A discrete Adomian decomposition method for discrete nonlinear Schrödinger equations

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Abstract

We present a new discrete Adomian decomposition method to approximate the theoretical solution of discrete nonlinear Schrödinger equations. The method is examined for plane waves and for single soliton waves in case of continuous, semi–discrete and fully discrete Schrödinger equations. Several illustrative examples and `Mathematica` program codes are presented.

Key words: Adomian decomposition method, discrete nonlinear Schrödinger equations, finite difference schemes, solitons, plane waves, Mathematica
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1 Introduction

In this work we want to describe a discrete version of the well–known *Adomian decomposition method* (ADM) applied to nonlinear Schrödinger equations. The ADM was introduced by Adomian [5], [6] in the early 1980s to solve nonlinear ordinary and partial differential equation. This method avoids artificial

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boundary conditions, linearization and yields an efficient numerical solution with high accuracy.

The *nonlinear cubic Schrödinger equation* (NLS) [12,34] is a typical dispersive nonlinear partial differential equation that plays a key role in a variety of areas in mathematical physics. It describes the spatio–temporal evolution of the complex field $u = u(x,t) \in \mathbb{C}$ and has the general form

$$i\partial_t u + \partial_x^2 u + q|u|^2 u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1a}$$
$$u(x,0) = f(x), \tag{1b}$$

where the parameter $q \in \mathbb{R}$ corresponds to a focusing ($q > 0$) or defocusing ($q < 0$) effect of the nonlinearity.

The NLS equation (1a) describes many problems in physics. The fields of application varies from optics [27], propagation of the electric field in optical fibers [23], self–focusing and collapse of Langmuir waves in plasma physics [42] to modelling deep water waves and freak waves (so–called rogue waves) in the ocean [30].

The theoretical solution for the NLS equation (1a) has been given among others [38–40]. Moreover, the NLS equation (1a) is completely S–integrable (in the sense of Calogero [11]) with the *inverse scattering method* (ISM) [3,7,41] and a *single soliton solution* is given by

$$u(x,t) = \left(\frac{2a}{q}\right)^{1/2} \exp\Big[i\Big(\frac{c}{2}x - \theta t\Big)\Big] \operatorname{sech}\Big[a^{1/2}(x - ct)\Big], \tag{2}$$

with $\theta = c^2/4 - a$. For fixed t the function u in (2) decays exponentially as $|x| \to \infty$. It travels with the envelope speed c and its amplitude is governed by the parameter $a \in \mathbb{R}$.

An *N–soliton solution* for $q \neq 0$ is given by the function [31]

$$u(x,t) = \left(\frac{2a}{q}\right)^{1/2} \sum_{p=1}^{N} \exp\Big[i\Big(\frac{c_p}{2}x_p - \theta_p t\Big)\Big] \operatorname{sech}\Big[a^{1/2}(x_p - c_p t)\Big], \tag{3}$$

with $\theta_p = c_p^2/4 - a$, the position x_p of the p–soliton and c_p its velocity.

Finally a particular simple form of solutions to the Schrödinger equation (1a) are the *plane wave solutions*

$$u(x,t) = \exp\Big[i\Big(\kappa x - \omega t\Big)\Big], \quad x \in \mathbb{R}, \quad t > 0, \tag{4}$$

where κ is the wave number and ω denotes the frequency. Substituting the ansatz (4) into the NLS (1a) yields the *dispersion relation*

$$\kappa^2 - \omega = q. \tag{5}$$

Since (1a) is S–integrable it is a Hamiltonian system with an infinite number of conserved quantities, cf. [36,37]. Here we will only present the two most important quantities. First the L^2-norm (mass, number of particles) is conserved:

$$N = \frac{2}{q}\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx = \text{const}\,. \tag{6}$$

Note that this conservation property (6) has an important meaning in physical applications. It can be interpreted as the conservation of the power of the beam in nonlinear optics and in Bose–Einstein condensation it denotes the conservation of the number of atoms in the condensate.

Another conserved quantity is the *Hamiltonian*

$$H = \int_{-\infty}^{\infty} \left[|\partial_x u(x,t)|^2 + \frac{q}{2}|u(x,t)|^4 \right] dx = \text{const}\,. \tag{7}$$

with $i\partial_t u = \partial_{u^*} H = \{H, u\}$, where the standard Poisson brackets have been used and * denotes complex conjugation. For more details on nonlinear Schrödinger equations and their conserved quantities we refer the reader to [12,34].

To the authors' knowledge, the Adomian decomposition method was regarded only for the continuous equation, cf. the articles [18,20,26] for the application of the ADM to the NLS (1). In this paper we will first review the basic ideas of the ADM for the NLS and show afterwards how a symbolic package like `Mathematica` can help using the ADM. Secondly, we will turn to the solution of spatially discrete Schrödinger–type equations by a *discrete ADM*. Finally, we end with the consideration of the fully discrete case.

2 The Adomian Decomposition Method

In this Section we shall sketch the ADM for partial differential equations applied to the cubic NLS (1). To this end, we consider (1a) written in operator form as

$$L_t u = i\partial_x^2 u + iqF(u), \quad x \in \mathbb{R},\ t > 0, \tag{8}$$

with the notation $L_t = \partial_t$ and the cubic nonlinear term $F(u) = |u|^2 u$. Then the inverse operator of L_t is defined by the indefinite integral

$$[L_t^{-1} v](t) = \int_0^t v(\tau)\, d\tau, \quad t > 0. \tag{9}$$

Now applying formally the inverse operator L_t^{-1} to (8) yields with the initial condition (1b) the *formal solution* to (1)

$$u(x,t) = f(x) + iL_t^{-1}\partial_x^2 u + iqL_t^{-1}F(u), \quad x \in \mathbb{R},\ t > 0. \tag{10}$$

The Adomian decomposition method [6] assumes a solution of the *series form* $u(x,t) = \sum_{l=0}^{\infty} u_l(x,t)$, where the components $u_l(x,t)$ are going to be determined recurrently. The nonlinear term $F(u)$ in (10) is decomposed into an infinite series of polynomials of the form $F(u) = \sum_{l=0}^{\infty} A_l(u)$, where the A_l are the so–called *Adomian polynomials*. Substituting these decomposition series into (10) gives

$$\sum_{l=0}^{\infty} u_l(x,t) = f(x) + i\sum_{l=0}^{\infty} L_t^{-1}\partial_x^2 u_l(x,t) + iq\sum_{l=0}^{\infty} L_t^{-1}A_l. \tag{11}$$

According to Adomian, $u_0(x,t)$ is identified with the initial data $f(x)$ and the following recurrence is proposed:

$$u_0(x,t) = f(x), \tag{12a}$$
$$u_{l+1}(x,t) = iL_t^{-1}\partial_x^2 u_l(x,t) + iqL_t^{-1}A_l, \quad l = 0,1,2,\dots. \tag{12b}$$

It remains to determine the Adomian polynomials A_l. They are defined by

$$A_l = \frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}\lambda^l}\Big[F\Big(\sum_{p=0}^{\infty}\lambda^p u_p\Big)\Big]_{\lambda=0}, \text{ for } l = 0,1,2,\dots \tag{13}$$

and constructed for all classes of nonlinearity according to algorithms given either by Adomian [6] or alternatively by Wazwaz [37]. To do so, we set $F(u) = u^2\bar{u}$ and obtain in a straight forward calculation

$$A_0 = u_0^2\bar{u}_0, \tag{14a}$$
$$A_1 = 2u_0u_1\bar{u}_0 + u_0^2\bar{u}_1, \tag{14b}$$
$$A_2 = 2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0 + 2u_0u_1\bar{u}_1 + u_0^2\bar{u}_2, \tag{14c}$$
$$A_3 = 2(u_0u_3\bar{u}_0 + u_1u_2\bar{u}_0 + u_0u_2\bar{u}_1 + u_0u_1\bar{u}_2) + u_1^2\bar{u}_1 + u_0^2\bar{u}_3, \tag{14d}$$
$$\dots.$$

The polynomials A_l, $l \geq 4$ can be computed in a similar manner.

Let us finally note that the convergence of this method was established in [13], [14] using a fixed point theorem. Since in practice not all terms (12) of the series $u(x,t) = \sum_{l=0}^{\infty} u_l(x,t)$ can be calculated we use a finite sum $U_L(x,t) = \sum_{l=0}^{L} u_l(x,t)$ to approximate the solution.

Remark 1 *There also exists the* modified decomposition technique *by Wazwaz [36] that accelerates the rapid convergence of the series solution without any* *need to use Adomian polynomials. The recursive relation reads*

$$u_0(x,t) = \tilde{f}(x), \tag{15a}$$
$$u_{l+1}(x,t) = \hat{f}(x) + iL_t^{-1}\partial_x^2 u_l(x,t) + iqL_t^{-1}F(u_l(x,t)), \quad l = 0,1,2,\dots, \tag{15b}$$

where the function $f(x)$ is properly decomposed (mainly on trial basis) as

$$f(x) = \tilde{f}(x) + \hat{f}(x).$$

3 The computation for the cubic Schrödinger equation

In this section we want to clarify the ADM approach using the Adomian polynomials (14) by two examples.

Example 2 *First we consider the simple example of a plane wave solution* (4)*. We obtain by the Adomian decomposition technique* (12)*:*

$$
\begin{aligned}
u_0(x,t) &= f(x) = e^{i\kappa x}, && (16a)\\
u_1(x,t) &= iL_t^{-1}\partial_x^2 u_0(x,t) + iqL_t^{-1}A_0 \\
&= iL_t^{-1}\partial_x^2 e^{i\kappa x} + iqL_t^{-1}[u_0^2\bar{u}_0] \\
&= -i\kappa^2 t e^{i\kappa x} + iqte^{i\kappa x} = -i(\kappa^2 - q)te^{i\kappa x}, && (16b)\\
u_2(x,t) &= iL_t^{-1}\partial_x^2 u_1(x,t) + iqL_t^{-1}A_1 \\
&= iL_t^{-1}\partial_x^2[-i(\kappa^2 - q)te^{i\kappa x}] + iqL_t^{-1}[2u_0u_1\bar{u}_0 + u_0^2\bar{u}_1] \\
&= -\frac{1}{2}\kappa^2(\kappa^2 - q)t^2e^{i\kappa x} + \frac{1}{2}q(\kappa^2 - q)t^2e^{i\kappa x} \\
&= -\frac{1}{2}(\kappa^2 - q)^2t^2e^{i\kappa x}, && (16c)\\
u_3(x,t) &= iL_t^{-1}\partial_x^2 u_2(x,t) + iqL_t^{-1}A_2 \\
&= iL_t^{-1}\partial_x^2[-\frac{1}{2}(\kappa^2 - q)^2t^2e^{i\kappa x}] \\
&\quad + iqL_t^{-1}[2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0 + 2u_0u_1\bar{u}_1 + u_0^2\bar{u}_2] \\
&= \frac{i}{6}(\kappa^2 - q)^3t^3e^{i\kappa x}. && (16d)
\end{aligned}
$$

Now summing up these components yields

$$
\begin{aligned}
u(x,t) &= \sum_{l=0}^{\infty} u_l(x,t) \\
&= e^{i\kappa x}\Big\{1 - i(\kappa^2 - q)t - \frac{1}{2}(\kappa^2 - q)^2t^2 + \frac{i}{6}(\kappa^2 - q)^3t^3 + \dots\Big\} \\
&= e^{i\kappa x}e^{-i\omega t} = e^{i(\kappa x - \omega t)},
\end{aligned}
$$

with $\omega = \kappa^2 - q$ already given in (5), i.e. the method converges to the exact solution.

Example 3 *Secondly, we consider the special case of a x–independent solution $u(t) = f\exp(iq|f|^2 t)$, where f denotes the constant initial value. We get by the ADM (12):*

$$
\begin{aligned}
u_0(t) &= f, && \text{(17a)}\\
u_1(t) &= iqL_t^{-1}A_0 = iqL_t^{-1}[u_0^2\bar{u}_0] = iq|f|^2 ft, && \text{(17b)}\\
u_2(t) &= iqL_t^{-1}A_1 = iqL_t^{-1}[2u_0u_1\bar{u}_0 + u_0^2\bar{u}_1] = -q^2|f|^4 f\,\frac{t^2}{2}, && \text{(17c)}\\
u_3(t) &= iqL_t^{-1}A_2 = iqL_t^{-1}[2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0 + 2u_0u_1\bar{u}_1 + u_0^2\bar{u}_2]\\
&= -iq^3|f|^6 f\,\frac{t^3}{6}. && \text{(17d)}
\end{aligned}
$$

Again, summing up these components yields obviously the exact solution.

Example 4 *In the third example we turn to the soliton solution (2). Following the above we get for the terms $u_l(x,t)$; $l = 1, 2, 3$*

$$u_0(x,t) = f(x) = \sqrt{\frac{2a}{q}}\, e^{\frac{i}{2}\,c\,x}\ \operatorname{sech}\left(\sqrt{a}\,x\right), \tag{18a}$$

$$u_1(x,t) = \frac{1}{4}\sqrt{\frac{2a}{q}}\, e^{\frac{i}{2}\,c\,x}\, t\, c_1 \operatorname{sech}^2\left(\sqrt{a}\,x\right), \tag{18b}$$

where

$$c_1 = i\,\left(4\,a - c^2\right)\cosh(\sqrt{a}\,x) + 4\sqrt{a}\,c\,\sinh(\sqrt{a}\,x),$$

$$u_2(x,t) = -\frac{1}{2^6}\sqrt{\frac{2a}{q}}\, e^{\frac{i}{2}\,c\,x}\, t^2\, c_2\, \operatorname{sech}^3(\sqrt{a}\,x), \tag{18c}$$

where

$$
\begin{aligned}
c_2 = {} & 16\,a^2 + 40\,a\,c^2 + c^4 + \left(16\,a^2 - 24\,a\,c^2 + c^4\right)\cosh(2\sqrt{a}\,x)\\
& - (8\,i)\,\sqrt{a}\,c\,\left(4\,a - c^2\right)\sinh(2\sqrt{a}\,x),
\end{aligned}
$$

$$u_3(x,t) = \frac{1}{3\cdot 2^9}\sqrt{\frac{2a}{q}}\, e^{\frac{i}{2}\,c\,x}\, t^3\, c_3 \operatorname{sech}^4(\sqrt{a}\,x), \tag{18d}$$

where

$$
\begin{aligned}
c_3 = {} & -3\,i\cosh(\sqrt{a}\,x)\left(64\,a^3 + 272a^2c^2 - 68\,a\,c^4 - c^6\right) - i\,\cosh(3\sqrt{a}\,x)\\
& \left(64\,a^3 - 240\,a^2\,c^2 + 60\,a\,c^4 - c^6\right) - 8\,\sqrt{a}\,c\ \ d_3\,\sinh(\sqrt{a}\,x),
\end{aligned}
$$

with

$$d_3 = 48\,a^2 + 152\,a\,c^2 + 3\,c^4 + \left(48\,a^2 - 40\,a\,c^2 + 3\,c^4\right)\cosh(2\sqrt{a}\,x).$$

This tedious calculation above was performed using the symbolic computing package `Mathematica`*. The code can be downloaded from the authors' homepages. Using the code the complicated* $u_4(x,t)$ *can be evaluated.*

Mathematica–Program 1 The Mathematica code

```
Clear["@"]

u[x_, t_]:=(2a/q)^(1/2) Exp[I(1/2cx - (c^2/4 - a)t)] * Sech[a^(1/2)(x - ct)]

u[x, t]/.Complex[0, n_]-> - Complex[0, n]

f[x_] = u[x, 0];

g[x_] = Simplify[D[u[x, t], t]/.t → 0];

u0[x_, t_] = f[x]

A0[x_] = Simplify[u0[x, t]^2 * (u0[x, t]/.Complex[0, n_]-> - Complex[0, n])];

u1[x_, t_] =Simplify[
Expand[I * (q * Integrate[A0[x], {t, 0, t}] + Integrate[D[u0[x, t], {x, 2}], {t, 0, t}])]]

A1[x_] = Simplify[2u0[x, t] * u1[x, t] * (u0[x, t]/.Complex[0, n_]-> - Complex[0, n])+
u0[x, t]^2 * (u1[x, t]/.Complex[0, n_]-> - Complex[0, n])]

u2[x_, t_] =
Simplify[I * (q * Integrate[A1[x], {t, 0, t}] + Integrate[D[u1[x, t], {x, 2}], {t, 0, t}])]

A2[x_] = Simplify[2u0[x, t] * u2[x, t] * (u0[x, t]/.Complex[0, n_]-> - Complex[0, n])+
u1[x, t]^2 * (u0[x, t]/.Complex[0, n_]-> - Complex[0, n])+
2 * u0[x, t] * u1[x, t] * (u1[x, t]/.Complex[0, n_]-> - Complex[0, n])+
u0[x, t]^2 * (u2[x, t]/.Complex[0, n_]-> - Complex[0, n])]

u3[x_, t_] =
Simplify[I * (q * Integrate[A2[x], {t, 0, t}] + Integrate[D[u2[x, t], {x, 2}], {t, 0, t}])]

A3[x_] = Simplify[2u0[x, t] * u3[x, t] * (u0[x, t]/.Complex[0, n_]-> - Complex[0, n])+
2u1[x, t] * u2[x, t] * (u0[x, t]/.Complex[0, n_]-> - Complex[0, n])+
2u0[x, t] * u2[x, t] * (u1[x, t]/.Complex[0, n_]-> - Complex[0, n])+
u1[x, t]^2 * (u1[x, t]/.Complex[0, n_]-> - Complex[0, n])+
2 * u0[x, t] * u1[x, t] * (u2[x, t]/.Complex[0, n_]-> - Complex[0, n])+
u0[x, t]^2 * (u3[x, t]/.Complex[0, n_]-> - Complex[0, n])]

u4[x_, t_] =
Simplify[I * (q * Integrate[A3[x], {t, 0, t}] + Integrate[D[u3[x, t], {x, 2}], {t, 0, t}])]
```

$\mathrm{solu}[\mathrm{x}\_, \mathrm{t}\_] = \mathrm{u0}[\mathbf{x}, \mathbf{t}] + \mathrm{u1}[\mathbf{x}, \mathbf{t}] + \mathrm{u2}[\mathbf{x}, \mathbf{t}] + \mathrm{u3}[\mathbf{x}, \mathbf{t}] + \mathrm{u4}[\mathbf{x}, \mathbf{t}];$

Now using the Adomian decomposition method a solution to the NLS (1) *is approximated by the following expansion*

$$u(x,t) \approx U_3(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t). \quad (19)$$

The approximating Adomian decomposition method was tested to the NLS equation (1) *for the single soliton wave to the problems proposed by Bratsos [9], [10] with the homogeneous boundaries at* $L_0 = -80$ *and* $L_1 = 100$*, and the theoretical solution given by* (2).

In Figure 1, the modulus $z = |u|$ *of the theoretical solution of NLS* (1) *with* $q = 1$*,* $a = 0.01$ *and velocity* $c = 0.1$ *for* $t \in [0, 108]$ *is presented. Whereas in Figure 2 the corresponding approximative solution* $Z = |U_3|$ *can be seen. Finally, in Figure 3 the corresponding error curve is plotted.*

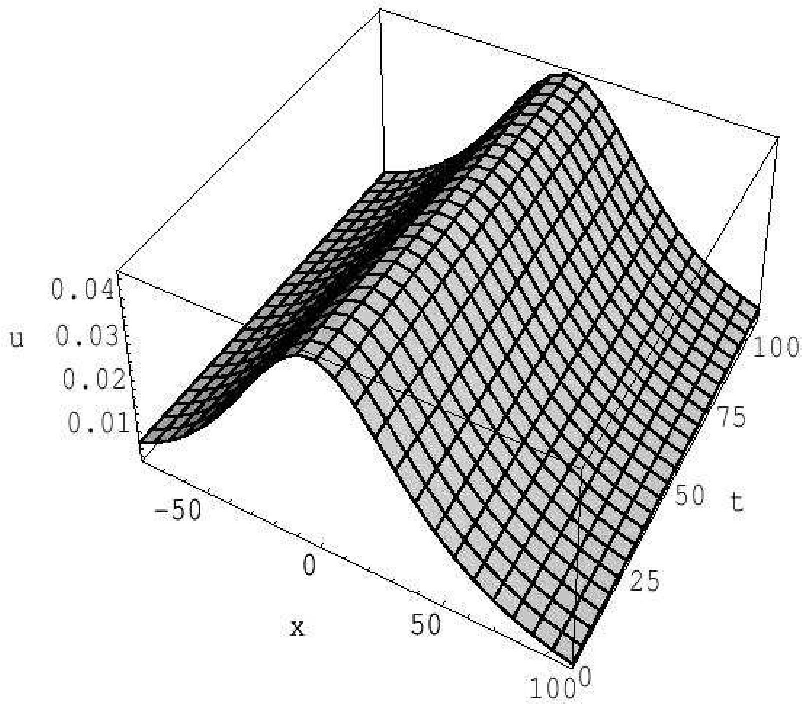

Fig. 1. The surface shows the solution $z = |u|$ for the NLS with $q = 1$, $a = 0.01$ and velocity $c = 0.1$ from $t = 0$ to $t = 108$.

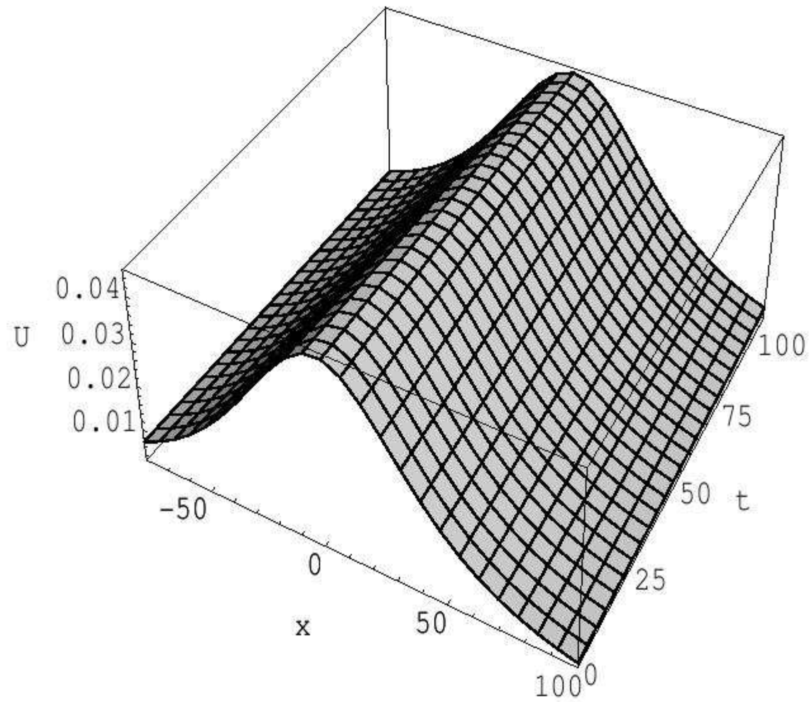

Fig. 2. The surface shows the solution $Z = |U_3|$ for the NLS with $q = 1$, $a = 0.01$ and velocity $c = 0.1$ from $t = 0$ to $t = 108$.

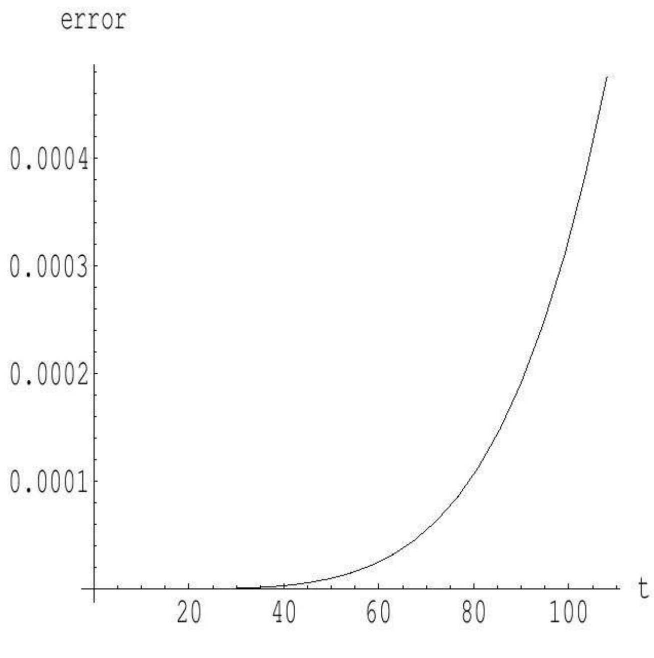

Fig. 3. The figure shows the error curve $|Z - z|$ for the NLS with $q = 1$, $a = 0.01$ and velocity $c = 0.1$ from $t = 0$ to $t = 108$.

4 Discrete nonlinear Schrödinger equations

Discrete nonlinear Schrödinger equations are omnipresent [25] in applied sciences, e.g. describing the propagation of electromagnetic waves in glass fibers, one–dimensional arrays of coupled optical waveguides [17] and light–induced photonic crystal lattices [16]. Moreover, they are used to describe Bose–Einstein condensates in optical lattices [35] and they are an established model for optical pulse propagation in various doped fibers [21], [22].

In this section we will consider the two most common discrete versions of the cubic NLS equation (1a) that arise from different spatial discretizations. These *discrete nonlinear Schrödinger equations* (DNLS) are also called *lattice NLS equations* and we refer the reader to [33, Chapter 5.2.2] for a concise discussion on this topic.

4.1 The standard discrete NLS

If one applies the standard spatial discretization to (1a) and replaces $F(u) = |u|^2 u$ with a *diagonal discretization* $F_D(u_j) = |u_j|^2 u_j$, we obtain the *usual DNLS equation*:

$$i\partial_t u_j + D_h^2 u_j + q|u_j|^2 u_j = 0, \quad j \in \mathbb{Z},\ t > 0, \tag{20a}$$

$$u_j(0) = f_j, \quad j \in \mathbb{Z}, \tag{20b}$$

with $u_j = u_j(t)$, $h = \Delta x$ and $D_h^2 u_j = (u_{j+1} - 2u_j + u_{j-1})/h^2$ denotes the standard second order difference quotient. The parameter $\varepsilon := h^{-2}$ is called *(discrete) dispersion* and the parameter q is called *anharmonicity*, since equation (20a) with $\varepsilon = 0$ describes a set of uncoupled anharmonic oscillators.

The DNLS equation (20a) has a *discrete conserved number* (mass, total excitation norm, power in nonlinear optics)

$$N_D = \frac{2}{q} \sum_{j\in\mathbb{Z}} |u_j|^2 \tag{21}$$

and the *discrete Hamiltonian*

$$H_D = -\sum_{j\in\mathbb{Z}} \Big[u_j^*(u_{j+1} + u_{j-1}) - 2|u_j|^2 + \frac{q}{2}|u_j|^4 \Big], \tag{22}$$

where * denotes the complex conjugate.

However, the standard DNLS equation (20a) is not an exactly integrable DNLS (if the spatial grid consists of more than 2 points) and thus less amenable to mathematical analysis. We can only give particular *discrete plane wave solutions* to the DNLS equation (20a) of the form

$$u_j(t) = \exp\Big[i\Big(j\kappa h - \omega t\Big)\Big], \quad j \in \mathbb{Z},\ t > 0. \tag{23}$$

Inserting (23) into the DNLS (20a) yields the *discrete dispersion relation*

$$\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big) - \omega = q, \tag{24}$$

which is obviously consistent with the continuous relation (5). Hence we will turn in the sequel to an integrable discrete NLS equation.

4.2 The Ablowitz–Ladik equation

After a discretization in space by replacing the cubic nonlinearity $F(u) = |u|^2 u$ in (1a) with an *off–diagonal discretization* $F_{AL}(u_j) = |u_j|^2(u_{j+1} + u_{j-1})/2$ and keeping the time variable continuous we obtain the *Ablowitz–Ladik (AL) equation* [1], [2]:

$$i\partial_t u_j + D_h^2 u_j + q|u_j|^2 \frac{u_{j+1} + u_{j-1}}{2} = 0, \quad j \in \mathbb{Z},\ t > 0, \tag{25a}$$

$$u_j(0) = f_j, \quad j \in \mathbb{Z}. \tag{25b}$$

Note that one term in (25a) can be removed through the transformation

$$u_j(t) = v_j(t)\exp(-i2t), \quad t > 0,$$

and equation (25a) reduces to the *normalized form*

$$i\partial_t v_j + \frac{v_{j+1} + v_{j-1}}{h^2} + q|v_j|^2 \frac{v_{j+1} + v_{j-1}}{2} = 0, \quad j \in \mathbb{Z},\ t > 0. \tag{26}$$

The AL equation has a *conserved number*

$$N_{AL} = \frac{2}{q}\sum_{j\in\mathbb{Z}} \log\Big(1 + \frac{q}{2}|u_j|^2\Big) \tag{27}$$

and the *Hamiltonian*

$$H_{AL} = -\sum_{j\in\mathbb{Z}}\Big[u_j^*(u_{j+1} + u_{j-1}) - \frac{4}{q}\log\Big(1 + \frac{q}{2}|u_j|^2\Big)\Big]. \tag{28}$$

The nonlinear differential–difference equation (25a) is the most famous integrable DNLS equation. As the AL equation (25a) is integrable it is possible to give exact *travelling–wave solutions* on the real line $j \in \mathbb{Z}$, including (cf. [4], [32], [33])

$$v_j(t) = A \exp\Big[i(\omega t + \alpha j + v_0)\Big] \operatorname{cn}\Big[\beta(j - vt); k\Big], \quad t > 0, \tag{29}$$

where $\operatorname{cn}[\cdot; k]$ is a *Jacobi elliptic function of modulus* k. For the case $h = 1$, $q = 2$ the parameters in (29) can be written as

$$A = \frac{k \operatorname{sn}[\beta; k]}{\operatorname{dn}[\beta; k]}, \qquad \omega = \frac{2 \operatorname{cn}[\beta; k] \cos\alpha}{\operatorname{dn}^2[\beta; k]}, \qquad v = \frac{2 \operatorname{sn}[\beta; k] \sin\alpha}{\beta \operatorname{dn}[\beta; k]},$$

where $-\pi \le \alpha \le \pi$, $\beta > 0$, $0 < k < 1$ are free parameters. In the limiting case $k \to 1$ (hyperbolic limit) we get for the Jacobi elliptic (sn, cn, dn) functions [28]:

$$\lim_{k\to 1} \operatorname{sn}[\beta; k] = \tanh\beta, \quad \lim_{k\to 1} \operatorname{cn}[\beta; k] = \lim_{k\to 1} \operatorname{dn}[\beta; k] = \operatorname{sech}\beta,$$

and obtain the *discrete soliton solution* of (26)

$$v_j(t) = \sinh\beta \exp\Big[-i(\omega t + \alpha j + v_0)\Big] \operatorname{sech}\Big[\beta(j - vt)\Big], \quad t > 0, \tag{30}$$

with

$$\omega = -2\cosh\beta\cos\alpha, \qquad v = -\frac{2}{\beta}\sinh\beta\sin\alpha,$$

that can travel at any velocity. It can be easily seen that the discrete soliton (30) is a fairly obvious discrete version of the continuous soliton solution (2).

There exist *discrete plane wave solutions* to the Ablowitz–Ladik equation (25a) of the form (23). Inserting (23) into the AL equation (25a) gives the *discrete dispersion relation*

$$\frac{4}{h^2}\sin^2\!\left(\frac{\kappa h}{2}\right) - \omega = q\cos(\kappa h). \tag{31}$$

Remark 5 *Let us remark that there also exists an explicit solution to the AL equation* (25a) *on a* periodic interval *[8].*

The main interest in the AL equation arises from mathematics (in contrast to the standard DNLS equation); only a few physical models [29] can be described by an AL–type equation.

Remark 6 *We note that there also exists another integrable DNLS equation, namely the* Izergin–Korepin *(IK) equation [24] that shares an important property with the continuous NLS: it has the same r–matrix [19]. However, the IK equation is a quite complicated system and no applications are known yet. Thus we will skip it here.*

5 The semi–discrete Adomian Decomposition Method

The analogue discrete steps to the continuous ADM of §2 are simply the formal solution to the DNLS equation (20) or the AL equation (25):

$$u_j(t) = f_j + iL_t^{-1}D_h^2 u_j + iqL_t^{-1}F_{D,AL}(u_j), \quad j \in \mathbb{Z},\ t > 0, \tag{32}$$

and the assumption that there exists a solution of the series form $u_j(t) = \sum_{l=0}^{\infty} u_{j,l}(t)$. The nonlinear term $F_{D,AL}(u_j)$ in (32) is decomposed into an infinite series of *discrete Adomian polynomials* $F_{D,AL}(u_j) = \sum_{l=0}^{\infty} A_l(u_j)$. Substituting these decompositions into (32) gives

$$\sum_{l=0}^{\infty} u_{j,l}(t) = f_j + i\sum_{l=0}^{\infty} L_t^{-1}D_h^2 u_j(t) + iq\sum_{l=0}^{\infty} L_t^{-1}A_l. \tag{33}$$

Again, $u_{j,0}(t)$ is identified with the initial data f_j and the following recurrence is proposed to determine the solution components $u_{j,l}(t)$:

$$u_{j,0}(t) = f_j, \tag{34a}$$
$$u_{j,l+1}(t) = iL_t^{-1}D_h^2 u_{j,l}(t) + iqL_t^{-1}A_l, \quad l = 0, 1, 2, \ldots \tag{34b}$$

For the standard DNLS equation the Adomian polynomials are the same as (14), but for the AL equation we write $F_{AL}(u_j) = |u_j|^2(u_{j+1} + u_{j-1})/2$ and obtain analogously to (14)

$$A_0 = u_{j,0}\frac{u_{j+1,0} + u_{j-1,0}}{2}\bar{u}_{j,0}, \tag{35a}$$

$$A_1 = \Big[u_{j,0}\frac{u_{j+1,1} + u_{j-1,1}}{2} + \frac{u_{j+1,0} + u_{j-1,0}}{2}u_{j,1}\Big]\bar{u}_{j,0} + u_{j,0}\frac{u_{j+1,0} + u_{j-1,0}}{2}\bar{u}_{j,1}, \tag{35b}$$

$$\begin{aligned} A_2 = {} & \Big[u_{j,0}\frac{u_{j+1,2} + u_{j-1,2}}{2} + \frac{u_{j+1,0} + u_{j-1,0}}{2}u_{j,2}\Big]\bar{u}_{j,0} + u_{j,1}\frac{u_{j+1,1} + u_{j-1,1}}{2}\bar{u}_{j,0} \\ & + \Big[u_{j,0}\frac{u_{j+1,1} + u_{j-1,1}}{2} + \frac{u_{j+1,0} + u_{j-1,0}}{2}u_{j,1}\Big]\bar{u}_{j,1} + u_{j,0}\frac{u_{j+1,0} + u_{j-1,0}}{2}\bar{u}_{j,2}, \end{aligned} \tag{35c}$$

$$\begin{aligned} A_3 = {} & \Big[u_{j,0}\frac{u_{j+1,3} + u_{j-1,3}}{2} + \frac{u_{j+1,0} + u_{j-1,0}}{2}u_{j,3}\Big]\bar{u}_{j,0} \\ & + \Big[u_{j,1}\frac{u_{j+1,2} + u_{j-1,2}}{2} + \frac{u_{j+1,1} + u_{j-1,1}}{2}u_{j,2}\Big]\bar{u}_{j,0} \\ & + \Big[u_{j,0}\frac{u_{j+1,2} + u_{j-1,2}}{2} + \frac{u_{j+1,0} + u_{j-1,0}}{2}u_{j,2}\Big]\bar{u}_{j,1} \\ & + \Big[u_{j,0}\frac{u_{j+1,1} + u_{j-1,1}}{2} + \frac{u_{j+1,0} + u_{j-1,0}}{2}u_{j,1}\Big]\bar{u}_{j,2} \\ & + u_{j,1}\frac{u_{j+1,1} + u_{j-1,1}}{2}\bar{u}_{j,1} + u_{j,0}\frac{u_{j+1,0} + u_{j-1,0}}{2}\bar{u}_{j,3}, \end{aligned} \tag{35d}$$

$$\ldots.$$

The polynomials A_l, $l \geq 4$ can be computed analogously in a tedious calculation. The calculation to obtain the Adomian polynomials for the AL equation (35) was performed using the following `Mathematica` code.

Mathematica–Program 2 We define the functions:

$\mathbf{F}[\mathrm{l\_}]{:=}\frac{\mathbf{1}}{\mathbf{2}}\mathrm{Expand}\Big[$
$\sum_{\mathbf{k=0}}^{\mathbf{l}}\mathbf{u}[\mathbf{k},\mathbf{t},\mathbf{j}]\sum_{\mathbf{k=0}}^{\mathbf{l}}\bar{\mathbf{u}}[\mathbf{k},\mathbf{t},\mathbf{j}]\left(\sum_{\mathbf{k=0}}^{\mathbf{l}}\mathbf{u}[\mathbf{k},\mathbf{t},\mathbf{j}+\mathbf{1}]+\sum_{\mathbf{k=0}}^{\mathbf{l}}\mathbf{u}[\mathbf{k},\mathbf{t},\mathbf{j}-\mathbf{1}]\right)\Big]$

$\bar{\mathbf{u}}[\mathrm{k\_},\mathrm{t\_},\mathrm{l\_}]{:=}\mathbf{u}[\mathbf{k},\mathbf{t},\mathbf{j}]/.\mathrm{Complex}[\mathbf{0},\mathrm{n\_}]\text{->}-\mathrm{Complex}[\mathbf{0},\mathbf{n}]$

$\mathbf{A}[\mathbf{0}]{:=}\mathbf{F}[\mathbf{0}]$

$\mathbf{A}[\mathrm{l\_}]{:=}\mathrm{Expand}\left[\mathbf{F}[\mathbf{l}]-\sum_{\mathbf{k=0}}^{\mathbf{l-1}}\mathbf{A}[\mathbf{k}]\right]$

Using the functions defined we can easily furnish the desired Adomian polynomials, e.g.

$\mathbf{A}[\mathbf{2}] =$

$$\begin{aligned}
&\tfrac{1}{2}u[0,t,j]^2u[2,t,-1+j] + u[0,t,j]u[1,t,j]u[2,t,-1+j] + \\
&\tfrac{1}{2}u[1,t,j]^2u[2,t,-1+j] + u[0,t,-1+j]u[0,t,j]u[2,t,j] + \\
&u[0,t,j]u[0,t,1+j]u[2,t,j] + u[0,t,j]u[1,t,-1+j]u[2,t,j] + \\
&u[0,t,-1+j]u[1,t,j]u[2,t,j] + u[0,t,1+j]u[1,t,j]u[2,t,j] + \\
&u[1,t,-1+j]u[1,t,j]u[2,t,j] + u[0,t,j]u[1,t,1+j]u[2,t,j] + \\
&u[1,t,j]u[1,t,1+j]u[2,t,j] + u[0,t,j]u[2,t,-1+j]u[2,t,j] + \\
&u[1,t,j]u[2,t,-1+j]u[2,t,j] + \tfrac{1}{2}u[0,t,-1+j]u[2,t,j]^2 + \\
&\tfrac{1}{2}u[0,t,1+j]u[2,t,j]^2 + \tfrac{1}{2}u[1,t,-1+j]u[2,t,j]^2 + \\
&\tfrac{1}{2}u[1,t,1+j]u[2,t,j]^2 + \tfrac{1}{2}u[2,t,-1+j]u[2,t,j]^2 + \\
&\tfrac{1}{2}u[0,t,j]^2u[2,t,1+j] + u[0,t,j]u[1,t,j]u[2,t,1+j] + \\
&\tfrac{1}{2}u[1,t,j]^2u[2,t,1+j] + u[0,t,j]u[2,t,j]u[2,t,1+j] + \\
&u[1,t,j]u[2,t,j]u[2,t,1+j] + \tfrac{1}{2}u[2,t,j]^2u[2,t,1+j]
\end{aligned}$$

$\mathbf{A}[\mathbf{3}] =$

$$\begin{aligned}
&\tfrac{1}{2}u[0,j]u[3,-1+j]\bar{u}[0,j] + \tfrac{1}{2}u[1,j]u[3,-1+j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[2,j]u[3,-1+j]\bar{u}[0,j] + \tfrac{1}{2}u[0,-1+j]u[3,j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[0,1+j]u[3,j]\bar{u}[0,j] + \tfrac{1}{2}u[1,-1+j]u[3,j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[1,1+j]u[3,j]\bar{u}[0,j] + \tfrac{1}{2}u[2,-1+j]u[3,j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[2,1+j]u[3,j]\bar{u}[0,j] + \tfrac{1}{2}u[3,-1+j]u[3,j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[0,j]u[3,1+j]\bar{u}[0,j] + \tfrac{1}{2}u[1,j]u[3,1+j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[2,j]u[3,1+j]\bar{u}[0,j] + \tfrac{1}{2}u[3,j]u[3,1+j]\bar{u}[0,j] + \\
&\tfrac{1}{2}u[0,j]u[3,-1+j]\bar{u}[1,j] + \tfrac{1}{2}u[1,j]u[3,-1+j]\bar{u}[1,j] + \\
&\tfrac{1}{2}u[2,j]u[3,-1+j]\bar{u}[1,j] + \tfrac{1}{2}u[0,-1+j]u[3,j]\bar{u}[1,j] +
\end{aligned}$$

$$
\begin{aligned}
&\tfrac{1}{2}u[0,1+j]u[3,j]\bar{u}[1,j] + \tfrac{1}{2}u[1,-1+j]u[3,j]\bar{u}[1,j] + \\
&\tfrac{1}{2}u[1,1+j]u[3,j]\bar{u}[1,j] + \tfrac{1}{2}u[2,-1+j]u[3,j]\bar{u}[1,j] + \\
&\tfrac{1}{2}u[2,1+j]u[3,j]\bar{u}[1,j] + \tfrac{1}{2}u[3,-1+j]u[3,j]\bar{u}[1,j] + \\
&\tfrac{1}{2}u[0,j]u[3,1+j]\bar{u}[1,j] + \tfrac{1}{2}u[1,j]u[3,1+j]\bar{u}[1,j] + \\
&\tfrac{1}{2}u[2,j]u[3,1+j]\bar{u}[1,j] + \tfrac{1}{2}u[3,j]u[3,1+j]\bar{u}[1,j] + \\
&\tfrac{1}{2}u[0,j]u[3,-1+j]\bar{u}[2,j] + \tfrac{1}{2}u[1,j]u[3,-1+j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[2,j]u[3,-1+j]\bar{u}[2,j] + \tfrac{1}{2}u[0,-1+j]u[3,j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[0,1+j]u[3,j]\bar{u}[2,j] + \tfrac{1}{2}u[1,-1+j]u[3,j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[1,1+j]u[3,j]\bar{u}[2,j] + \tfrac{1}{2}u[2,-1+j]u[3,j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[2,1+j]u[3,j]\bar{u}[2,j] + \tfrac{1}{2}u[3,-1+j]u[3,j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[0,j]u[3,1+j]\bar{u}[2,j] + \tfrac{1}{2}u[1,j]u[3,1+j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[2,j]u[3,1+j]\bar{u}[2,j] + \tfrac{1}{2}u[3,j]u[3,1+j]\bar{u}[2,j] + \\
&\tfrac{1}{2}u[0,-1+j]u[0,j]\bar{u}[3,j] + \tfrac{1}{2}u[0,j]u[0,1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,j]u[1,-1+j]\bar{u}[3,j] + \tfrac{1}{2}u[0,-1+j]u[1,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,1+j]u[1,j]\bar{u}[3,j] + \tfrac{1}{2}u[1,-1+j]u[1,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,j]u[1,1+j]\bar{u}[3,j] + \tfrac{1}{2}u[1,j]u[1,1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,j]u[2,-1+j]\bar{u}[3,j] + \tfrac{1}{2}u[1,j]u[2,-1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,-1+j]u[2,j]\bar{u}[3,j] + \tfrac{1}{2}u[0,1+j]u[2,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[1,-1+j]u[2,j]\bar{u}[3,j] + \tfrac{1}{2}u[1,1+j]u[2,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[2,-1+j]u[2,j]\bar{u}[3,j] + \tfrac{1}{2}u[0,j]u[2,1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[1,j]u[2,1+j]\bar{u}[3,j] + \tfrac{1}{2}u[2,j]u[2,1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,j]u[3,-1+j]\bar{u}[3,j] + \tfrac{1}{2}u[1,j]u[3,-1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[2,j]u[3,-1+j]\bar{u}[3,j] + \tfrac{1}{2}u[0,-1+j]u[3,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,1+j]u[3,j]\bar{u}[3,j] + \tfrac{1}{2}u[1,-1+j]u[3,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[1,1+j]u[3,j]\bar{u}[3,j] + \tfrac{1}{2}u[2,-1+j]u[3,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[2,1+j]u[3,j]\bar{u}[3,j] + \tfrac{1}{2}u[3,-1+j]u[3,j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[0,j]u[3,1+j]\bar{u}[3,j] + \tfrac{1}{2}u[1,j]u[3,1+j]\bar{u}[3,j] + \\
&\tfrac{1}{2}u[2,j]u[3,1+j]\bar{u}[3,j] + \tfrac{1}{2}u[3,j]u[3,1+j]\bar{u}[3,j]
\end{aligned}
$$

Example 7 *First we consider the simple example of a plane wave solution* (23) *to the DNLS equation* (20a)*. We obtain by the Adomian decomposition technique* (34) *with the Adomian polynomials* (14) *and the semi–discrete dis-*

persion relation $\omega = (4/h^2)\sin^2\Big(\kappa h/2\Big) - q$ *given in* (24)*:*

$$
\begin{aligned}
u_{j,0}(t) &= f_j = e^{ij\kappa h}, && \text{(36a)}\\
u_{j,1}(t) &= iL_t^{-1}D_h^2 u_{j,0}(t) + iqL_t^{-1}A_0 = iL_t^{-1}D_h^2 e^{ij\kappa h} + iqL_t^{-1}[u_{j,0}^2\bar{u}_{j,0}]\\
&= -i\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big)te^{ij\kappa h} + iqte^{ij\kappa h} = -i\omega te^{ij\kappa h}, && \text{(36b)}\\
u_{j,2}(t) &= iL_t^{-1}D_h^2 u_{j,1}(t) + iqL_t^{-1}A_1\\
&= iL_t^{-1}D_h^2[-i\omega te^{ij\kappa h}] + iqL_t^{-1}[2u_{j,0}u_{j,1}\bar{u}_{j,0} + u_{j,0}^2\bar{u}_{j,1}]\\
&= -\frac{1}{2}\omega\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big)t^2e^{ij\kappa h} + \frac{1}{2}q\omega t^2 e^{i\kappa x} = -\frac{1}{2}\omega^2t^2e^{i\kappa x}, && \text{(36c)}\\
u_{j,3}(t) &= iL_t^{-1}D_h^2 u_{j,2}(t) + iqL_t^{-1}A_2\\
&= iL_t^{-1}D_h^2[-\frac{1}{2}\omega^2t^2e^{ij\kappa h}]\\
&\quad + iqL_t^{-1}[2u_{j,0}u_{j,2}\bar{u}_{j,0} + u_{j,1}^2\bar{u}_{j,0} + 2u_{j,0}u_{j,1}\bar{u}_{j,1} + u_{j,0}^2\bar{u}_{j,2}]\\
&= \frac{i}{6}\omega^3t^3e^{ij\kappa h}. && \text{(36d)}
\end{aligned}
$$

Finally summing up the iterates yields

$$
\begin{aligned}
u_j(t) &= \sum_{l=0}^{\infty} u_{j,l}(t) = e^{ij\kappa h}\Big\{1 - i\omega t - \frac{1}{2}\omega^2t^2 + \frac{i}{6}\omega^3t^3 + \dots\Big\}\\
&= e^{ij\kappa h}e^{-i\omega t} = e^{i(j\kappa h - \omega t)}.
\end{aligned}
$$

Example 8 *Secondly, we want to compute the* x*–independent solution* $u_j(t) = f\exp(iq|f|^2t)$ *(cf. Example 3). In this special case the DNLS equation* (20) *and the AL equation* (25) *coincide and it is fairly easy to see that the ADM yields exactly the desired solution.*

Example 9 *Now we consider the AL equation* (25a) *and a plane wave solution* (23)*. We get using the Adomian decomposition technique* (34) *and the*

Adomian polynomials (35)*:*

$$
\begin{aligned}
u_{j,0}(t) &= f_j = e^{ij\kappa h}, && \text{(37a)}\\
u_{j,1}(t) &= iL_t^{-1}D_h^2 u_{j,0}(t) + iqL_t^{-1}A_0\\
&= iL_t^{-1}D_h^2 e^{ij\kappa h} + iqL_t^{-1}[u_{j,0}\frac{u_{j+1,0}+u_{j-1,0}}{2}\bar{u}_{j,0}]\\
&= -i\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big)te^{ij\kappa h} + iqt\cos(\kappa h) = -i\omega te^{ij\kappa h}, && \text{(37b)}\\
u_{j,2}(t) &= iL_t^{-1}D_h^2 u_{j,1}(t) + iqL_t^{-1}A_1\\
&= iL_t^{-1}D_h^2[-i\omega te^{ij\kappa h}] + iqL_t^{-1}\Big[u_{j,0}\frac{u_{j+1,0}+u_{j-1,0}}{2}\bar{u}_{j,1}\\
&\quad + \Big[u_{j,0}\frac{u_{j+1,1}+u_{j-1,1}}{2} + \frac{u_{j+1,0}+u_{j-1,0}}{2}u_{j,1}\Big]\bar{u}_{j,0}\Big]\\
&= -\frac{1}{2}\omega\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big)t^2e^{ij\kappa h} + \frac{1}{2}q\cos(\kappa h)\omega t^2e^{i\kappa x} = -\frac{1}{2}\omega^2t^2e^{i\kappa x}, && \text{(37c)}\\
u_{j,3}(t) &= iL_t^{-1}D_h^2 u_{j,2}(t) + iqL_t^{-1}A_2\\
&= iL_t^{-1}D_h^2[-\frac{1}{2}\omega^2t^2e^{ij\kappa h}] + iqL_t^{-1}\Big[u_{j,1}\frac{u_{j+1,1}+u_{j-1,1}}{2}\bar{u}_{j,0} && \text{(37d)}\\
&\quad + \Big[u_{j,0}\frac{u_{j+1,2}+u_{j-1,2}}{2} + \frac{u_{j+1,0}+u_{j-1,0}}{2}u_{j,2}\Big]\bar{u}_{j,0}\\
&\quad + \Big[u_{j,0}\frac{u_{j+1,1}+u_{j-1,1}}{2} + \frac{u_{j+1,0}+u_{j-1,0}}{2}u_{j,1}\Big]\bar{u}_{j,1}\\
&\quad + u_{j,0}\frac{u_{j+1,0}+u_{j-1,0}}{2}\bar{u}_{j,2}\Big]\\
&= \frac{i}{6}\omega^3t^3e^{ij\kappa h}, && \text{(37e)}
\end{aligned}
$$

where $\omega = (4/h^2)\sin^2\big(\kappa h/2\big) - q\cos(\kappa h)$ *is given in* (31)*. Finally summing up the iterates yields again the exact plane wave solution*

$$
u_j(t) = \sum_{l=0}^{\infty} u_{j,l}(t) = e^{ij\kappa h}e^{-i\omega t} = e^{i(j\kappa h-\omega t)}.
$$

6 Fully discrete nonlinear Schrödinger equations

In this section we consider equations with a discrete time variable, i.e. fully discrete NLS equations.

A fully discrete NLS is the implicit *Durán–Sanz-Serna finite difference scheme* [15] which is a modification of the usual Crank–Nicolson scheme. This scheme is very well designed for computing soliton solutions [15]. It is given by

$$iD_\tau^+ u_j^n + D_h^2 u_j^{n+\frac{1}{2}} + q\left|u_j^{n+\frac{1}{2}}\right|^2 u_j^{n+\frac{1}{2}} = 0, \quad j \in \mathbb{Z},\ n \in \mathbb{N}_0, \tag{38a}$$

$$u_j^0 = f_j, \quad j \in \mathbb{Z}, \tag{38b}$$

with $u_j^n \sim u(jh, n\tau)$, $\tau = \Delta t$, where $D_\tau^+ u_j^n = (u_j^{n+1} - u_j^n)/\tau$ denotes the standard forward–in–time difference quotient and we used in (38a) the time average in $u_j^{n+\frac{1}{2}} = (u_j^{n+1} + u_j^n)/2$. The cubic nonlinearity is discretized by $F_{DSS}(u_j^{n+\frac{1}{2}}) = \left|u_j^{n+\frac{1}{2}}\right|^2 u_j^{n+\frac{1}{2}}$.

Since (38a) is not integrable we can only give particular *discrete plane wave solutions* to the DSS–scheme (38) of the form

$$u_j^n = \exp\Big[i\Big(j\kappa h - n\omega\tau\Big)\Big], \quad j \in \mathbb{Z},\ n \in \mathbb{N}. \tag{39}$$

Inserting the solution ansatz (39) into (38a) yields

$$i\,\frac{e^{-i\omega\tau} - 1}{\tau} - \frac{2}{h^2}(e^{-i\omega\tau} + 1)\sin^2\Big(\frac{\kappa h}{2}\Big) + q\,\frac{|e^{-i\omega\tau} + 1|^2}{4}\,\frac{e^{-i\omega\tau} + 1}{2} = 0,$$

i.e. we obtain the *discrete dispersion relation*

$$\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big) - \frac{2}{\tau}\tan\Big(\frac{\omega\tau}{2}\Big) = q\,\frac{|e^{-i\omega\tau} + 1|^2}{4}. \tag{40}$$

Finally, using the series representation of the tan function in (40) we get

$$\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big) - \frac{2}{\tau}\Big(\frac{\omega\tau}{2} + \frac{(\omega\tau)^3}{24} + \dots\Big) = q\Big(1 - \frac{(\omega\tau)^2}{4} + \frac{(\omega\tau)^4}{48} + \dots\Big), \tag{41}$$

which is consistent with the semi–discrete relation (24) of the DNLS equation.

7 The fully discrete Adomian Decomposition Method

Finally we adopt the steps of the semi–discrete ADM of §5 to solve fully discrete equations. To do so, we assume a formal solution to the Durán–Sanz-Serna scheme (38):

$$u_j^n = f_j + i(D_\tau^+)^{-1} D_h^2 u_j^{n+\frac{1}{2}} + iq(D_\tau^+)^{-1} F_{DSS}(u_j^{n+\frac{1}{2}}), \quad j \in \mathbb{Z},\ n \in \mathbb{N}_0, \tag{42}$$

where the inverse discrete operator is given by

$$(D_\tau^+)^{-1} v^n = \tau \sum_{m=0}^{n-1} v^m, \quad n \in \mathbb{N}_0. \tag{43}$$

Note that using this definition (43) we get

$$(D_\tau^+)^{-1} D_\tau^+ u_j^n = u_j^n - u_j^0.$$

We assume that there exists a solution of the series form $u_j^n = \sum_{l=0}^{\infty} u_{j,l}^n$, where the components $u_{j,l}^n$ are going to be determined recurrently. Again, the nonlinear term $F_{DSS}(u_j^{n+\frac{1}{2}})$ in (42) is decomposed into an infinite series of *discrete Adomian polynomials* $F_{DSS}(u_j) = \sum_{l=0}^{\infty} A_l(u_j)$. Substituting these decompositions into (42) gives

$$\sum_{l=0}^{\infty} u_{j,l}^n = f_j + i \sum_{l=0}^{\infty} (D_\tau^+)^{-1} D_h^2 u_j^{n+\frac{1}{2}} + iq \sum_{l=0}^{\infty} (D_\tau^+)^{-1} A_l. \tag{44}$$

Again, $u_{j,0}^n$ is identified with the initial data f_j and the following recurrence is proposed to determine the solution components $u_{j,l}^{n+\frac{1}{2}}$:

$$u_{j,0}^n = f_j, \tag{45a}$$

$$u_{j,l+1}^n = i (D_\tau^+)^{-1} D_h^2 u_{j,l}^{n+\frac{1}{2}} + iq (D_\tau^+)^{-1} A_l, \quad l = 0, 1, 2, \ldots \tag{45b}$$

Note that the Adomian polynomials for the DSS–scheme are the same as (14) and for the DNLS equation.

Example 10 *We consider the example of a plane wave solution* (39) *to the Durán–Sanz-Serna equation* (38)*. We obtain by the Adomian decomposition technique* (45) *with the Adomian polynomials* (14) *and the discrete dispersion relation* (41) *rewritten in the form ($\tau = \Delta t$):*

$$\frac{4}{h^2} \sin^2\Big(\frac{\kappa h}{2}\Big) - q = \underbrace{w\Big(1 + \frac{(\omega\tau)^2}{12} + \ldots\Big) - q \frac{(\omega\tau)^2}{4}\Big(1 - \frac{(\omega\tau)^2}{12} + \ldots\Big)}_{=:\omega(\kappa)},$$

$$u_{j,0}^n = f_j = e^{ij\kappa h}, \tag{46a}$$

$$\begin{aligned} u_{j,1}^n &= i(D_\tau^+)^{-1} D_h^2 e^{ij\kappa h} + iq(D_\tau^+)^{-1}\Big[\Big(u_{j,0}^{n+\frac{1}{2}}\Big)^2 \bar{u}_{j,0}^{n+\frac{1}{2}}\Big] \\ &= -i\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big) t_n e^{ij\kappa h} + iqt_n e^{ij\kappa h} \\ &= -it_n\Big[\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big) - q\Big] e^{ij\kappa h}, \end{aligned} \tag{46b}$$

$$\begin{aligned} u_{j,2}^n &= i(D_\tau^+)^{-1} D_h^2 u_{j,1}^{n+\frac{1}{2}} + iq(D_\tau^+)^{-1} A_1 \\ &= (D_\tau^+)^{-1} D_h^2 \Big[t_{n+\frac{1}{2}}\omega(\kappa) e^{ij\kappa h}\Big] \\ &\quad + iq(D_\tau^+)^{-1}\Big[2u_{j,0}^{n+\frac{1}{2}} u_{j,1}^{n+\frac{1}{2}} \bar{u}_{j,0}^{n+\frac{1}{2}} + \Big(u_{j,0}^{n+\frac{1}{2}}\Big)^2 \bar{u}_{j,1}^{n+\frac{1}{2}}\Big] \\ &= -\frac{t_n^2}{2}\omega(\kappa)\frac{4}{h^2}\sin^2\Big(\frac{\kappa h}{2}\Big) e^{ij\kappa h} + \frac{t_n^2}{2} q\omega(\kappa) e^{i\kappa x} = -\frac{t_n^2}{2}\omega(\kappa)^2 e^{i\kappa x}, \end{aligned} \tag{46c}$$

$$\begin{aligned} u_{j,3}^n &= i(D_\tau^+)^{-1} D_h^2 u_{j,2}^{n+\frac{1}{2}} + iq(D_\tau^+)^{-1} A_2 \\ &= i(D_\tau^+)^{-1} D_h^2 [-\frac{t_n^2}{2}\omega(\kappa)^2 e^{ij\kappa h}] + iq(D_\tau^+)^{-1}\Big[2u_{j,0}^{n+\frac{1}{2}} u_{j,2}^{n+\frac{1}{2}} \bar{u}_{j,0}^{n+\frac{1}{2}} \\ &\quad + \Big(u_{j,1}^{n+\frac{1}{2}}\Big)^2 \bar{u}_{j,0}^{n+\frac{1}{2}} + 2u_{j,0}^{n+\frac{1}{2}} u_{j,1}^{n+\frac{1}{2}} \bar{u}_{j,1}^{n+\frac{1}{2}} + \Big(u_{j,0}^{n+\frac{1}{2}}\Big)^2 \bar{u}_{j,2}^{n+\frac{1}{2}}\Big] \\ &= i\frac{n(n^2+\frac{1}{2})}{6}\tau^3\omega(\kappa)^3 e^{ij\kappa h} = i\frac{1}{6}(t_n^3 + t_n\tau^2)\,\omega(\kappa)^3 e^{ij\kappa h}. \end{aligned} \tag{46d}$$

For the discrete time integration in (46) *we used well–known formulas for finite sums like* $\sum_{m=1}^n (2m-1) = n^2$, $\sum_{m=1}^n m^2 = n(n+1)(2n+1)/6$, *etc. The results in* (46) *look similar to the semi–discrete ones of Example 7. However, there is a small error of order* $t_n\tau^2$ *introduced in* $u_{j,3}^n$.

Conclusions

In this work we have shown how the well–known Adomian decomposition technique can be adapted to use for (semi)–discrete equations. We applied our findings to continuous, (semi)–discrete and fully discrete nonlinear Schrödinger equations and presented some illustrative examples including two `Mathematica` program codes.

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