Technische Universität Berlin

FAKULTÄT II
MATHEMATIK UND
NATURWISSENSCHAFTEN
Institut für Mathematik

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No. 007/2006

Reducing the Optimality Gap of Strictly Fundamental Cycle Bases in Planar Grids*

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Abstract. The Minimum Cycle Basis (MCB) Problem is a classical problem in combinatorial optimization. An $O(m^2 n + mn^2 \log n)$-algorithm for this problem is known. Much faster heuristics have been examined in the context of several practical applications. These heuristics restrain the solution space to strictly fundamental cycle bases, hereby facing a significant loss in quality. We complement these experimental studies by giving theoretical evidence *why* strictly fundamental cycle bases (SFCB) in general must be much worse than general MCB.
Alon et al. (1995) provide the first non-trivial lower bound for the minimum SFCB problem, which in general is NP-hard. For unweighted planar square grid graphs they achieve a lower bound of $\frac{\ln 2}{2048} n \log_2 n - O(n)$, where $\frac{\ln 2}{2048} \approx \frac{1}{2955}$.
Using a new recursive approach, we are able to establish a substantially better lower bound. Our explicit method yields a lower bound of only $\frac{1}{12} n \log_2 n - O(n)$. In addition, we provide an exact way of counting a short SFCB that was presented by Alon et al. In particular, we improve their upper bound from $2n \log_2 n + o(n \log n)$ to only $\frac{4}{3} n \log_2 n - \Theta(n)$. We thus reduce the optimality gap for the MSFCB problem on planar square grids to a factor of 16—compared to about 5900 being the former state-of-the-art.
As a consequence, we conclude that for unweighted planar square grid graphs the ratio of the length of a minimum SFCB over a general MCB is $\Theta(\log n)$.

1 Introduction

The cycle space of an undirected graph G is the vector space spanned by the $\{0, 1\}$-incidence vectors of the circuits of G. Prominent—though specialized—cycle bases are the ones that are induced by the chords of spanning trees. These particular cycle bases are called strictly fundamental cycle bases.

Typically, cycle bases serve as input for algorithms that solve various practical applications. These arise in the fields of chemistry ([3]), electrical engineering ([6]), or periodic scheduling ([16]). More precisely, traffic light scheduling ([12,15]) and railway timetabling ([19]) are prominent applications of periodic scheduling. In general, the computation time of the algorithms for solving the above practical problems increases with the length of the used cycle basis, i.e. the sum over the weights of all the edges of the basic circuits. Hence, one is seeking for minimum cycle bases as a preprocessing step for solving these important real-world applications.

In 1987, Horton ([13]) presented the first polynomial algorithm for the minimum cycle basis (MCB) problem. Its complexity of $O(m^3 n)$ has been reduced in a series of recent contributions ([4,7,11]). The presently fastest algorithm has been presented in 2004, and it takes $O(m^2 n + mn^2 \log n)$ ([14]). Despite this improvement, there are real-world applications for which the computation time is still enormous. Therefore, several research groups seek for faster algorithms. Recently a new exact algorithm has been presented ([18]). But it has worse asymptotic

* Partially supported by the DFG Research Center MATHEON in Berlin and by the DFG Research Training Group GK-621 "Stochastic Modelling and Quantitative Analysis of Complex Systems in Engineering" (MAGSI).

complexity. Yet, in an empirical study it outperforms earlier algorithms, however, only on a very specific class of graphs. Alternatively, heuristics were considered. Speed-ups from those heuristics for the MCB problem would pay off, if the applications only face a minor slow down when fed with slightly worse cycle bases.

There are many heuristics for the MCB problem ([2,9,10,13]) that are much faster than the best exact algorithm. However, each of them is limited to only a subset of cycle bases of undirected graphs. More precisely, these heuristics produce weakly fundamental cycle bases—as they have been introduced by Whitney ([22]) in 1935—or even strictly fundamental cycle bases. The complete map of subclasses of cycle bases is drawn in [17].

There have been several empirical studies in which heuristics for short cycle bases were compared. It was observed that heuristics that are restricted to strictly fundamental cycle bases perform much worse than heuristics that also consider weakly fundamental cycle bases ([10]). Similarly, in [2] it was observed that the gap between a minimum weakly fundamental cycle basis and the strictly fundamental cycle bases that were generated by different heuristics may become very large. So far, in the context of MCB no theoretic explanation for these effects had been given.

It was shown already in 1982 that it is NP-hard to compute a minimum strictly fundamental cycle basis (MSFCB) of an (unweighted) undirected graph ([8]). In 1995, Alon, Karp, Peleg, and West ([1]) established that an MSFCB of the planar square grid G on n vertices has length $\Omega(n \log n)$. In more detail, its length is bounded from below by about $\frac{1}{2955} n \log_2 n$. Initially, the result of Alon et al. ([1]) has been obtained in the context of a graph game in which tree spanners are constructed. On the positive side, Alon et al. ([1]) introduce a family of spanning trees which induce short SFCB. They conjecture that these spanning trees are "essentially optimal, with $2n \log_2 n + o(n \log n)$" being their length.

Contribution. We present a new way of computing lower bounds on the length of a MSFCB of planar grids. With our recursive method we substantially improve on the presently known lower bound: from $\frac{1}{2955} n \log_2 n - O(n)$ to only $\frac{1}{12} n \log_2 n + O(n)$.[3] Notice that in Section 3.1 of this paper we conduct a concise analysis providing a lower bound of $\frac{1}{16} n \log_2 n + O(n)$, while a better—though more technical—approach is the main issue of Section 3.2.

Moreover, in Section 4 we perform an accurate counting for the spanning trees that have been considered in [1]. We find out that the exact length of these trees is $\frac{4}{3} n \log_2 n - \Theta(n)$, compared to $2n \log_2 n + o(n \log n)$, which has been conjectured in [1] being "essentially optimal." Hereby, we cut the optimality gap from about 5900 to only 16.

Finally, comparing our results and [1] to the unique general minimal cycle basis of the planar square grid—having length $\Theta(n)$—there is a non-constant asymptotic gap. We conclude that the bad quality experienced in approximating minimum cycle bases through the use of heuristics focused on strictly fundamental cycle bases ([2,10]) is rather due to the structure of strictly fundamental cycle bases than due to a fault of the particular heuristics.

2 Preliminaries

We consider cycle bases of a 2-connected simple undirected graph $G = (V, E)$. Define $n = |V|$, $m = |E|$, and $\nu = m - n + 1$, where ν is the *cyclomatic number* of G. Let C be a circuit (cf. [20, Ch. 3]) in G and denote by γ_C its $\{0,1\}$-incidence vector. The *cycle space* $\mathcal{C}$ of G is the following vector subspace over GF(2),

$$\mathcal{C} := \text{span}\left(\{\gamma_C \mid C \text{ circuit in } G\}\right).$$

[3] Note that the authors of [1] were not trying to optimize the constants.

A *cycle basis* B of G is a set of ν circuits of G whose incidence vectors are a basis of $\mathcal{C}$. The *length* $\Phi(B)$ of a cycle basis of an unweighted graph is defined as $\Phi(B) = \sum_{C \in B} |C|$. A *minimum cycle basis* (MCB) of a graph G is a cycle basis of G of minimum length.

A set of circuits $\{C_1, \dots, C_\nu\}$ such that

$$C_i \setminus (C_1 \cup \cdots \cup C_{i-1}) \neq \emptyset, \quad \forall i = 2, \dots, \nu$$

is clearly a cycle basis. We call such a basis *weakly fundamental*. Notice that these were already considered by Whitney ([22]) in 1935.

Let T be some spanning tree of G. Depending on the context, we either regard T as a subgraph of G or as a set of edges $T \subset E$. For $e \in E \setminus T$, we denote by $C_T(e)$—or C_e for short—the *fundamental circuit* that e induces with respect to T, i.e. the unique circuit in $T \cup \{e\}$. To T are associated ν fundamental circuits. These form a cycle basis which is called *strictly fundamental*. Here, we may write $\Phi(T)$ instead of $\Phi(B)$. A *minimum strictly fundamental cycle basis* (MSFCB) has minimum length among the set of strictly fundamental cycle bases.

In general, strictly fundamental cycle bases are a proper subset of weakly fundamental cycle bases, which in turn are a proper subset of general cycle bases of undirected graphs. Moreover, in general none of the three corresponding minimization problems coincide ([17]).

At the same time, given a spanning tree T of G and any edge $f \in T$, the graph $T_f := T \setminus \{f\}$ is a forest comprising precisely two trees with vertex sets S_f and $\overline{S}_f$ respectively. We denote by $\delta(S_f)$ the set of edges in E with precisely one end-vertex in S_f. This set $\delta(S_f)$ is called the *fundamental cut* of f with respect to T. To T are associated $n-1$ fundamental cuts. These form a cut basis (or co-cycle basis) which is called *strictly fundamental*. We denote by $\Psi(T) := \sum_{f \in T} |\delta(S_f)|$ the length of this strictly fundamental cut basis.

With $N \in \mathbb{N}$, the planar *grid graph* $G_{N,N}$ is the graph on $V = \{1, \dots, N\} \times \{1, \dots, N\}$ with

$$E = \{\{(i,j),(i',j')\} : |i-i'| + |j-j'| = 1\} = \{\{u,v\} : ||u-v||_1 = 1\}.$$

In a graphical representation, e.g. in an embedding into $\mathbb{Z}^2$, the first index of a vertex represents its x-coordinate, the second index its y-coordinate. The graph $G_{N,N}$ has $n = N^2$ vertices and contains $m = 2 \cdot N \cdot (N-1)$ edges. Its cyclomatic number ν is $(N-1)^2$. We collect some well-known simple properties of the cycle space of such grids.

Proposition 1. *The planar grid graph $G_{N,N}$ has a unique minimum cycle basis B. In B each basic circuit contains precisely four edges, thus $\Phi(B) = 4\,\nu = \Theta(n)$. The basis B is weakly fundamental. But for $N \geq 4$ B is* not *strictly fundamental.*

Now, consider the dual of an embedded planar graph G, which we will denote by G^*. For a primal grid of dimension $N \times N$ embedded into $\mathbb{Z}^2$, the graph G^* is again the graph of a square $(N-1) \times (N-1)$ grid plus a further vertex F^∞, which corresponds to the outer face of the initial embedded planar graph. This vertex is adjacent to all border-vertices of G^*. For the corner vertices $(1,1)$, $(1, N-1)$, $(N-1, 1)$, and $(N-1, N-1)$ there exist two parallel edges with the other endpoint being F^∞. Recall from [20, Ch. 3] that the edge set of G can be identified with the edge set of G^*.

Consider a spanning tree T of $G_{N,N}$ and its dual counterpart, that we denote by T^*. In fact, T^* can be understood as the complement of T, as it contains the counterpart in G^* of each edge in $E(G_{N,N}) \setminus T$. The graph T^* is a spanning tree of G^*, although it is not necessarily connected when restricted to $G^* \setminus \{F^\infty\}$.

The following key observation is well known (e.g. cf. [20, Ch. 3]). There is a one-to-one correspondence between fundamental circuits w.r.t. T in $G_{N,N}$ and fundamental cuts w.r.t T^*

in G^*. More precisely, $F \subseteq E(G_{N,N})$ is a fundamental circuit w.r.t. T in $G_{N,N}$ if and only if F itself is a fundamental cut w.r.t T^* in G^*. Therefore, bounding sizes of cuts in the dual is the same as bounding sizes of circuits in the primal, in particular

$$\Phi(T) = \Psi(T^*). \tag{1}$$

3 Lower bound

In this section we first show that every strictly fundamental cycle basis B of the square $N \times N$ grid with $n = N^2$ vertices satisfies $\Phi(B) \geq \frac{1}{16} n \log_2 n + O(n)$. Hereby, our direct approach substantially improves the lower bound that has been obtained in [1, Thm. 6.6]—by a factor of more than 245. In Sect. 3.2 we go one step further and establish a lower bound of $\frac{1}{12} n \log_2 n + O(n)$.

In contrast to [1] we decided to tackle the lower bound problem from the dual side. Here, some structural coherences, e.g. as elaborated in Lemma 2, can be seen better. For sake of convenience, we only consider grids of dimension $N - 1 = 2^k + 1$, with k integer. Note that this is the dual dimension, and $|V(G^*)| = N^2 - 2N + 2$. The corresponding primal grid is of size $n = (2^k + 2)^2$. With this particular definition of N it is much easier to follow the recursive approach that is to be explained.

The main ideas of our proof are as follows. We consider an arbitrary spanning tree T of the primal grid $G_{N,N}$. Instead of counting the length of the strictly fundamental cycle basis that it induces, we look at the length $\Psi(T^*)$ of the strictly fundamental cut basis of its dual tree T^*. In several iterations—which will be organized in levels—we consider sub-paths of T^* that start at certain specified vertices of the dual grid. Each edge of these paths induces a fundamental cut. Yet, we consider only those fundamental cuts that are induced by specific subsets of the edges of these paths. We will denote these subsets as pseudo-paths. For one such cut, Lemma 2 provides us with a lower bound on its contribution to $\Psi(T^*)$. As pseudo-paths of different levels do in general intersect, in Corollary 5 we finally identify values that we may sum over *all* levels.

As a first important tool we introduce pseudo-paths, the above mentioned subsets of paths. Consider two vertices $u = (i, j)$ and $v = (i', j')$ in $G^* \setminus \{F^\infty\}$ such that the unique u, v-path P in T^* does not contain F^∞ and $i \leq i'$. We now define a vertical and a horizontal *pseudo-path*, which exclusively consist of vertical and horizontal edges, respectively, that "lead from u to v". More precisely, to obtain the horizontal pseudo-path $P^H_{u,v}$ of P, we check whether $i = i'$, in which case we set $P^H_{u,v} = \emptyset$. Otherwise, starting from u we traverse the path P until we reach the first edge f with end-vertices $w_1 = (i_1, j_1)$ and $w_2 = (i_2, j_2)$ such that $i = i_1$, $j_1 = j_2$, and $|i_2 - i'| = |i_1 - i'| - 1$. Now we recursively define $P^H_{u,v} := \{f\} \cup P^H_{w_2,v}$. We define the *position* of an edge f' in $P^H_{u,v}$ as

$$\text{pos}(f', P^H_{u,v}) = \begin{cases} 1, & \text{if } f' = f, \\ \text{pos}(f', P^H_{w_2,v}) + 1, & \text{otherwise, i.e. } f' \in P^H_{w_2,v}. \end{cases}$$

An equivalent procedure defines the vertical pseudo-path $P^V_{u,v}$. Observe that in general $P^H_{u,v} \cup P^V_{u,v} \neq P$.

As an example, consider the dual graph T^* in Fig. 1(a) and the black vertex u in the center of the grid. Let v be the penultimate vertex of the u, F^∞-path. With this, $P^V_{u,v}$ exactly consists of the black edges, highlighted in Fig. 1(e).

Lemma 2. *Let $u = (i, j) \in V(G^*) \setminus \{F^\infty\}$ be some vertex in the dual grid and let v be a vertex on the (unique) path P between u and F^∞ in T^*. Further, let $P_{u,v} \subseteq P$ be a pseudo-path between u and v. For the sizes of the fundamental cuts there holds*

$$|\delta(S_f)| \geq 2 \cdot \textit{pos}(f, P_{u,v}), \quad \forall f \in P_{u,v}. \tag{2}$$

Proof: Without loss of generality, regard $P_{u,v}$ as a horizontal pseudo-path, and assume $v = (i', j')$ where $i' = i + |P_{u,v}|$.

Consider some edge $f \in P_{u,v}$ and the induced set $S_f \subset V(G^*)$ such that $u \in S_f$ and $F^\infty \in \overline{S}_f$, where $\delta(S_f)$ is the corresponding fundamental cut. As we consider a horizontal pseudo-path the y-coordinate of both vertices of f is equal and their x-coordinates are $i + \text{pos}(f, P_{u,v}) - 1$ and $i + \text{pos}(f, P_{u,v})$, respectively. Remember that $P_{u,v}$ is contained in the unique u, F^∞-path P of T^*. Therefore, all vertices between u and f are contained in S_f. In particular, for each integer α with $i \leq \alpha < i + \text{pos}(f, P_{u,v})$, there exists a vertex in S_f with α as x-coordinate.

Out of those vertices in S_f with x-coordinate α consider the vertex $w_\alpha^{\max}$ ($w_\alpha^{\min}$) with maximal (minimal) y-coordinate. Note that $w_\alpha^{\max}$ and $w_\alpha^{\min}$ may coincide. Now, to $w_\alpha^{\max}$ ($w_\alpha^{\min}$) one edge in the cut $\delta(S_f)$ can be assigned, because the dual vertex directly above (below)—at the latest F^∞—is not included in S_f. Hence, for each α we get a contribution of two distinct edges and therefore a lower bound of $2 \cdot \text{pos}(f, P_{u,v})$ on $|\delta(S_f)|$ in total. □

Note that in general (2) does not hold when choosing the vertex v such that it is *not* contained in the unique u, F^∞-path in the dual tree. Furthermore, (2) does not hold either when considering all the edges of an *ordinary* path P instead of one of its two pseudo-paths. Moreover, the estimate in (2) can be far from being tight. Consider in Fig. 1(f) the vertex u having Cartesian coordinates $(16, 2)$. In a vertical pseudo-path that starts at u the first edge only contributes with 2, although it induces a cut of length 18.

In order to employ this powerful tool for estimating sizes of cuts, we introduce some more definitions. An important concept for our approach is the distance between two dual vertices. Let the grid-graph $G^* \setminus \{F^\infty\}$ be embedded in $\mathbb{Z}^2$ in the straightforward way and let $u = (i, j)$ and $v = (i', j')$ be vertices of it. Then the *distance* $d_{u,v}$ is defined as $\max\{|i - i'|, |j - j'|\}$, or $||u - v||_\infty$. It is a simple observation that for any two distinct vertices u, v that are connected by a path in $T^* \setminus \{F^\infty\}$ at least one of the two pseudo-paths from u to v has precisely $d_{u,v}$ edges.

Next, we a priori tag specific vertices which are organized in what we will call levels. In a dual grid with $(N-1)^2 = \left(2^k + 1\right) \cdot \left(2^k + 1\right)$ vertices we establish k different levels of vertices as follows. The *level* k only contains the unique grid's center vertex. The center-vertices of the four quarters of the grid (which overlap on their borders) constitute level $k - 1$. Recursively, each of these four quarters is again subdivided into four new quarters whose centers define the next levels. Hence, level $1 \leq \ell \leq k$ consists of $4^{k-\ell}$ vertices.

We next assign boxes to level-vertices. These boxes are exactly the quarters which were used to define their center-vertices as belonging to a certain level—technically, for a vertex u of level ℓ with $u = (i, j) \in V(G^*) \setminus \{F^\infty\}$, we define its *box* as the set of dual vertices $B_u = \left\{v : d_{u,v} \leq 2^{\ell-1}\right\}$. Further, we call the set $\left\{v : d_{u,v} = 2^{\ell-1}\right\}$ the *border* of B_u.

We illustrate the arrangement of the levels in Fig. 1(a). There, the $64 = 4^{4-1}$ level-1 vertices are marked as small light-grey circles. With increasing level index the (every time fewer) level-vertices are sketched with increasing intensity culminating with the one level-4 vertex in the grid's center. In Fig. 1(b)-1(d), the boxes of the level-vertices are indicated as thin lines. In the figures these are levels 1, 2 and 3. In Fig. 1(e), the box of the level-4 vertex constitutes the whole dual graph, except for F^∞.

We count along the following pseudo-paths. Every level-vertex u serves as the starting point of one pseudo-path. We then consider the unique u, F^∞-path P in G^*. Every such path has to intersect with the border of the box B_u. With v being the first such border vertex, for every level-vertex u we denote by $P_{u,v}$ the longer one of the two pseudo-paths from u to v. If u is a level-ℓ vertex, then $|P_{u,v}| = 2^{\ell-1} = d_{u,v}$.

3.1 A simple $n \log n$ lower bound

Lemma 2 suggests that we can count for every edge $e \in P_{u,v}$ a contribution of $2 \cdot \text{pos}(e, P_{u,v})$ to the global lower bound. However, as pseudo-paths of different levels may intersect (cf. Figure 1(c)) this may over-estimate the lower bound. In fact, there even exist spanning trees such that one edge is contained in a pseudo-path of *every* single level.

We solve this major inconvenience by voluntarily counting less for every occurrence of an edge on any pseudo-path. On the one hand, in a sense the estimate in Lemma 2 is tight, because there exist spanning trees such that for every level there exists an edge e for which we only need to add two and our estimate on e meets the size of the actual cut, cf. Fig. 3(a). On the other hand, with our reduced estimate we may eventually sum over *every* occurrence of an edge on some pseudo-path. The following two lemmas are the key observations to justify this approach.

Lemma 3. *Let $u \neq u'$ be two level-vertices of levels ℓ and ℓ', respectively, such that their pseudo-paths $P_{u,v}$ and $P_{u',v'}$ share some edge e. Then $\ell \neq \ell'$.*

Proof: The claim follows from two facts: First, every pseudo-path only consists of edges within its box. Second, boxes of the same level only intersect on their borders. □

Lemma 4. *Let $u \neq u'$ be two level-vertices of levels ℓ and ℓ', respectively, such that their pseudo-paths $P_{u,v}$ and $P_{u',v'}$ share some edge e. Assuming w.l.o.g. that $1 \leq \ell \leq \ell' \leq k$, there holds*

$$\textit{pos}(e, P_{u',v'}) \geq 2 \cdot \textit{pos}(e, P_{u,v}). \tag{3}$$

Proof: Since u is a level-ℓ vertex, there holds $\text{pos}(e, P_{u,v}) \leq 2^{\ell-1}$. Hence, it suffices to show that $\text{pos}(e, P_{u',v'}) \geq \text{pos}(e, P_{u,v}) + 2^{\ell-1}$.

By Lemma 3 we know that in fact $\ell < \ell'$. Denote by (i,j) the coordinates of u in the dual grid. Without loss of generality we assume $P_{u,v}$ to be a horizontal pseudo-path leaving its box B_u at the eastern border, i.e. at x-coordinate $i + 2^{\ell-1}$. Then the endpoints of e have x-coordinates $i + \text{pos}(e, P_{u,v}) - 1$ and $i + \text{pos}(e, P_{u,v})$, respectively.

A simple but important observation is that the u-F^∞ and u', F^∞-paths coincide precisely from their first common vertex on, at the latest from the endpoints of e on. In particular, they traverse their common edges in the very same direction. But so do the pseudo-paths. Hence, $P_{u',v'}$ is a horizontal pseudo-path leaving its box $B_{u'}$ at its eastern border, too.

As $e \in P_{u,v} \cap P_{u',v'}$ and $\ell' > \ell$ we obtain $B_u \subset B_{u'}$. On the one hand, by the definition of $P_{u,v}$ this path contains only edges with x-coordinates at least as large as those of the center vertex u of its box. On the other hand, because of $B_u \subset B_{u'}$ the pseudo-path $P_{u',v'}$ contains $2^{\ell-1}$ edges with both of their x-coordinates in the set $\{i - 2^{\ell-1}, \ldots, i\}$, thus not being contained in $P_{u,v}$. Hence, $\text{pos}(e, P_{u',v'}) \geq \text{pos}(e, P_{u,v}) + 2^{\ell-1}$, which proves (3). □

Corollary 5. *Let e be an edge which is contained in pseudo-paths $P^{\ell_1}, \ldots, P^{\ell_s}$ of levels $\ell_1, \ldots, \ell_s$, where $\ell_s = \max\{\ell_1, \ldots, \ell_s\}$. There holds*

$$\textit{pos}(e, P^{\ell_s}) \geq \sum_{i=1}^{s-1} \textit{pos}(e, P^{\ell_i}). \tag{4}$$

Proof: The claim follows simply by applying Lemma 4 inductively. □

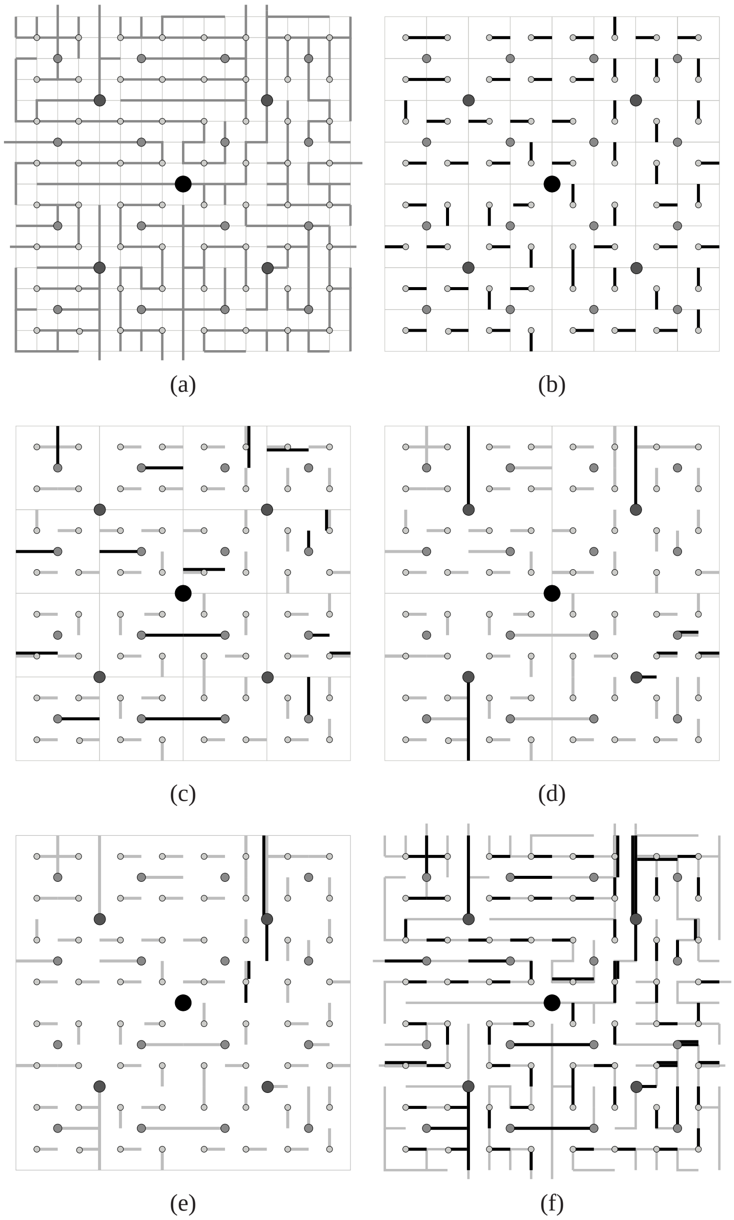


Fig. 1. In (a) a dual tree T^* is sketched. The edges that overhang the grid indicate the connections to F^∞. In (b)-(e) one can see which parts of T^* are used for bounding the according cut-lengths in each level-iteration. Those are depicted using black lines, whereas the grey parts stand for pseudo-paths of previous levels. Note that the pseudo-paths do not necessarily start from the vertex they belong to. In addition, the boxes of the vertices of each iteration are illustrated. In (f) we illustrate what a small part of the tree is actually taken into consideration to obtain the desired lower bound.

Now, recall from Lemma 2 that for every edge e twice the position value in particular on the path of the maximal level ℓ_s that e occurs on is a valid lower bound for the length of the fundamental cut $\delta(S_e)$ induced by e. Finally, by Corollary 5 we are maintaining a valid lower bound when summing over *every* pseudo-path P^{ℓ_i} that the edge e occurs on, but only its undoubled position values $\mathrm{pos}(e, P^{\ell_i})$, i.e.

$$|\delta(S_e)| \overset{(2)}{\geq} 2 \cdot \mathrm{pos}(e, P^{\ell_s}) \overset{(4)}{\geq} \sum_{i=1}^{s} \mathrm{pos}(e, P^{\ell_i}). \tag{5}$$

Theorem 6. *Let $G_{N,N}$ be the planar grid graph with $n = N^2 = (2^k + 2)^2$ vertices. For every spanning tree T of $G_{N,N}$ there holds*

$$\Phi(T) \geq \frac{1}{16} n \log_2 n + O(n). \tag{6}$$

Proof: Let $E(\mathcal{P})$ be the set of edges that appear on some pseudo-path corresponding to the dual tree T^* to T. Then we can conclude

$$\begin{aligned}\Phi(T) = \Psi(T^*) := \sum_{e \in E} |\delta(S_e)| \geq \sum_{e \in E(\mathcal{P})} |\delta(S_e)| = \sum_{\ell=1}^{k} \sum_{\substack{P_{u,v} \\ u \text{ level-}\ell \text{ vertex}}} \sum_{\substack{e \in P_{u,v} \\ \ell \text{ max. level for } e}} |\delta(S_e)| \\ \overset{(5)}{\geq} \sum_{\ell=1}^{k} \sum_{\substack{P_{u,v} \\ u \text{ level-}\ell \text{ vertex}}} \sum_{e \in P_{u,v}} \mathrm{pos}(e, P_{u,v}).\end{aligned}$$

Since on level ℓ there exist $4^{k-\ell}$ pseudo-paths of length $2^{\ell-1}$ each, we finally conclude

$$\begin{aligned}\Phi(T) &\geq \sum_{\ell=1}^{k} 4^{k-\ell} \cdot \left(\sum_{i=1}^{2^{\ell-1}} i \right) = \sum_{\ell=1}^{k} 4^{k-\ell} \cdot \left(\frac{1}{8} \cdot 4^{\ell} + 2^{\ell-2} \right) \\ &= \frac{1}{8} \cdot 4^k \cdot k + \frac{1}{4} \left(4^k - 2^k \right) = \frac{1}{8} (N-2)^2 \log_2 (N-2) + O(N^2) \\ &= \frac{1}{16} n \log_2 n + O(n).\end{aligned}$$

□

3.2 A refined analysis

In this section we perform a refined analysis using our concept of counting along pseudo-paths. More precisely, we will show that

$$\Phi(T) \geq \frac{1}{12} n \log n + O(n).$$

In order to obtain a simple asymptotic proof of the $n \log n$ lower bound in Section 3.1 we decided not to take the risk of over-estimating contributions of edges. In fact, in this section we will show that we do not have to abandon the factor of 2 (cf. Lemma 2) as we did before. Of course, this requires a more detailed examination on the occurrences of edges in different pseudo-paths.

The following Lemma 7 quantifies how much the edges of a pseudo-path of some level ℓ vertex u can contribute to our objective function. This follows directly from the argumentation in the previous section. We will denote this amount by $p(\ell)$. Thereafter, we introduce a correction term ensuring that these very edges do not contribute at a different level as well.

Lemma 7. *For a vertex u of level ℓ there exists a vertex v and a pseudo-path $P_{u,v} = (f_1, \ldots, f_d)$ of length $2^{\ell-1}$. Then, the sum of the sizes of the fundamental cuts induced by the edges of $P_{u,v}$ is at least*

$$p(\ell) := \sum_{i=1}^{2^{\ell-1}} 2 \cdot pos(f_i, P_{u,v}) = \frac{1}{4} \cdot 4^{\ell} + 2^{\ell-1}.$$

□

The crucial point in the refined analysis is now the quantification of the correction term; in detail, this will be discussed in Lemma 9. Consider the vertex u on level ℓ. In order to count the whole pseudo-path of u as proposed in Lemma 7, i.e. without “loosing” the factor of 2 as in the previous section we will show (Lemma 9) that it suffices to subtract

$$\sum_{i=1}^{\ell-1} p(i) \tag{7}$$

where $p(i)$ is exactly the contribution of an entire pseudo-path at level i. Here, ‘suffices’ means that by subtracting (7) we ensure that no edge on $P_{u,v}$ is charged with more than its actual contribution. We have to take care of this, since edges of $P_{u,v}$ may been considered and counted, respectively, in previous levels before ℓ.

With this, Lemma 7 and term (7), we then deduce in the following way.

$$\begin{aligned}
\Phi(T) = \Psi(T^*) := \sum_{e \in E} |\delta(S_e)| &\geq \sum_{e \in E(\mathcal{P})} |\delta(S_e)| \geq \sum_{\ell=1}^{k} 4^{k-\ell} \left(p(\ell) - \sum_{i=1}^{\ell-1} p(i) \right) \\
&\geq \sum_{\ell=1}^{k} 4^{k-\ell} \left(\sum_{i=1}^{2^{\ell-1}} 2i - \sum_{j=1}^{\ell-1} \sum_{i=1}^{2^{j-1}} 2i \right) \\
&= \sum_{\ell=1}^{k} 4^{k-\ell} \left(\frac{1}{6} \cdot 4^{\ell} - 2^{\ell-2} + 16 \right) \\
&= \frac{1}{6} \cdot 4^{k} \cdot k + O(4^{k}) \\
&= \frac{1}{6} \cdot (N-2)^2 \log_2 (N-2) + O(N^2) \\
&= \frac{1}{12} \cdot n \cdot \log_2 n + O(n)
\end{aligned} \tag{8}$$

Thus, we strengthened the bound of Theorem 6.

Theorem 8. *Let $G_{N,N}$ be the planar grid graph with $n = N^2 = (2^k + 2)^2$ vertices. For every spanning tree T of $G_{N,N}$ there holds*

$$\Phi(T) \geq \frac{1}{12} n \log_2 n + O(n).$$

□

Still, it remains to show that the term stated in Eq. (7) indeed has the desired property. In particular, the second inequality in (8) has to be proven.

Lemma 9. *Let u be a vertex of level ℓ. Further, let $P_{u,v}$ be the pseudo-path of vertex u of length $2^{\ell-1}$, where v denotes the border vertex of u. Then, when charging $p(\ell)$ as the contribution of $P_{u,v}$ and subtracting $\sum_{i=1}^{\ell-1} p(i)$, no edge of $P_{u,v}$ is charged with more than its actual contribution to $\Psi(T^*)$.*

Proof: At first, we assume w.l.o.g. that $P_{u,v}$ is a vertical pseudo-path that leaves its box B_u via its northern border. All other cases follow the very same argumentation with adjustments of directions.

For providing a more accessible line of argumentation it is helpful to introduce two new definitions. First, let $NH(u)$ define the "northern hemisphere" of the box B_u of $u = (i, j)$. Thus, $NH(u) = \{v = (i', j') : d_{u,v} \leq 2^{\ell-1}, j' \geq j\}$.

The second new definition now provides subsets of the northern hemisphere $NH(u)$. Let $S_0 = \emptyset$ and define sets S_i for $i = 1, \ldots, \ell - 1$ recursively as follows.

$$S_{i+1} = NH(u) \quad \cap \bigcup_{\substack{z \text{ is} \\ \text{level}-(\ell-i)\text{vertex,} \\ z \notin S_i}} NH(z)$$

Geometrically, the S_i can be seen as "stripes" within $NH(u)$ lying one upon the other each time doubling their height, when decreasing i by 1.

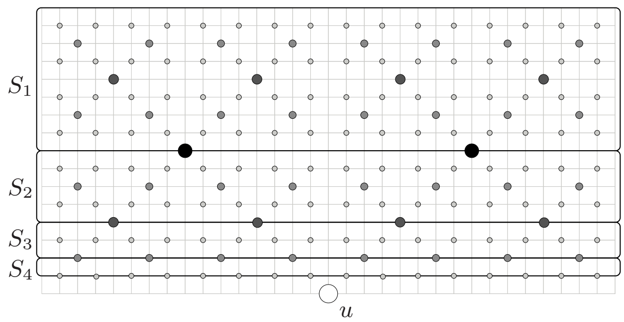


Fig. 2. An illustration of the sets S_i, $i = 1, \ldots, 4$, within the northern hemisphere $NH(u)$ of the level 5 vertex u.

Now, we come back to the actual argumentation. Remember, we are considering the pseudo-path $P_{u,v}$ of the level-ℓ vertex u. If no edge of $P_{u,v}$ has contributed before, we surely do not overestimated the contribution of the edges of $P_{u,v}$ when even subtracting the correction term. And, as stated before in Lemma 3, edges of $P_{u,v}$ cannot be contained in other pseudo-paths of level ℓ. So, we can assume that some edges were included in previous pseudo-paths.

At this time, one has to observe the following.

Fact 10 *Let f be an edge of $P_{u,v}$ that is contained in a previous pseudo-path P'. Then P' must be a vertical pseudo-path, too.*

Whereas Fact 10 simply states a trivial observation, the following Fact 11 follows directly from the definition of pseudo-paths.

Fact 11 *It holds that $P_{u,v} \subseteq NH(u)$. Furthermore, let f be an edge of $P_{u,v}$ that is contained in a previous pseudo-path P' of vertex z'. Then, $f \in NH(z')$.*

We will now argue that according to its occurrence in one of the sets S_i we can bound from above what an edge f of $P_{u,v}$ may have contributed before. As already mentioned, $P_{u,v}$ contains $2^{\ell-1}$ edges. By the definition of the northern hemispheres each set S_i is a union of northern hemispheres of vertices of previous levels. With this and Fact 11 it follows that f must lie within one of the S_i, unless $pos(f, P_{u,v}) = 1$. In general, it is easy to notice the following.

Fact 12 *Out of the $2^{\ell-1}$ edges of $P_{u,v}$ exactly $2^{\ell-2}$ are contained in S_1, $2^{\ell-3}$ edges in $S_2, \ldots,$ 1 edge in $S_{\ell-1}$. Moreover, it remains 1 edge—the first one on $P_{u,v}$—that is not contained in any of the S_i.*

What is still missing is the concrete relation between former levels j, $1 \leq j \leq \ell - 1$, and the sets S_i. This is explained in the next fact which itself is a consequences of Fact 11 and the definitions of the northern hemispheres and the levels in general.

Fact 13 *Let f be an edge of $P_{u,v}$ that is contained in the set S_j for some $j \in \{1, \ldots, \ell - 1\}$. Then, the highest level in which the edge f may have contributed is level $\ell - j$.*

Because of Lemma 4 we need only be interested in the highest occurrence (w.r.t. the level index) of each $f \in P_{u,v}$ individually, since all other occurrences are dominated by this term. Thus, we deduce from Facts 12 and 13 that only $2^{\ell-2}$ edges out of the $2^{\ell-1}$ edges of $P_{u,v}$ may have contributed within level $\ell - 1$. Further, only $2^{\ell-3}$ of the remaining edges of $P_{u,v}$ may have been counted within level $\ell - 2$ and so on until we observe that there exists just one edge which has been counted within level 1. It even remains one edge that cannot have been counted before at all. More formally, $2^{\ell-j-1}$ or less edges contributed within level $\ell - j$. So, we can bound from above the sum of the previous contributions of edges of $P_{u,v}$ from within all the levels $1 \leq j \leq \ell - 1$, or within $S_{\ell-j}$, respectively. Finally, observe that all the $2^{\ell-j-1}$ edges in $S_{\ell-j}$ must have had a different position within their pseudo-path—just because otherwise there would exist "parallel" edges in $P_{u,v}$. Hence, it follows that from within each level j, $1 \leq j \leq \ell - 1$, the edges of $P_{u,v}$ may have contributed only $p(j)$ before. This completes the proof of Lemma 9. □

4 Provably Good Spanning Trees

Recall that in [8] it had been proven that the minimum strictly fundamental cycle basis (MSFCB) problem is NP-hard for general graphs. For planar grid graphs the first family of spanning trees that induce SFCB of length $6n \log_2 n + O(n)$ was published in [21]. Later, in [1, Sect. 6.1] other spanning trees have been considered. We will recapitulate their construction in this section. The authors of [1] prove that the length of their corresponding cycle bases is bounded from above by $2n \log_2 n + o(n \log_2 n)$. Moreover, they conjecture, that this is "essentially optimal" ([1]). Though, in the remainder we perform a careful counting of these spanning trees and show that the lengths of their cycle bases are in fact only

$$\frac{4}{3} n \log_2 n - \Theta(n). \tag{9}$$

In combination with the results presented in Section 3 we conclude that these spanning trees miss the optimum only by a factor of 16, compared to about 5900 achieved by [1].

As mentioned before, also several heuristics for the MSFCB problem on general graphs have been proposed (see [2,8,9,10]). One of these approaches is based on a very natural local search neighborhood ([2]). Here, we establish that an algorithm that optimizes over this neighborhood can end in local optima that miss the global optimum by a factor of $\Theta\left(\frac{\sqrt{n}}{\log n}\right)$.

In [2], the following edge swap operation is defined: Let T be some spanning tree of G and let $e \in T$. If $f \in E \setminus T$ is contained in the fundamental cut induced by e with respect to T, then the tree $T \cup \{f\} \setminus \{e\}$ is said to be obtained by an *edge swap*. The neighborhood of a spanning tree T is then simply the set of trees that can be derived by applying one edge swap operation to T.

Unfortunately, this simple neighborhood is not exact, i.e. there are local optima that do not constitute a global optimum. We provide one such example in Fig. 3(a). One can easily check that no edge swap will ever decrease the length of the corresponding strictly fundamental cycle basis. In particular, for N odd the length of these strictly fundamental cycle bases is

$$4 \cdot \left(2 \cdot \sum_{i=1}^{\frac{N-3}{2}} \sum_{j=1}^{i} (2j+2) + \sum_{i=1}^{\frac{N-1}{2}} (2i+2) \right) = \frac{1}{3}N^3 + 2N^2 - \frac{13}{3}N + 2 = \Theta\left(n^{\frac{3}{2}}\right).$$

By (9) and Theorem 6 or, alternatively, with the asymptotic results in [1], we conclude that the edge swap neighborhood could leave an optimality gap of $\Theta\left(\frac{\sqrt{n}}{\log n}\right)$ on planar grids.

Now we will present the aforementioned concise counting of the SFCB that are induced by the spanning trees proposed in [1]. Hereby, we improve the initial estimate of $2n\log_2 n + o(n \log n)$ to only $\frac{4}{3}n\log_2 n - \Theta(n)$. Our notion of these trees is twofold: As they are recursively defined, they are well structured. However, they do not even constitute local optima with respect to the edge swap neighborhood. Thus they might not be the first configuration coming into mind—in particular when considering small dimensions.

The definition of the provably good spanning trees, as they have been introduced in [1], is somewhat similar to the approach that we followed in Section 3 to establish a lower bound on the value of an MSFCB. Note that this time we find it more convenient to set the dimension of the grid to $N = 2^k$, where $k \geq 1$ is an integer, and, instead of looking at the dual grid, we keep a primal perspective. Again, we assume $G_{N,N}$ to be embedded into $\mathbb{Z}^2$.

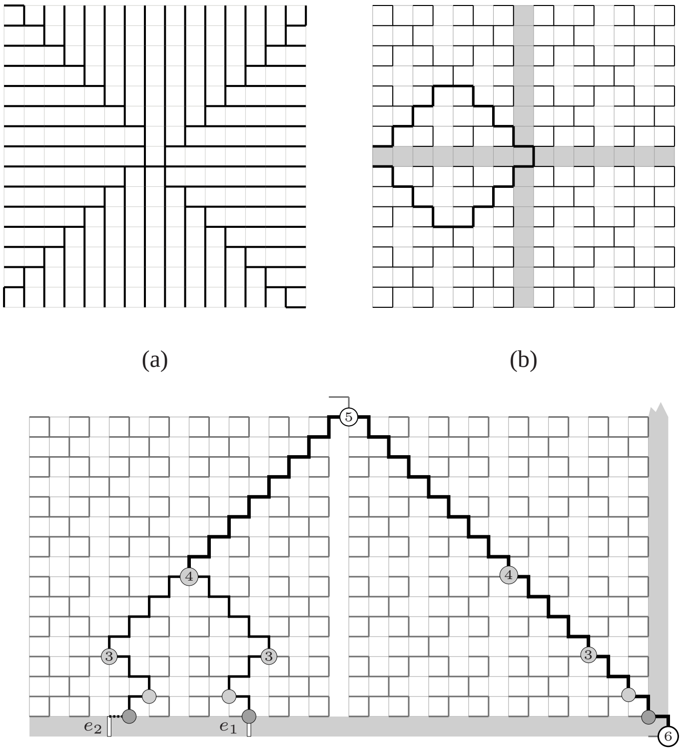
For $N = 2$ the only edge that the spanning tree T_2 does not contain is $\{(1,1),(1,2)\}$. For each integer $k \geq 2$ and $N = 2^k$ we define the spanning tree T_N recursively. Consider the four sub-grids $G_{\frac{N}{2},\frac{N}{2}}$ which partition the original grid's vertex set. To define T_N, we first adopt the edges of the four copies of $T_{\frac{N}{2}}$. We have to add three more edges to connect these four connected components. We do so by placing the tree T_2 into the center of the grid $G_{N,N}$. We refer to the bottom-right vertex u of the copy of T_2 as the *beacon vertex* of this recursive step, and label it with the grid's level k, $BV(u) := k$. Observe that a beacon vertex u with label $BV(u) = \ell$ is adjacent with a face whose dual vertex served as a level-ℓ vertex in the previous section.

Fact 14 *T_N is symmetric with respect to the central horizontal axis, having height $\frac{N+1}{2}$ when considering $\mathbb{R}^2$.*

We partition the set of edges of $G_{N,N}$ into two subsets: those which have both endpoints in the same $\frac{N}{2} \times \frac{N}{2}$ subgrid, and those having the endpoints in different subgrids. The fundamental circuits that are induced by the former set of edges exclusively consist of edges of the particular subgrid. Hence, to compute $\Phi(T_N)$ we may make use of the recursive structure of T_N. To that end, denote by E_M the set of edges $e = \{u, v\}$ for which u and v are contained in different subgrids of $G_{N,N}$, i.e. the "middle cross" in Fig. 3(b). With $f(N) := \sum_{e \in E_M} |C_{T_N}(e)|$ there holds

$$\Phi(T_N) = \begin{cases} 4, & \text{if } N = 2 \text{, and} \\ 4 \cdot \Phi\left(T_{\frac{N}{2}}\right) + f(N), & \text{otherwise (i.e. } N = 2^k,\ k \geq 2). \end{cases} \tag{10}$$

For an edge $e \in E_M$ we will partition its fundamental circuit $C_{T_N}(e)$ into paths between beacon vertices of adjacent levels. Due to space limitations we have to focus on presenting properties of $C_{T_N}(e)$ rather than proving them in detail.



(a) (b) (c)

Fig. 3. (a) The spanning trees that were investigated in [5] turn out to be locally optimal with respect to the edge swap neighborhood; (b) the representative T_{16} of a family of trees that induce SFCB with length $4/3 \cdot n \log n + O(n)$ as proposed in [1, Sect. 6.1]; (c) a detailed view on parts of T_{64}. The "northern parts" of the fundamental circuits $C_{T_{64}}(e_i)$ visit beacon vertices that have adjacent labels.

Fact 15 *Let u, v be two beacon vertices with $BV(u) = \ell$ and $BV(v) = \ell + 1$ such that u is contained in the subgrid that we associate with the vertex v. For the unique u, v-path $P_{u,v}$ in T_N there holds $|P_{u,v}| = 2^\ell$.*

For an edge $e \in E_M$ we denote by $\ell^{\max}(e)$ the maximal value strictly smaller than k such that the fundamental circuit $C_{T_N}(e)$ contains some beacon vertex u with $BV(u) = \ell^{\max}(e)$. To assess the value of $f(N)$ we start by considering the set $E_L \subseteq E_M$ of edges whose vertices both have x-coordinates at most $\frac{N}{2}$, i.e. the "left arm" of E_M. We define $\tilde{f}(N) := \sum_{e \in E_L} |C_{T_N}(e)|$.

Fact 16 *Let $e = \{(i, j), (i, j')\} \in E_L$ such that $j = j' + 1$, i.e. the vertex (i, j) is contained in the "north-west" subgrid of $G_{N,N}$, cf. Fig. 3(c). Let π be the sequence of beacon vertices that are contained in the unique $(i, j), (i', j')$-path P in T_N. There exists a subsequence $\sigma = (u_1, u_2, \dots)$ of π such that*

i	1	$\overset{(+1)}{\dots}$	$\ell^{\max}(e)$	$\overset{(+1)}{\dots}$	$2 \cdot \ell^{\max}(e) - 1$	$2 \cdot \ell^{\max}(e)$
$BV(u_i)$	1	$\overset{(+1)}{\dots}$	$\ell^{\max}(e)$	$\overset{(-1)}{\dots}$	1	k

and the unique paths in T_N between subsequent beacon vertices in σ partition the set of edges of P that occur until the vertex $u_{2 \cdot \ell^{\max}(e)}$—only the first edge of this subset of P possibly is not covered by one of these paths, cf. the circuit $C_{T_{64}}(e_2)$ in Fig. 3(c).

Combining Facts 14, 15, and 16, we are in the position to compute the length of the fundamental circuit $C_{T_N}(e)$ induced by an edge $e \in E_L$ simply in function of $\ell^{\max}(e)$.

Fact 17 *For $e \in E_L$ we have $|C_{T_N}(e)| = 4 \cdot \left(\sum_{\ell=1}^{\ell^{\max}(e)-1} 2^\ell\right) + 5 \pm 1 \leq 2^{\ell^{\max}(e)+2} - 2$.*

Fact 18 *There are $2^{\ell-1}$ nontree edges $e \in E_L$ for which $\ell^{\max}(e) = \ell$, where $1 \leq \ell < k$.*

Together, these two facts provide us with

$$\tilde{f}(N) := \sum_{e \in E_L} |C_{T_N}(e)| \leq \left(\sum_{\ell=1}^{k-1} 2^{\ell-1} \cdot \left(2^{\ell+2} - 2\right)\right) + 4 = \frac{2}{3}4^k - 2^k + \frac{10}{3} = \frac{2}{3}N^2 - N + \frac{10}{3}. \quad (11)$$

Now, observe that every circuit that is induced by an edge $e \in E_M \setminus E_L$ can easily be associated with a fundamental circuit that is induced by an edge $e \in E_L$. It can be verified that for any of the $3 \cdot \frac{N}{2}$ so-obtained pairs of circuits, their lengths differ by at most one. In other words, $4 \cdot \tilde{f}(N) \approx f(N)$, or more precisely,

$$4 \cdot \left(\tilde{f}(N) - \frac{N}{2} - 4\right) \leq f(N) \leq 4 \cdot \tilde{f}(N). \quad (12)$$

Plugging (11) and (12) into (10), for $N \geq 4$ we obtain

$$\Phi(T_N) \leq 4 \cdot \Phi\left(T_{\frac{N}{2}}\right) + \frac{8}{3}N^2 - 4N + \frac{40}{3} \leq 4 \cdot \Phi\left(T_{\frac{N}{2}}\right) + \frac{8}{3}N^2 + 15.$$

Solving the recursion for $\Phi(T_N)$ we conclude

$$\Phi(T_N) \leq \frac{8}{3}N^2 \log_2 N - \frac{5}{12}N^2 - 5 = \frac{4}{3}n \log_2 n - \frac{5}{12}n - 5.$$

Finally, observe that the errors that we make during our analysis in Fact 17 and Equation (12) do not affect the coefficient of the $n \log n$ term, in particular $\Phi(T_N) \geq \frac{8}{3}N^2 \log_2 N - \frac{44}{9}N^2$.

Theorem 19. *The spanning trees T_N for the planar square grid graph with $n = N^2$ vertices induce strictly fundamental cycle bases with length $\frac{4}{3} \cdot n \log_2 n - \Theta(n)$.*

5 Conclusions

We presented a new technique for computing lower bounds for the minimum strictly fundamental cycle basis (MSFCB) problem on planar square grids. Moreover, we performed an accurate counting of the length of the SFCB that is induced by spanning trees that have been introduced in [1]. In total, we reduce the optimality gap for the MSFCB problem on planar square grids to a factor of only 16—compared to about 5900 being the state-of-the-art so far. We suppose that stronger lower bounds require case distinctions for different types of spanning trees.

We concluded that approximating minimum cycle bases (MCB) through SFCB in general has to leave a gap of $\Omega(\log n)$. This might be critical for several practical applications that require a short cycle basis as their input.

Notice that in [2] compact representability has been identified as an additional feature of SFCB, when comparing with MCB. However, it is a simple observation that WFCB can be represented much more compact than general cycle bases, too.

Hence, to approximate MCB we strongly encourage the use of heuristics that also take into account weakly fundamental cycle bases (WFCB). Hereby, discovering the complexity of the minimum weakly fundamental cycle basis problem becomes even more important.

Acknowledgments

We thank Edoardo Amaldi, Marco Lübbecke, Rolf Möhring, and Guido Schäfer for numerous intensive discussions.

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