On Linear-Quadratic Control Theory of Implicit Difference Equations

Master Thesis

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Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den 13. August 2016

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Abstract

In this master's thesis we adapt recent results on optimal control of continuous-time linear differential-algebraic equations to the discrete-time case of implicit difference equations. First, we adapt equivalent characterizations of solvability of the so-called Kalman-Yakubovich-Popov inequality for differential-algebraic equations to the case of implicit difference equations. That is, we relate the solvability of a certain matrix inequality to the positivity of the Popov function on the unit circle. An essential difference between the continuous-time and the discrete-time linear-quadratic optimal control problem is due to different structures occurring during the analysis in the form of even or palindromic matrix pencils, respectively. Therefore, with the help of certain structured Kronecker canonical forms, we adapt characterizations of inertia of even matrix pencils to palindromic matrix pencils. To this end, we first introduce a suitable notion of inertia for palindromic matrix pencils. These results are used – analogously to the continuous-time case – to characterize solvability of Lur'e equations equivalently by the existence of certain deflating subspaces of the palindromic matrix pencil. Then we use these findings to describe feasibility and the structure of solutions of the linear-quadratic control problem with infinite time horizon. Finally, these results are illustrated by means of an example.

Zusammenfassung

In dieser Masterarbeit werden aktuelle Ergebnisse zur Optimalsteuerung zeitkontinuierlicher linearer differentiell-algebraischer Gleichungen auf den zeitdiskreten Fall impliziter Differenzengleichungen übertragen. Zunächst werden äquivalente Charakterisierungen zur Lösung der sogenannten Kalman-Yakubovich-Popov-Ungleichung für differentiell-algebraische Gleichungen auf den Fall impliziter Differenzengleichungen übertragen, d. h. die Lösung einer bestimmten linearen Matrixungleichung wird mit der Positivität der Popov-Funktion auf dem Einheitskreis in Verbindung gebracht. Ein wesentlicher Unterschied zwischen dem zeitdiskreten und dem zeitkontinuierlichen linear-quadratischen Optimalsteuerungsproblem besteht in der bei der Analyse auftretenden Struktur in Form eines geraden bzw. palindromischen Matrizenbüschels. Anschließend werden daher – mithilfe geeigneter strukturierter Kronecker-Normalformen – Charakterisierungen über die Trägheit für gerade Matrizenbüschel auf palindromische Matrizenbüschel übertragen. Dazu wird zunächst ein geeigneter Trägheitsbegriff für palindromische Matrizenbüschel eingeführt. Diese Ergebnisse werden benutzt, um – analog zum zeitkontinuierlichem Fall – die Lösbarkeit von Lur'e-Gleichungen äquivalent durch die Existenz bestimmter invarianter Unterräume des palindromischen Matrizenbüschels zu beschreiben. Danach werden die gewonnenen Resultate genutzt, um Zulässigkeit und die Struktur von Lösungen des linear-quadratischen Optimalsteuerungsproblems für implizite Differenzengleichungen mit unendlichem Zeithorizont zu untersuchen. Abschließend werden die Ergebnisse anhand eines Beispiels verdeutlicht.

Nomenclature

$\varnothing$	empty set
$\mathbb{N}$	$= \{1, 2, \ldots\}$; set of natural numbers
$\mathbb{N}_0$	$= \mathbb{N} \cup \{0\}$
$\mathbb{R}$	field of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}_0^+$	set of non-negative real numbers
$\mathbb{C}$	field of complex numbers
$\Re(\alpha)$	real part of a complex number $\alpha \in \mathbb{C}$
$\Im(\alpha)$	imaginary part of a complex number $\alpha \in \mathbb{C}$
$\mathbb{K}$	$\in \{\mathbb{C}, \mathbb{R}\}$
$K^{\mathbb{N}_0}$	set of all sequences $x = (x_j)_j$ whose components lie in the space K
$\mathbb{K}^{m\times n}$	set of m by n matrices over $\mathbb{K}$
$A^{\frac{1}{2}}$	the unique square root of a positive semi-definite matrix $A \in \mathbb{K}^{n\times n}$
$\det A$	determinant of a matrix $A \in \mathbb{K}^{n\times n}$
$\operatorname{tr} A$	trace of a matrix $A \in \mathbb{K}^{n\times n}$
A^*	complex transpose of a matrix $A \in \mathbb{K}^{m\times n}$
A^+	Moore-Penrose pseudo inverse of a matrix $A \in \mathbb{K}^{m\times n}$
A^g	a generalized inverse of a matrix $A \in \mathbb{K}^{m\times n}$, i. e., $AA^gA = A$
A^{-*}	conjugate transpose of the inverse of an invertible matrix $A \in \mathbb{K}^{m\times n}$
$\operatorname{In}(A)$	$= (n_+, n_0, n_-)$; inertia of a quasi-Hermitian $A \in \mathbb{K}^{n\times n}$, where n_+, n_0, and n_- denote the number of eigenvalues $\lambda = r\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}$ where r is positive, zero, or negative, respectively, $\omega \in [0, 2\pi)$

$\mathbb{K}^{m\times n}[z]$	ring of polynomial matrices with coefficients in $\mathbb{K}$
$\mathbb{K}^{m\times n}(z)$	field of rational matrices with coefficients in $\mathbb{K}$
$\operatorname{rk}_{\mathbb{K}(z)} A(z)$	rank of a rational matrix $A(z) \in \mathbb{K}^{m\times n}(z)$
$G^{\sim}(z)$	$:= G\left(\overline{z}^{-1}\right)^*$ for the rational matrix $G(z) \in \mathbb{K}^{n\times n}(z)$
$\|x\|_2$	2-norm of a vector $x \in \mathbb{K}^n$
$\ell^2(\mathbb{K}^n)$	space of quadratic-summable sequences $x \in (\mathbb{K}^n)^{\mathbb{N}_0}$, i. e., $\sum_{k=0}^{\infty} \|x_j\|_2 < \infty$
$\|x\|_{\ell^2}$	$= (\sum_{k=0}^{\infty} \|x_j\|_2)^{\frac{1}{2}}$; ℓ^2-norm of a sequence $x \in \ell^2(\mathbb{K}^n)$
$\frac{\mathrm{d}}{\mathrm{d}t}$	differentiation operator, i. e., $\frac{\mathrm{d}}{\mathrm{d}t}x(\cdot) = \dot{x}(\cdot)$
Δ_h	discretization operator, see (2.10)
σ	shift operator, i. e., $\sigma(x_j)_j = (x_{j+1})_j$
$\mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$	set of all $[\,x^*\ \ u^*\,]^*$ which solve the discrete-time IDE (2.6) for some consistent $x^0 \in \mathbb{K}^n$
$\mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,\,A,\,B)}$	set of all $[\,x^*\ \ u^*\,]^*$ which solve the continuous-time DAE (2.8) for some consistent $x^0 \in \mathbb{K}^n$
$\Sigma_{m,n}(\mathbb{K})$	set of all $(E,\,A,\,B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ with regular $zE - A$
$\Sigma^{w}_{m,n}(\mathbb{K})$	set of all $(E,\,A,\,B,\,Q,\,S,\,R) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{m\times m}$, where $(E,\,A,\,B) \in \Sigma_{m,n}(\mathbb{K})$ and Q and R are Hermitian
$\mathcal{W}^c$	set of $x^0 \in \mathbb{K}^n$ such that there exist $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ so that the IDE (2.6) is solvable
$\mathcal{J}^{\sigma}(x,u)$	objective function of the discrete-time optimal control problem (2.27)
$\mathcal{J}^{\frac{\mathrm{d}}{\mathrm{d}t}}(x,u)$	objective function of the continuous-time optimal control problem (2.28)
$\mathcal{W}^{\sigma}$	set of all $x^0 \in \mathbb{K}^n$ such that there exists $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$ with $Ex_0 = Ex^0$
$\mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}$	set of all $x^0 \in \mathbb{K}^n$ such that there exists $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,\,A,\,B)}$ with $Ex(0) = Ex^0$

1 Introduction

Differential-algebraic equations arise when modeling the behavior of dynamical systems [Lue77; ND89]. For instance, these can be electrical circuits [Ebe08; Rei10] or mechanical multibody systems [Ste06]. For the analysis of these mostly nonlinear problems one usually employs some linearization techniques [Cam95; KM06]. If one chooses a stationary reference solution one ends up in a linear system of the form

$$E\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $E, A \in \mathbb{K}^{n\times n}, B \in \mathbb{K}^{n\times m}$ are some matrices, $x(t) \in \mathbb{K}^n$ is *state*, and $u(t) \in \mathbb{K}^m$ denotes the *input*. This – in addition to the fact that some applications can even be modeled directly by a linear system as in (1.1) – makes it important to understand the structure of these linear systems. However, in practice it is not always possible to reflect the continuous-time character of equation (1.1). Measurements are usually made at *discrete* time points only. This leads to *implicit difference equations* of the form

$$E\sigma x_j = Ax_j + Bu_j, \tag{1.2}$$

where σ denotes the shift operator, i. e., $\sigma x_j = x_{j+1}$ and $(x_j)_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$, $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ are some sequences.

A particularly important problem for systems (1.1) and (1.2) is the linear-quadratic optimal control problem [Bac06; KM04; LR95; Meh91], that means finding an input u such that a certain functional given by

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t \tag{1.3}$$

subject to (1.1) in the continuous-time case or

$$\sum_{j=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} \tag{1.4}$$

subject to (1.2) in the discrete-time case, respectively, is minimized. Directly connected to the optimal control problem are specially structured matrix pencils in the form of even matrix pencils in the continuous-time case or BVD or palindromic matrix pencils in the discrete-time case.

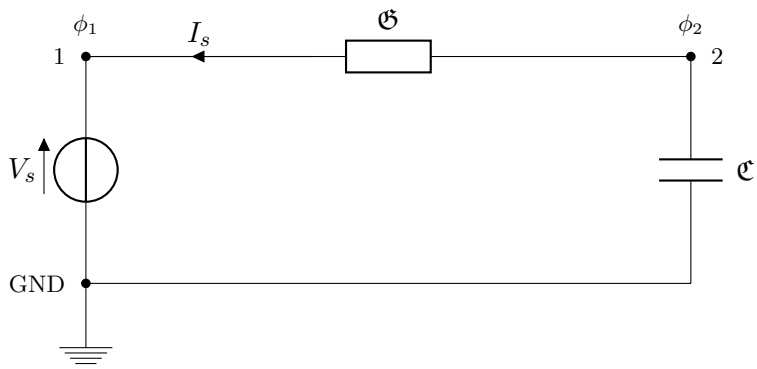

Figure 1.1: Simple Electrical Circuit

As an instructive example we consider the simple electrical circuit as in Figure 1.1 consisting of a voltage source V_s, a conductance $\mathfrak{G} > 0$, and a capacitor $\mathfrak{C} > 0$; all connected in series. Using the modified nodal analysis [ET00], i. e., writing down Kirchhoff's current law for nodes 1 and 2, we obtain the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{C} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{bmatrix} = \begin{bmatrix} -\mathfrak{G} & \mathfrak{G} \\ \mathfrak{G} & -\mathfrak{G} \end{bmatrix} \begin{bmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} I_s, \tag{1.5}$$

where ϕ_1, ϕ_2 denote the node potentials and I_s is the input current from the voltage source. Set $x = [\,\phi_1^* \;\; \phi_2^*\,]^*$ and $u = I_s$. Then, discretization of this equation with the explicit Euler method with stepsize h, i. e., replacing $\dot{x}(t_j)$, where $t_j = t_{j-1} + h$, by

$$\frac{x(t_{j+1}) - x(t_j)}{h}, \tag{1.6}$$

leads to the system

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{C} \end{bmatrix} \sigma \begin{bmatrix} (\phi_{1,j})_j \\ (\phi_{2,j})_j \end{bmatrix} = \begin{bmatrix} -h\mathfrak{G} & h\mathfrak{G} \\ h\mathfrak{G} & \mathfrak{C} - h\mathfrak{G} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} -h \\ 0 \end{bmatrix} I_s. \tag{1.7}$$

We will be interested in finding solutions to this system such that the expression

$$\sum_{j=0}^{\infty} \|x_j\|_2^2 + \|u_j\|_2^2 \tag{1.8}$$

is minimized, i. e., we set $Q = I_2$, $S = 0$, and $R = 1$ in (1.3).

The emphasis in this thesis is to provide analogous results in the discrete-time setting to what was obtained in [Rei11; RRV15; Voi15] for continuous-time systems. Therefore,

at first, in Chapter 2 we recap basic matrix and control theoretic notations and results. In Chapter 3 we introduce a variant of the Kalman-Yakubovich-Popov inequality for implicit difference equations given by

$$\begin{bmatrix} A^*PA - E^*PE + Q & A^*PB + S \\ B^*PA + S^* & B^*PB + R \end{bmatrix} \succeq_{\mathcal{V}^\Sigma} 0, \qquad P = P^*, \tag{1.9}$$

a discrete-time version of the inequality introduced in [RRV15]. We show statements which relate the solvability of this inequality on a certain subspace $\mathcal{V}^\Sigma$ to the non-negativity of the Popov function on the unit circle, a certain rational matrix function defined by

$$\Phi(z) := \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix}^\sim \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}^{m\times m}(z), \tag{1.10}$$

where $G^\sim(z) := G\left(\overline{z}^{-1}\right)^*$ for a rational matrix $G(z) \in \mathbb{K}^{n\times n}(z)$.

In Chapter 4 we introduce the notion of inertia for palindromic matrix pencils evaluated at the unit circle and provide spectral characterizations regarding positivity of the Popov function, similar to the characterizations which were obtained in [Rei11] and [Voi15] for even matrix pencils in the continuous-time case.

In Chapter 5 we investigate Lur'e equations – a more general form of algebraic Riccati equations [LR95; Meh91] – for the discrete-time control problem, where for $q := \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$ a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ with $X = X^*$ fulfills

$$\begin{bmatrix} A^*XA - E^*XE + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} =_{\mathcal{V}^\Sigma} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}. \tag{1.11}$$

We show that solvability of this equation – in analogy to the results in [Rei11; RRV15] for even matrix in the continuous-time case – can be related to the existence of certain deflating subspaces of corresponding BVD matrix pencils or palindromic matrix pencils, respectively.

Finally, in Chapter 6 we apply these results to the optimal control problem and describe feasibility and the structure of optimal solutions.

2 Mathematical Preliminaries

In this chapter we introduce basic notions that we need for the investigations in this thesis.

2.1 Matrix Theoretic Concepts

First we recall some fundamentals of matrices, matrix polynomials, and rational matrices.

2.1.1 Matrix Inequalities

Matrix inequalities play an important role in optimal and robust control. Here we consider matrix inequalities on a certain subspace.

Definition 2.1. Let $A \in \mathbb{K}^{n\times n}$ and $\mathcal{V} \subseteq \mathbb{K}^n$ be some subspace. Then we call A positive (semi-)definite on $\mathcal{V}$ if $x^*Ax > 0$ ($x^*Ax \geq 0$) for all $x \in \mathcal{V} \setminus \{0\}$ and we write $A \succ_{\mathcal{V}} 0$ ($A \succeq_{\mathcal{V}} 0$). Furthermore, A is called negative (semi-)definite if $-A$ is positive (semi-)definite and we write $A \prec_{\mathcal{V}} 0$ ($A \preceq_{\mathcal{V}} 0$). If A is positive and negative semi-definite we write $A =_{\mathcal{V}} 0$.

Further, let $B \in \mathbb{K}^{n\times n}$. By $A \succeq_{\mathcal{V}} B$, $A \succ_{\mathcal{V}} B$, or $A =_{\mathcal{V}} B$ we mean $A - B \succeq_{\mathcal{V}} 0$, $A - B \succ_{\mathcal{V}} 0$, and $A - B =_{\mathcal{V}} 0$, respectively.

Note that for $\mathcal{V} = \mathbb{K}^n$ this reduces to the standard definition of definiteness. If this is the case then we omit the subscript $\mathcal{V}$ in the inequalities.

Lemma 2.2. *Let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times k}$, and $\mathcal{V} \subseteq \mathbb{K}^n$ be some subspace with $\operatorname{im} B \subseteq \mathcal{V}$. Then we have:*

(a) For $A \succeq_{\mathcal{V}} 0$ it follows that $B^*AB \succeq 0$.

(b) If $\operatorname{im} B = \mathcal{V}$, then $B^*AB \succeq 0$ implies $A \succeq_{\mathcal{V}} 0$.

(c) Let $\tilde{\mathcal{V}} \subseteq \mathbb{K}^n$ be a subspace such that $\tilde{\mathcal{V}} = T^{-1}\mathcal{V}$ for some invertible $T \in \mathbb{K}^{n\times n}$. Then we have $A \succeq_{\mathcal{V}} 0$ if and only if $T^*AT \succeq_{\tilde{\mathcal{V}}} 0$.
In particular, if $\mathcal{V} = \tilde{\mathcal{V}} = \mathbb{K}^n$ then for every invertible $T \in \mathbb{K}^{n\times n}$ this means that $A \succeq 0$ if and only if $T^*AT \succeq 0$.

Proof. First we show statement (a). Thus, let $y = Bx \in \operatorname{im} B \subseteq \mathcal{V}$ for some arbitrary $x \in \mathbb{K}^k$ be given. Then we obtain

$$x^*(B^*AB)x = (Bx)^*A(Bx) \geq 0$$

and thus $B^*AB \succeq 0$.

For statement (b) assume that $y \in \mathcal{V}$ is given. We can write $y = Bx$ for some $x \in \mathbb{K}^k$ and thus

$$y^*Ay = (Bx)^*A(Bx) = x^*(B^*AB)x \geq 0.$$

Hence, $A \succeq_{\mathcal{V}} 0$.

Now we show part (c). Let $B \in \mathbb{K}^{n\times k}$ be some matrix with $\operatorname{im} B = \mathcal{V}$. Then $\tilde{B} := T^{-1}B$ fulfills $\operatorname{im} \tilde{B} = \tilde{\mathcal{V}}$. Assume that $A \succeq_{\mathcal{V}} 0$. By (a) we obtain that $B^*AB \succeq 0$ and thus also $\tilde{B}^*(T^*AT)\tilde{B} \succeq 0$. Hence, by (b) we have $T^*AT \succeq_{\tilde{\mathcal{V}}} 0$. The converse direction follows analogously. □

We have seen that transformations of the form T^*AT for matrices $A \in \mathbb{K}^{n\times n}$ and invertible $T \in \mathbb{K}^{n\times n}$ can be used in matrix inequalities. Thus, we introduce the following notation.

Definition 2.3. Let $A, B \in \mathbb{K}^{n\times n}$ be given. Then A and B are *congruent* if there exists an invertible $T \in \mathbb{K}^{n\times n}$ such that

$$T^*AT = B.$$

If this is the case then we write $A \overset{T}{\sim} B$ and if the transformation matrix T is not of interest we may also write $A \sim B$.

A useful tool for characterizing positivity of a block matrix is the *Schur complement*.

Proposition 2.4. *[HZ05, Section 1.6] Let an Hermitian matrix*

$$X = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{K}^{n+m\times n+m}$$

be given and denote by A^g and C^g generalized inverses of A and C satisfying $AA^gA = A$ and $CC^gC = C$, respectively. Then the following are equivalent.

(a) We have that $X \succeq 0$.

(b) It holds that $A \succeq 0$, $C - B^*A^gB \succeq 0$, and $(I_n - AA^g)B = 0$.

(c) It holds that $C \succeq 0$, $A - BC^gB^* \succeq 0$, and $(I_n - CC^g)B^* = 0$.

The matrices $C - B^*A^gB$ and $A - BC^gB^*$ are called *generalized Schur complement* with respect to A and C, respectively.

2.1.2 Matrix Pencils

In this subsection we introduce well-known concepts for general matrix pencils $zE - A \in \mathbb{K}^{m\times n}[z]$. These are matrix polynomials of degree less or equal one.

Definition 2.5. A matrix pencil $zE - A \in \mathbb{K}^{m\times n}[z]$ is called *regular* if $m = n$ and

$$\det(zE - A) \in \mathbb{K}[z] \setminus \{0\}.$$

Otherwise it is called *singular*.

For a regular pencil $zE - A \in \mathbb{K}^{n\times n}[z]$ we define eigenvalues and eigenvectors.

Definition 2.6. Let $zE - A \in \mathbb{K}^{n\times n}[z]$ be regular. A scalar $\lambda \in \mathbb{C}$ is called *(finite) eigenvalue* of the pencil $zE - A \in \mathbb{K}^{n\times n}[z]$ if

$$\det(\lambda E - A) = 0.$$

In addition, $\lambda = \infty$ is called *(infinite) eigenvalue* of $zE - A$ if $\det(E) = 0$. Vectors $0 \neq x \in \mathbb{K}^n$ such that $(\lambda E - A)x = 0$ in the case of finite eigenvalues or $Ex = 0$ in the case of infinite eigenvalues, respectively, are called *eigenvectors* of the pencil $zE - A \in \mathbb{K}^{n\times n}[z]$.

The definition of infinite eigenvalues ∞ is justified by the fact that every finite eigenvalue also fulfills

$$\det\left(\frac{1}{\lambda}(\lambda E - A)\right) = 0$$

and

$$\lim_{\lambda\to\infty} \det\left(\frac{1}{\lambda}(\lambda E - A)\right) = \lim_{\lambda\to\infty} \det\left(E - \frac{1}{\lambda}A\right) = \det(E).$$

Thus, whenever we say that some statement is valid at ∞, we actually mean it is valid at 0 for the reverse pencil $E - zA \in \mathbb{K}^{n\times n}[z]$.

The spectral structure of a matrix pencil is obtained from the so-called *Kronecker canonical form (KCF)*. A matrix pencil $zE - A \in \mathbb{K}^{m\times n}[z]$ is said to be in KCF if it can be written as

$$\operatorname{diag}\left(\mathcal{K}_1(z), \ldots, \mathcal{K}_l(z)\right), \qquad l \in \mathbb{N},$$

where each block $\mathcal{K}_j(z)$ is in one of the following forms:

Type K1:

$$\begin{bmatrix} z-\lambda & 1 & & & \\ & z-\lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & z-\lambda \end{bmatrix} \in \mathbb{K}^{k_j\times k_j}[z],\ k_j \in \mathbb{N},\ \lambda \in \mathbb{C};$$

Type K2:

$$\begin{bmatrix} -1 & z & & & \\ & -1 & z & & \\ & & \ddots & \ddots & \\ & & & \ddots & z \\ & & & & -1 \end{bmatrix} \in \mathbb{K}^{k_j \times k_j}[z],\ k_j \in \mathbb{N};$$

Type K3:

$$\begin{bmatrix} -1 & z & & & \\ & -1 & z & & \\ & & \ddots & \ddots & \\ & & & -1 & z \end{bmatrix} \in \mathbb{K}^{k_j-1 \times k_j}[z],\ k_j \in \mathbb{N};$$

Type K4:

$$\begin{bmatrix} -1 & & & & \\ z & -1 & & & \\ & z & \ddots & & \\ & & \ddots & -1 & \\ & & & z & \end{bmatrix} \in \mathbb{K}^{k_j \times k_j-1}[z],\ k_j \in \mathbb{N}.$$

Blocks of type K1 and K2 correspond to finite eigenvalues and infinite eigenvalues, respectively. Blocks of these types and combinations of them are regular. Blocks of types K3 and K4 are rectangular and thus not regular. Note that we allow for blocks of type K3 or K4 to have zero rows or zero columns, respectively. Such blocks represent a zero row or zero column, respectively, in the KCF of $zE - A$.

We have the following theorem.

Theorem 2.7. *[Gan60] For every matrix pencil $zE - A \in \mathbb{K}^{m \times n}[z]$, there exist invertible matrices $W \in \mathbb{C}^{m \times m}$, $T \in \mathbb{C}^{n \times n}$ such that the pencil*

$$W(zE - A)T \in \mathbb{K}^{m \times n}[z]$$

is in KCF. The KCF is unique up to permutations of the blocks.

In the case of a regular pencil $zE - A \in \mathbb{K}^{n \times n}[z]$, i. e., blocks of type K3 and K4 are not present in its KCF, we obtain the following simplification.

Theorem 2.8 (Weierstrass canonical form)**.** *For every regular matrix pencil $zE - A \in \mathbb{K}^{n \times n}[z]$, there exist invertible matrices W, $T \in \mathbb{C}^{n \times n}$ such that*

$$W(zE - A)T = \begin{bmatrix} zI_{n_1} - J & 0 \\ 0 & zN - I_{n-n_1} \end{bmatrix}, \tag{2.1}$$

where $n_1 \in \mathbb{N}_0$, $J \in \mathbb{K}^{n_1 \times n_1}$ is in Jordan canonical form, and $N \in \mathbb{K}^{n-n_1 \times n-n_1}$ is in Jordan canonical form and nilpotent.

If we want transformation matrices $W, T \in \mathbb{K}^{n \times n}$, i. e., W and T lie in the same field as E and A, we can use another form which we obtain at the cost of losing the Jordan structure of J and N.

Theorem 2.9 (Quasi-Weierstrass form [BIT12])**.** *For every regular matrix pencil $zE - A \in \mathbb{K}^{n \times n}[z]$, there exist invertible matrices $W, T \in \mathbb{K}^{n \times n}$ such that*

$$W(zE - A)T = \begin{bmatrix} zI_{n_1} - J & 0 \\ 0 & zN - I_{n-n_1} \end{bmatrix}, \tag{2.2}$$

where $n_1 \in \mathbb{N}_0$, $J \in \mathbb{K}^{n_1 \times n_1}$, and $N \in \mathbb{K}^{n-n_1 \times n-n_1}$ is nilpotent.

When characterizing the eigenstructure of matrices $A \in \mathbb{K}^{n \times n}$, often invariant subspaces are involved, i. e., subspaces $\mathcal{V} \subseteq \mathbb{K}^n$ such that $A\mathcal{V} \subseteq \mathcal{V}$. The generalization of invariant subspaces to matrix pencils $zE - A \in \mathbb{K}^{m \times n}[z]$ are so-called deflating subspaces. Here, we are using a general definition which is also suitable for singular matrix pencils, see [Doo83; Voi15].

Definition 2.10 (Basis matrix, deflating subspaces)**.** Let $zE - A \in \mathbb{K}^{n \times n}[z]$ and some subspace $\mathcal{Y} \subseteq \mathbb{K}^n$ be given.

(a) A matrix $Y \in \mathbb{C}^{n \times k}$ with full column rank such that $\mathcal{Y} = \operatorname{im} Y$ is called *basis matrix* of $\mathcal{Y}$.

(b) If for a basis matrix $Y \in \mathbb{C}^{n \times k}$ of $\mathcal{Y}$ there exist $W \in \mathbb{C}^{n \times p}$ and $z\hat{E} - \hat{A} \in \mathbb{C}^{p \times n}[z]$ such that

$$(zE - A)V = W(z\hat{E} - \hat{A})$$

and $\operatorname{rk}_{\mathbb{C}(z)}(z\hat{E} - \hat{A}) = p$, then $\mathcal{Y}$ is called *deflating subspace* of $zE - A$.

Indeed, every invariant subspace $\mathcal{V} \subseteq \mathbb{K}^n$ of $A \in \mathbb{K}^{n \times n}$ with basis matrix $V \in \mathbb{K}^{n \times k}$ describes a deflating subspace for the associated matrix pencil $zI_n - A$ by setting $W = V$ and $(z\hat{E} - \hat{A}) = zI_k - \Lambda$, where $\Lambda \in \mathbb{K}^{k \times k}$ fulfills $AV = V\Lambda$.

An important property that deflating subspaces might have is E-neutrality.

Definition 2.11 (E-neutrality)**.** [GLR06; Rei11] Let $E \in \mathbb{K}^{n \times n}$ and some subspace $\mathcal{Y} \subseteq \mathbb{K}^n$ be given. Then $\mathcal{Y}$ is called *E-neutral* if for all $x, y \in \mathcal{Y}$ it holds that

$$x^* E y = 0.$$

It is called *maximally* E-neutral if every proper superspace $\mathcal{W} \supsetneq \mathcal{Y}$ is not E-neutral.

For a subspace $\mathcal{Y} \subseteq \mathbb{K}^n$ we can check E-neutrality by testing whether

$$Y^* E Y = 0, \tag{2.3}$$

where $Y \in \mathbb{K}^{n\times k}$ is given such that $\operatorname{im} Y = \mathcal{Y}$. In addition, we have the following lemma.

Lemma 2.12. *Let $Y \in \mathbb{K}^{n\times k}$ be given and let $W \in \mathbb{K}^{n\times n}$, $T \in \mathbb{K}^{k\times k}$ be some invertible matrices. Further, let $E \in \mathbb{K}^{n\times n}$ and $\hat{Y} := WYT$. Then $\mathcal{Y}$ is E-neutral if and only if $\operatorname{im} \hat{Y}$ is $W^{-*}EW^{-1}$-neutral.*

Proof. We have that $\operatorname{im} Y$ is E-neutral if and only if $Y^*EY = 0$ which holds if and only if

$$\hat{Y}^*(W^{-*}EW^{-1})\hat{Y} = T^*Y^*W^*W^{-*}EW^{-1}WYT = 0$$

which is valid if and only if $\operatorname{im} \hat{Y}$ is $W^{-*}EW^{-1}$-neutral. □

We now provide a result on how to check for maximality of an E-neutral space.

Lemma 2.13. *Let X, $Y \in \mathbb{K}^{n\times r}$ be given such that*

$$\operatorname{im} \begin{bmatrix} X \\ Y \end{bmatrix}$$

is E-neutral with

$$E = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Then it follows that

$$\operatorname{rk} \begin{bmatrix} X \\ Y \end{bmatrix} \leq n.$$

Proof. From E-neutrality we obtain that

$$\begin{bmatrix} Y^* & -X^* \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0$$

and thus

$$\dim \ker \begin{bmatrix} Y^* & -X^* \end{bmatrix} \geq \operatorname{rk} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Hence, the rank nullity theorem [Mey00, p. 199] implies

$$2n = \dim \ker \begin{bmatrix} Y^* & -X^* \end{bmatrix} + \operatorname{rk} \begin{bmatrix} Y^* & -X^* \end{bmatrix} \geq 2 \operatorname{rk} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

□

2.1.3 Polynomial and Rational Matrices

We now turn to matrix polynomials of arbitrary degrees, i. e., we consider matrices $A(z) \in \mathbb{K}^{m\times n}[z]$. This part is based on [Kac07; PW98].

Definition 2.14. A matrix $U(z) \in \mathbb{K}^{n\times n}[z]$ is called *unimodular* if it is invertible over $\mathbb{K}[z]$, i. e., its inverse exists and is again a polynomial in $\mathbb{K}$.

Unimodular matrices can be characterized as follows.

Lemma 2.15. *[PW98, Section 2.5] Let $U(z) \in \mathbb{K}^{n\times n}[z]$ be given. Then $U(z)$ is unimodular if and only if its determinant is a constant.*

We have the following canonical form under unimodular transformations for polynomial matrices $A(z) \in \mathbb{K}[z]$.

Theorem 2.16 (Smith form)**.** *[PW98, Section 2.5] Let $A(z) \in \mathbb{K}^{m\times n}[z]$ be given. Then there exist unimodular $U(z) \in \mathbb{K}^{m\times m}[z]$, $V(z) \in \mathbb{K}^{n\times n}[z]$ such that*

$$U(z)^{-1}A(z)V(z)^{-1} = \begin{bmatrix} p_1(z) & & & & & \\ & \ddots & & & & \\ & & p_r(z) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \tag{2.4}$$

where $p_j \in \mathbb{K}[z] \setminus \{0\}$ and p_j divides p_{j+1} for $j = 1, \dots, r-1$ and $r \in \mathbb{N}_0$.

Similar statements as for matrix polynomials can be obtained for rational matrices. These are matrices $A(z) \in \mathbb{K}^{m\times n}(z)$, i. e., matrices whose coefficients lie in the field of rational functions.

Theorem 2.17 (Smith-McMillan form)**.** *[Kac07, Section 2.6] Let $A(z) \in \mathbb{K}^{m\times n}(z)$ be given. Then there exist unimodular $U(z) \in \mathbb{K}^{m\times m}[z]$, $V(z) \in \mathbb{K}^{n\times n}[z]$ such that*

$$U(z)^{-1}A(z)V(z)^{-1} = \begin{bmatrix} \frac{p_1(z)}{q_1(z)} & & & & & \\ & \ddots & & & & \\ & & \frac{p_r(z)}{q_r(z)} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \tag{2.5}$$

where $p_j, q_j \in \mathbb{K}[z]\backslash\{0\}$ with leading coefficient one such that p_j and q_j have no nontrivial common divisor, p_j divides p_{j+1}, and q_{j+1} divides q_j for $j = 1, \dots, r-1$ and $r \in \mathbb{N}_0$.

By $\mathfrak{P}(A)$ we denote the set of poles of $A(z)$, i. e., $\mathfrak{P}(A)$ is the set of all $\lambda \in \mathbb{C}$ with $q_1(\lambda) = 0$.

Note that polynomial matrices are canonically embedded in the field of rational matrices, i. e., the following statements are also valid for polynomial matrices.

Definition 2.18. Let $A(z) \in \mathbb{K}^{m\times n}(z)$ be given. The *normal rank* of $A(z)$ is defined as the rank over the field $\mathbb{K}^{m\times n}(z)$ and denoted by $\operatorname{rk}_{\mathbb{K}(z)} A(z)$. The rational matrix $A(z)$ is *rank deficient*, if

$$\operatorname{rk}_{\mathbb{K}(z)} A(z) < \min\{m, n\}.$$

Moreover, $A(z)$ has a *rank drop at* $\lambda \in \mathbb{C}$ if

$$\operatorname{rk} A(\lambda) < \operatorname{rk}_{\mathbb{K}(z)} A(z).$$

Lemma 2.19. *[Kac07, Section 1.6] Let $A \in \mathbb{K}^{m\times n}(z)$ be given. Then the normal rank of A is invariant under multiplications with unimodular matrices.*

We have an immediate consequence of the Smith-McMillan form.

Corollary 2.20. *For a rational matrix $A(z) \in \mathbb{K}^{m\times m}(z)$ with set of poles $\mathfrak{P}(A)$ the mapping*

$$\mathbb{K} \setminus \mathfrak{P}(A) \ni \lambda \mapsto \operatorname{rk} A(\lambda)$$

is lower semi-continuous*, i. e., for all $\lambda_0 \in \mathbb{K} \setminus \mathfrak{P}(A)$ we have*

$$\operatorname{rk} A(\lambda_0) \leq \liminf_{\lambda\to\lambda_0} \operatorname{rk} A(\lambda).$$

In particular, rank drops only occur at isolated points.

Proof. First transform $A(z)$ into Smith-McMillan form which by Lemma 2.19 has the same normal rank as $A(z)$. The diagonal structure and the fact that rational functions only have rank drops at isolated points then completes the proof. □

2.2 Control Theoretic Concepts

Let E, $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{m\times n}$ be given matrices. Then

$$E\sigma(x_j)_j = A\left((x_j)_j\right) + B\left((u_j)_j\right), \qquad x_0 = x^0, \tag{2.6}$$

where $x = (x_j)_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$ and $u = (u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$, is called *linear-implicit difference equation* – or short – *implicit difference equation (IDE)*. Here, σ denotes the shift operator mapping a sequence $(x_j)_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$ to $(x_{j+1})_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$ and $x^0 \in \mathbb{K}^n$ is the initial condition. For every $j \in \mathbb{N}_0$ (2.6) reads as

$$Ex_{j+1} = Ax_j + Bu_j, \qquad x_0 = x^0. \tag{2.7}$$

In the case where $E = I_n$ we also call (2.6) an *explicit difference equation (EDE)*.

By replacing the shift operator σ by the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}t}$ we obtain a *linear differential-algebraic equation (DAE)*, also called *descriptor systems* – the continuous-time counterpart of IDEs – which have the form

$$E\frac{\mathrm{d}}{\mathrm{d}t}x(\cdot) = A\left(x(\cdot)\right) + B\left(u(\cdot)\right), \qquad x(0) = x^0, \tag{2.8}$$

where $x : \mathbb{I} \to \mathbb{K}^n$, $u : \mathbb{I} \to \mathbb{K}^m$, and $\mathbb{I} = [0, \infty) \subseteq \mathbb{R}$. For every $t \in \mathbb{I}$ equation (2.8) has the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{2.9}$$

with the initial condition $x(0) = x^0$. In the case where $E = I_n$ we also call (2.8) an *ordinary differential equation (ODE)*.

IDEs and DAEs are strongly related. For instance, by sampling we can construct an IDE out of a DAE. To see this, we first construct a sequence $(x_j)_j$ from $x(\cdot)$ by setting $x_j := x(t_j)$, where $t_j = t_{j-1} + h$, and $h \in \mathbb{R}^+$ denotes a *step size*. The quantity $1/h$ is called *sampling rate*. Next, we introduce the discretization operator $\Delta_h := (\sigma - I_n)/h$ corresponding to a discretization with the explicit Euler method. With a slight abuse of notation we also allow the application of Δ_h (and σ) to single elements x_j. This has to be understood as taking the n-th element of the sequence obtained by evaluating $\Delta_h(x_j)_j$ (or $\sigma(x_j)_j$). This operator maps every element x_j to a first order forward difference, i. e.,

$$\Delta_h x_j = \frac{x_{j+1} - x_j}{h}. \tag{2.10}$$

Letting h tend towards 0, we see that $\Delta_h x_j$ is an approximation to $\dot{x}(t_j)$. Thus, replacing $\dot{x}(t_j)$ in (2.9) by $\Delta_h x_j$ we obtain for every $j \in \mathbb{N}_0$ that

$$E\sigma x_j = (E + A_h)x_j + B_h u_j,$$

where $A_h := hA$ and $B_h := hB$.

Another approach to discretize the DAE (2.9) is using the implicit Euler method, i. e., we approximate $\dot{x}(t_j)$ by

$$\frac{x_j - x_{j-1}}{h}. \tag{2.11}$$

This leads to the discretized system

$$\begin{bmatrix} E - A_h & -B_h \end{bmatrix} \sigma \begin{bmatrix} x_j \\ u_j \end{bmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}. \tag{2.12}$$

Yet another approach to discretize the DAE (2.9) using the trapezoidal rule [HNW93, p. 204] is approximating $\dot{x}(t_{j+1}) + \dot{x}(t_j)$ by $2/h(x_{j+1} + x_j)$. Thus, summing up equation (2.9) at $t = t_j$ and $t = t_{j+1}$ and applying the trapezoidal rule gives

$$\begin{bmatrix} E - \frac{h}{2}A & -\frac{h}{2}B \end{bmatrix} \sigma \begin{bmatrix} x_j \\ u_j \end{bmatrix} = \begin{bmatrix} E + \frac{h}{2}A & \frac{h}{2}B \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}. \tag{2.13}$$

The discretizations presented here will appear at different stages in the forthcoming chapters.

2.2.1 Solution Theory

In this subsection we characterize solutions of the IDE (2.6). For given $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ and $x^0 \in \mathbb{K}^n$ we say that (2.6) is *solvable* if we have $(x_j)_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$ such that (2.6) is fulfilled. If such $(x_j)_j$ is unique it is also called *uniquely solvable* for $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ and $x^0 \in \mathbb{K}^n$. We call $x^0 \in \mathbb{K}^n$ a *consistent* initial value if there exists $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ such that the IDE (2.6) is solvable, otherwise x^0 is called *inconsistent*. Furthermore, we denote by $\mathcal{W}^c$ the set of consistent initial values, i. e., the set of all $x^0 \in \mathbb{K}^n$ such there exist $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ so that the IDE (2.6) is solvable.

We have the following well known result, see [Brü07; Sty03].

Lemma 2.21. *Let matrices $(E, A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ be given. Then the IDE (2.6) is uniquely solvable for every $(u_j)_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$ and consistent $x^0 \in \mathbb{K}^n$ if and only if $zE - A$ is regular.*
If this is the case then there are $T, W \in \mathbb{K}^{n\times n}$ leading to quasi-Weierstrass form (2.2) with some $J \in \mathbb{K}^{n_1\times n_1}$ and nilpotent $N \in \mathbb{K}^{n-n_1\times n-n_1}, n_1 \in \mathbb{N}_0$. Further, let ν be the index of nilpotency of N, i. e.,

$$\nu = \begin{cases} \min\left\{k \in \mathbb{N} \,\middle|\, N^{k-1} \neq 0,\, N^k = 0\right\}, & n - n_1 \neq 0, \\ 0, & n_1 = n, \end{cases}$$

with the convention that $0^0_{n\times n} := I_n$.

Then the unique solution $(x_j)_j \in (\mathbb{K}^n)^{\mathbb{N}_0}$ for given $(u_j)_j \in (\mathbb{K}^m)^{\mathbb{N}_0}$ and consistent initial value $x^0 \in \mathbb{K}^n$ can be written as

$$x_j = F_j E x^0 + \sum_{k=0}^{j+\nu-1} F_{j-k-1} B u_k, \tag{2.14}$$

where the coefficients F_j are defined by

$$F_j = \begin{cases} T \begin{bmatrix} J^j & 0 \\ 0 & 0 \end{bmatrix} W, & j = 0, 1, \ldots, \\ T \begin{bmatrix} 0 & 0 \\ 0 & -N^{-j-1} \end{bmatrix} W, & j = -1, -2, \ldots. \end{cases}$$

For the rest of this thesis we will restrict to the case where $zE - A$ is regular and denote the set of all such $(E, A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ with regular $zE - A$ by $\Sigma_{m,n}(\mathbb{K})$. This is no restriction in practice, since by reinterpretation of variables we can

always extract a regular system from an IDE, see [CKM12]. The index of nilpotency ν as defined in Lemma 2.21 is characteristic for the system $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$, i. e., it is independent of the choice of particular transformation matrices that lead to quasi-Weierstrass form (2.2). Thus, we will refer to ν as *the index* of such a system (E, A, B). Moreover, we denote by $\mathfrak{B}^{\sigma}_{(E,A,B)}$ the *(discrete-time) behavior* of (E, A, B), i. e., the linear space of all

$$\begin{bmatrix} (x_j)_j \\ (u_j)_j \end{bmatrix} \in (\mathbb{K}^n)^{\mathbb{N}_0} \times (\mathbb{K}^m)^{\mathbb{N}_0}$$

which solve (2.6) for some consistent $x^0 \in \mathbb{K}^n$. In addition, we call $\mathcal{W}^{\sigma}$ the set of *consistent initial shift variables*, i. e., the set of all $x^0 \in \mathbb{K}^n$ such that there exists

$$\begin{bmatrix} (x_j)_j \\ (u_j)_j \end{bmatrix} \in \mathfrak{B}^{\sigma}_{(E,A,B)}$$

with $Ex_0 = Ex^0$.

We have the following characterization of $\mathcal{W}^{\sigma}$ which is an adaption of the characterization [Ber14, Lemma 3.1.4(c)] to the discrete-time case.

Lemma 2.22. *Let $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ with corresponding IDE (2.6) be given. Further, let $\mathcal{W}^c$ be the set of all consistent initial values $x^0 \in \mathbb{K}^n$, i. e., for all $x^0 \in \mathcal{W}^c$ there exists $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,A,B)}$ with $x_0 = x^0$. Then for $\mathcal{W}^{\sigma}$ we have that*

$$\mathcal{W}^{\sigma} = \mathcal{W}^c + \ker E.$$

Proof. First we show the inclusion $\subseteq$. Thus, let $x^0 \in \mathcal{W}^{\sigma}$. Then there exists $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,A,B)}$ with $Ex_0 = Ex^0$. Since $x_0 \in \mathcal{W}^c$, this means we can write $x^0 = x_0 + (x^0 - x_0)$, where $x^0 - x_0 \in \ker E$.

For the inclusion $\supseteq$ let $x = x_v + x_e \in \mathcal{W}^c + \ker E$. Since $x_v \in \mathcal{W}^c$ we can find $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,A,B)}$ such that $x_0 = x_v$ and thus $Ex = E(x_v + x_e) = Ex_v = Ex_0$. $\square$

In the continuous-time case, similar statements are valid, see, e. g., [Ber14; KM06]. We denote by $\mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,A,B)}$ the *(continuous-time) behavior* of (E, A, B), i. e., the linear space of all $[\,x^* \;\; u^*\,]^*$ in some proper function space which solve (2.8) for some consistent $x^0 \in \mathbb{K}^n$, see [Ber14] for details. Note that the set of initial values x^0 such that there exists $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,A,B)}$ with $x_0 = x^0$ coincides with $\mathcal{W}^c$, see, e. g., [Dai89]. In addition, we call $\mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}$ the set of *consistent initial differential variables*, i. e., the set of all $x^0 \in \mathbb{K}^n$ such that there exists

$$\begin{bmatrix} x(\cdot) \\ u(\cdot) \end{bmatrix} \in \mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,A,B)}$$

with $Ex_0 = Ex^0$. It coincides with $\mathcal{W}^{\sigma}$ by Lemma 2.22.

2.2.2 Feedback Equivalence

In this subsection we introduce an equivalence relation on the set $\Sigma_{m,n}(\mathbb{K})$ which will be particularly useful in Chapters 3 and 5. This section is mainly based on [RRV15, Section 2.3].

Definition 2.23 (Feedback equivalence)**.** Two systems $(E_i,\, A_i,\, B_i) \in \Sigma_{m,n}(\mathbb{K})$, $i = 1, 2$, are said to be *feedback equivalent* if there exist invertible matrices $W,\, T \in \mathbb{K}^{n\times n}$ and a feedback matrix $F \in \mathbb{K}^{m\times n}$ such that

$$\begin{bmatrix} zE_2 - A_2 & -B_2 \end{bmatrix} = W \begin{bmatrix} zE_1 - A_1 & -B_1 \end{bmatrix} \mathcal{T}_F,$$

where

$$\mathcal{T}_F = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}.$$

If this is the case we say that $(E_1,\, A_1,\, B_1)$ is feedback equivalent to $(E_2,\, A_2,\, B_2)$ via W and $\mathcal{T}_F$.

Note that in the behavior sense, i. e., looking at the system defined by $z\mathcal{E} - \mathcal{A}$, where

$$\mathcal{E} := \begin{bmatrix} E & 0 \end{bmatrix}, \qquad \mathcal{A} := \begin{bmatrix} A & B \end{bmatrix},$$

feedback equivalence corresponds to strong equivalence as introduced in [KM06]. In particular, this means that feedback equivalence is indeed an equivalence relation, see [KM06, Lemma 2.2.].

Given such an equivalence relation, one is usually interested in some condensed form. The following result provides such a form.

Theorem 2.24 (Feedback equivalence form)**.** *[IR14, Proposition 2.12] Let the system $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$ be given. Then $(E,\, A,\, B)$ is feedback equivalent to $(E_F,\, A_F,\, B_F)$ via some W and $\mathcal{T}_F$, where*

$$\begin{bmatrix} zE_F - A_F & B_F \end{bmatrix} = \left[\begin{array}{ccc|c} zI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & zE_{23} & B_2 \\ 0 & 0 & zE_{33} - I_{n_3} & 0 \end{array}\right], \tag{2.15}$$

$n_1, n_2, n_3 \in \mathbb{N}_0$, and E_{33} is nilpotent.

A similar form has also been achieved in [BGM97, Theorem 4.1] via unitary transformations.

Example 2.25 (Simple circuit revisited)**.** Consider the electrical circuit as in (1.7). For the sake of simplicity we now set $h = 1$, $\mathfrak{G} = 1$, and $\mathfrak{C} = 1$. Then the system is given by

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{2.16}$$

We obtain that the system is feedback equivalent to

$$\begin{bmatrix} zE_F - A_F & -B_F \end{bmatrix} = W \begin{bmatrix} zE - A & -B \end{bmatrix} \mathcal{T}_F = \begin{bmatrix} z-1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \tag{2.17}$$

via zero feedback, i. e., $F = 0$ and

$$W = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{T}_F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.18}$$

Thus, we have $n_1 = n_2 = 1$ and $n_3 = 0$ in (2.15).

Proposition 2.26. *Let the system $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$ be feedback equivalent to the system $(E_F,\, A_F,\, B_F) \in \Sigma_{m,n}(\mathbb{K})$ in feedback equivalence form (2.15). Further, denote by*

$$(I_{n_1},\, A_{11},\, B_1) \in \Sigma_{m,n}(\mathbb{K})$$

the associated EDE system. Then for $\lambda \in \mathbb{C}$ we have $\det(\lambda E_F - A_F) \neq 0$ if and only if $\det(\lambda I_{n_1} - A_{11}) \neq 0$.

Proof. Note that every nilpotent matrix $E_{33} \in \mathbb{K}^{n_3 \times n_3}$ has only eigenvalues zero and thus $\det(sE_{33} - I_{n_3}) = (-1)^{n_3}$. Then the assertion follows immediately from the block-diagonal structure of $zE_F - A_F$. □

2.2.3 System Space

In this subsection we investigate properties of the solution space of the IDEs given by a system $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$. This section is based on [RRV15, Chapter 3].

Definition 2.27. Let $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$. The smallest subspace $\mathcal{V}^\Sigma \subseteq \mathbb{K}^{n+m}$ such that

$$\begin{bmatrix} x_j \\ u_j \end{bmatrix} \in \mathcal{V}^\Sigma$$

for all $j \in \mathbb{N}_0$ and for all $[\, x^* \;\; u^* \,]^* \in \mathfrak{B}^\sigma_{(E,\,A,\,B)}$ is called system space of $(E,\, A,\, B)$.

Lemma 2.28. *Let $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$. Further, assume that $(E_F,\, A_F,\, B_F) \in \Sigma_{m,n}(\mathbb{K})$ is feedback equivalent to $(E,\, A,\, B)$ via W and $\mathcal{T}_F$. Then the system spaces $\mathcal{V}^\Sigma$ and $\mathcal{V}^\Sigma_F$ of $(E,\, A,\, B)$ and $(E_F,\, A_F,\, B_F)$, respectively, are related via*

$$\mathcal{V}^\Sigma = \mathcal{T}_F \mathcal{V}^\Sigma_F. \tag{2.19}$$

Proof. The assertion has been shown in [RRV15, Lemma 3.2]. □

Proposition 2.29. *Let $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ with the system space $\mathcal{V}^\Sigma$ be given. Further, assume that $(E_F, A_F, B_F) \in \Sigma_{m,n}(\mathbb{K})$ with corresponding system space $\mathcal{V}_F^\Sigma$ is feedback equivalent to (E, A, B) via W and $\mathcal{T}_F$ such that (E_F, A_F, B_F) is in feedback equivalence form (2.15). Then we have:*

(a) It holds that $\mathcal{V}_F^\Sigma = \operatorname{im} V_F^\Sigma$, where

$$V_F^\Sigma := \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \in \mathbb{K}^{n+m \times n+m}. \tag{2.20}$$

(b) It holds that

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix} = z \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix}. \tag{2.21}$$

(c) For all $\lambda \in \mathbb{C}$ with $\det(\lambda E - A) \neq 0$ it holds that

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \subseteq \mathcal{V}^\Sigma. \tag{2.22}$$

(d) Consider V_F^Σ as in (a) and let $V^\Sigma := \mathcal{T}_F V_F^\Sigma$. Then

$$V_F^\Sigma \mathcal{V}_F^\Sigma = \mathcal{V}_F^\Sigma$$

and

$$V^\Sigma \mathcal{T}_F^{-1} \mathcal{V}^\Sigma = \mathcal{V}^\Sigma.$$

Proof. Assertion (a) is shown in the proof of [RRV15, Proposition 3.3]. Assertion (c) is shown in [RRV15, Lemma 3.5], where part (b) is obtained in the proof of [RRV15, Lemma 3.5]. For part (d) we have

$$V_F^\Sigma V_F^\Sigma = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} = V_F^\Sigma.$$

Since $\operatorname{im} V_F^\Sigma = \mathcal{V}_F^\Sigma$ this shows the first statement. Thus, with Lemma 2.28 we obtain

$$V^\Sigma(\mathcal{T}_F^{-1}\mathcal{V}^\Sigma) = (\mathcal{T}_F V_F^\Sigma)\mathcal{V}_F^\Sigma = \mathcal{T}_F(V_F^\Sigma \mathcal{V}_F^\Sigma) = \mathcal{T}_F \mathcal{V}_F^\Sigma = \mathcal{V}^\Sigma. \qquad \square$$

2.2.4 Controllability

Before we introduce the linear-quadratic optimal control problem, we first need to recap several concepts of controllability for the system given by $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$. These concepts are similar to the continuous-time as in [BBMN99; Dai89] and are discussed in, e. g., [Dai89; Sty03].

Definition 2.30. The system $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$ is called

(a) *completely controllable (C-controllable)* if for every initial point $x^0 \in \mathbb{K}^n$ and every final point $x^f \in \mathbb{K}^n$ there exist $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$ such that $x_0 = x^0$ and $x_{j_f} = x^f$ at some timepoint $j_f \in \mathbb{N}_0$;

(b) *controllable on the reachable set (R-controllable)* if for every consistent initial point $x^0 \in \mathbb{K}^n$ and every consistent final point $x_f \in \mathbb{K}^n$ there exist $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$ such that $x_{j_f} = x^f$ at some timepoint $j_f \in \mathbb{N}_0$;

(c) *impulse controllable (I-controllable)* if for every initial point $x^0 \in \mathbb{K}^n$ there exists $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$ such that $Ex_0 = Ex^0$, i. e., $\mathcal{W}^{\sigma} = \mathbb{K}^n$.

In the case where $E = I_n$, the notions R-controllability and C-controllability coincide and thus for systems of the form $(I_n,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$ we omit the prefix R or C and say that they are *controllable*.

We have the following well-known characterizations of the different controllability notions [Ber14; Dai89; Sty03].

Proposition 2.31. *Let the system $(E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$ be given. Then we have:*

(a) The system $(E,\, A,\, B)$ is R-controllable if and only if

$$\operatorname{rk}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = n$$

for all $\lambda \in \mathbb{C}$.

(b) The system $(E,\, A,\, B)$ is C-controllable if and only if it is R-controllable and in addition

$$\operatorname{rk}\begin{bmatrix}E & B\end{bmatrix} = n.$$

(c) The system $(E,\, A,\, B)$ is I-controllable if and only if

$$\operatorname{rk}\begin{bmatrix}E & AS_\infty & B\end{bmatrix} = n,$$

where S_∞ is a basis of $\ker E$.

(d) *The system* (E, A, B) *is I-controllable if and only if there exist* W *and* $\mathcal{T}_F$ *such that for the system in feedback equivalence form* $(E_F, A_F, B_F) \in \Sigma_{m,n}(\mathbb{K})$ *as in* (2.15) *it holds that* $n_3 = 0$.

Proof. Parts (a) and (b) follow from the discussion in [Sty03]. Part (c) is shown in [Ber14] and part (d) is shown in [BBMN99]. □

According to Proposition 2.31 eigenvalues $\lambda \in \mathbb{C}$ of (E, A) such that

$$\operatorname{rk} \begin{bmatrix} \lambda E - A & B \end{bmatrix} \neq n$$

destroy the controllability property and thus are referred to as *uncontrollable modes at* λ; otherwise they are called *controllable modes at* λ.

Lemma 2.32. *Let the system* $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ *be feedback equivalent to the system* $(E_F, A_F, B_F) \in \Sigma_{m,n}(\mathbb{K})$ *in feedback equivalence form* (2.15). *Furthermore, denote by* $(I_{n_1}, A_{11}, B_1) \in \Sigma_{m,n_1}(\mathbb{K})$ *the associated EDE system. Then we have:*

(a) *Let* $\lambda \in \mathbb{C}$. *Then the system* (E_F, A_F, B_F) *has an uncontrollable mode at* λ *if and only if the system* (I_{n_1}, A_{11}, B_1) *has an uncontrollable mode at* λ.

(b) *The system* (E_F, A_F, B_F) *is R-controllable if and only if* (I_{n_1}, A_{11}, B_1) *is controllable.*

Proof. Assertion (a) is shown in [RRV15, Lemma 2.9(c)]. Then assertion (b) is an immediate consequence of (a), since by Proposition 2.31 the system (E_F, A_F, B_F) is R-controllable if and only if no $\lambda \in \mathbb{C}$ is an uncontrollable mode of (E_F, A_F, B_F). □

Controllable modes at some $\lambda \in \mathbb{C}$ can be used to change the spectral properties of (E, A).

Proposition 2.33 (Pole placement)**.** *[Cob81] Let* $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ *be given. Further, let the system* (E, A, B) *have a controllable mode at some* $\lambda \in \mathbb{C}$. *Then for any* $\mu \in \mathbb{C}$ *there exists a feedback matrix* $F \in \mathbb{K}^{m\times n}$ *such that in* $zE-(A+BF)$ *the eigenvalue* λ *is replaced by* μ, *where meanwhile all other eigenvalues of* $zE - A$ *are unchanged.*

2.2.5 Asymptotic Stability

Asymptotic stability of a system $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ is another important property of dynamical systems. It guarantees that with zero input $u = (u_j)_j$ the state $x = (x_j)_j$ tends towards zero as j goes to infinity. Since the input u does not contribute to the analysis we may instead consider systems $(E, A) \in \Sigma_{0,n}(\mathbb{K}) =: \Sigma_n(\mathbb{K})$ without input and corresponding behavior $\mathfrak{B}^{\sigma}_{(E,A)}$. This subsection is based on [Ela05].

Definition 2.34. Let $(E,\, A) \in \Sigma_n(\mathbb{K})$ be given. Then $(E,\, A)$ is called *asymptotically stable* if for $x \in \mathfrak{B}^{\sigma}_{(E,\, A)}$ we have that $\lim\limits_{j\to\infty} Ex_j = 0$.

Let $W, T \in \mathbb{K}^{n\times n}$ be the transformation matrices leading to quasi-Weierstrass form (2.2). Then, from (2.14) we obtain that $x \in \mathfrak{B}^{\sigma}_{(E,\, A)}$ is given by

$$x_j = T \begin{bmatrix} J^j & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0^1 \\ 0 \end{bmatrix}$$

and some $x_0^1 \in \mathbb{K}^{n_1}$. Thus,

$$WEx_j = W \begin{bmatrix} J^j x_0^1 \\ 0 \end{bmatrix}$$

and asymptotic stability of $(E,\, A)$ is equivalent to asymptotic stability of the EDE system $(I_{n_1},\, J) \in \Sigma_{n_1}(\mathbb{K})$. We have the following algebraic characterization of asymptotic stability.

Proposition 2.35. *Let $(E,\, A) \in \Sigma_n(\mathbb{K})$ be given. Then $(E,\, A)$ is asymptotically stable if and only if all finite eigenvalues $\lambda \in \mathbb{C}$ of $zE - A$ lie inside the unit disk, i. e., $|\lambda| < 1$.*

Proof. Let $W, T \in \mathbb{K}^{n\times n}$ be the transformation matrices leading to quasi-Weierstrass form (2.1). The assertion is an immediate consequence of [Ela05, Theorem 4.13] and the fact that the finite eigenvalues of $(E,\, A)$ correspond to the eigenvalues of $(I_{n_1},\, J)$. □

Another characterization is obtained via *Lyapunov equations*.

Proposition 2.36. *Let $(E,\, A) \in \Sigma_n(\mathbb{K})$ be given with corresponding system space $\mathcal{V}^{\Sigma}$. Then $(E,\, A)$ is asymptotically stable if and only if for every Hermitian $Q \in \mathbb{K}^{n\times n}$ with $Q >_{\mathcal{V}^{\Sigma}} 0$ there exists an Hermitian $P \in \mathbb{K}^{n\times n}$ with $P >_{\mathcal{V}^{\Sigma}} 0$ such that*

$$A^*PA - E^*PE + Q =_{\mathcal{V}^{\Sigma}} 0. \tag{2.23}$$

Proof. Let $W, T \in \mathbb{K}^{n\times n}$ be the transformation matrices leading to quasi-Weierstrass form (2.1). Since we do not consider inputs we can apply Lemma 2.28 and Proposition 2.29(a) to obtain

$$T\mathcal{V}^{\Sigma} = \operatorname{im} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}.$$

Set

$$W^{-*}PW^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}, \qquad T^*QT = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}. \tag{2.24}$$

Thus, by Lemma 2.2(c), equation (2.23) is equivalent to

$$\begin{aligned} 0 =& \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}^* T^* \left((A^*W^*)(W^{-*}PW^{-1})WA - (E^*W^*)(W^{-*}PW^{-1})WE + Q \right) T \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \\ =& J^* P_{11} J - P_{11} + Q_{11}, \end{aligned}$$

where $P_{11} \succ 0$, $Q_{11} \succ 0$. This equality holds if and only if (I_n, J) is asymptotically stable [Ela05, Theorem 4.30]. □

In the case where $E = I_n$ we obtain for asymptotically stable A that for every positive definite Q we have the existence of positive definite P such that

$$A^*PA - P + Q = 0. \tag{2.25}$$

If Q is positive semi-definite, then P as in (2.25) can be constructed by [Ela05]

$$P = \sum_{k=0}^{\infty} (A^*)^k Q A^k. \tag{2.26}$$

For the IDE case this means, that a solution P as in (2.23) can be constructed by setting $P_{11} = \sum_{k=0}^{\infty} (J^*)^k Q_{11} J^k$, $P_{12} = 0$, and $P_{22} = 0$ in (2.24).

It is also of interest, if for a system $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ we can find a feedback such that the resulting system is asymptotically stable.

Definition 2.37. Let $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ be given. Then (E, A, B) is called *stabilizable* if there exists a feedback $F \in \mathbb{K}^{m \times n}$ such that the system given by $(E, A + BF) \in \Sigma_n(\mathbb{K})$ is asymptotically stable.

Stabilizability can be characterized algebraically.

Lemma 2.38. *Let $(E, A, B) \in \Sigma_{m,n}(\mathbb{K})$ be given. Then (E, A, B) is stabilizable if and only if*

$$\operatorname{rk} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$.

Proof. The result is an immediate consequence of Propositions 2.33 and 2.35. □

2.2.6 Linear-Quadratic Optimal Control

One main goal of this thesis is to provide tools for analyzing the discrete-time infinite horizon linear-quadratic control problem [Bac06; KM04; Meh91]. It is given by:

For $x^0 \in \mathcal{W}^\sigma$ find $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^\sigma_{(E,\,A,\,B)}$ such that $Ex_0 = Ex^0$, $\lim\limits_{j\to\infty} Ex_j = 0$, and the *objective function*

$$\mathcal{J}^\sigma(x,u) := \sum_{j=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} \tag{2.27}$$

is minimized. In other words, we are interested in the value of the functional $\mathcal{W}^\sigma_+(Ex^0)\colon E\mathcal{W}^\sigma \to \mathbb{R}^+_0 \cup \{\pm\infty\}$ defined by

$$\mathcal{W}^\sigma_+(Ex^0) := \inf\left\{ \mathcal{J}^\sigma(x,u) \,\middle|\, [\,x^* \;\; u^*\,]^* \in \mathfrak{B}^\sigma_{(E,\,A,\,B)},\; Ex_0 = Ex^0,\; \lim_{j\to\infty} Ex_j = 0 \right\}.$$

The problem is called *feasible* if $\infty > \mathcal{W}^\sigma_+(Ex^0) > -\infty$. It is called *solvable* if the infimum is actually a minimum. Note that for $x^0 \in \mathcal{W}^\sigma$ the existence of $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^\sigma_{(E,\,A,\,B)}$ such that $Ex_0 = Ex^0$ is guaranteed by the definition of $\mathcal{W}^\sigma$. If further $(E,\,A,\,B)$ is stabilizable we can choose u such that in addition $\lim\limits_{j\to\infty} Ex_j = 0$, i. e., $\mathcal{W}^\sigma_+(Ex^0) < \infty$.

In the continuous-time case this problem is formulated as:

For $x^0 \in \mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}$ find $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,\,A,\,B)}$ such that $Ex(0) = Ex^0$, $\lim\limits_{t\to\infty} Ex(t) = 0$ and the objective function

$$\mathcal{J}^{\frac{\mathrm{d}}{\mathrm{d}t}}(x,u) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t \tag{2.28}$$

is minimized. Again, we are interested in the value of the functional $\mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}_+(Ex^0) : E\mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}} \to \mathbb{R}^+_0 \cup \{\pm\infty\}$ defined by

$$\mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}_+(Ex^0) := \inf\left\{ \mathcal{J}^{\frac{\mathrm{d}}{\mathrm{d}t}}(x,u) \,\middle|\, [\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\frac{\mathrm{d}}{\mathrm{d}t}}_{(E,\,A,\,B)},\; Ex(0) = Ex^0,\; \lim_{t\to\infty} Ex(t) = 0 \right\}.$$

As before, the problem is called *feasible* if $\infty > \mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}_+(Ex^0) > -\infty$. It is called *solvable* if the infimum is actually a minimum. By $\Sigma^w_{m,n}(\mathbb{K})$ we denote the set of all

$$(E,\,A,\,B,\,Q,\,S,\,R) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{m\times m}, \tag{2.29}$$

where $(E,\,A,\,B) \in \Sigma_{m,n}(\mathbb{K})$ and Q and R are Hermitian.

Now we show that the objective function $\mathcal{J}^\sigma(x,u)$ does not change if we perform a feedback transformation of $(E,\,A,\,B)$ and appropriately transform Q, S, R. To this end, assume that the system $(E_F,\,A_F,\,B_F) \in \Sigma_{m,n}(\mathbb{K})$ is equivalent to $(E,\,A,\,B)$ via W and

$\mathcal{T}_F$. Applying the transformation $\mathcal{T}_F$ to $[\,x^* \;\; u^*\,]^*$ in $\mathcal{J}^\sigma(x,u)$ leads to

$$\begin{aligned}
\mathcal{J}^\sigma(x_F,u_F) &= \sum_{j=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^* \mathcal{T}_F^{-1} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix} \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} x_{F,j} \\ u_{F,j} \end{bmatrix}^* \begin{bmatrix} T^*(Q+F^*S^*) & T^*(S+F^*R) \\ S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} x_{F,j} \\ u_{F,j} \end{bmatrix} \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} x_{F,j} \\ u_{F,j} \end{bmatrix}^* \begin{bmatrix} T^*(Q+F^*S^*+SF+F^*RF)T & T^*(S+F^*R) \\ (S^*+RF)T & R \end{bmatrix} \begin{bmatrix} x_{F,j} \\ u_{F,j} \end{bmatrix},
\end{aligned}$$

where

$$\begin{bmatrix} x_{F,j} \\ u_{F,j} \end{bmatrix} := \mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix}.$$

Thus, $\mathcal{J}^\sigma(x,u)$ does not change under feedback transformations when we use the modified weights

$$\begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix} := \begin{bmatrix} T^*(Q+F^*S^*+SF+F^*RF)T & T^*(S+F^*R) \\ (S^*+RF)T & R \end{bmatrix}. \tag{2.30}$$

Analogously, one can show that $\mathcal{J}^{\frac{\mathrm{d}}{\mathrm{d}t}}(x,u)$ does not change under feedback transformations when we use the modified weights as in (2.30).

In the following we assume that the system $(E,\,A,\,B,\,Q,\,S,\,R) \in \Sigma_{m,n}^{w}(\mathbb{K})$ is impulse controllable and that

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \succeq 0. \tag{2.31}$$

It is well-known that in this case solutions of the optimal control problem can be characterized via certain structured matrix pencils, see, [BMMX09; KM08; Meh91; MMMM06]. In the continuous-time case, applying Pontryagin's maximum principle [Meh91] leads to the necessary optimality conditions

$$\begin{aligned}
\begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} &= \begin{bmatrix} 0 & A & B \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}, \\
Ex(0) = Ex^0, \qquad \lim_{t\to\infty} E^*\mu(t) &= 0,
\end{aligned} \tag{2.32}$$

where $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}_{(E,\,A,\,B)}^{\frac{\mathrm{d}}{\mathrm{d}t}}$, $x^0 \in \mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}$, and $\mu : \mathbb{I} \to \mathbb{K}^n$ denotes some Lagrange multiplier. In [RRV15] the necessary optimality conditions (2.32) are also derived for

the case where (2.31) does not hold. Moreover, the behavior of this DAE can be described by the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \in \mathbb{K}^{2n+m \times 2n+m}[s]. \tag{2.33}$$

It has the special structure that $\mathcal{E}^* = -\mathcal{E}$ and $\mathcal{A}^* = \mathcal{A}$. Such pencils are called *even* matrix pencils [MMMM06].

In the discrete-time case, applying Pontryagin's maximum principle [Meh91] leads to

$$\begin{bmatrix} 0 & E & 0 \\ A^* & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} \sigma \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ E^* & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}, \qquad Ex_0 = Ex^0, \qquad \lim_{j\to\infty} E^*\mu_j = 0, \tag{2.34}$$

where $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$, $x^0 \in \mathcal{W}^{\frac{\mathrm{d}}{\mathrm{d}t}}$, and $\mu \in (\mathbb{K}^n)^{\mathbb{N}_0}$ denote some Lagrange multipliers. The behavior of this IDE can be described by the matrix pencil

$$z\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & zE - A & -B \\ zA^* - E^* & -Q & -S \\ zB^* & -S^* & -R \end{bmatrix} \in \mathbb{K}^{2n+m \times 2n+m}[z], \tag{2.35}$$

a so-called *BVD-pencil*; here BVD is an acronym for ***B**oundary **V**alue problem for the optimal control of **D**iscrete systems*. The structure of this pencil is not invariant under unitary transformations which leads to problems in the numerical treatment [BMMX09]. In [BMMX09; Sch08] it is shown how we can achieve a structured version if we introduce new Lagrange multipliers

$$m_j := \mu_j - \mu_{j+1}. \tag{2.36}$$

This reformulation yields

$$\begin{bmatrix} 0 & E & 0 \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ E^* & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}, \qquad Ex_0 = Ex^0, \qquad \sum_{j=0}^{\infty} E^* m_j = E^*\mu_j \tag{2.37}$$

with corresponding matrix pencil

$$z\mathcal{E} - \mathcal{A} = z\mathcal{A}^* - \mathcal{A} \begin{bmatrix} 0 & zE - A & -B \\ zA^* - E^* & (z-1)Q & (z-1)S \\ zB^* & (z-1)S^* & (z-1)R \end{bmatrix} \in \mathbb{K}^{2n+m \times 2n+m}[z]. \tag{2.38}$$

This pencil has the special property of being *palindromic*, i. e., $\mathcal{E} = \mathcal{A}^*$. This structure is preserved under congruence transformation and thus is suitable for the numerical treatment [BMMX09; Sch08]. In Section 4.2 we discuss in more detail properties of palindromic pencils and their relation to even matrix pencils. However, in [MS14] it is shown that in an abstract Banach space setting the operator associated to the palindromic pencil (2.38) is not self-adjoint; in contrast to the operator associated to the even pencil (2.33), see also [KMS14]. One way out is to use a second order palindromic matrix polynomial $z^2\mathcal{A}_2 + z\mathcal{A}_1 + \mathcal{A}_0 \in \mathbb{K}^{2n+m\times 2n+m}[z]$. It has the form

$$z^2\mathcal{A}_2 + z\mathcal{A}_1 + \mathcal{A}_0 = z^2 \begin{bmatrix} 0 & E & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & -A & -B \\ -A^* & Q & S \\ -B^* & S^* & R \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with the special property that $\mathcal{A}_2^* = \mathcal{A}_0$ and $\mathcal{A}_1^* = \mathcal{A}_1$.

We show in Chapters 5 and 6 that in analogy to the continuous-time case in [RRV15], also in the discrete-time case we can drop assumption (2.31) to obtain the necessary optimality conditions (2.34) and (2.37).

3 Kalman-Yakubovich-Popov Lemma

Consider the weighted system $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (2.29) and corresponding system space $\mathcal{V}^\Sigma$. In this chapter we relate positive semi-definiteness on the unit circle of the Popov function – a specific rational matrix function – to the solvability of a certain matrix inequality, namely the Kalman-Yakubovich-Popov inequality. We will see in Chapter 6 that positive semi-definiteness on the unit circle of the Popov function is sufficient for feasibility of the optimal control problem (2.27).

First, we reconsider the well-known results for explicit difference equations. Then we generalize these results to IDEs in a similar way as it was done in [RRV15] for the generalization of the ODE case to the DAE case.

Definition 3.1. Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given. Then the rational matrix function

$$\Phi(z) := \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix}^\sim \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (zE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}^{m\times m}(z)$$

is called *Popov function* of (E, A, B, Q, S, R).
Here, for a given rational matrix function $G(z) \in \mathbb{K}^{m\times n}[z]$ the rational matrix $G^\sim(z)$ is defined by $G^\sim(z) = G(\overline{z}^{-1})^*$. Note that for values z on the unit circle this notation coincides with the conjugate transpose. In the continuous-time case the definition of $G^\sim(z)$ is altered in such a way that $G^\sim(z)$ coincides with the conjugate transpose of $G(z)$ on the imaginary axis.

The Popov function is important for characterizing the solvability of the Kalman-Yakubovich-Popov inequality.

Definition 3.2. Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given. Consider $P = P^* \in \mathbb{K}^{n\times n}$ and

$$\mathcal{M}(P) := \begin{bmatrix} A^*PA - E^*PE + Q & A^*PB + S \\ B^*PA + S^* & B^*PB + R \end{bmatrix}. \tag{3.1}$$

If $\mathcal{M}(P) \succeq_{\mathcal{V}^\Sigma} 0$, then P is called solution of the discrete-time *Kalman-Yakubovich-Popov (KYP)* inequality

$$\mathcal{M}(P) \succeq_{\mathcal{V}^\Sigma} 0, \qquad P^* = P. \tag{3.2}$$

Throughout this chapter we will make use of the system $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$ being transformed to $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$ via feedback equivalence, i. e., we have invertible $W,\, T \in \mathbb{K}^{n\times n}$ and a feedback matrix $F \in \mathbb{K}^{m\times n}$ such that

$$\begin{aligned} E_F = WET, \qquad A_F = W(A+BF)T, \qquad B_F = WB, \\ Q_F = T^*(Q+SF+F^*S^*+F^*RF)T, \qquad S_F = T^*(S+F^*R), \qquad R_F = R. \end{aligned} \tag{3.3}$$

These transformations will allow us to extract an EDE formulation from the IDE problem. The next results are crucial for the proof of the KYP Lemma in the EDE case as well as in the IDE case and are mainly adaptions of the corresponding results in [RRV15, Section 4].

Lemma 3.3. *Let $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$. Then we have*

$$\begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix}^\sim \begin{bmatrix} A^*PA - E^*PE & A^*PB \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix} = 0. \tag{3.4}$$

Proof. By using (2.21) we obtain

$$\begin{aligned} & \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix}^\sim \begin{bmatrix} A^*PA - E^*PE & A^*PB \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix} \\ = & \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix}^\sim \left(\begin{bmatrix} A^* \\ B^* \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} E^* \\ 0 \end{bmatrix} P \begin{bmatrix} E & 0 \end{bmatrix} \right) \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix} \\ \overset{(2.21)}{=} & \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix}^\sim \left(\overline{z}^{-*} \begin{bmatrix} E^* \\ 0 \end{bmatrix} P \begin{bmatrix} E & 0 \end{bmatrix} z - \begin{bmatrix} E^* \\ 0 \end{bmatrix} P \begin{bmatrix} E & 0 \end{bmatrix} \right) \begin{bmatrix} (zE-A)^{-1}B \\ I_m \end{bmatrix} = 0. \end{aligned}$$

□

Lemma 3.4. *Let $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$ with corresponding feedback equivalent system $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$ as in (3.3) be given. Further, let $P = P^* \in \mathbb{K}^{n\times n}$ and set $P_F = W^{-*}PW^{-1}$ and*

$$\mathcal{T}_F = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}.$$

Then

$$\mathcal{M}_F(P_F) = \mathcal{T}_F^* \mathcal{M}(P) \mathcal{T}_F, \tag{3.5}$$

where $\mathcal{M}_F(P_F)$ is the matrix in (3.2) corresponding to $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$.

Proof. We have

$$\begin{aligned}
\mathcal{T}_F^* \mathcal{M}(P) \mathcal{T}_F &= \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}^* \begin{bmatrix} A^*PA - E^*PE + Q & A^*PB + S \\ B^*PA + S^* & B^*PB + R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \\
&= \begin{bmatrix} A_F^* P_F W A - E_F^* P_F W E & A_F^* P_F B_F \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} + \begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix} \\
&= \begin{bmatrix} A_F^* P_F A_F - E_F^* P_F E_F & A_F^* P_F B_F \\ B_F^* P_F A_F & B_F^* P_F B_F \end{bmatrix} + \begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix} \\
&= \mathcal{M}_F(P_F).
\end{aligned}$$

□

3.1 Explicit Difference Equations

In this section we consider weighted explicit difference equations (EDEs), i. e., systems $(E, A, B, Q, S, R) \in \Sigma_{m,n}^w(\mathbb{K})$ with $E = I_n$. These systems have been studied a lot in the last decades. The famous Kalman-Yakubovich-Popov Lemma for ODEs goes back to the 1950s, see [Kal63; Pop61; Yak62]. A version for the EDE case can be found in [Ran96; ZDG96]. Note that for the system space $\mathcal{V}^\Sigma$ of EDEs it holds that $\mathcal{V}^\Sigma = \mathbb{K}^{n+m}$ and the KYP inequality thus asks for $P = P^*$ such that $\mathcal{M}(P) \succeq 0$. Since this result is of major importance for the considerations in this thesis, we present the main ideas of the proof by Rantzer [Ran96] for the discrete-time case.

First we need the following lemmas.

Lemma 3.5. *Let $M, N \in \mathbb{C}^{p\times q}$ be some matrices. Then the following holds:*

(a) We have $MM^* = NN^*$ if and only if there exists $U \in \mathbb{C}^{q\times q}$ such that $UU^* = I_q$ and $M = NU$.

(b) If further $q = 1$ then $MM^* = NN^*$ is equivalent to the existence of $\omega \in [0, 2\pi)$ such that $M = \mathrm{e}^{\mathrm{i}\omega} N$.

Proof. Assertion (a) is shown in [Ran96, Lemma 3(i)]. Then (b) is an immediate consequence of (a) since numbers $U \in \mathbb{C}$ fulfilling $UU^* = 1$ lie on the unit circle. □

Lemma 3.6. *Let $M, \tilde{M} \in \mathbb{C}^{p\times q}$ be given. If $MWM^* - \tilde{M}W\tilde{M}^* = 0$ for some positive semi-definite Hermitian $W \in \mathbb{C}^{q\times q}$, then W is of the form*

$$W = \sum_{k=1}^{q} w_k w_k^*, \tag{3.6}$$

where $w_k \in \mathbb{C}^q$ fulfill $Mw_k w_k^* M^* - \tilde{M} w_k w_k^* \tilde{M}^* = 0$ for $k = 1, \ldots, q$.

Proof. The proof is similar to [Ran96, Lemma 5.]. By Lemma 3.5(a) we obtain a unitary matrix $U \in \mathbb{C}^{q\times q}$ such that

$$\tilde{M}W^{\frac{1}{2}} = MW^{\frac{1}{2}}U.$$

Rewrite U as $U = \sum_{k=1}^{q} \mathrm{e}^{\mathrm{i}\omega_k} v_k v_k^*$ such that $\sum_{k=1}^{q} v_k v_k^* = I_q$ and $v_k^* v_j = 0$ for $j \neq k$. Set $w_k := W^{\frac{1}{2}} v_k$. Then for $k = 1, \dots, q$ we obtain

$$Mw_k \mathrm{e}^{\mathrm{i}\omega_k} = MW^{\frac{1}{2}}Uv_k = \tilde{M}W^{\frac{1}{2}}v_k = \tilde{M}w_k$$

and thus we have $Mw_k w_k^* M^* - \tilde{M}w_k w_k^* \tilde{M}^* = 0$. □

Then we obtain the following theorem.

Theorem 3.7 (KYP Lemma)**.** *Let $(I_n, A, B, Q, S, R) \in \Sigma_{m,n}^{w}(\mathbb{K})$ and the Popov function $\Phi(z) \in \mathbb{K}^{m\times m}(z)$ be given. Then the following statements hold:*

(a) Assume that (I_n, A, B) is controllable. If for all $\omega \in \mathbb{R}$ with $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$ we have

$$\Phi(\mathrm{e}^{\mathrm{i}\omega}) = \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix} \succeq 0, \tag{3.7}$$

then there exists some $P = P^* \in \mathbb{K}^{n\times n}$ such that

$$\mathcal{M}(P) = \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} + \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \succeq 0. \tag{3.8}$$

(b) If, on the other hand, we have some $P = P^* \in \mathbb{K}^{n\times n}$ such that (3.8) is valid, then (3.7) holds for all $\omega \in \mathbb{R}$ with $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$.

Proof. First we show statement (a). Thus, assume that for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$ we have $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$. This is equivalent to the fact that for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$ we have

$$\begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \succeq 0$$

for all $u \in \mathbb{K}^m$ and $x = (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} Bu \in \mathbb{K}^n$. By using Lemma 3.5(b) with $M = Ax + Bu$ and $N = x$ this again is equivalent to $\Xi \cap \Upsilon = \varnothing$, where

$$\Xi := \left\{ \left(\begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, (Ax + Bu)(Ax + Bu)^* - xx^* \right) \middle| \, [\, x^* \;\; u^* \,]^* \in \mathbb{K}^{n+m} \right\}$$

and

$$\Upsilon := \{(s, 0_{n\times n}) \mid s < 0\}.$$

Every element in the convex hull of Ξ can be written as

$$\sum_{k=1}^{N}\left(\begin{bmatrix}x_k\\u_k\end{bmatrix}^*\begin{bmatrix}Q & S\\S^* & R\end{bmatrix}\begin{bmatrix}x_k\\u_k\end{bmatrix},\ (Ax_k+Bu_k)(Ax_k+Bu_k)^*-x_kx_k^*\right), \tag{3.9}$$

where $[\,x_k^*\ \ u_k^*\,]^*\in\mathbb{C}^{n+m}$, $k=1,\dots,N$. Equivalently, using the switching invariance of the trace operator, i. e., $\operatorname{tr}(C_1C_2)=\operatorname{tr}(C_2C_1)$ for all matrices $C_1\in\mathbb{K}^{q_1\times q_2}$, $C_2\in\mathbb{K}^{q_2\times q_1}$, every element in the convex hull of Ξ can be reformulated as

$$\left(\operatorname{tr}\left(W\begin{bmatrix}Q & S\\S^* & R\end{bmatrix}\right),\ \begin{bmatrix}A & B\end{bmatrix}W\begin{bmatrix}A^*\\B^*\end{bmatrix}-\begin{bmatrix}I_n & 0\end{bmatrix}W\begin{bmatrix}I_n\\0\end{bmatrix}\right),$$

where $W=\sum_{k=1}^{N}[\,x_k^*\ \ u_k^*\,]^*[\,x_k\ \ u_k\,]^*\succeq 0$. If such an element also lies in Υ, Lemma 3.6 implies that we can redefine $[\,x_k^*\ \ u_k^*\,]^*$ in such a way that the second component of each term in (3.9) is zero and $N=n+m$. Thus, there has to be at least one index j such that

$$\begin{bmatrix}x_j\\u_j\end{bmatrix}^*\begin{bmatrix}Q & S\\S^* & R\end{bmatrix}\begin{bmatrix}x_j\\u_j\end{bmatrix}<0.$$

This shows that we have $\Xi\cap\Upsilon=\varnothing$ if and only if $\operatorname{conv}\Xi\cap\Upsilon=\varnothing$, where $\operatorname{conv}\Xi$ denotes the convex hull of Ξ.

Now we take the final step of the proof. If $\operatorname{conv}\Xi\cap\Upsilon=\varnothing$, then by the separation theorem for convex sets [Bre10, Section 1.2] there exists $(0,0)\neq(p,P)\in\mathbb{R}\times\mathbb{K}^{n\times n}$ with $P=P^*$ such that

$$\Re(py+\operatorname{tr}PV)\geq 0$$

for all $(y,V)\in\Xi$ and

$$\Re(py+\operatorname{tr}PV)\leq 0$$

for all $(y,V)\in\Upsilon$. This means that $p\geq 0$ and

$$\begin{aligned}0&\leq p\begin{bmatrix}x\\u\end{bmatrix}^*\begin{bmatrix}Q & S\\S^* & R\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}+\operatorname{tr}\left(P((Ax+Bu)(Ax+Bu)^*-xx^*)\right)\\&=\begin{bmatrix}x\\u\end{bmatrix}^*\left(p\begin{bmatrix}Q & S\\S^* & R\end{bmatrix}+\begin{bmatrix}A^*PA-P & A^*PB\\B^*PA & B^*PB\end{bmatrix}\right)\begin{bmatrix}x\\u\end{bmatrix}\end{aligned} \tag{3.10}$$

for all $[\,x^*\ \ u^*\,]^*\in\mathbb{K}^{n+m}$.

If $p=0$ then equation (3.10) implies that

$$\begin{bmatrix}A^*PA-P & A^*PB\\B^*PA & B^*PB\end{bmatrix}\succeq 0. \tag{3.11}$$

This inequality is invariant under feedback by Lemma 3.4. Thus, by controllability of (I_n, A, B, Q, S, R) and Proposition 2.33 we can assume without loss of generality that A is asymptotically stable. Hence, by (2.26) we have $P \preceq 0$. Let

$$\tilde{P} := U^* P U = \operatorname{diag}(P_1, 0) \preceq 0$$

for some unitary $U \in \mathbb{K}^{n\times n}$, where $P_1 \in \mathbb{K}^{r\times r}$ is invertible. We have that $r > 0$ since $P \neq 0$. Furthermore, let

$$\tilde{A} = U^* A U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{B} = U^* A U = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

be partitioned accordingly.

Thus, with (3.11) and Lemma 3.4 we have

$$\begin{bmatrix} A_{11}^* P_1 A_{11} - P_1 & A_{11}^* P_1 A_{12} & A_{11}^* P_1 B_1 \\ A_{12}^* P_1 A_{11} & A_{12}^* P_1 A_{12} & A_{12}^* P_1 B_1 \\ B_1^* P_1 A_{11} & B_1^* P_1 A_{12} & B_1^* P_1 B_1 \end{bmatrix} \succeq 0. \tag{3.12}$$

Therefore, by Proposition 2.4 we obtain $A_{12}^* P_1 A_{12} = 0$ and $B_1^* P_1 B_1 = 0$ and thus $A_{12} = 0$ and $B_1 = 0$. This is contradicting the controllability of (I_n, A, B), since every eigenvalue of A_{11} at the same time is an uncontrollable mode of $(I_r, A_{11}, B_1) \in \Sigma_{m,r}(\mathbb{K})$ and due to the block diagonal structure also of (I_n, A, B). Hence, $p > 0$. Thus, dividing equation (3.10) by p leads to (3.8).

If, on the other hand, statement (b) holds, we have that (3.10) is valid for $p = 1$. Thus, another application of the separation theorem immediately shows that $\operatorname{conv} \Xi \cap \Upsilon = \varnothing$, which – as shown in the first part of the proof – holds if and only if (3.7) is true for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$. □

Note that in the proof of Theorem 3.7 we did not use the controllability assumption on $(I_n, A, B, Q, S, R) \in \Sigma_{m,n}^w(\mathbb{K})$ for showing statement (b).

3.2 Implicit Difference Equations

For the generalization of the KYP inequality to implicit difference equations we first need to understand relations between the different Popov functions and KYP inequalities corresponding to systems $(E, A, B, Q, S, R) \in \Sigma_{m,n}^w(\mathbb{K})$ and

$$(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma_{m,n}^w(\mathbb{K})$$

as in (3.3) and how they are related to explicit difference equations.

The associated EDE part is given by

$$(I_{n_1}, A_s, B_s, Q_s, S_s, R_s) \in \Sigma_{m,n_1}^w(\mathbb{K})$$

which is defined by

$$\begin{aligned} & A_s = A_{11}, \qquad B_s = B_1, \\ Q_s = Q_{11}, \qquad S_s = S_1 - Q_{12}B_2, \qquad & R_s = B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R. \end{aligned} \tag{3.13}$$

Proposition 3.8. *Consider the Popov function*

$$\Phi_F(z) := \begin{bmatrix} (zE_F - A_F)^{-1}B_F \\ I_m \end{bmatrix}^{\sim} \begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix} \begin{bmatrix} (zE_F - A_F)^{-1}B_F \\ I_m \end{bmatrix} \in \mathbb{K}^{m\times m}(z)$$

of the system $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$ *as in* (3.3).

(a) The Popov functions $\Phi_F(z)$ *and* $\Phi(z)$ *are related via*

$$\Phi_F(z) = \Theta_F^{\sim}(z)\Phi(z)\Theta_F(z), \tag{3.14}$$

where $\Theta_F(z) = I_m + FT(zE_F - A_F)^{-1}B_F \in \mathbb{K}^{m\times m}(z)$ *is invertible.*

(b) Further, assume that $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$ *is given in feedback equivalence form as in* (2.15) *and partitioned accordingly. Then it holds that*

$$\Phi_F(z) = \Phi_s(z),$$

where $\Phi_s(z)$ *is the Popov function corresponding to the EDE part*

$$(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$$

as in (3.13).

Proof. Relation (3.14) is shown in [Voi15, Proposition 3.2.2 a)] with the help of the fact that

$$T(\lambda E_F - A_F)^{-1}B_F = (\lambda E - A)^{-1}B\Theta_F(\lambda) \tag{3.15}$$

for all $\lambda \in \mathbb{C}$ such that $\det(\lambda E_F - A_F) \neq 0$ and $\det(\lambda E - A) \neq 0$. Furthermore, by using (3.15) we obtain

$$\Theta_F(\lambda) = I_m + FT(\lambda E_F - A_F)^{-1}B_F = I_m + F(\lambda E - A)^{-1}B\Theta_F(\lambda)$$

and thus

$$(I_m - F(\lambda E - A)^{-1}B)\Theta_F(\lambda) = I_m.$$

Hence, $\Theta_F(s)$ is invertible.

For part (b) see [RRV15, Lemma 4.2(a)]. □

We now turn to a reduction of the problem for systems $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ in feedback equivalence form as in (3.3) to the corresponding EDE system

$$(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$$

as in (3.13).

Lemma 3.9. *Assume that $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3) is given in feedback equivalence form as in (2.15) and partitioned accordingly. Further, consider the corresponding EDE part*

$$(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$$

as in (3.13) and partition the Hermitian matrix

$$P_F = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} \in \mathbb{K}^{n\times n}$$

accordingly. Then $P_{11} \in \mathbb{K}^{n_1\times n_1}$ is a solution of the KYP inequality (3.2) corresponding to the EDE part $(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s)$ if and only if P_F is a solution of the KYP inequality (3.2) corresponding to $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$.

Proof. We have

$$\begin{aligned}
&\begin{bmatrix} A_F^*P_FA_F - E_F^*P_FE_F + Q_F & A_F^*P_FB_F + S_F \\ B_F^*P_FA_F + S_F^* & B_F^*P_FB_F + R_F \end{bmatrix} \\
&= \left[\begin{array}{ccc|c} A_{11}^*P_{11}A_{11} - P_{11} & A_{11}^*P_{12} & M_{13} & A_{11}^*P_{11}B_1 + A_{11}^*P_{12}B_2 \\ P_{12}^*A_{11} & P_{22} & M_{23} & P_{12}^*B_1 + P_{22}B_2 \\ M_{13}^* & M_{23}^* & M_{33} & M_{34} \\ \hline B_1^*P_{11}A_{11} + B_2^*P_{12}^*A_{11} & B_1^*P_{12} + B_2^*P_{22} & M_{34}^* & M_{44} \end{array}\right] \\
&\qquad + \left[\begin{array}{ccc|c} Q_{11} & Q_{12} & Q_{13} & S_1 \\ Q_{12}^* & Q_{22} & Q_{23} & S_2 \\ Q_{13}^* & Q_{23}^* & Q_{33} & S_3 \\ \hline S_1^* & S_2^* & S_3^* & R \end{array}\right]
\end{aligned} \tag{3.16}$$

for some $M_{13} \in \mathbb{K}^{n_1\times n_3}$, $M_{23} \in \mathbb{K}^{n_2\times n_3}$, $M_{33} \in \mathbb{K}^{n_3\times n_3}$, $M_{34} \in \mathbb{K}^{n_3\times m}$, and

$$M_{44} = B_1^*P_{11}B_1 + B_1^*P_{12}B_2 + B_2^*P_{22}B_2 + B_2^*P_{12}^*B_1 \in \mathbb{K}^{m\times m}.$$

Let $[\, x^* \;\; u^* \,]^* \in \mathcal{V}_F^\Sigma$. Thus, by (2.20) there exists an $x_1 \in \mathbb{K}^{n_1}$ such that

$$x = \begin{bmatrix} x_1 \\ -B_2u \\ 0_{n_3\times 1} \end{bmatrix}. \tag{3.17}$$

Then we obtain

$$
\begin{aligned}
&\begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} A_F^* P_F A_F - E_F^* P_F E_F + Q_F & A_F^* P_F B_F + S_F \\ B_F^* P_F A_F + S_F^* & B_F^* P_F B_F + R_F \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\
= \quad & x_1^* \Big((A_{11}^* P_{11} A_{11} - P_{11} + Q_{11}) x_1 - (A_{11}^* P_{12} + Q_{12}) B_2 u \\
& \qquad + (A_{11}^* P_{11} B_1 + A_{11}^* P_{12} B_2 + S_1) u \Big) \\
& - u^* B_2^* \Big((P_{12}^* A_{11} + Q_{12}^*) x_1 - (P_{22} + Q_{22}) B_2 u + (P_{12}^* B_1 + P_{22} B_2 + S_2) u \Big) \\
& + u^* \Big((B_1^* P_{11} A_{11} + B_2^* P_{12}^* A_{11} + S_1^*) x_1 - (B_1^* P_{12} + B_2^* P_{22} + S_2^*) B_2 u \\
& \qquad + (B_1^* (P_{11} B_1 + P_{12} B_2) + B_2^* (P_{22} B_2 + P_{12}^* B_1) + R) u \Big) \\
= & \begin{bmatrix} x_1 \\ u \end{bmatrix}^* \begin{bmatrix} A_s^* P_{11} A_s - P_{11} + Q_s & A_s^* P_{11} B_s + S_s \\ B_s^* P_{11} A_s + S_s^* & B_s^* P_{11} B_s + R_s \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix}.
\end{aligned}
\tag{3.18}
$$

Thus

$$
\begin{bmatrix} x_1 \\ u \end{bmatrix}^* \begin{bmatrix} A_s^* P_{11} A_s - P_{11} + Q_s & A_s^* P_{11} B_s + S_s \\ B_s^* P_{11} A_s + S_s^* & B_s^* P_{11} B_s + R_s \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix}^* \geq 0 \tag{3.19}
$$

for all $[\, x_1^* \;\; u^* \,]^* \in \mathbb{K}^{n_1+m}$ if and only if

$$
\begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} A_F^* P_F A_F - E_F^* P_F E_F + Q_F & A_F^* P_F B_F + S_F \\ B_F^* P_F A_F + S_F^* & B_F^* P_F B_F + R_F \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0
$$

for all $[\, x^* \;\; u^* \,]^* \in \mathcal{V}_F^\Sigma$. Hence, P_{11} is a solution of the KYP inequality (3.2) corresponding to the EDE part if and only if P_F solves (3.2) corresponding to $(E_F,\, A_F,\, B_F)$. □

We are now ready to state the generalization of the KYP Lemma for IDEs.

Theorem 3.10 (KYP Lemma for IDEs)**.** *Let* $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma_{m,n}^w(\mathbb{K})$ *and the system space* $\mathcal{V}^\Sigma$ *be given with corresponding Popov function* $\Phi(z) \in \mathbb{K}^{m\times m}(z)$.

(a) If there exists some $P \in \mathbb{K}^{n\times n}$ *that is a solution of* (3.2)*, then* $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ *for all* $\omega \in \mathbb{R}$ *with* $\det(\mathrm{e}^{\mathrm{i}\omega} E - A) \neq 0$.

(b) If on the other hand (E, A, B) *is R-controllable and* $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ *for all* $\omega \in \mathbb{R}$ *with* $\det(\mathrm{e}^{\mathrm{i}\omega} E - A) \neq 0$*, then there exists a solution* $P \in \mathbb{K}^{n\times n}$ *of* (3.2).

Proof. We first show assertion (a). Assume that $P \in \mathbb{K}^{n\times n}$ fulfills the KYP inequality (3.2), i. e., $\mathcal{M}(P) \succeq_{\mathcal{V}^\Sigma} 0$. Further, let $\omega \in \mathbb{R}$ be such that $\det(\mathrm{e}^{\mathrm{i}\omega} E - A) \neq 0$. By

Lemma 3.3, together with Lemma 2.2(b) and Proposition 2.29(b), statement (a) then follows due to

$$\begin{aligned}\Phi(\mathrm{e}^{\mathrm{i}\omega}) &= \begin{bmatrix}(\mathrm{e}^{\mathrm{i}\omega}E-A)^{-1}B\\ I_m\end{bmatrix}^{\sim}\begin{bmatrix}Q & S\\ S^* & R\end{bmatrix}\begin{bmatrix}(\mathrm{e}^{\mathrm{i}\omega}E-A)^{-1}B\\ I_m\end{bmatrix}\\ &= \begin{bmatrix}(\mathrm{e}^{\mathrm{i}\omega}E-A)^{-1}B\\ I_m\end{bmatrix}^{*}\mathcal{M}(P)\begin{bmatrix}(\mathrm{e}^{\mathrm{i}\omega}E-A)^{-1}B\\ I_m\end{bmatrix}\succeq 0.\end{aligned} \tag{3.20}$$

For part (b) assume that $\Phi(\mathrm{e}^{\mathrm{i}\omega})\succeq 0$ for all $\omega\in\mathbb{R}$ with $\det(\mathrm{e}^{\mathrm{i}\omega}E-A)\neq 0$. For the system in feedback equivalence form $(E_F,\,A_F,\,B_F,\,Q_F,\,S_F,\,R_F)\in\Sigma^w_{m,n}(\mathbb{K})$ and corresponding Popov function $\Phi_F(z)\in\mathbb{K}^{m\times m}(z)$ we obtain from Proposition 3.8(b) that $\Phi_F(\mathrm{e}^{\mathrm{i}\omega})\succeq 0$ for all $\omega\in\mathbb{R}$ also fulfilling $\det(\mathrm{e}^{\mathrm{i}\omega}E_F-A_F)\neq 0$. In particular, by Proposition 2.26 for such ω we have $\det(\mathrm{e}^{\mathrm{i}\omega}I_{n_1}-A_{11})\neq 0$. Furthermore, by Proposition 2.32(b) the associated EDE system $(I_{n_1},\,A_{11},\,B_1)\in\Sigma_{m,n}(\mathbb{K})$ is controllable.

This means we are in the situation of Theorem 3.7 for the EDE system

$$(I_{n_1},\,A_s,\,B_s,\,Q_s,\,S_s,\,R_s)\in\Sigma^w_{m,n_1}(\mathbb{K})$$

as in (3.13). Thus, applying Lemma 3.9 gives a solution P_F of the KYP inequality (3.2) corresponding to the system $(E_F,\,A_F,\,B_F,\,Q_F,\,S_F,\,R_F)$. Then, using Lemma 3.4 and Lemma 2.2(c) completes the proof. □

Example 3.11 (Example 2.25 revisited)**.** Consider the system $(E,\,A,\,B)$ as in Example 2.25. From its feedback equivalence form as in (2.17) we obtain with (2.20) that

$$V_F^{\Sigma}=\begin{bmatrix}1&0&0\\0&0&-1\\0&0&1\end{bmatrix}$$

spans the system space $\mathcal{V}_F^{\Sigma}$ and thus

$$\mathcal{V}^{\Sigma}=\mathcal{T}_F\mathcal{V}_F^{\Sigma}=\operatorname{im}\begin{bmatrix}1&0&-1\\1&0&0\\0&0&1\end{bmatrix}$$

with

$$\mathcal{T}_F=\begin{bmatrix}1&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

From

$$\begin{bmatrix}Q&S\\S^*&R\end{bmatrix}=I_3$$

we obtain as modified weights

$$\begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix} = \begin{bmatrix} T^*(Q + F^*S^* + SF + F^*RF)T & T^*(S + F^*R) \\ (S^* + RF)T & R \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, the associated EDE part as in (3.13) is given by

$$A_s = 1, \quad B_s = -1, \quad Q_s = 2, \quad S_s = -1, \quad R_s = 2. \tag{3.21}$$

Thus, by Proposition 2.4, P_{11} solves the KYP inequality

$$\begin{bmatrix} 2 & -P_{11} - 1 \\ -P_{11} - 1 & P_{11} + 2 \end{bmatrix} \succeq 0 \tag{3.22}$$

if and only if

$$-\sqrt{3} \leq P_{11} \leq \sqrt{3}.$$

Therefore, choosing $P_{11} = -1$, we have that

$$P = W^* P_F W = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

solves the KYP inequality (3.2). In particular, by Theorem 3.10 we obtain that for the Popov functions $\Phi_F(z) \in \mathbb{K}(z)$ and $\Phi(z) \in \mathbb{K}(z)$ we have $\Phi_F(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ and $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$.

The next remark is based on an idea presented in [SW10] for the case where $E = I_n$.

Remark 3.12. Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$. Then we have

$$\begin{aligned} &\left[\begin{array}{cc} E & 0 \\ A & B \\ \hline I_n & 0 \\ 0 & I_n \end{array}\right]^* \left[\begin{array}{cc|cc} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^* & R \end{array}\right] \left[\begin{array}{cc} E & 0 \\ A & B \\ \hline I_n & 0 \\ 0 & I_n \end{array}\right] \\ &= \begin{bmatrix} -E^*P & A^*P \\ 0 & B^*P \end{bmatrix} \begin{bmatrix} E & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \\ &= \begin{bmatrix} A^*PA - E^*PE + Q & A^*PB + S \\ B^*PA + S^* & B^*PB + R \end{bmatrix} = \mathcal{M}(P). \end{aligned} \tag{3.23}$$

Thus, the existence of a solution $P = P^* \in \mathbb{K}^{n \times n}$ to (3.2) is equivalent to P solving the matrix inequality for the extended equation (3.23). Furthermore, replacing

$$\begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix}$$

by

$$\begin{aligned}\begin{bmatrix} 0 & hP \\ hP & 0 \end{bmatrix} &= \begin{bmatrix} I_n & I_n \\ -hI_n/2 & hI_n/2 \end{bmatrix} \begin{bmatrix} -P & hP/2 \\ P & hP/2 \end{bmatrix} \\ &= \begin{bmatrix} I_n & I_n \\ -hI_n/2 & hI_n/2 \end{bmatrix} \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I_n & -hI_n/2 \\ I_n & hI_n/2 \end{bmatrix}\end{aligned} \tag{3.24}$$

in (3.23), for $h = 1$ we obtain

$$\begin{aligned}&\left[\begin{array}{cc} E & 0 \\ A & B \\ \hline I_n & 0 \\ 0 & I_n \end{array}\right]^* \left[\begin{array}{cc|cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^* & R \end{array}\right] \left[\begin{array}{cc} E & 0 \\ A & B \\ \hline I_n & 0 \\ 0 & I_n \end{array}\right] \\ &= \begin{bmatrix} A^*P & E^*P \\ B^*P & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \\ &= \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix},\end{aligned}$$

which corresponds to the matrix $\mathcal{M}(P)$ in the continuous-time KYP inequality, see [RRV15].

The replacement (3.24) corresponds to a discretization of the DAE (2.9) with the trapezoidal rule (2.13). To see this, consider

$$\begin{aligned}&h \begin{bmatrix} E & 0 \\ A & B \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ A & B \end{bmatrix} \\ &\quad = \begin{bmatrix} E & 0 \\ A & B \end{bmatrix}^* \begin{bmatrix} I_n & I_n \\ -hI_n/2 & hI_n/2 \end{bmatrix} \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I_n & -hI_n/2 \\ I_n & hI_n/2 \end{bmatrix} \begin{bmatrix} E & 0 \\ A & B \end{bmatrix} \\ &\quad = \begin{bmatrix} E - \frac{h}{2}A & -\frac{h}{2}B \\ E + \frac{h}{2}A & \frac{h}{2}B \end{bmatrix}^* \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} E - \frac{h}{2}A & -\frac{h}{2}B \\ E + \frac{h}{2}A & \frac{h}{2}B \end{bmatrix},\end{aligned}$$

where

$$\begin{bmatrix} \sigma I_n & -I_n \end{bmatrix} \begin{bmatrix} E - \frac{h}{2}A & -\frac{h}{2}B \\ E + \frac{h}{2}A & \frac{h}{2}B \end{bmatrix} \begin{bmatrix} x_n \\ u_n \end{bmatrix} = 0$$

corresponds to the system equations obtained from the trapezoidal rule (2.13).

Remark 3.13. The result of Theorem 3.10 is analogous to the continuous-time result in [RRV15]. To see this, replace positivity of the Popov function on the unit circle by positivity on the imaginary axis in (a) and replace $\mathcal{M}(P)$ by its continuous-time analog. However, in [RRV15] the assumption of R-controllability was alternatively replaced by

the condition that the Popov function has full rank and $(E,\ A,\ B)$ is *sign-controllable*. To adapt this to the discrete-time setting we would need a discrete-time analog of [CALM97, Theorem 6.1], which provides the characterizations via sign-controllability in the ODE case.

4 Structure of Palindromic Matrix Pencils

In this chapter we are concerned with specially structured matrix pencils $z\mathcal{E} - \mathcal{A} \in \mathbb{K}^{n\times n}[z]$, in particular palindromic matrix pencils. For the investigation of these palindromic matrix pencils we first introduce so-called quasi-Hermitian matrices. Then we present the connection to even matrix pencils and we show characterizations of the inertia of palindromic matrix pencils similar to what was done in [CALM97; CG89; Rei11; Voi15] in the even case.

4.1 Quasi-Hermitian Matrices

Here we introduce the concept of quasi-Hermitian matrices, which is an extension to the notion of Hermitian and skew-Hermitian matrices.

Definition 4.1. A matrix $\mathcal{A} \in \mathbb{K}^{n\times n}$ with

$$\mathcal{A} = \mathrm{e}^{\mathrm{i}\omega}\mathcal{A}^*$$

and $\omega \in [0, 2\pi)$ is called quasi-Hermitian.

Lemma 4.2. *Let $\mathcal{A} \in \mathbb{K}^{n\times n}$ be quasi-Hermitian with angle $\omega \in [0, 2\pi)$. Then all eigenvalues λ of $\mathcal{A}$ lie on a line with angle $\frac{\omega}{2}$ through the origin, i. e.,*

$$\lambda = r\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}$$

for some $r \in \mathbb{R}$.

Proof. Let $\lambda = r\mathrm{e}^{\mathrm{i}\theta}$, with $\theta \in [0, \pi)$ and $r \in \mathbb{R}$, be an eigenvalue of $\mathcal{A}$ and $x \in \mathbb{K}^n$ a corresponding eigenvector with $\|x\|_2 = 1$. Then it holds that

$$\begin{aligned} re^{i\theta} &= \lambda = \lambda x^* x = x^*\mathcal{A}x = \mathrm{e}^{\mathrm{i}\omega}x^*\mathcal{A}^*x \\ &= \mathrm{e}^{\mathrm{i}\omega}r\mathrm{e}^{-\mathrm{i}\theta}. \end{aligned}$$

Hence, we obtain $\theta = \frac{\omega}{2}$. □

We can extend the notion of inertia for Hermitian matrices to quasi-Hermitian matrices.

Definition 4.3. Let $\mathcal{A} \in \mathbb{K}^{n\times n}$ be quasi-Hermitian with angle $\omega \in [0, 2\pi)$. Then the inertia of $\mathcal{A}$ along $\frac{\omega}{2}$ is

$$\operatorname{In}(\mathcal{A}) := \operatorname{In}_{\frac{\omega}{2}}(\mathcal{A}) := (n_+,\, n_0,\, n_-), \tag{4.1}$$

where n_+, n_0, and n_- denote the number of eigenvalues $\lambda = r\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}$ where r is positive, zero, or negative, respectively. We omit the subscript $\frac{\omega}{2}$ in $\operatorname{In}_{\frac{\omega}{2}}(\mathcal{A})$ if the angle is clear from the context.

Similar to the Hermitian case, also in the quasi-Hermitian case we have a canonical form under congruence transformations.

Theorem 4.4. *The inertia of a quasi-Hermitian matrix is invariant under congruence transformations. On the other hand, for two matrices $\mathcal{A}, \mathcal{B} \in \mathbb{K}^{n\times n}$ with the same inertia there exists some invertible $U \in \mathbb{K}^{n\times n}$ such that*

$$U^*\mathcal{A}U = \mathcal{B},$$

i. e., $\mathcal{A}$ and $\mathcal{B}$ are congruent.

Proof. In [Ikr01] it is shown that two normal matrices $\mathcal{A}, \mathcal{B} \in \mathbb{K}^{n\times n}$ are congruent if and only if for every angle $\omega \in [0, 2\pi)$ they have the same number of nonzero eigenvalues λ_i and μ_i, respectively, with $\arg \lambda_i = \arg \mu_i = \omega$ and, in addition, have the same number of zero eigenvalues.
This proves the statement, since in particular $\mathcal{A}$ is normal, i. e., $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$, and from Lemma 4.2 we obtain that all eigenvalues of $\mathcal{A}$ lie on a line through the origin which is reflected in the definition of $\operatorname{In}(A)$. □

We conclude this section with some results on inertia of some specially structured quasi-Hermitian matrices. Note that here, adding two tuples of inertia $(n_1^+,\, n_1^0,\, n_1^-)$ and $(n_2^+,\, n_2^0,\, n_2^-)$ has to be understood component-wise, i. e.,

$$(n_1^+,\, n_1^0,\, n_1^-) + (n_2^+,\, n_2^0,\, n_2^-) := (n_1^+ + n_2^+,\, n_1^0 + n_2^0,\, n_1^- + n_2^-).$$

Lemma 4.5. *Consider a quasi-Hermitian matrix $\mathcal{A} \in \mathbb{K}^{n+l+m\times n+l+m}$ of the form*

$$\mathcal{A} = \mathrm{e}^{\mathrm{i}\omega} \begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ C^* & 0 & 0 \end{bmatrix} \tag{4.2}$$

for some $B = B^* \in \mathbb{K}^{n\times n}$, $C \in \mathbb{K}^{l\times m}$ and $\omega \in [0, 2\pi)$. Then it holds that

$$\operatorname{In}(\mathcal{A}) = \operatorname{In}(B) + (\operatorname{rk} C, l + m - 2\operatorname{rk} C, \operatorname{rk} C)\,.$$

Proof. The assertion is shown in [Brü11, Lemma 3.10] for $\omega = 0$ with congruence transformations. By applying the same congruence transformations, by Theorem 4.4 it also holds for general $\omega \in [0, 2\pi)$. □

Lemma 4.6. *Consider a quasi-Hermitian matrix $\mathcal{A} \in \mathbb{K}^{n+2m \times n+2m}$ of the form*

$$\mathcal{A} = \mathrm{e}^{\mathrm{i}\omega} \begin{bmatrix} 0 & 0 & C \\ 0 & B & D \\ C^* & D^* & E \end{bmatrix} \tag{4.3}$$

for some $B = B^* \in \mathbb{K}^{n \times n}$, invertible $C \in \mathbb{K}^{m \times m}$, $D \in \mathbb{K}^{m \times n}$, $E = E^* \in \mathbb{K}^{n \times n}$, and $\omega \in [0, 2\pi)$. Then it holds that

$$\mathrm{In}(\mathcal{A}) = \mathrm{In}(B) + (m, 0, m)\,.$$

Proof. Applying congruence transformations to $\mathcal{A}$ via

$$U = \begin{bmatrix} I_n & -C^{-*}D^* & -\frac{C^{-*}E}{2} \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{bmatrix}$$

we obtain that

$$U^*\mathcal{A}U = \begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ C^* & 0 & 0 \end{bmatrix}.$$

Thus, the assertion is an immediate consequence of Lemma 4.5 by using Theorem 4.4. □

Lemma 4.7. *Consider a quasi-Hermitian matrix $\mathcal{A} \in \mathbb{K}^{2m \times 2m}$ of the form*

$$\mathcal{A} = \mathrm{e}^{\mathrm{i}\omega} \begin{bmatrix} 0 & C \\ C^* & E \end{bmatrix} \tag{4.4}$$

for some $E = E^* \in \mathbb{K}^{m \times m}$, invertible $C \in \mathbb{K}^{m \times m}$ and $\omega \in [0, 2\pi)$. Then it holds that

$$\mathrm{In}(\mathcal{A}) = (m, 0, m)\,.$$

Proof. The assertion is an immediate consequence of Lemma 4.6. □

4.2 Palindromic and Even Matrix Pencils

In this section we consider so-called even and palindromic matrix pencils, where we will focus on the latter ones. Both are matrix polynomials of degree one of the form

$$z\mathcal{E} - \mathcal{A} \in \mathbb{K}^{n \times n}[z], \tag{4.5}$$

where $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{n\times n}$. Even matrix pencils arise in the continuous-time optimal control problem and are characterized by the fact that $\mathcal{E}^* = -\mathcal{E}$ and $\mathcal{A}^* = \mathcal{A}$, whereas palindromic matrix pencils are obtained in the discrete-time setting with the property that $\mathcal{E} = \mathcal{A}^*$. Note that congruence transformations preserve the palindromic and even structure, respectively: For invertible $U \in \mathbb{K}^{n\times n}$ the matrix pencil

$$U^*(z\mathcal{E} - \mathcal{A})U = z(U^*\mathcal{E}U) - U^*\mathcal{A}U$$

is still palindromic or even, respectively. Palindromic matrix pencils are directly connected to even matrix pencils via the *generalized Cayley transform* $\mathbf{c}$ [Cay46; Meh96; Sch08], which is defined by

$$\mathbf{c}(z\mathcal{E} - \mathcal{A}) := z(\mathcal{E} + \mathcal{A}) - (\mathcal{E} - \mathcal{A}).$$

Since

$$\mathbf{c}\left(\mathbf{c}(z\mathcal{E} - \mathcal{A})\right) = 2z\mathcal{E} - 2\mathcal{A}$$

we immediately see that the image of $\mathbf{c}$ under palindromic pencils is the set of even pencils and vice versa. In particular, one can show that the eigenvalue ∞ is mapped uniquely to the eigenvalue one. We are now interested in a structure-preserving canonical form revealing the eigenstructure of a palindromic matrix pencil.

Theorem 4.8 (Palindromic Kronecker canonical form)**.** *[Sch08] Let $z\mathcal{A}^* - \mathcal{A} \in \mathbb{K}^{n\times n}[z]$ be a palindromic matrix pencil. Then there exists some invertible $U \in \mathbb{C}^{n\times n}$ such that*

$$U^*(z\mathcal{A}^* - \mathcal{A})U = \operatorname{diag}\left(D_1(z), \ldots, D_k(z)\right) \tag{4.6}$$

for some $k \in \mathbb{N}$ is in palindromic Kronecker canonical form (PKCF)*, where each block $D_j(z) \in \mathbb{C}^{k_j\times k_j}[z]$, $k_j \in \mathbb{N}$, is of one of the following forms:*

Type P1: For $\lambda_j \in \mathbb{C}$ with $|\lambda_j| < 1$ and k_j even $D_j(z)$ is of the form

$$D_j(z) = \left[\begin{array}{cccc|cccc}
 & & & & & & & z-\lambda_j \\
 & & & & & & \iddots & -1 \\
 & & & & & \iddots & \iddots & \\
 & & & & z-\lambda_j & -1 & & \\
\hline
 & & & z\overline{\lambda_j}-1 & & & & \\
 & & \iddots & z & & & & \\
 & \iddots & \iddots & & & & & \\
z\overline{\lambda_j}-1 & z & & & & & &
\end{array}\right].$$

Type P2: For $\lambda_j = \mathrm{e}^{\mathrm{i}\theta_j} \in \mathbb{C}$ *with* $\theta_j \in [0, 2\pi)$*, i. e.,* $|\lambda_j| = 1$*,* $\varepsilon_j \in \{-1, 1\}$*, and* k_j *odd* $D_j(z)$ *is of the form*

$$D_j(z) = \varepsilon_j \left[\begin{array}{ccccc|c|cccc}
 & & & & & & & & & z-\lambda_j \\
 & & & & & & & & \iddots & -1 \\
 & & & & & & & \iddots & \iddots & \\
 & & & & & & z-\lambda_j & -1 & & \\
\hline
 & & & & & z\mathrm{e}^{-\mathrm{i}\theta_j/2} - \mathrm{e}^{\mathrm{i}\theta_j/2} & -1 & & & \\
\hline
 & & & & z\overline{\lambda_j} - 1 & z & & & & \\
 & & & \iddots & z & & & & & \\
 & & \iddots & \iddots & & & & & & \\
z\overline{\lambda_j} - 1 & z & & & & & & & &
\end{array}\right].$$

Type P3: For $\lambda_j = \mathrm{e}^{\mathrm{i}\theta_j} \in \mathbb{C}$ *with* $\theta_j \in (0, 2\pi)$*, i. e.,* $|\lambda_j| = 1$*,* $\varepsilon_j \in \{-1, 1\}$*, and* k_j *even* $D_j(z)$ *is of the form*

$$D_j(z) = \varepsilon_j \left[\begin{array}{ccccc|cccc}
 & & & & & & & & z-\lambda_j \\
 & & & & & & & \iddots & -1 \\
 & & & & & & \iddots & \iddots & \\
 & & & & & z-\lambda_j & -1 & & \\
\hline
 & & & & z\overline{\lambda_j} - 1 & z-1 & & & \\
 & & & \iddots & z & & & & \\
 & & \iddots & \iddots & & & & & \\
z\overline{\lambda_j} - 1 & z & & & & & & &
\end{array}\right].$$

Type P4: For $\lambda_j = 1 = \mathrm{e}^{\mathrm{i}\theta_j} \in \mathbb{C}$*, i. e.,* $\theta_j = 0$*,* $\varepsilon_j \in \{-1, 1\}$*, and* k_j *even* $D_j(z)$ *is of the form*

$$D_j(z) = \varepsilon_j \left[\begin{array}{ccccc|cccc}
 & & & & & & & & z-1 \\
 & & & & & & & \iddots & -1 \\
 & & & & & & \iddots & \iddots & \\
 & & & & & z-1 & -1 & & \\
\hline
 & & & & z-1 & \mathrm{i}z + \mathrm{i} & & & \\
 & & & \iddots & z & & & & \\
 & & \iddots & \iddots & & & & & \\
z-1 & z & & & & & & &
\end{array}\right].$$

Type P5: For k_j odd $D_j(z)$ is of the form

$$D_j(z) = \left[\begin{array}{cccc|cccc} & & & & & & & z \\ & & & & & & \iddots & -1 \\ & & & & & \iddots & & \\ & & & & z & & & \\ & & & & -1 & & & \\ \hline & & -1 & z & & & & \\ & \iddots & \iddots & & & & & \\ -1 & z & & & & & & \end{array}\right].$$

The PKCF is unique up to permutations of the blocks, and the quantities $\varepsilon_j \in \{-1, 1\}$ are called sign-characteristics.

A closely related version of the above theorem was developed simultaneously in [HS06].

Remark 4.9. We have multiplied the sign-characteristics of the blocks of type P4 occurring in [Sch08] with -1 in order to simplify some of the upcoming results. This is justified by the fact that if $\tilde{D}_j(z)$ with sign-characteristic $\tilde{\varepsilon}_j$ corresponds to a block of type P4 introduced in [Sch08], then $D_j(z) = -U^* \tilde{D}_j(z) U$ with

$$U = \mathrm{i} \begin{bmatrix} I_{k_j/2} & \\ & -I_{k_j/2} \end{bmatrix} \in \mathbb{C}^{k_j \times k_j}$$

is a block of type P4 with sign-characteristic $\varepsilon_j = -\tilde{\varepsilon}_j$ according to Theorem 4.8.

Remark 4.10. By analyzing the eigenstructure of the blocks in the form (4.6) we obtain:

(a) Blocks of type P1 correspond to eigenvalues λ and $1/\overline{\lambda}$ with $|\lambda| \neq 1$, i. e., these eigenvalues occur in pairs $\left\{\lambda, \frac{1}{\overline{\lambda}}\right\}$. In particular, this holds for the pairing $\{0, \infty\}$.

(b) Blocks of type P2, P3, and P4 correspond to eigenvalues λ with $|\lambda| = 1$.

(c) Blocks of type P5 correspond to rank deficiency of the pencil, i. e., they correspond to singular blocks.

Consider the palindromic matrix pencil $\mathcal{P}(z) = z\mathcal{A}^* - \mathcal{A} \in \mathbb{K}^{n \times n}[z]$. By inserting $\mathrm{e}^{\mathrm{i}\omega}$ for the polynomial variable z we obtain

$$\mathcal{P}(\mathrm{e}^{\mathrm{i}\omega}) = \mathrm{e}^{\mathrm{i}\omega}\mathcal{A}^* - \mathcal{A} = \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}(\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}\mathcal{A} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}\mathcal{A}^*) \tag{4.7}$$

and hence

$$\mathcal{P}(\mathrm{e}^{\mathrm{i}\omega})^* = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}(\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}\mathcal{A} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}\mathcal{A}^*) = (-\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}})^2\mathcal{P}(\mathrm{e}^{\mathrm{i}\omega}) = -\mathrm{e}^{-\mathrm{i}\omega}\mathcal{P}(\mathrm{e}^{\mathrm{i}\omega}).$$

Thus, $\mathcal{P}(\mathrm{e}^{\mathrm{i}\omega})$ is quasi-Hermitian and has a well-defined inertia. Investigating the block structure of the PKCF leads to the following result.

Lemma 4.11. *Assume that* $z\mathcal{A}^* - \mathcal{A} \in \mathbb{K}^{n\times n}[z]$ *is in PKCF, i. e., it holds that* $z\mathcal{A}^* - \mathcal{A} = \operatorname{diag}(D_1(z), \ldots, D_k(z))$ *for some* $k \in \mathbb{N}$. *Then the inertia pattern of each block* $D_j(z) \in \mathbb{C}^{k_j\times k_j}[z]$, $k_j \in \mathbb{N}$, *is given as follows:*

(a) If $D_j(z)$ *is of type P1, then for all* $\omega \in [0, 2\pi)$ *it holds that*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j}{2}, 0, \frac{k_j}{2}\right).$$

(b) If $D_j(z)$ *is of type P2 and* $\theta_j \in (0, 2\pi)$, *then for all* $\omega \in [0, 2\pi)$ *it holds that*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j-1}{2}, 0, \frac{k_j-1}{2}\right) + \operatorname{In}\left(\varepsilon_j(\omega - \theta_j)\right).$$

(c) If $D_j(z)$ *is of type P2 and* $\theta_j = 0$, *then for all* $\omega \in [0, 2\pi)$ *it holds that*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j-1}{2}, 0, \frac{k_j-1}{2}\right) + \operatorname{In}\left(\varepsilon_j\omega\right).$$

(d) If $D_j(z)$ *is of type P3 and* $\theta_j \in (0, 2\pi)$, *then for* $\omega \in [0, 2\pi)$ *it holds that*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j}{2}, 0, \frac{k_j}{2}\right)$$

if $\omega \neq \theta_j$, *and*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j}{2} - 1, 1, \frac{k_j}{2} - 1\right) + \operatorname{In}\left(\varepsilon_j\right)$$

if $\omega = \theta_j$.

(e) If $D_j(z)$ *is of type P4 with* $\theta_j = 0$, *then for* $\omega \in [0, 2\pi)$ *it holds that*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j}{2}, 0, \frac{k_j}{2}\right)$$

if $\omega \neq 0$, *and*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j}{2} - 1, 1, \frac{k_j}{2} - 1\right) + \operatorname{In}\left(\varepsilon_j\right)$$

if $\omega = 0$.

(f) If $D_j(z)$ *is of type P5, then for all* $\omega \in [0, 2\pi)$ *it holds that*

$$\operatorname{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j-1}{2}, 1, \frac{k_j-1}{2}\right).$$

Proof. First we show statement (a). Thus, consider a block $D_j(z)$ of type P1. Inserting $\mathrm{e}^{\mathrm{i}\omega}$ for z and using (4.7) yields

$$D_j(\mathrm{e}^{\mathrm{i}\omega}) = \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}$$

with some $C \in \mathbb{C}^{k_j/2 \times k_j/2}$. Note that C is invertible since $|\lambda_j| < 1$. Hence, applying Lemma 4.5, claim (a) follows.
To show parts (b) and (c) we have to consider blocks $D_j(z)$ of type P2, i. e., we have

$$D_j(\mathrm{e}^{\mathrm{i}\omega}) = \varepsilon_j \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \left[\begin{array}{cccc|c|cccc} & & & & & & & & \alpha \\ & & & & & & & \iddots & \gamma \\ & & & & & & \iddots & \iddots & \\ & & & & & \alpha & \gamma & & \\ \hline & & & & \beta & \gamma & & & \\ \hline & & & \overline{\alpha} & \overline{\gamma} & & & & \\ & & \iddots & \overline{\gamma} & & & & & \\ & \iddots & \iddots & & & & & & \\ \overline{\alpha} & \overline{\gamma} & & & & & & & \end{array}\right],$$

where $\alpha = \mathrm{i}\left(-\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} + \mathrm{e}^{-\mathrm{i}\left(\frac{\omega}{2}-\theta_j\right)}\right)$, $\beta = \mathrm{i}\left(\mathrm{e}^{-\mathrm{i}(\omega-\theta_j)/2} - \mathrm{e}^{\mathrm{i}(\omega-\theta_j)/2}\right) = 2\sin\left(\frac{\omega-\theta_j}{2}\right)$, $\gamma = \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega/2}$. First assume that $w = \theta_j$, i. e., $\alpha = \beta = 0$. Then we have

$$D_j(\mathrm{e}^{\mathrm{i}\omega}) = \varepsilon_j \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \left[\begin{array}{c|c} 0 & \\ \hline & \gamma I_{(k_j-1)/2} \\ & \overline{\gamma} I_{(k_j-1)/2} \end{array}\right].$$

Thus, the block structure together with Lemma 4.5 implies

$$\mathrm{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j-1}{2}, 1, \frac{k_j-1}{2}\right). \tag{4.8}$$

Now let $\omega \neq \theta_j$, i. e., $\alpha, \beta \neq 0$. Thus, we are in the situation of Lemma 4.6 and since $\beta > 0$ if and only if $w > \theta_j$, with (4.8) this concludes parts (b) and (c).

Now we show parts (d) and (e). If we have a block of type P3 or P4, then the matrix

$D_j(\mathrm{e}^{\mathrm{i}\omega})$ is structured as

$$D_j(\mathrm{e}^{\mathrm{i}\omega}) = \varepsilon_j \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \left[\begin{array}{ccccc|ccccc} & & & & & & & & & \alpha \\ & & & & & & & & \iddots & \gamma \\ & & & & & & & \iddots & \iddots & \\ & & & & & \alpha & \gamma & & & \\ \hline & & & & \overline{\alpha} & \beta & & & & \\ & & & \iddots & \overline{\gamma} & & & & & \\ & & \iddots & \iddots & & & & & & \\ \overline{\alpha} & \overline{\gamma} & & & & & & & & \end{array}\right],$$

where $\alpha = \mathrm{i}\left(-\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} + \mathrm{e}^{-\mathrm{i}\left(\frac{\omega}{2}-\theta_j\right)}\right)$, $\gamma = \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega/2}$ and $\beta = \mathrm{i}\left(\mathrm{e}^{-\mathrm{i}\omega/2} - \mathrm{e}^{\mathrm{i}\omega/2}\right) = 2\sin\left(\frac{\omega}{2}\right)$ or $\beta = \left(\mathrm{e}^{\mathrm{i}\omega/2} + \mathrm{e}^{-\mathrm{i}\omega/2}\right) = 2\cos\left(\frac{\omega}{2}\right)$ for type P3 or P4, respectively. First assume that $\omega = \theta_j$, i. e., $\alpha = 0$ and $\beta > 0$. Then we have

$$D_j(\mathrm{e}^{\mathrm{i}\omega}) = \varepsilon_j \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \left[\begin{array}{c|ccc} 0 & & & \\ \hline & & & \gamma I_{k_j/2-1} \\ & & \beta & \\ & \overline{\gamma} I_{k_j/2-1} & & \end{array}\right].$$

Thus, the block structure together with Lemma 4.5 implies that

$$\mathrm{In}\left(D_j(\mathrm{e}^{\mathrm{i}\omega})\right) = \left(\frac{k_j}{2} - 1, 1, \frac{k_j}{2} - 1\right) + \mathrm{In}\left(\varepsilon_j\right),$$

since for all possible values of $\theta_j \in [0, 2\pi)$ the corresponding value of β is strictly greater than zero. Now assume that $\omega \neq \theta_j$, i. e., α is nonzero. Then we are in the situation of Lemma 4.7 and the assertion follows immediately.
Finally, we show statement (f). If we have a block of type P5, then we have to consider

$$D_j(\mathrm{e}^{\mathrm{i}\omega}) = \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix},$$

where $\omega \in [0, 2\pi)$,

$$C = \begin{bmatrix} & & -\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \\ & \iddots & \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}} \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} & \iddots & \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}} & & \end{bmatrix} \in \mathbb{C}^{(k_j-1)/2+1\times(k_j-1)/2},$$

and $\operatorname{rk} C = \frac{k_j-1}{2}$. Lemma 4.5 then implies the desired result. □

Remark 4.12. The results from Lemma 4.11 can be used to determine the block structure of a pencil $z\mathcal{A}^* - \mathcal{A} \in \mathbb{K}^{n\times n}[z]$ in the form (4.6), given the inertia patterns for $\omega \in [0, 2\pi)$. For specific patterns all possible combinations of blocks can be found in Table 4.1. Note that blocks of type P1 have a very simple inertia pattern and thus from a general pattern

$$\mathrm{In}(\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}^* - \mathcal{A}) = (k_1, k_2, k_3)$$

– except for the case where $k_2 = 0$, i. e., all blocks are of type P1 – we cannot tell if and how many blocks of type P1 there are. For this reason in Table 4.1 we omit combinations of blocks with type P1 except for the pattern made up solely from blocks of type P1.

4.3 Inertia of Palindromic Pencils in Optimal Control

Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given. We consider palindromic matrix pencils arising in the optimal control problem as in (2.38) of the form

$$z\mathcal{A}^* - \mathcal{A} = \begin{bmatrix} 0 & zE - A & -B \\ zA^* - E^* & (z-1)Q & (z-1)S \\ zB^* & (z-1)S^* & (z-1)R \end{bmatrix} \in \mathbb{K}^{2n+m\times 2n+m}[z]. \tag{4.9}$$

If we insert $\mathrm{e}^{\mathrm{i}\omega}$ into (2.38) for z we obtain the quasi-Hermitian matrix

$$\mathcal{D}(\omega) := \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}(\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}\mathcal{A} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}\mathcal{A}^*) = \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix} 0 & E_\omega - A_\omega & B_\omega \\ E_\omega^* - A_\omega^* & Q_\omega & S_\omega \\ B_\omega^* & S_\omega^* & R_\omega \end{bmatrix} \in \mathbb{C}^{2n+m\times 2n+m} \tag{4.10}$$

with $E_\omega = -\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}E$, $A_\omega = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}A$, $B_\omega = \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}B$, $Q_\omega = s_\omega Q$, $S_\omega = s_\omega S$ and $R_\omega = s_\omega R$, where $s_\omega = \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}} - \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} = 2\sin\left(\frac{\omega}{2}\right)$.

Lemma 4.13. *Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ and consider the matrix $\mathcal{D}(\omega)$ as in (4.10) with ω such that $\det(E_\omega - A_\omega) \neq 0$. Furthermore, let*

$$U = \begin{bmatrix} I_n & 0 & (E_\omega^* - A_\omega^*)^{-1}(Q_\omega(E_\omega - A_\omega)^{-1}B_\omega - S_\omega) \\ 0 & I_n & -(E_\omega - A_\omega)^{-1}B_\omega \\ 0 & 0 & I_m \end{bmatrix} \in \mathbb{C}^{2n+m\times 2n+m}. \tag{4.11}$$

Then $\mathcal{D}(\omega)$ is congruent to

$$U^*\mathcal{D}(\omega)U = \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix} 0 & E_\omega - A_\omega & 0 \\ E_\omega^* - A_\omega^* & Q_\omega & 0 \\ 0 & 0 & 2\sin\left(\frac{\omega}{2}\right)\Phi\left(\mathrm{e}^{\mathrm{i}\omega}\right) \end{bmatrix}. \tag{4.12}$$

Table 4.1: Correspondence between specific inertia patterns for $\omega \in [0, 2\pi)$ and some fixed $\theta \in [0, 2\pi)$ of a matrix pencil in the form (4.6) and all possible block combinations that might lead to such a pattern. The signs + or − denote the sign-characteristic of a block, if applicable. For combinations including blocks of type P4, i. e., $\theta = 0$, the first column must be ignored. Blocks of type P5 do not have an associated angle θ, i. e., their pattern is independent of ω. For example, the pattern in the 8th entry can be achieved by either combining a block with positive sign-characteristic of type P2 with a block with positive sign-characteristic of type P3 or combining a block with positive sign-characteristic of type P2 with a block with positive sign-characteristic of type P4

	Inertia pattern for			
	$\omega < \theta$	$\omega = \theta$	$\omega > \theta$	Possible block combinations
1	$(k, 0, k)$	$(k, 0, k)$	$(k, 0, k)$	P1
2	$(k, 1, k)$	$(k, 1, k)$	$(k, 1, k)$	P5
3	$(k, 0, k)$	$(k-1, 2, k-1)$	$(k, 0, k)$	+P2 −P2, +P3 −P3, +P4 −P4
4	$(k, 0, k)$	$(k, 1, k-1)$	$(k, 0, k)$	+P3, +P4
5	$(k, 0, k)$	$(k-1, 1, k)$	$(k, 0, k)$	−P3, −P4
6	$(k, 0, k+1)$	$(k, 1, k)$	$(k+1, 0, k)$	+P2
7	$(k+1, 0, k)$	$(k, 1, k)$	$(k, 0, k+1)$	−P2
8	$(k, 0, k+1)$	$(k, 2, k-1)$	$(k+1, 0, k)$	+P2 +P3, +P2 +P4
9	$(k+1, 0, k)$	$(k-1, 2, k)$	$(k, 0, k+1)$	−P2 −P3, −P2 −P4
10	$(k, 0, k+1)$	$(k-1, 2, k)$	$(k+1, 0, k)$	+P2 −P3, +P2 −P4
11	$(k+1, 0, k)$	$(k, 2, k-1)$	$(k, 0, k+1)$	−P2 +P3, −P2 +P4

Proof. We have

$$
\begin{aligned}
&\begin{bmatrix}0 & 0 & I_m\end{bmatrix} U^*\mathcal{D}(\omega)U \begin{bmatrix}0\\0\\I_m\end{bmatrix}\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \left(B_\omega^*(E_\omega^* - A_\omega^*)^{-1}(Q_\omega(E_\omega - A_\omega)^{-1}B_\omega - S_\omega) - S_\omega^*(E_\omega - A_\omega)^{-1}B_\omega + R_\omega\right)\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix}(E_\omega - A_\omega)^{-1}B_\omega\\ -I_m\end{bmatrix}^* \begin{bmatrix}Q_\omega & S_\omega\\ S_\omega^* & R_\omega\end{bmatrix} \begin{bmatrix}(E_\omega - A_\omega)^{-1}B_\omega\\ -I_m\end{bmatrix}\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix}\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}(\mathrm{e}^{\mathrm{i}\omega}E - A)^{-1}\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}B\\ -I_m\end{bmatrix}^* s_\omega \begin{bmatrix}Q & S\\ S^* & R\end{bmatrix} \begin{bmatrix}\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}}(\mathrm{e}^{\mathrm{i}\omega}E - A)^{-1}\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega}{2}}B\\ -I_m\end{bmatrix}\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix}-(\mathrm{e}^{\mathrm{i}\omega}E - A)^{-1}B\\ -I_m\end{bmatrix}^* 2\sin\left(\frac{\omega}{2}\right) \begin{bmatrix}Q & S\\ S^* & R\end{bmatrix} \begin{bmatrix}-(\mathrm{e}^{\mathrm{i}\omega}E - A)^{-1}B\\ -I_m\end{bmatrix}\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \cdot 2\sin\left(\frac{\omega}{2}\right)\Phi\left(\mathrm{e}^{\mathrm{i}\omega}\right).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\begin{bmatrix}I_n & 0 & 0\\ 0 & I_n & 0\end{bmatrix} U^*\mathcal{D}(\omega)U = \begin{bmatrix}I_n & 0 & 0\\ 0 & I_n & 0\end{bmatrix}\mathcal{D}(\omega)U\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix}0 & E_\omega - A_\omega & B_\omega\\ E_\omega^* - A_\omega^* & Q_\omega & S_\omega\end{bmatrix} U\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix}0 & E_\omega - A_\omega & -B_\omega + B_\omega\\ E_\omega^* - A_\omega^* & Q_\omega & (Q_\omega(E_\omega - A_\omega)^{-1}B_\omega - S_\omega) - Q_\omega(E_\omega - A_\omega)^{-1}B_\omega + S_\omega\end{bmatrix}\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\omega}{2}} \begin{bmatrix}0 & E_\omega - A_\omega & 0\\ E_\omega^* - A_\omega^* & Q_\omega & 0\end{bmatrix}.
\end{aligned}
$$

The remaining components of the desired matrix can then be obtained by the fact that $U^*\mathcal{D}(\omega)U$ is Hermitian. □

Theorem 4.14. *Let $(E, A, B, Q, S, R) \in \Sigma_{m,n}^w(\mathbb{K})$ be given with corresponding Popov function $\Phi(z) \in \mathbb{K}^{m\times m}(z)$ and $\operatorname{rk}_{\mathbb{K}(z)} \Phi(z) = q$ for some $q \in \mathbb{N}_0$. Assume that (E, A, B) has no uncontrollable modes on the unit circle and let $r = \operatorname{rk}[E - A \;\; B]$. Then the following are equivalent:*

(a) The Popov function $\Phi(z)$ is positive semi-definite on the unit circle, i. e., $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$.

(b) The following conditions for the PKCF of $z\mathcal{A}^* - \mathcal{A}$ as in (4.6) hold:

(i) There are no blocks of type P2 corresponding to eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \neq 0$, and all blocks of type P3 have negative sign-characteristic.

(ii) The number of blocks of type P2 corresponding to an eigenvalue $\lambda = 1$ with positive sign-characteristic is greater by q than the number of those with negative sign-characteristic.

(c) The following conditions for the PKCF of $z\mathcal{A}^* - \mathcal{A}$ as in (4.6) hold:

(i') There are no blocks of type P2 corresponding to eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \neq 0$.

(ii') The number of blocks of type P2 corresponding to an eigenvalue $\lambda = 1$ with positive sign-characteristic is greater by q than the number of those with negative sign-characteristic.

Proof. The strategy of the proof is similar to the one in [Voi15, Theorem 3.4.2] for the continuous-time case. First note that since (E, A, B) has no uncontrollable modes on the unit circle, by Proposition 2.33 we can find a feedback matrix $F \in \mathbb{K}^{m\times n}$ such that $(E, A + BF)$ has no eigenvalues on the unit circle. Then by Lemma 3.4 and the fact that the palindromic pencil $z\mathcal{A}_F^* - \mathcal{A}_F$ corresponding to $(E, A + BF, B)$ is connected to $z\mathcal{A}^* - \mathcal{A}$ via $\mathcal{A}_F = U_F^*\mathcal{A}U_F$, where

$$U_F := \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & F & I_m \end{bmatrix} \in \mathbb{K}^{2n+m\times 2n+m},$$

we can assume without loss of generality that (E, A) has no eigenvalues on the unit circle.

Now we show that (a) implies (b). Therefore, assume that $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$. Then in particular we have

$$\mathrm{In}(\Phi(\mathrm{e}^{\mathrm{i}\omega})) = (q - a(\omega), m - q + a(\omega), 0)$$

for all $\omega \in (0, 2\pi)$, where $a : (0, 2\pi) \to \mathbb{N}_0$ is some function which is zero for almost all $\omega \in (0, 2\pi)$. Hence, by Lemma 4.13, Lemma 4.7 and Theorem 4.4 we obtain

$$\mathrm{In}\left(\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}^* - \mathcal{A}\right) = (n + q - a(\omega), m - q + a(\omega), n)$$

for $\omega \in (0, 2\pi)$. Again, by Theorem 4.4 the inertia of $\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}^* - \mathcal{A}$ coincides with the inertia of the PKCF of $z\mathcal{A}^* - \mathcal{A}$ as in (4.6) evaluated at $\mathrm{e}^{\mathrm{i}\omega}$. Since by Theorem 4.8 the block structure of the PKCF is uniquely determined, we can proceed by identifying blocks by their inertia patterns.

Note that $\mathrm{rk}_{\mathbb{K}(z)}(z\mathcal{A}^* - \mathcal{A}) = 2n + q$, since $z\mathcal{A}^* - \mathcal{A}$ can only have a finite amount of rank drops and due to Lemma 4.13 and the regularity of $zE - A$ there exist infinitely many values $\lambda \in \mathbb{C}$ for which $\mathrm{rk}(\lambda\mathcal{A}^* - \mathcal{A}) = 2n + q$. From Lemma 4.11 we can infer

that we have exactly $2n + m - (2n + q) = m - q$ blocks of type P5 in the PKCF of $z\mathcal{A}^* - \mathcal{A}$, since these are the only rank deficient blocks.
Thus, since $\operatorname{rk}(\mathcal{A}^* - \mathcal{A}) = 2r$, the number of blocks of type P2 or P4 corresponding to an eigenvalue $\lambda = 1$ is exactly $2(n - r) + q$.
Then, removing the blocks of type P5 from the PKCF of $z\mathcal{A}^* - \mathcal{A}$ yields a matrix pencil $z\mathcal{A}_1^* - \mathcal{A}_1 \in \mathbb{K}^{2n_1+q\times 2n_1+q}[z]$ in PKCF with full normal rank and inertia

$$\operatorname{In}(\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}_1^* - \mathcal{A}_1) = (n_1 + q - a(\omega), a(\omega), n_1)$$

on $(0, 2\pi)$. Then, by Lemma 4.11, there are q blocks of type P2 with corresponding eigenvalue $\lambda = 1$ and positive sign-characteristic, since these are – according to Table 4.1, entries $6, 8, 10$ – present in every combination of blocks with an inertia pattern of the form

$$(k + 1, 0, k)$$

independent of $\omega > 0$. Removing these blocks leads to the pencil $z\mathcal{A}_2^* - \mathcal{A}_2 \in \mathbb{K}^{2n_2\times 2n_2}[z]$ in PKCF with inertia

$$\operatorname{In}(\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}_2^* - \mathcal{A}_2) = (n_2 - a(\omega), a(\omega), n_2)$$

on $(0, 2\pi)$. Furthermore, from Lemma 4.11 and Table 4.1, entry 6, we deduce that there are no blocks of type P2 corresponding to eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \neq 0$. Thus, all blocks of type P3 have negative sign-characteristic, since these are – according to Table 4.1, entry 5 – the only blocks with an inertia pattern of the form

$$(k - 1, 1, k)$$

for exactly one value of $\omega > 0$. This shows statement (i). Removing these blocks, we obtain a matrix pencil $z\mathcal{A}_3^* - \mathcal{A}_3 \in \mathbb{K}^{2n_3\times 2n_3}[z]$ in PKCF with inertia

$$\operatorname{In}(\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}_3^* - \mathcal{A}_3) = (n_3, 0, n_3)$$

on $(0, 2\pi)$. The inertia of $z\mathcal{A}_3^* - \mathcal{A}_3$ together with Lemma 4.11 and Table 4.1, entries $3, 6, 7$, reveals that the remaining blocks of type P2 corresponding to an eigenvalue $\lambda = 1$ are split up equally into those with positive and those with negative sign-characteristic. Since there are exactly $2(n - r) + q$ blocks corresponding to an eigenvalue $\lambda = 1$, this shows (ii) and thus statement (b).

The proof that (c) follows from (b) is clear, since condition (i') follows immediately from condition (i) and conditions (ii) and (ii') coincide.

Now let the conditions (i'), and (ii') hold. Again, by Lemma 4.13, Lemma 4.7 and Theorem 4.4, for $\omega \in (0, 2\pi)$ we obtain

$$\begin{aligned}\operatorname{In}\left(\mathrm{e}^{\mathrm{i}\omega}\mathcal{A}^* - \mathcal{A}\right) &= (n, 0, n) + \operatorname{In}\left(\Phi(\mathrm{e}^{\mathrm{i}\omega})\right)\\ &= (n + m_1 - a_1(\omega), m - m_1 - m_2 + a_1(\omega) + a_2(\omega), n + m_2 - a_2(\omega))\end{aligned}$$

and functions $a_i : (0, 2\pi) \to \mathbb{N}_0$, $i = 1, 2$, which are zero for almost all $\omega \in (0, 2\pi)$ such that $m_1 + m_2 = q$. We now have to show that $m_2 = a_2(\omega) = 0$. By Table 4.1, entry 6, blocks of type P2 with positive sign-characteristic are the only ones leading to an inertia pattern of the form

$$(k + 1, 0, k)$$

for $\omega > \theta$. The only blocks that could compensate the additional positive eigenvalue for $\omega > \theta$ are – according to Table 4.1, entries $3, 6$ – blocks of type P2 with negative sign-characteristic. By condition (i') we are only allowed to take such blocks with $\theta = 0$. By condition (ii') then we obtain that $n + m_1 = (n + m_2) + q$ and thus $m_2 = 0$, $m_1 = q$. Hence, we have

$$\operatorname{In}(\Phi(\mathrm{e}^{\mathrm{i}\omega})) = (q - a_1(\omega), m - q + a_1(\omega) + a_2(\omega), -a_2(\omega)) \,.$$

Since the inertia of a quasi-Hermitian matrix by definition is a triple of non-negative integers, this implies $a_2 \equiv 0$ and thus $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in (0, 2\pi)$. Then, by continuity, also $\Phi(1) \succeq 0$. □

Example 4.15 (Example 2.25 revisited)**.** We consider the system (E, A, B, Q, S, R) with corresponding system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form as in (2.16), (2.17), and Example 3.11. The associated palindromic pencil $z\mathcal{A}^* - \mathcal{A} \in \mathbb{K}^{5\times 5}[z]$ as in (2.38) is given by

$$z\mathcal{A}^* - \mathcal{A} = \left[\begin{array}{cc|cc|c} 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & z & 0 \\ \hline -z & z & z-1 & 0 & 0 \\ z & -1 & 0 & z-1 & 0 \\ \hline z & 0 & 0 & 0 & z-1 \end{array}\right] . \tag{4.13}$$

Transforming the matrix $\mathcal{A}$ to the corresponding matrix $\mathcal{A}_F$ of the system in feedback equivalence form (2.17) via

$$U_F := \begin{bmatrix} W^* & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix} = \left[\begin{array}{cc|cc|c} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \in \mathbb{K}^{5\times 5}$$

we obtain

$$\mathcal{A}_F = U_F^* \mathcal{A} U_F = \left[\begin{array}{cc|cc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ \hline 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] .$$

The matrix $\mathcal{A}_F$ can can be further transformed to

$$U^*(z\mathcal{A}_F^* - \mathcal{A}_F)U = \left[\begin{array}{cc|cc|c} 0 & -1 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & z-(2+\sqrt{3}) & 0 \\ 0 & 0 & (2+\sqrt{3})z-1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & z-1 \end{array}\right] \tag{4.14}$$

in PKCF as in (4.6) via

$$U = \begin{bmatrix} 0 & -1 & -1+\sqrt{3} & -\frac{3}{2}-\sqrt{3} & 0 \\ 1 & -1 & -1+\frac{1}{\sqrt{3}} & -1-\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{4}\left(1+\sqrt{3}\right) & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1-\frac{1}{\sqrt{3}} & \frac{1}{2}\left(2+\sqrt{3}\right) & 0 \\ 0 & 0 & -1+\frac{1}{\sqrt{3}} & -1-\frac{\sqrt{3}}{2} & 0 \end{bmatrix}.$$

From (4.14) we see that the PKCF of $z\mathcal{A}_F^* - \mathcal{A}_F$ and thus also of $z\mathcal{A}^* - \mathcal{A}$ consists of a 2×2 block of type P1 corresponding to the eigenvalues $\{0, \infty\}$, a 2×2 block of type P1 corresponding to the eigenvalues $\{2+\sqrt{3}, 2-\sqrt{3}\}$, and a 1×1 block of type P2 corresponding to the eigenvalue 1. Furthermore, for the Popov function $\Phi_F(z)$ it holds that $\operatorname{rk}_{\mathbb{K}(z)} \Phi_F(z) = 1$. Thus, we have shown that the assumptions of Theorem 4.14(b) are fulfilled and hence $\Phi_F(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$. Thus, we have confirmed the result obtained in Example 3.11.

Remark 4.16. The result of Theorem 4.14 is related to [Voi15, Theorem 3.4.2]. Here, we replaced positivity of the Popov function on the imaginary axis by positivity of the Popov function on the unit circle.

Moreover, the blocks of the PKCF 4.6 have some correspondence to the blocks of type E1–E4 of the *even Kronecker canonical form* [Tho76] of an even matrix pencil. With the notation as introduced in [Voi15, Theorem 2.1.13] we obtain that blocks of type E1 correspond to finite eigenvalues not on the imaginary axis, blocks of type E2 correspond to eigenvalues on the imaginary axis, blocks of type E3 correspond to infinite eigenvalues, and blocks of type E4 correspond to rank deficiency of the matrix pencil.

Then, comparing the respective results on the inertia patterns of these blocks in Lemma 4.11 and [Voi15, Lemma 2.1.15] we find that blocks of type P1 are analogs of blocks of type E1, blocks of type P2 corresponding to eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \neq 0$ are analogs of blocks of type E2 of odd size, blocks of type P3 are analogs of blocks of type E2 of even size, blocks of type P4 corresponding to an eigenvalue $\lambda = 1$ are analogs of blocks of type E3 of even size, blocks of type P2 corresponding to an eigenvalue $\lambda = 1$ are analogs of blocks of type E3 of odd size, and blocks of type P5 are analogs of blocks of type E4.

5 Lur'e Equations

In this chapter we characterize solvability of Lur'e equations for explicit as well as for implicit difference equations in a similar way as in [RRV15] for continuous-time systems. Finding a solution of the Lur'e equation means finding $X = X^* \in \mathbb{K}^{n\times n}$, $K \in \mathbb{K}^{q\times n}$, and $L \in \mathbb{K}^{q\times m}$ such that

$$\mathcal{M}(X) = \begin{bmatrix} A^*XA - E^*XE + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} =_{\mathcal{V}^\Sigma} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \tag{5.1}$$

where $q := \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$.

If X is a solution of the KYP inequality (3.2), then we can always find $K \in \mathbb{K}^{p\times n}$ and $L \in \mathbb{K}^{p\times m}$ for some $p \in \mathbb{N}_0$ such that (5.1) holds. To see this, let V^Σ be a basis of the system space $\mathcal{V}^\Sigma \in \mathbb{K}^{n+m\times n_1+m}$. Thus, by Lemma 2.2 we have $(V^\Sigma)^*\mathcal{M}(X)V^\Sigma \succeq 0$, where p denotes the number of positive eigenvalues. This means that we can find $U_{11} \in \mathbb{K}^{p\times n_1}$, $U_{12} \in \mathbb{K}^{p\times m}$, $U_{21} \in \mathbb{K}^{n_1+m-p\times n_1}$, and $U_{22} \in \mathbb{K}^{n_1+m-p\times m}$ such that

$$(V^\Sigma)^*\mathcal{M}(X)V^\Sigma = \begin{bmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

Neglecting the parts corresponding to the zero eigenvalues leads to

$$(V^\Sigma)^*\mathcal{M}(X)V^\Sigma = \begin{bmatrix} U_{11}^* \\ U_{12}^* \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \end{bmatrix}.$$

Set

$$\begin{bmatrix} K & L \end{bmatrix} := \begin{bmatrix} U_{11} & U_{12} \end{bmatrix} (V^\Sigma)^+,$$

where $(V^\Sigma)^+ \in \mathbb{K}^{n_1+m\times n+m}$ denotes the Moore-Penrose left inverse of V^Σ. Hence, we have found K and L such that

$$\mathcal{M}(X) =_{\mathcal{V}^\Sigma} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}.$$

The next result shows that for such solutions it holds that $p \geq q$. Thus, in other words, we are interested in the existence of solutions of (5.1) with minimal rank q.

Proposition 5.1. *Let* $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$ *be given and let* $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$. *Further, let*

$$(X,\, K,\, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$$

be a solution of the Lur'e equation (5.1) *and assume, that for* $M \in \mathbb{K}^{p\times n}$ *and* $N \in \mathbb{K}^{p\times m}$ *also the triple* $(X,\, M,\, N)$ *fulfills* (5.1). *Then we have* $q \le p$ *and*

$$\operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} zE - A & -B \\ (z-1)K & (z-1)L \end{bmatrix} = n + q.$$

Proof. Let $\omega_0 \in [0, 2\pi)$ be given with $\det(\mathrm{e}^{\mathrm{i}\omega_0}E - A) \neq 0$ and $\operatorname{rk} \Phi(\mathrm{e}^{\mathrm{i}\omega_0}) = q$. Then, from Lemma 3.3, for the Popov function we obtain

$$\begin{aligned}
\Phi(\mathrm{e}^{\mathrm{i}\omega_0}) &= \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix} \\
&= \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*XA - E^*XE + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix} \\
&= \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix} \\
&= \left(\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix} \right)^* \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega_0}E - A)^{-1}B \\ I_m \end{bmatrix} \\
&= Z(\mathrm{e}^{\mathrm{i}\omega_0})^* Z(\mathrm{e}^{\mathrm{i}\omega_0}),
\end{aligned}$$

where $Z(z) = N + M(zE - A)^{-1}B \in \mathbb{K}^{p\times m}(z)$. Hence, we have

$$q = \operatorname{rk} \Phi(\mathrm{e}^{\mathrm{i}\omega_0}) = \operatorname{rk} Z(\mathrm{e}^{\mathrm{i}\omega_0})^* Z(\mathrm{e}^{\mathrm{i}\omega_0}) = \operatorname{rk} Z(\mathrm{e}^{\mathrm{i}\omega_0}) \le p.$$

Analogously, we obtain for all $\omega \in [0, 2\pi)$ with $\det(\mathrm{e}^{\mathrm{i}\omega_0}E - A) \neq 0$ that

$$\Phi(\mathrm{e}^{\mathrm{i}\omega}) = W(\mathrm{e}^{\mathrm{i}\omega})^* W(\mathrm{e}^{\mathrm{i}\omega}),$$

where $W(z) = L + K(zE - A)^{-1}B \in \mathbb{K}^{q\times m}(z)$. Since

$$q \ge \operatorname{rk} \Phi(\mathrm{e}^{\mathrm{i}\omega}) = \operatorname{rk} W(\mathrm{e}^{\mathrm{i}\omega})^* W(\mathrm{e}^{\mathrm{i}\omega}) = \operatorname{rk} W(\mathrm{e}^{\mathrm{i}\omega})$$

and $\operatorname{rk} W(\mathrm{e}^{\mathrm{i}\omega_0}) = q$ it follows that $\operatorname{rk}_{\mathbb{K}(z)} W(z) = q$. Then we obtain

$$\begin{aligned}
n + q &= \operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} I_n & 0 \\ 0 & (z-1)I_q \end{bmatrix} \begin{bmatrix} I_n & 0 \\ K(zE - A)^{-1} & I_q \end{bmatrix} \begin{bmatrix} zE - A & -B \\ 0 & W(z) \end{bmatrix} \\
&= \operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} zE - A & -B \\ (z-1)K & (z-1)L \end{bmatrix}.
\end{aligned}$$

□

In the following we will derive certain deflating subspaces of BVD and palindromic matrix pencils, respectively, from a solution of the Lur'e equation (5.1). First, we do this for the case of explicit difference equations. Afterwards, based on these results, we do the generalization to the implicit case with the help of feedback transformations similar to the approach in Chapter 3.

5.1 Explicit Difference Equations

In the EDE case, i. e., systems $(I_n, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ finding a solution of the Lur'e equation (5.1) reduces to:

For $q := \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$ find $X \in \mathbb{K}^{n\times n}$, $K \in \mathbb{K}^{q\times n}$, and $L \in \mathbb{K}^{q\times m}$ such that

$$\mathcal{M}(X) = \begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}. \tag{5.2}$$

The next result is an analogous version of [Rei11, Lemma 12] for the discrete-time case.

Lemma 5.2. *Let $(I_n, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given and let $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$. Furthermore, let $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$ and let an Hermitian $X \in \mathbb{K}^{n\times n}$ be given with*

$$\operatorname{rk} \mathcal{M}(X) = q.$$

Then (5.2) has a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$.

Proof. Let $\omega \in [0, 2\pi)$ be such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$ and $\operatorname{rk} \Phi(\mathrm{e}^{\mathrm{i}\omega}) = q$. From Lemma 3.3 applied for $E = I_n$ we obtain that

$$\Phi(\mathrm{e}^{\mathrm{i}\omega}) = \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix}^* \mathcal{M}(X) \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix} \succeq 0.$$

Assume that

$$\operatorname{In}(\mathcal{M}(X)) = (q_+, m - q_+ - q_-, q_-)$$

with $q_+ + q_- = q$. Using Theorem 4.4 and neglecting the parts of the transformation matrix corresponding to the zero eigenvalues we find $U_{11} \in \mathbb{K}^{q_+\times n}$, $U_{12} \in \mathbb{K}^{q_+\times m}$, $U_{21} \in \mathbb{K}^{q_-\times n}$, and $U_{22} \in \mathbb{K}^{q_-\times m}$ such that

$$\mathcal{M}(X) = \begin{bmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{bmatrix} \begin{bmatrix} I_{q_+} & 0 \\ 0 & -I_{q_-} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

Suppose now that X is not a solution of the KYP inequality (3.8), i. e., that $q_- > 0$. Then for $G_1(\mathrm{e}^{\mathrm{i}\omega}) = U_{11}(\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B + U_{12}$ and $G_2(\mathrm{e}^{\mathrm{i}\omega}) = U_{21}(\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B + U_{22}$ it follows that

$$0 \preceq \Phi(\mathrm{e}^{\mathrm{i}\omega}) = G_1^*(\mathrm{e}^{\mathrm{i}\omega}) G_1(\mathrm{e}^{\mathrm{i}\omega}) - G_2^*(\mathrm{e}^{\mathrm{i}\omega}) G_2(\mathrm{e}^{\mathrm{i}\omega}).$$

Let $x \in \ker G_1(\mathrm{e}^{\mathrm{i}\omega})$. Then

$$0 \leq -x^* G_2^*(\mathrm{e}^{\mathrm{i}\omega}) G_2(\mathrm{e}^{\mathrm{i}\omega}) x \leq 0$$

and thus $x \in \ker G_2(\mathrm{e}^{\mathrm{i}\omega})$. Therefore, $\operatorname{im} G_2^*(\mathrm{e}^{\mathrm{i}\omega}) \subseteq \operatorname{im} G_1^*(\mathrm{e}^{\mathrm{i}\omega})$ and hence, we obtain

$$q = \operatorname{rk} \Phi(\mathrm{e}^{\mathrm{i}\omega}) \leq \operatorname{rk} G_1(\mathrm{e}^{\mathrm{i}\omega}) \leq q_+ < q.$$

This is a contradiction and thus $q_+ = q$. But then (X, U_{11}, U_{12}) is a solution of the Lur'e equation (5.2). □

Example 5.3 (Example 2.25 revisited)**.** Consider the system (E, A, B, Q, S, R) as in (2.16) and Example 3.11. We have seen in Example 3.11 that with

$$\mathcal{M}_s(P_s) = \begin{bmatrix} 2 & -P_s - 1 \\ -P_s - 1 & P_s + 2 \end{bmatrix}$$

$P_s = \sqrt{3}$ solves the KYP inequality (3.2) for the EDE system $(I_{n_1}, A_s, B_s, Q_s, S_s, R_s)$ as in (3.21). In particular, we have that $\operatorname{rk} \mathcal{M}_s(P_s) = 1 = \operatorname{rk}_{\mathbb{K}(z)} \Phi_s(z)$ for the Popov function $\Phi_s(z) \in \mathbb{K}(z)$ of the EDE system. Thus we obtain

$$\mathcal{M}_s(P_s) = \begin{bmatrix} \sqrt{2} & 0 \\ -\frac{\sqrt{3}+1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{3}+1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix}$$

and hence,

$$(P_s, K_s, L_s) = \left(\sqrt{3}, \sqrt{2}, -\frac{\sqrt{3}+1}{\sqrt{2}} \right)$$

is a solution of the Lur'e equation (5.2).

Now we are ready to show that the existence of a solution of the Lur'e equation (5.2) is equivalent to the existence of a certain deflating subspace of the palindromic matrix pencil as in (2.38). This result is the continuous-time analog of [Rei11, Theorem 11].

Theorem 5.4. *Let $(I_n, A, B, Q, S, R) \in \Sigma_{m,n}^w(\mathbb{K})$ be given and consider the associated palindromic pencil $z\mathcal{A}^* - \mathcal{A}$ as in (2.38). Further, let $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$ and assume that $\operatorname{rk} \begin{bmatrix} I_n - A & -B \end{bmatrix} = n$. Then the following are equivalent:*

(a) There exists a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \in \mathbb{K}^{q\times m}$ of (5.2).

(b) It holds that $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$. Furthermore, there exist matrices $Y_\mu, Y_x \in \mathbb{K}^{n\times n+m}$, $Y_u \in \mathbb{K}^{m\times n+m}$ and $Z_\mu, Z_x \in \mathbb{K}^{n\times n+q}$, $Z_u \in \mathbb{K}^{m\times n+q}$ such that for

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix}$$

the following holds:

(i) The matrix

$$\begin{bmatrix} I_n - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

has full row rank n.

(ii) The space $\mathcal{Y} = \operatorname{im} Y$ is maximally $(\mathcal{A}^* - \mathcal{A})$-neutral.

(iii) There exist $\tilde{E}$, $\tilde{A} \in \mathbb{K}^{n+q\times n+m}$ such that $(z\mathcal{A}^* - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A})$.

Proof. Denote by $C \in \mathbb{K}^{n+m\times n}$ and $C_c \in \mathbb{K}^{n+m\times m}$ the right inverse and a basis matrix of the kernel of

$$\begin{bmatrix} I_n - A & -B \end{bmatrix},$$

respectively. Further let

$$\begin{bmatrix} C_1^- \\ C_2^- \end{bmatrix} := \begin{bmatrix} C_{11}^- & C_{12}^- \\ C_{21}^- & C_{22}^- \end{bmatrix} := \begin{bmatrix} C & C_c \end{bmatrix}^{-1},$$

where $C_1^- = [\, I_n - A \quad -B \,] \in \mathbb{K}^{n\times n+m}$, $C_2^- \in \mathbb{K}^{m\times n+m}$, $C_{11}^- = I_n - A \in \mathbb{K}^{n\times n}$, $C_{12}^- = -B \in \mathbb{K}^{n\times m}$, $C_{21}^- \in \mathbb{K}^{m\times n}$, and $C_{22}^- \in \mathbb{K}^{m\times m}$.

First assume that there exists a solution (X, K, L) of (5.2). Then by Theorem 3.7 we have $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$. Set

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} X(A - I_n) & XB \\ I_n & 0 \\ 0 & I_m \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ (I_n - A^*)X & K^* \\ -B^*X & L^* \end{bmatrix}, \tag{5.3}$$

and

$$z\tilde{E} - \tilde{A} = \begin{bmatrix} zI_n - A & -B \\ (z-1)K & (z-1)L \end{bmatrix}. \tag{5.4}$$

Property (i) follows, since

$$\operatorname{rk} \begin{bmatrix} I_n - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix} = \operatorname{rk} \begin{bmatrix} I_n - A & -B \end{bmatrix} = n$$

by assumption. For property (ii) we first note that for

$$V := \begin{bmatrix} I_n & 0 & 0 \\ 0 & C_{11}^- & C_{12}^- \\ 0 & C_{21}^- & C_{22}^- \end{bmatrix} \in \mathbb{K}^{2n+m\times 2n+m} \tag{5.5}$$

we have

$$V^{-*}(\mathcal{A}^* - \mathcal{A})V^{-1} = \mathcal{E}, \tag{5.6}$$

where $\mathcal{E} \in \mathbb{K}^{2n+m\times 2n+m}$ as in (2.33). Then by Lemma 2.12, $\operatorname{im} Y$ is maximally $(\mathcal{A}^* - \mathcal{A})$-neutral if and only if $\operatorname{im} \hat{Y}$ is maximally $\mathcal{E}$-neutral, where

$$\hat{Y} := VY \begin{bmatrix} C & C_c \end{bmatrix} = \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}. \tag{5.7}$$

On the one hand, $\operatorname{im} \hat{Y}$ is $\mathcal{E}$-neutral, since

$$\hat{Y}^* \mathcal{E} \hat{Y} = \begin{bmatrix} -X + X & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

On the other hand, we have that $n+m = \operatorname{rk} \hat{Y}$ and by Lemma 2.13, the rank of every $\mathcal{E}$-neutral space is bounded from above by $n+m$. Therefore, $\operatorname{im} \hat{Y}$ is maximally $\mathcal{E}$-neutral which shows (ii). Finally, we have (iii) by

$$\begin{aligned}
&(z\mathcal{A}^* - \mathcal{A})Y \\
=& \begin{bmatrix} zI_n - A & -B \\ z\left(A^*X(A-I_n)+Q\right) - X(A-I_n) - Q & z(A^*XB+S) - XB - S \\ z\left(B^*X(A-I_n)+S^*\right) - S^* & z(B^*XB+R) - R \end{bmatrix} \\
=& \begin{bmatrix} zI_n - A & -B \\ z\left((I_n - A^*)X + K^*K\right) - (I_n - A^*)XA - K^*K & zK^*L - (I_n - A^*)XB - K^*L \\ z\left(-B^*X + L^*K\right) - B^*XA - L^*K & zL^*L + B^*XB - L^*L \end{bmatrix} \\
=& Z(z\tilde{E} - \tilde{A}).
\end{aligned}$$

Now assume that we are in the situation of (b). Then by (ii), $\operatorname{im} \hat{Y}$ is maximally $\mathcal{E}$-neutral for

$$\hat{Y} := \begin{bmatrix} \hat{Y}_\mu \\ \hat{Y}_x \\ \hat{Y}_u \end{bmatrix} = VY \begin{bmatrix} C & C_c \end{bmatrix}$$

and V and $\mathcal{E}$ as in (5.5) and (5.6). By property (i) we obtain

$$\operatorname{rk} \hat{Y}_x = \operatorname{rk} \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} \hat{Y}_x \\ \hat{Y}_u \end{bmatrix} = \operatorname{rk} \begin{bmatrix} I_n - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix} \begin{bmatrix} C & C_c \end{bmatrix} = n.$$

Thus, there exists an invertible $T_1 \in \mathbb{K}^{n+m\times n+m}$ such that

$$\hat{Y}T_1 = \begin{bmatrix} \hat{Y}_{\mu_1} & \hat{Y}_{\mu_2} \\ I_n & 0 \\ \hat{Y}_{u_1} & \hat{Y}_{u_2} \end{bmatrix}.$$

By Lemma 2.12, $\hat{Y}T_1$ is still maximally $\mathcal{E}$-neutral and we obtain

$$0 = (\hat{Y}T_1)^*\mathcal{E}\hat{Y}T_1 = \begin{bmatrix} I_n & -\hat{Y}_{\mu_1}^* & 0 \\ 0 & -\hat{Y}_{\mu_2}^* & 0 \end{bmatrix} \hat{Y}T_1 = \begin{bmatrix} \hat{Y}_{\mu_1} - \hat{Y}_{\mu_1}^* & \hat{Y}_{\mu_2} \\ -\hat{Y}_{\mu_2}^* & 0 \end{bmatrix};$$

in particular $X := -\hat{Y}_{\mu_1}$ is Hermitian. Hence, maximal $\mathcal{E}$-neutrality implies full rank of $\hat{Y}_{u_2}$. Applying another column transformation to $\hat{Y}$ via an invertible $T_2 \in \mathbb{K}^{n+m\times n+m}$ yields

$$\hat{Y}T_1T_2 = \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Doing the backtransformation for Y we obtain

$$Y = V^{-1}\hat{Y}T_1T_2 \begin{bmatrix} C & C_c \end{bmatrix}^{-1} \hat{T}, \tag{5.8}$$

where

$$\hat{T} := \begin{bmatrix} C & C_c \end{bmatrix} (T_1T_2)^{-1} \begin{bmatrix} C & C_c \end{bmatrix}^{-1}.$$

This implies

$$Y\hat{T}^{-1} = \begin{bmatrix} X(A - I_n) & XB \\ I_n & 0 \\ 0 & I_m \end{bmatrix}. \tag{5.9}$$

We partition $z\hat{E} - \hat{A} := (z\tilde{E} - \tilde{A})\hat{T}^{-1}$ into

$$z\hat{E} - \hat{A} = \begin{bmatrix} z\hat{E}_1 - \hat{A}_1 & z\hat{E}_2 - \hat{A}_2 \end{bmatrix},$$

where $z\hat{E}_1 - \hat{A}_1 \in \mathbb{K}^{n+q\times n}[z]$ and $z\hat{E}_2 - \hat{A}_2 \in \mathbb{K}^{n+q\times m}[z]$. Then property (iii) implies

$$\begin{aligned} &\begin{bmatrix} zI_n - A & -B \\ z\left(A^*X(A - I_n) + Q\right) - X(A - I_n) - Q & z(A^*XB + S^*) - XB - S^* \\ z\left(B^*X(A - I_n) + S^*\right) - S^* & z(B^*XB + R) - R \end{bmatrix} \\ &= \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \begin{bmatrix} z\hat{E}_1 - \hat{A}_1 & z\hat{E}_2 - \hat{A}_2 \end{bmatrix}, \end{aligned}$$

yielding $I_n = Z_\mu\hat{E}_1$ and thus $\operatorname{rk} Z_\mu = n$. Therefore, there exists invertible $T_3 \in \mathbb{K}^{n+m\times n+m}$ such that $Z_\mu T_3 = \begin{bmatrix} I_n & 0 \end{bmatrix}$. Then for

$$ZT_3 =: \begin{bmatrix} I_n & 0 \\ Z_{x_1} & Z_{x_2} \\ Z_{u_1} & Z_{u_2} \end{bmatrix}, \qquad T_3^{-1}(z\hat{E} - \hat{A}) =: \begin{bmatrix} z\hat{E}_{11} - \hat{A}_{11} & z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{21} - \hat{A}_{21} & z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix}$$

partitioned accordingly, we obtain

$$\begin{aligned}&\begin{bmatrix} zI_n - A & -B \\ z\left(A^*X(A-I_n)+Q\right) - X(A-I_n) - Q & z(A^*XB+S^*) - XB - S^* \\ z\left(B^*X(A-I_n)+S^*\right) - S^* & z(B^*XB+R) - R \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ Z_{x_1} & Z_{x_2} \\ Z_{u_1} & Z_{u_2} \end{bmatrix} \begin{bmatrix} z\hat{E}_{11} - \hat{A}_{11} & z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{21} - \hat{A}_{21} & z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix}.\end{aligned} \tag{5.10}$$

Thus, the first equation gives $\hat{E}_{11} = I_n$, $\hat{A}_{11} = A$, $\hat{E}_{12} = 0$, and $\hat{A}_{12} = B$. For $z = 1$ we obtain from (5.10) that

$$\begin{bmatrix} I_n - A & -B \\ (A^* - I_n)X(A - I_n) & (A^* - I_n)XB \\ B^*X(A - I_n) & B^*XB \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ Z_{x_1} & Z_{x_2} \\ Z_{u_1} & Z_{u_2} \end{bmatrix} \begin{bmatrix} I_n - A & -B \\ \hat{E}_{21} - \hat{A}_{21} & \hat{E}_{22} - \hat{A}_{22} \end{bmatrix}.$$

Multiplying from the right with C results in

$$\begin{bmatrix} (I_n - A^*)X & 0 \\ 0 & -B^*X \end{bmatrix} \begin{bmatrix} I_n - A & -B \\ I_n - A & -B \end{bmatrix} C = \begin{bmatrix} Z_{x_1} \\ Z_{u_1} \end{bmatrix} + \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \begin{bmatrix} \hat{E}_{21} - \hat{A}_{21} & \hat{E}_{22} - \hat{A}_{22} \end{bmatrix} C$$

and thus

$$\begin{bmatrix} Z_{x_1} \\ Z_{u_1} \end{bmatrix} = \begin{bmatrix} (I_n - A^*)X \\ -B^*X \end{bmatrix} - \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \begin{bmatrix} \hat{E}_{21} - \hat{A}_{21} & \hat{E}_{22} - \hat{A}_{22} \end{bmatrix} C.$$

Inserting this relation into (5.10) for $z = \infty$ gives

$$\begin{aligned}&\begin{bmatrix} A^*X(A - I_n) + Q & A^*XB + S \\ B^*X(A - I_n) + S^* & B^*XB + R \end{bmatrix} \\ &= \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \begin{bmatrix} \hat{E}_{21} & \hat{E}_{22} \end{bmatrix} + \left(\begin{bmatrix} (I_n - A^*)X \\ -B^*X \end{bmatrix} - \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \begin{bmatrix} \hat{E}_{21} - \hat{A}_{21} & \hat{E}_{22} - \hat{A}_{22} \end{bmatrix} C \right) \begin{bmatrix} I_n & 0 \end{bmatrix},\end{aligned}$$

which leads to

$$\begin{aligned}\mathcal{M}(X) &= \begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} \\ &= \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \left(\begin{bmatrix} \hat{E}_{21} & \hat{E}_{22} \end{bmatrix} - \begin{bmatrix} \hat{E}_{21} - \hat{A}_{21} & \hat{E}_{22} - \hat{A}_{22} \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right).\end{aligned}$$

Thus we have

$$\operatorname{rk} \mathcal{M}(X) \leq q. \tag{5.11}$$

Further, by Lemma 3.3, for $\omega \in [0, 2\pi)$ we can rewrite $\Phi(\mathrm{e}^{\mathrm{i}\omega})$ as

$$\Phi(\mathrm{e}^{\mathrm{i}\omega}) = \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix}^* \mathcal{M}(X) \begin{bmatrix} (\mathrm{e}^{\mathrm{i}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix}$$

and thus in (5.11) we even have equality. Therefore, we can apply Lemma 5.2 and hence, we have shown that (a) holds.

□

In the case of a BVD pencil we can prove a similar statement.

Theorem 5.5. *Let $(I_n, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given and consider the associated BVD pencil $z\mathcal{E} - \mathcal{A}$ as in (2.35). Further, let $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$. Then the following are equivalent:*

(a) There exists a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ of (5.2).

(b) It holds that $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$. Furthermore, there exist matrices $Y_\mu, Y_x \in \mathbb{K}^{n\times n+m}$, $Y_u \in \mathbb{K}^{m\times n+m}$ and $Z_\mu, Z_x \in \mathbb{K}^{n\times n+q}$, $Z_u \in \mathbb{K}^{m\times n+q}$ such that for

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix}$$

the following hold:

(i) The matrix

$$Y_x = \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

has full row rank n.

(ii) The space $\mathcal{Y} = \operatorname{im} Y$ is maximally $\mathcal{E}^e$-neutral, where

$$\mathcal{E}^e := \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

corresponds to $\mathcal{E}$ in the even pencil in (2.33).

(iii) There exist $\tilde{E}, \tilde{A} \in \mathbb{K}^{n+q\times n+m}$ such that $(z\mathcal{E} - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A})$.

Proof. First assume that there exists a solution (X, K, L) of (5.2). Then by Theorem 3.7 $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega} I_n - A) \neq 0$. Set

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -A^*X & -K^* \\ -B^*X & -L^* \end{bmatrix},$$

and

$$z\tilde{E} - \tilde{A} = \begin{bmatrix} zI_n - A & -B \\ K & L \end{bmatrix}.$$

This immediately shows property (i). Note that $\operatorname{im} Y$ is exactly $\operatorname{im} \hat{Y}$ in the proof of Theorem 5.4, see (5.7). Thus, following the lines of this proof we obtain property (ii). Finally, we have (iii) by

$$\begin{aligned}(z\mathcal{E} - \mathcal{A})Y &= \begin{bmatrix} zI_n - A & -B \\ -zA^*X - Q + X & -S \\ -zB^*X - S^* & -R \end{bmatrix} \\ &= \begin{bmatrix} zI_n - A & -B \\ -zA^*X + A^*XA - K^*K & A^*XB - K^*L \\ -zB^*X + B^*XA - L^*K & B^*XB - L^*L \end{bmatrix} \\ &= Z(z\tilde{E} - \tilde{A}).\end{aligned}$$

Now assume that we are in the situation of (b). Then, by again looking at the proof of Theorem 5.4, properties (i) and (ii) imply that there exists a transformation matrix $T_1 \in \mathbb{K}^{n+m\times n+m}$ such that

$$YT_1 = \begin{bmatrix} -X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}$$

and X is Hermitian. Partition $z\hat{E} - \hat{A} := (z\tilde{E} - \tilde{A})T_1^{-1}$ into

$$z\hat{E} - \hat{A} = \begin{bmatrix} z\hat{E}_1 - \hat{A}_1 & z\hat{E}_2 - \hat{A}_2 \end{bmatrix},$$

where $z\hat{E}_1 - \hat{A}_1 \in \mathbb{K}^{n+q\times n}[z]$ and $z\hat{E}_2 - \hat{A}_2 \in \mathbb{K}^{n+q\times m}[z]$. Then property (iii) implies

$$\begin{bmatrix} zI_n - A & -B \\ -zA^*X - (Q - X) & -S \\ -zB^*X - S^* & -R \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \begin{bmatrix} z\hat{E}_1 - \hat{A}_1 & z\hat{E}_2 - \hat{A}_2 \end{bmatrix},$$

yielding $I_n = Z_\mu \hat{E}_1$ and thus $\operatorname{rk} Z_\mu = n$. Therefore, there exists invertible $T_2 \in \mathbb{K}^{n+m\times n+m}$ such that $Z_\mu T_2 = \begin{bmatrix} I_n & 0 \end{bmatrix}$. Then for

$$ZT_2 =: \begin{bmatrix} I_n & 0 \\ Z_{x_1} & Z_{x_2} \\ Z_{u_1} & Z_{u_2} \end{bmatrix}, \qquad T_2^{-1}(z\hat{E} - \hat{A}) =: \begin{bmatrix} z\hat{E}_{11} - \hat{A}_{11} & z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{21} - \hat{A}_{21} & z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix}$$

accordingly partitioned, we obtain

$$\begin{bmatrix} zI_n - A & -B \\ -zA^*X - (Q - X) & -S \\ -zB^*X - S^* & -R \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ Z_{x_1} & Z_{x_2} \\ Z_{u_1} & Z_{u_2} \end{bmatrix} \begin{bmatrix} z\hat{E}_{11} - \hat{A}_{11} & z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{21} - \hat{A}_{21} & z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix}. \tag{5.12}$$

Thus, the first equation gives $\hat{E}_{11} = I_n$, $\hat{A}_{11} = A$, $\hat{E}_{12} = 0$, and $\hat{A}_{12} = B$. For $z = \infty$ we obtain from (5.12) that

$$\begin{bmatrix} I_n & 0 \\ -A^*X & 0 \\ -B^*X & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ Z_{x_1} & Z_{x_2} \\ Z_{u_1} & Z_{u_2} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \hat{E}_{21} & \hat{E}_{22} \end{bmatrix}.$$

Thus we have

$$\begin{bmatrix} Z_{x_1} \\ Z_{u_1} \end{bmatrix} = \begin{bmatrix} -A^*X \\ -B^*X \end{bmatrix} - \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \hat{E}_{21}.$$

Inserting this relation into (5.12) for $z = 0$ gives

$$\begin{bmatrix} -X + Q & S \\ S^* & R \end{bmatrix} = \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \begin{bmatrix} \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} + \left(\begin{bmatrix} -A^*X \\ -B^*X \end{bmatrix} - \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \hat{E}_{21} \right) \begin{bmatrix} A & B \end{bmatrix},$$

which finally leads to

$$\mathcal{M}(X) = \begin{bmatrix} A^*XA - X + Q & A^*XB + S \\ B^*XA + S^* & B^*XB + R \end{bmatrix} = \begin{bmatrix} Z_{x_2} \\ Z_{u_2} \end{bmatrix} \begin{bmatrix} \hat{A}_{21} - \hat{E}_{21}A & \hat{A}_{22} - \hat{E}_{21}B \end{bmatrix}.$$

Thus

$$\operatorname{rk} \mathcal{M}(X) \leq q$$

and the result follows as in Theorem 5.4 by using Lemma 5.2. □

5.2 Implicit Difference Equations

In this section we generalize the results from the previous section to implicit difference equations. As for the KYP inequality we need relations between the Lur'e equation (5.1) corresponding to the original system and the associated equation corresponding to the feedback equivalent system $(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3). These findings are related to the results in [RRV15] in the continuous-time case.

Lemma 5.6. *Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given and $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$. Then $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ is a solution of (5.1) if and only if*

$$(X_F, K_F, L_F) := (W^{-*}XW^{-1}, KT + LFT, L) \tag{5.13}$$

is a solution of (5.1) associated to the feedback system

$$(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$$

as in (3.3), i. e.,

$$\mathcal{M}_F(X_F) = \begin{bmatrix} A_F^*X_FA_F - E_F^*X_FE_F + Q_F & A_F^*X_FB_F + S_F \\ B_F^*X_FA_F + S_F^* & B_F^*X_FB_F + R_F \end{bmatrix} =_{\mathcal{V}_F^\Sigma} \begin{bmatrix} K_F^* \\ L_F^* \end{bmatrix} \begin{bmatrix} K_F & L_F \end{bmatrix}.$$

Proof. First note that for

$$\mathcal{T}_F = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}$$

we have

$$\begin{bmatrix} K & L \end{bmatrix} \mathcal{T}_F = \begin{bmatrix} KT + LFT & L \end{bmatrix}. \tag{5.14}$$

In addition, by Proposition 3.8(a) we obtain that $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z) = \operatorname{rk}_{\mathbb{K}(z)} \Phi_F(z)$. Thus, Lemma 3.4 and Lemma 2.2(c) immediately yield the assertion. □

Moreover, we now characterize the connection between the Lur'e equation (5.1) corresponding to the system $(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ in feedback equivalence form as in (3.3) and the Lur'e equation (5.2) corresponding to the associated EDE part as in (3.13).

Lemma 5.7. *Let $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ with the system space $\mathcal{V}^\Sigma$ be given and consider the system $(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3) in feedback equivalence form (2.15) with system space $\mathcal{V}^\Sigma_F$. Further, consider (X_F, K_F, L_F) as in (5.13) partitioned according to the block structure of the feedback equivalence form.*

Then with $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$ we have that $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ is a solution of (5.1) if and only if $(X_{11}, K_1, L - K_2B_2) \in \mathbb{K}^{n_1\times n_1} \times \mathbb{K}^{q\times n_1} \times \mathbb{K}^{q\times m}$ is a solution of (5.2) for the EDE system

$$(I_{n_1}, A_s, B_s, Q_s, S_s, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$$

as in (3.13).

Proof. Consider $[\, x^* \;\; u^* \,]^* \in \mathcal{V}^\Sigma_F$, i. e.,

$$x = \begin{bmatrix} x_1 \\ -B_2u \\ 0_{n_3\times 1} \end{bmatrix},$$

with $x_1 \in \mathbb{K}^{n_1}$. Then

$$\begin{bmatrix} K_F & L_F \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} K_1 & K_2 & K_3 & L \end{bmatrix} \begin{bmatrix} x_1 \\ -B_2u \\ 0_{n_3\times 1} \\ u \end{bmatrix} = \begin{bmatrix} K_1 & L - K_2B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix}.$$

Furthermore, by Proposition 3.8(b) we obtain that $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi_s(z) = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$, where $\Phi_s(z)$ denotes the Popov function corresponding to the EDE system. Thus, together with (3.18), Lemma 3.4 and Lemma 2.2(c) immediately yield the assertion. □

For the rest of this chapter we assume that $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$ is impulse controllable, i. e., there exists a feedback such that the system

$$(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$$

as in (3.3) is in feedback equivalence form such that $n_3 = 0$. This is justified by the fact that by Lemma 2.21 the subsystem described by $(E_{33},\, I_{n_3},\, 0) \in \Sigma_{m,n_3}(\mathbb{K})$ obtained from the feedback equivalence form (2.15) has only the zero solution and thus does not contribute to the dynamics of the system. Indeed, in the proofs of Lemma 3.9 and Lemma 5.7 the parts of $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$ corresponding to the last n_3 variables do not contribute to the analysis. The following proposition makes this precise, using the same projection ansatz as in [RRV15, Theorem 5.9].

Proposition 5.8. *Let $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^w_{m,n}(\mathbb{K})$ with the system space $\mathcal{V}^\Sigma$ be given and consider the system $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3) in feedback equivalence form (2.15). Further, let $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$. Define the projector*

$$\Pi := W^{-1} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W \in \mathbb{K}^{n\times n}. \tag{5.15}$$

Then we have

$$\operatorname{im} \Pi = E\mathcal{W}^\sigma \tag{5.16}$$

and the following statements hold:

(a) The projected system $(\Pi E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$ is impulse controllable and

$$\mathfrak{B}^\sigma_{(E,\, A,\, B)} = \mathfrak{B}^\sigma_{(\Pi E,\, A,\, B)}.$$

In particular, the system space of $(\Pi E,\, A,\, B)$ is $\mathcal{V}^\Sigma$.

(b) There exists a solution $P \in \mathbb{K}^{n\times n}$ of the KYP inequality (3.2), i. e., $\mathcal{M}(P) \succeq_{\mathcal{V}^\Sigma} 0$, if and only if $\mathcal{M}_\Pi(P) \succeq_{\mathcal{V}^\Sigma} 0$, where $\mathcal{M}_\Pi(P)$ is the matrix in (3.2) corresponding to the projected system $(\Pi E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$.

(c) There exists a solution $(X,\, K,\, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ of the Lur'e equation (5.1) if and only if $(X,\, K,\, L)$ also fulfills the Lur'e equation (5.1) corresponding to the projected system $(\Pi E,\, A,\, B) \in \Sigma_{m,n}(\mathbb{K})$.

Proof. Part (a) and (5.16) follow with the algebraic manipulations mentioned in the proof of [RRV15, Theorem 5.9].

Now set

$$\Pi_F := W\Pi W^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For parts (b) and (c) note that the system $(\Pi_F E_F,\, A_F,\, B_F) \in \Sigma_{m,n}(\mathbb{K})$ is in feedback equivalence form (2.15) where compared to $(E_F,\, A_F,\, B_F) \in \Sigma_{m,n}(\mathbb{K})$ the matrices E_{23} and E_{33} are set to zero. Looking carefully at the proofs of Lemma 3.9 and Lemma 5.7 we see that these matrices have no effect in the respective results and thus the assertion follows. □

As a next step, we perform transformations of the palindromic or BVD pencils corresponding to the system $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ in feedback equivalence form as in (3.3) such that we obtain the respective palindromic or BVD pencils corresponding to the EDE system $(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$ as in (3.13) in the first diagonal block of the transformed pencil.

Lemma 5.9. *Let $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3) be given in feedback equivalence form (2.15) such that $n_3 = 0$. Further, let the corresponding palindromic pencil $z\mathcal{A}_F^* - \mathcal{A}_F$ as in (2.38) be given. Denote by $z\mathcal{A}_s^* - \mathcal{A}_s$ the pencil corresponding to the EDE system $(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$ as in (3.13), i. e.,*

$$z\mathcal{A}_s^* - \mathcal{A}_s = \begin{bmatrix} 0 & zI_{n_1} - A_s & -B_s \\ zA_s^* - I_{n_1} & (z-1)Q_s & (z-1)S_s \\ zB_s^* & (z-1)S_s^* & (z-1)R_s \end{bmatrix} \in \mathbb{K}^{n_1+m \times n_1+m}[z]. \tag{5.17}$$

Then there exists an invertible $\hat{U} \in \mathbb{K}^{2n+m \times 2n+m}$ such that

$$\hat{U}^*(z\mathcal{A}_F^* - \mathcal{A}_F)\hat{U} = \left[\begin{array}{c|cc} z\mathcal{A}_s^* - \mathcal{A}_s & zD & 0 \\ \hline -D^* & 0 & zI_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right], \tag{5.18}$$

where

$$D = \begin{bmatrix} 0 \\ Q_{12} \\ S_2^* - B_2^* Q_{22} \end{bmatrix} \tag{5.19}$$

and

$$\begin{aligned} \hat{U} &= \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & -Q_{12}^* & -S_2 + Q_{22}B_2 & -Q_{22} & I_{n_2} \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & -B_2 & I_{n_2} & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix}}_{=:\tilde{P}} \underbrace{\begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & -B_2 & I_{n_2} & 0 \\ 0 & -Q_{12}^* & -S_2 + Q_{22}B_2 & -Q_{22} & I_{n_2} \end{bmatrix}}_{=:\tilde{U}}. \end{aligned} \tag{5.20}$$

Proof. We prove the assertion by subsequent congruence transformations, where the first transformation consists of permutations only. Set

$$U_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & -Q_{22} & I_{n_2} \end{bmatrix}, \quad U_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & -B_2 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix}, \tag{5.21}$$

and

$$U_3 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & -Q_{12}^* & -S_2 & 0 & I_{n_2} \end{bmatrix}. \tag{5.22}$$

Then we have

$$z\mathcal{A}_F^* - \mathcal{A}_F = \begin{bmatrix} 0 & 0 & zI_{n_1} - A_{11} & 0 & -B_1 \\ 0 & 0 & 0 & -I_{n_2} & -B_2 \\ zA_{11}^* - I_{n_1} & 0 & (z-1)Q_{11} & (z-1)Q_{12} & (z-1)S_1 \\ 0 & zI_{n_2} & (z-1)Q_{12}^* & (z-1)Q_{22} & (z-1)S_2 \\ zB_1^* & zB_2^* & (z-1)S_1^* & (z-1)S_2^* & (z-1)R \end{bmatrix}$$

$$\overset{\tilde{P}}{\sim} \begin{bmatrix} 0 & zI_{n_1} - A_{11} & -B_1 & 0 & 0 \\ zA_{11}^* - I_{n_1} & (z-1)Q_{11} & (z-1)S_1 & (z-1)Q_{12} & 0 \\ zB_1^* & (z-1)S_1^* & (z-1)R & (z-1)S_2^* & zB_2^* \\ 0 & (z-1)Q_{12}^* & (z-1)S_2 & (z-1)Q_{22} & zI_{n_2} \\ 0 & 0 & -B_2 & -I_{n_2} & 0 \end{bmatrix}$$

$$\overset{U_1}{\sim} \begin{bmatrix} 0 & zI_{n_1} - A_{11} & -B_1 & 0 & 0 \\ zA_{11}^* - I_{n_1} & (z-1)Q_{11} & (z-1)S_1 & (z-1)Q_{12} & 0 \\ zB_1^* & (z-1)S_1^* & (z-1)R & z(S_2^* - B_2^*Q_{22}) - S_2^* & zB_2^* \\ 0 & (z-1)Q_{12}^* & zS_2 - (S_2 - Q_{22}B_2) & 0 & zI_{n_2} \\ 0 & 0 & -B_2 & -I_{n_2} & 0 \end{bmatrix}$$

$$\overset{U_2}{\sim} \begin{bmatrix} 0 & zI_{n_1} - A_{11} & -B_1 & 0 & 0 \\ zA_{11}^* - I_{n_1} & (z-1)Q_s & (z-1)S_s & (z-1)Q_{12} & 0 \\ zB_1^* & (z-1)S_s^* & (z-1)R_s & z(S_2^* - B_2^*Q_{22}) - S_2^* & 0 \\ 0 & (z-1)Q_{12}^* & zS_2 - (S_2 - Q_{22}B_2) & 0 & zI_{n_2} \\ 0 & 0 & 0 & -I_{n_2} & 0 \end{bmatrix}$$

$$\overset{U_3}{\sim} \begin{bmatrix} 0 & zI_{n_1} - A_{11} & -B_1 & 0 & 0 \\ zA_{11}^* - I_{n_1} & (z-1)Q_s & (z-1)S_s & zQ_{12} & 0 \\ zB_1^* & (z-1)S_s^* & (z-1)R_s & z(S_2^* - B_2^*Q_{22}) & 0 \\ 0 & -Q_{12}^* & -(S_2 - Q_{22}B_2) & 0 & zI_{n_2} \\ 0 & 0 & 0 & -I_{n_2} & 0 \end{bmatrix}.$$

Thus we obtain $\tilde{U} = U_1 U_2 U_3$. □

Corollary 5.10. *Let* $(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ *as in* (3.3) *be given in feedback equivalence form* (2.15) *such that* $n_3 = 0$. *Further, let the corresponding BVD pencil* $z\mathcal{E}_F - \mathcal{A}_F$ *as in* (2.35) *be given. Denote by* $z\mathcal{E}_s - \mathcal{A}_s$ *the BVD pencil corresponding to the EDE system* $(I_{n_1}, A_s, B_s, Q_s, S_s, R_s) \in \Sigma^w_{m,n_1}(\mathbb{K})$ *as in* (3.13), *i. e.,*

$$z\mathcal{E}_s - \mathcal{A}_s = \begin{bmatrix} 0 & zI_{n_1} - A_s & -B_s \\ zA_s^* - I_{n_1} & -Q_s & -S_s \\ zB_s^* & -S_s^* & -R_s \end{bmatrix} \in \mathbb{K}^{n_1+m \times n_1+m}[z]. \tag{5.23}$$

Consider D, $\hat{U}$, $\tilde{P}$, *and* U_2 *as in* (5.19)–(5.22). *Then together with*

$$\check{U} := \tilde{P}U_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & -B_2 & I_{n_2} & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix} \tag{5.24}$$

we have

$$\hat{U}^*(z\mathcal{E} - \mathcal{A})\check{U} = \left[\begin{array}{c|cc} z\mathcal{E}_s - \mathcal{A}_s & 0 & 0 \\ \hline -D^* & 0 & zI_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right]. \tag{5.25}$$

Proof. Note that $z\mathcal{E}_F - \mathcal{A}_F$ as in (2.35) distinguishes from $z\mathcal{A}_F^* - \mathcal{A}_F$ as in (2.38) by not incorporating terms made up from

$$z \begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix}.$$

Thus, proceeding as in the proof of Lemma 5.9 we see that the term zD in (5.18) cannot be present in (5.25). □

Now we are able to prove a generalization of Theorem 5.4. This result is related to the result in [RRV15] in the continuous-time case.

Theorem 5.11. *Let* $(E_F, A_F, B_F, Q_F, S_F, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ *as in* (3.3) *be given in feedback equivalence form* (2.15) *such that* $n_3 = 0$. *Further, let the corresponding palindromic pencil* $z\mathcal{A}_F^* - \mathcal{A}_F$ (2.38) *be given. In addition, let* $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$ *and assume that* $\operatorname{rk}\begin{bmatrix} E - A & B \end{bmatrix} = n$. *Then the following are equivalent:*

(a) There exists a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ *of the Lur'e equation* (5.1).

(b) *It holds that* $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ *for all* $\omega \in [0, 2\pi)$ *such that* $\det(\mathrm{e}^{\mathrm{i}\omega}E - A) \neq 0$. *Furthermore, there exist matrices* Y_μ, $Y_x \in \mathbb{K}^{n\times n+m}$, $Y_u \in \mathbb{K}^{m\times n+m}$ *and* Z_μ, $Z_x \in \mathbb{K}^{n\times n+q}$, $Z_u \in \mathbb{K}^{m\times n+q}$ *such that for*

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix}$$

the following holds:

(i) *The matrix*

$$\begin{bmatrix} E - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

has rank n_1.

(ii) *The space* $\mathcal{Y} = \operatorname{im} Y$ *is of dimension* $n + m$ *and* $(\mathcal{A}^* - \mathcal{A})$*-neutral.*

(iii) *It holds that*

$$\mathcal{V}^\Sigma = \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}.$$

(iv) *There exist* $\tilde{E}$, $\tilde{A} \in \mathbb{K}^{n+q\times n+m}$ *such that* $(z\mathcal{A}^* - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A})$.

Proof. First we show that the statement is invariant under feedback transformations. Therefore, assume we have given the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form as in (3.3) with corresponding transformation matrices W and $\mathcal{T}_F$ and corresponding palindromic pencil $z\mathcal{A}_F^* - \mathcal{A}_F$ as in (2.38). Then by Lemma 5.6, part (a) is equivalent to the existence of a solution (X_F, K_F, L_F) as in (5.13) of the Lur'e equation (5.1) corresponding to $(E_F, A_F, B_F, Q_F, S_F, R_F)$.

To show the equivalence of statement (b) to according statements for the system in feedback equivalence form let

$$U_F := \begin{bmatrix} W^* & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix} \in \mathbb{K}^{2n+m\times 2n+m}$$

and set

$$Y_F := \begin{bmatrix} Y_{\mu,F} \\ Y_{x,F} \\ Y_{u,F} \end{bmatrix} := U_F^{-1}Y, \quad Z_F := \begin{bmatrix} Z_{\mu,F} \\ Z_{x,F} \\ Z_{u,F} \end{bmatrix} = U_F^* Z. \tag{5.26}$$

Then $\mathcal{A}_F = U_F^* \mathcal{A} U_F$ and statement (i) is equivalent to

$$\begin{aligned} &\operatorname{rk} \begin{bmatrix} E_F - A_F & -B_F \end{bmatrix} \begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix} \\ &= \operatorname{rk} \begin{bmatrix} E - A & -B \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}^{-1} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix} = n_1. \end{aligned}$$

Furthermore, we have that $\operatorname{rk} Y_F = \operatorname{rk} Y = n+m$ and by Lemma 2.12, $\operatorname{im} Y$ is $(\mathcal{A}^* - \mathcal{A})$-neutral if and only if $\operatorname{im} Y_F$ is $(\mathcal{A}_F^* - \mathcal{A}_F)$-neutral, i. e., there is no larger space that is $(\mathcal{A}_F^* - \mathcal{A}_F)$-neutral. In addition, by Proposition 2.29(a) we obtain that (iii) is equivalent to

$$\mathcal{V}_F^{\Sigma} = \begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix}.$$

Finally, statement (iv) is equivalent to $(z\mathcal{A}_F^* - \mathcal{A}_F) Y_F = Z_F (z\tilde{E} - \tilde{A})$ by the definition of $\mathcal{A}_F$, Y_F and Z_F. Hence, we have shown that it is sufficient to prove the equivalence between (a) and (b) for the system $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$ in feedback equivalence form.

Now we show that statement (b) follows from statement (a). From Lemma 5.7 we infer that $(X_{11},\, K_1,\, L - K_2 B_2)$ is a solution of the EDE Lur'e equation (5.2) for the EDE system

$$(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma_{m,n_1}^{w}(\mathbb{K})$$

as in (3.13). By denoting the corresponding palindromic pencil arising in the optimal control problem by $z\mathcal{A}_s^* - \mathcal{A}_s$ as in (5.17), Theorem 5.4 implies the existence of

$$Y_s = \begin{bmatrix} X_{11}(A_{11} - I_{n_1}) & X_{11} B_1 \\ I_{n_1} & 0 \\ 0 & I_m \end{bmatrix}, \qquad Z_s = \begin{bmatrix} I_{n_1} & 0 \\ (I_{n_1} - A_{11}^*) X_{11} & K_1^* \\ -B_1^* X_{11} & (L - K_2 B_2)^* \end{bmatrix}$$

as in (5.3) and

$$z\hat{E}_s - \hat{A}_s = \begin{bmatrix} zI_{n_1} - A_{11} & -B_1 \\ (z-1)K_1 & (z-1)(L - K_2 B_2) \end{bmatrix}$$

as in (5.4) such that $(z\mathcal{A}_s^* - \mathcal{A}_s) Y_s = Z_s (z\hat{E}_s - \hat{A}_s)$. Note that as in Theorem 5.4, $\operatorname{im} Y_s$ is maximally $(\mathcal{A}_s^* - \mathcal{A}_s)$-neutral.

From Lemma 5.9 we obtain an invertible transformation matrix $\hat{U} \in \mathbb{K}^{2n+m \times 2n+m}$ as in (5.20) such that

$$z\hat{\mathcal{A}}^* - \hat{\mathcal{A}} := \hat{U}^* (z\mathcal{A}_F^* - \mathcal{A}_F) \hat{U} = \left[\begin{array}{c|cc} z\mathcal{A}_s^* - \mathcal{A}_s & zD & 0 \\ \hline -D^* & 0 & zI_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right] \tag{5.27}$$

with

$$D = \begin{bmatrix} 0 \\ Q_{12} \\ S_2^* - B_2^* Q_{22} \end{bmatrix} \in \mathbb{K}^{2n_1+m\times n_2}.$$

By inspecting the proof of Theorem 5.4 we find that

$$(z\hat{\mathcal{A}}^* - \hat{\mathcal{A}})\hat{Y} = \hat{Z}(z\hat{E} - \hat{A}), \tag{5.28}$$

where

$$\hat{Y} = \left[\begin{array}{c|c} Y_s & 0 \\ \hline 0 & 0 \\ 0 & I_{n_2} \end{array}\right], \qquad \hat{Z} = \left[\begin{array}{c|c} Z_s & 0 \\ \hline 0 & I_{n_2} \\ 0 & 0 \end{array}\right], \qquad z\hat{E} - \hat{A} = \left[\begin{array}{c|c} z\hat{E}_s - \hat{A}_s & 0 \\ \hline -D^* Y_s & zI_{n_2} \end{array}\right].$$

Thus we have

$$\begin{aligned}
\hat{Y}^*(\hat{\mathcal{A}}^* - \hat{\mathcal{A}})\hat{Y} &= \left[\begin{array}{c|cc} Y_s^* & 0 & 0 \\ \hline 0 & 0 & I_{n_2} \end{array}\right] \left[\begin{array}{c|cc} \mathcal{A}_s^* - \mathcal{A}_s & D & 0 \\ \hline -D^* & 0 & I_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right] \left[\begin{array}{c|c} Y_s & 0 \\ \hline 0 & 0 \\ 0 & I_{n_2} \end{array}\right] \\
&= \left[\begin{array}{c|cc} Y_s^*(\mathcal{A}_s^* - \mathcal{A}_s) & Y_s^* D & 0 \\ \hline 0 & -I_{n_2} & 0 \end{array}\right] \left[\begin{array}{c|c} Y_s & 0 \\ \hline 0 & 0 \\ 0 & I_{n_2} \end{array}\right] \\
&= \left[\begin{array}{c|c} Y_s^*(\mathcal{A}_s^* - \mathcal{A}_s)Y_s & 0 \\ \hline 0 & 0 \end{array}\right] = 0,
\end{aligned}$$

and we obtain that $\operatorname{im}\hat{Y}$ is $n+m$ dimensional and $(\hat{\mathcal{A}}^* - \hat{\mathcal{A}})$-neutral. Set

$$\hat{V} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_m \\ Q_{12}^* & -I_{n_2} & -B_2 + S_2 - Q_{22}B_2 \end{bmatrix}.$$

Transforming the quantities in (5.28) to feedback equivalence form (3.3) we obtain

$$(z\mathcal{A}_F^* - \mathcal{A}_F)Y_F\hat{V} = Z_F(z\tilde{E} - \tilde{A}), \tag{5.29}$$

where

$$Y_F\hat{V} = \begin{bmatrix} Y_{\mu,F} \\ Y_{x,F} \\ Y_{u,F} \end{bmatrix} \hat{V} := \begin{bmatrix} Y_{\mu_1,F} \\ Y_{\mu_2,F} \\ Y_{x_1,F} \\ Y_{x_2,F} \\ Y_{u,F} \end{bmatrix} := \hat{U}\hat{Y}\hat{V} = \left[\begin{array}{ccc} X_{11}(A_{11} - I_{n_1}) & 0 & X_{11}B_1 \\ 0 & -I_{n_2} & -B_2 \\ \hline I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{array}\right], \tag{5.30}$$

$Z_F := \hat{U}^{-*}\hat{Z}$, and

$$(z\tilde{E}-\tilde{A}) := (z\hat{E}-\hat{A})\hat{V} = \begin{bmatrix} zI_{n_1} - A_{11} & 0 & -B_1 \\ (z-1)K_1 & 0 & (z-1)(L-K_2B_2) \\ (z-1)Q_{12}^* & -zI_{n_2} & -zB_2 + (z-1)(S_2 - Q_{22}B_2) \end{bmatrix}. \tag{5.31}$$

Then we obtain property (i) by

$$\begin{aligned} \operatorname{rk}\begin{bmatrix} E_F - A_F & -B_F \end{bmatrix}\begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix} &= \operatorname{rk}\begin{bmatrix} I_{n_1} - A_{11} & 0 & -B_1 \\ 0 & -I_{n_2} & -B_2 \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & -B_2 & 0 \\ 0 & I_m & 0 \end{bmatrix} \\ &= \operatorname{rk}\begin{bmatrix} I_{n_1} - A_{11} & -B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = n_1. \end{aligned}$$

Property (ii) follows from the fact that $\operatorname{im}\hat{Y}$ is $n+m$ dimensional and $(\hat{\mathcal{A}}^* - \hat{\mathcal{A}})$-neutral and Lemma 2.12. Furthermore, by Proposition 2.29(a) we have property (iii). Altogether, this shows statement (b).

Now assume that (b) holds for the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form, i. e., properties (i)–(iv) are satisfied. From these properties we construct a deflating subspace for the palindromic pencil $z\mathcal{A}_s^* - \mathcal{A}_s$ such that we can apply Theorem 5.4. Therefore, partition Y_F into

$$Y_F = \begin{bmatrix} Y_{\mu,F} \\ Y_{x,F} \\ Y_{u,F} \end{bmatrix} := \begin{bmatrix} Y_{\mu_1,F} \\ Y_{\mu_2,F} \\ Y_{x_1,F} \\ Y_{x_2,F} \\ Y_{u,F} \end{bmatrix}$$

and denote by $\hat{\mathcal{A}}$ and $\hat{U}$ the matrices we obtain from Lemma 5.9 such that (5.27) holds. Then, for $\hat{Y} := \hat{U}^{-1}Y_F$ we have

$$\hat{Y} = \hat{U}^{-1}Y_F = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ 0 & I_{n_2} & Q_{12}^* & Q_{22} & S_2 \end{bmatrix}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{\mu_2,F} \\ Y_{x_1,F} \\ Y_{x_2,F} \\ Y_{u,F} \end{bmatrix} =: \begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \\ \hat{Y}_{\mu_2,F} \\ \hat{Y}_{x_2,F} \end{bmatrix}$$

for some $\hat{Y}_{\mu_2,F}, \hat{Y}_{x_2,F} \in \mathbb{K}^{n_2\times n+m}$. Thus, $\operatorname{im}\hat{Y}$ is $n+m$ dimensional by property (ii) and $(\hat{\mathcal{A}}^* - \hat{\mathcal{A}})$-neutral by Lemma 2.12.

Now we show that

$$\operatorname{rk}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} = n_1 + m. \tag{5.32}$$

From property (iii) and Proposition 2.29(a) we obtain that

$$\operatorname{rk}\begin{bmatrix} Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} = n_1 + m.$$

Furthermore,

$$\operatorname{im}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix}$$

is $(\mathcal{A}_s^* - \mathcal{A}_s)$-neutral and thus with the same argumentation as in the proof of Theorem 5.4 this means that

$$\operatorname{rk}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} \leq n_1 + m.$$

Altogether, we thus have

$$n_1 + m = \operatorname{rk}\begin{bmatrix} Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} \leq \operatorname{rk}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} \leq n_1 + m.$$

The relation (5.32) implies that

$$\operatorname{im}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix}$$

is also maximally $(\mathcal{A}_s^* - \mathcal{A}_s)$-neutral. This, together with the fact that $\operatorname{rk}\hat{Y} = n + m$, allows us to perform a column transformation of $\hat{Y}$ via $T_1 \in \mathbb{K}^{n+m\times n+m}$ such that

$$\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \\ \hat{Y}_{\mu_2,F} \\ \hat{Y}_{x_2,F} \end{bmatrix} T_1 = \hat{Y}T_1 = \left[\begin{array}{cc|c} X_{11}(A_{11} - I_{n_1}) & X_{11}B_1 & 0 \\ I_{n_1} & 0 & 0 \\ 0 & I_m & 0 \\ \hline 0 & 0 & Y_{x_2} \\ 0 & 0 & Y_{\mu_2} \end{array}\right]$$

with Hermitian $X_{11} \in \mathbb{K}^{n_1\times n_1}$, some Y_{x_2}, $Y_{\mu_2} \in \mathbb{K}^{n_2\times n_2}$, and

$$\operatorname{rk}\begin{bmatrix} Y_{x_2} \\ Y_{\mu_2} \end{bmatrix} = n_2. \tag{5.33}$$

Set

$$Y_s := \begin{bmatrix} X_{11}(A_{11} - I_{n_1}) & X_{11}B_1 \\ I_{n_1} & 0 \\ 0 & I_m \end{bmatrix}.$$

Additionally, from the $(\hat{\mathcal{A}}^* - \hat{\mathcal{A}})$-neutrality of $\operatorname{im} \hat{Y} T_1$ in particular we obtain

$$
\begin{aligned}
0 &= \begin{bmatrix} Y_s^* & 0 & 0 \end{bmatrix} \left[\begin{array}{c|cc} \mathcal{A}_s^* - \mathcal{A}_s & D & 0 \\ \hline -D^* & 0 & I_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right] \begin{bmatrix} 0 \\ Y_{x_2} \\ Y_{\mu_2} \end{bmatrix} = \begin{bmatrix} Y_s^* & 0 & 0 \end{bmatrix} \begin{bmatrix} DY_{x_2} \\ Y_{\mu_2} \\ -Y_{x_2} \end{bmatrix} \\
&= \begin{bmatrix} (A_{11}^* - I_{n_1})X_{11} & I_{n_1} & 0 & 0 & 0 \\ B_1^* X_{11} & 0 & I_m & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Q_{12} Y_{x_2} \\ (S_2^* - B_2^* Q_{22}) Y_{x_2} \\ Y_{\mu_2} \\ -Y_{x_2} \end{bmatrix} \\
&= \begin{bmatrix} Q_{12} Y_{x_2} \\ (S_2^* - B_2^* Q_{22}) Y_{x_2} \end{bmatrix}.
\end{aligned}
$$

Hence we have $DY_{x_2} = 0$.

From property (iv) we obtain

$$
\left[\begin{array}{c|cc} z\mathcal{A}_s^* - \mathcal{A}_s & zD & 0 \\ \hline -D^* & 0 & zI_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right] \left[\begin{array}{c|c} Y_s & 0 \\ \hline 0 & Y_{x_2} \\ 0 & Y_{\mu_2} \end{array}\right] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} z\hat{E}_{11} - \hat{A}_{11} & z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{21} - \hat{A}_{21} & z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix}, \tag{5.34}
$$

where $Z_{11} \in \mathbb{K}^{2n_1+m \times n_1+q}$, $Z_{12} \in \mathbb{K}^{2n_1+m \times n_2}$, $Z_{21} \in \mathbb{K}^{2n_2 \times n_1+q}$, $Z_{22} \in \mathbb{K}^{2n_2 \times n_2}$, $z\hat{E}_{11} - \hat{A}_{11} \in \mathbb{K}^{n_1+q \times n_1+m}[z]$, $z\hat{E}_{12} - \hat{A}_{12} \in \mathbb{K}^{n_1+q \times n_2}[z]$, $z\hat{E}_{21} - \hat{A}_{21} \in \mathbb{K}^{n_2 \times n_1+m}[z]$, and $z\hat{E}_{22} - \hat{A}_{22} \in \mathbb{K}^{n_2 \times n_2}[z]$. From the last block column and block row of (5.34) we obtain

$$
\begin{bmatrix} zY_{\mu_2} \\ -Y_{x_2} \end{bmatrix} = \begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix} \tag{5.35}
$$

and thus by (5.33) we have

$$
\operatorname{rk} \begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} = n_2.
$$

Therefore, we can determine a transformation matrix $T_2 \in \mathbb{K}^{n+q \times n+q}$ such that

$$
\operatorname{rk} \begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} T_2 = \begin{bmatrix} 0 & \tilde{Z}_{22} \end{bmatrix}
$$

for some $\tilde{Z}_{22} \in \mathbb{K}^{2n_2 \times n_2}$. Set

$$
\begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ 0 & \tilde{Z}_{22} \end{bmatrix} := \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} T_2
$$

and

$$
\begin{bmatrix} z\tilde{E}_{11} - \tilde{A}_{11} & z\tilde{E}_{12} - \tilde{A}_{12} \\ z\tilde{E}_{21} - \tilde{A}_{21} & z\tilde{E}_{22} - \tilde{A}_{22} \end{bmatrix} := T_2^{-1} \begin{bmatrix} z\hat{E}_{11} - \hat{A}_{11} & z\hat{E}_{12} - \hat{A}_{12} \\ z\hat{E}_{21} - \hat{A}_{21} & z\hat{E}_{22} - \hat{A}_{22} \end{bmatrix},
$$

accordingly partitioned. Thus reevaluating (5.35) for the transformed matrices we also obtain full normal rank n_2 of $z\tilde{E}_{22} - \tilde{A}_{22}$. Hence there exists some $\lambda_0 \in \mathbb{C}$ such that $\lambda_0\tilde{E}_{22} - \tilde{A}_{22}$ is invertible. From the last block column and first block row of (5.34) we infer

$$0 = \lambda_0 DY_{x_2} = \tilde{Z}_{11}(\lambda_0\tilde{E}_{12} - \tilde{A}_{12}) + \tilde{Z}_{12}(\lambda_0\tilde{E}_{22} - \tilde{A}_{22}).$$

Thus, $\tilde{Z}_{12}$ can be expressed as

$$\tilde{Z}_{12} = -\tilde{Z}_{11}(\lambda_0\tilde{E}_{12} - \tilde{A}_{12})(\lambda_0\tilde{E}_{22} - \tilde{A}_{22})^{-1}.$$

Inserting this relation into the first block row and block column of (5.34) we have

$$(z\mathcal{A}_s^* - \mathcal{A}_s)Y_s = \tilde{Z}_{11}\left(z\tilde{E}_{11} - \tilde{A}_{11} - (\lambda_0\tilde{E}_{12} - \tilde{A}_{12})(\lambda_0\tilde{E}_{22} - \tilde{A}_{22})^{-1}(z\tilde{E}_{21} - \tilde{A}_{21})\right).$$

Hence, we are finally in the position to apply Theorem 5.4. From this we obtain a solution (X_s, K_s, L_s) of (5.2) corresponding to the system $(I_{n_1}, A_s, B_s, Q_s, S_s, R_s)$. By Lemma 5.7 we then also find a solution (X_F, K_F, L_F) of (5.1) corresponding to the system in feedback equivalence form $(E_F, A_F, B_F, Q_F, S_F, R_F)$. □

Remark 5.12. Let an impulse controllable system $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given and let $z\mathcal{A}^* - \mathcal{A}$ be the palindromic pencil as in (2.38). Further, assume that there exists a solution (X, K, L) of the Lur'e equation (5.1).

(a) The matrix pencil $(z\tilde{E} - \tilde{A}) \in \mathbb{K}^{n+q\times n+m}[z]$ that we have obtained in the proof of Theorem 5.11 fulfills $\operatorname{rk}_{\mathbb{K}(z)}(z\tilde{E} - \tilde{A}) = n + q$ by Proposition 5.1, since

$$\begin{aligned} n+q &= \operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} zE - A & -B \\ (z-1)K & (z-1)L \end{bmatrix} \\ &= \operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} zE - A & -B \\ (z-1)K & (z-1)L \end{bmatrix} \mathcal{T}_F \\ &= \operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} zI_{n_1} - A_{11} & 0 & -B_1 \\ 0 & -I_{n_2} & -B_2 \\ (z-1)K_1 & (z-1)K_2 & (z-1)L \end{bmatrix} \\ &= \operatorname{rk}_{\mathbb{K}(z)} \begin{bmatrix} zI_{n_1} - A_{11} & 0 & -B_1 \\ 0 & -I_{n_2} & 0 \\ (z-1)K_1 & 0 & (z-1)(L - K_2B_2) \end{bmatrix}. \end{aligned} \tag{5.36}$$

This means in particular that the existence of solutions of (5.1) implies the existence of a deflating subspace of the palindromic pencil $z\mathcal{A}^* - \mathcal{A}$.

(b) In the proof of Theorem 5.11 we have constructed a deflating subspace $\operatorname{im} Y_F$ as in (5.30) for the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form

(3.3) from a solution $(X_F,\, K_F,\, L_F)$ of the Lur'e equation (5.1). From here we can construct a deflating subspace $\operatorname{im} Y$ for the original system by using (5.26). By Lemma 5.7 it is justified to set

$$W^{-*}XW^{-1} = X_F := \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.37}$$

Thus, we have

$$\begin{aligned}
Y :=& U_F Y_F \hat{V} \mathcal{T}_F^{-1} \\
=& \left[\begin{array}{c|cc} W^* & 0 & 0 \\ \hline 0 & T & 0 \\ 0 & FT & I_m \end{array}\right] \left[\begin{array}{ccc} X_{11}(A_{11} - I_{n_1}) & 0 & X_{11}B_1 \\ 0 & -I_{n_2} & -B_2 \\ \hline I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{array}\right] \mathcal{T}_F^{-1} \\
=& \left[\begin{array}{c|cc} W^* & 0 & 0 \\ \hline 0 & T & 0 \\ 0 & FT & I_m \end{array}\right] \left[\begin{array}{c|cc} -X_F + (I_n - E_F) & 0 & 0 \\ \hline 0 & I_n & 0 \\ 0 & 0 & I_m \end{array}\right] \left[\begin{array}{ccc} I_{n_1} - A_{11} & 0 & -B_1 \\ 0 & -I_{n_2} & -B_2 \\ \hline I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{array}\right] \mathcal{T}_F^{-1} \\
=& \begin{bmatrix} X(A - E) + G_1 & XB + G_2 \\ V_1^\Sigma & V_2^\Sigma \end{bmatrix},
\end{aligned}$$

where

$$\operatorname{im} \begin{bmatrix} G_1 & G_2 \end{bmatrix} = \operatorname{im} W^* \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} W \begin{bmatrix} E - A & -B \end{bmatrix} \subseteq \ker E^*, \tag{5.38}$$

$$\begin{bmatrix} V_1^\Sigma & V_2^\Sigma \end{bmatrix} := \mathcal{T}_F V_F^\Sigma \mathcal{T}_F^{-1}, \tag{5.39}$$

and

$$V_F^\Sigma := \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{bmatrix} \tag{5.40}$$

spans the system space $\mathcal{V}_F^\Sigma$, see Proposition 2.29(a). Altogether, this leads to $(z\mathcal{A}^* - \mathcal{A})Y = Z(z\check{E} - \check{A})$, where $Z = U_F^* Z_F$ and $z\check{E} - \check{A} := (z\tilde{E} - \tilde{A})\mathcal{T}_F^{-1}$.

Example 5.13 (Example 2.25 revisited)**.** Consider the system $(E,\, A,\, B,\, Q,\, S,\, R)$ as in (2.16) and Example 3.11. Note that since $n_3 = 0$ in (2.17) the system $(E,\, A,\, B)$ is impulse controllable according to Proposition 2.31(d). We have seen in Example 5.3 that

$$(X_s,\, K_s,\, L_s) = \left(\sqrt{3},\, \sqrt{2},\, -\frac{\sqrt{3}+1}{\sqrt{2}}\right)$$

is a solution of the Lur'e equation (5.2) corresponding to the EDE system

$$(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s)$$

as in (3.21). By Lemma 5.7 we obtain that

$$X_F = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{bmatrix}, \quad K_F = \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix}, \quad L_F = -\frac{\sqrt{3}+1}{\sqrt{2}} \tag{5.41}$$

solves the Lur'e equation of the system in feedback equivalence form. Therefore, by Lemma 5.6 we see that

$$\begin{aligned} X &= W^* X_F W = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} \end{bmatrix}, \\ K &= K_F T^{-1} \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix}, \quad L = -\frac{\sqrt{3}+1}{\sqrt{2}} \end{aligned} \tag{5.42}$$

solves the Lur'e equation (5.1) corresponding to the original system.

Thus, according to Remark 5.12 the matrix $Y \in \mathbb{K}^{5\times 3}$ defined by

$$Y = \begin{bmatrix} X(A-E)+G_1 & XB+G_2 \\ V_1^{\Sigma} & V_2^{\Sigma} \end{bmatrix} = \left[\begin{array}{cc|c} 0+1 & 0-1 & -\sqrt{3}+1 \\ 0+0 & 0+0 & -\sqrt{3}+0 \\ \hline 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{5.43}$$

is a deflating subspace of the palindromic pencil as in (4.13).

As in the EDE case we can show a similar statement for BVD pencils as in (2.35).

Theorem 5.14. *Let $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3) be given in feedback equivalence form (2.15) such that $n_3 = 0$. Further, let the corresponding BVD pencil $z\mathcal{E}_F - \mathcal{A}_F$ as in (2.38) be given. In addition, let $q = \operatorname{rk}_{\mathbb{K}(z)} \Phi(z)$. Then the following are equivalent:*

(a) There exists a solution $(X,\, K,\, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ of the Lur'e equation (5.1).

(b) It holds that $\Phi(\mathrm{e}^{\mathrm{i}\omega}) \succeq 0$ for all $\omega \in [0, 2\pi)$ such that $\det(\mathrm{e}^{\mathrm{i}\omega}E - A) \neq 0$. Furthermore, there exist matrices $Y_\mu,\, Y_x \in \mathbb{K}^{n\times n+m}$, $Y_u \in \mathbb{K}^{m\times n+m}$ and $Z_\mu,\, Z_x \in \mathbb{K}^{n\times n+q}$, $Z_u \in \mathbb{K}^{m\times n+q}$ such that for

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix}$$

the following holds:

(i) *The matrix*

$$\begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

has rank n_1.

(ii) *The space* $\mathcal{Y} = \operatorname{im} Y$ *is of dimension* $n+m$ *and* $\mathcal{E}^e$*-neutral, where*

$$\mathcal{E}^e := \begin{bmatrix} 0 & -E & 0 \\ E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

corresponds to $\mathcal{E}$ *in the even pencil in* (2.33).

(iii) *It holds that*

$$\mathcal{V}^\Sigma = \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}.$$

(iv) *There exist* $\tilde{E}$, $\tilde{A} \in \mathbb{K}^{n+q \times n+m}$ *such that* $(z\mathcal{E} - \mathcal{A})Y = Z(z\tilde{E} - \tilde{A})$.

Proof. First we show that the statement is invariant under feedback transformations. Therefore, assume we have given the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form as in (3.3) with corresponding transformation matrices W and $\mathcal{T}_F$ and corresponding BVD pencil $z\mathcal{E}_F - \mathcal{A}_F$ as in (2.35). Then by Lemma 5.6, part (a) is equivalent to the existence of a solution (X_F, K_F, L_F) as in (5.13) of the Lur'e equation (5.1) corresponding to $(E_F, A_F, B_F, Q_F, S_F, R_F)$.

To show the equivalence of statement (b) to according statements for the system in feedback equivalence form let

$$U_F := \begin{bmatrix} W^* & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix} \in \mathbb{K}^{2n+m \times 2n+m}$$

and set

$$Y_F := \begin{bmatrix} Y_{\mu,F} \\ Y_{x,F} \\ Y_{u,F} \end{bmatrix} := U_F^{-1} Y, \quad Z_F := \begin{bmatrix} Z_{\mu,F} \\ Z_{x,F} \\ Z_{u,F} \end{bmatrix} = U_F^* Z. \tag{5.44}$$

Then $z\mathcal{E}_F - \mathcal{A}_F = U_F^*(z\mathcal{E} - \mathcal{A})U_F$, and statement (i) is equivalent to

$$\begin{aligned} &\operatorname{rk} \begin{bmatrix} E_F & 0 \end{bmatrix} \begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix} \\ &= \operatorname{rk} \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}^{-1} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix} = n_1. \end{aligned}$$

Furthermore, we have that $\operatorname{rk} Y_F = \operatorname{rk} Y = n+m$ and by Lemma 2.12, $\operatorname{im} Y$ is $\mathcal{E}^e$-neutral if and only if $\operatorname{im} Y_F$ is $\mathcal{E}_F^e$-neutral, where

$$\mathcal{E}_F^e := U_F^* \mathcal{E}^e U_F = \begin{bmatrix} 0 & 0 & -I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In addition, by Proposition 2.29(a) we obtain that (iii) is equivalent to

$$\mathcal{V}_F^\Sigma = \begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix}.$$

Finally, statement (iv) is equivalent to $(z\mathcal{E}_F - \mathcal{A}_F)Y_F = Z_F(z\tilde{E} - \tilde{A})$ by the definition of $z\mathcal{E}_F - \mathcal{A}_F$, Y_F and Z_F. Hence, we have shown that it is sufficient to prove the equivalence between (a) and (b) for the system in feedback equivalence form $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F)$.

Now we show that statement (b) follows from statement (a). From Lemma 5.7 we infer that $(X_{11},\, K_1,\, L - K_2B_2)$ is a solution of the EDE Lur'e equation (5.2) for the EDE system

$$(I_{n_1},\, A_s,\, B_s,\, Q_s,\, S_s,\, R_s) \in \Sigma_{m,n_1}^w(\mathbb{K})$$

as in (3.13). By denoting the corresponding BVD pencil arising in the optimal control problem by $z\mathcal{E}_s - \mathcal{A}_s$ as in (5.17), Theorem 5.5 implies the existence of

$$Y_s = \begin{bmatrix} -X_{11} & 0 \\ I_{n_1} & 0 \\ 0 & I_m \end{bmatrix}, \qquad Z_s = \begin{bmatrix} I_{n_1} & 0 \\ -A_{11}^* X_{11} & -K_1^* \\ -B_1^* X_{11} & -(L - K_2B_2)^* \end{bmatrix}$$

as in (5.3), and

$$z\hat{E}_s - \hat{A}_s = \begin{bmatrix} zI_{n_1} - A_{11} & -B_1 \\ K_1 & L - K_2B_2 \end{bmatrix}$$

as in (5.4) such that $(z\mathcal{E}_s - \mathcal{A}_s)Y_s = Z_s(z\hat{E}_s - \hat{A}_s)$. Note that as in Theorem 5.5 $\operatorname{im} Y_s$ is maximally $\mathcal{E}_s^e$-neutral. From Corollary 5.10 we obtain invertible transformation matrices $\hat{U}, \check{U} \in \mathbb{K}^{2n+m \times 2n+m}$ such that

$$z\hat{\mathcal{E}} - \hat{\mathcal{A}} := \hat{U}^*(z\mathcal{E}_F - \mathcal{A}_F)\check{U} = \left[\begin{array}{c|cc} z\mathcal{E}_s - \mathcal{A}_s & 0 & 0 \\ \hline -D^* & 0 & zI_{n_2} \\ 0 & -I_{n_2} & 0 \end{array}\right] \tag{5.45}$$

with

$$D = \begin{bmatrix} 0 \\ Q_{12} \\ S_2^* - B_2^* Q_{22} \end{bmatrix} \in \mathbb{K}^{2n_1+m \times n_2},$$

and

$$\hat{\mathcal{E}}^e := \hat{U}^*\mathcal{E}_F^e\check{U} = \left[\begin{array}{c|cc} \mathcal{E}_s^e & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & -I_{n_1} & 0 & 0 & 0 \\ I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{5.46}$$

Inspecting the proof of Theorem 5.5 we find that

$$(z\hat{\mathcal{E}} - \hat{\mathcal{A}})\hat{Y} = \hat{Z}(z\hat{E} - \hat{A}), \tag{5.47}$$

where

$$\hat{Y} = \left[\begin{array}{c|c} Y_s & 0 \\ \hline 0 & 0 \\ 0 & I_{n_2} \end{array}\right], \qquad \hat{Z} = \left[\begin{array}{c|c} Z_s & 0 \\ \hline 0 & I_{n_2} \\ 0 & 0 \end{array}\right], \qquad z\hat{E} - \hat{A} = \left[\begin{array}{c|c} zE_s - A_s & 0 \\ \hline -D^*Y_s & zI_{n_2} \end{array}\right].$$

Thus we have

$$\hat{Y}^*\hat{\mathcal{E}}^e\hat{Y} = \left[\begin{array}{c|cc} Y_s^* & 0 & 0 \\ \hline 0 & 0 & I_{n_2} \end{array}\right] \left[\begin{array}{c|cc} \mathcal{E}_s^e & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c|c} Y_s & 0 \\ \hline 0 & 0 \\ 0 & I_{n_2} \end{array}\right] = 0,$$

and we obtain that $\operatorname{im}\hat{Y}$ is $n+m$ dimensional and $\hat{\mathcal{E}}^e$-neutral. Set

$$\hat{V} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_m \\ 0 & -I_{n_2} & -B_2 \end{bmatrix}.$$

Transforming the quantities in (5.47) to feedback equivalence form (3.3), we obtain

$$(z\mathcal{E}_F - \mathcal{A}_F)Y_F\hat{V} = Z_F(z\tilde{E} - \tilde{A}), \tag{5.48}$$

where

$$Y_F\hat{V} = \begin{bmatrix} Y_{\mu,F} \\ Y_{x,F} \\ Y_{u,F} \end{bmatrix}\hat{V} := \begin{bmatrix} Y_{\mu_1,F} \\ Y_{\mu_2,F} \\ Y_{x_1,F} \\ Y_{x_2,F} \\ Y_{u,F} \end{bmatrix} := \check{U}\hat{Y}\hat{V} = \left[\begin{array}{ccc} -X_{11} & 0 & 0 \\ 0 & -I_{n_2} & -B_2 \\ \hline I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{array}\right], \tag{5.49}$$

$Z_F := \hat{U}^{-*}\hat{Z}$, and

$$(z\tilde{E} - \tilde{A}) := (z\hat{E} - \hat{A})\hat{V} = \begin{bmatrix} zI_{n_1} - A_{11} & 0 & -B_1 \\ K_1 & 0 & L - K_2B_2 \\ -Q_{12}^* & -zI_{n_2} & -zB_2 + Q_{22}B_2 - S_2 \end{bmatrix}. \tag{5.50}$$

Then we obtain property (i) by

$$\operatorname{rk}\begin{bmatrix} E_F & 0 \end{bmatrix}\begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix} = \operatorname{rk}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{bmatrix}$$
$$= \operatorname{rk}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = n_1.$$

Property (ii) follows from the fact that $\operatorname{im}\hat{Y}$ is $n+m$ dimensional and $\hat{\mathcal{E}}^e$-neutral and Lemma 2.12. Furthermore, by Proposition 2.29(a) we have property (iii). Altogether, this shows statement (b).

Now assume that (b) holds for the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form, i. e., properties (i)–(iv) are satisfied. From these properties we construct a deflating subspace for the BVD pencil $z\mathcal{E}_s - \mathcal{A}_s$ such that we can apply Theorem 5.5. Therefore, partition Y_F into

$$Y_F = \begin{bmatrix} Y_{\mu,F} \\ Y_{x,F} \\ Y_{u,F} \end{bmatrix} := \begin{bmatrix} Y_{\mu_1,F} \\ Y_{\mu_2,F} \\ Y_{x_1,F} \\ Y_{x_2,F} \\ Y_{u,F} \end{bmatrix}$$

and denote by $\hat{\mathcal{E}}, \hat{\mathcal{A}}$, $\hat{U}$, and $\check{U}$ the matrices we obtain from Corollary 5.10 such that (5.45) is fulfilled. Then, for $\hat{Y} := \check{U}^{-1}Y_F$ we have

$$\hat{Y} = \check{U}^{-1}Y_F = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ 0 & I_{n_2} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{\mu_2,F} \\ Y_{x_1,F} \\ Y_{x_2,F} \\ Y_{u,F} \end{bmatrix} =: \begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \\ \hat{Y}_{\mu_2,F} \\ \hat{Y}_{x_2,F} \end{bmatrix}$$

for some $\hat{Y}_{\mu_2,F}, \hat{Y}_{x_2,F} \in \mathbb{K}^{n_2 \times n+m}$. Then, $\operatorname{im}\hat{Y}$ is $n+m$ dimensional by property (ii) and $\hat{\mathcal{E}}^e$-neutral by Lemma 2.12.

We first show that

$$\operatorname{rk}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} = n_1 + m. \tag{5.51}$$

From property (iii) and Proposition 2.29(a) we obtain that

$$\operatorname{rk}\begin{bmatrix} Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} = n_1 + m.$$

Furthermore,

$$\operatorname{im}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix}$$

is $\mathcal{E}_s^e$-neutral for $\mathcal{E}_s^e$ as in (5.46) and thus with the same argumentation as in the proof of Theorem 5.5 this means that

$$\operatorname{rk}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} \leq n_1 + m.$$

Altogether, we thus have

$$n_1 + m = \operatorname{rk}\begin{bmatrix} Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} \leq \operatorname{rk}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix} \leq n_1 + m.$$

The relation (5.51) implies that

$$\operatorname{im}\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \end{bmatrix}$$

is also maximally $\mathcal{E}_s^e$-neutral. This, together with the fact that $\operatorname{rk}\hat{Y} = n + m$, allows us to perform a column transformation of $\hat{Y}$ via $T_1 \in \mathbb{K}^{n+m\times n+m}$ such that

$$\begin{bmatrix} Y_{\mu_1,F} \\ Y_{x_1,F} \\ Y_{u,F} \\ \hat{Y}_{\mu_2,F} \\ \hat{Y}_{x_2,F} \end{bmatrix} T_1 = \hat{Y}T_1 = \left[\begin{array}{cc|c} -X_{11} & 0 & 0 \\ I_{n_1} & 0 & 0 \\ 0 & I_m & 0 \\ \hline 0 & 0 & Y_{x_2} \\ 0 & 0 & Y_{\mu_2} \end{array}\right]$$

with Hermitian $X_{11} \in \mathbb{K}^{n_1\times n_1}$, some Y_{x_2}, $Y_{\mu_2} \in \mathbb{K}^{n_2\times n_2}$, and

$$\operatorname{rk}\begin{bmatrix} Y_{x_2} \\ Y_{\mu_2} \end{bmatrix} = n_2. \tag{5.52}$$

Then with property (iv) proceeding as in the proof of Theorem 5.4 we find $Z_s \in \mathbb{K}^{2n_1+m\times n_1+q}$, $zE_s - A_s \in \mathbb{K}^{n_1+q\times n_1+m}[z]$ such that we are in the position to apply Theorem 5.4. In this way we obtain a solution (X_s, K_s, L_s) of (5.2) corresponding to the system $(I_{n_1}, A_s, B_s, Q_s, S_s, R_s)$. By Lemma 5.7 we then also find a solution (X_F, K_F, L_F) of (5.1) corresponding to the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form. □

Remark 5.15. Let an impulse controllable system $(E, A, B, Q, S, R) \in \Sigma^w_{m,n}(\mathbb{K})$ be given and let $z\mathcal{E} - \mathcal{A}$ be the BVD pencil as in (2.35). Further, assume that there exists a solution (X, K, L) of the Lur'e equation (5.1).

(a) The matrix pencil $(z\tilde{E} - \tilde{A}) \in \mathbb{K}^{n+q\times n+m}[z]$ we obtain in the proof of Theorem 5.14 fulfills $\operatorname{rk}_{\mathbb{K}(z)}(z\tilde{E} - \tilde{A}) = q$, see (5.36). This means in particular that the existence of solutions of (5.1) implies the existence of a deflating subspace of the BVD pencil $z\mathcal{E} - \mathcal{A}$.

(b) In the proof of Theorem 5.14 we have constructed a deflating subspace Y_F as in (5.49) for the system $(E_F, A_F, B_F, Q_F, S_F, R_F)$ in feedback equivalence form (3.3) from a solution (X_F, K_F, L_F) of the Lur'e equation (5.1). From here we can construct a deflating subspace Y for the original system by using (5.44). By Lemma 5.7 it is justified to set

$$W^{-*}XW^{-1} = X_F := \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.53}$$

Thus we have

$$\begin{aligned} Y :=& U_F Y_F \hat{V} \mathcal{T}_F^{-1} \\ =& \left[\begin{array}{c|cc} W^* & 0 & 0 \\ \hline 0 & T & 0 \\ 0 & FT & I_m \end{array}\right] \left[\begin{array}{ccc} -X_{11} & 0 & 0 \\ 0 & -I_{n_2} & -B_2 \\ \hline I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{array}\right] \mathcal{T}_F^{-1} = \begin{bmatrix} -XE + G_1 & G_2 \\ V_1^\Sigma & V_2^\Sigma \end{bmatrix}, \end{aligned}$$

where V_1^Σ, V_2^Σ are as in (5.39) and

$$\operatorname{im}\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \operatorname{im} W^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I_{n_2} & -B_2 \end{bmatrix} \mathcal{T}_F^{-1} \subseteq \ker E^*. \tag{5.54}$$

Altogether, this leads to $(z\mathcal{E} - \mathcal{A})Y = Z(z\check{E} - \check{A})$, where $Z = U_F^* Z_F$ and $z\check{E} - \check{A} := (z\tilde{E} - \tilde{A})\mathcal{T}_F^{-1}$.

Remark 5.16. A major difference between Theorem 5.11 and Theorem 5.14 or Theorem 5.4 and Theorem 5.5 is that in the BVD case we do not need the artificial assumption

$$\operatorname{rk}\begin{bmatrix} E - A & -B \end{bmatrix} = n, \tag{5.55}$$

or equivalently

$$\operatorname{rk}\begin{bmatrix} I_{n_1} - A_{11} & -B_1 \end{bmatrix} = n_1, \tag{5.56}$$

i. e., controllability at one. If the system (E, A, B) is obtained by discretization with the implicit Euler method as in (2.12), we see that in the limiting case $h \to 0$ this corresponds to

$$\operatorname{rk} \lim_{h \to 0} \begin{bmatrix} I_{n_1} - hA_{11} & -hB_1 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} = n_1,$$

which is trivially fulfilled. Therefore, for sufficiently small h we may assume validity of this assumption.

Remark 5.17. The result of Theorem 5.11 is completely analogous to [RRV15, Theorem 6.2]. Here, we again replace positivity of the Popov function on the imaginary axis by positivity on the unit circle. Furthermore, instead of a deflating subspace for the associated even pencil as in (2.33), we here find a deflating subspace of the associated palindromic pencil as in (2.38). Moreover, in condition (b)(i) we replace

$$\begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

by

$$\begin{bmatrix} E - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix},$$

or in other words, instead of evaluating the pencil

$$\begin{bmatrix} zE - A & -B \end{bmatrix} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

at ∞, we evaluate it at one. Finally, we replace the condition of $\mathcal{E}$-neutrality of the deflating subspace $\mathcal{Y}$ of (2.33) by $(\mathcal{A}^* - \mathcal{A})$-neutrality of the deflating subspace $\mathcal{Y}$ of (2.38). In other words, the associated matrix pencil in the one case is evaluated at ∞, and in the other case it is evaluated at one.

6 Application to Optimal Control

In this chapter we discuss implications for the feasibility and the structure of solutions of the discrete-time optimal control problem (2.27) corresponding to the system $(E,\, A,\, B,\, Q,\, S,\, R) \in \Sigma^{w}_{m,n}(\mathbb{K})$ based on the results from the previous chapters. Here we assume that the system $(E,\, A,\, B,\, Q,\, S,\, R)$ is stabilizable.

First, let $P \in \mathbb{K}^{n\times n}$ be a solution of the KYP inequality (3.2) and let $x^0 \in \mathcal{W}^{\sigma}$. Thus, we have $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^{\sigma}_{(E,\,A,\,B)}$ with $Ex_0 = Ex^0$ and $\lim_{j\to\infty} Ex_j = 0$. Then by the definition of the system space $\mathcal{V}^{\Sigma} \subseteq \mathbb{K}^{n+m}$, we obtain $[\,x_j^*\ \ u_j^*\,]^* \in \mathcal{V}^{\Sigma}$ for all $j \in \mathbb{N}_0$. Thus, for $j_2 \geq j_1$ we have that

$$\begin{aligned}
&x_{j_2}^* E^* P E x_{j_2} - x_{j_1}^* E^* P E x_{j_1} \\
&= \sum_{k=j_1}^{j_2-1} \Delta_1(x_k^* E^* P E x_k) \\
&= \sum_{k=j_1}^{j_2-1} \sigma(x_k^* E^* P E x_k) - x_k^* E^* P E x_k \\
&= \sum_{k=j_1}^{j_2-1} (Ax_k + Bu_k)^* P (Ax_k + Bu_k) - x_k^* E^* P E x_k \\
&= \sum_{k=j_1}^{j_2-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} A^*PA - E^*PE & A^*PB \\ B^*PA & B^*PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\
&\geq - \sum_{k=j_1}^{j_2-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}.
\end{aligned} \tag{6.1}$$

For $j_1 = 0$, $j_2 \to \infty$ we thus obtain for the objective function $\mathcal{J}^{\sigma}(x,u)$ that

$$x_0^* E^* P E^* x_0 \leq \mathcal{J}^{\sigma}(x,u)$$

and thus

$$x_0^* E^* P E^* x_0 \leq \mathcal{W}_+^{\sigma}(Ex^0) < \infty. \tag{6.2}$$

Hence, if $P \in \mathbb{K}^{n\times n}$ solves the KYP inequality (3.2) we have feasibility of the optimal control problem for every $x^0 \in \mathcal{W}^{\sigma}$, i. e., $\infty > \mathcal{W}_+^{\sigma}(Ex^0) > -\infty$. In addition, we obtain that the Popov function $\Phi(z) \in \mathbb{K}^{m\times m}[z]$ is positive semi-definite on the unit circle by

Theorem 3.10. On the other hand, if the system $(E,\, A,\, B)$ is R-controllable and the Popov function $\Phi(z)$ is positive semi-definite on the unit circle, then by Theorem 3.10 we have the existence of a solution $P \in \mathbb{K}^{n\times n}$ of the KYP inequality (3.2) and thus feasibility of the optimal control problem.

If further we have a solution $(X,\, K,\, L)$ of the Lur'e equation (5.1), then for every $x^0 \in \mathcal{W}^\sigma$ and $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^\sigma_{(E,\,A,\,B)} \cap \ell^2(\mathbb{K}^{n+m})$ with $Ex_0 = Ex^0$ and $\lim\limits_{j\to\infty} Ex_j = 0$ we obtain in (6.1) that

$$\begin{aligned}
-x_0^* E^* X E x_0 &= \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \left(\mathcal{M}(X) - \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \right) \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} - \mathcal{J}^\sigma(x,u)
\end{aligned}$$

and thus $x_0^* E^* X E^* x_0 + \|Kx + Lu\|_{\ell^2}^2 = \mathcal{J}^\sigma(x,u)$.

In the following we characterize the structure of an optimal solution if it exists. First, we impose the assumption that $(X,\, K,\, L)$ even fulfills

$$(x^0)^* E^* X E x^0 = \mathcal{W}_+^\sigma(Ex^0) \tag{6.3}$$

for all $x^0 \in \mathcal{W}^\sigma$.

If $x^0 \in \mathcal{W}^\sigma$ is given, then $[\,x^*\ \ u^*\,]^* \in \mathfrak{B}^\sigma_{(E,\,A,\,B)}$ with $Ex_0 = Ex^0$ and $\lim\limits_{j\to\infty} Ex_j = 0$ is an optimal solution if and only if $\|Kx + Lu\|_{\ell^2} = 0$. If this is the case, then $[\,x^*\ \ u^*\,]^*$ fulfills

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{j+1} \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix},$$

$$Ex_0 = Ex^0, \qquad \lim_{j\to\infty} Ex_j = 0.$$

For an impulse controllable system $(E_F,\, A_F,\, B_F,\, Q_F,\, S_F,\, R_F) \in \Sigma^w_{m,n}(\mathbb{K})$ as in (3.3) in feedback equivalence form with corresponding transformation matrices W, T, F we set

$$X_F = W^{-*} X W^{-1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Since

$$\begin{aligned}
E^* X E &= T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \\
&= T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1},
\end{aligned}$$

by Lemma 5.7 we can without loss of generality set $X_{12} = 0$ and $X_{22} = 0$.

In addition, from Theorem 5.14 we obtain a deflating subspace $Y \in \mathbb{K}^{2n+m\times n+m}$ of the BVD pencil $z\mathcal{E} - \mathcal{A}$ as in (2.35), i. e., we have $Z \in \mathbb{K}^{2n+m\times n+q}$ and a matrix pencil $z\check{E} - \check{A} \in \mathbb{K}^{n+q\times n+m}[z]$ such that $(z\mathcal{E} - \mathcal{A})Y = Z(z\check{E} - \check{A})$. It can be constructed as in Remark 5.15(b). Inserting σ for z this leads to

$$\left[\begin{array}{cc|c} 0 & \sigma E - A & -B \\ \hline \sigma A^* - E^* & -Q & -S \\ \sigma B^* & -S^* & -R \end{array}\right] \left[\begin{array}{c|c} -XE + G_1 & G_2 \\ \hline V_1^\Sigma & V_2^\Sigma \end{array}\right] \begin{bmatrix} x_j \\ u_j \end{bmatrix}$$
$$= Z \begin{bmatrix} \sigma I_{n_1} - A_{11} & 0 & -B_1 \\ K_1 & 0 & L - K_2 B_2 \\ -Q_{12}^* & -\sigma I_{n_2} & -\sigma B_2 + Q_{22} B_2 - S_2 \end{bmatrix} \mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix}, \quad (6.4)$$

where $\operatorname{im}[\,G_1 \;\; G_2\,] \subseteq \ker E^*$, see (5.54),

$$\begin{bmatrix} V_1^\Sigma & V_2^\Sigma \end{bmatrix} := \mathcal{T}_F V_F^\Sigma \mathcal{T}_F^{-1},$$

and

$$V_F^\Sigma := \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & -B_2 \\ 0 & 0 & I_m \end{bmatrix}.$$

Since $[\,x_j^* \;\; u_j^*\,]^* \in \mathcal{V}^\Sigma$, it follows with Proposition 2.29(a) that

$$\mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix} = \begin{bmatrix} x_{1,j} \\ -B_2 u_j \\ u_j \end{bmatrix}$$

for some $x_{1,j} \in \mathbb{K}^{n_1}$. Then by Lemma 5.7 and Lemma 5.6 we have that

$$\begin{bmatrix} K_1 & L - K_2 B_2 \end{bmatrix} \begin{bmatrix} x_{1,j} \\ u_j \end{bmatrix} = \begin{bmatrix} K_F & L_F \end{bmatrix} \mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix} = \begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} = 0. \quad (6.5)$$

In addition, from (3.16) we obtain

$$\begin{bmatrix} Q_{12}^* & Q_{22} & S_2 \end{bmatrix} = K_2^* \begin{bmatrix} K_1 & K_2 & L \end{bmatrix}$$

and thus

$$\begin{aligned} \begin{bmatrix} Q_{12}^* & S_2 - Q_{22} B_2 \end{bmatrix} \begin{bmatrix} x_{1,j} \\ u_j \end{bmatrix} &= \begin{bmatrix} Q_{12}^* & Q_{22} & S_2 \end{bmatrix} \mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix} = K_2^* \begin{bmatrix} K_1 & K_2 & L \end{bmatrix} \mathcal{T}_F^{-1} \begin{bmatrix} x_j \\ u_j \end{bmatrix} \\ &= K_2^* \begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} = 0. \end{aligned} \quad (6.6)$$

Thus, by equations (6.5) and (6.6) the right-hand-side of (6.4) is zero. Furthermore, by Proposition 2.29(d) we have that

$$V^{\Sigma}\begin{bmatrix} x_j \\ u_j \end{bmatrix} = \begin{bmatrix} x_j \\ u_j \end{bmatrix}.$$

Set

$$\mu_j := \begin{bmatrix} -XE + G_1 & G_2 \end{bmatrix}\begin{bmatrix} x_j \\ u_j \end{bmatrix}. \tag{6.7}$$

Thus

$$\lim_{j\to\infty} E^*\mu_j = \lim_{j\to\infty} -E^*XEx_j = 0,$$

and hence, $(\mu_j)_j$ is part of a solution of the boundary value problem

$$\begin{bmatrix} 0 & E & 0 \\ A^* & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix}\sigma\begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ E^* & Q & S \\ 0 & S^* & R \end{bmatrix}\begin{bmatrix} \mu \\ x \\ u \end{bmatrix},$$

$$Ex_0 = Ex^0, \qquad \lim_{j\to\infty} E^*\mu_j = 0.$$

Moreover, for an impulse controllable system (E, A, B, Q, S, R) we can take the same approach for a deflating subspace $Y \in \mathbb{K}^{2n+m\times n+m}$ of the palindromic pencil $z\mathcal{A}^* - \mathcal{A}$ as in (2.38) obtained in Theorem 5.11. There we have $Z \in \mathbb{K}^{2n+m\times n+q}$ and a matrix pencil $z\check{E} - \check{A} \in \mathbb{K}^{n+q\times n+m}[z]$ such that $(z\mathcal{A}^* - \mathcal{A})Y = Z(z\check{E} - \check{A})$. It can be constructed as in Remark 5.12(b). Inserting σ for z leads to

$$\left[\begin{array}{c|cc} 0 & \sigma E - A & -B \\ \hline \sigma A^* - E^* & (\sigma-1)Q & (\sigma-1)S \\ \sigma B^* & (\sigma-1)S^* & (\sigma-1)R \end{array}\right]\left[\begin{array}{cc} -X(E-A)+G_1 & -XB+G_2 \\ \hline V_1^{\Sigma} & V_2^{\Sigma} \end{array}\right]\begin{bmatrix} x_j \\ u_j \end{bmatrix}$$
$$= Z\begin{bmatrix} \sigma I_{n_1} - A_{11} & 0 & -B_1 \\ (\sigma-1)K_1 & 0 & (\sigma-1)(L - K_2B_2) \\ (\sigma-1)Q_{12}^* & -\sigma I_{n_2} & -\sigma B_2 + (\sigma-1)(S_2 - Q_{22}B_2) \end{bmatrix}\mathcal{T}_F^{-1}\begin{bmatrix} x_j \\ u_j \end{bmatrix}, \tag{6.8}$$

where $\operatorname{im}[\,G_1 \;\; G_2\,] \subseteq \ker E^*$, see (5.38).

Again by equations (6.5) and (6.6) the right-hand-side of (6.8) is 0. Set

$$m_j := \begin{bmatrix} X(A-E) + G_1 & XB + G_2 \end{bmatrix}\begin{bmatrix} x_j \\ u_j \end{bmatrix}. \tag{6.9}$$

Thus

$$\begin{aligned}\sum_{k=0}^{\infty} E^* m_k &= \sum_{k=0}^{\infty} -E^* X E x_k + E^* X \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} -E^* X E (x_k - x_{k+1}) \\ &= - E^* X E x_0 = E^* \mu_0\end{aligned}$$

and hence, $(m_j)_j$ is part of a solution of the boundary value problem

$$\begin{gathered}\begin{bmatrix} 0 & E & 0 \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \sigma \begin{bmatrix} m \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ E^* & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} m \\ x \\ u \end{bmatrix}, \\ E x_0 = E x^0, \qquad \sum_{k=0}^{\infty} E^* m_k = E^* \mu_0.\end{gathered} \tag{6.10}$$

Example 6.1 (Example 2.25 revisited)**.** Consider the system (E, A, B, Q, S, R) as in (2.16) and Example 3.11. In Example 5.13 we have seen that

$$(X, K, L) = \left(\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix}, -\frac{\sqrt{3}+1}{\sqrt{2}} \right)$$

is a solution of the Lur'e equation (5.1). We have

$$E^* X E = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

and thus by (6.2) for every

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \in \mathcal{W}^\sigma$$

the optimal value $\mathcal{W}_+^\sigma(Ex^0)$ is bounded from below by $\sqrt{3}\,|x_2^0|^2$.

Indeed, setting

$$u_j = \frac{2}{\sqrt{3}+1} \left(1 - \frac{2}{\sqrt{3}+1}\right)^j x_2^0,$$

we obtain – according to (2.14) – that

$$x_j = \begin{bmatrix} 1 - \frac{2}{\sqrt{3}+1} \\ 1 \end{bmatrix} \left(1 - \frac{2}{\sqrt{3}+1}\right)^j x_2^0$$

solves (1.7) with

$$\begin{aligned}\mathcal{J}^\sigma(x,u) =& \sum_{j=0}^{\infty} \|x_j\|^2 + \|u_j\|^2 \\ =& |x_2^0|^2 \left(\left(1 - \frac{2}{\sqrt{3}+1}\right)^2 + 1 + \left(\frac{2}{\sqrt{3}+1}\right)^2 \right) \sum_{j=0}^{\infty} \left(1 - \frac{2}{\sqrt{3}+1}\right)^{2j} \\ =& |x_2^0|^2 \frac{12 - 6\sqrt{3}}{1 - \left(1 - \frac{2}{\sqrt{3}+1}\right)^2} = |x_2^0|^2 \sqrt{3},\end{aligned}$$

i. e., (x,u) is an optimal solution fulfilling $Ex_0 = Ex^0$ and $\lim_{j\to\infty} Ex_j = 0$.

In particular, from (5.43) we obtain that

$$m_j = \begin{bmatrix} 1 & -1 & -\sqrt{3}+1 \\ 0 & 0 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix} = -\sqrt{3}\frac{2}{\sqrt{3}+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(1 - \frac{2}{\sqrt{3}+1}\right)^j x_2^0$$

fulfills the boundary value problem (6.10), where

$$\mu_0 = -\sqrt{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2^0.$$

Remark 6.2. The results of this Chapter are analogous to [RRV15, Section 7] if we replace the respective continuous-time objects by their discrete-time analogs. For instance, the continuous-time analog of (6.1) is obtained by replacing the operator Δ_1 by $\frac{\mathrm{d}}{\mathrm{d}t}$ and replacing the sum by an integral.

However, the assumption made in (6.3) is somewhat restrictive at first glance. To justify this choice, it would be sufficient to show that for $\varepsilon > 0$ there exists $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^\sigma_{(E,A,B)}$ with $Ex_0 = Ex^0$ and $\lim_{j\to\infty} Ex_j = 0$ such that $\|Kx + Lu\|_{\ell^2} < \varepsilon$. This means that we can find $[\,x^* \;\; u^*\,]^* \in \mathfrak{B}^\sigma_{(E,A,B)}$ with $Ex_0 = Ex^0$ and $\lim_{j\to\infty} Ex_j = 0$ such that we get arbitrarily close to the optimal value, although an optimal control might not exist.

In the continuous-time case, the validity of this assumption is guaranteed by an application of [IR14, Theorem 6.6] to the closed-loop system obtained from a stabilizing solution of the continuous-time Lur'e equation. To achieve a similar result in the discrete-time case we would need to investigate properties and existence of stabilizing solutions of the discrete-time Lur'e equation in analogy to [Rei11; RRV15]. Stabilizing solutions of the Lur'e equation (5.1) are those which fulfill

$$\operatorname{rk} \begin{bmatrix} \lambda E - A & -B \\ (\lambda - 1)K & (\lambda - 1)L \end{bmatrix} = n + q$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Furthermore, we would need an analogous discrete-time result for [IR14, Theorem 6.6].

7 Conclusions and Outlook

We have discussed several problems arising in the discrete-time linear-quadratic optimal control problem and we have seen their relations to the results that have been obtained in the continuous-time setting. In Chapter 3 we have discussed an extension of the Kalman-Yakubovich-Popov inequality for standard difference equations to the case of implicit difference equations. The characterizations are analogous to what was obtained in [RRV15] in the continuous-time case. For an analogous relaxation of the controllability assumption to sign-controllability we would need the discrete-time analog of of [CALM97, Theorem 6.1].

In Chapter 4 we further related the spectral properties of the palindromic pencil associated to the discrete-time optimal control problem (2.27) to the positivity of the Popov function on the unit circle. To this end, we introduced the notion of quasi-Hermitian matrices which allows for a generalization of the concept of inertia. Also these results are related to the results in [Voi15] for the continuous-time case.

In Chapter 5 we introduced Lur'e equations for explicit as well as for implicit difference equations. We have shown that solvability of these equations is equivalent to the existence of certain deflating subspaces of the BVD and palindromic pencil arising in the discrete-time control problem (2.27). In the palindromic case we needed the additional assumption that the given system is controllable at the eigenvalue one, which can always be achieved for discrete-time systems originating from discretization. It is an open question whether this condition can be dropped if the latter is not the case. Moreover, the results of this chapter are related to the results obtained in [RRV15; Voi15] in the continuous-time case. Nonetheless, some more technical difficulties had to be tackled.

In Chapter 6 we have seen how we can use these results to characterize the solutions of the boundary value problems associated to the discrete-time optimal control problem (2.27). However, the assumption made in (6.3) is somewhat restrictive. In the continuous-time case discussed in [RRV15], the validity of this assumption is guaranteed by an application of [IR14, Theorem 6.6] to the closed-loop system obtained from a stabilizing solution of the continuous-time Lur'e equation. To achieve a similar result in the discrete-time case we would need to investigate stabilizing solutions of the discrete-time Lur'e equation in analogy to [Rei11; RRV15]. Furthermore we would need an analogous discrete-time result for [IR14, Theorem 6.6].

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