TECHNISCHE UNIVERSITÄT BERLIN

MINIMAL LAGRANGIAN SUBMANIFOLDS WITH CONSTANT SECTIONAL CURVATURE IN INDEFINITE COMPLEX SPACE FORMS

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Preprint No. 644/1999

PREPRINT REIHE MATHEMATIK

FACHBEREICH 3

Minimal Lagrangian submanifolds with constant sectional curvature in indefinite complex space forms

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October 8, 1999

Abstract

We study minimal Lagrangian immersions from an indefinite real space form $M_s^n(c)$ into an indefinite complex space form $\tilde{M}_s^n(4\tilde{c})$. Provided that $c \neq \tilde{c}$, we show that M^n has to be flat and we obtain an explicit description of the immersion. In the case the metric is positive definite or Lorentzian, this result was respectively obtained by Ejiri [4] and by Kriele and the author [5]. In the case that $c = \tilde{c}$, this theorem is no longer true, see for instance the examples discovered in [3] by Chen and the author.

Subject class: 53B35, 53B30

Keywords: Lagrangian, constant sectional curvature, complex space forms

1 Introduction

Whereas Lagrangian submanifolds of complex Riemannian space forms are widely studied, see a.o. [6] and the references contained therein, not much is known about Lagrangian submanifolds of indefinite complex space forms. As far as we know most results about submanifolds of indefinite complex space forms, see for example [1], [9] or [8] deal with complex submanifolds. In this paper we study minimal Lagrangian submanifolds $M_s^n(c)$ with constant sectional curvature of indefinite complex space forms $\tilde{M}_s^n(4\tilde{c})$. Provided $c \neq \tilde{c}$, we obtain a complete classification and show amongst others that M has to be flat. The corresponding theorem for Lagrangian immersions in Riemannian or Lorentzian complex space forms were obtained respectively by Ejiri in [4] and Kriele and the author in [5]

The paper is organized as follows. In Section 2 we recall the basic models of indefinite complex space forms and we give the basic formulas for Lagrangian submanifolds. In particular, We notice that the basic formulas are similar to those of affine hyperspheres with constant sectional curvature. This similarity allows us to apply the results of [10] to obtain an intrinsic characterization of these hypersurfaces. In Section 3, we start by recalling Reckziegel result [7] which allows us, using the Hopf fibration, to consider the horizontal lifts of these immersions into indefinite real space forms. Combining then in Section 3, Reckziegel's result with the intrinsic characterization, we obtain explicit formulas for the immersion and prove the classification theorem.

2 Indefinite complex space forms and their Lagrangian submanifolds

In this section, we briefly recall some facts about indefinite complex space forms. For more details, we refer the reader to [2]. Let $\tilde{M}_k^n(4\tilde{c})$ be a complex space form of complex dimension

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†Partially supported by a research fellowship of the Alexander von Humboldt Stiftung (Germany)

n and complex index k. The complex index is defined as the (complex) dimension of the largest complex negative definite vector space of the tangent space. The curvature tensor $\tilde{R}$ of $\tilde{M}(4\tilde{c})$ is given by

$$\tilde{R}(X,Y)Z = \tilde{c}\big(\langle v,w\rangle\, u - \langle u,w\rangle\, v + \langle Jv,w\rangle\, Ju - \langle Ju,w\rangle\, Jv + 2\,\langle u,Jv\rangle\, Jw\big),$$

where J denotes the complex structure. We refer to [2] for the construction of the standard models of indefinite complex space forms $\mathbb{C}P^n_s(4\tilde{c})$, when $\tilde{c}>0$, $\mathbb{C}H^n_s(4\tilde{c})$, when $\tilde{c}<0$ and $\mathbb{C}^n_s$. For our purposes it is sufficient to know that there exist pseudo-Riemannian submersions, called Hopf fibrations,

$$\Pi : S^{2n+1}_{2s}(\tilde{c}) \to \mathbb{C}P^n_s(4\tilde{c}) : z \mapsto z \cdot \mathbb{C}^\star$$

if $\tilde{c}>0$ and if $\tilde{c}<0$ by

$$\Pi : H^{2n+1}_{2s+1}(\tilde{c}) \to \mathbb{C}H^n_s(4\tilde{c}) : z \mapsto z \cdot \mathbb{C}^\star,$$

where $S^{2n+1}_{2s}(\tilde{c}) = \{z \in \mathbb{C}^{n+1} | b_{s,n+1}(z,z) = \frac{1}{\tilde{c}}\}$ and $H^{2n+1}_{s+1}(\tilde{c}) = \{z \in \mathbb{C}^{n+1} | b_{s+1,n+1}(z,z) = \frac{1}{\tilde{c}}\}$ and $b_{p,q}$ is the standard Hermitian form with index p on $\mathbb{C}^q$. For our convenience, we will assume that we have chosen a basis such that the first p odd terms appear with a minus sign.

In [2] it is shown that locally any indefinite complex space form is holomorphically isometric to either $\mathbb{C}^n_s$, $\mathbb{C}P^n_s(4\tilde{c})$, or $\mathbb{C}H^n_s(4\tilde{c})$. Remark that, by replacing the metric $<.,.>$ by $-<.,.>$, we have that $\mathbb{C}H^n_s(4\tilde{c})$ is congruent with $\mathbb{C}P^n_{n-s}(-4\tilde{c})$. For that purpose, we may assume that $n-2s \geq 0$ and if $n-2s=0$, we only need to consider $\mathbb{C}^n_s$ and $\mathbb{C}P^n_s(4\tilde{c})$.

Next, we consider Lagrangian submanifolds. A submanifold M of a Kähler manifold is Lagrangian if and only if J interchanges the tangent and the normal space. Hence a Lagrangian submanifold of an indefinite complex space form of index s, has real index s. From now on let $\tilde{M}^n_s(4\tilde{c})$ be a complex space form of curvature $4\tilde{c}$. We denote by $\tilde{\nabla}$ the Levi Civita connection of the metric $\langle .,. \rangle$ on $\tilde{M}^n_s(4\tilde{c})$. Then, the formulas of Gauss and Weingarten are respectively given by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X,Y),\\ \tilde{\nabla}_X JY &= -A_{JY}X + \nabla^\perp_X JY,\end{aligned}$$

defining the induced connection ∇, the second fundamental form h, the Weingarten operator A and the normal connection $\nabla^\perp$. Since J is parallel, we deduce as in the Riemannian case that

$$\begin{aligned}\nabla^\perp_X JY &= J\nabla_X Y\\ A_{JY}X &= -Jh(X,Y).\end{aligned}$$

We now introduce a tensor K on M by $K(X,Y) = K_X Y = -Jh(X,Y) = A_{JY}X$. It follows from the above equations that $\langle K(X,Y),Z\rangle$ is totally symmetric. A straightforward computation shows that the equations of Gauss, Codazzi and Ricci for Lagrangian submanifolds are equivalent to

$$R(X,Y)Z = \tilde{c}(\langle Y,Z\rangle\, X - \langle X,Z\rangle\, Y) + [K_X,K_Y]Z \tag{1}$$
$$(\nabla_X K)(Y,Z) = (\nabla_Y K)(X,Z) \tag{2}$$

A Lagrangian submanifold M with constant sectional curvature c is minimal if and only if

$$\operatorname{trace} K_X = 0 \tag{3}$$

for every tangent tangent vector field X. The Gauss and Codazzi equations reduce to

$$\begin{aligned}[K_X,K_Y]Z &= a(\langle Y,Z\rangle\, X - \langle X,Z\rangle\, Y),\\ (\nabla_X K)(Y,Z) &= (\nabla_Y K)(X,Z),\end{aligned}$$

where $a = c - \tilde{c}$.

Remark that, since ∇ is the Levi Civita connection of the metric $\langle .,. \rangle$, this is exactly the situation which also appears in affine differential geometry when one investigates affine hyperspheres with constant sectional curvature, see [10]. Therefore, applying the results of [10], we obtain in particular the following:

Theorem 1 *Let $\phi : M_s^n(c) \to \tilde{M}_s^n(4\tilde{c})$ be a minimal, Lagrangian isometric immersion. Suppose that $n - 2s \geq 0$ and that $c \neq \tilde{c}$. Then, if $n - 2s > 1$, we must have that $c < \tilde{c}$. Moreover, let $p \in M_s^n(c)$. Then there exists local vectorfields $\{U_1, V_1, \dots, U_s, V_s, E_1, \dots, E_r\}$, where $r = n - 2s$ such that*

$$\begin{aligned} &< U_m, U_j >=< V_m, V_j >= 0, && (4)\\ &< E_k, E_\ell >= \delta_{k\ell}, && (5)\\ &< U_m, V_j >= \delta_{mj}, && (6)\\ &< U_j, E_k >=< V_j, E_k >= 0, && (7)\end{aligned}$$

where $j, m = 1, \dots, s$ and $k, \ell = 1, \dots, r$. Assuming now that $\ell < k$ and $m < j$, we introduce a multilinear map K by

$$\begin{aligned} K(U_m, U_j) &= \lambda_m U_j && (8)\\ K(U_m, V_j) &= \lambda_m V_j && (9)\\ K(U_m, E_k) &= \lambda_m E_k && (10)\\ K(V_m, U_j) &= \lambda_m(\lambda_m - \alpha_m) U_j && (11)\\ K(V_m, V_j) &= \lambda_m(\lambda_m - \alpha_m) V_j && (12)\\ K(V_m, E_k) &= \lambda_m(\lambda_m - \alpha_m) E_k && (13)\\ K(U_j, U_j) &= \alpha_j U_j + V_j && (14)\\ K(U_j, V_j) &= \sum_{m=1}^{j-1} (\lambda_m V_m + (\lambda_m - \alpha_m)\lambda_m U_m) + \beta_j U_j + \alpha_j V_j && (15)\\ K(V_j, V_j) &= \gamma_j U_j + \beta_j V_j && (16)\\ K(E_k, E_\ell) &= -\mu_\ell E_k && (17)\\ K(E_k, E_k) &= \sum_{j=1}^{s} (\lambda_j V_j + (\lambda_j - \alpha_j)\lambda_j U_j) - \sum_{\ell=1}^{k-1} \mu_\ell E_\ell + (r-k)\mu_k E_k, && (18)\end{aligned}$$

where the $a_j, \alpha_j, \beta_j, \gamma_j, \mu_k$ and λ_j are constants determined by

$$\begin{aligned} \lambda_j^3 &= \lambda_1^3 \tfrac{n(n+1)(n-1)}{(n-2j+2)(n-2j+1)(n-2j+3)}, && (19)\\ \alpha_j &= -\tfrac{n-2j}{2}\lambda_j, && (20)\\ \beta_j &= -\tfrac{1}{4}(n-2j)(n-2j+2)\lambda_j^2, && (21)\\ \gamma_j &= \tfrac{\lambda_j^3}{8}(n-2j+2)^3, && (22)\\ a_j &= -\tfrac{\lambda_j^3}{2}(n-2j+2)(n-2j+1), && (23)\\ a_1 &= a, && (24)\\ \mu_1^2 &= -\tfrac{a_1(n+1)}{r(r+1)}, && (25)\\ \mu_{\ell+1}^2 &= \tfrac{(r-\ell+2)}{(r-\ell)}\mu_\ell^2. && (26)\end{aligned}$$

Remark that in the special case that $n - 2s = 1$, μ_1 is not needed and therefore can be defined arbitrarily. Using the fact that K is a Codazzi tensor with respect to ∇ it is then straightforward, see also [10] to show the following:

Lemma 1 *Let $\{U_1, V_1, \dots, U_s, V_s, W_1, \dots, W_r\}$ be the frame constructed before. Then all connection coefficients (with respect to ∇) vanish. In particular M has flat affine metric, i.e $c = 0$.*

3 Classification results

Using the results of the previous section, we already obtain the following corollaries:

Corollary 1 *Let M_s^n be a minimal Lagrangian submanifold with constant sectional curvature c of $\mathbb{C}H_s^n(4\tilde{c})$, $n-2s>1$, where $\tilde{c}<0$, then $c=\tilde{c}$.*

Proof: Suppose that $c \neq \tilde{c}$. In that case, $a \neq 0$. Since $n-2s>1$, it follows from the previous theorem that a has to be negative. Since, from the previous lemma, we know that our Lagrangian submanifold has to be flat, we find that $\tilde{c}$ is positive. This is a contradiction. ∎

Of course, also in the flat case, a similar theorem can be obtained:

Corollary 2 *Let M_s^n be a minimal Lagrangian submanifold with constant sectional curvature c of $\mathbb{C}_s^n$, then $c=0$.*

Proof: Suppose that $c \neq 0$. Then $a \neq 0$. Consequently, from the previous lemma it follows that $c=0$, which is a contradiction. ∎

In general, applying the results of the previous chapters, a classification result can be proved. However, in order to obtain explicit equations, we first recall some basic facts from [7] which relate Lagrangian submanifolds of respectively $\mathbb{C}P_s^n(4\tilde{c})$ and $\mathbb{C}H_s^n(4\tilde{c})$ to horizontal immersions in respectively $S_{2s}^{2n+1}(\tilde{c})$ and $H_{2s+1}^{2n+1}(\tilde{c})$. Here a horizontal immersion $f\colon M \to S_{2s}^{2n+1}(\tilde{c})$ (respectively, $f\colon M \to H_{2s+1}^{2n+1}(\tilde{c})$) is an immersion which satisfies $if(x) \perp f_*(T_xM)$ for all $x \in M$.

Theorem 2 **([7])** *Let $\tilde{c}>0$ and let $\Pi : S_{2s}^{2n+1}(\tilde{c}) \to \mathbb{C}P_s^n(4\tilde{c})$ be the Hopf fibration. If $f: M^n \to S_{2s}^{2n+1}(\tilde{c})$ is a horizontal immersion, then $F = \Pi \circ f : M^n \to \mathbb{C}P_s^n(4\tilde{c})$ is a Lagrangian immersion.*

Conversely, let M^n be a simply connected manifold and let $F : M^n \to \mathbb{C}P_s^n(4\tilde{c})$ be a Lagrangian immersion. Then there exist a 1-parameter family of horizontal lifts $f : M \to S_{2s}^{2n+1}(\tilde{c})$ such that $F = \Pi \circ f$. Any two such lifts f_1 and f_2 are related by $f_1 = e^{i\theta} f_2$, where θ is a constant.

The analogous statement for $\tilde{c}<0$ also holds if one replaces $S_{2s}^{2n+1}(\tilde{c})$ by $H_{2s+1}^{2n+1}(\tilde{c})$ and $\mathbb{C}P_s^n(4\tilde{c})$ by $\mathbb{C}H_s^n(4\tilde{c})$.

Then, if we denote by $\mathbb{M}_s^n(4\tilde{c}) = \mathbb{C}P_s^n(4\tilde{c})$ if $\tilde{c}>0$ and $\mathbb{M}_s^n(4\tilde{c}) = \mathbb{C}H_s^n(4\tilde{c})$ if $\tilde{c}<0$ we have the following result:

Theorem 3 *Let $\phi : M_n^s(c) \to \mathbb{M}(4\tilde{c})$ be a minimal isometric Lagrangian immersion into $\tilde{M}_s^n(4\tilde{c})$, $n-2s \geq 0$. Suppose that $c \neq \tilde{c}$. Then $c=0$, and if $n-2s>1$ then $\tilde{c}>0$. Then, we get using the Hopf fibration that $\phi(M_s^n)$ is congruent the immersion x, inductively defined by*

$$\begin{aligned}
x &= x^1 \\
x^1 &= (z_{11} + \tfrac{1}{2a_1(n+1)} z_{12}, z_{11} - \tfrac{1}{2a_1(n+1)} z_{12}, x^2 e^{i\lambda_1 u_1 + i(\lambda_1 - \alpha_1)\lambda_1 v_1}) \\
x^j &= (z_{j1} + \tfrac{1}{2a_j(n-2j+3)} z_{j2}, z_{j1} - \tfrac{1}{2a_j(n-2j+31)} z_{j2}, x^{j+1} e^{i\lambda_j u_j + i(\lambda_j - \alpha_j)\lambda_j v_j}),
\end{aligned}$$

where the definition x^{s+1} depends on s. In particular if $n-2s=1$ and $\tilde{c}<0$,

$$x^{s+1}(w_1) = \tfrac{1}{b}(\cosh w_1, \sinh w_1), \qquad b^2 = a_{s+1}.$$

If $n-2s = 0$, 1 or r with $r>1$, then x^{s+1} is repectively defined by

$$\begin{aligned}
x^{s+1} &= \sqrt{-\tfrac{1}{a_{s+1}}}, \\
x^{s+1}(w_1) &= \tfrac{1}{\sqrt{2}b}(e^{ibw_1}, e^{-ibw_1}), \qquad b^2 = -a_{s+1} \\
x^{s+1}(\tilde{w}_1, \dots, \tilde{w}_r) &= \tfrac{1}{\sqrt{r(r+1)}} \tfrac{1}{\mu_1}(e^{i\tilde{w}_1}, \dots, e^{i\tilde{w}_r}, e^{-i\tilde{w}_1 - \dots - i\tilde{w}_r}).
\end{aligned}$$

In the above we have that $1 \leq j \leq s$, a_j, α_j and lambda$_j$ are as defined before and z_{j1} and z_{j2} are defined by

$$z_{j1} = e^{i(\eta_{j1}u_j+(\eta_{j1}^2-\eta_{j2}^2-\alpha_j\eta_{j1})v_j)}e^{(\eta_{j2}u_j+(2\eta_{j1}\eta_{j2}-\alpha_j\eta_{j2})v_j)}$$
$$z_{j2} = e^{i(\eta_{j1}u_j+(\eta_{j1}^2-\eta_{j2}^2-\alpha_j\eta_{j1})v_j)}e^{-(\eta_{j2}u_j+(2\eta_{j1}\eta_{j2}-\alpha_j\eta_{j2})v_j)}.$$

where

$$\eta_{j1} = -\tfrac{1}{2}\lambda_j(n-2j+1)$$
$$\eta_{j2} = \tfrac{1}{2}\lambda_j\sqrt{(n-2j+1)(n-2j+3)}.$$

Proof: From the previous lemma, we know that there exists coordinates $u_1, v_1, \ldots, u_s, v_s$ on M^n such that

$$U_i = \frac{\partial}{\partial u_i} \tag{27}$$
$$V_i = \frac{\partial}{\partial v_i} \tag{28}$$
$$E_k = \frac{\partial}{\partial w_k} \tag{29}$$

where $i = 1, \ldots, s$ and $k = 1, \ldots, r$. We denote the Hopf lift of the immersion of M^n into $\mathbb{M}_s^n(4\tilde{c})$ by x. We then get that x is characterized by the following system of differential equations:

$$x_{u_j u_m} = i\lambda_j x_{u_m}, \quad m > j \tag{30}$$
$$x_{u_j v_m} = i\lambda_j x_{v_m}, \quad m > j \tag{31}$$
$$x_{u_j w_k} = i\lambda_j x_{w_k} \tag{32}$$
$$x_{v_j u_m} = i\lambda_j(\lambda_j - \alpha_j)x_{u_m}, \quad m > j \tag{33}$$
$$x_{v_j v_m} = i\lambda_j(\lambda_j - \alpha_j)x_{v_m}, \quad m > j \tag{34}$$
$$x_{v_j w_k} = i\lambda_j(\lambda_j - \alpha_j)x_{w_k} \tag{35}$$
$$x_{u_j u_j} = i\alpha_j x_{u_j} + i x_{v_j} \tag{36}$$
$$x_{u_j v_j} = i\sum_{m=1}^{j-1}(\lambda_m x_{v_m} + (\lambda_m - \alpha_m)\lambda_m x_{u_m}) + i\beta_j x_{u_j} + i\alpha_j x_{v_j} + ax \tag{37}$$
$$x_{v_j v_j} = i\gamma_j x_{u_j} + i\beta_j x_{v_j} \tag{38}$$
$$x_{w_k w_\ell} = -i\mu_\ell x_{w_k}, \quad k > \ell \tag{39}$$
$$x_{w_k w_k} = i\sum_{j=1}^{s}(\lambda_j x_{v_j} + (\lambda_j - \alpha_j)\lambda_j x_{u_j}) - i\sum_{\ell=1}^{k-1}\mu_\ell x_{w_\ell} + i(r-k)\mu_k x_{w_k} + ax, \tag{40}$$

where the $a_j, \alpha_j, \beta_j, \gamma_j, \mu_k$ and λ_j are the constants defined earlier and where $a = -\tilde{c}$.
In particular, we have that

$$x_{u_1 u_1} = i\alpha_1 x_{u_1} + i x_{v_1}, \tag{41}$$
$$x_{u_1 v_1} = i\beta_1 x_{u_1} + i\alpha_1 x_{v_1} + ax, \tag{42}$$
$$x_{v_1 v_1} = i\gamma_1 x_{u_1} + i\beta_1 x_{v_1}, \tag{43}$$

From these equations we deduce that

$$\begin{aligned} x_{u_1 u_1 u_1} &= i\alpha_1 x_{u_1 u_1} + i x_{u_1 v_1} \\ &= i\alpha_1 x_{u_1 u_1} - \beta_1 x_{u_1} + i\alpha_1(x_{u_1 u_1} - i\alpha_1 x_{u_1}) + iax \\ &= 2\alpha_1 i x_{u_1 u_1} - (\beta_1 - \alpha_1^2)x_{u_1} + a_1 i x \end{aligned}$$

We now look at the corresponding equation of degree 3,

$$t^3 - 2\alpha_1 i t^2 + (\beta_1 - \alpha_1^2)t - a_1 i = 0. \tag{44}$$

It is easy to see that (44) has one purely imaginary root, namely $\lambda_1 i$ and two complex roots $i\eta_{11} - \eta_{12}$ and $i\eta_{11} + \eta_{12}$ which are determined by

$$\begin{aligned} \eta_{11} &= -\tfrac{1}{2}\lambda_1(n-1) \\ \eta_{12} &= \tfrac{1}{2}\lambda_1\sqrt{(n-1)(n+1)}. \end{aligned}$$

Using now oncemore our system of differential equations, it follows that we can write

$$x = A(u_i, v_j, w_k)e^{i\lambda_1 u_1} + C_1(v_1)z_{11} + D_1(v_1)z_{12} \tag{45}$$

where we have written

$$\begin{aligned} z_{11} &= e^{i(\eta_{11}u_1 + (\eta_{11}^2 - \eta_{12}^2 - \alpha_1\eta_{11})v_1)}e^{(\eta_{12}u_1 + (2\eta_{11}\eta_{12} - \alpha_1\eta_{12})v_1)} \\ z_{12} &= e^{i(\eta_{11}u_1 + (\eta_{11}^2 - \eta_{12}^2 - \alpha_1\eta_{11})v_1)}e^{-(\eta_{12}u_1 + (2\eta_{11}\eta_{12} - \alpha_1\eta_{12})v_1)} \end{aligned}$$

Substituting now the above expression (45) for x into

$$x_{v_1} = -ix_{u_1u_1} - \alpha_1 x_{u_1},$$

and using the fact that $e^{i\lambda_1 u_1}$, z_{11} and z_{12} are linearly independent functions, we obtain the following system of differential equations for A, C_1 and D_1:

$$\begin{aligned} A_{v_1} &= (\lambda_1 - \alpha_1)\lambda_1 iA, \\ (C_1)_{v_1} &= 0, \\ (D_1)_{v_1} &= 0, \end{aligned}$$

from which it follows that

$$\begin{aligned} A &= x^2(u_2, v_2, \ldots, u_s, v_s, w_1, \ldots, w_r)e^{i(\lambda_1 - \alpha_1)\lambda_1 v_1} \\ C_1(v_2) &= C_1 \\ D_1(v_2) &= D_1 \end{aligned}$$

A straightforward computation shows that $a_2 = a - 2\lambda_1^2(\lambda_1 - \alpha_1)$ and

$$\begin{aligned} 0 &= \lambda_1(\eta_{11}^2 - \eta_{12}^2) - \alpha_1\eta_{11} + \lambda_1(\lambda_1 - \alpha_1)\eta_{11} - a \\ 0 &= \lambda_1(2\eta_{11}\eta_{12} - \alpha_1\eta_{12}) + \lambda_1(\lambda_1 - \alpha_1)\eta_{12} \end{aligned}$$

Therefore, we get that

$$i(\lambda_1 x_{v_1} + (\lambda_1 - \alpha_1)\lambda_1 x_{u_1}) + ax = a_2 x^2 e^{i\lambda_1 u_1 + i(\lambda_1 - \alpha_1)\lambda_1 v_1}$$

Using the above, we obtain by substituting the found expression of $x = x^1$ into the system of differential equations, that x^2 satisfies a similar system of differential equations.

Moreover, if we define

$$y_1 = \frac{e^{\frac{1}{4}\lambda_1(2((n-1)i - \sqrt{n^2-1})u) + \lambda_1 n((n-1)i + \sqrt{n^2-1})v}}{2\lambda_1^2 n(n+1)\sqrt{n^2-1}}, \qquad y_2 = \frac{e^{\frac{1}{4}\lambda_1(2((n-1)i + \sqrt{n^2-1})u) + \lambda_1 n((n-1)i - \sqrt{n^2-1})v}}{2\lambda_1^2 n(n+1)\sqrt{n^2-1}},$$

we get that

$$\begin{aligned} C_1 &= y_1(\lambda_1 n((n+1) + i\sqrt{n^2-1})U_1 + 2(-(n+1) + i\sqrt{n^2-1})V_1 + 2\lambda_1^2 n\sqrt{n^2-1}x), \\ D_1 &= y_2(\lambda_1 n(-(n+1) + i\sqrt{n^2-1})U_1 + 2((n+1) + i\sqrt{n^2-1})V_1 + 2\lambda_1^2 n\sqrt{n^2-1}x), \\ x^2 &= \frac{e^{-\frac{1}{2}i\lambda_1(2u_1 + \lambda_1 n v_1)}}{\lambda_1^2 n(n+1)}(-i\lambda_1 nU_1 - 2iV_1 + \lambda_1^2 n(n-1)x) \end{aligned}$$

From this it follows that

$$\begin{aligned}
< C_1, C_1 > &= 0, \\
< D_1, D_1 > &= 0, \\
< C_1, D_1 > &= \tfrac{-1}{(a_1(n+1))}, \\
< C_1, iD_1 > &= 0, \\
< x^2, C_1 > &= 0, \\
< x^2, D_1 > &= 0, \\
< x^2, iC_1 > &= 0, \\
< x^2, iD_1 > &= 0, \\
< x^2, x^2 > &= -\tfrac{1}{a_2}.
\end{aligned}$$

Therefore, we get that we have an orthogonal decomposition $\mathbb{C}^{n+1} = \mathbb{C}^2 \oplus \mathbb{C}^{n-1}$, where C_{11} and C_{12} span $\mathbb{C}^2$ and x^2 defines an immersion into $\mathbb{C}^{n-1}$ which is minimal in $S^{2n-3}_{2s-2}(-a_2)$, if a_1 (and therefore also a_2) is negative or in $H^{2n-3}_{2s-1}(-a_2)$ if a_1 is positive. Since x^2 satisfies a similar system of differential equations and since x^1 is horizontal, it is easy to check that x^2 defines a flat (provided $n-2 \geq 2$), minimal horizontal immersion into $S^{2n-3}_{2s-2}(-a_2)$ or $H^{2n-3}_{2s-1}(-a_2)$. Thus we can write

$$x^1 = (z_{11} + \tfrac{1}{2a_1(n+1)} z_{12}, z_{11} - \tfrac{1}{2a_1(n+1)} z_{12}, x^2 e^{i\lambda_1 u_1 + i(\lambda_1 - \alpha_1)\lambda_1 v_1}).$$

Proceeding now by induction we can define flat horizontal immersions x^j into $\mathbb{C}^{n-2j+3}$ which are minimal in $S^{2n-4j+5}_{2s+2-2j}(-a_j)$ if a_1 is negative or in $H^{2n-4j+5}_{2s+3-2j}(-a_j)$ if a_1 is positive such that

$$x^j = (z_{j1} + \tfrac{1}{2a_j(n-2j+3)} z_{j2}, z_{j1} - \tfrac{1}{2a_j(n-2j+31)} z_{j2}, x^{j+1} e^{i\lambda_j u_j + i(\lambda_j - \alpha_j)\lambda_j v_j}),$$

where we have written

$$\begin{aligned}
z_{j1} &= e^{i(\eta_{j1} u_j + (\eta_{j1}^2 - \eta_{j2}^2 - \alpha_j \eta_{j1}) v_j)} e^{(\eta_{j2} u_j + (2\eta_{j1}\eta_{j2} - \alpha_j \eta_{j2}) v_j)} \\
z_{j2} &= e^{i(\eta_{j1} u_j + (\eta_{j1}^2 - \eta_{j2}^2 - \alpha_j \eta_{j1}) v_j)} e^{-(\eta_{j2} u_j + (2\eta_{j1}\eta_{j2} - \alpha_j \eta_{j2}) v_j)}.
\end{aligned}$$

In particular, we have that the immersion x^{s+1} satisfies the following system of differential equations:

$$\begin{aligned}
x^{s+1}_{w_k w_\ell} &= -i\mu_\ell x^{s+1}_{w_k}, \quad k > \ell \\
x^{s+1}_{w_k w_k} &= -i \sum_{\ell=1}^{k-1} \mu_\ell x^{s+1}_{w_\ell} + (r-k) i\mu_k x^{s+1}_{w_k} + a_{s+1} x^{s+1},
\end{aligned}$$

where

$$a_{s+1} = a_s - 2\lambda_s^3 + 2\alpha_s \lambda_s^2 = a_1 \tfrac{(n+1)}{(r+1)}$$

Now, we have to consider different cases depending on the value of $n - 2s$. If $n - 2s = 0$, in which case by the assumption a_1 is negative and x^{s+1} is a constant vector lying in $S^1(-a_{s+1})$.

If $n - 2s = 1$ and $a_1 < 0$, we have that x^{s+1} defines a horizontal minimal immersion into $S^3(-a_{s+1})$. We introduce a number b such that $b^2 = -a_{s+1}$. It is then well known that x^{s+1} is congruent with

$$x^{s+1}(w_1) = \tfrac{1}{\sqrt{2}} \tfrac{1}{b} (e^{ibw_1}, e^{-ibw_1}).$$

In the case that $n - 2s = 1$ but $a_1 > 0$, we have that x^{s+1} defines a horizontal minimal immersion ino $H^3(-a_{s+1})$. We again introduce a number b such that $b^2 = a_{s+1}$. Then it follows that we can write

$$\begin{aligned}
x^{s+1}(w_1) &= C_{s+1} e^{bw_1} + D_{s+1} e^{-bw_1}, \\
W_1 &= bC_{s+1} e^{bw_1} - bD_{s+1} e^{-bw_1}.
\end{aligned}$$

Hence

$$2bC_{s+1}e^{bw_1} = bx^{s+1} + W_1$$
$$2bD_{s+1}e^{-bw_1} = bx^{s+1} - W_1,$$

from which it follows that C_{s+1} and D_{s+1} are null vectors with $< C_{s+1}, D_{s+1} >= -\frac{1}{2a_{s+1}}$. Therefore, it follows that

$$x^{s+1}(w_1) = \tfrac{1}{b}(\cosh w_1, \sinh w_1).$$

Finally, we consider the case that $n - 2s > 1$. In this case, we know that a, and thus also $a_{s+1} = a_1\frac{n+1}{r+1}$ is negative. We also have that $a_{s+1} = -r\mu_1^2$. In this case, from the system of differential equations satisfied by x^{s+1}, it is clear that x^{s+1} defines a flat minimal horizontal immersion into $S^{2r+1}(\frac{1}{r\mu_1^2})$. It is well known, see [11] that after a change of coordinates such an immersion is congruent with

$$x^{s+1} = \sqrt{\frac{1}{r(r+1)}}\frac{1}{\mu_1}(e^{i\tilde{w}_1}, \dots, e^{i\tilde{w}_r}, e^{-i\tilde{w}_1-\dots-i\tilde{w}_r}).$$

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