Technische Universität Berlin
Institut für Mathematik

Controller design for the Navier-Stokes system.
Part I: Concept & numerical results

Michael Hinze and Daniel Wachsmuth

Preprint 11-2003

Preprint-Reihe des Instituts für Mathematik
Technische Universität Berlin

Report 11-2003 June 2003

Controller design for the Navier-Stokes system.
Part I: Concept & numerical results

M. Hinze
Technische Universität Dresden
Institut für Numerische Mathematik
Zellescher Weg 12-14
D-01062 Dresden, Germany
email: hinze@math.tu-dresden.de

D. Wachsmuth
Technische Universität Berlin
Institut für Mathematik
Straße des 17. Juni 136
D-10623 Berlin
email: wachsmut@math.tu-berlin.de

Abstract

We present a widely applicable construction recipe for closed-loop feedback controllers of nonlinear dynamical systems. Its basic idea consists in approximately solving certain instantaneous optimization problems for the discrete-in-time dynamical system. Easy incorporation of control constraints is one key feature of the recipe. The instationary Navier-Stokes equations serve as model application.

In the first part of the work, we introduce the basic construction recipes and present numerical results for several closed-loop feedback control laws derived with the recipe.

The stability analysis is contained in the second part of the work.

Keywords: Instantaneous control, Closed loop control, Navier-Stokes equations, control constraints.

1 Introduction

This research is devoted to the construction, numerical validation and stability analysis of nonlinear feedback control policies for the instationary Navier-Stokes system. The governing equations in the primitive setting are given by

$$\textbf{(P)} \qquad \begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = \mathcal{B}u & \text{in } (0,T)\times\Omega, \\ -\text{div } y = 0 & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = \phi & \text{in } \Omega. \end{cases}$$

The control target is to match the given desired state z in the L^2-sense by adjusting the body force $\mathcal{B}u$. In this context $\mathcal{B}$ denotes an abstract control extension operator and $\Omega\subset\mathbb{R}^2$ denotes a bounded domain.

In the first part of this work we present a recipe for the construction of nonlinear feedback control laws of the form

$$\mathcal{B}u = K(y)$$

and numerically illustrate their performance. The second part of the work is devoted to the stability analysis of the controlled systems.

The construction principle works as follows. The uncontrolled Navier-Stokes system is discretized with respect to time. Then, at selected time instances an appropriate cost functional is approximately minimized with respect to a stationary quasi-(Navier-)Stokes system, whose structure depends on the chosen time discretization method. The obtained control is used to steer the system to the next time instance, where the procedure is repeated. We note that this approach is related to model prediction control techniques, see [6].

Main result: Given a sufficiently smooth desired state z, and a time discretization scheme for the Navier-Stokes system, the above described construction process can be regarded as time discretization of a closed loop feedback policy K, i.e. with S denoting the Stokes operator and $b(y)$ the nonlinearity of the Navier-Stokes equations we get the controlled system

$$y_t + \nu Sy + b(y) = K(y).$$

Under certain assumptions on the initial states the controller K steers the Navier-Stokes system exponentially fast to z. To be more precise, the solution of this system satisfies $|y(t) - z(t)|_{H^1} \leq ce^{-\kappa t}$ with some positive constants c and κ.

It turns out that instantaneous control [9, 12] is a special case of our approach. For applications of instantaneous control we refer to [1, 2, 3, 8, 11, 16, 17, 20], stability analysis of the method is presented in [9, 10, 12]. Further contributions to closed loop control of the Navier-Stokes system can be found in [7], where linear body force feedback control was applied to control the system. The analysis of special case of modelpredictive control of the Navier-Stokes equations can be found in [14, 15].

The paper is organized as follows. Section 2 contains the analytical preliminaries. In Section 3 we introduce the basic construction recipe which lead to certain discretized closed-loop control laws. These are related to continuous closed-loop control laws, whose stability properties are stated in Section 4. Finally, in Section 5 numerical examples are presented, which illustrate the theoretical results.

Throughout this work c and C denote global generic constants whose dependencies are mentioned when necessary.

2 Analytical preliminaries and time discretization

For given $T > 0$ let $Q = (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. We set $V = \{v \in H^1_0(\Omega)^2,\ \text{div}\ v = 0\}$, $H = \text{clos}_{L^2(\Omega)^2}\{v \in C^\infty_0(\Omega)^2,\ \text{div}\ v = 0\}$ and identify the Hilbert space H with its dual H'. The dual space of V is defined to get a Gelfand-triple $V \hookrightarrow H \hookrightarrow V'$. On H the common inner product is used, and V is endowed with the inner product

$$(\varphi, \psi)_V = (\varphi', \psi')_H \text{ for } \varphi, \psi \in V.$$

Moreover, with Z denoting a Hilbert space, $L^p(Z)$ $(1 \leq p \leq \infty)$ denotes the space of measurable abstract functions $\varphi : (0,T) \to Z$, which are p-integrable $(1 \leq p < \infty)$, or essentially bounded on $(0,T)$ $(p = \infty)$, respectively.

As control space $L^2(\mathcal{U})$ is taken, where $\mathcal{U}$ denotes the Hilbert space of abstract controls. The space $\mathcal{U}$ also is identified with its dual. Furthermore,

$$\mathcal{B} : \mathcal{U} \to V' \tag{1}$$

denotes the control extension operator which is assumed to be bounded. The set of admissible controls is denoted by $\mathcal{U}_{ad} \subseteq \mathcal{U}$ and is required to be closed, convex and bounded. In order to formulate the weak form of the instationary Navier-Stokes equations let

$$W := W(V) = \{\varphi \in L^2(V) : \varphi_t \in L^2(V')\}$$

supplied with the common inner product. Further, we define

$$H^{2,1}(Q) := \{\varphi \in L^2(V \cap H^2(\Omega)), \varphi_t \in L^2(H)\}.$$

For convenience we introduce the tri-linear form

$$b(u, v, w) \ := \int_\Omega (u \cdot \nabla)v\, w\, dx.$$

Now, for $y \in L^2(V)$ the function $b(y)$ defined by

$$\langle b(y), v\rangle_{V',V} := -b(y,y,v) \quad \text{for all } v \in V \tag{2}$$

is an element of V' for almost all $t \in (0,T)$ and $b(y) \in L^1(V)$ [18, Lemma 3.1]. If in addition $y \in L^\infty(H)$ holds then $b(y)$ is an element of $L^2(V')$. This statement is true especially for functions $y \in W$, since W is continuously imbedded in $L^\infty(H)$, confer [5].

For controls $u \in L^2(\mathcal{U})$ the solenoidal form of the Navier-Stokes equations reads: Find the state $y \in W$ such that

$$\begin{aligned} &\frac{d}{dt}\,(y(t),\varphi)_H + \nu(y(t),\varphi)_V \\ &\qquad = \langle b(y) + \mathcal{B}u(t), \varphi\rangle_{V',V} \text{ for all } \varphi \in V \text{ and a.e. } t \in [0,T] \end{aligned} \tag{3a}$$

and

$$(y(0),\chi)_H = (\phi,\chi)_H \text{ for all } \chi \in H. \tag{3b}$$

With Re denoting the Reynolds number, 1/Re=: $\nu > 0$ is the viscosity parameter. The proof of the following well-known existence theorem can be found in [18].

Theorem 2.1. *For any $\phi \in H$ and for every control $u \in L^2(\mathcal{U})$ equations (3) admit a unique weak solution $y \in W$.*

2.1 Time discretization

For notational purposes let $P : L^2(\Omega)^2 \to H$ denote the Leray projector [4, Remark 1.10]. Then, the Stokes operator S is given by

$$S : \mathcal{D}(S) \subset H \to H, \quad S := -P\Delta, \quad \mathcal{D}(S) = H^2(\Omega)^2 \cap V.$$

Now define

$$A \ := \ \nu S.$$

In this setting the Navier-Stokes equations (3) for $u = 0$ in variational formulation may be rewritten as Burgers equation in the space V',

$$\begin{aligned} y_t + Ay &= b(y), \\ y(0) &= \phi, \end{aligned}$$

where the nonlinearity $b(y)$ is defined in (2). For $m \in \mathbb{N}$ an equidistant discretization of the time interval $(0,T)$ is defined by $h = \frac{T}{m}$ and $t_k = kh$, $k = 0,1,\ldots,m$. Now let $z \in H^{2,1}(Q)$ the desired state. We define

$$J^k : V \times \mathcal{U} \to \mathbb{R}, \quad (y,u) \mapsto \frac{1}{2}\,|y - z^k|_H^2 + \frac{\gamma}{2}\,|u|_{\mathcal{U}}^2,$$

where

$$z^k = \frac{1}{h}\int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} z(s,\cdot)\,ds \tag{4}$$

and $z(t,\cdot) = 0$ for $t > T$. Finally, for $k = 1,\ldots,m$ and $i = 1,2$ introduce the operators $e_i^k : V \times \mathcal{U} \to V'$ by

$$e_1^k(y,u) = (I + hA)y \ - \ hb(y^{k-1}) - y^{k-1} - \mathcal{B}u,$$

and

$$e_2^k(y,u) = (I + hA)y \; - \; hb(y) - y^{k-1} - \mathcal{B}u$$

and where y^{k-1} denotes the state at the previous time slice.

The instantaneous optimal control problem for the semi-implicit time integration is given by

$$(\mathbf{P}^k) \qquad \text{minimize } J^k(y,u) \text{ subject to } e_1^k(y,u) = 0 \text{ in } V', \quad u \in \mathcal{U}_{ad},$$

where $y^0 = \phi$. The initial value ϕ now is required to be an element of the space V. For given y^{k-1} a pair (y^k, u^k) satisfies the subsidiary condition $e_1^k(y,u) = 0$ in V' if and only if

$$(y^k, v)_H + \nu h \, (y^k, v)_V = (y^{k-1}, v)_H + \langle \mathcal{B}u^k + hb(y^{k-1}), v \rangle)_{V',V} \quad \forall \, v \in V. \tag{5}$$

Since $\phi \in V$ holds, the right-hand side in this linear equation defines a bounded linear functional on V. Thus, for every $u^k \in \mathcal{U}$ Eq. (5) admits a unique solution $y^k \in V$ which satisfies the a-priori estimate

$$|y^k|_V \leq \frac{C}{\nu h} \left(|y^{k-1}|_H + h|y^{k-1}|^2{}_V + |u^k|_{\mathcal{U}} \right).$$

Since J^k is quadratic, e_1^k is linear and $\mathcal{U}_{ad}$ is closed and convex every problem $(\mathbf{P}^k)$, $k = 1, \ldots, m$, admits a unique solution $(y_*^k, u_*^k) \in V \times \mathcal{U}$. Furthermore, the unique Lagrange multiplier $\lambda_*^k \in V$ together with the solution (y_*^k, u_*^k) satisfies the first-order necessary optimality conditions (note that A is selfadjoint)

$$(I + hA)y = \mathcal{B}u + y^{k-1} + hb(y^{k-1}), \tag{6a}$$

$$(I + hA)\lambda = -(y - z^k), \tag{6b}$$

$$(\gamma u - \mathcal{B}^\star \lambda, v - u) \geq 0 \text{ for all } v \in \mathcal{U}_{ad}, \tag{6c}$$

where we have set $(y, u, \lambda) = (y_*^k, u_*^k, \lambda_*^k)$. Furthermore, the second-order sufficient optimality condition holds on the whole space $V \times \mathcal{U} \times V$. Hence, the solution (y_*^k, u_*^k) of (6) is the minimum for $(\mathbf{P}^k)$.

The optimal control problem $(\mathbf{P}^k)$ is equivalent with respect to existence to the control-constrained minimization of the functional

$$\hat{J}^k(u) \;\; = \;\; J^k(y(u), u) \tag{7}$$

over $\mathcal{U}_{ad}$, where for a control $u \in \mathcal{U}$ the state $y(u) \in V$ is given as the unique solution to (5) (indexes dropped). The gradient of $\hat{J}^k$ at u is given by

$$\nabla \hat{J}^k(u) = \gamma u \; - \; \mathcal{B}^\star \lambda,$$

where for given u the function λ is obtained by first solving the linear quasi-Stokes problem (6a) for the state y, and then solving (6b) for λ.

From now onwards let $B := (I + hA)^{-1}$ denote the solution operator of the time-discrete equation (6a), i.e. $e_1^k(y,u) = 0$ implies $y = B(y^{k-1} + hb(y^{k-1}) + \mathcal{B}u)$.

Remark 2.1. If one would use implicit time integration in problem $(\mathbf{P}^k)$, i.e. in the subsidiary condition the operator e_1^k is replaced by e_2^k, the adjoint equation (6b) alters to

$$(I + hA)\lambda - b'(y)^\star \lambda = -(y - z^k). \tag{8}$$

Thus, in this case the gradient $\nabla \hat{J}^k(u)$ at a control u depends on the observation $y^k - z^k$, which occurs as right-hand side in the adjoint equation for the computation of the auxiliary variable λ, and also on the whole state y^k in terms of the coefficient $b'(y^k)^\star$ which enters into the adjoint equation for λ. This structure remains valid even in the case of boundary observation, where the observation enters as boundary condition into the adjoint equation, but the whole state y^k again as a coefficient function. As a consequence, in this case computation of gradient information for $\hat{J}^k$ can not be based on observations alone.

On the other hand the adjoint equation (6b) only depends on the observation $y^k - z^k$. Therefore, gradient information for the functional $\hat{J}^k$ is available utilizing the observations only. In the particular case of boundary observation no information of the state in the whole computational domain is needed at all.

We will now apply the instantaneous control strategy to derive a feedback controller.

3 The closed-loop feedback recipe

The feedback recipe is formulated in terms of a pseudo-algorithm. The particular form of the feedback strategy depends on the oracle $RECIPE$ called in step 2.) of the following algorithm.

Algorithm 1. Feedback recipe.

1.) Set $y^0 = \phi$, $k = 0$ and $t_0 = 0$.

2.) Given an initial control u_0^k, set

$$u^{k+1} = RECIPE(u_0^k, y^k, z^k, t_k)$$

3.) Solve

$$(I + hA)y^{k+1} = y^k + hb(y^k) + \mathcal{B}u^{k+1}.$$

4.) Set $t_{k+1} = t_k + h$, $k = k + 1$. If $t_k < T$ goto 2.

Next we discuss the $RECIPE$s which are investigated in the present work. We use the instantaneous control problem (**P**) to define the first feedback law. For a given initial control u_0^k one can use a gradient step in direction $-\nabla \hat{J}(u_0^k)$ given by (7). Then one gets the following recipe, already investigated in [9].

RECIPE 1. (Instantaneous control)
For the instantaneous control strategy [9] the oracle RECIPE is defined by

$$u = RECIPE(v, y^k, z, t)$$

iff

- Solve $(I + hA)y = y^k + hb(y^k) + \mathcal{B}v$,
- solve $(I + hA)\lambda = -(y - z)$,
- set $d = \gamma v - \mathcal{B}^\star \lambda$.
- determine $\rho > 0$,
- set $RECIPE = v - \rho d$.

The choice of the stepsize ρ in RECIPE 1 is crucial. Since $(\mathbf{P}^k)$ is quadratic with linear constraints, the optimal choice ρ^* can be computed exactly by utilizing only the solution of one additional auxiliary problem. To see this decompose the function $y(u + \rho d)$, see (5), into its affine part $y(u)$ and its homogeneous part $y(d)$, i.e. , write $y(u + \rho d) = y(u) + \rho y(d)$ and set

$$h(\rho) = J(y(u + \rho d), u + \rho d).$$

Then h is a quadratic polynomial in ρ which takes its minimum value at

$$\rho^* = -\frac{(y(u)-z, y(d))_H + \gamma(u,d)_{\mathcal{U}}}{|y(d)|_H^2 + \gamma\ |d|_{\mathcal{U}}^2}. \tag{9}$$

The computation of ρ^* requires only the additional computation of the auxiliary function $y(d)$.

Let us a further look at the previous RECIPE. In [9] the following interpretation of RECIPE 1 is given.

Theorem 3.1. For $u_0^k = 0$ RECIPE 1 is equivalent to the semi-implicit time discretization with discretization step size h

$$(I + hA)y^{k+1} = y^k + hb(y^k) - \rho BB(y^k - z^k) - h\rho BB(b(y^k) - Az^k), \quad y^k = \phi, \tag{10}$$

of the dynamical system

$$\dot{y} + Ay = b(y) - \frac{\rho}{h} BB(y-z) - \rho BB(b(y) - Az), \quad y(0) = \phi. \tag{11}$$

Due to Theorem 3.1 the term

$$Ky = -\frac{\rho}{h} BB(y-z) - \rho BB(b(y) - Az) \tag{12}$$

in (11) can be interpreted as a non-linear closed-loop control policy for the Navier-Stokes equations. It is important to note that the discretization step-size h and the descent parameter ρ of RECIPE 1 in the continuous case (11) may now be regarded as parameters defining the controller.

In order to further improve the controller derived in Theorem 3.1, suppose$K$ steers y to z, eq. (11) necessarily implies that the desired state z would have to satisfy

$$z_t + Az - b(z) = -\rho BB(b(z) - Az), \quad z(0) = \phi.$$

This suggests to generalize the control law (12) to

$$Ky = -\frac{\rho}{h} BB(y-z) - \rho BB(b(y) - b(z)) + z_t + Az - b(z). \tag{13}$$

With this control law the controlled Navier-Stokes equations become the form

$$y_t + Ay - b(y) = Ky \quad \text{in } L^2(V') \text{ and } \quad y(0) = \phi. \tag{14}$$

This system has the desired stabilizing property. The controller (13) can be derived from Algorithm (1) with a particular choice of the initial control u_0^k.

Lemma 3.1. Choosing the initial control u_0^k in Algorithm 1 with RECIPE 1 as solution of

$$\left(I - \frac{\rho}{1-\rho\gamma} BB\right) u_0^k = \frac{1}{1-\rho\gamma}\ \left(z^{k+1} - z^k + Az^{k+1} - b(z^k) + \rho BB(b(z^k) - Az^k)\right)$$

in Theorem 3.1 one would end up with control law (13) instead of control law (12). The analogon to (10) with $w = y - z$ is given by

$$(I + hA)w^{j+1} = w^j + h\left(b(y^j) - b(z^j)\right) - \rho BBw^j - \rho hBB\left(b(y^j) - b(z^j)\right), \quad w^0 = \phi - z(0). \tag{15}$$

The related discrete controller is given by

$$K^D y^j = -\frac{\rho}{h} BB(y^j - z^j) - \rho BB(b(y^j) - b(z^j)) + \frac{z^{j+1} - z^j}{h} + Az^{j+1} - b(z^j), \tag{16}$$

The stability analysis of RECIPE 1 is given in [9], for the result see also the next section.

The RECIPE 1 is easy to adapt to the constrained optimization problem where $u \in \mathcal{U}_{ad}$ is required.

RECIPE 2. (Constrained instantaneous control)
For the instantaneous control strategy [9] with control constraints of the form $u \in \mathcal{U}_{ad}$ the oracle RECIPE is defined by

$$u = RECIPE(v, y^k, z, t)$$

iff

- Solve $(I + hA)y = y^k + hb(y^k) + \mathcal{B}v$,
- solve $(I + hA)\lambda = -(y - z)$,
- set $d = \gamma v - \mathcal{B}^\star \lambda$.
- determine $\rho > 0$,
- set $RECIPE = \mathcal{P}_{\mathcal{U}_{ad}}(v - \rho d)$.

Here $P_{\mathcal{U}_{ad}}$ denotes the projection onto the admissible set of controls, $\mathcal{U}_{ad}$.

We now construct a recipe which applied in Algorithm 1 realizes a full optimization step for problem $(\mathbf{P}^k)$. For this purpose first let $\mathcal{U} = L^2(\Omega)^2$. Then the operator $\mathcal{B}$ is defined by

$$\langle \mathcal{B}u, v\rangle_{V',V} = (u, v).$$

We choose the control u^k to be the solution of $(\mathbf{P}^k)$ for the unconstrained case, i.e. $\mathcal{U}_{ad} = \mathcal{U}$. Then u^k solves the optimality system, compare (6),

$$\begin{aligned} (I + hA)y^{k+1} &= y^k + hb(y^k) + u^k \\ (I + hA)\lambda^k &= z^k - y^{k+1} \\ \gamma u^k - \lambda^k &= 0. \end{aligned} \tag{17}$$

It is easy to see that

$$u^k = -(BB + \gamma I)^{-1} B(B(y^k + hb(y^k)) - z^k)$$

holds. Now let

$$S = \gamma(BB + \gamma I)^{-1} BB, \tag{18}$$

which defines a continuous linear operator in $L(H, H)$. Further properties of S are investigated in [13]. Exploiting the relation $Bz^k = BB(z^k + hAz^k)$ we get

$$(I + hA)y^{k+1} = y^k + hb(y^k) - \frac{1}{\gamma} S(y^k - z^k + hb(y^k) - hAz^k), \quad y^0 = \phi, \tag{19}$$

which suggest to define a feedback recipe that realizes the term $-\frac{1}{\gamma}S(y^k - z^k + hb(y^k) - hAz^k)$.

RECIPE 3. (Suboptimal controll or $(\delta t, 1)$ model predictive control)
For suboptimal control the oracle RECIPE is defined by

$$u = RECIPE(v, y^k, z, t)$$

iff

- Solve the optimality system for u

$$\begin{aligned} (I + hA)y &= y^k + hb(y^k) + u \\ (I + hA)\lambda &= z - y \\ \gamma u - \lambda &= 0. \end{aligned}$$

- set $RECIPE = u$

In each step of Algorithm 1 with for RECIPE 3, a optimization problem has to be solved. If the solution form the previous time-step is used as initial guess, the method can be efficiently implemented. Similar arguments as those leading to Theorem 3.1 yield that equation (19) in y is the semidiscretization with stepsize h of the dynamical system

$$y_t + Ay - b(y) = -\frac{1}{\gamma h} S(y - z + hb(y) - hAz), \quad y(0) = \phi. \tag{20}$$

In order to further improve this control law we argue as in the case of RECIPE 1. Suppose that $y \to z$ for $t \to \infty$ in (20). Then the desired state z by necessity has to satisfy

$$z_t + Az - b(z) = -\frac{1}{\gamma h} S(b(z) - Az).$$

This suggest to modify (20) to

$$y_t + Ay - b(y) = -\frac{1}{\gamma h} S(y - z + hb(y) - hb(z)) + z_t + Az - b(z), \quad y(0) = \phi. \tag{21}$$

This system now for arbitrary (but smooth enough) desired states z is consistent with respect to convergence of $y \to z$ for $t \to \infty$. In section 4 we state the stabilizing properties of this equation. Its semidiscretization is given by

$$\begin{aligned}(I + hA)y^{k+1} = y^k + hb(y^k) - \frac{1}{\gamma} S(y^k - z^k) - \frac{h}{\gamma} S(b(y^k) - b(z^k))\\ +z^{k+1} - z^k + hAz^k - hb(z^k).\end{aligned} \tag{22}$$

If we define the feedback control operator K by

$$u = K(y) = -\frac{1}{\gamma h} S(y - z + hb(y) - hb(z)) + z_t + Az - b(z) \tag{23}$$

we end up with a closed loop control interpretation of Algorithm 1 with RECIPE 3.

Let us note that one would obtain the feedback operator K in (23) also by investigating the optimal control problem

$$\min J(v^k) = \frac{1}{2} \int_\Omega |w^{k+1}| + \frac{\gamma}{2} |v^k|^2, \tag{$\tilde{\mathbf{P}}^k$}$$

subject to

$$(I + hA)w^{k+1} = w^k + hb(w^k + z^k) - hb(z^k) + v^k.$$

Here $w = y - z$ denotes the difference of the state and the desired state. This point of view is useful to derive controllers for constraint control, see the exposition below. It further leads us to the following definition

RECIPE 4. (Suboptimal control 2)
We define the extended oracle RECIPE by

$$u = RECIPE(v, y^k, z^k, z^{k+1}, t)$$

iff

- Solve the optimality system for u

$$\begin{aligned}(I + hA)y &= y^k - z^k + hb(y^k) - hb(z^k) + (I + hA)z^{k+1} + u\\ (I + hA)\lambda &= z^{k+1} - y\\ \gamma u - \lambda &= 0.\end{aligned}$$

- set $RECIPE = u$

Now let us also impose constraints on the controls in RECIPEs 3 and 4, i.e. we now require $u \in \mathcal{U}_{ad}$ for the optimization problem $(\mathbf{P}^k)$, where $\mathcal{U}_{ad}$ is non-empty, convex, bounded and closed. If, for given y^k, z^k we denote the solution operator of this optimal control problem by $\mathcal{S}$, we have $u^k = \mathcal{S}(y^k + hb(y^k), z^k)$ and similar arguments as those leading to (23) now yield the dynamical system

$$y_t + Ay - b(y) = \mathcal{S}(y + hb(y), z), \quad y(0) = \phi. \tag{24}$$

If instead we take u^k to be the solution of $(\tilde{\mathbf{P}}^k)$ subject to the constraint $u \in \mathcal{U}_{ad}$, we end up with a controlled system similar to (21).

To anticipate parts of the discussion contained in part II of this work let us note concerning existence of solutions of (24) that the right hand side is Lipschitz with respect to y. However, concerning stability we cannot expect further stability properties of controlled system without further assumptions on $\mathcal{U}_{ad}$. In fact, if we choose $\mathcal{U}_{ad} = \{0\}$ the system is uncontrolled.

So far, we have assumed that observations can be obtained in the whole domain and also, that control can be applied in the whole of Ω. In the following we derive feedback control policies in the presence of control and observation operators. More precisely we investigate the case where observation and control act only on subdomains of Ω. For this purpose denote by Ω_o the domain where the state can be observed, and by Ω_c the control domain. For the derivation of the control policies we now proceed as in the case $\Omega_o = \Omega_c = \Omega$. At time instance t_k we the consider control problem

$$\min J(u^k) = \frac{1}{2}|y^{k+1} - z^k|^2_{L^2(\Omega_o)} + \frac{\gamma}{2}|u^k|^2_{L^2(\Omega_c)} \tag{25}$$

subject to

$$(I + hA)y^{k+1} = y^k + hb(y^k) + C^* u^k. \tag{26}$$

The control u^k is an element of $\mathcal{U} = L^2(\Omega_c)$ and the operator $C^* : L^2(\Omega_c) \mapsto L^2(\Omega)$ defined by

$$(C^* u)(x) = \begin{cases} u(x) & x \in \Omega_c \\ 0 & x \in \Omega \setminus \Omega_c. \end{cases}$$

is an extension operator. Its dual therefore is the truncation operator $C : L^2(\Omega) \mapsto L^2(\Omega_c)$. Observe that $CC^* = \mathrm{Id}_{L^2(\Omega_c)}$. Similarly we define the operators $D : L^2(\Omega) \mapsto L^2(\Omega_o)$ and $D^* : L^2(\Omega_o) \mapsto L^2(\Omega)$. Then we can write the functional J in (25) as

$$\min J(u^k) = \frac{1}{2}|D^*(Dy^{k+1} - z^k)|^2_{L^2(\Omega)} + \frac{\gamma}{2}|C^* u^k|^2_{L^2(\Omega)}. \tag{27}$$

Let u denote the solution of (26),(27). Then it solves the optimality system

$$\begin{aligned} (I + hA)y^{k+1} \quad &= y^k + hb(y^k) + C^* u \\ (I + hA)\lambda^k \quad &= -D^*(DD^*)(Dy^{k+1} - z^k) \\ \gamma(CC^*)u - C\lambda^k &= 0. \end{aligned} \tag{28}$$

Now let us define the control u^k according to Algorithm 1 with RECIPE 1 and $u_0^k = 0$, i.e. we obtain u^k by applying one gradient step to approximately solve (26),(27). This results in

$$u^k = -\rho\nabla J(0) = -\rho CBD^*DB\left(y^k - D^*z^k + hb(y^k) - hAD^*z^k\right),$$

were we utilized the identity $z^k = DB(I + hA)D^*z^k$. Re-arranging all identities in a suitable manner we arrive at the discrete controlled system

$$(I + hA)y^{k+1} = y^k + hb(y^k) - \rho C^*CBD^*DB\left(y^k - D^*z^k + hb(y^k) - hAD^*z^k\right),$$

which in turn is the semidiscretization of

$$y_t + Ay - b(y) = -\frac{\rho}{h} C^*CBD^*DB\left(y - D^*z + hb(y) - hAD^*z\right), \quad y(0) = \phi.$$

However, since in general $\Omega_o \neq \Omega_c$ we cannot apply the arguments which lead to the control law (13). But instead we can use the ideas related to the modified control problem ($\tilde{\mathbf{P}}^k$) which resulted in RECIPE 4. To begin with denote by $w^k = y^k - D^*z^k$ the difference of the state and the extended desired state at time t_k. Consider now the control problem

$$\min J(v^k) = \frac{1}{2}|Dw^{k+1}|_{L^2(\Omega)} + \frac{\gamma}{2}|C^*v^k|^2_{L^2(\Omega)}, \tag{29}$$

subject to

$$(I + hA)w^{k+1} = w^k + hb(w^k + D^*z^k) - hb(D^*z^k) + C^*v^k.$$

The latter equation is equivalent to

$$(I + hA)y^{k+1} = y^k + hb(y^k) - D^*z^k - hb(D^*z^k) + (I + hA)D^*z^{k+1} + C^*v^k.$$

Since the term $-D^*z^k - hb(D^*z^k) + (I + hA)D^*z^{k+1}$ should serve as a control it has to be restricted to Ω_c. This can be accomplished by inserting the operator C^*C, i.e.

$$(I + hA)y^{k+1} = y^k + hb(y^k) - C^*C\left(D^*z^k + hb(D^*z^k) - (I + hA)D^*z^{k+1}\right) + C^*v^k.$$

Now let us insert

$$v^k := -\rho CBD^*DB\left(y^k - D^*z^k + hb(y^k) - hAD^*z^k\right)$$

which would be obtained after application of one gradient step with $u_0^k = 0$ to the numerical solution of the control problem above. Finally we end up with the linear system

$$\begin{aligned}(I + hA)y^{k+1} = y^k + hb(y^k) &- \rho C^*CBD^*DB\left(y^k - D^*z^k + hb(y^k) - hAD^*z^k\right)\\ &- C^*C\left(D^*z^k + hb(D^*z^k) - (I + hA)D^*z^{k+1}\right), \quad y^0 = \phi,\end{aligned} \tag{30}$$

and its continuous equivalent

$$\begin{aligned}y_t + Ay - b(y) = &- \rho C^*CBD^*DB\left(y - D^*z + hb(y) - hAD^*z\right)\\ &+ \ C^*C\left((D^*z^k)_t + AD^*z - hb(D^*z)\right), \quad y(0) = \phi,\end{aligned} \tag{31}$$

Here we observe that we need information of the state y in the whole domain to compute the controller which acts only on smaller subdomain. In practical applications this would require appropriate state estimations from state observations. However our numerical results indicate good stabilizing properties for several combinations of control domains Ω_c and observation domains Ω_o.

If $\Omega = \Omega_o = \Omega_c$ holds then we have $C = C^* = D = D^* = \mathrm{Id}_{L^2(\Omega)^2}$, and the controlled systems (30), (31) and (21), (22) are equal.

4 Existence and stability results

The stability results stated in this section for the control law (13) are proven in [9]. A proof of stability for the suboptimal controllers obtained by RECIPEs 3 and 4 will be presented in part II of this work, see [13]. We begin with stating the result for (13). For this purpose define the constant ρ_0 by

$$\rho_0 = \frac{\nu^2}{8\nu^2 + 4|\phi|_H^2},$$

and the parameter ρ implicitly by the condition

$$0 < \rho \leq \rho_1 := \min\left(\rho_0, \frac{\nu^2}{2\nu^2 + e^{\frac{4+\rho}{\nu}|z|^2_{L^2(V)}}|\phi|_H^2 + |z|^2_{L_\infty(H)}}\right). \tag{32}$$

Theorem 4.1. *Let $\phi \in H$ be a given initial state and $z \in W$ the desired state. The parameter $h > 0$ is fixed. Then for every $0 < \rho \le \rho_1$ with ρ_1 given by (32) the system*

$$\begin{aligned} y_t + Ay - b(y) &= -\tfrac{\rho}{h} BB(y-z) - \rho BB(b(y) - b(z)) + z_t + Az - b(z), \\ y(0) &= \phi. \end{aligned}$$

has a unique solution $y \in W$. In particular for the difference $w = y - z$ the following decay estimates are fulfilled:

$$|w(t)|_H^2 \le C e^{-\frac{\rho}{h} t},$$

where C is a positive constant.

For a proof see [9, Theorem 6.1] where also a-priori estimates for the decay of the V-norm $|y-z|_V$ are proven under further assumptions on ρ, and $z \in H^{2,1}(Q)$, $z(0) - \phi \in V$.

The stability results for system (21) are slightly different. Here we have to impose a condition on the size of the regularization parameter γ.

Theorem 4.2. *Given initial state $\phi \in H$ and desired state $z \in W$. Define γ implicitly by*

$$\gamma > \max\left(\frac{4}{\nu^2} \left(e^{\frac{2\gamma+1}{\nu\gamma} |z|^2_{L^2(V)}} |w(0)|_H^2 + |z|^2_{L_\infty(H)} \right),\ \frac{9}{2\nu^2} |w(0)|_H^2 \right). \tag{33}$$

Then the controlled system

$$\begin{aligned} y_t + Ay - b(y) &= -\tfrac{1}{\gamma h} S(y - z + hb(y) - hb(z)) + z_t + Az - b(z), \\ y(0) &= \phi \end{aligned} \tag{34}$$

admits a unique solution $y \in W$. For the difference $w = y - z$ we obtain the decay

$$|w(t)|_H^2 \le C\, e^{-\frac{\alpha(\gamma)}{h} t} |w(0)|_H^2 \quad \forall t \in [0, T],$$

where $\alpha(\gamma)$ is defined by

$$\alpha(\gamma) = \frac{\gamma}{(1+\gamma)^2}$$

and C is a generic positive constant.

The condition given by (33) requires either a large value of the regularizing parameter γ or a large viscosity ν. However, the numerical results show good stabilizing properties of the controlled system also for small values of γ. Even for $\gamma = 0$ exponential decay was observed.

This was our starting point to investigating the operator S for γ tending to zero. Passing formally to the limit $\gamma \to 0$, we obtain

$$\frac{1}{\gamma} S = (BB + \gamma I)^{-1} BB \xrightarrow{\gamma \to 0} I.$$

The controlled system (21) for $\gamma \to 0$ then transforms into

$$\begin{aligned} y_t + Ay &= -\tfrac{1}{h}(y - z) + z_t + Az \\ y(0) &= \phi, \end{aligned} \tag{35}$$

which is in fact linear. The following theorem now is easy to establish:

Theorem 4.3. *Given initial state $\phi \in H$ and desired state $z \in W$. Then the controlled system (35) admits a unique solution $y \in W$. Furthermore, for the difference $w = y - z$ we now have the decay estimate*

$$|w(t)|_H^2 \le C\, e^{-\frac{2}{h} t} |w(0)|_H^2 \quad \forall t \in [0, T],$$

where C is a generic positive constant.

5 Numerical results

In this section, we numerically investigate the stabilizing properties of Algorithm 1 with RECIPEs 1, 2, 4. Further numerical results for RECIPE 1 can be found in [9].

In the control problem considered here we intend to track a desired time dependent state z in the L^2-sense, i.e. our instantaneous cost functional has the form

$$J(y,u) = \frac{1}{2}\int_0^T \int_{\Omega_o} |y(x,t) - z(x,t)|^2 dxdt + \frac{\gamma}{2}\int_0^T \int_{\Omega_c} |u(x,t)|^2 dxdt,$$

where Ω_o denotes the observation domain, and Ω_c the control domain, respectively. State and control are coupled by the instationary Navier-Stokes system (**P**). We realize the control u by the feedback control laws derived in the RECIPEs 1, 2 and 4.

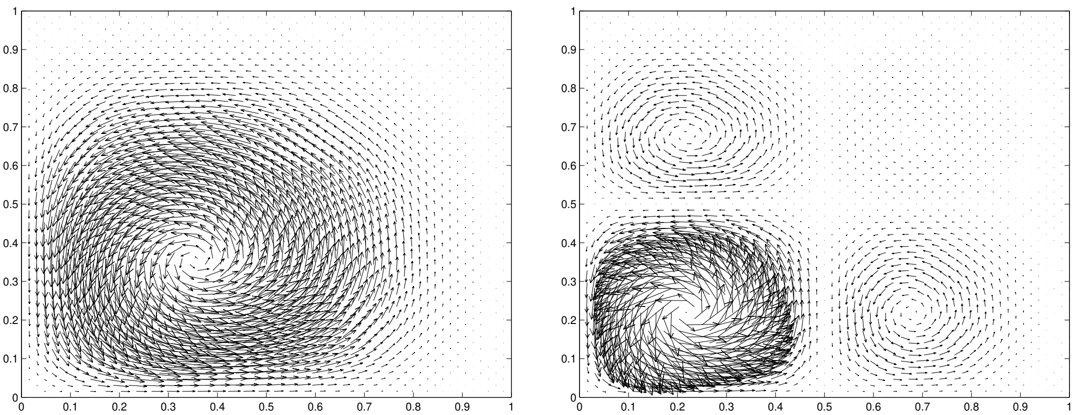

Figure 1: Desired flow at $T = 1$ and $T = 2$

Let us specify the problem setting. The computation domain is the unit square $\Omega = [0,1]^2$. The initial value is chosen as

$$y(x,0) = \phi(x) = e \begin{pmatrix} (\cos 2\pi x_1 - 1)\sin 2\pi x_2 \\ -(\cos 2\pi x_2 - 1)\sin 2\pi x_1 \end{pmatrix},$$

where e is the Euler number, and the desired flow is time-dependent and defined by

$$z(t,x) = \begin{pmatrix} \psi_{x_2}(t,x_1,x_2) \\ -\psi_{x_1}(t,x_1,x_2) \end{pmatrix},$$

where ψ is given through the stream function

$$\psi(t,x_1,x_2) = \theta(t,x_1)\theta(t,x_2)$$

with

$$\theta(t,y) = (1-y)^2(1-\cos 2\pi y t).$$

The Reynolds number is set to 10, which results in a viscosity coefficient of $\nu = 1/10$. The final time is chosen as $T = 2$. For the discretization in time a equidistant grid with stepsize $\delta t = 0.01$ is used, whereas for the spatial discretization the Taylor-Hood finite element is applied on a grid containing 1024 triangles with 2113 velocity and 545 pressure nodes.

The time discretization of the continuous controlled systems is always chosen in such a way that the related discrete-in-time controlled systems are obtained, i.e. the discretization of (14) gives (15). We note that this discretization leads to exponential stable discrete-in-time controlled systems. This can be shown utilizing the techniques of [9], where exponential stability of the discrete-in-time controlled system is proven for Algorithm 1 with RECIPE 1.

Control in the whole domain

At first, we compare the results of the RECIPEs 1 and 2. The step-size of the gradient iteration was set to $\rho = 0.1$. The evolution of the costs is shown in Figure 2 for the unconstrained and the box-constrained case, where $|u(x,t)| \leq 1e-3$ is required. As one could expect, both controllers give the same output if the control constraint is not active.

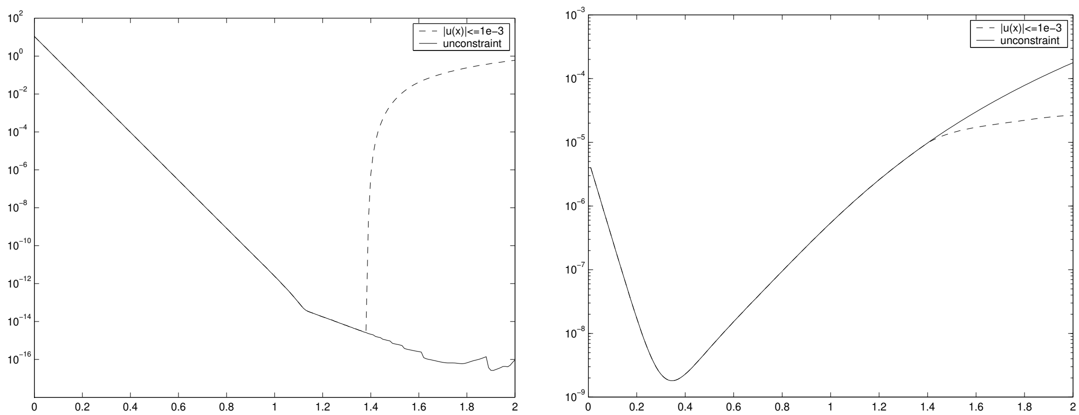

Figure 2: Evolution of $|y(t) - z(t)|_H^2$ and of $|u(t)|_{L^2(\Omega)}^2$ for unconstrained and constrained control

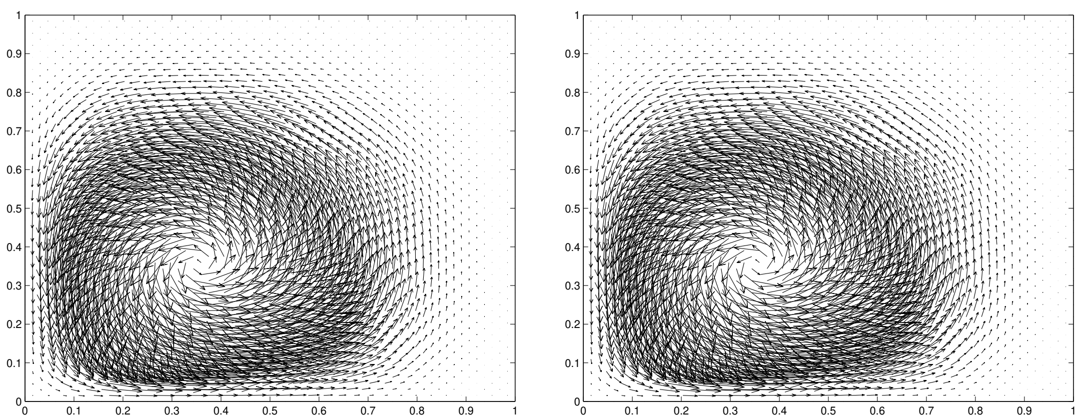

Figure 3: Flow at $T = 1$ for RECIPEs 1 and 2

Since the desired state becomes more and more dynamic, the constraint controller can not adapt to this situation, and the distance between state and desired state grows for $t \geq 1.4$. On the other hand, the control costs per time step remain constant, see the right-hand diagram in Fig. 2. This fact is also illustrated by the Figures 3 and 4.

Secondly, we present some results for the feedback controller given by RECIPE 4. In Fig. 5 the evolution of the L^2-Cost $|y(t) - z(t)|_H$ is shown for different values of γ. Exponential decay for both large and small values of the regularization parameter is observed numerically, although the theory is only satisfying for large values. In each application of the control RECIPE 4, an optimization problem has to be solved. It turns out, that with the solution from the previous time-step as initial guess in the cg-method, very few conjugate gradient steps are needed. The optimization process is stopped if either the relative residual norm is less than $1e-4$ or the absolute residual norm is less than $1e-8$.

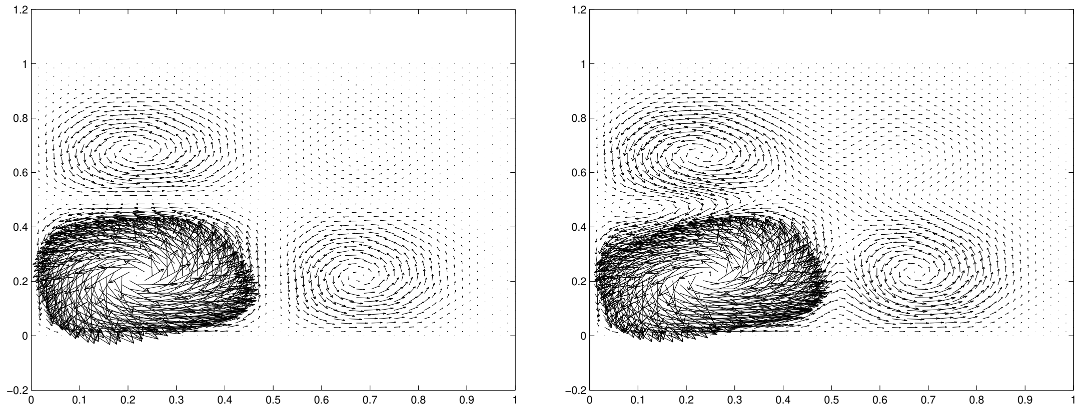

Figure 4: Flow at $T = 2$ for RECIPEs 1 and 2

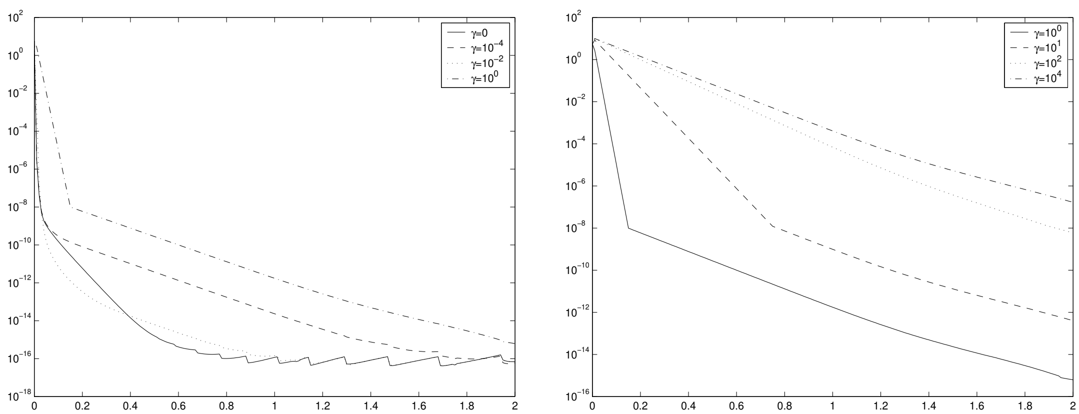

Figure 5: Evolution of $|y(t) - \bar{y}(t)|_H^2$ for different values of γ

Subdomain controller

Finally, we present numerical results for the controllers (30), (31). They apply for the case where observation and control domain are strict subdomains of the computational domain Ω. We consider rectangular domains $\Omega_o = [x_{1,1}, x_{1,2}] \times [x_{2,1}, x_{2,2}]$ and Ω_c analogously. The desired state is linearly transformed into the observation domain by

$$\bar{z}(x_1, x_2, t) = z\left(\frac{x_1 - x_{1,1}}{x_{1,2} - x_{1,1}}, \frac{x_2 - x_{2,1}}{x_{2,2} - x_{2,1}}, t\right).$$

The results for the following configurations are shown in Figure 6. In the two first examples Ω_c and Ω_o are equal, whereas in Example 3 and 4 the domains have a non-empty difference.

Example 1. $\Omega_c = \Omega_o = [0, 0.5]^2$.

Example 2. $\Omega_c = \Omega_o = [0, 0.8]^2$.

Example 3. $\Omega_c = [0, 1] \times [0.5, 1]$ and $\Omega_o = [0, 0.5] \times [0, 1]$.

Example 4. $\Omega_c = [0.5, 0.9]^2$ and $\Omega_o = [0.2, 0.7]^2$.

The results are similar to the results obtained for constraint control. We observe exponential decay of $|y(t) - z(t)|_{L^2(\Omega_o)}$ in all 4 cases for $t < 0.2$. For later times t the norm

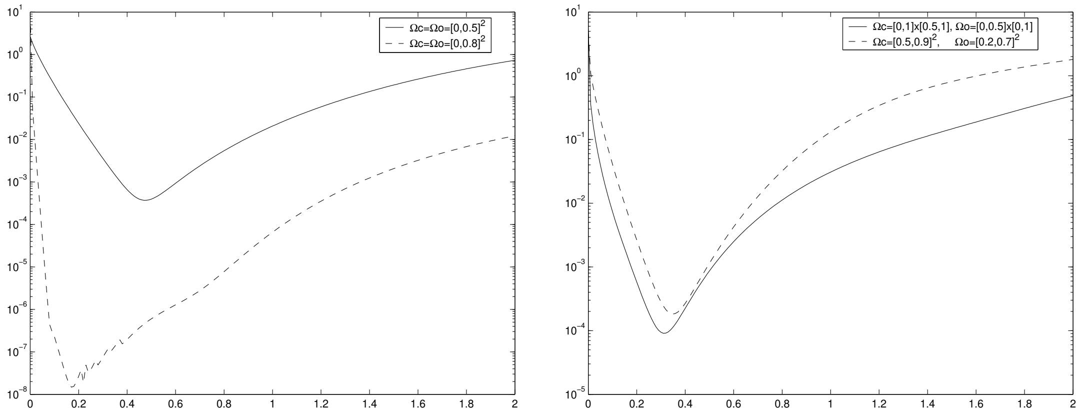

Figure 6: Evolution of $|y(t) - z(t)|^2_{L^2(\Omega_o)}$

of $|z(t)|_{L^2(\Omega_o)}$ becomes larger, and the disturbing effects of the flow outside the observation domain can not be compensated by the controller, so that the distance between the state of the system and the desired state increases.

We also run tests with a stationary desired state. To compare these with the previous results the desired state is chosen as a snapshot of $\bar{z}$ at time $t = 0.4$, i.e. $\tilde{z}(x,t) = \bar{z}(x, 0.4)$. In Figure 7 the distance of state and desired state is plotted for the configurations of Examples 1–4. The resulting values at time $T = 2$ are very similar to the distances at time $t = 0.4$ plotted in Fig. 6.

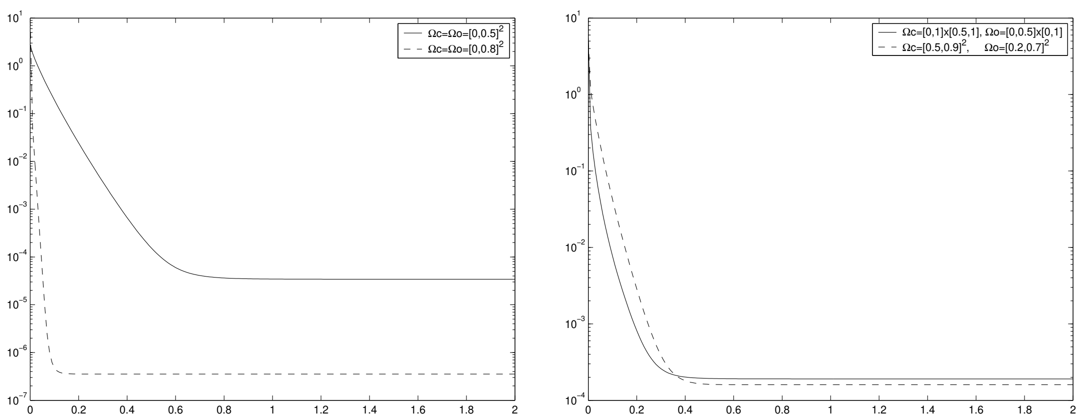

Figure 7: Evolution of $|y(t) - z(t)|^2_{L^2(\Omega_o)}$

We should note that the computational time required for the application of the recipe discussed in this research is in the range of 2.5 to 5 times the computational time required for one forward solve of the uncontrolled time dependent Navier-Stokes system, which in fact is *fast* compared to optimal control, see [10].

References

[1] **Choi, H.** Suboptimal Control of Turbulent Flow Using Control Theory. In *Proceedings of the International Symposium. on Mathematical Modelling of Turbulent Flows*, 1995. Tokyo, Japan.

[2] **Choi, H.; Hinze, M.** & **Kunisch, K.** Instantaneous control of backward-facing-step flows. *Applied Numerical Mathematics*, 31:133–158, 1999.

[3] **Choi, H; Temam, R.; Moin, P.** & **Kim, J.** Feedback control for unsteady flow and its application to the stochastic Burgers equation. *J. Fluid Mech.*, 253:509–543, 1993.

[4] **Constantin, P.** & **Foias, C.** *Navier-Stokes Equations.* The University of Chicago Press, 1988.

[5] **Dautray, R.** & **Lions, J. L.** *Evolution problems I*, volume 5 of *Mathematical analysis and numerical methods for science and technology.* Springer, Berlin, 1992.

[6] **García, C.E.; Prett, D.M.** & **Morari, M.** Model predictive control: Theory and practice - a survey. *Automatica*, 25(3):335–348, 1989.

[7] **Gunzburger, M.D.** & **Manservisi, S.** Analysis and approximation for linear feedback control for tracking the velocity in Navier-Stokes flows. *Comput. Methods Appl. Mech. Eng.*, 189:803–823, 2000.

[8] **Hill, D.C.** Drag reduction strategies. CTR Annual Research Briefs, 1993. Center for Turbulence Research, Stanford University/NASA Ames Research Center, 3-14.

[9] **Hinze, M.** Instantaneous closed loop control of the Navier-Stokes system. Preprint MATH-NM-09-2002. Institut für Numerische Mathematik, Technische Universität Dresden, Germany.

[10] **Hinze, M.** Optimal and instantaneous control of the instationary Navier-Stokes equations. Habilitationsschrift, 1999. Fachbereich Mathematik, Technische Universität Berlin, revised version see http://www.math.tu-dresden.de/~hinze/publications.html.

[11] **Hinze, M.** & **Kunisch, K.** Control strategies for fluid flows - optimal versus suboptimal control. In H.G.Bock et al., editor, *ENUMATH 97*, pages 351–358. World Scientific, 1997. Singapore.

[12] **Hinze, M.** & **Volkwein, S.** Instantaneous control for the Burgers equation: Convergence analysis and numerical implementation. *Nonlinear Analysis T.M.A.*, 50:1–26, 2002.

[13] **Hinze, M.** & **Wachsmuth, D.** Controller design for the Navier-Stokes system. Part II: Stability Analysis.

[14] **Hou, L.S.** & **Yan, Y.** Dynamics and approximations of a velocity tracking problem for the Navier-Stokes flows with piecewise distributed controls. *SIAM J. Control Optim.*, 35:1847–1185, 1997.

[15] **Hou, L.S.** & **Yan, Y.** Dynamics for controlled Navier-Stokes Systems with distributed controls. *SIAM J. Control Optim.*, 35:654–677, 1997.

[16] **Lee, C.; Kim, J.** & **Choi, H.** Suboptimal control of turbulent channel flow for drag reduction. *J. Fluid Mech.*, 358:245–258, 1998.

[17] **Min, C.** & **Choi, H.** Suboptimal feedback control of vortex shedding at low Reynolds nunmbers. *J. Fluid Mech.*, 401:123 – 156, 1999.

[18] **Temam, R.** *Navier-Stokes Equations.* North-Holland, 1979.

[19] **Temam, R.** *Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Second Edition.* Springer, 1997.

[20] **Tröltzsch, F.** & **Unger, A.** Fast solution of optimal control problems in selective cooling of steel. *ZAMM*, 81:447–456, 2001.