministic and deemphasizes the effect of receiver noise, it is able to capture some important basic features of wireless systems, namely the *superposition property* and the *broadcast property* of electromagnetic wave propagation. Hence, in multi-user systems where interference is one of the most important limiting factors on system performance, the model can also be useful to devise effective coding and interference mitigation techniques. There are many examples where a linear deterministic analysis can be successfully carried over to coding schemes for the physical (Gaussian) models or be used for approximative capacity or (generalized) degrees of freedom determination.

From a practical viewpoint, *cellular systems* are of great interest. Generally, a cellular system consists of a set of base stations each communicating with a distinct set of (mobile) users. Effective coding and interference mitigation schemes for such systems are still an active area of research. Approximative approaches such as the LDM might help to gain more insight into these problems. The work presented in this chapter takes a step into this direction and investigates a cellular setup using the LDM. More precisely, we study the LDM for a two-user multiple-access channel interfering with a point-to-point link. We remark that this channel also serves as a basic model for device-to-device communication underlaying an uplink transmission in a cell, where both the cellular network and the device-to-device communication use the same resources and which has recently drawn increasing attention [DRW$^+$09]. We derive LDM capacity results under various interference scenarios and give, for the Gaussian channel, lower bounds on the generalized degrees of freedom for rational link ratio parameters. Note that the cellular channel has also been studied in [SW11], which derives the degrees of freedom of cellular (MIMO) systems in a *fading* environment and gives interference alignment algorithms that achieve these degrees of freedom under some conditions.

5.1 The Linear Deterministic Model

In the LDM, the channel is given by a deterministic mapping that operates on bit vectors and mimics the effect the physical channel and interfering signals have on the binary expansion of the transmitted symbols. More precisely, consider the basic example of a real point-to-point AWGN (additive white Gaussian noise) channel with system equation

$$Y = \sqrt{\mathrm{SNR}}X + Z, \tag{5.1}$$

where the input signal X is subject to an average power constraint $\mathbf{E}[X^2] \leq 1$, the noise is $Z \sim \mathcal{N}(0,1)$ and $\mathrm{SNR} \geq 0$ represents the signal-to-noise ratio. Assuming $X, Z \in [0,1]$ and writing the input symbol and the noise term in binary expansion as $X = \sum_{k=1}^{\infty} x_k 2^{-k}$ and $Z = \sum_{k=1}^{\infty} z_k 2^{-k}$ with $x_k, z_k \in \mathbb{F}_2$, one can show that the channel is approximately described by a deterministic channel that has bit vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^q$ with entries from the binary finite field $\mathbb{F}_2$ as input and output symbols, respectively. The input and output vectors are related as

$$\mathbf{y} = \mathbf{S}^{q-n}\mathbf{x}, \tag{5.2}$$

where $q \geq n$, $n = \frac{1}{2}(\lceil \log(\mathrm{SNR}) \rceil)^+$ and

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{F}_2^{q \times q} \tag{5.3}$$

is the *downshift matrix*. The entries of these bit vectors are also referred to as *bit levels*: the effect of the channel is to erase the $q-n$ lowest bit levels that are, after passing through the channel, below the noise level at the receiver, whereas the first n levels are received unaltered.

Clearly, the capacity C_{det} of the channel given by (5.2) is

$$C_{\mathrm{det}} = n = \frac{1}{2}(\lceil \log(\mathrm{SNR}) \rceil)^+ \text{ bits/channel use.} \tag{5.4}$$

Hence, the LDM approximates the capacity of the AWGN channel

$$C_{\text{AWGN}} = \frac{1}{2}\log(1+\text{SNR}) \text{ bits/channel use} \tag{5.5}$$

to within $\frac{1}{2}$ bits/channel use.

The model can easily be extended to more general network structures. Let the network be given by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. For each edge $(i, j) \in \mathcal{E}$, a channel parameter n_{ji} is specified that determines the number of bits that can be passed over that edge. As above, this number roughly corresponds to the channel gain of the link in logarithmic scale. Let $\mathbf{x}_i(t) \in \mathbb{F}_2^q$ be the signal transmitted by node $i \in \mathcal{V}$ at (discrete) time t. Then the received signal $\mathbf{y}_j(t)$ at node $j \in \mathcal{V}$ at time t is given by the superposition

$$\mathbf{y}_j(t) = \sum_{i:(i,j)\in\mathcal{E}} \mathbf{S}^{q-n_{ji}}\mathbf{x}_i(t), \tag{5.6}$$

where summation is in the finite field $\mathbb{F}_2$ and $q = \max\limits_{(i,j\in\mathcal{E})} n_{ji}$ is the reference vector length.

Moreover, a set $\mathcal{M} \subseteq \{(s,d) : s,d \in \mathcal{V}, s \neq d\}$ of source-destination pairs for the messages is specified with corresponding messages W_{sd} for $(s,d) \in \mathcal{M}$. The size of the message set for message W_{sd} is denoted by w_{sd}, and each message W_{sd} can be represented by a bit vector $\mathbf{m}_{sd} \in \mathbb{F}_2^{\lceil\log(w_{sd})\rceil}$. The notion of codes and transmission rates are defined in the usual way [CT06]: during each coding period of length $N \geq 1$, the transmitted symbol at the node j at time m is a function $f_j^{(m)}$ of the symbols received during the past m time instances and the source messages $\mathbf{m}_{jd}$ with $(j,d) \in \mathcal{M}$ originating from node j. Stacking all the input vectors, the encoding function can be assumed to also operate on bit vectors, and the transmit symbol $\mathbf{x}_j(m)$ of node j at time $m \in \{0, \ldots, N-1\}$ is given by

$$\mathbf{x}_j(m) = f_j^{(m)}\left([(\mathbf{y}_j(t))_{t=0}^m; (\mathbf{m}_{jd})_{d:(j,d)\in\mathcal{M}}]\right). \tag{5.7}$$

Similarly, decoding is performed for each message W_{sd} at the corresponding destination node $d \in \mathcal{V}$ by a decoding function g_{sd}, which takes all the symbols previously received at node d in the current transmission period as input and outputs the estimate

$$\widehat{W}_{sd} = g_{sd}\left((\mathbf{y}_d(t))_{t=0}^{N-1}\right) \tag{5.8}$$

of the message W_{sd}. The transmission rate corresponding to the message W_{sd} is $R_{sd} = \frac{\log(w_{sd})}{N}$. Since the channel is deterministic, it suffices to use a zero error probability criterion (also see e.g. [AK11]): the rates $r_{ds}, (s,d) \in \mathcal{M}$ are *achievable* if there exists a period length N, encoding functions $f_j^{(m)}$ and decoding functions g_{sd} such that for all $(s,d) \in \mathcal{M}$ it holds that $W_{sd} = \widehat{W}_{sd}$ and $r_{ds} \leq R_{sd}$. Again, the *capacity region* is defined

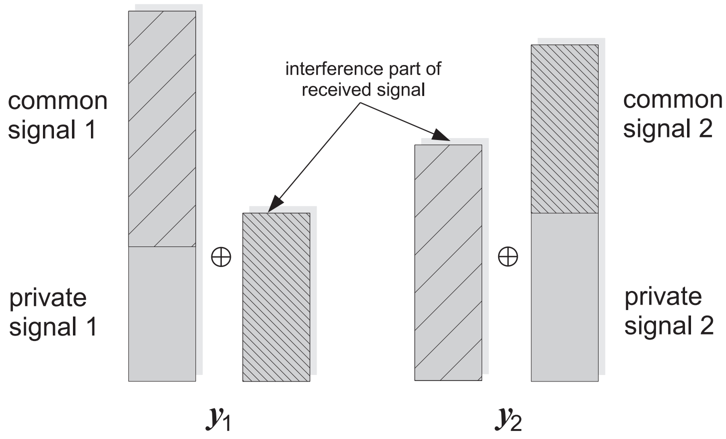

Figure 5.1: Illustration of the received signals (bit vectors) $\mathbf{y}_1$ and $\mathbf{y}_2$ for the linear deterministic two-user interference channel. The bars represent the bit vectors as seen at the two receivers; the zero parts due to the channel shifts are not displayed. A capacity-achieving coding scheme is of Han-Kobayashi type where the transmit signals are split into private and common information parts.

as the closure of the set of achievable rate vectors, taken over all possible encoding and decoding functions. We say that the rates r_{ds} are achieved by *linear coding* if the encoding functions $f_j^{(m)}$ and the decoding functions g_{sd} are linear, i.e., $f_j^{(m)}(\mathbf{x}) = \mathbf{A}^{(m,j)}\mathbf{x}$ and $g_{sd}(\mathbf{x}) = \mathbf{B}^{(s,d)}\mathbf{x}$ for matrices $\mathbf{A}^{(m,j)}$, $\mathbf{B}^{(s,d)}$ of appropriate sizes. As a special case of linear coding, we use the term *orthogonal coding* if the transmission is uncoded and the bit levels are used in an orthogonal fashion in the sense that at each receiving node, the incoming signals do not interfere on any bit level. More precisely, we require that $N = 1$ and for all codewords, for each receiving node j (i.e., a node that is a message destination) and for each bit level k, there is at most one node i with $(i, j) \in \mathcal{E}$ and for which the signal component $\mathbf{S}^{q-n_{ji}}\mathbf{x}_i(1)$ has a 1 at level k. Hence, each link $(i, j) \in \mathcal{E}$ has a dedicated set of receiver bit levels at node j. Moreover, for each message $(s, j) \in \mathcal{M}$, the corresponding encoding function operates as the identity on the bit levels reserved for the link (s, j).

Clearly, this model is considerably simpler than the Gaussian system it mimics. For this reason, it is often considerably easier to characterize capacity-optimal operating points for LDMs. Interestingly, it is frequently the case that the schemes constructed for the LDM can guide the development of techniques for the Gaussian case as well. As mentioned above, even though the receiver noise is neglected here, the model captures some important basic features of wireless systems, namely the superposition and the broadcast property [ADT11] of electromagnetic wave propagation. As an example, consider the deterministic two-user interference channel where there are two pairs of transmitters and receivers that mutually interfere: the LDM for this channel is given by

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{S}^{q-n_{11}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{12}}\mathbf{x}_2, \\ \mathbf{y}_2 &= \mathbf{S}^{q-n_{21}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{22}}\mathbf{x}_2. \end{aligned} \tag{5.9}$$

Here, a natural communication scheme consists in splitting the transmit signal into two parts, namely a *common* part and a *private* part: the common part is the portion of the signal that is seen as interference at the non-intended receiver, while the private part is only seen at the intended receiver (cf. Figure 5.1). It can be shown [BT08] that a capacity-achieving scheme can always be constructed in this way such that each receiver is able to decode both common messages and his own private message. This scheme essentially implements the well-known Han-Kobayashi scheme [HK81], which for the Gaussian channel is close to optimal [ETW08]. In this example, a seemingly complex strategy for the Gaussian channel naturally emerges from the analysis of the linear deterministic counterpart [BT08]. What is more, a characterization of the capacity within a constant number of bits (i.e., independent of the channel parameters) can be found for the Gaussian channel, as is also shown in [BT08]. The capacity region of the interference channel in the LDM is given by the set of points $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy

$$R_i \leq n_{ii},\ i = 1, 2 \tag{5.10}$$

$$R_1 + R_2 \leq (n_{11} - n_{12})^+ + \max(n_{22}, n_{12}) \tag{5.11}$$

$$R_1 + R_2 \leq (n_{22} - n_{21})^+ + \max(n_{11}, n_{21}) \tag{5.12}$$

$$R_1 + R_2 \leq \max(n_{21}, (n_{11} - n_{12})^+) + \max(n_{12}, (n_{22} - n_{21})^+) \tag{5.13}$$

$$2R_1 + R_2 \leq \max(n_{11}, n_{21}) + (n_{11} - n_{12})^+ + \max(n_{12}, (n_{22} - n_{21})^+) \tag{5.14}$$

$$R_1 + 2R_2 \leq \max(n_{22}, n_{12}) + (n_{22} - n_{21})^+ + \max(n_{21}, (n_{11} - n_{12})^+). \tag{5.15}$$

5.1.1 Related Work

In the literature, there is a large amount of work concerning the LDM and examples for which a linear deterministic analysis can be successfully carried over to coding schemes for the Gaussian models or be used for approximative capacity or degrees of freedom determination. Here, the optimal transmission schemes often also involve *interference alignment* as introduced in [CJ08], where the transmitted signals are designed such that at the receivers, the undesired part of the signal that is due to the different interfering signals aligns in a certain subspace of the receive space. A considerable portion of this work deals with relaying structures where a message is to be transported to its destination through a network of several intermediate nodes. Actually, such a situation was the motivation for the introduction of the LDM in [ADT11]. This work studies relay networks in which there is one message source and one or multiple destination nodes for this message, which is relayed along the (acyclic) network. It is shown that the capacity is given by an extension of the classical *max-flow-min-cut theorem* for wired networks [FF58]. More precisely, the capacity equals the minimum of the rank of all transfer matrices $\mathbf{G}_{\Omega,\Omega^c}$ corresponding to the possible node partitions (cuts) (Ω, Ω^c) into two subsets with one containing the source and the other containing the destination:

$$C = \min_{\text{cuts } (\Omega,\Omega^c)} \text{rank}(\mathbf{G}_{\Omega,\Omega^c}). \tag{5.16}$$

Achievability of this rate is shown using random coding, i.e., by choosing the encoding matrices at the relaying nodes randomly. In addition, a connection to the Gaussian channel is made in [ADT11]: using the insight from the deterministic analysis, a new relaying scheme—*quantize-map-and-forward*—for the Gaussian channel is proposed. The performance of this scheme can be shown to be within a constant gap from the optimum, that is, the gap is independent of the channel parameters. Other work in this direction includes extensions to multiple sources (with a single destination) [BCM10], fading channels [JC09], nodes with half-duplex constraints [KHK09] and code construction algorithms and connections to algebraic network coding [KEYM11], [YS10], [GIZ09].

Another line of work concerning the LDM considers different variations of relaying problems, such as bi-directional relaying or relaying with multiple source-destination pairs. The reference [AST10] studies the linear deterministic bi-directional relaying channel where two nodes A and B *each* have a message to be transmitted to the other receiver. A relay node in between helps the communication. In this work, full-duplex operation is assumed, i.e., all nodes can transmit and receive at the same time. In [AST10], the capacity for the LDM is derived and used to provide a relaying scheme for the Gaussian channel that is within at most 3 bits from the optimum, regardless of the channel gains. An extension of this scenario to the case of two communicating pair and a single full-duplex relay is carried out in [HSKA09] and [AKSH09]. A related system model is investigated in [MDFT08] [MTD09], [SMT10]. Here, there is a two-hop chain of interference channels, i.e., there are two communicating transmitter-receiver pairs and two relay nodes. Again, the deterministic capacity and constant-gap approximations for the Gaussian case are derived for some specific configurations of the channel gains. Apart from relaying systems, the deterministic model has been successfully applied to a multitude of communication problems and network structures. An interesting direction of research are studies of the impact of *cooperation* among nodes [PV11b], [PV11a], [WT11a], and [WT11b]. These works will be elaborated on below in the context of duality results. An (incomplete) list of references of work on some other network structures includes the 3- user interference channel [SB10], the K-user interference channel [CJS09], the X-channel [HCJ09], the Z channel [CJV09], fading broadcast channels without channel state information at the transmitter [TY09], many-to-one and one-to-many interference channels [BPT10], interference channels with feedback [SAYS10], [VA10], cognitive radio channels [RTD10], and cyclically symmetric deterministic interference channels [BEGVV09]. The work in [MM11] studies the generalized degrees of freedom that are achieved by a greedy algorithm for LDM for the symmetric K-user interference channel. Another interesting research direction has been taken by the work in [SCAL10], which investigates cross-layer utility optimization for wireless networks based on the LDM.

5.1.2 Limitations and Extensions

As a final remark concerning the background and related work, we briefly point to the work in [AK09], which discusses some of the limitations of the LDM approach. Here, it is demonstrated that the gap between the capacities of a Gaussian relay network and the corresponding LDM can be unbounded in certain cases. In other words, the LDM is not always a good approximation to the physical channel. One of the major reasons is that the LDM fails to capture the phase of received signals. This also renders the application of the LDM to MIMO systems infeasible. The *discrete superposition model (DSM)* is proposed in [AK09] as an alternative deterministic approximation approach. An interesting property of the DSM is shown in [AK11] for relay networks with a single source-destination pair, interference networks, multicast networks and the MIMO versions of these networks: any code for the DSM can be lifted to a code for the corresponding Gaussian network such that the difference between the rate in the Gaussian channel and in the DSM is at most a constant number of bits. This constant depends only on the number of network nodes, but not on the channel gains or SNR. Moreover, the capacities of the Gaussian network and the DSM are within a constant of each other.

It is worth mentioning that while the discrete superposition model has better approximation properties as compared to the LDM, this comes with the price of a higher complexity: finding the capacity of discrete superposition networks is much more difficult than for the LDM—this is even true for the basic example of the MAC [SS11].

5.2 A Cellular System: MAC and Point-to-Point Link

We now turn to the core subject of this chapter: the system we study here represents a basic version of the uplink of cellular system and consists of three transmitters (mobile users) Tx_1, Tx_2 and Tx_3 and two receivers (base stations) Rx_1, Rx_2 (cf. Figure 5.2).

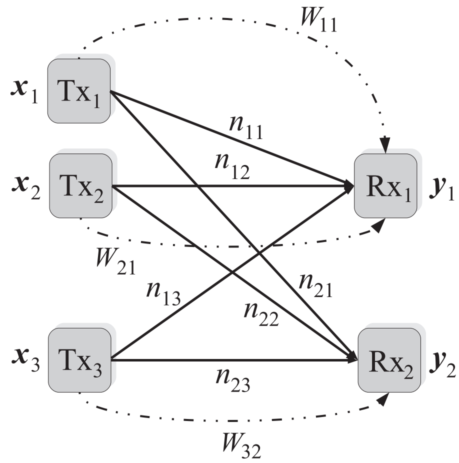

Figure 5.2: The system model: a MAC interfering with a point-to-point link.

The transmitters Tx_1, Tx_2 (in cell 1) communicate with the base station Rx_1, mutually interfering with the cell 2 user Tx_3 which is transmitting to the base station Rx_2. In other words, the system consists of a MAC interfering with a point-to-point link.

The system is modeled using the LDM [ADT11] as described in the previous section. The input symbol at transmitter Tx_i is given by a bit vector $\mathbf{x}_i \in \mathbb{F}_2^q$ and the output bit vectors $\mathbf{y}_j$ at Rx_j are deterministic functions of the inputs: the input-output equations of the system are given by

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{S}^{q-n_{11}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{12}}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_{13}}\mathbf{x}_3, \\ \mathbf{y}_2 &= \mathbf{S}^{q-n_{21}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{22}}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_{23}}\mathbf{x}_3. \end{aligned} \tag{5.17}$$

Again, q is chosen arbitrarily such that $q \geq \max_{j,i}\{n_{ji}\}$, $\mathbf{S} \in \mathbb{F}_2^{q\times q}$ is the downshift matrix given by (5.3) and n_{ji} determines the number of bits that can be passed over the link between Tx_i and Rx_j.

There are three messages W_{11}, W_{21} and W_{32} to be transmitted in the system, where W_{ij} denotes the message from transmitter Tx_i to the intended receiver Rx_j. The definitions of (block) codes, achievable rates and the capacity region are as given above in Section 5.1. In what follows, the transmission rate corresponding to message W_{11} is represented by R_1, the rate corresponding to W_{21} by R_2 and the rate for W_{32} by R_3.

In the following, we assume without loss of generality that $n_{11} \geq n_{12}$. Moreover, we write $n_1 := n_{11}, n_2 := n_{12}, n_3 := n_{23}, n_D := n_{13}$ and $\Delta := n_1 - n_2$. Also, we may choose $q = \max\{n_1, n_3\}$. Furthermore, we restrict ourselves to the case $n_{21} = n_{22} =: n_M$, to which we refer to as *symmetric MAC interference*. Note that this property holds approximately

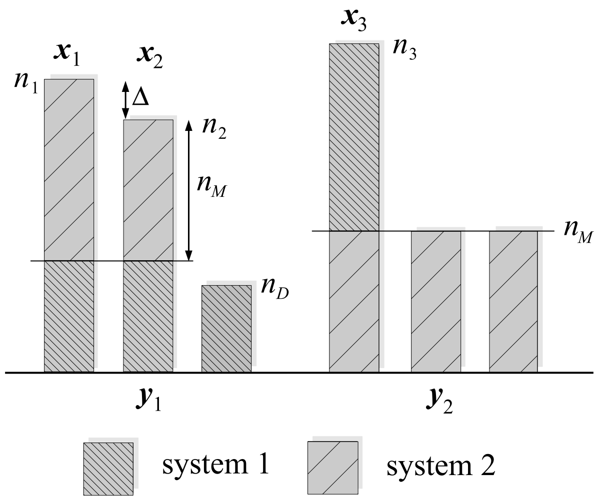

Figure 5.3: Illustration of the split of the system in terms of the bit vectors as seen at the receivers.

if the distance between the two cells is comparatively large. We will refer to the channel defined above as the *MAC-P2P*.

5.2.1 The Weak Interference Case

We start with the case of limited interference strength and assume the *weak interference condition*, which requires the sum of the two interference strength values not to be greater than the smallest direct link strength, i.e.,

$$n_D + n_M \leq \min\{n_2, n_3\}. \tag{5.18}$$

The MAC-P2P that satisfies this property is called *W-MAC-P2P* in the following.

An Achievable Region

We now derive an achievable rate region for the case of weak interference. For this, we split the system into two subsystems and derive achievable regions for each of them, denoted as $\mathcal{R}_{\text{ach}}^{(1)}$ and $\mathcal{R}_{\text{ach}}^{(2)}$, respectively. The sum of these two regions then results in an achievable region $\mathcal{R}_{\text{ach}}$ for the overall system. We begin by analyzing each of the two subsystems and the corresponding achievable rate regions. The first system is given by the equations

$$\begin{aligned}\mathbf{y}_1^{(1)} &= \mathbf{S}^{q^{(1)}-n_1^{(1)}}\mathbf{x}_1^{(1)} \oplus \mathbf{S}^{q^{(1)}-n_2^{(1)}}\mathbf{x}_2^{(1)} \oplus \mathbf{S}^{q^{(1)}-n_D}\mathbf{x}_3^{(1)},\\ \mathbf{y}_2^{(1)} &= \mathbf{S}^{q^{(1)}-n_3^{(1)}}\mathbf{x}_3^{(1)},\end{aligned} \tag{5.19}$$

where $n_1^{(1)} = n_2^{(1)} = n_2 - n_M, n_3^{(1)} = n_3 - n_M$ and $q^{(1)} = \max\{n_1^{(1)}, n_3^{(1)}\}$. The second system is defined by

$$\begin{aligned}
\mathbf{y}_1^{(2)} &= \mathbf{S}^{q^{(2)}-n_1^{(2)}}\mathbf{x}_1^{(2)} \oplus \mathbf{S}^{q^{(2)}-n_2^{(2)}}\mathbf{x}_2^{(2)}, \\
\mathbf{y}_2^{(2)} &= \mathbf{S}^{q^{(2)}-n_M}\mathbf{x}_1^{(2)} \oplus \mathbf{S}^{q^{(2)}-n_M}\mathbf{x}_2^{(2)} \oplus \mathbf{S}^{q^{(2)}-n_3^{(2)}}\mathbf{x}_3^{(2)},
\end{aligned} \tag{5.20}$$

with $q^{(2)} = n_1^{(2)} = n_M + \Delta$, $n_2^{(2)} = n_M$ and $n_3^{(2)} = n_M$. Note that this split is possible due to the weak interference condition $n_D + n_M \leq \min\{n_2, n_3\}$: roughly speaking, it ensures that, at each receiver, there is no overlap of the private signal part with the common part of the interfering transmitter(s) in the other cell. The corresponding transmitters and receivers are denoted as $Tx_i^{(s)}$ and $Rx_i^{(s)}$ in the subsystem $s \in \{1, 2\}$. Figure 5.3 and Figure 5.4 illustrate the split of the system.

We define $\mathcal{R}_{\text{ach}}^{(1)}$ as the set of points $\left(R_1^{(1)}, R_2^{(1)}, R_3^{(1)}\right)$ satisfying

$$R_3^{(1)} \leq n_3^{(1)} \tag{5.21}$$

$$R_1^{(1)} + R_2^{(1)} \leq n_1^{(1)} \tag{5.22}$$

$$R_1^{(1)} + R_2^{(1)} + R_3^{(1)} \leq n_2 + n_3 - n_D - 2n_M. \tag{5.23}$$

Similarly, $\mathcal{R}_{\text{ach}}^{(2)}$ is defined by the set of equations

$$R_1^{(2)} + R_2^{(2)} \leq n_1^{(2)} \tag{5.24}$$

$$R_1^{(2)} + R_3^{(2)} \leq n_1^{(2)} \tag{5.25}$$

$$R_2^{(2)} + R_3^{(2)} \leq n_M \tag{5.26}$$

$$R_1^{(2)} + R_2^{(2)} + R_3^{(2)} \leq n_M + \varphi(n_M, \Delta). \tag{5.27}$$

Here, the function φ for $p, q \in \mathbb{N}$ is defined as

$$\varphi(p, q) := \begin{cases} q + \frac{l(p,q)q}{2}, & \text{if } \ l(p,q) \ \text{ is even} \\ p - \frac{(l(p,q)-1)q}{2}, & \text{if } \ l(p,q) \ \text{ is odd.} \end{cases} \tag{5.28}$$

where $l(p, q) := \left\lfloor \frac{p}{q} \right\rfloor$ for $q > 0$ and $l(p, 0) = 0$.

We now show that the rate regions $\mathcal{R}_{\text{ach}}^{(1)}$ and $\mathcal{R}_{\text{ach}}^{(2)}$ are achievable in the respective subsystems. Both regions can be achieved by an orthogonal bit level assignment (and possibly time-sharing). More precisely, for each transmitter, a set of bit levels to be used for data transmission is specified such that at the intended receiver, there is no overlap of these levels with levels used by any other transmitter (cf. the definition of orthogonal

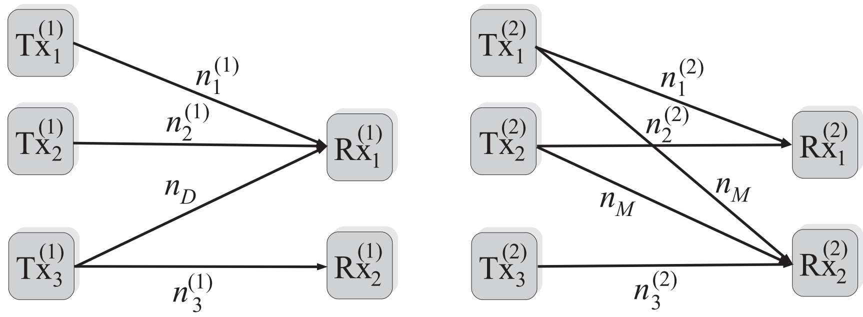

Figure 5.4: Resulting system diagrams after splitting the system: in system 1, only receiver $Rx_1^{(2)}$ suffers from interference, while in system 2 only receiver $Rx_2^{(2)}$ is subject to interference.

coding in Section 5.1). In this way, the achievability of the points in $\mathcal{R}_{\text{ach}}^{(1)}$ follows directly from the results on the interference channel: if we first consider the one-sided interference channel obtained by removing (or silencing) transmitter $Tx_2^{(1)}$, the points $\left(R_\Sigma^{(1)}, R_3^{(1)}\right)$ with $R_3^{(1)} \leq n_3^{(1)}, R_\Sigma^{(1)} \leq n_1^{(1)}, R_\Sigma^{(1)} + R_3^{(1)} \leq n_2 + n_3 - n_D - 2n_M$ can be achieved in this one-sided interference channel using orthogonal coding, which follows from the results in [BT08] [see also (5.10)-(5.15)]. The same rates can be achieved for the interference channel obtained by removing $Tx_1^{(1)}$. Then, since the signals $\mathbf{x}_1^{(1)}$ and $\mathbf{x}_2^{(1)}$ are shifted by the same amount at the first receiver, it is clear that in system 1, we can achieve $R_1^{(1)}$ and $R_1^{(2)}$ such that $R_1^{(1)} + R_2^{(1)} \leq R_\Sigma^{(1)}$, which implies the achievability of region $\mathcal{R}_{\text{ach}}^{(1)}$.

To show the achievability of $\mathcal{R}_{\text{ach}}^{(2)}$, let $\mathbf{a} \in \mathbb{F}_2^{n_M}$ specify the levels used for encoding (in an orthogonal fashion) the message from transmitter $Tx_3^{(2)}$ to receiver $Rx_2^{(2)}$, where $a_i = 1$ if level i is used and $a_i = 0$ otherwise. Moreover, we define $\gamma(\mathbf{a}) = \mathbf{1}_{n_M} - \mathbf{a}$, $\gamma_1(\mathbf{a}) = [\gamma(\mathbf{a}); \mathbf{1}_\Delta]$ and $\gamma_2(\mathbf{a}) = [\mathbf{0}_\Delta; \gamma(\mathbf{a})]$. Then we can achieve $R_3^{(2)} = |\mathbf{a}|$ and all rates $\left(R_1^{(2)}, R_2^{(2)}\right)$ with

$$R_1^{(2)} \leq |\gamma_1(\mathbf{a})| = n_M + \Delta - |\mathbf{a}| \tag{5.29}$$

$$R_2^{(2)} \leq |\gamma_2(\mathbf{a})| = n_M - |\mathbf{a}| \tag{5.30}$$

$$R_1^{(2)} + R_2^{(2)} \leq |\gamma_1(\mathbf{a})| + |\gamma_2(\mathbf{a})| - \rho(|\mathbf{a}|), \tag{5.31}$$

where

$$\rho(x) := \min_{\mathbf{a} \in \mathbb{F}_2^{n_M} : |\mathbf{a}| = x} \gamma_1(\mathbf{a})^T \gamma_2(\mathbf{a}). \tag{5.32}$$

An assignment vector $\mathbf{a}$ solving (5.32) for a given x can be shown to be of the form described in the following. Let $l = n_M \text{ div } \Delta$ and $Q = n_M \text{ mod } \Delta$, i.e., $n_M = l\Delta + Q$. We

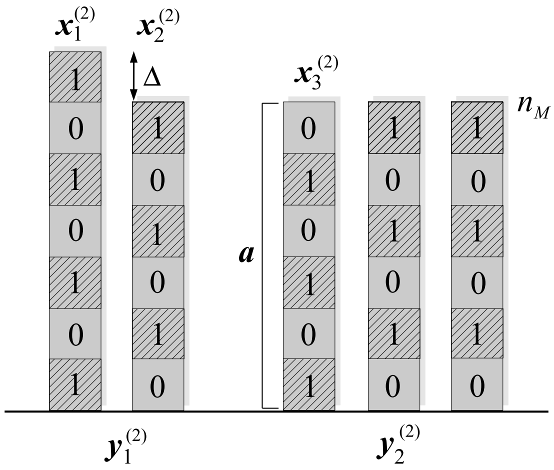

Figure 5.5: Example for optimal bit level assignment: the MAC interference aligns at receiver $Rx_2^{(2)}$.

subdivide $\mathbf{a}$ into $l\Delta$ subsequences (blocks) of length Δ and one remainder block of length Q. We distribute ones over $\mathbf{a}$ until x entries in $\mathbf{a}$ have been set to 1: we start with the even-numbered blocks, followed by the remainder block. If l is even, we finally distribute over the odd-numbered blocks. If l is odd, we also fill the odd-numbered blocks, but in reversed (decreasing) order. To be precise, we define for the case that l is even

$$\mathbf{A}_{\text{even}} := \left(\mathbf{0}_{1\times\frac{l}{2}}; \mathbf{e}_k\right)_{k=1}^{l/2} \otimes \mathbf{I}_\Delta, \tag{5.33}$$

$$\mathbf{A}_{\text{odd}} := \left(\mathbf{e}_k; \mathbf{0}_{1\times\frac{l}{2}}\right)_{k=1}^{l/2} \otimes \mathbf{I}_\Delta \tag{5.34}$$

and

$$\mathbf{A}_{\text{even}} := \left[\left(\mathbf{0}_{1\times\frac{(l-1)}{2}}; \mathbf{e}_k\right)_{k=1}^{(l-1)/2}; \mathbf{0}_{1\times\frac{l-1}{2}}\right] \otimes \mathbf{I}_\Delta, \tag{5.35}$$

$$\mathbf{A}_{\text{odd}} := \mathbf{M}_{l\Delta}\left(\left[\left(\mathbf{e}_k; \mathbf{0}_{1\times\frac{(l+1)}{2}}\right)_{k=1}^{(l-1)/2}; \mathbf{e}_{\frac{l+1}{2}}\right] \otimes \mathbf{I}_\Delta\right) \tag{5.36}$$

for odd l. Here, $\otimes$ denotes the Kronecker product, $\mathbf{e}_k$ the unit row vector of appropriate size with 1 at position k and $\mathbf{M}_N = (\mathbf{e}_{N-k+1})_{k=1}^N$ is the flip matrix. Then, defining the matrix $\mathbf{P} := [\mathbf{A}_{\text{even}}|\mathbf{0}_{l\Delta\times Q}|\mathbf{A}_{\text{odd}}]$, we obtain an optimal assignment $\mathbf{a}$ by setting $\mathbf{a} = \mathbf{P}[\mathbf{1}_x; \mathbf{0}_{n_M-x}]$. We remark that this assignment is not the unique optimal one. Furthermore, it can be interpreted as interference alignment [CJ08] at the receiver $Rx_2^{(2)}$: The bit levels are chosen such that the interference caused by $\mathbf{x}_1^{(2)}$ and $\mathbf{x}_2^{(2)}$ (MAC interference) aligns at $Rx_2^{(2)}$ as much as possible in the levels that are unused by $\mathbf{x}_3^{(2)}$. Figure 5.5 displays the optimal assignment for the case $l=6, Q=0$ and $x=R_3^{(2)}=3\Delta$. The rates achieved here for the first two transmitters are $R_1^{(2)}=4\Delta$ and $R_2^{(2)}=3\Delta$. The optimal assignment described above for the case l even results in the following form of the function ρ:

$$\rho(x)=\begin{cases} n_M-2x, & \text{if } 0\le x\le \frac{l\Delta}{2} \\ Q+\frac{l\Delta}{2}-x, & \text{if } \frac{l\Delta}{2}\le x\le \frac{l\Delta}{2}+Q \\ 0, & \text{if } \frac{l\Delta}{2}+Q\le x\le n_M. \end{cases} \tag{5.37}$$

In the case that l is odd, we similarly obtain

$$\rho(x)=\begin{cases} n_M-2x, & \text{if } 0\le x\le Q+\frac{\Delta(l-1)}{2} \\ \frac{\Delta(l+1)}{2}-x, & \text{if } Q+\frac{\Delta(l-1)}{2}\le x\le \frac{\Delta(l+1)}{2} \\ 0, & \text{if } \frac{\Delta(l+1)}{2}\le x\le n_M. \end{cases} \tag{5.38}$$

Substituting into (5.31), it is easy to verify that all points in $\mathcal{R}_{\text{ach}}^{(2)}$ are achievable.

Finally, we obtain an achievable region $\mathcal{R}_{\text{ach}}=\mathcal{R}_{\text{ach}}^{(1)}+\mathcal{R}_{\text{ach}}^{(2)}$ for the overall system. This sum region can be computed in explicit form by employing the Fourier-Motzkin elimination algorithm [Sch98]. The resulting region is stated in the following proposition:

Proposition 4. *An achievable region $\mathcal{R}_{ach}$ for the W-MAC-P2P is given by the set of points $(R_1,R_2,R_3)\in\mathbb{R}^3$ satisfying the constraints*

$$R_1\le n_1 \tag{5.39}$$
$$R_2\le n_2 \tag{5.40}$$
$$R_3\le n_3 \tag{5.41}$$
$$R_1+R_2\le n_1 \tag{5.42}$$
$$R_1+R_3\le n_1+n_3-n_D-n_M \tag{5.43}$$
$$R_2+R_3\le n_2+n_3-n_D-n_M \tag{5.44}$$
$$R_1+R_2+R_3\le n_2+n_3-n_D-n_M+\varphi(n_M,\Delta) \tag{5.45}$$
$$R_1+R_2+2R_3\le n_1+2n_3-n_D-n_M. \tag{5.46}$$

Capacity Region

In what follows, we show that the achievable region $\mathcal{R}_{\text{ach}}$ actually constitutes the capacity region $\mathcal{C}_{\text{MAC-P2P}}$ of the W-MAC-P2P. For the proof, we will need the following lemma:

Lemma 6. *For two independent random matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{F}_2^{n+\Delta \times m}$ *with* $m, n, \Delta \in \mathbb{N}, m \geq 1$*, it holds that*

$$H(\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) - H(\mathbf{S}^{\Delta}\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) \leq m\varphi(n, \Delta), \tag{5.47}$$

$$H(\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) - 2H(\mathbf{S}^{\Delta}\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) \leq m\Delta. \tag{5.48}$$

Proof. In order to show (5.47), we let $l = n \text{ div } \Delta, Q = n \bmod \Delta$ and introduce the following labels for blocks of rows of the matrices $\mathbf{A}$ and $\mathbf{B}$: $\mathbf{A} = [\mathbf{U}; (\mathbf{A}_k)_{k=1}^{l+1}], \mathbf{B} = [\mathbf{T}; (\mathbf{B}_k)_{k=1}^{l+1}]$, where $\mathbf{U}, \mathbf{T} \in \mathbb{F}_2^{Q \times m}$ and $\mathbf{A}_k, \mathbf{B}_k \in \mathbb{F}_2^{\Delta \times m}$. Then the shifted versions of $\mathbf{A}$ and $\mathbf{B}$ are $\mathbf{S}^{\Delta}\mathbf{A} = [\mathbf{0}_{\Delta \times m}; \mathbf{U}; (\mathbf{A}_k)_{k=1}^{l}]$ and $\mathbf{S}^{\Delta}\mathbf{B} = [\mathbf{0}_{\Delta \times m}; \mathbf{T}; (\mathbf{B}_k)_{k=1}^{l}]$, respectively.

First consider the case that l is even. Then we have

$$\begin{aligned}
&H(\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) - H(\mathbf{S}^{\Delta}\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) \\
&= H\left[\mathbf{U}; \mathbf{A}_1 \oplus [\mathbf{0}_{\Delta - Q \times m}; \mathbf{T}]; (\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l}\right] - H\left[\mathbf{U} \oplus \mathbf{T}; (\mathbf{A}_k \oplus \mathbf{B}_k)_{k=1}^{l}\right] \\
&\leq H\left[\mathbf{U}; \mathbf{A}_1 \oplus [\mathbf{0}_{\Delta - Q \times m}; \mathbf{T}]; (\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l}\right] \\
&\quad - H\left[\mathbf{U} \oplus \mathbf{T}; (\mathbf{A}_k \oplus \mathbf{B}_k)_{k=1}^{l} \middle| \mathbf{T}, (\mathbf{A}_{2k-1})_{k=1}^{l/2}, (\mathbf{B}_{2k})_{k=1}^{l/2}\right] \\
&= H\left[\mathbf{U}; \mathbf{A}_1 \oplus [\mathbf{0}_{\Delta - Q \times m}; \mathbf{T}]; (\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l}\right] - H\left[\mathbf{U}; (\mathbf{A}_{2k})_{k=1}^{l/2}; (\mathbf{B}_{2k-1})_{k=1}^{l/2}\right] \\
&\leq m\Delta\left(1 + \frac{l}{2}\right) + H\left[\mathbf{U}; (\mathbf{A}_{2k} \oplus \mathbf{B}_{2k-1})_{k=1}^{l/2}\right] - H\left[\mathbf{U}; (\mathbf{A}_{2k})_{k=1}^{l/2}; (\mathbf{B}_{2k-1})_{k=1}^{l/2}\right] \\
&\leq m\Delta\left(1 + \frac{l}{2}\right) = m\varphi(n, \Delta).
\end{aligned} \tag{5.49}$$

where the last inequality is due to the independence of $\mathbf{A}$ and $\mathbf{B}$.

For odd l, we can bound the expression as follows:

$$\begin{aligned}
&H(\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) - H(\mathbf{S}^{\Delta}\mathbf{A} \oplus \mathbf{S}^{\Delta}\mathbf{B}) \\
&= H\left[\mathbf{U}; \mathbf{A}_1 \oplus [\mathbf{0}_{\Delta - Q \times m}; \mathbf{T}]; (\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l}\right] \\
&\quad - H\left[\mathbf{U} \oplus \mathbf{T}; (\mathbf{A}_k \oplus \mathbf{B}_k)_{k=1}^{l}\right] \\
&\leq H\left[\mathbf{U}; \mathbf{A}_1 \oplus [\mathbf{0}_{\Delta - Q \times m}; \mathbf{T}]; (\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l}\right] \\
&\quad - H\left[\mathbf{U} \oplus \mathbf{T}; (\mathbf{A}_k \oplus \mathbf{B}_k)_{k=1}^{l} \middle| \mathbf{U}, (\mathbf{A}_{2k})_{k=1}^{(l-1)/2}, (\mathbf{B}_{2k-1})_{k=1}^{(l+1)/2}\right] \\
&= H\left[\mathbf{U}; \mathbf{A}_1 \oplus [\mathbf{0}_{\Delta - Q \times m}; \mathbf{T}]; (\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l}\right] \\
&\quad - H\left[\mathbf{T}; (\mathbf{A}_{2k-1})_{k=1}^{(l+1)/2}; (\mathbf{B}_{2k})_{k=1}^{(l-1)/2}\right]
\end{aligned} \tag{5.50}$$

$$
\begin{aligned}
&\leq m\frac{\Delta(l+1)}{2} + mQ + H\left[\mathbf{A}_1 \oplus [\mathbf{0}_{\Delta-Q\times m};\mathbf{T}];(\mathbf{A}_{2k+1}\oplus\mathbf{B}_{2k})_{k=1}^{(l-1)/2}\right]\\
&\quad -H\left[\mathbf{T};(\mathbf{A}_{2k-1})_{k=1}^{(l+1)/2};(\mathbf{B}_{2k})_{k=1}^{(l-1)/2}\right]\\
&\leq m\frac{\Delta(l+1)}{2} + mQ = m\varphi(n,\Delta).
\end{aligned}
$$

The bound (5.48) follows from

$$
\begin{aligned}
&H(\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) - 2H(\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) \qquad (5.51)\\
&\quad\leq H(\mathbf{A}) + H(\mathbf{S}^{\Delta}\mathbf{B}) - 2H(\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B})\\
&\quad\leq m\Delta + H(\mathbf{S}^{\Delta}\mathbf{A}) + H(\mathbf{S}^{\Delta}\mathbf{B}) - 2H(\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B})\\
&\quad= m\Delta + H(\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) + H(\mathbf{S}^{\Delta}\mathbf{A}|\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) - 2H(\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B})\\
&\quad= m\Delta + H(\mathbf{S}^{\Delta}\mathbf{A}|\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) - H(\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B})\\
&\quad= m\Delta + H(\mathbf{S}^{\Delta}\mathbf{A}|\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) - H(\mathbf{S}^{\Delta}\mathbf{A}) + H(\mathbf{S}^{\Delta}\mathbf{B}|\mathbf{S}^{\Delta}\mathbf{A}\oplus\mathbf{S}^{\Delta}\mathbf{B}) - H(\mathbf{S}^{\Delta}\mathbf{B})\\
&\quad\leq m\Delta,
\end{aligned}
$$

where we used $H(\mathbf{A}) + H(\mathbf{B}) = H(\mathbf{A}\oplus\mathbf{B}) + H(\mathbf{A}|\mathbf{A}\oplus\mathbf{B})$. □

We are now ready to prove the main result for the weak interference case:

Theorem 7. *The capacity region $\mathcal{C}_{\text{W-MAC-P2P}}$ of the W-MAC-P2P is given by $\mathcal{R}_{ach}$ [defined by (5.39)-(5.46)].*

Proof. Consider the interference channel formed by Tx_1, Tx_3, Rx_1 and Rx_2 with corresponding capacity region $\mathcal{C}_{\text{IC}1/3}$, the interference channel build from Tx_2, Tx_3, Rx_1 and Rx_2 with capacity region $\mathcal{C}_{\text{IC}2/3}$ and the multiple-access channel consisting of Tx_1, Tx_2 and Rx_1 with capacity region $\mathcal{C}_{\text{MAC}}$. Then, it is clear that if $(R_1, R_2, R_3) \in \mathcal{C}_{\text{W-MAC-P2P}}$, we must have $(R_1, R_2) \in \mathcal{C}_{\text{MAC}}, (R_1, R_3) \in \mathcal{C}_{\text{IC}1/3}$ and $(R_2, R_3) \in \mathcal{C}_{\text{IC}2/3}$. Evaluating the corresponding capacity regions [BT08], this implies the bounds (5.39) - (5.44).

In order to prove (5.45), we apply Fano's inequality: for each triple of achievable rates $(R_1, R_2, R_3) \in \mathcal{C}_{\text{W-MAC-P2P}}$, Fano's inequality implies that there exists a sequence ε_N with $\varepsilon_N \to 0$ for $N \to \infty$ and a sequence of joint factorized distributions on the input sequences such that for all $N \in \mathbb{N}, N \geq 1$

$$R_1 + R_2 \leq \frac{1}{N} I(\mathbf{x}_1^N, \mathbf{x}_2^N; \mathbf{y}_1^N) + \varepsilon_N, \qquad (5.52)$$

$$R_3 \leq \frac{1}{N} I(\mathbf{x}_3^N; \mathbf{y}_2^N) + \varepsilon_N. \qquad (5.53)$$

We let $\beta = n_2 - n_D - n_M, \epsilon = n_3 - n_M - n_D$, $T_k = [\mathbf{0}_{q-k\times m}; \mathbf{1}_{k\times m}]$ and define the following partial matrices of the components of the received signals (note that $\beta, \epsilon \geq 0$ from the weak interference condition):

$$\begin{aligned} \widehat{\mathbf{x}}_1 &:= \mathbf{S}^{q-n_M}\mathbf{x}_1^N, \mathbf{x}_1^{\uparrow} := \mathbf{S}^{q-n_M-\Delta}\mathbf{x}_1^N, \mathbf{x}_1^{\downarrow} := T_{n_D+\beta}\mathbf{x}_1^N, \\ \widehat{\mathbf{x}}_2 &:= \mathbf{S}^{q-n_M}\mathbf{x}_2^N, \mathbf{x}_2^{\downarrow} := T_{n_D+\beta}\mathbf{x}_2^N, \\ \widehat{\mathbf{x}}_3 &:= \mathbf{S}^{q-n_D}\mathbf{x}_3^N, \mathbf{x}_3^{\uparrow} := \mathbf{S}^{q-n_D-\epsilon}\mathbf{x}_3^N, \mathbf{x}_3^{\downarrow} := T_{n_M}\mathbf{x}_3^N. \end{aligned} \tag{5.54}$$

Then we have the chain of inequalities

$$\begin{aligned} &(R_1+R_2+R_3)N - \varepsilon_N N \\ &\leq I(\mathbf{x}_1^N, \mathbf{x}_2^N; \mathbf{y}_1^N) + I(\mathbf{x}_3^N; \mathbf{y}_2^N) \\ &= H(\mathbf{y}_1^N) - H(\mathbf{y}_1^N|\mathbf{x}_1^N, \mathbf{x}_2^N) + H(\mathbf{y}_2^N) - H(\mathbf{y}_2^N|\mathbf{x}_3^N) \\ &= H(\mathbf{y}_1^N) - H(\widehat{\mathbf{x}}_3) + H(\mathbf{y}_2^N) - H(\widehat{\mathbf{x}}_1 \oplus \widehat{\mathbf{x}}_2) \\ &\leq H(\mathbf{x}_1^{\uparrow} \oplus \widehat{\mathbf{x}}_2) + H(\mathbf{x}_1^{\downarrow} \oplus \mathbf{x}_2^{\downarrow} \oplus \widehat{\mathbf{x}}_3) - H(\widehat{\mathbf{x}}_3) \\ &\quad + H(\mathbf{x}_3^{\uparrow}) + H(\mathbf{x}_3^{\downarrow} \oplus \widehat{\mathbf{x}}_1 \oplus \widehat{\mathbf{x}}_2) - H(\widehat{\mathbf{x}}_1 \oplus \widehat{\mathbf{x}}_2) \\ &\overset{(a)}{\leq} H(\mathbf{x}_1^{\uparrow} \oplus \widehat{\mathbf{x}}_2) - H(\widehat{\mathbf{x}}_1 \oplus \widehat{\mathbf{x}}_2) + N(n_2 + n_3 - n_M - n_D) \\ &\overset{(b)}{\leq} N\varphi(n_M, \Delta) + N(n_2 + n_3 - n_M - n_D), \end{aligned} \tag{5.55}$$

where (a) follows from

$$H(\mathbf{x}_3^{\downarrow} \oplus \widehat{\mathbf{x}}_1 \oplus \widehat{\mathbf{x}}_2) \leq N n_M, \tag{5.56}$$

$$H(\mathbf{x}_1^{\downarrow} \oplus \mathbf{x}_2^{\downarrow} \oplus \widehat{\mathbf{x}}_3) \leq N(n_2 - n_M, \tag{5.57}$$

$$H(\mathbf{x}_3^{\uparrow}) - H(\widehat{\mathbf{x}}_3) \leq N(n_3 - n_M - n_D) \tag{5.58}$$

and (b) is obtained by applying Lemma 6. A similar argument, using the second part of Lemma 6, shows the bound (5.46). □

As an example, Figure 5.6 shows the capacity region for $n_1 = 18$, $n_2 = 16$, $n_3 = 14$, $n_M = 6$ and $n_D = 7$.

5.2.2 Sum Capacity for Arbitrary Interference

We now relax the constraints on the interference strength and study the sum capacity of the system. We retain the same assumptions and notation as above (without the weak interference constraint). However, in order to keep the presentation clear and reduce the number of cases to be distinguished, we make some further assumptions on the channel gains: we let the direct link strength for the point-to-point link to be equal the strongest

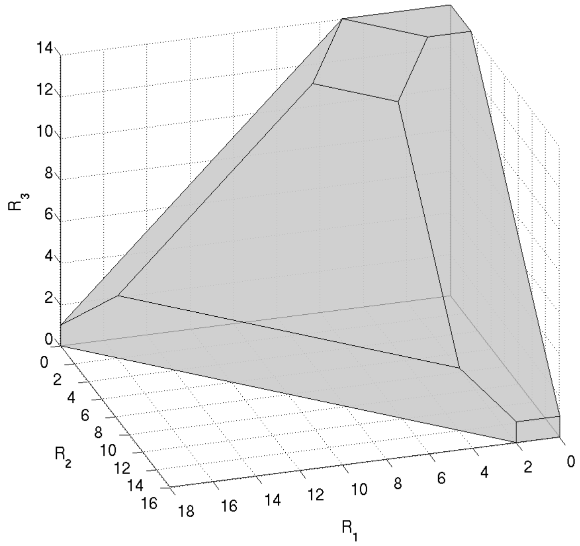

Figure 5.6: Capacity region of the W-MAC-P2P for channel parameters $n_1 = 18$, $n_2 = 16$, $n_3 = 14$, $n_M = 6$ and $n_D = 7$.

direct MAC link, i.e., $n_{23} = n_1$ (recall that we assumed $n_1 \geq n_2$). In addition, we assume identical interference links $n_i := n_{21} = n_{22} = n_{13}$. We refer to this channel as the *A-MAC-P2P* in what follows. We remark that these restrictions can easily be relaxed and the techniques applied in the following extend to more general cases as well. To be specific, the input-output equations of the A-MAC-P2P are given by

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{S}^{q-n_1}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_2}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_i}\mathbf{x}_3, \\ \mathbf{y}_2 &= \mathbf{S}^{q-n_i}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_i}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_1}\mathbf{x}_3. \end{aligned} \tag{5.59}$$

Again, we write $\Delta = n_1 - n_2$ for the direct link strength difference in the MAC. For achievable rates R_1, R_2, R_3, we denote the resulting sum rate as $R_\Sigma = R_1 + R_2 + R_3$.

The corresponding (real) Gaussian channel is defined by output symbols

$$\begin{aligned} Y_1 &= h_1X_1 + h_2X_2 + h_iX_3 + Z_1, \\ Y_2 &= h_iX_1 + h_iX_2 + h_1X_3 + Z_2 \end{aligned} \tag{5.60}$$

with (constant) channel coefficients $h_1, h_2, h_i \in \mathbb{R}_+$ satisfying $h_1 \geq h_2$, input symbols X_i subject to power constraints $\mathbf{E}[X_i^2] \leq P_i$ and noise $Z_i \sim \mathcal{N}(0, 1)$. Note that a related model with identical channel gains in the two-user cell ($h_1 = h_2$) is studied in [CS10]. This work derives a capacity outer bound which is achievable in some cases.

For the weak interference case (which is characterized by the condition $2n_i \leq n_2$ here), the capacity region for the deterministic model has been characterized in the previous section, and from these results, it can be seen that the maximum sum rate is given by

$$R_\Sigma \leq n_1 + n_2 - 2n_i + \varphi(n_i, \Delta). \tag{5.61}$$

We now derive outer bounds on the achievable sum rate for scenarios with arbitrarily strong interference.

Outer Bounds on Sum Rate

Before stating the outer bounds, we introduce some definitions. Throughout the following, we let $\alpha := \frac{n_i}{n_1}, \beta := \frac{n_2}{n_1} \leq 1$, $\sigma := 2n_i - n_1$ and $\tau := 2n_i - n_2$. Furthermore, recall that for $p, q \in \mathbb{R}_+$, we defined $l(p,q) = \left\lfloor \frac{p}{q} \right\rfloor$ for $q > 0$ and $l(p,0) = 0$. Moreover, we define the function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as (note that 0 is considered an even number)

$$\omega(p,q) := \begin{cases} p - \frac{l(p,q)q}{2}, & \text{if } \ l(p,q) \ \text{ is even} \\ \frac{(l(p,q)+1)q}{2}, & \text{if } \ l(p,q) \ \text{ is odd.} \end{cases} \tag{5.62}$$

Recall that the function φ is defined in a similar manner in (5.28). Moreover, let $\overline{\alpha} := \min\left(1 - \frac{\beta}{2}, \frac{2}{3}\right)$. Then, we have the following outer bound on the achievable sum rate:

Proposition 5. *For $\alpha < \beta$, the achievable sum rate for the A-MAC-P2P is bounded by*

$$R_\Sigma \leq \begin{cases} n_1 + n_2 - 2n_i + \varphi(n_i, \Delta), & \alpha \in \left[0, \frac{1}{2}\right] \\ 2n_i + \omega(n_2 - n_i, \Delta), & \alpha \in \left(\frac{1}{2}, \overline{\alpha}\right) \\ 2n_i + \omega(2n_1 - 3n_i, \Delta), & \alpha \in \left[\overline{\alpha}, \frac{2}{3}\right) \\ \min\left(2n_1, \max(n_1, n_i) + (n_1 - n_i)^+\right), & \alpha \in \left[\frac{2}{3}, \infty\right). \end{cases} \tag{5.63}$$

For $\alpha \geq \beta$, it holds that

$$R_\Sigma \leq \min\left[\max(n_1, n_i) + (n_1 - n_i)^+, 2 \cdot \max(n_i, (n_1 - n_i)^+)\right]. \tag{5.64}$$

We remark that the bounds for $\alpha \geq \beta$ and $\alpha < \beta, \alpha \geq \frac{2}{3}$ coincide with the sum capacity for the interference channel formed by Tx_1, Tx_3 and Rx_1, Rx_2.

Proof. In order to prove the outer bounds on the sum rate, we again apply Fano's inequality: for each triple of achievable rates (R_1, R_2, R_3), Fano's inequality implies that there exists a sequence ε_N with $\varepsilon_N \to 0$ for $N \to \infty$ and a sequence of joint factorized

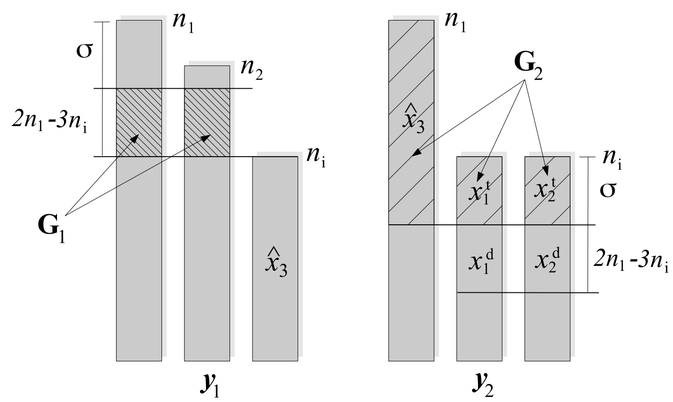

Figure 5.7: Illustration of received signals and the signal parts used for upper bounding the sum rate for the case $\alpha \in \left[\overline{\alpha}, \frac{2}{3}\right)$ (and $\alpha < \beta$).

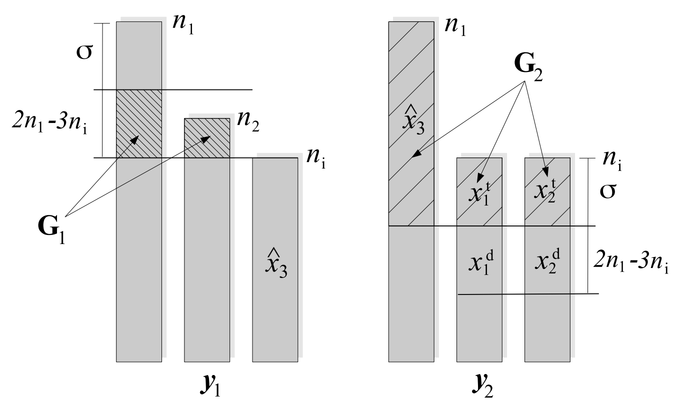

Figure 5.8: Illustration of received signals and the signal parts used for upper bounding the sum rate for the case $\alpha \in \left(\frac{1}{2}, \overline{\alpha}\right)$ (and $\alpha < \beta$).

distributions on the input sequences such that for all $N \in \mathbb{N}, N \geq 1$

$$R_1 + R_2 \leq \frac{1}{N} I(\mathbf{x}_1^N, \mathbf{x}_2^N; \mathbf{y}_1^N) + \varepsilon_N, \tag{5.65}$$

$$R_3 \leq \frac{1}{N} I(\mathbf{x}_3^N; \mathbf{y}_2^N) + \varepsilon_N. \tag{5.66}$$

The upper bound obtained in this way is bounded further in different ways for the different channel gain regimes determined by α and β.

i) The case $\alpha < \beta$**:** We start with the case $\alpha \geq \frac{2}{3}$. Here, the bound is obtained by letting a genie provide $\mathbf{x}_1^N$ and $\mathbf{x}_2^N$ to receiver Rx_2, which results in

$$\begin{aligned}&(R_1+R_2+R_3)N-\varepsilon_N N \\ &\le I(\mathbf{x}_1^N,\mathbf{x}_2^N;\mathbf{y}_1^N)+I(\mathbf{x}_3^N;\mathbf{y}_2^N,\mathbf{x}_1^N,\mathbf{x}_2^N) \\ &= H(\mathbf{y}_1^N)+H(\mathbf{x}_3^N)-H(\mathbf{y}_1^N|\mathbf{x}_1^N,\mathbf{x}_2^N) \\ &\le N\min\left(2n_1,\max(n_1,n_i)+(n_1-n_i)^+\right).\end{aligned} \tag{5.67}$$

For $\alpha\in\left(\frac{1}{2},\frac{2}{3}\right)$, we write $\mathbf{G}_1 := \mathbf{y}_1^N[\sigma+1:\sigma+\tau]$, $\mathbf{G}_2 := \mathbf{y}_2^N[1:n_i]$, $\widehat{\mathbf{x}}_3 := \mathbf{x}_3^N[1:n_i]$, $\mathbf{x}_1^d := \mathbf{x}_1^N[\sigma+1:n_1-n_i], \mathbf{x}_2^d := \mathbf{x}_2^N[\sigma+1:n_1-n_i]$, $\mathbf{x}_1^t := \mathbf{x}_1^N[1:\sigma]$ and $\mathbf{x}_2^t := \mathbf{x}_2^N[1:\sigma]$ (cf. Figure 5.8 and Figure 5.7). Then, we have

$$\begin{aligned}&(R_1+R_2+R_3)N-\varepsilon_N N \\ &\le I(\mathbf{x}_1^N,\mathbf{x}_2^N;\mathbf{y}_1^N,\mathbf{G}_1)+I(\mathbf{x}_3^N;\mathbf{y}_2^N,\mathbf{G}_2) \\ &= H(\mathbf{G}_1)+H(\mathbf{y}_1^N|\mathbf{G}_1)-H(\mathbf{y}_1^N|\mathbf{G}_1,\mathbf{x}_1^N,\mathbf{x}_2^N)+H(\mathbf{G}_2) \\ &\quad -H(\mathbf{G}_2|\mathbf{x}_3^N)+H(\mathbf{y}_2^N|\mathbf{G}_2)-H(\mathbf{y}_2^N|\mathbf{G}_2,\mathbf{x}_3^N) \\ &= H(\mathbf{G}_1)+H(\mathbf{y}_1^N|\mathbf{G}_1)-H(\widehat{\mathbf{x}}_3)+H(\mathbf{G}_2) \\ &\quad -H(\mathbf{x}_1^t\oplus\mathbf{x}_2^t)+H(\mathbf{y}_2^N|\mathbf{G}_2)-H(\mathbf{y}_2^N|\mathbf{G}_2,\mathbf{x}_3^N) \\ &\le 2n_iN+H(\mathbf{G}_1)-H(\mathbf{x}_1^d\oplus\mathbf{x}_2^d),\end{aligned} \tag{5.68}$$

where we used $H(\mathbf{y}_1^N|\mathbf{G}_1)\le N(\sigma+n_i)$, $H(\mathbf{y}_2^N|\mathbf{G}_2)\le N(n_1-n_i)$, $H(\mathbf{G}_2)-H(\widehat{\mathbf{x}}_3)\le H(\widehat{\mathbf{x}}_3)+H(\mathbf{x}_1^t\oplus\mathbf{x}_2^t)-H(\widehat{\mathbf{x}}_3)=H(\mathbf{x}_1^t\oplus\mathbf{x}_2^t)$ and $H(\mathbf{y}_2^N|\mathbf{G}_2,\mathbf{x}_3^N)\ge H(\mathbf{x}_1^d\oplus\mathbf{x}_2^d)$. Now

$$H(\mathbf{G}_1)-H(\mathbf{x}_1^d\oplus\mathbf{x}_2^d)\le\begin{cases}N\omega(2n_1-3n_i,\Delta), & \text{if } \sigma\ge\Delta \\ N\omega(n_2-n_i,\Delta), & \text{if } \sigma<\Delta,\end{cases} \tag{5.69}$$

which for $\sigma\ge\Delta$ follows from applying Lemma 7 given below. A slight modification of Lemma 7 shows the other case. Note that $\sigma<\Delta$ corresponds to $\alpha\in\left(\frac{1}{2},\overline{\alpha}\right)$ and $\sigma\ge\Delta$ to $\alpha\in\left[\overline{\alpha},\frac{2}{3}\right)$.

The bound for $\alpha\in\left[0,\frac{1}{2}\right)$ is proved by a similar argument, which we sketch only here: $\mathbf{G}_1$ and $\mathbf{G}_2$ are taken as $\mathbf{G}_1 := \mathbf{y}_1^N[1:n_1-n_i]$ and $\mathbf{G}_2 := \mathbf{y}_2^N[1:n_i]$, respectively and by an appropriate adjustment of Lemma 7, the bound follows. We remark that for $\alpha\in\left[0,\frac{\beta}{2}\right)$, the outer bound also follows from the results for the weak interference case.

ii) The case $\alpha\ge\beta$: The bound $R_\Sigma\le 2\cdot\max(n_i,(n_1-n_i)^+)$ follows easily by providing the genie information $\mathbf{G}_1 := \mathbf{x}_1^N[1:\min(n_1,n_i)]$ and $\mathbf{G}_2 := \mathbf{x}_1^N[1:\min(n_1,n_i)]$ to receiver Rx_1 and Rx_2, respectively. The other bound $R_\Sigma\le\min(2n_1,\max(n_1,n_i)+(n_1-n_i)^+)$ has already been shown above in (5.67). □

Lemma 7. *Let* $\mathbf{A} \in \mathbb{F}_2^{n\times m}$, $\mathbf{B} \in \mathbb{F}_2^{n+\Delta\times m}$ *be independent random matrices with* $m, n, \Delta \in \mathbb{N}, m, n \geq 1$ *and* $\mathbf{B}' := \mathbf{B}[1:n], \mathbf{B}'' := \mathbf{B}[\Delta+1:n+\Delta]$. *Then, it holds that*

$$H(\mathbf{A} \oplus \mathbf{B}') - H(\mathbf{A} \oplus \mathbf{B}'') \leq m\omega(n, \Delta). \tag{5.70}$$

Proof. We let $l := n \text{ div } \Delta, Q := n \bmod \Delta$ and introduce the following labels for blocks of rows of the matrices $\mathbf{A}, \mathbf{B}'$ and $\mathbf{B}''$: $\mathbf{A} = [(\mathbf{A}_k)_{k=1}^{l}; \mathbf{Q}_\mathbf{A}], \mathbf{B}' = [(\mathbf{B}_k)_{k=0}^{l-1}; \mathbf{Q}'_\mathbf{B}], \mathbf{B}'' = [(\mathbf{B}_k)_{k=1}^{l}; \mathbf{Q}''_\mathbf{B}]$, where $\mathbf{Q}_\mathbf{A}, \mathbf{Q}'_\mathbf{B}, \mathbf{Q}''_\mathbf{B} \in \mathbb{F}_2^{Q\times m}$ and $\mathbf{A}_k, \mathbf{B}_k \in \mathbb{F}_2^{\Delta\times m}$. First consider the case that l is even. Here, we have

$$\begin{aligned}
&H(\mathbf{A} \oplus \mathbf{B}') - H(\mathbf{A} \oplus \mathbf{B}'') \qquad (5.71)\\
&\leq m\Delta + H\left[(\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l-1}; \mathbf{Q}_\mathbf{A} \oplus \mathbf{Q}'_\mathbf{B}\right] - H\left[(\mathbf{A}_k \oplus \mathbf{B}_k)_{k=1}^{l}; \mathbf{Q}_\mathbf{A} \oplus \mathbf{Q}''_\mathbf{B}\right]\\
&\leq m\Delta + H\left[(\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l-1}; \mathbf{Q}_\mathbf{A} \oplus \mathbf{Q}'_\mathbf{B}\right] - H\left[(\mathbf{A}_k \oplus \mathbf{B}_k)_{k=1}^{l} \,\middle|\, (\mathbf{A}_{2k-1})_{k=1}^{l/2}, (\mathbf{B}_{2k})_{k=1}^{l/2}\right]\\
&= m\Delta + H\left[(\mathbf{A}_{k+1} \oplus \mathbf{B}_k)_{k=1}^{l-1}; \mathbf{Q}_\mathbf{A} \oplus \mathbf{Q}'_\mathbf{B}\right] - H\left[(\mathbf{A}_{2k})_{k=1}^{l/2}; (\mathbf{B}_{2k-1})_{k=1}^{l/2}\right]\\
&\leq m\frac{l\Delta}{2} + mQ + H\left[(\mathbf{A}_{2k} \oplus \mathbf{B}_{2k-1})_{k=1}^{l/2}\right] - H\left[(\mathbf{A}_{2k})_{k=1}^{l/2}; (\mathbf{B}_{2k-1})_{k=1}^{l/2}\right]\\
&\leq m\frac{l\Delta}{2} + mQ = m\omega(n, \Delta),
\end{aligned}$$

where the last inequality is due to the independence of $\mathbf{A}$ and $\mathbf{B}$. A similar line of argument can be applied for the case that l is odd. □

Achievability

We now describe how the outer bounds given in Proposition 5 can be achieved. It turns out that it suffices to restrict to *linear coding* over a single symbol period (cf. the definitions in Section 5.1). For this, each transmitter Tx_i chooses a coding matrix $\mathbf{V}_i \in \mathbb{F}_2^{k_i\times q}$ and transmits the signal $\mathbf{V}_i\mathbf{x}_i$, where $\mathbf{x}_i \in \mathbb{F}_2^{k_i\times 1}$ represents the message. Let us write $\mathbf{A} := \mathbf{V}_1$, $\mathbf{B} := \mathbf{S}^{q-n_2}\mathbf{V}_2$, $\mathbf{C} := \mathbf{S}^{q-n_i}\mathbf{V}_3$, $\mathbf{D} := \mathbf{S}^{q-n_i}\mathbf{V}_1$, $\mathbf{E} := \mathbf{S}^{q-n_i}\mathbf{V}_2$ and $\mathbf{F} := \mathbf{V}_3$. Then the following rate R_1 is achievable:

$$\begin{aligned}
R_1 &= \dim\left(\langle\mathbf{A}\rangle + \langle\mathbf{B}\rangle + \langle\mathbf{C}\rangle\right) - \dim\left(\langle\mathbf{B}\rangle + \langle\mathbf{C}\rangle\right) &(5.72)\\
&= \operatorname{rank}([\mathbf{A}\ \mathbf{B}\ \mathbf{C}]) - \operatorname{rank}([\mathbf{B}\ \mathbf{C}]). &(5.73)
\end{aligned}$$

Similarly, we can achieve the rates

$$\begin{aligned}
R_2 &= \operatorname{rank}([\mathbf{A}\ \mathbf{B}\ \mathbf{C}]) - \operatorname{rank}([\mathbf{A}\ \mathbf{C}]), &(5.74)\\
R_3 &= \operatorname{rank}([\mathbf{D}\ \mathbf{E}\ \mathbf{F}]) - \operatorname{rank}([\mathbf{D}\ \mathbf{E}]). &(5.75)
\end{aligned}$$

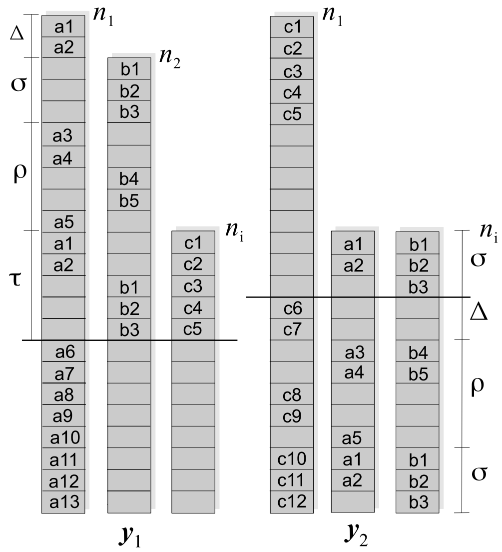

Figure 5.9: Illustration of the code resulting from the coding matrices given in (5.76) for $n_1 = 23, n_2 = 21, n_i = 13$, achieving the outer bound $R_\Sigma = 30$ (with $R_1 = 13, R_2 = 5, R_3 = 12$). Again, the bars represent the bit vectors as seen at the two receivers and a_i, b_i and c_i are the data bits for Tx_1, Tx_2 and Tx_3, respectively.

In the following, we sketch how to construct coding matrices $\mathbf{V}_i$ that achieve the outer bounds. The construction of the coding matrices again depends on the channel parameters.

For the cases $\alpha \geq \beta$ and $\alpha < \beta, \alpha \geq \frac{2}{3}$, the maximum sum-rate can be achieved by keeping transmitter Tx_2 silent and applying the sum-rate optimal interference channel code for the interference channel formed by Tx_1, Tx_3 and Rx_1, Rx_3. The results from [BT08] show that in this way, the bounds can be achieved (by linear coding).

For $\alpha < \beta, \alpha \in [0, \frac{\beta}{2}]$, an optimal construction follows from the weak interference case, and the extension to the case $\alpha < \beta, \alpha \in (\frac{\beta}{2}, \frac{1}{2}]$ is straightforward. Here, the maximum sum rate can be achieved by orthogonal coding, where for each transmitter, a set of bit levels to be used for data transmission is specified such that at the intended receiver, there is no overlap of these levels with levels used by any other transmitter. Recall that the assignment can be interpreted as interference alignment: the bit levels are chosen such that the interference caused by Tx_1 and Tx_2 aligns at Rx_2 as much as possible in the levels unused by Tx_3.

For the remaining case $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, the maximum sum rate can not be obtained by orthogonal coding; instead, *coding across levels* is necessary (as is for the interference channel in a certain interference range [BT08]). However, interference alignment still plays a key role. To describe the construction, we let $\rho \coloneqq n_i - \sigma - \tau$ in what follows. For the construction, three cases have to be distinguished: a) $\rho < 0$, b) $\rho \geq 0$ and $\alpha \in \left(\frac{1}{2}, \overline{\alpha}\right)$ and c) $\rho \geq 0$ and $\alpha \in \left[\overline{\alpha}, \frac{2}{3}\right)$. We only consider case c) here in detail; the constructions for a) and b) are similar. For $n, \Delta \in \mathbb{N}$, we let $\mathcal{K}_n^{\Delta} \coloneqq \{k \in \{1, \dots, n\} : \mathrm{mod}(k, 2\Delta) < \Delta\}$ and for an element $k \in \mathcal{K}_n^{\Delta}$, $i_n^{\Delta}(k)$ denotes the position of the element in the sequence of increasingly ordered elements in $\mathcal{K}_n^{\Delta}$. Then, coding matrices achieving the outer bound can be constructed as follows:

$$\begin{aligned}
\mathbf{V}_1(k,k) &= 1, 1 \leq k \leq \Delta \\
\mathbf{V}_1(n_1 - n_i + k, k) &= 1, 1 \leq k \leq \Delta \\
\mathbf{V}_1(\tau + k, \Delta + i_{\rho}^{\Delta}(k)) &= 1, k \in \mathcal{K}_{\rho}^{\Delta} \\
\mathbf{V}_1(n_i + \Delta + k, \Delta + |\mathcal{K}_{\rho}^{\Delta}| + k) &= 1, 1 \leq k \leq n_2 - n_i \\
\mathbf{V}_2(k,k) &= 1, 1 \leq k \leq \sigma \\
\mathbf{V}_2(n_1 - n_i + k, k) &= 1, 1 \leq k \leq \sigma \\
\mathbf{V}_2(\tau + k, \sigma + i_{\rho-\Delta}^{\Delta}(k)) &= 1, k \in \mathcal{K}_{\rho-\Delta}^{\Delta} \\
\mathbf{V}_3(k,k) &= 1, 1 \leq k \leq \tau \\
\mathbf{V}_3(n_i + k, \tau + i_{\rho+\Delta}^{\Delta}(k)) &= 1, k \in \mathcal{K}_{\rho+\Delta}^{\Delta} \\
\mathbf{V}_3(2(n_1 - n_i) + k, \tau + |\mathcal{K}_{\rho+\Delta}^{\Delta}| + k) &= 1, 1 \leq k \leq \sigma.
\end{aligned} \tag{5.76}$$

An example for this coding scheme is given in Figure 5.9 for the A-MAC-P2P with parameters $n_1 = 23, n_2 = 21$ and $n_i = 13$. We summarize the achievability results in the following theorem:

Theorem 8. *The sum capacity of the A-MAC-P2P equals the sum rate bounds given in Proposition 5.*

5.2.3 Generalized Degrees of Freedom

In this section, we investigate the connection to the Gaussian channel given in (5.60). The input-output equations can equivalently be written as

$$\begin{aligned}
Y_1 &= \sqrt{p}X_1 + \sqrt{p^b}X_2 + \sqrt{p^a}X_3 + Z_1, \\
Y_2 &= \sqrt{p^a}X_1 + \sqrt{p^a}X_2 + \sqrt{p}X_3 + Z_2
\end{aligned} \tag{5.77}$$

with $p, a, b \in \mathbb{R}_+$, $b \leq 1$, power constraints $P_i = 1$ and Gaussian noise $Z_i \sim \mathcal{N}(0,1)$. In this model, the parameter $a \in \mathbb{R}_+$ specifies the interference strength, relative to the stronger direct link in the MAC. On the other hand, $b \in [0,1]$ determines the relation of the link strengths of the two direct links in the MAC. Then, the *generalized degrees of freedom* [ETW08] are defined as

$$d(a,b) := \limsup_{p\to\infty} \frac{C_\Sigma(p,a,b)}{\frac{1}{2}\log(p)}, \tag{5.78}$$

where $C_\Sigma(p,a,b)$ is the sum capacity of the channel. This measure represents a high SNR description of the system, where the channel gains are kept in constant relation (determined by a and b) in the dB scale, which allows a more detailed asymptotic description of the system as opposed to the degrees of freedom characterization (also see e.g. [BT08]).

The technique given in [JV10], [CJS09] (and also applied in e.g. [HCJ09]) allows to transfer an achievable scheme for the deterministic model to a coding scheme for the Gaussian channel. The basic idea is to inscribe the code obtained for the LDM into a Q-ary expansion of the transmit signals, asymptotically mimicking the behavior of the deterministic channel. The information about the message is contained in the digits of the transmitted symbols: generally, each symbol of a block code for the LDM is encoded into a real number which represents the corresponding code symbol in the Gaussian channel. Note that since the scheme presented above for the A-MAC-P2P operates on a single time slot (i.e., block length one), the corresponding Gaussian codewords consisting of merely one symbol (real number).

This code construction method can, in principle, also be applied for the channel at hand. However, we restrict to rational parameters $a, b \in \mathbb{Q}$ here. We outline the code construction and the achievability proof in the following. For this, we write $a, b \in \mathbb{Q}$ as

$$a = \frac{s_a}{t},\ b = \frac{s_b}{t}. \tag{5.79}$$

with $s_a, s_b, t \in \mathbb{N}$. We assume $a \leq 1$; for $a > 1$, the bound given in Proposition 6 follows from the generalized degrees of freedom results [ETW08] for the interference channel consisting of Tx_1, Tx_3 and Rx_1, Rx_2. We define a sequence of channels, each specified by an integer k: we define

$$\varrho_k := Q^{2kt} \tag{5.80}$$

for some (large) constant expansion base $Q \in \mathbb{N}$. Then

$$\frac{\sqrt{\varrho_k^a}}{\sqrt{\varrho_k}} = Q^{k(s_a - t)} \text{ and } \frac{\sqrt{\varrho_k^b}}{\sqrt{\varrho_k}} = Q^{k(s_b - t)}. \tag{5.81}$$

We note that the technique given in [JV10] and [HCJ09] cannot be applied directly if a or b are irrational numbers, i.e., if $a \in \mathbb{R} \setminus \mathbb{Q}$ or $b \in \mathbb{R} \setminus \mathbb{Q}$. This is because in these cases,

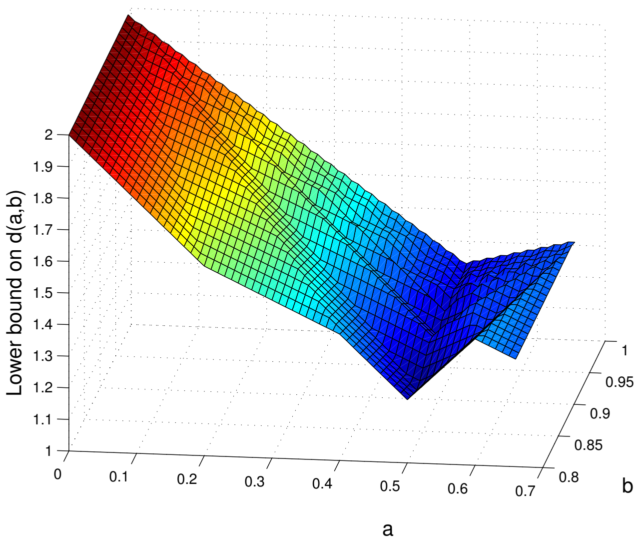

Figure 5.10: Achievable generalized degrees of freedom for a range of parameters $a, b \in \mathbb{Q}$.

we have to operate with rational approximation sequences converging to a and b: $\frac{s_a^k}{t^k} \to a$ and $\frac{s_b^k}{t^k} \to b$. Proving the convergence to a channel that asymptotically behaves as a LDM using the techniques from [JV10] seems not to be possible immediately.

Now, it is clear that the coding scheme for the LDM described above can easily be extended from operating on $\mathbb{F}_2$ to a greater field size $\mathbb{F}_{\widetilde{Q}}$. Consider the (linear deterministic) A-MAC-P2P specified by channel parameters $n_1 = kt, n_2 = ks_b$ and interference strength $n_i = ks_a$. Then a coding scheme that achieves the sum capacity has transmit signals $\mathbf{x}_i \in \mathbb{F}_{\widetilde{Q}}^{kt}$ for $i \in \{1, 2, 3\}$, with the lth bit level being referred to as $x_i(l) \in \mathbb{F}_{\widetilde{Q}}$. As in [HCJ09], we choose $\widetilde{Q}$ sufficiently small in order to avoid carry-overs form interfering terms and form the actual (physical) transmit symbol X_i for transmitter Tx_i by inscribing the LDM code in the Q-ary expansion of the signal:

$$X_i = \frac{1}{\sqrt{\varrho_k}} \sum_{l=0}^{kt-1} x_i(l) Q^l =: \frac{1}{\sqrt{\varrho_k}} \widetilde{X}_i. \tag{5.82}$$

Again as in [HCJ09], each transmit signal X_i satisfies the power constraint $\mathbf{E}[X_i^2] \leq 1$, and decoding at the receiver side is performed by reconstructing the Q-ary expansion of

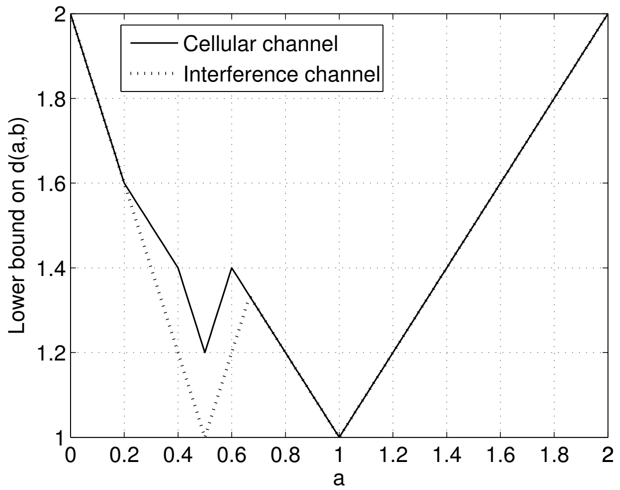

Figure 5.11: Achievable generalized degrees of freedom for $b = 0.8$ and $a \in \mathbb{Q}$.

the received signal. The received signals Y_1 and Y_2 in the Gaussian channel are

$$Y_1 = \widetilde{X}_1 + Q^{k(s_b-t)}\widetilde{X}_2 + Q^{k(s_a-t)}\widetilde{X}_3 + Z_1, \tag{5.83}$$

$$Y_2 = Q^{k(s_a-t)}\widetilde{X}_1 + Q^{k(s_a-t)}\widetilde{X}_2 + \widetilde{X}_3 + Z_2. \tag{5.84}$$

In the asymptotic limit (for $k \to \infty$), the effects of noise terms Z_1, Z_2 disappear, and the effective asymptotic behavior of the channel is

$$Y_1 = \widetilde{X}_1 + Q^{k(s_b-t)}\widetilde{X}_2 + Q^{k(s_a-t)}\widetilde{X}_3, \tag{5.85}$$

$$Y_2 = Q^{k(s_a-t)}\widetilde{X}_1 + Q^{k(s_a-t)}\widetilde{X}_2 + \widetilde{X}_3. \tag{5.86}$$

This essentially implements the LDM model (on the bits of the Q-ary expansion of the signals), in which we can achieve the sum rate

$$R_\Sigma^{\mathrm{LDM}}(kt, ks_a, ks_b)\log_Q \widetilde{Q} = kR_\Sigma^{\mathrm{LDM}}(t, s_b, s_b)\log_Q \widetilde{Q}, \tag{5.87}$$

where $R_\Sigma^{\mathrm{LDM}}(n_1, n_2, n_i)$ denotes the sum capacity of the A-MAC-P2P with parameters n_1, n_2, n_i and which is given in Theorem 8. In the Gaussian channel, we can achieve the sum rate (using the same argument as in [HCJ09])

$$R_\Sigma = kR_\Sigma^{\mathrm{LDM}}(t, s_b, s_a)\log_Q \widetilde{Q} + o(k), \tag{5.88}$$

implying the generalized degrees of freedom bound

$$d(a,b) \geq \limsup_{k\to\infty} \frac{kR_{\Sigma}^{\mathrm{LDM}}(t,s_b,s_a)\log_Q \widetilde{Q} + o(k)}{\frac{1}{2}\log_Q \varrho_k} \tag{5.89}$$

$$= \limsup_{k\to\infty} \frac{R_{\Sigma}^{\mathrm{LDM}}(t,s_b,s_a)\log_Q \widetilde{Q} + o(k)}{t} \tag{5.90}$$

$$= \frac{1}{t} R_{\Sigma}^{\mathrm{LDM}}(t,s_b,s_a)\log_Q \widetilde{Q}. \tag{5.91}$$

Considering that Q can be chosen arbitrarily large and plugging in the sum capacity expression given by Theorem 8, one obtains the following lower bound on the generalized degrees of freedom:

Proposition 6. *Let $a, b \in \mathbb{Q}$ and $b \leq 1$. For $a < b$, it holds that*

$$d(a,b) \geq \begin{cases} 1 + b - 2a + \varphi(a, 1-b), & a \in \left[0, \frac{1}{2}\right] \\ 2a + \omega(b-a, 1-b), & a \in \left(\frac{1}{2}, \overline{a}\right) \\ 2a + \omega(2-3a, 1-b), & a \in \left[\overline{a}, \frac{2}{3}\right) \\ \min\left(2, \max(1,a) + (1-a)^+\right), & a \in \left[\frac{2}{3}, \infty\right). \end{cases} \tag{5.92}$$

For $a \geq b$,

$$d(a,b) \geq \min\left(\max(1,a) + (1-a)^+, 2\max(a,(1-a)^+)\right). \tag{5.93}$$

Figure 5.10 displays the lower bound on $d(a,b)$ in the range $a \in [0,0.7] \cap \mathbb{Q}, b \in [0.8,1] \cap \mathbb{Q}$. For $b = 0.8$, the achievable generalized degrees of freedom are shown in Figure 5.11 for different a values, together with the generalized degrees of freedom for the interference channel consisting of only Tx_1, Tx_3 and Rx_1, Rx_2. Note that the latter one represents the well-known *W curve* [ETW08] of the generalized degrees of freedom for the (symmetric) interference channel. For $a < \frac{2}{3}$, the channel gain difference in the two-user cell can be exploited for interference alignment, pushing the achievable generalized degrees of freedom higher than the W curve, whereas for $a \geq \frac{2}{3}$, the lower bound can be achieved by coding only for the interference channel consisting of Tx_1, Tx_3 and Rx_1, Rx_2. Note that, for a fixed b and for $a < \frac{2}{3}$, the achievable generalized degrees of freedom have a "zigzag" shape, which is due to the definitions of the functions φ and ω, respectively. From these definitions, and as can also be observed in Figure 5.10, the period of the "zigzag" oscillation becomes smaller when b approaches 1. This reflects the fact that the closer b is to 1, the "finer" is the resolution of the Δ-blocks used to implement interference alignment in the LDM.

5.3 Duality for Linear Deterministic Channels

In this section, we return to the topic of Chapter 4, namely duality relations between different network structures. There are some interesting aspects to be considered in the context of the LDM: first of all, a quite general duality holds for arbitrary networks under the restriction to linear coding. Secondly, there are duality results for some specific LDM networks. Finally, the LDM can also be applied in order to obtain approximate duality results for Gaussian channels; here, duality holds to within a constant number of bits.

The work in [RPV09] studies general networks defined by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. As a generalization of the LDM network as in (5.6), each link $(i, j) \in \mathcal{E}$ is specified by a channel gain matrix $G_{ji} \in \mathbb{F}_p^{q\times q}$, and the received signal at (discrete) time t at node j is given by

$$\mathbf{y}_j(t) = \sum_{i:(i,j)\in\mathcal{E}} G_{ji}\mathbf{x}_i(t) \tag{5.94}$$

for transmit signals $\mathbf{x}_i(t)$. Encoding, decoding and achievable rates are defined as in Section 5.1 for the LDM. The *dual* or *reciprocal* network is obtained as $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ by reversing the link and message directions, i.e., $(i, j) \in \mathcal{E}' :\Leftrightarrow (j, i) \in \mathcal{E}$ and $(i, j) \in \mathcal{M}' :\Leftrightarrow (j, i) \in \mathcal{M}$ (cf. Section 5.1). Moreover, the channel gain matrix G'_{ij} for link $(i, j) \in \mathcal{E}'$ is given by $G'_{ji} = G^T_{ij}$. Then, duality holds in the following sense [RPV09]:

Any rate vector that is achievable by linear coding is also achievable by linear coding in the corresponding dual network.

Note that setting $G_{ji} = \mathbf{S}^{q-n_{ji}}$, one obtains the LDM (5.6). The dual network as defined above operates with upshift matrices instead of downshift matrices. However, this is just a technical issue since the direction of the shift is unimportant—duality also holds with respect to the regular (downshift) dual model [RPV09].

The duality described above is shown under the restriction to linear coding strategies. In the literature, duality results for the LDM with respect to the capacity region have also been reported for some specific networks. We briefly discuss some of these results here.

The first set of duality results of interest here is concerned with *cooperation* in interference channels. These models extend the interference channel as described above (cf. Section 5.1) and allow the receiving nodes or the transmitting node to cooperate. These channels are motivated by the current research for cooperating base stations. The work in [WT11a] and [WT11b] deals with the case of *out-of-band* cooperation, where a dedicated cooperation channel of finite capacity is reserved for information exchange between the transmitters and the receivers, respectively. For transmitter cooperation, a so-called conference over the cooperation link is carried out between the two source nodes during

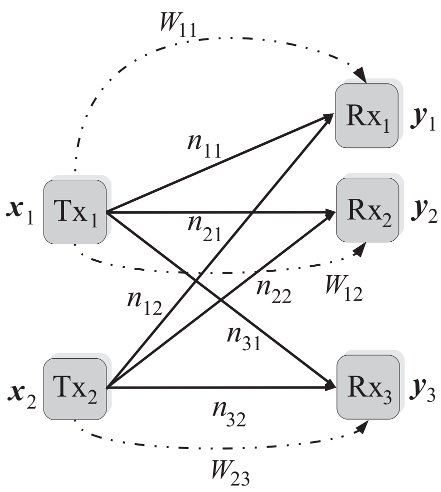

Figure 5.12: The dual channel for the MAC-P2P: a BC interfering with a point-to-point link.

the encoding process. For receiver cooperation, the destination nodes similarly exchange cooperative messages prior to decoding. The network for transmitter and receiver cooperation are duals of each other in the sense as described above. Concerning duality, the work in [WT11a] and [WT11b] shows that the LDM capacity regions for transmitter cooperation and receiver cooperation are identical. Moreover, it provides a characterization of the capacity region for the corresponding Gaussian channel within a constant number of bits, also both for transmit and receive cooperation. Interestingly, a *constant-bit duality approximation* is shown as well: independent of the channel parameters, the capacity region of the Gaussian channels for transmitter cooperation and receiver cooperation are within a constant number of bits. Closely related to this work are the results in [PV11b] and [PV11a], which also study cooperation in the interference channel. The model differs from the one previously described in that the cooperation is performed in an *in-band* fashion, i.e., the cooperative links share the same band as the actual data transmission and hence interfere with the signals used for data communication. Besides providing constant-gap approximations for the sum capacity in the Gaussian channel, a duality result is given as well: the sum capacity for the LDM is the same both for transmitter and receiver cooperation. A different model is considered in [BPT10], namely the *many-to-one interference channel*. This is a K-user interference channel where only one receiver suffers from interference. The capacity region for the LDM is derived, along with a constant-gap capacity approximation for the Gaussian channel. The dual setup is the *one-to-many interference channel*, in which only one transmitter causes interference at other nodes. Again, duality holds for the LDM: the capacity for the many-to-one and the one-to-many interference channel are identical.

In the light of all these dualities known for the LDM, it is of interest to study the dual setup corresponding to the MAC-P2P channel as investigated in the previous section. The dual channel is given by a BC and a point-to-point link (cf. Figure 5.12) and has input-output equations

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{S}^{q-n_{11}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{12}}\mathbf{x}_2, \\
\mathbf{y}_2 &= \mathbf{S}^{q-n_{21}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{22}}\mathbf{x}_2, \\
\mathbf{y}_3 &= \mathbf{S}^{q-n_{31}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{32}}\mathbf{x}_2.
\end{aligned} \tag{5.95}$$

In the previous section, we have derived the capacity region for the case of weak interference (i.e., the W-MAC-P2P) and the sum capacity for arbitrary interference (for the A-MAC-P2P). These capacities can all be achieved by linear coding—in some cases, it even suffices to use orthogonal coding. Under the corresponding restrictions, we refer to the dual channels as the *W-BC-P2P* and the *A-BC-P2P*, respectively. Applying the above duality result for linear coding [RPV09], one immediately obtains the following achievable rates for the dual channels:

Corollary 3. *An achievable region for the W-BC-P2P is given by the set of points $(R_1, R_2, R_3) \in \mathbb{R}^3$ satisfying the constraints*

$$\begin{aligned}
R_i &\le n_i, \; i = 1, 2, 3 \\
R_1 + R_2 &\le n_1 \\
R_1 + R_3 &\le n_1 + n_3 - n_D - n_M \\
R_2 + R_3 &\le n_2 + n_3 - n_D - n_M \\
R_1 + R_2 + R_3 &\le n_2 + n_3 - n_D - n_M + \varphi(n_M, \Delta) \\
R_1 + R_2 + 2R_3 &\le n_1 + 2n_3 - n_D - n_M.
\end{aligned} \tag{5.96}$$

Corollary 4. *The following sum rates are achievable for the A-BC-P2P:*
For $\alpha < \beta$,

$$R_\Sigma = \begin{cases} n_1 + n_2 - 2n_i + \varphi(n_i, \Delta), & \alpha \in \left[0, \frac{1}{2}\right] \\ 2n_i + \omega(n_2 - n_i, \Delta), & \alpha \in \left(\frac{1}{2}, \overline{\alpha}\right) \\ 2n_i + \omega(2n_1 - 3n_i, \Delta), & \alpha \in \left[\overline{\alpha}, \frac{2}{3}\right) \\ \min\left(2n_1, \max(n_1, n_i) + (n_1 - n_i)^+\right), & \alpha \in \left[\frac{2}{3}, \infty\right). \end{cases} \tag{5.97}$$

For $\alpha \geq \beta$,

$$R_\Sigma = \min\left[\max(n_1, n_i) + (n_1 - n_i)^+, 2 \cdot \max(n_i, (n_1 - n_i)^+)\right]. \tag{5.98}$$

We conjecture that these achievable rates cannot be exceeded, i.e., actually give the capacities. However, the proof for this statement is still open. The difficulty is in finding appropriate modifications to Lemma 6 and Lemma 7 for bounding the rates in the dual setup.

5.4 Summary and Conclusions

In view of the tremendous difficulties involved with an *exact* information-theoretic analysis of multi-user systems, there is a recent trend to an approximative characterization of channel capacity. One of the research avenues that arose in this context is the linear deterministic model introduced in [ADT11], here briefly referred to as LDM. This approach has its origin in looking at the alternation of the individual bits in the binary expansion of the transmit symbols when passing over an AWGN channel. The resulting simplified model describes the channel as a deterministic bit pipe, where a certain number of bits are passed over the channel unchanged, whereas the other bits are erased completely. The number of bits that are transferred over the channel roughly corresponds to one half of the signal-to-noise ratio in logarithmic scale. Moreover, superposition of signals received is modeled by modulo-2 addition.

In this chapter, we first gave a brief introduction and review of the LDM and then analyzed a specific channel using the LDM: we discussed the LDM for the MAC-P2P, i.e., the two-user multiple access channel mutually interfering with a point-to-point link. This constitutes a basic model for a cellular channel: there are two users (in cell 1) transmitting to a receiver (base station 1), mutually interfering with a third transmitter (in cell 2) communicating with a second base station (base station 2). Note that this setup can also be regarded as a model for the situation of device-to-device communication underlaying a cell.

We first studied the case of symmetric weak interference where the interference links from cell 1 user to the cell 2 base station are identical and the sum of the interference gains are less or equal then the smallest direct link (the W-MAC-P2P). We derived the capacity region and the corresponding transmission scheme. Here, all the points in the capacity region can be achieved by orthogonal coding. Even though the system resembles the interference channel, the presence of the second link in cell 1 offers additional potential for aligning the interference caused by the two users at the receiver in the second cell. For this, the transmitted signals in cell 1 are chosen such that the interference at the second receiver aligns on the part of the signal that is unused by the transmitter in cell 2 as much as possible.

Furthermore, for the case of unrestricted interference and under certain symmetry assumptions on the channel gains (the A-MAC-P2P), we derived the sum capacity and the corresponding transmission schemes. These schemes also make use of interference alignment and linear coding, possibly across bit levels. While for a large parameter range, the sum capacity is identical to the sum capacity of the interference channel obtained by silencing the weaker user in the MAC, for a certain parameter range, the channel gain difference in the MAC allows to get a higher sum rate using interference alignment. From these results, a lower bound on the generalized degrees of freedom for the Gaussian channel was given for rational link ratio parameters a and b, increasing the so-called W curve for the interference channel in a certain interference range.

Although we made some restrictions on the channel models here, we believe that the achievability and converse arguments presented here give valuable insights for the consideration of less restricted systems. Future work should study extensions to the more general case with additional users. Another interesting direction for future investigations is to further explore the connections to the Gaussian equivalent of the channel, specifically concerning outer bounds on the generalized degrees of freedom of the system and approximate capacity characterizations.

Finally, it is still an interesting open issue if duality to the dual setup, i.e., the case of a BC interfering with a point-to-point link, holds. Since the coding schemes constructed for the W-MAC-P2P and the A-MAC-P2P are linear, existing duality results directly imply achievable rates for the corresponding dual channels. However, deriving the outer bounds required to show the duality conjecture is an interesting open task for future work. Related to this, it would be interesting to investigate if a constant-bit duality result (as it holds for the cooperating interference channel [WT11a], [WT11b]) can be derived for the corresponding Gaussian channels.

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