Pesin's Formula for Translation Invariant Random Dynamical Systems

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Zusammenfassung

Pesins Formel besagt, dass die metrische Entropie eines dynamischen Systems gleich der Summe seiner positiven Lyapunov Exponenten ist, wobei die metrische Entropie die Chaotizität des Systems beschreibt und Lyapunov Exponenten die asymptotische exponentielle Rate der Trennung benachbarter Trajektorien messen. Es ist bekannt, dass diese Formel für dynamische Systeme auf einer kompakten Riemannschen Mannigfaltigkeit mit invariantem Wahrscheinlichkeitsmaß gilt.

Translationsinvariante Brownsche Flüsse sind eine spezifische Klasse stochastischer Flüsse auf $\mathbb{R}^d$ mit unabhängigen stationären Inkrementen und einer Verteilung, die im Bezug auf Translationen im $\mathbb{R}^d$ unveränderlich ist. Sie haben ein Lyapunov Spektrum, aber kein invariantes Wahrscheinlichkeitsmaß. Wir repräsentieren translationsinvariante Brownsche Flüsse als zufällige dynamische Systemen im Sinne von [18] und [25]. Außerdem definieren wir die Entropie für translationsinvariante (in der Verteilung gegenüber Translationen im $\mathbb{R}^d$) zufällige dynamische Systeme, wobei die Definition auf den Einheitswürfel beschränkt wird. Es stellt sich heraus, dass diese Definition aufgrund der Translationinvarianz der Systeme sinnvoll ist. Danach zeigen wir, dass für translationsinvariante zufällige dynamische Systeme die definierte Entropie kleiner oder gleich der Summe ihrer positiven Lyapunov Exponenten ist. Außerdem legen wir Pesins Formel für den Fall fest, wenn das System das Volumen beibehält. Dies impliziert auch die jeweiligen Ergebnisse für translationsinvariante Brownsche Flüsse.

Wir diskutieren auch einen alternativen Ansatz zur Definition von Entropie. Wir definieren die Entropie für zufällige dynamische Systeme mit festem Ursprung mit Ideen von Brin und Katok, siehe [9]. Danach beweisen wir Ruelles Ungleichung mit dieser Definition, d.h. wir schätzen von oben her die definierte Entropie durch die Summe der positiven Lyapunov Exponenten der Systeme ab. Dies impliziert das jeweilige Ergebnis für translationsinvariante zufällige dynamische Systeme und translationsinvariante Brownsche Flüsse.

Abstract

Pesin's formula asserts that metric entropy of a dynamical system is equal to the sum of its positive Lyapunov exponents, where metric entropy measures the chaoticity of the system, whereas Lyapunov exponents measure the asymptotic exponential rate of separation of nearby trajectories. It is well known, that this formula holds for dynamical systems on a compact Riemannian manifold with an invariant probability measure.

Translation invariant Brownian flows is a specific class of stochastic flows on $\mathbb{R}^d$ with independent and stationary increments and with a distribution, which is invariant with respect to translations in $\mathbb{R}^d$. They have a Lyapunov spectrum but do not have an invariant probability measure. We represent translation invariant Brownian flows as random dynamical systems in the sense of [18] and [25]. Further, we define entropy for translation invariant (in distribution with respect to translations in $\mathbb{R}^d$) random dynamical systems restricting the definition to the unit cube. It turns out that this definition makes sense because of the translation invariance of the systems. After that, we show that for translation invariant random dynamical systems the defined entropy is less then or equal to the sum of their positive Lyapunov exponents. Moreover, we establish Pesin's formula in the case when the system preserves the volume. This also implies the respective results for translation invariant Brownian flows.

We also discuss an alternative approach to the definition of entropy. We define entropy for random dynamical systems with the fixed origin using ideas of Brin and Katok, see [9]. After that we prove Ruelle's inequality with respect to this definition, i.e. we bound from above the defined entropy by the sum of positive Lyapunov exponents of the systems. This implies the respective result for translation invariant random dynamical systems and translation invariant Brownian flows.

Contents

Chapter 1

Introduction

In the thesis we deal with certain random dynamical systems (RDSs) on $\mathbb{R}^d$, which are translation invariant in distribution. We provide a way to define entropy for such systems without assuming the existence of an invariant probability measure. Further, we estimate the defined entropy from below and above in terms of certain local characteristics of the systems that are called Lyapunov exponents. Later in the chapter we provide an introduction to the notions of entropy, Lyapunov exponents, what is known about the estimates (from below and above) in the literature and our results. We start from a motivating example.

One of the essential topics of stochastic analysis is the analysis of stochastic differential equations (SDEs) of the type

$$\phi_{s,t}(x) = x + \int_s^t b(\phi_{s,u}(x))du + \int_s^t \sigma(\phi_{s,u}(x))dW_u, \quad 0 \leq s \leq t,\, x \in \mathbb{R}^d, \tag{1.1}$$

where $W = (W^1, \ldots, W^k)$ denotes a k-dimensional Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ denote appropriate drift and diffusion functions. The existence and uniqueness of solutions of different types of this equation was already studied, see e.g. [15], Chapter IV. Moreover, under some smoothness assumptions on the functions b and σ (see for example [15], Chapter V.2), the solution of the SDE (1.1) generates a stochastic flow of homeomorphisms, that is a family $\{\phi_{s,t} : s, t \in [0,\infty)\}$ of random diffeomorphisms on $\mathbb{R}^d$ that satisfies almost surely

i) $\phi_{u,t} \circ \phi_{s,u} = \phi_{s,t}$ for all $s, t, u \in [0,\infty)$;

ii) $\phi_{s,s} = \mathbf{id}|_{\mathbb{R}^d}$ for all $s \in [0,\infty)$;

iii) $(s, t, x) \mapsto \phi_{s,t}$ is continuous.

However, it turns out that not every stochastic flow is generated by an SDE of the type (1.1). Roughly speaking, some of them involve too much randomness for only finitely many Brownian motions. An example is translation invariant Brownian flows (TIBFs) that will be introduced in the next chapter. However, one can observe a one-to-one correspondence between the solution of SDEs and stochastic flows, considering the definition of SDEs in the sense of Kunita [20]. He introduced a more general class of SDEs (see Section 2.1):

$$\phi_{s,t}(x) = x + \int_s^t F(\phi_{s,u}(x), du) \quad 0 \leq s \leq t,\, x \in \mathbb{R}^d,$$

where $F : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ is a continuous semimartingale field (see Section 2.1). Kunita [20] proved that there is a one-to-one correspondence between the solutions of SDEs of Kunita-type and stochastic flows of homeomorphisms. We will state some of these results in Section 2.1.

An important class of stochastic flows, which will be the focus of interest, are translation invariant Brownian flows (introduced in Section 2.1). These stochastic flows have the additional property that the homeomorphisms on disjoint time intervals are independent, their distributions are homogeneous in time and invariant under translations in space. A particular subclass of TIBFs which is called isotropic Brownian flows (that are additionally invariant in distribution with respect to rotations) was extensively studied in the 1980s by Le Jan [23] and Baxandale and Harris [5]. In particular, they have calculated the Lyapunov exponents of these flows in terms of the isotropic covariance function. Lyapunov exponents describe the exponential rate of separation in a certain (usually random) direction of infinitesimally close trajectories. These exponents crucially affect the global behaviour of the flow. Existence of a finite Lyapunov spectrum (which means that all the exponents are finite) intuitively tells us that the flows are not too chaotic. Indeed, there are a lot of results in the literature which bound chaoticity of certain systems from above and even measure it in terms of Lyapunov exponents. In fact, it often happens that a smooth dynamical system (DS) has entropy, which is equal to the sum of positive Lyapunov exponents of the system. Usually, these results are called Ruelle's inequality if we measure entropy from above by this sum, and Pesin's formula if we prove the equality. Such results will be discussed later in the introduction. However, it turns out that we can not define entropy for TIBFs because they have no invariant probability measure, which is a crucial restriction in the classical definition of entropy. As we will see later, it turns out that one can define entropy for the flows using their translation invariance and then prove the analogues of Ruelle's inequality and Pesin's formula for TIBFs. Now let us finish the discussion of the motivating example and provide more details of the main objects and results of the thesis.

The standard quantity to measure chaoticity or uncertainty is the notion of entropy. For dynamical systems, one can consider the notion of the so-called metric entropy (or sometimes called Kolmogorov-Sinaĭ entropy). It was introduced by Kolmogorov [19] and Sinaĭ [39], and later was studied by many authors, for example [6], [34], [30], [44], First of all, let us explain the meaning of entropy is for a deterministic dynamical system with an invariant probability measure. The entropy of such a system, given a partition of the space, is the asymptotic exponential rate of yes-no questions (with respect to the invariant probability measure) necessary to encrypt the trajectory of a particle evolving with this system with respect to the partition. Taking the supremum over all appropriate partitions then provides the entropy of the system.

Now let us introduce the notion of random dynamical systems. A random dynamical system is the discrete evolution process generated by the composition of independent, identically distributed (i.i.d.) random diffeomorphisms acting on some state space. This notion follows [18] and [25], that studied these systems on a compact state space. We will see that stochastic flows with independent and stationary increments after discretization in time can be seen as such random dynamical systems, see Section 2.2. Let us remark that Arnold introduced in [1] (see Section 1.1.1) a more general class of random dynamical systems. It has been shown by Arnold and Scheutzow [2] that under some mild assumptions there exists even a one-to-one correspondence between RDSs in the sense of [1] and stochastic flows. However, the independence of increments of stochastic flows is essential for us, so we stick to a more restrictive notion of RDSs from [18] and [25].

In the thesis, all the main results are represented for RDSs. In Chapter 2 we provide a way to represent TIBFs as translation invariant random dynamical systems (TIRDSs) (i.e. RDSs which are invariant in distribution with respect to translations). Further, it turns out that all the main results for RDSs can be translated to TIBFs, see Corollary 4.1.1, Corollary 5.1.1 and Remark 6.1.1.

Kifer [18] extended the notion of entropy to random dynamical systems: a probability measure is said to be invariant for RDS if the average over all possible diffeomorphisms preserves the measure. Hence, entropy of a RDS given a partition of the state space is defined as for deterministic DSs, but the number of yes-no questions is additionally averaged with respect to randomness. Again taking supremum over all appropriate partitions yields the entropy of the RDS (see [18], Section 2.1). Thus, entropy can be seen as a description of the chaotic behaviour of typical random trajectories generated by the system. However, TIRDSs have no invariant probability measure, which is essential for the definition of entropy. To resolve this problem, we repeat the arguments of Kifer, but consider only periodic partitions and observe only the dynamics in the unit cube $[0, 1)^d$. It turns out that because of translation invariance of the systems we can in a similar way define the notion of entropy and enjoy its properties such as scalability in time or stability with respect to the sequence of approximating partitions (see Lemma 3.3.5 and Lemma 3.3.6), which we need to prove further results. We mainly follow here [18], Section 2.1. The details can be found in Chapter 3.

Alternatively to the notion of entropy, one can measure chaoticity of a DS on the local level defining the notion of Lyapunov exponents. These values intuitively provide the rate of separation of infinitesimally close trajectories. More precisely, Lyapunov exponents provide the exponential rate of growth of the derivative of the composed maps of the DS. There are two famous formulas relating entropy with the Lyapunov exponents of DSs. They are called Ruelle's inequality and Pesin's formula. We provide a brief introduction to these two formulas in the next two paragraphs.

Ruelle's inequality (or sometimes called Margulis-Ruelle inequality) states that metric entropy of a (random) dynamical system is bounded from above by the sum of its positive Lyapunov exponents. The first result of this sort was obtained for C^1 maps by Ruelle [35]. The first formulation for RDSs appears in [18] (Theorem V.1.4). This proof contained a mistake, and later Liu and Qian ([25])

and Bahnmüller and Bogenschütz ([4]) independently provided the corrections to the proof. Later van Bargen [42] and Biskamp [8] proved Ruelle's inequality for certain RDSs on $\mathbb{R}^d$. However, they still imposed the existence of an invariant probability measure for the RDSs. It turns out that our definition of entropy lets us essentially repeat the proof of Ruelle's inequality from [42] to respectively obtain Ruelle's inequality for TIRDSs. The details can be found in Chapter 4.

Pesin's formula asserts that the entropy of a dynamical system equals the sum of its positive Lyapunov exponents. Hence, Pesin's formula is an improvement of Ruelle's inequality. This remarkable formula was first established for deterministic DSs on a compact Riemannian manifold preserving a smooth measure (see [31], [32] and [33]). For some cases, it was generalized to deterministic DSs that preserve only a Borel measure (see [36], [13]) and to DSs with singularities, see [17]. The first result for RDSs was obtained by Ledrappier and Young [22]. Let us note that Pesin's formula typically requires more regularity then Ruelle's inequality. For example, let us compare the first results in this direction, obtained by Ruelle [35] and Pesin [32]. Both results concern deterministic dynamical systems on a compact Riemannian manifold. However, Ruelle's inequality and Pesin's formula require C^1 and C^2 smoothness respectively. That is a typical situation, i.e. Pesin's formula is a stronger result, which however holds for a smaller class of systems. It turns out that our definition of entropy lets us apply Mañé's ideas, see [26] to prove Pesin's formula for volume preserving TIRDSs. The details can be found in Chapter 5.

Another approach to define entropy for TIRDSs is to use another notion of entropy, which appears in the literature, and to connect it with Pesin's formula. Perhaps the most natural idea in this direction is to use Brin and Katok's definition of local entropy, that defined a way to locally measure chaoticity of a deterministic DS on a compact metric space. They define local entropy in the following way. For a given point x they consider the Bowen ball with radius r around the point, i.e. the set of points that stay with the trajectory of x during first n iterations of the DS. Then they measure the exponential rate of decay of measures of such sets in terms of $\liminf$ and $\limsup$. Finally, it turns out that adding additional limit in space, i.e. when $r \to 0+$, they obtain the same limit, which coincides a.e. with the Kolmogorov-Sinaĭ entropy, see [9]. That shows that local entropy and Kolmogorov-Sinaĭ entropy are (at least in some cases) similar objects, so one can try to use local entropy for the definition of entropy for TIRDSs.

Formally we can in the same way define "lower" local entropy, which corresponds to $\liminf$ and "upper" local entropy, which corresponds to $\limsup$. However, the lack of compactness and the absence of an invariant probability measure do not give us a chance to apply Brin and Katok's ideas. It is even unclear if the "lower" and "upper" local entropies coincide. However, it turns out that for a RDS with the fixed origin we can estimate the defined local entropy, which corresponds to $\limsup$, from above by the sum of positive Lyapunov exponents, obtaining some analogue of Ruelle's inequality for the RDSs, see Theorem 6.1.1. Further, this theorem implies the respective result for TIRDSs, see Corollary 6.1.1, and also for TIBFs, see Remark 6.1.1. Surprisingly enough, we can apply some ideas from Mañé's paper, see [26], even though in the paper he estimates entropy from below by the sum of the positive Lyapunov exponents. The details can be found in Chapter 6.

To the knowledge of the author, this is the first case of a direct connection between Brin-Katok entropy and Lyapunov exponents, without using metric entropy. Note that Duc and Siegmund (see [12]) defined local metric entropy for certain dynamical systems and directly connected it with their Lyapunov exponents. However, they defined it only for systems with finite time horizon, and it turns out that their approach can not be applied to our case.

Chapter 2

Preliminaries

In this chapter we will provide an introduction to stochastic flows in the sense of Kunita [20]. In particular, we will state the main definitions and some previous results we will use in the thesis. We also provide a brief introduction to random dynamical systems and Lyapunov exponents.

The chapter is organized as follows. In Section 2.1.1 we define the notions of driving fields and local characteristics. In Section 2.1.2 we introduce Kunita-type integrals. In Section 2.1.3 we define stochastic flows and TIBFs. In Section 2.1.4 we state the representation theorems for stochastic flows via stochastic differential equations of a Kunita-type. In Section 2.1.5 we obtain certain integrability and regularity properties of TIBFs. In Section 2.2 we give a short introduction to random dynamical systems and describe how TIBFs can be seen as such an evolution process. In Section 2.3 we establish Lyapunov spectrum for RDSs and TIBFs.

2.1 Stochastic Flows

In this section we give an introduction to stochastic flows, following mainly [20], Chapters 3 and 4, and [7], Chapter 2.

2.1.1 Driving Fields and Local Characteristics

We provide a brief introduction to driving fields and local characteristics following mainly [20], Section 3.1, and [7], Section 2.2.1.

For $m \in \mathbb{N}_0$ we denote by C^m the set of m-times continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}^d$. In the case $m = 0$ we will often denote C^0 by C. For $f \in C^m$ define

$$\|f\|_m := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{1+|x|} + \sum_{1\le|\alpha|\le m} \sup_{x\in\mathbb{R}^d} |D^\alpha f(x)|,$$

and denote $C_b^m := \{f \in C^m : \|f\|_m < \infty\}$. Then C_b^m with the norm $\|\cdot\|_m$ is a Banach space. For $\delta \in (0, 1]$ we denote by $C^{m,\delta}$ the set of functions $f \in C^m$ such that $D^\alpha f$ for $|\alpha| = m$ are δ-Hölder continuous. Introducing for $f \in C^m$

$$\|f\|_{m+\delta} := \|f\|_m + \sum_{|\alpha|=m} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\delta}$$

the space $C_b^{m,\delta} := \{f \in C^m : \|f\|_{m+\delta} < \infty\}$ with the norm $\|\cdot\|_{m+\delta}$ is again a Banach space.

We say that a continuous function $f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$; $(x,t) \mapsto f(x,t)$ belongs to $C_b^{m,\delta}$ if $f(t) \equiv f(\cdot,t)$ is an element of $C_b^{m+\delta}$ for any $t \in [0,\infty)$ and for any $T < \infty$

$$\int_0^T \|f(t)\|_{m+\delta} dt < \infty.$$

If $\|f(t)\|_{m+\delta}$ is uniformly bounded in t then we say that f belongs to the class $C_{ub}^{m,\delta}$.

Now let us denote for $m \in \mathbb{N}_0$ the space $\tilde{C}^m$ which consists of functions $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ that are m-times continuously differentiable with respect to each spatial variable. For $g \in \tilde{C}^m$ define

$$\|g\|_m^\sim := \sup_{x,y\in\mathbb{R}^d} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x\in\mathbb{R}^d} |D_1^\alpha D_2^\alpha g(x,y)|$$

and for $\delta \in (0,1]$

$$\|g\|_{m+\delta}^\sim := \|g\|_m^\sim + \sum_{1\leq|\alpha|\leq m} \sup_{x\in\mathbb{R}^d} |D_1^\alpha D_2^\alpha g\|_\delta^\sim,$$

where

$$\|g\|_\delta^\sim := \sup_{x\neq x', y\neq y'} \frac{|g(x,y) - g(x',y) - g(x,y') + g(x',y')|}{|x-x'|^\delta |y-y'|^\delta}.$$

Then we can define

$$\tilde{C}_b^m := \left\{ g \in \tilde{C}^m : \|g\|_m^\sim < \infty \right\}$$

and

$$\tilde{C}_b^{m,\delta} := \left\{ g \in \tilde{C}^m : \|g\|_{m+\delta}^\sim < \infty \right\}.$$

We say that a continuous function $g : \mathbb{R}^d \times \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$; $(x,y,t) \mapsto g(x,y,t)$ belongs to $\tilde{C}_b^{m,\delta}$, if $g(t) \equiv g(\cdot,\cdot,t)$ is an element of $\tilde{C}_b^{m,\delta}$ for any $t \in [0,\infty)$ and for any $T < \infty$

$$\int_0^T \|g(t)\|_{m+\delta}^\sim dt < \infty.$$

If additionally $\|g(t)\|_{m+\delta}^\sim$ is uniformly bounded in t then we say that g belongs to the class $C_{ub}^{m,\delta}$.

Let us now consider a family $\{F(x,t)\}_{t\geq 0}$ of $\mathbb{R}^d$-valued continuous semimartingales, where $x \in \mathbb{R}^d$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Further, consider the canonical decomposition of the semimartingale

$$F(x,t) = M(x,t) + V(x,t)$$

into a local martingale $M(x,t)$ and a process $V(x,t)$ of locally bounded variation. The process $F(x,t)$ is called a continuous semimartingale with values in $C^{m,\delta}$ (or simply a continuous $C^{m,\delta}$-semimartingale) if $t \mapsto M(x,t)$ is a continuous local martingale with values in $C^{m,\delta}$ (or simply a continuous $C^{m,\delta}$-local martingale) and $V(x,t)$ is a continuous $C^{m,\delta}$ process, such that $D_x^\alpha V(x,t)$, $|\alpha| \leq m$ are all processes of locally bounded variation.

We assume that there exists a covariance function $a : \mathbb{R}^d \times \mathbb{R}^d \times [0,+\infty) \times \Omega \to \mathbb{R}^{d\times d}$ and a drift function $b : \mathbb{R}^d \times [0,+\infty) \times \Omega \to \mathbb{R}^d$ such that

$$\langle M_i(x,\cdot), M_j(y,\cdot)\rangle_t = \int_0^t a_{i,j}(x,y,u)du, \qquad V_i(x,t) = \int_0^t b_i(x,u)du,$$

where $\langle\cdot,\cdot\rangle$ denotes the quadratic variation process at time t. We call the pair (a,b) the local characteristics of the family of semimartingales $F(x,t)$, $x \in \mathbb{R}^d$. Also a and b are called infinitesimal covariance and infinitesimal mean of the family of semimartingales $F(x,t)$ respectively.

The infinitesimal covariance $a(x,y,t)$ is said to belong to the class $B_b^{m,\delta}$ if $a(x,y,t)$ has a modification that is a predictable process with values in $\tilde{C}_b^{m,\delta}$ and for all $T < \infty$

$$\int_0^T \|a(t)\|_{m,\delta}^{\sim} dt < \infty \quad \mathbb{P}\text{-almost surely.} \tag{2.1}$$

Analogously, the infinitesimal mean $b(x,t)$ belongs to $B_b^{m',\delta'}$ if $b(x,t)$ has a modification that is a predictable process with values in $C_b^{m',\delta'}$ and for all $T < \infty$

$$\int_0^T \|b(t)\|_{m',\delta'} dt < \infty \quad \mathbb{P}\text{-almost surely.} \tag{2.2}$$

In this case we say the pair (a,b) belongs to the class $(B_b^{m,\delta}, B_b^{m',\delta'})$. The pair (a,b) belongs to the class $(B_{ub}^{m,\delta}, B_{ub}^{m',\delta'})$ if (2.1) is replaced by

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{0\leq t\leq T} \|a(t)\|_{m+\delta}^{\sim} < \infty$$

and (2.2) by

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{0\leq t\leq T} \|b(t)\|_{m'+\delta'}^{\sim} < \infty.$$

If $m = m'$ and $\delta = \delta'$ the pair (a,b) is said to belong to the class $B_b^{m,\delta}$ (or $B_{ub}^{m,\delta}$). We simply write $F \in B_b^{m,\delta}$ (or $F \in B_{ub}^{m,\delta}$) to indicate that the local characteristics of the semimartingales $F(x,t), x \in \mathbb{R}^d$ belong to the class $B_b^{m,\delta}$ (or $B_{ub}^{m,\delta}$).

2.1.2 Kunita-Type Integrals

We mainly follow here [20], Section 3.2, and [7], Section 2.2.1.

Let $F(x,t)$, $x \in \mathbb{R}^d$ be a family of continuous C-martingales such that local characteristics (a,b) belongs to the class $B_b^{0,\delta}$ for some $\delta > 0$. Further, let $\{f_t\}_{t\geq 0}$ be a predictable $\mathbb{R}^d$-valued process such that for all $T < \infty$ $\mathbb{P}$-almost surely

$$\int_0^T a(f_s, f_s, s)ds < +\infty, \qquad \int_0^T b(f_s, s)ds < +\infty. \tag{2.3}$$

If f is a simple process, i.e. there exists $n \in \mathbb{N}$, $0 = t_0 < \ldots < t_n < +\infty$ and functions $f_{t_i} \in C$, $0 \leq i \leq n$ satisfying

$$f_t = \sum_{i=0}^{n-1} f_{t_i} 1_{[t_i, t_{i+1})}(t) + f_{t_n} 1_{[t_n, +\infty)}(t),$$

then the Itô-Kunita stochastic integral of f with respect to the local martingale field $M(x,t)$ is defined in the following way

$$\int_0^t M(f_s, ds) := \sum_{i=0}^{n} \{M(f_{t_i \wedge t}, t_{i+1} \wedge t)\} - M(f_{t_i \wedge t}, t_i \wedge t)\}.$$

Let now f_t be a general predictable process that satisfies (2.3). Then there exists a Cauchy-sequence $\{f^n\}$ of simple predictable processes such that for any $m, n \to \infty$ and $T < \infty$ we have $\mathbb{P}$-almost surely

$$\int_0^T a(f_s^n, f_s^n, s) - 2a(f_s^n, f_s^m, s) + a(f_s^m, f_s^m, s)ds \to 0.$$

Further, one can show (see [20], Section 3.2) that the sequence $\left\{ \int_0^t M(f_s^n, ds) \right\}_n$ converges uniformly in t on compact subsets of $[0,\infty)$ in probability. The limit is called the Itô-Kunita stochastic integral of f with respect to the local martingale field $M(x,t)$ and is denoted by $\int_0^t M(f_s, ds)$. Thus the Itô-Kunita stochastic integral of f with respect to the semimartingale field $F(x,t)$ is defined by its canonical decomposition, i.e. for any $T < \infty$

$$\int_0^T F(f_s, ds) := \int_0^T M(f_s, ds) + \int_0^T b(f_s, s)\, ds.$$

Note that analogously one can define a Stratonovich-Kunita integral (see [20], Section 2.3).

2.1.3 Stochastic Flows and Translation Invariant Brownian Flows

We provide a brief introduction to stochastic flows and TIBFs following mainly [20], Section 4.1, and [7], Section 2.2.1.

First of all we define the notion of a stochastic flow.

Definition 2.1.1. *A family of random homeomorphisms $\{\phi_{s,t} : s, t \in [0, \infty)\}$ on $\mathbb{R}^d$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a stochastic flow of homeomorphisms if almost surely*

i) $\phi_{s,t} = \phi_{u,t} \circ \phi_{s,u}$ for all $s, t, u \in [0, \infty)$;

ii) $\phi_{s,s} = \mathbf{Id}|_{\mathbb{R}^d}$ for all $s \in [0, \infty)$;

iii) $(s, t, x) \mapsto \phi_{s,t}(x)$ is continuous.

It is called a stochastic flow of C^k-diffeomorphisms, if additionally almost surely

iv) $\phi_{s,t}(x)$ is k times differentiable with respect to x for all $s, t \in [0, \infty)$ and the derivatives are continuous in (s, t, x).

Properties $i)$ and $ii)$ immediately imply that $\phi_{s,t}(\omega)^{-1}$ is given by $\phi_{t,s}(\omega)$. This fact together with condition $iii)$ yields that $\phi_{s,t}(\omega)^{-1}(x)$ is also continuous in (s, t, x). Condition $iv)$ shows that $\phi_{s,t}(\omega)^{-1}(x)$ is k times continuously differentiable with respect to x. Therefore $\phi_{t,s}(\omega)$ is indeed a C^k-diffeomorphism for all $s, t \in [0, \infty)$.

Let us denote by G the set of homeomorphisms on $\mathbb{R}^d$. This set forms a group with respect to the composition of maps. Further, it can be equipped with the metric

$$d_0(\phi, \phi') := \rho(\phi, \phi') + \rho(\phi^{-1}, (\phi')^{-1})$$

where

$$\rho(\phi, \phi') := \sum_{N \geq 1} 2^{-N} \frac{\sup_{|x| \leq N} |\phi(x) - \phi'(x)|}{1 + \sup_{|x| \leq N} |\phi(x) - \phi'(x)|}.$$

The metric ρ induces the so called topology of uniform convergence on compact sets. Then the set (G, d_0) is a complete separable topological group. A stochastic flow of homeomorphisms can be regarded as a G-valued continuous random process with index set $[0, \infty) \times [0, \infty)$ which satisfies $i)$ and $ii)$. We call it a stochastic flow with values in G.

For a multi index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_i \in \mathbb{N}_0$, $i = 1, \dots, d$ we denote $|\alpha| := \sum_{i=1}^d |\alpha_i|$. Further, we denote the spatial partial differential operator with respect to index α by D^α. More precisely

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Finally, denote by $D_x f$ the differential of a function f evaluated at point $x \in \mathbb{R}^d$.

Let $G^k \subset G$ be the set of all C^k-diffeomorphisms on $\mathbb{R}^d$. It is a subgroup of G, and moreover, it is again a complete separable topological group with respect to the metric

$$d_k(\phi, \phi') := \sum_{|\alpha| \le k} \rho(D^\alpha \phi, D^\alpha \phi') + \sum_{|\alpha| \le k} \rho(D^\alpha \phi^{-1}, D^\alpha (\phi')^{-1}).$$

A stochastic flow of C^k-diffeomorphisms can be seen as a G^k-valued continuous random process with index set $[0, \infty) \times [0, \infty)$ satisfying properties $i)$ and $ii)$. Analogously, we call it a stochastic flow with values in G^k.

Now let us define the class of translation invariant Brownian flows.

Definition 2.1.2. *A stochastic flow ϕ with values in G^2 is called*

i) a Brownian flow if any $n \in \mathbb{N}$, $0 \le t_0 < \ldots \le t_n < \infty$ the random variables $\{\phi_{t_{i-1}, t_i}\}_{i=\overline{1,n}}$ are independent;

ii) a homogeneous Brownian flow, if additionally for any $h \ge 0$ the laws of $\{\phi_{s,t} : 0 \le s \le t < \infty\}$ and $\{\phi_{s+h,t+h} : 0 \le s \le t < \infty\}$ coincide;

iii) a translation invariant stochastic flow, if the distributions of $\phi_{s,t}(\cdot + a)$ and $\phi_{s,t} + a$ coincide for all $s, t \in \mathbb{R}_+$ and $a \in \mathbb{R}^d$;

iv) a translation invariant Brownian flow if conditions i)-iii) are satisfied and ϕ is a solution of a SDE

$$\phi_{s,t}(x) = x + \int_s^t F(du, \phi_{s,u}(x)), \quad \text{for all } t \ge s \ge 0.$$

where $F(t, x, \omega) : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a continuous semimartingale with values in C with $F \in B^{2,1}_{ub}$.

Note that TIBFs were already discussed in [37], see Remark on p. 50. In that article condition $iv)$ was not included in the definition. In fact, there was a similar restriction, which however imposed less regularity on the infinitesimal mean and covariance. Here we use a stronger condition $iv)$ to be able to obtain Ruelle's inequality and Pesin's formula for TIBFs, see Corollary 4.1.1, Corollary 5.1.1 and Remark 6.1.1.

2.1.4 Representation of Stochastic Flows

In Section 2.1.3 TIBFs are defined as solutions of certain Kunita-type SDEs, which seems to be a serious restriction. However, it turns out that this is not the case thanks to the results in [20]. Now let us provide more details. We follow here [20], Chapter 4, and [7], Section 2.2.1.

In this section we discuss the connection between stochastic flows and SDEs of the type

$$dX_t = F(X_t, dt), \quad t \ge s \ge 0, \tag{2.4}$$

where s is a fixed positive number and F is a semimartingale field.

For fixed $s \in [0,\infty)$ and $x \in \mathbb{R}^d$ a continuous $\mathbb{R}^d$-valued process $\phi_{s,t}(x)$, $0 \le s \le t < \infty$ adapted to $\{\mathcal{F}_t\}$ is called a solution of SDE (2.4) starting at time s in point x if it satisfies

$$\phi_{s,t}(x) = x + \int_s^t F(\phi_{s,u}(x), du), \quad \text{for all } t \ge s. \tag{2.5}$$

The existence and uniqueness of a solution is shown in [20], Theorem 3.4.1:

Theorem 2.1.1. *Let $F(x,t)$ be a continuous semimartingale with values in C with local characteristics belonging to the class $B_b^{0,1}$. Then for each s and x the equation (2.5) has a unique solution.*

Consider a stochastic flow $\{\phi_{s,t} : s,t \in [0,\infty)\}$ with values in G^k, $k \in \mathbb{N}_0$. Let $\{\mathcal{F}_{s,t} : 0 \le s \le t < \infty\}$ be the filtration generated by the flow, which is for $s < t$ the least σ-algebra $\mathcal{F}_{s,t}$ containing all null sets and $\cap_{\epsilon>0}\sigma(\phi_{u,v} : s-\epsilon \le u \le v \le t+\epsilon)$. The forward part $\{\phi_{s,t} : 0 \le s \le t < \infty\}$ is called a forward $C^{k,\delta}$-semimartingale flow, if for every s the stochastic process $\{\phi_{s,t} : 0 \le s \le t < \infty\}$ is a continuous $C^{k,\delta}$-semimartingale adapted to $\{\mathcal{F}_{s,t} : t \in [s,\infty)\}$.

Then any sufficiently smooth forward semimartingale flow the following result (see [20], Theorem 4.4.1) provides the existence and uniqueness of a continuous semimartingale field satisfying (2.4):

Theorem 2.1.2. *Let $\{\phi_{s,t} : 0 \le s \le t < \infty\}$ be a forward $C^{k,\delta}$-semimartingale flow for some $k \ge 0$ and $\delta > 0$ such that for every s the local characteristics belongs to the class $B_b^{k,\delta}$. Then there exists a unique continuous $C^{k,\epsilon}$-semimartingale $F(x,t)$ with $F(x,0) = 0$ (for all $\epsilon < \delta$) with local characteristics belonging to the class $B_b^{k,\delta}$ such that for each s and x the process $\{\phi_{s,t}, t \in [s,\infty)\}$ satisfies (2.5).*

Proof. See [20], Theorem 4.4.1. □

On the other hand the following statement (see [20], Theorem 4.6.5) yields that under certain smoothness assumptions on a semimartingale F there exists a solution of SDE (2.4), which forms a forward stochastic flow of diffeomorphisms.

Theorem 2.1.3. *Let $F(x,t)$ be a continuous C-semimartingale whose local characteristics belongs to the class $B_b^{k,\delta}$ for some $k \ge 1$ and $\delta > 0$. Then the solution of the stochastic differential equation (2.4) based on F has a modification $\{\phi_{s,t} : 0 \le s \le t < \infty\}$ such that it is a forward stochastic flow of C^k-diffeomorphisms. Further, it is a forward $C^{k,\epsilon}$-semimartingale for any $\epsilon < \delta$.*

Theorem 2.1.2 and Theorem 2.1.3 provide the correspondence between stochastic flows and semimartingale fields by the SDE (2.4). Note that in [20] all the above is done only on a finite time interval, i.e. when $0 \le s \le t \le T$ for some fixed $T < +\infty$. However, a standard localizing argument for local martingales provides the results as stated above.

2.1.5 Regularity Properties of Translation Invariant Brownian Flows

The aim of the section is to show that TIBFs fulfill certain regularity assumptions, that we will use later to obtain the main results.

The following result provides integrability of derivatives of ϕ. Note that a similar result with also a similar proof was established in [8], Section 9, pp. 140-141.

Lemma 2.1.1. *Let ϕ be a TIBF. Then we have*

$$\int \log^+ \sup_{v \in B(0,1)} \|D_v \phi_{0,n}\| \, d\mathbb{P} < \infty, \quad \forall n \in \mathbb{N}, \tag{2.6}$$

$$\int \log^+ \sup_{v \in B(0,1)} \left\|D_v(\phi_{0,n}^{-1})\right\| d\mathbb{P} < \infty, \quad \forall n \in \mathbb{N}, \tag{2.7}$$

and

$$\int \log^+ \sup_{v \in B(0,1)} \left\|D_v^2 \phi_{0,n}\right\| d\mathbb{P} < \infty, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Proof. First of all, let us quote the following result by Imkeller and Scheutzow (see [16], Theorem 2.2). For the sake of convenience we formulate it for TIBFs.

Theorem 2.1.4. *Let ϕ be a TIBF. Then for all $T \geq 0$, there exist $c, \gamma > 0$ such that for all $1 \leq |\alpha| \leq 2$ the random variable*

$$Y_\alpha := \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s,t \leq T} \|D_y^\alpha \phi_{s,t}\| \exp\{-\gamma(\log^+ |y|)^{1/2}\}$$

is Φ_c-integrable, where

$$\Phi_c : [0, +\infty) \to [0, +\infty); \quad x \mapsto \int_1^\infty \exp(-ct^2) x^t \, dt.$$

By [16], Lemma 1.1 (left inequality in (4)) we have for $z \geq 1$ the inequality

$$\exp((\log z)^2/4c - (\log K)^2/4c) \leq \Phi_c(z). \tag{2.9}$$

Now let us show (2.8). Fix $n \in \mathbb{N}$. Let α be a multi index with $|\alpha| = 2$. Then Theorem 2.1.4 for $T = n$ and $s = 0$ implies

$$\begin{aligned}
&\int \log^+ \sup_{v \in B(0,1)} \|D_v^\alpha \phi_{0,n}\| \, d\mathbb{P} \leq \int \log^+ Y_\alpha d\mathbb{P} = \int \log(Y_\alpha \vee 1) d\mathbb{P} \\
=&2\sqrt{c} \int \log(Y_\alpha \vee 1)/2\sqrt{c} d\mathbb{P};
\end{aligned}$$

Note that $z \leq \exp\{z^2\}$, and so

$$\begin{aligned}
&2\sqrt{c}\int \log(Y_\alpha \vee 1)/2\sqrt{c}d\mathbb{P} \\
\leq &2\sqrt{c}\int \exp((\log(Y_\alpha \vee 1))^2/4c)d\mathbb{P} \\
\leq &2\sqrt{c}\exp\{(\log K)^2/4c\}\int \exp((\log(Y_\alpha \vee 1))^2/4c - (\log K)^2/4c)d\mathbb{P} \\
\stackrel{(2.9)}{\leq} &2\sqrt{c}\exp\{(\log K)^2/4c\}\int \Phi_c(Y_\alpha \vee 1)d\mathbb{P} \\
\leq &2\sqrt{c}\exp\{(\log K)^2/4c\}\int (\Phi_c(Y_\alpha) + \Phi_c(1))d\mathbb{P} < \infty,
\end{aligned}$$

which completes the proof of (2.8). In the same way one can prove (2.6) and (2.7). Note that in the case of (2.7) we additionally use that $\phi_{0,n}^{-1} = \phi_{n,0}$. □

2.2 Random Dynamical Systems

In this section we introduce the notion of random dynamical systems introduced in [18], Section 1.2 and [25], Chapter 1, §1. We mainly follow here [25], Chapter 1, §1.

It the thesis we deal with random dynamical systems generated by i.i.d. maps. More precisely, a random dynamical system for us always is the discrete-time evolution process generated by superpositions of some random diffeomorphisms on $\mathbb{R}^d$. These diffeomorphisms will be assumed to be i.i.d. according to a certain distribution on the set of diffeomorphisms. Note that, as it was mentioned in the introduction, this view is quite restricted and usually random dynamical systems are defined as in [1], Section 1.1.1.

For the sake of convenience we consider the space of space of possible diffeomorphisms as the initial probability space. More precisely, recall that (see Section2.1.3) the space G^2 is the space of 2-times continuously differentiable diffeomorphisms on $\mathbb{R}^d$. Now denote G^2 by $\tilde{\Omega}$. Then it is (see Section 2.1.3) a complete separable topological group w.r.t. the topology of uniform convergence on compact sets for all derivatives up to order two. Further, denote by $\mathcal{B}(\tilde{\Omega})$ the Borel σ-algebra on $\tilde{\Omega}$. Now fix a probability measure $\tilde{\nu}$ on $\mathcal{B}(\tilde{\Omega})$, according to which we will chose the diffeomorphic maps. Further, let

$$(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{N}}, \tilde{\nu}^{\mathbb{N}}) = \prod_{i=1}^{+\infty}(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), \tilde{\nu})$$

be the infinite product of copies of the probability space $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), \tilde{\nu})$. Denote by $\psi_i : \tilde{\Omega}^{\mathbb{N}} \to \tilde{\Omega}$ the i-th coordinate function on the sequence space $\tilde{\Omega}^{\mathbb{N}}$. Let us define for every $\tilde{\omega} = (\psi_0(\tilde{\omega}), \psi_1(\tilde{\omega})), \ldots) \in \tilde{\Omega}^{\mathbb{Z}}$ and $n \in \mathbb{N}$

$$\begin{aligned}
\psi_{0,\tilde{\omega}} =&\mathbf{Id}|_{\mathbb{R}^d}, \\
\psi_{n,\tilde{\omega}} =&\psi_{n-1}(\tilde{\omega}) \circ \psi_{n-2}(\tilde{\omega}) \ldots \circ \psi_0(\tilde{\omega}).
\end{aligned}$$

The one-sided RDS generated by these composed maps, that is $\{\psi_{n,\tilde{\omega}} : n \in \mathbb{N}_0, \tilde{\omega} \in (\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{N}}, \tilde{\nu}^{\mathbb{N}})\}$, will be referred to as ψ.

Now we define the notion of two-sided RDS. The main difference is that two-sided RDSs are defined also for negative times. Let

$$(\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{Z}}, \tilde{\nu}^{\mathbb{Z}}) = \prod_{i=-\infty}^{+\infty} (\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), \tilde{\nu})$$

be the infinite product of copies of the probability space $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), \tilde{\nu})$. Again denote by $\psi_i : \tilde{\Omega}^{\mathbb{Z}} \to \tilde{\Omega}$ the i-th coordinate function on the sequence space $\tilde{\Omega}^{\mathbb{Z}}$. Let us define for every $\tilde{\omega} = (\ldots, \psi_{-1}(\tilde{\omega}), \psi_0(\tilde{\omega}), \psi_1(\tilde{\omega})), \ldots) \in \tilde{\Omega}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$

$$\begin{aligned}\psi_{0,\tilde{\omega}} =& \mathbf{Id}|_{\mathbb{R}^d},\\ \psi_{n,\tilde{\omega}} =& \psi_{n-1}(\tilde{\omega}) \circ \psi_{n-2}(\tilde{\omega}) \ldots \circ \psi_0(\tilde{\omega}),\\ \psi_{-n,\tilde{\omega}} =& \psi_{-n}^{-1}(\tilde{\omega}) \circ \psi_{-n+1}^{-1}(\tilde{\omega}) \ldots \circ \psi_{-1}^{-1}(\tilde{\omega}).\end{aligned}$$

The two-sided random dynamical system generated by these composed maps, that is $\{\psi_{n,\tilde{\omega}} : n \in \mathbb{Z}, \tilde{\omega} \in (\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{Z}}, \tilde{\nu}^{\mathbb{Z}})\}$, will also be referred to as ψ.

Now we define the notion of invariant measure of RDSs. Intuitively a measure is invariant for RDS if and only if it is preserved under the action of the system on average. Note that the definition is borrowed from [25], Chapter I, Definition 1.1; see also [18], Section 1.2 ($\mathbf{P}^*$-invariance).

Definition 2.2.1. *i) Let ψ be a one-sided RDS defined on a probability space $(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{N}}, \tilde{\nu}^{\mathbb{N}})$. Then a Borel measure $\tilde{\mu}$ on $\mathbb{R}^d$ is called an invariant measure of ψ if*

$$\int \tilde{\mu}(\psi_{1,\omega}^{-1}(A))\, d\tilde{\nu}^{\mathbb{N}} = \tilde{\mu}(A), \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

ii) Let ψ be a two-sided RDS on a probability space $(\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{Z}}, \tilde{\nu}^{\mathbb{Z}})$. Then a Borel measure $\tilde{\mu}$ on $\mathbb{R}^d$ is called an invariant measure of ψ if

$$\int \tilde{\mu}(\psi_{1,\omega}^{-1}(A))\, d\tilde{\nu}^{\mathbb{Z}} = \tilde{\mu}(A), \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

Now define for a two-sided RDS ψ by θ the left shift operator on $\tilde{\Omega}$, namely

$$\psi_n(\theta\tilde{\omega}) = \psi_{n+1}(\tilde{\omega})$$

for all $\tilde{\omega} = (\ldots, \psi_{-1}(\tilde{\omega}), \psi_0(\tilde{\omega}), \psi_1(\tilde{\omega}), \ldots) \in \tilde{\Omega}^{\mathbb{Z}}$. Note that θ is measurable and with a measurable inverse. Moreover, θ is a measure-preserving transformation on $(\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{Z}}, \tilde{\nu}^{\mathbb{Z}})$. Finally, let us note that θ is ergodic, since $\ldots, \psi_{1,\theta^{-1}\omega}, \psi_{1,\omega}, \psi_{1,\theta\omega}, \ldots$ are independent and identically distributed.

Further, for a two-sided RDS ψ define the skew product shift $\Theta : \mathbb{R}^d \times \tilde{\Omega}^{\mathbb{Z}} \to \mathbb{R}^d \times \tilde{\Omega}^{\mathbb{Z}}$ as

$$\Theta(x, \tilde{\omega}) := (\psi_0(\tilde{\omega})x, \theta\tilde{\omega}).$$

Note that Θ is measurable.

In the same way for a one-sided system ψ define the left shift operators θ_+ and the skew product shift Θ_+. Then θ_+ is measurable measure-preserving ergodic transformation on $(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{N}}, \tilde{\nu}^{\mathbb{N}})$, where Θ_+ is measurable.

Now let us define the notion of translation invariant random dynamical system.

Definition 2.2.2. *A (one-sided or two-sided) RDS ψ is called translation invariant random dynamical system (TIRDS) if the distributions of $\psi_{1,\omega}(\cdot + a)$ and $\psi_{1,\omega}(\cdot) + a$ coincide for all $a \in \mathbb{R}^d$.*

To the knowledge of the author this is the first definition of TIRDSs in such a sense. However, translation invarince in the same sense was already defined for Brownian flows, see [37], Remark on p. 50; see also [10], Section 1.2, p.17, property *(iii)*.

An example of TIRDSs is discretized in time TIBFs, see the end of the section.

Now we establish the invariance of the Lebesgue measure for TIRDSs.

Proposition 2.2.1. *Let ψ be a two-sided (one-sided) TIRDS on a probability space $(\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{Z}}, \tilde{\nu}^{\mathbb{Z}})$ $((\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{N}}, \tilde{\nu}^{\mathbb{N}}))$. Then the Lebesgue measure μ is invariant for ψ.*

Proof. We prove the lemma only for two-sided TIRDSs. The proof for one-sided TIRDSs is the same. We have

$$\begin{aligned}
\int \mu(\psi_{1,\omega}^{-1}(A))\, d\tilde{\nu}^{\mathbb{Z}} &= \int \int 1_{\psi_{1,\omega}(x) \in A}(x)\, d\mu(x)\, d\tilde{\nu}^{\mathbb{Z}} \\
&= \int \int 1_{\psi_{1,\omega}(x) \in A}(x)\, d\tilde{\nu}^{\mathbb{Z}}\, d\mu(x) \\
&= \int \tilde{\nu}^{\mathbb{Z}}(\psi_{1,\omega}(x) \in A)\, d\mu(x) \\
&= \int \tilde{\nu}^{\mathbb{Z}}(\zeta_x \in A - x)\, d\mu(x),
\end{aligned}$$

where $\zeta_x := \psi_{1,\omega}(x) - x$. Now because of translation invariance of ψ, the distribution of ζ_x does not depend on x, and therefore

$$\begin{aligned}
\int \tilde{\nu}^{\mathbb{Z}}(\zeta_x \in A - x)\, d\mu(x) &= \int \tilde{\nu}^{\mathbb{Z}}(\zeta_0 \in A - x)\, d\mu(x) \\
&= \int \int 1_{\zeta_0 \in A - x}(x)\, d\tilde{\nu}^{\mathbb{Z}}\, d\mu(x) \\
&= \int \int 1_{x \in A - \zeta_0}(x)\, d\mu(x)\, d\tilde{\nu}^{\mathbb{Z}} \\
&= \int \mu(A - \zeta_0)\, d\tilde{\nu}^{\mathbb{Z}} = \mu(A),
\end{aligned}$$

as required. □

Now we formulate a consequence of Proposition I.2.3 from [18], but for TIRDSs.

Proposition 2.2.2. *Let ψ be a one-sided TIRDS on a probability space $(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega})^{\mathbb{N}}, \tilde{\nu}^{\mathbb{N}})$. Then the measure $\mathbf{M}^{+} := \mu \times \tilde{\nu}^{\mathbb{N}}$ (defined on $\mathbb{R}^d \times \tilde{\Omega}^{\mathbb{N}}$) is invariant for Θ_{+}.*

Proof. See [18], Proposition I.2.3 together with Proposition 2.2.1. □

Our aim now is to construct from a homogeneous Brownian flow ϕ a translation invariant random dynamical system φ, such that $\{\varphi_{n,\theta^m\omega}, m, n \in \mathbb{N}_0\}$ coincides in distribution with $\{\phi_{m,m+n}, m, n \in \mathbb{N}_0\}$.

Define

$$\hat{\Omega} = G^2; \quad \hat{\nu}(A) = \mathbb{P}(\phi_{0,1} \in A),$$

where $A \in \mathcal{B}(G^2)$. The procedure from the beginning of the section generates the triple $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$ and the corresponding one-sided random dynamical system, denote it by φ. Then it is easy to check that indeed

$$\{\varphi_{n,\theta^m\omega}, m, n \in \mathbb{N}_0\} \stackrel{d}{=} \{\phi_{m,m+n}, m, n \in \mathbb{N}_0\}.$$

Alternatively, on the last step of the described procedure we generate the triple $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$ and the corresponding two-sided random dynamical system, denote it also by φ. In any case if ϕ is a TIBF, then φ is a TIRDS. In the thesis we will use these procedures only in Corollary 4.1.1, which provides Ruelle's inequality for TIBFs, and in Corollary 5.1.1 (only the procedure of generating two-sided system), which provides Pesin's formula for TIBFs.

2.3 Lyapunov Spectrum of Translation Invariant Brownian Flows and Random Dynamical Systems

Let ξ be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the expected value of ξ, denoted by $\mathbb{E}^{\mathbb{P}}\xi$, is defined as the Lebesgue integral

$$\mathbb{E}^{\mathbb{P}}\xi = \int_{\Omega} \xi(\omega) d\mathbb{P}(\omega).$$

From now on we will abbreviate $\mathbb{E}$ instead of $\mathbb{E}^{\mathbb{P}}$ if there is no risk of ambiguity.

Now we state assumptions for a RDS ψ, which correspond to integrability conditions (2.6), (2.7), (2.8).

Assumption 1: ψ is a one-sided (two-sided) RDS on a probability space $(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega}^{\mathbb{N}}), \tilde{\nu}^{\mathbb{N}})$ $((\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega}^{\mathbb{Z}}), \tilde{\nu}^{\mathbb{Z}}))$ and satisfies

$$\mathbb{E} \log^{+} \sup_{v \in B(0,1)} \|D_v \psi_{n,\omega}\| < \infty, \quad \forall n \in \mathbb{N}.$$

Assumption 2: ψ is a one-sided (two-sided) RDS on a probability space $(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega}^{\mathbb{N}}), \tilde{\nu}^{\mathbb{N}})$ $((\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega}^{\mathbb{Z}}), \tilde{\nu}^{\mathbb{Z}}))$ and satisfies

$$\mathbb{E}\log^+ \sup_{v\in B(0,1)} \left\|D_v(\psi_{n,\omega}^{-1})\right\| < \infty, \quad \forall n\in\mathbb{N}.$$

Assumption 3: ψ is a one-sided (two-sided) RDS on a probability space $(\tilde{\Omega}^{\mathbb{N}}, \mathcal{B}(\tilde{\Omega}^{\mathbb{N}}), \tilde{\nu}^{\mathbb{N}})$ $((\tilde{\Omega}^{\mathbb{Z}}, \mathcal{B}(\tilde{\Omega}^{\mathbb{Z}}), \tilde{\nu}^{\mathbb{Z}}))$ and satisfies

$$\mathbb{E}\log^+ \sup_{v\in B(0,1)} \left\|D_v^2\psi_{n,\omega}\right\| < \infty, \quad \forall n\in\mathbb{N}.$$

Now we connect TIBFs with Assumptions 1-3.

Theorem 2.3.1. *Let ϕ be a TIBF and φ be the respective one-sided (two-sided) TIRDS on a probability space $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega}^{\mathbb{N}}), \hat{\nu}^{\mathbb{N}})$ (respectively $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega}^{\mathbb{Z}}), \hat{\nu}^{\mathbb{Z}})$) ,i.e. is constructed as in Section 2.2. Then φ satisfies Assumptions 1-3.*

Proof. Assumption 1, 2 and 3 holds because of (2.6), (2.7), and (2.8) respectively. □

Note that one can construct also other RDSs, that are translation invariant and satisfy Assumptions 1-3. The following example is proposed by M. Scheutzow.

Example. Let a one-sided TIRDS φ on a probability space $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$ corresponds to a TIBF ϕ, i.e. is constructed as in Section 2.2. Recall that $\hat{\nu}(A) = \mathbb{P}(\phi_{0,1}\in A)$, where $A\in\mathcal{B}(\hat{\Omega})$. Now define another measure $\check{\nu}$ on $\hat{\Omega}^{\mathbb{N}}$ in the following way

$$\check{\nu}(A) = \begin{cases} \frac{1}{2}\hat{\nu}(A), & A\in\mathcal{B}(\hat{\Omega}) \text{ and } \mathbf{id}_{\mathbb{R}^d}\notin A,\\ \frac{1}{2}, & \{\mathbf{id}_{\mathbb{R}^d}\} = A.\end{cases}$$

This procedure generates the triple $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \check{\nu}^{\mathbb{N}})$ and the corresponding random dynamical system, denote it by $\check{\varphi}$. Intuitively, one can explain $\check{\varphi}$ in the following way: we consider φ, but before each iteration we toss a symmetric coin. Then we apply the dynamics of φ in the case of heads and force the flow stay the same in the case of tails. Then $\check{\varphi}$ does not correspond to a TIBF, because $\check{\varphi}_{1,\omega}(0)$ is not a normal random variable. However, it is easy to check that $\check{\varphi}$ is translation invariant and satisfies Assumptions 1-3.

Now we state two results, which show the existence of a finite Lyapunov spectrum of one-sided RDSs with the fixed origin and of one-sided TIRDSs respectively. Note that the result below is an analogue of Theorem 1.6 from [35].

Theorem 2.3.2. *Let ψ be a one-sided random dynamical system on a probability space $(\overline{\Omega}^{\mathbb{N}}, \mathcal{B}(\overline{\Omega})^{\mathbb{N}}, \overline{\nu}^{\mathbb{N}})$, which satisfies Assumption 1, and also has the fixed origin, i.e. $\psi_{1,\omega}(0)=0$, $\forall\omega$. Then there exist numbers $\lambda_1>\lambda_2>\ldots>\lambda_p$ and a forward θ_+-invariant measurable set $\overline{\Omega}_1^{\mathbb{N}}$ with $\overline{\nu}^{\mathbb{N}}(\overline{\Omega}_1^{\mathbb{N}})=1$, such that for all $\omega\in\overline{\Omega}_1^{\mathbb{N}}$ there exists a measurable splitting*

$$\mathbb{R}^d = E_1^{\psi}(\omega)\oplus\ldots\oplus E_p^{\psi}(\omega)$$

of $\mathbb{R}^d$ *over* $\overline{\Omega}_1^{\mathbb{N}}$ *into random subspaces* $E_i^{\psi}(\omega)$ *(so-called Oseledets spaces) with dimension* $\dim E_i^{\psi}(\omega) = d_i$*,* $i = \overline{1,p}$ *(so-called Oseledets splitting) with the following properties*

i) we have

$$\lim_{n\to\infty} \frac{1}{n} \log |(D_0\psi_{0,t})v| = \lambda_i \iff v \in V_i^{\psi}(\omega)\backslash V_{i+1}^{\psi}(\omega),$$

where $V_{p+1} := \{0\}$ *and for* $i = \overline{1,p}$

$$V_i^{\psi}(\omega) := E_p^{\psi}(\omega) \oplus \ldots \oplus E_i^{\psi}(\omega).$$

ii) the subspaces V_i^{ψ} *are* θ_+*-invariant, i.e.* $(D_0\psi_{n,\omega})V_i^{\psi}(\omega) = V_i^{\psi}(\theta_+^n\omega)$*;*

The numbers λ_i *and* d_i *are called Lyapunov exponents of the RDS* ψ *and their multiplicities respectively.*

Proof. Equality i) for ψ holds because of [35], Theorem 1.6, where τ and T should be substituted by θ_+ and $D_0\psi_{1,\omega}$ respectively. Note that the integrability conditions of the theorem hold because of Assumption 1 and ergodicity of θ. Finally, ii) is a trivial consequence of i). □

Note that several statements below in this section also provide numbers λ_i and d_i. In every case these numbers are also called (as in Theorem 2.3.2) Lyapunov exponents and their multiplicities respectively.

Theorem 2.3.3. *Let* ψ *be a one-sided translation invariant random dynamical system on a probability space* $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$*, which satisfies Assumption 1. Then there exist numbers* $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ *and a* θ_+*-invariant measurable set* $\hat{\Omega}_1^{\mathbb{N}}$ *with* $\hat{\nu}^{\mathbb{N}}(\hat{\Omega}_1^{\mathbb{N}}) = 1$*, such that for all* $\omega \in \hat{\Omega}_1^{\mathbb{N}}$ *there exists a measurable splitting*

$$\mathbb{R}^d = E_1^{\psi}(\omega) \oplus \ldots \oplus E_p^{\psi}(\omega)$$

of $\mathbb{R}^d$ *over* $\hat{\Omega}_1^{\mathbb{N}}$ *into random subspaces* $E_i^{\psi}(\omega)$ *(so-called Oseledets spaces) with dimension* $\dim E_i^{\psi}(\omega) = d_i$*,* $i = \overline{1,p}$ *(so-called Oseledets splitting) with the following properties*

i) we have

$$\lim_{n\to\infty} \frac{1}{n} \log |(D_0\psi_{0,t})v| = \lambda_i \iff v \in V_i^{\psi}(\omega)\backslash V_{i+1}^{\psi}(\omega),$$

where $V_{p+1} := \{0\}$ *and for* $i = \overline{1,p}$

$$V_i^{\psi}(\omega) := E_p^{\psi}(\omega) \oplus \ldots \oplus E_i^{\psi}(\omega).$$

ii) the subspaces V_i^{ψ} *are* θ_+*-invariant, i.e.* $(D_0\psi_{n,\omega})V_i^{\psi}(\omega) = V_i^{\psi}(\theta_+^n\omega)$*;*

Proof. It suffices to prove the theorem for another one-sided RDS $\overline{\psi}$ on a probability space $(\overline{\Omega}^{\mathbb{N}}, \mathcal{B}(\overline{\Omega})^{\mathbb{N}}, \overline{\nu}^{\mathbb{N}})$ generated by i.i.d. mappings

$$\psi_{1,\omega} - \psi_{1,\omega}(0), \psi_{1,\theta\omega} - \psi_{1,\theta\omega}(0)\ldots,$$

because random matrices $D_0\psi_{1,\omega}$ and $D_0\overline{\psi}_{1,\omega}$ have the same distribution. For $\overline{\psi}$ the theorem holds because of Theorem 2.3.2. $\square$

In fact, TIBFs have a finite Lyapunov spectrum even in continuous time. The following proposition provides the precise statement.

Proposition 2.3.1. *Let ϕ be a TIBF. Then there exist numbers $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ and measurable set Ω_1 with $\mathbb{P}(\Omega_1) = 1$, such that for all $\omega \in \Omega_1$ there exists a measurable splitting*

$$\mathbb{R}^d = E_1^\phi(\omega) \oplus \ldots \oplus E_p^\phi(\omega)$$

of $\mathbb{R}^d$ over Ω_1 into random subspaces $E_i^\phi(\omega)$ (so-called Oseledets spaces) with dimension $\dim E_i^\phi(\omega) = d_i$, $i = \overline{1,p}$ (so-called Oseledets splitting) with the following property

$$\lim_{t\to\infty} \frac{1}{t} \log |(D_0\phi_{0,t})v| = \lambda_i \iff v \in V_i^\phi(\omega) \backslash V_{i+1}^\phi(\omega)$$

(here $t \in \mathbb{R}_+$), where $V_{p+1} := \{0\}$ and for $i = \overline{1,p}$

$$V_i^\phi(\omega) := E_p^\phi(\omega) \oplus \ldots \oplus E_i^\phi(\omega).$$

Proof. We can consider ϕ as a one-sided RDS in the sense of Arnold, see [1], Section 1.1.1. Then we can apply [1], Theorem 3.4.1 (C). The integrability condition of Theorem 3.4.1 (C) holds because as in the proof of Lemma 2.1.1 we can show that

$$\log^+ \sup_{t\in[0,1]} \|D_0\phi_{0,t}\| \in L^1(\mathbb{P}).$$

$\square$

Remark 2.3.1. *From now on we stop discussing TIBFs. The only exceptions are Corollary 4.1.1, Corollary 5.1.1 and Remark 6.1.1, which formulate the main results of the thesis in terms of TIBFs.*

Now we state two results, which show the existence of a finite Lyapunov spectrum of two-sided RDSs with the fixed origin and of two-sided TIRDSs respectively. Note that the result below is an analogue of Theorem 3.1 from [35].

Theorem 2.3.4. *Let ψ be a two-sided random dynamical system on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$, which satisfies Assumptions 1 and 2, and also has the fixed origin, i.e. $\psi_{1,\omega}(0) = 0$, $\forall\omega$. Then there exist numbers $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ and a θ-invariant measurable set $\overline{\Omega}_1^{\mathbb{Z}}$ with $\overline{\nu}^{\mathbb{Z}}(\overline{\Omega}_1^{\mathbb{Z}}) = 1$, such that for all $\omega \in \overline{\Omega}_1^{\mathbb{Z}}$ there exists a measurable splitting*

$$\mathbb{R}^d = E_1^\psi(\omega) \oplus \ldots \oplus E_p^\psi(\omega)$$

of $\mathbb{R}^d$ over $\overline{\Omega}_1^{\mathbb{Z}}$ into random subspaces $E_i^\psi(\omega)$ (so-called Oseledets spaces) with dimension $\dim E_i^\psi(\omega) = d_i$, $i = \overline{1,p}$ (so-called Oseledets splitting) with the following properties

i) the subspaces E_i^ψ are θ-invariant, i.e. $(D_0\psi_{n,\omega})E_i^\psi(\omega) = E_i^\psi(\theta^n\omega)$;
ii) $\lim\limits_{n\to\pm\infty} \frac{1}{n}\log|(D_0\psi_{n,\omega})v| = \lambda_i \iff v \in E_i^\psi(\omega)\backslash\{0\}$.

Proof. Equality ii) for ψ holds because of [35], Theorem 3.1, where τ and T should be substituted by θ and $D_0\psi_{1,\omega}$ respectively. Note that the integrability conditions of the theorem hold because of Assumptions 1 and 2, and ergodicity of θ. Finally, i) is a trivial consequence of ii). □

Theorem 2.3.5. *Let ψ be a two-sided translation invariant random dynamical system on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$, which satisfies Assumptions 1 and 2. Then there exist numbers $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ and a θ-invariant measurable set $\hat{\Omega}_1^{\mathbb{Z}}$ with $\hat{\nu}^{\mathbb{Z}}(\hat{\Omega}_1^{\mathbb{Z}}) = 1$, such that for all $\omega \in \hat{\Omega}_1^{\mathbb{Z}}$ there exists a measurable splitting*

$$\mathbb{R}^d = E_1^\psi(\omega) \oplus \ldots \oplus E_p^\psi(\omega)$$

of $\mathbb{R}^d$ over $\hat{\Omega}_1^{\mathbb{Z}}$ into random subspaces $E_i^\psi(\omega)$ (so-called Oseledets spaces) with dimension $\dim E_i^\psi(\omega) = d_i$, $i = \overline{1,p}$ (so-called Oseledets splitting) with the following properties
i) the subspaces E_i^ψ are θ-invariant, i.e. $(D_0\psi_{n,\omega})E_i^\psi(\omega) = E_i^\psi(\theta^n\omega)$;
ii) $\lim\limits_{n\to\pm\infty} \frac{1}{n}\log|(D_0\psi_{n,\omega})v| = \lambda_i \iff v \in E_i^\psi(\omega)\backslash\{0\}$.

Proof. It suffices to prove the theorem for another two-sided RDS $\overline{\psi}$ on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$ generated by i.i.d. mappings

$$\ldots \psi_{1,\theta^{-1}\omega} - \psi_{1,\theta^{-1}\omega}(0), \psi_{1,\omega} - \psi_{1,\omega}(0), \psi_{1,\theta\omega} - \psi_{1,\theta\omega}(0) \ldots,$$

because random matrices $D_0\psi_{1,\omega}$ and $D_0\overline{\psi}_{1,\omega}$ have the same distribution. For $\overline{\psi}$ the theorem holds because of Theorem 2.3.4. □

Remark 2.3.2. *One can ask if two-sided and one-sided systems based on the same probability space (say $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), \tilde{\nu})$) have the same Lyapunov spectrum. In fact it is true and is a trivial corollary of the asymptotic behaviour of $\frac{1}{n}\log|(D_0\psi_{n,\omega})v|$ when $n \to +\infty$.*

For a two-sided RDS ψ as in Theorem 2.3.4 or Theorem 2.3.5 define $i_0 := \max\{i \in \mathbb{N} : \lambda_i > 0\}$. For $\omega \in \hat{\Omega}_1^{\mathbb{Z}}$ define the following linear subspaces by

$$S^\psi(\omega) := E_p^\psi(\omega) \oplus \ldots \oplus E_{i_0+1}^\psi(\omega), \tag{2.10}$$

$$U^\psi(\omega) := E_1^\psi(\omega) \oplus \ldots \oplus E_{i_0}^\psi(\omega). \tag{2.11}$$

From now on we will abbreviate E_i^ψ, V_i^ψ, S^ψ, and U^ψ by E_i, V_i, S, and U respectively if there is no risk of ambiguity.

Remark 2.3.3. *Note that the subspaces S and U are defined with respect to the origin. In the same way we can define subspaces S_x and U_x with respect to x, i.e. considering spatial derivatives at x.*

Lemma 2.3.1. *Let ψ be a two-sided RDS on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$, which satisfies Assumption 1 and 2, has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$, and also has the fixed origin, i.e. $\psi_{1,\omega}(0) = 0$, $\forall \omega$. Then there exists an invariant set of a full measure $\overline{\Omega}_1^{\mathbb{Z}}$ such that for every $\omega \in \overline{\Omega}_1^{\mathbb{Z}}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}|_{U(\omega)}]| = \sum_{i=1}^{p} d_i \lambda_i^+.$$

Proof. Theorem 2.3.5 implies

$$\limsup_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}|_{S(\omega)}]| \le \sum_{i=1}^{p} d_i \lambda_i - \sum_{i=1}^{p} d_i \lambda_i^+, \tag{2.12}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}|_{U(\omega)}]| \le \sum_{i=1}^{p} d_i \lambda_i^+. \tag{2.13}$$

Moreover,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}|_{U(\omega)}]| \\
\ge &\liminf_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}] / \det[D_0\psi_{n,\omega}|_{S(\omega)}]| \\
\ge &\liminf_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}]| \\
- &\limsup_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}|_{S(\omega)}]| \\
\overset{(2.12)}{\ge} &\liminf_{n\to\infty} \frac{1}{n} \log |\det[D_0\psi_{n,\omega}]| - \sum_{i=1}^{p} d_i \lambda_i + \sum_{i=1}^{p} d_i \lambda_i^+ = \sum_{i=1}^{p} d_i \lambda_i^+,
\end{aligned}$$

where the last equality holds by Furstenberg-Kesten Theorem, see [1], Theorem 3.3.3. This together with (2.13) completes the proof of the lemma. □

Remark 2.3.4. *Let G be a subset of $\mathbb{R}^d$. From now on we will sometimes omit brackets and abbreviate $\psi_{n,\omega} G$ instead of $\psi_{n,\omega}(G)$ if there is no risk of ambiguity.*

Chapter 3

Definition of Entropy

Kifer in [18] successfully defined the notion of entropy for RDS (in the case of invariant probability measure). However, it turns out that some basic properties of entropy in Kifer's setting, such as stability with respect to the sequence of approximating partitions (see [18], Corollary II.2.1), can not be proved in the same way as in deterministic dynamics. To resolve the problem, he defines entropy of the respective skew product and then connects the entropy of the RDS with the entropy of the skew product. The whole procedure is described in [18], Section 2.1.

The definition of entropy in our case is even a more challenging task because TIRDSs have no invariant probability measure, but the Lebesgue measure, which is an infinite invariant measure. To define entropy in our case, we consider only periodic partitions and look at the dynamics of the system only in the fixed cube $[0,1)^d$. At the same time we also, following Kifer, define the notion of entropy of the skew product and, as in [18], connect the defined entropy with the entropy of one-sided TIRDSs.

Let us note that we impose on the partitions of the state space of the skew product certain specific assumptions, see Definition 3.2.2. These assumptions, in particular, imply translation invariance of the partitions in some sense, which lets us show invariance of the entropy with respect to the skew product and establish the desired properties of the entropy, see Section 3.3.

Finally, in Section 3.4 we define entropy for volume preserving two-sided TIRDSs, which is basically the adaptation of the arguments from Section 3.3. Note that here we restrict ourselves to the volume preserving case because later (in the proof of Pesin's formula) it will be important to have the invariance of conditional measures (see Theorem 3.4.1) with respect to the whole randomness, i.e. with respect to $\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}$, and in this case the preservation of the volume is essential.

3.1 Definition of Entropy of Partitions

We provide a short introduction to entropy and conditional entropy of partitions, mainly following [25], Chapter 0, §3, and also [43], Section 6.2.

We put $0 \log 0 := 0$. Now let us start the section with the definition of the entropy of a partition.

Definition 3.1.1. *Let $\mathcal{P}$ be a countable measurable partition of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. The conditional entropy of $\mathcal{P}$ given $\mathcal{G}$ is the number*

$$H_{\mathbb{P}}(\mathcal{P}|\mathcal{G}) := -\int_{\Omega} \sum_{A \in \mathcal{P}} \mathbb{P}(A|\mathcal{G}) \log \mathbb{P}(A|\mathcal{G}) d\mathbb{P} \in [0, \infty].$$

The number

$$H_{\mathbb{P}}(\mathcal{P}) := -\sum_{A \in \mathcal{P}} \mathbb{P}(A) \log \mathbb{P}(A) \in [0, \infty]$$

is called the entropy of $\mathcal{P}$.

Note that since $0 \log 0 = 0$, the sums in the latter definition always make sense.

For two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ we denote their common refinement by $\mathcal{P}_1 \vee \mathcal{P}_2$. Note also that $\sigma(\mathcal{P})$ denotes the σ-algebra generated by the elements of $\mathcal{P}$. Finally, $\mathcal{P}_1 \prec \mathcal{P}_2$ means that $\sigma(\mathcal{P}_1) \subset \sigma(\mathcal{P}_2)$.

Now we provide some basic properties of the defined entropy.

Lemma 3.1.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be countable measurable partitions of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let further $\mathcal{G}, \mathcal{G}' \subset \mathcal{F}$ be σ-algebras, and $f : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ be a measure preserving measurable map. Then the following holds true*

1. $H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G}) \geq 0$.
2. $H_{\mathbb{P}}(\mathcal{P}_1 \vee \mathcal{P}_2|\mathcal{G}) = H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G}) + H_{\mathbb{P}}(\mathcal{P}_2|\sigma(\mathcal{P}_1) \vee \mathcal{G})$.
3. $H_{\mathbb{P}}(\mathcal{P}_1 \vee \mathcal{P}_2) = H_{\mathbb{P}}(\mathcal{P}_1) + H_{\mathbb{P}}(\mathcal{P}_2|\sigma(\mathcal{P}_1))$.
4. $\mathcal{P}_1 \prec \mathcal{P}_2$ *implies* $H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G}) \leq H_{\mathbb{P}}(\mathcal{P}_2|\mathcal{G})$.
6. $\mathcal{P}_1 \prec \mathcal{P}_2$ *implies* $H_{\mathbb{P}}(\mathcal{P}_1) \leq H_{\mathbb{P}}(\mathcal{P}_2)$.
7. $H_{\mathbb{P}}(\mathcal{P}_1) \geq H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G})$.
8. $\mathcal{G} \subset \mathcal{G}'$ *implies* $H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G}) \geq H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G}')$.
9. $H_{\mathbb{P}}(\mathcal{P}_1 \vee \mathcal{P}_2|\mathcal{G}) \leq H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G}) + H_{\mathbb{P}}(\mathcal{P}_2|\mathcal{G})$.
10. $H_{\mathbb{P}}(\mathcal{P}_1 \vee \mathcal{P}_2) \leq H_{\mathbb{P}}(\mathcal{P}_1) + H_{\mathbb{P}}(\mathcal{P}_2)$.
11. $H_{\mathbb{P}}(f^{-1}\mathcal{P}_1|f^{-1}\mathcal{G}) = H_{\mathbb{P}}(\mathcal{P}_1|\mathcal{G})$.
12. $H_{\mathbb{P}}(f^{-1}\mathcal{P}_1) = H_{\mathbb{P}}(\mathcal{P}_1)$.

Proof. The same as in [18], Remark II.1.1 and [18], Lemma II.1.2. Note that in [18] partitions are finite, but it does not alter the proof.

□

Now we formulate a trivial upper bound on the entropy of a partition.

Lemma 3.1.2. *The (conditional) entropy of a partition is at most the logarithm of its cardinality.*

Proof. The same as in [18], Corollary II.1.1. □

From now on in this chapter consider a one-sided TIRDS ψ on a probability space $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$ (the only exception is Section 3.4, where we discuss entropy for two-sided systems). Recall that μ is the Lebesgue measure on $\mathbb{R}^d$. Further, for $m, n \in \mathbb{R}$, $m < n$ denote $\mu_{m,n} := \mu|_{[m,n)^d}$ the restriction of μ to the cube $[m,n)^d$. Recall that $\mathbf{M}^+ := \mu \times \hat{\nu}^{\mathbb{N}}$ is invariant for Θ_+, see Proposition 2.2.2. Finally, denote

$$\mathbf{M}^+_{0,1} := \mathbf{M}^+|_{[0,1)^d \times \hat{\Omega}},$$

where the measure $\mathbf{M}^+_S$ is the restriction of the measure $\mathbf{M}^+$ to the subset S. Note that $\mu_{0,1}$ is a probability measure, so entropies $H_{\mu_{0,1}}$ (with respect to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu_{0,1})$) and $H_{\mathbf{M}^+_{0,1}}$ (with respect to $(\mathbb{R}^d \times \hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \mathbf{M}^+_{0,1})$) perfectly make sense.

3.2 Class of 1-periodic in Distribution Sets

In this section we present a certain class of subsets of $\mathbb{R}^d$, which enlarges the class of 1-periodic sets (the definition see below) by certain random sets (i.e. by certain subsets of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{N}}$), which we need for the proof of Pesin's formula. Now let us define the notion of 1-periodic set and 1-periodic partition.

Definition 3.2.1. *A set $A \subset \mathbb{R}^d$ is called* 1-periodic, *if for every $v \in \mathbb{Z}^d$ we have*

$$A + v = A.$$

Definition 3.2.2. *A countable measurable partition $\mathcal{P}$ of $\mathbb{R}^d$ is called* 1-periodic *if every element of $\mathcal{P}$ is a 1-periodic set.*

Denote by $\mathcal{A}_{tr}$ the class of elements of $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}$ of the form $A \times \hat{\Omega}^{\mathbb{N}}$, where A is a 1-periodic set.

Further, denote

$$\mathcal{B}_{tr} := \left\{ \bigcap_{i=1}^{m} \Theta_+^{-n_i} \mathbf{A}_i \,\middle|\, \mathbf{A}_i \in \mathcal{A}_{tr}, n_i \in \mathbb{N}_0, m \in \mathbb{N} \right\}.$$

Finally, let us define the notion of 1-periodic in distribution set and 1-periodic in distribution partition.

Definition 3.2.3. *A set $\mathbf{B} \subset \mathbb{R}^d \times \hat{\Omega}^{\mathbb{N}}$ is called* 1-periodic in distribution, *if $\mathbf{B} \in \mathcal{B}_{tr}$.*

Definition 3.2.4. *A countable measurable partition $\mathcal{P}$ of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{N}}$ is called* 1-periodic in distribution *if every element of $\mathcal{P}$ is a 1-periodic in distribution set.*

3.3 Metric Entropy of Translation Invariant Random Dynamical Systems

We start the section with a crucial result: we prove invariance of the (conditional) measure $\mathbf{M}^+_{0,1}$ of a 1-periodic in distribution partition with respect to the skew product Θ_+. This result lets us define entropy of ψ and then develop entropy theory (with respect to 1-periodic in distribution partitions) in a similar way to the case of standard settings.

Recall that in this chapter one-sided TIRDS ψ is defined on a probability space $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$.

Theorem 3.3.1. *Let $\mathbf{A}$ be a measurable 1-periodic in distribution set. Then we have*

$$\mathbf{M}^+_{0,1}(\Theta_+^{-1}\mathbf{A}|\Theta_+^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}})) = \mathbf{M}^+_{0,1}(\mathbf{A}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}) \circ \Theta_+, \tag{3.1}$$

and

$$\mathbf{M}^+_{0,1}(\Theta_+^{-1}\mathbf{A}) = \mathbf{M}^+_{0,1}(\mathbf{A}). \tag{3.2}$$

Proof. Let us show (3.1). For a measurable subset $\mathbf{A}$ of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{N}}$ denote by $\mathbf{A}_\omega$ the restriction of $\mathbf{A}$ to $\mathbb{R}^d \times \{\omega\}$.

Then by translation invariance of ψ, evaluation of the left hand side for given ω provides

$$\begin{aligned}
LHS(\omega) =& \mathbb{E}[\mu_{0,1}(\psi^{-1}_{1,\omega}(\mathbf{A}_{\theta_+\omega}))|\Theta_+^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}})] \\
=& \mathbb{E}\left[\sum_{v\in\mathbb{Z}^d} \mu_{0,1}(\psi^{-1}_{1,\omega}(\mathbf{A}_{\theta_+\omega} \cap (v + [0,1)^d)))|\Theta_+^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}})\right] \\
=& \mathbb{E}\left[\sum_{v\in\mathbb{Z}^d} \mu_{[0,1)^d - v}(\psi^{-1}_{1,\omega}(\mathbf{A}_{\theta_+\omega} \cap [0,1)^d))|\Theta_+^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}})\right] \\
=& \mathbb{E}\left[\mu(\psi^{-1}_{1,\omega}(\mathbf{A}_{\theta_+\omega} \cap [0,1)^d))|\Theta_+^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}})\right] =: I.
\end{aligned}$$

Now by invariance of $\mathbf{M}^+$, see Proposition 2.2.2, we have

$$I = \mu(\mathbf{A}_{\theta_+\omega} \cap [0,1)^d) = \mu_{0,1}(\mathbf{A}_{\theta_+\omega}) = RHS(\omega),$$

as required. The proof of (3.2) is the same (in fact is even easier).

□

Now we prove invariance of entropy of a 1-periodic in distribution partition with respect to the skew product Θ_+.

Theorem 3.3.2. *Let $\mathcal{P}$ be a countable measurable 1-periodic in distribution partition of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{N}}$ with finite entropy, i.e. $H_{\mathbf{M}^+_{0,1}}(\mathcal{P}) < \infty$. Then we have*

$$H_{\mathbf{M}^+_{0,1}}(\Theta_+^{-1}\mathcal{P}|(\Theta_+^{-1}(\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}))=H_{\mathbf{M}^+_{0,1}}(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}), \tag{3.3}$$

and

$$H_{\mathbf{M}^+_{0,1}}(\Theta_+^{-1}\mathcal{P})=H_{\mathbf{M}^+_{0,1}}(\mathcal{P}). \tag{3.4}$$

Proof. It suffices to show the theorem for finite partitions, because entropy of an infinite partition $\{\mathbf{C}_1,\mathbf{C}_2,\ldots\}$ with finite entropy can be approximated by entropies of finite partitions $\{\mathbf{C}_1,\ldots,\mathbf{C}_n\}$, when n goes to infinity. For finite partitions the theorem is a direct corollary of Theorem 3.3.1.

□

The following lemma defines metric entropy of the skew product.

Lemma 3.3.1. *Let ξ be a countable measurable 1-periodic in distribution partition with finite entropy. Then there exist*

$$h_{\mathbf{M}^+}(\Theta_+,\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}):=\lim_{n\to\infty}\frac{1}{n}H_{\mathbf{M}^+_{0,1}}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right) \tag{3.5}$$

and

$$h_{\mathbf{M}^+}(\Theta_+,\xi):=\lim_{n\to\infty}\frac{1}{n}H_{\mathbf{M}^+_{0,1}}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi\right). \tag{3.6}$$

The numbers

$$h_{\mathbf{M}^+}(\Theta_+|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}):=\sup_{\xi}h_{\mathbf{M}^+_{0,1}}(\Theta_+,\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})$$

and

$$h_{\mathbf{M}^+}(\Theta_+):=\sup_{\xi}h_{\mathbf{M}^+}(\Theta_+,\xi)$$

are called metric entropy of Θ_+ given randomness and metric entropy of Θ_+ respectively. The supremum is taken over all finite 1-periodic in distribution partitions.

Proof. Note that the proof of the theorem is similar to the proof of Theorem II.1.1 from [18].

Denote by

$$a_n:=H_{\mathbf{M}^+_{0,1}}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi\,\middle|\,\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)$$

Then by Statement 9 from Lemma 3.1.1 we have

$$
\begin{aligned}
a_{n+m} =& H_{\mathbf{M}_{0,1}^+}\left(\bigvee_{i=0}^{n+m-1}\Theta_+^{-i}\xi\middle|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)\\
=& H_{\mathbf{M}_{0,1}^+}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi\middle|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)\\
+& H_{\mathbf{M}_{0,1}^+}\left(\Theta_+^{-n}\bigvee_{i=0}^{m-1}\Theta_+^{-i}\xi\middle|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)\\
=& H_{\mathbf{M}_{0,1}^+}\left(\Theta_+^{-n}\bigvee_{i=0}^{m-1}\Theta_+^{-i}\xi\middle|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)+a_n := I;
\end{aligned}
$$

now Statement 8 from Lemma 3.1.1 implies

$$
I \leq H_{\mathbf{M}_{0,1}^+}\left(\Theta_+^{-n}\bigvee_{i=0}^{m-1}\Theta_+^{-i}\xi\middle|\Theta_+^{-n}(\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})\right)+a_n = a_m + a_n,
$$

where the last equality holds by Statement 11 from Lemma 3.1.1. Thus, we obtain

$$
a_{m+n} \leq a_m + a_n, \qquad m,n\in\mathbb{N}. \tag{3.7}
$$

Then (3.7) together with subadditivity arguments provides the existence of the left hand side of (3.5). Moreover, (3.7) implies for all positive integers n inequality $a_n \leq na_1$, which means that the left hand side of (3.5) is finite. Analogously, one can show that the left hand side of (3.6) exists and finite.

□

Lemma 3.3.2. *If $\xi = \{A_1,\dots,A_k\}$ and $\eta = \{B_1,\dots,B_m\}$ are finite measurable partitions of $\mathbb{R}^d$ and $\hat{\Omega}^{\mathbb{N}}$ respectively, then*

$$
\mathbb{E}H_{\mu_{0,1}}\left(\bigvee_{i=0}^{n-1}\psi_{i,\omega}^{-1}\xi\right) = H_{\mathbf{M}_{0,1}^+}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}(\xi\times\eta)|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right),
$$

where $\xi\times\eta := \{A_i\times B_j : 1\leq i\leq k, 1\leq j\leq m\}$.

Proof. The proof is similar to a part of the proof of Theorem II.1.4 (i) from [18].

Fix $n\in\mathbb{N}$. Further, fix sets $i_0,\dots,i_{n-1},j_0,\dots j_{n-1}\in\mathbb{N}$. Then we have

$$
\mathbf{M}_{0,1}^+\left(\bigcap_{k=0}^{n-1}\Theta_+^{-k}(A_{i_k}\times B_{j_k})|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right) = \mathbf{M}_{0,1}^+(\mathbf{C}_n|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}),
$$

where $\mathbf{C}_n := \bigcap_{k=0}^{n-1}\{(x,\omega) : \psi_{k,\omega}(x)\in A_{i_k},\ \text{and } \theta^k\in B_{j_k}\}$. Further

$$
\mathbf{M}_{0,1}^+(\mathbf{C}_n|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}) = \mu_{0,1}\left(\bigcap_{k=0}^{n-1}\psi_{k,\omega}^{-1}(A_{i_k})\right)1_{B_{i_0}}(\omega)\dots 1_{B_{i_{n-1}}}(\theta^{n-1}\omega).
$$

Thus

$$\mathbf{M}^+_{0,1}\left(\bigcap_{k=0}^{n-1}\Theta_+^{-k}(A_{i_k}\times B_{j_k})|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)=\mu_{0,1}\left(\bigcap_{k=0}^{n-1}\psi_{k,\omega}^{-1}(A_{i_k})\right)$$
$$\times 1_{B_{i_0}}(\omega)\dots 1_{B_{i_{n-1}}}(\theta^{n-1}\omega),$$

and the latter equality directly implies the lemma. □

Remark 3.3.1. *Lemma 3.3.2 implies that for a finite* 1*-periodic measurable partition* $\xi=\{A_1,\dots,A_k\}$ *of* $\mathbb{R}^d$ *and a finite measurable partition* $\eta=\{B_1,\dots,B_m\}$ *of* $\hat{\Omega}^{\mathbb{N}}$*, the number*

$$H_{\mathbf{M}^+_{0,1}}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}(\xi\times\eta)|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)$$

does not depend on η*. Therefore, we also can define* $h_{\mathbf{M}^+}(\Theta_+,\xi\times\eta|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})$ *(which is equal to* $h_{\mathbf{M}^+}(\Theta_+,\xi\times\hat{\Omega}^{\mathbb{N}}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})$*).*

Now we are ready for the main definition of the thesis. The following theorem provides the definition of entropy for TIRDSs.

Theorem 3.3.3. *Let* $\mathcal{P}$ *be a finite* 1*-periodic partition of* $\mathbb{R}^d$*. Then there exists*

$$h_\mu(\psi,\mathcal{P}):=\lim_{n\to\infty}\mathbb{E}\frac{1}{n}H_{\mu_{0,1}}\left(\bigvee_{i=0}^{n-1}\psi_{i,\omega}^{-1}\mathcal{P}\right).$$

The number

$$h_\mu(\psi):=\sup_{\mathcal{P}}h_\mu(\psi,\mathcal{P})$$

is called metric entropy of ψ*. The supremum is taken over all finite* 1*-periodic partitions.*

Proof. Because of Lemma 3.3.1 and Lemma 3.3.2 (with $\eta=\hat{\Omega}^{\mathbb{N}}$) we know that

$$h_\mu(\psi,\mathcal{P})=h_{\mathbf{M}^+}(\Theta_+,\mathcal{P}\times\hat{\Omega}^{\mathbb{N}}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}),\tag{3.8}$$

and the right hand side exists because of Lemma 3.3.1. The lemma is proven. □

Now we develop entropy theory as in [18], Section 2.1. We start from the following result.

Lemma 3.3.3. *If* $\xi=\{A_1,\dots,A_k\}$ *and* $\eta=\{B_1,\dots,B_m\}$ *are finite measurable partitions of* $\mathbb{R}^d$ *and* $\hat{\Omega}^{\mathbb{N}}$ *respectively. Further, let* ξ *be* 1*-periodic. Then we have*

$$h_\mu(\psi,\xi)=h_{\mathbf{M}^+}(\Theta_+,\xi\times\eta|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}),$$

where $\xi\times\eta:=\{A_i\times B_j:1\le i\le k,1\le j\le m\}$*.*

Proof. This is a straight consequence of Lemma 3.3.1, Lemma 3.3.2 and Theorem 3.3.3. □

Lemma 3.3.4. *We have*

$$h_\mu(\psi) = h_{\mathbf{M}^+}(\Theta_+|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}).$$

Proof. Because of Remark 3.3.1 the proof can be done in the same way as in Kifer's book, see [18], Theorem II.1.4 (ii).

□

For a positive integer n denote by ψ^n the random dynamical system with rescaled time, i.e. $\psi^n_{m,\omega}(x) := \psi_{nm,\omega}(x)$. Indeed, then it is easy to check that ψ^n is a one-sided TIRDS.

Lemma 3.3.5. *For any $n \in \mathbb{N}$ we have*

$$h_\mu(\psi^n) = nh_\mu(\psi).$$

Proof. The same as in [18], Lemma II.1.4. □

Lemma 3.3.6. *If $\mathcal{P}_1$, $\mathcal{P}_2, \ldots$ is a sequence of finite 1-periodic partitions such that $\lim_{n\to\infty} \mathbf{diam}(\mathcal{P}_n \cap [0,1)^d) = 0$ wherein $\mathbf{diam}(\mathcal{P}) = \sup_{C\in\mathcal{P}} \mathbf{diam}(C)$ is the diameter of the partition $\mathcal{P}$. Then*

$$h_\mu(\psi) = \lim_{n\to\infty} h_\mu(\psi, \mathcal{P}_n).$$

Proof. Because of Lemma 3.3.4 and (3.8) it suffices to show that

$$\lim_{n\to\infty} h_{\mathbf{M}^+}(\Theta_+, \mathcal{P}_n \times \hat{\Omega}^{\mathbb{N}}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}) = h_{\mathbf{M}^+}(\Theta_+|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}),$$

and this equality can be obtained in the same way as Lemma 3.3.4.

□

Remark 3.3.2. *It is natural to expect that one can alternatively define entropies of Θ_+ using the dynamics of the whole space. Now let us be more precise. Let ξ be a finite 1-periodic in distribution partition. Denote*

$$\mathbf{M}^+_c := \frac{1}{(2c)^d}\mathbf{M}^+|_{[-c,c)^d\times\hat{\Omega}}, \quad \textit{and} \quad \mathbf{M}^+_{0,1,v} := \frac{1}{(2c)^d}\mathbf{M}^+|_{([0,1)^d+v)\times\hat{\Omega}},$$

where $v \in \mathbb{R}^d$. Then $\mathbf{M}^+_c$ and $\mathbf{M}^+_{0,1,v}$ are probability measures, so entropies $H_{\mathbf{M}^+_c}$ (with respect to $(\mathbb{R}^d\times\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \mathbf{M}^+_c$ and $(\mathbb{R}^d\times\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \mathbf{M}^+_{0,1,v})$ perfectly make sense.

Then one could try to define entropy of Θ_+ with respect to ξ in one of the following ways:

$$h'_{\mathbf{M}^+}(\Theta_+, \xi|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}) := \lim_{n\to\infty}\lim_{N\to\infty} \frac{1}{n} H_{\mathbf{M}^+_N}\left(\bigvee_{i=0}^{n-1} \Theta_+^{-i}\xi|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right),$$

or

$$h''_{\mathbf{M}^+}(\Theta_+,\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}):=\lim_{N\to\infty}\lim_{n\to\infty}\frac{1}{n}H_{\mathbf{M}_N^+}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right);$$

moreover, it is natural to expect that the right hand sides of the latter two definitions coincide with $h_{\mathbf{M}^+}(\Theta_+,\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})$*. However, it is not clear if the limits in these definitions exist and if the right hand sides (in the case of existence) coincide with the defined entropy. However, one can show that*

$$h_{\mathbf{M}^+}(\Theta_+,\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})\leq\liminf_{n\to\infty}\liminf_{N\to\infty}\frac{1}{n}H_{\mathbf{M}_N^+}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right),\tag{3.9}$$

and

$$h_{\mathbf{M}^+}(\Theta_+,\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}})\leq\liminf_{N\to\infty}\liminf_{n\to\infty}\frac{1}{n}H_{\mathbf{M}_N^+}\left(\bigvee_{i=0}^{n-1}\Theta_+^{-i}\xi|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right).\tag{3.10}$$

Indeed, by Jensen's inequality, applied to $g(x):=-x\log x$*, we have* $\forall\mathbf{A}\in\mathcal{P}$

$$\sum_{v\in\mathbb{Z}^d\cap[-N,N)^d}\int_{\mathbb{R}^d\times\hat{\Omega}^{\mathbb{Z}}}g(\mathbf{M}_{0,1,v}^+(\mathbf{A}|\mathcal{G}))d\mathbf{M}_{0,1,v}^+\leq(2N)^d\int_{\mathbb{R}^d\times\hat{\Omega}^{\mathbb{Z}}}g(\mathbf{M}_N^+(\mathbf{A}|\mathcal{G}))d\mathbf{M}_N^+,$$

where $\mathcal{G}:=\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}$*. This yields*

$$\sum_{v\in\mathbb{Z}^d\cap[-N,N)^d}H_{\mathbf{M}_{0,1,v}^+}\left(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)\leq(2N)^dH_{\mathbf{M}_N^+}\left(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right),$$

Finally, since for every finite 1*-periodic in distribution partition* $\mathcal{P}$ *and* $\forall v,w\in\mathbb{Z}^d$ *we have*

$$H_{\mathbf{M}_{0,1,v}^+}\left(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)=H_{\mathbf{M}_{0,1,w}^+}\left(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right),$$

we obtain

$$H_{\mathbf{M}_{0,1}^+}\left(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right)\leq H_{\mathbf{M}_N^+}\left(\mathcal{P}|\mathbb{R}^d\times\mathcal{B}(\hat{\Omega})^{\mathbb{N}}\right).$$

The latter inequality implies (3.9) and (3.10) as required.

3.4 Entropy for two-sided systems

Consider a two-sided TIRDS ψ defined on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$. Further, let ψ be volume preserving, i.e. the measure $\mathbf{M} := \mu \times \hat{\nu}^{\mathbb{Z}}$ (defined on $\mathbb{R}^d \times \tilde{\Omega}^{\mathbb{Z}}$) is invariant for the skew product Θ. Now we briefly provide the same procedure as in Section 3.3, but for two-sided systems.

Denote by $\mathcal{A}_{tr}^{\mathbb{Z}}$ the class of elements of $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}$ of the form $A \times \hat{\Omega}^{\mathbb{Z}}$, where A is a 1-periodic set. Further, denote

$$\mathcal{B}_{tr}^{\mathbb{Z}} := \left\{ \bigcap_{i=1}^{m} \Theta^{-n_i} \mathbf{A}_i \,\middle|\, \mathbf{A}_i \in \mathcal{A}_{tr}, n_i \in \mathbb{N}_0, m \in \mathbb{N} \right\};$$

we say that a partition $\mathcal{P}$ is called 1-periodic in distribution if $\mathcal{P} \in \mathcal{B}_{tr}^{\mathbb{Z}}$. Define $\mathbf{M}_{0,1} := \mathbf{M}|_{[0,1)^d \times \hat{\Omega}}$. Now let us formulate an analogue of Theorem 3.3.1 for two-sided systems.

Theorem 3.4.1. *Let $\mathbf{A} \in \mathcal{B}_{tr}^{\mathbb{Z}}$. Then we have*

$$\mathbf{M}_{0,1}(\Theta^{-1}\mathbf{A} | \Theta^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}})) = \mathbf{M}_{0,1}(\mathbf{A} | \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}) \circ \Theta, \tag{3.11}$$

and

$$\mathbf{M}_{0,1}(\Theta^{-1}\mathbf{A}) = \mathbf{M}_{0,1}(\mathbf{A}). \tag{3.12}$$

Proof. Let us show (3.11). Indeed, by translation invariance of ψ, evaluation of the left hand side for given ω provides

$$\begin{aligned}
LHS(\omega) =& \mathbb{E}[\mu_{0,1}(\psi_{1,\omega}^{-1}(\mathbf{A}_{\theta\omega})) | \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}] \\
=& \mathbb{E}\left[\sum_{v \in \mathbb{Z}^d} \mu_{0,1}(\psi_{1,\omega}^{-1}(\mathbf{A}_{\theta\omega} \cap (v + [0,1)^d))) | \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}\right] \\
=& \mathbb{E}\left[\sum_{v \in \mathbb{Z}^d} \mu_{[0,1)^d - v}(\psi_{1,\omega}^{-1}(\mathbf{A}_{\theta\omega} \cap [0,1)^d)) | \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}\right] \\
=& \mathbb{E}\left[\mu(\psi_{1,\omega}^{-1}(\mathbf{A}_{\theta\omega} \cap [0,1)^d)) | \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}\right] =: I.
\end{aligned}$$

Now by invariance of $\mathbf{M}$ we have

$$I = \mu(\mathbf{A}_{\theta\omega} \cap [0,1)^d)) = \mu_{0,1}(\mathbf{A}_{\theta\omega}) = RHS(\omega),$$

as required. The proof of (3.12) is the same (in fact is even easier). $\square$

Remark 3.4.1. *All the results from Section 3.3 also hold for two-sided systems with respect to $\mathcal{B}_{tr}^{\mathbb{Z}}$ (and $\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}$). To prove them, it suffices to repeat the arguments from Section 3.3. In particular, entropy $h_\mu(\psi)$ is well defined. For the sake of completeness we formulate the definition of $h_\mu(\psi)$. Note that it is the same as in Theorem 3.3.3.*

Definition 3.4.1. *Let $\mathcal{P}$ be a finite* 1*-periodic partition of* $\mathbb{R}^d$*. Then there exists*

$$h_\mu(\psi, \mathcal{P}) := \lim_{n\to\infty} \mathbb{E}\frac{1}{n} H_{\mu_{0,1}}\left(\bigvee_{i=0}^{n-1} \psi_{1,\omega}^{-1}\mathcal{P}\right).$$

The number

$$h_\mu(\psi) := \sup_{\mathcal{P}} h_\mu(\psi, \mathcal{P})$$

is called metric entropy of ψ*. The supremum is taken over all finite* 1*-periodic partitions.*

Chapter 4

Ruelle's Inequality for Translation Invariant Random Dynamical Systems

Ruelle's inequality is a relation between entropy of a DS and its Lyapunov exponents. Namely, it states that the entropy is less than or equal to the sum of the positive Lyapunov exponents of the system. The formula was first established by Ruelle for deterministic DSs acting on a compact Riemannian manifold, see [35]. Later different authors proved Ruelle's inequality in different settings, see e.g. the discussion of the inequality in the introduction. Nevertheless, to the best of our knowledge, this formula has never been established before for systems without invariant probability measure. The problem is that entropy in this case is ill-posed. It is ill-posed for TIRDSs as well, because these systems do not have an invariant probability measure, but the Lebesgue measure, which is an infinite invariant measure.

In this chapter we use the definition of entropy as in Theorem 3.3.3. This definition lets us prove Ruelle's inequality for one-sided TIRDSs repeating the standard arguments. In this chapter we follow closely van Bargen, see [42] (and also [43], Section 6.3), that proved Ruelle's inequality for certain stochastic flows on $\mathbb{R}^d$.

4.1 Main Result

In this section we prove the following theorem

Theorem 4.1.1. *Let ψ be a one-sided TIRDS defined on a probability space $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$, which satisfies Assumptions 1 and 2, and has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Then we have*

$$h_\mu(\psi) \leq \sum_{i=1}^{p} d_i \lambda_i^+.$$

Note that in the same way, as we prove Theorem 4.1.1, we can prove Ruelle's inequality for two-sided volume preserving TIRDSs. Note that this will be important only for Remark 5.1.1. Now let us formulate the respective result.

Proposition 4.1.1. *Let ψ be a two-sided TIRDS on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$, which satisfies Assumptions 1 and 2, and has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Further, let the measure $\mathbf{M} = \mu \times \hat{\nu}^{\mathbb{Z}}$ be invariant for the skew product Θ. Then we have*

$$h_\mu(\psi) \leq \sum_{i=1}^{p} d_i \lambda_i^+.$$

Theorem 4.1.1 immediately imply Ruelle's inequality for TIBFs. Let us formulate the result precisely.

Corollary 4.1.1. *Let ϕ be a TIBF and φ be the respective one-sided TIRDS on a probability space $(\hat{\Omega}^{\mathbb{N}}, \mathcal{B}(\hat{\Omega})^{\mathbb{N}}, \hat{\nu}^{\mathbb{N}})$ (i.e. is constructed as in Section 2.2), which has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Then we have*

$$h_\mu(\varphi) \leq \sum_{i=1}^{p} d_i \lambda_i^+.$$

Now let us come back to Theorem 4.1.1. Note that the proof of the theorem is similar to the proof of Theorem 4.1 in [42].

For a $r > 0$ and $S \subset \mathbb{R}^d$ denote by $B_r(S)$ the r-neighbourhood of S, i.e.

$$B_r(S) := \bigcup_{x \in S} B(x, r);$$

now we formulate the following purely deterministic result from geometry.

Lemma 4.1.1. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear mapping and let $\mathbb{R}^d$ be equipped with the usual Euclidean norm $|\cdot|$. Let further $\delta_1(A) \geq \ldots \geq \delta_d(A)$ denote the singular values of A. Then there exists a constant $C(d)$ which only depends on d such that for any $\epsilon > 0$ the number of disjoint balls with radius $\frac{\epsilon}{2}$, which can intersect $B_{2\epsilon}(AB_\epsilon(0))$ does not exceed*

$$C(d) \prod_{u=1}^{d} (\delta_u(A) \vee 1).$$

Proof. See [21], Lemma II.2.3. □

Denote

$$\mathcal{P}_k := \{v + [0, 2^{-k})^d + \mathbb{Z}^d | v \in 2^{-k}\mathbb{Z}^d \cap [0,1)^d\};$$

finally, we state the following lemma

Lemma 4.1.2. *For every positive integers k and n we have*

$$h_\mu(\psi^n, \mathcal{P}_k) \le \mathbb{E}H_{\mu_{0,1}}(\psi_{n,\omega}^{-1}\mathcal{P}_k|\sigma(\mathcal{P}_k)).$$

Proof. By Proposition 2.2.2, the proof is the same as in [4], Corollary 1.

□

4.2 Proof of Ruelle's inequality

In this section we prove Theorem 4.1.1.

Fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$ define the set $\hat{\Omega}_k^{\mathbb{N}}$ of ω for which we have the following statement: for any $\epsilon \le \sqrt{d}2^{-k}$ and $x, y \in [0,1)^d$ the inequality $|x-y| \le \epsilon$ implies

$$|\psi_{n,\omega}(x) - \psi_{n,\omega}(y) - (D_x\psi_{n,\omega})(x-y)| \le \epsilon. \tag{4.1}$$

Note that for $k \in \mathbb{N}$ we have trivial inclusions

$$\hat{\Omega}_k^{\mathbb{N}} \subset \hat{\Omega}_{k+1}^{\mathbb{N}},\ k \in \mathbb{N}, \quad \text{and} \quad \hat{\Omega}^{\mathbb{N}} = \bigcup_{k=1}^{\infty} \hat{\Omega}_k^{\mathbb{N}}.$$

We have

$$nh_\mu(\psi) = h_\mu(\psi^n) = \lim_{k\to\infty} h_\mu(\psi^n, \mathcal{P}_k) \le \liminf_{k\to\infty} \mathbb{E}H_{\mu_{0,1}}(\psi_{n,\omega}^{-1}\mathcal{P}_k|\sigma(\mathcal{P}_k)),$$

where the last inequality holds by Lemma 4.1.2. Now let us enumerate the elements of $\mathbf{P}_k$ as $\mathbf{P}_{k,1}, \dots, \mathbf{P}_{k,2^{kd}}$. Then

$$\begin{aligned}
&\mathbb{E}H_{\mu_{0,1}}(\psi_{n,\omega}^{-1}\mathcal{P}_k|\sigma(\mathcal{P}_k)) = -\mathbb{E}\sum_{i=1}^{2^{kd}} \mu_{0,1}(\mathbf{P}_{k,i}) \\
&\times \sum_{j=1}^{2^{kd}} \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \log \mu_{0,1}(\psi_{n,\omega}^{-1}\mathcal{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \\
&= -\,\mathbb{E}2^{-kd}\sum_{i=1}^{2^{kd}}\sum_{j=1}^{2^{kd}} \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \log \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})).
\end{aligned}$$

Now we can estimate

$$
\begin{aligned}
nh_\mu(\psi) \le & \liminf_{k\to\infty} \mathbb{E} H_{\mu_{0,1}}(\psi_{n,\omega}^{-1}\mathcal{P}_k|\sigma(\mathcal{P}_k)) \\
\le & \limsup_{k\to\infty} -\mathbb{E}1_{\hat{\Omega}_k^{\mathbb{N}}} 2^{-kd} \\
& \times \sum_{i=1}^{2^{kd}} \sum_{j=1}^{2^{kd}} \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \log \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \\
& + \limsup_{k\to\infty} -\mathbb{E}1_{\hat{\Omega}^{\mathbb{N}}\setminus\hat{\Omega}_k^{\mathbb{N}}} 2^{-kd} \\
& \times \sum_{i=1}^{2^{kd}} \sum_{j=1}^{2^{kd}} \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \log \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) := I_1 + I_2.
\end{aligned}
$$

Now let us enumerate all the vectors $v \in 2^{-k}\mathbb{Z}^d$ as $v(1) := 0, v(2), v(3), \ldots$ Fix a positive integer i with $v(i) \subset [0,1)^d$. We will estimate the number of sets $v(j) + [0, 2^{-k})^d$ that intersect $\psi_{n,\omega}(v(i) + [0, 2^{-k})^d)$ to estimate I_1 via Lemma 3.1.2. Note that $\mathbf{diam}(\mathcal{P}_k) = \sqrt{d}2^{-k}$, and hence for every $\omega \in \hat{\Omega}_k^{\mathbb{N}}$ by (4.1) we have

$$
\psi_{n,\omega}(v(i) + [0, 2^{-k})^d) \subset \psi_{n,\omega}(v(i)) + B_{\sqrt{d}2^{-k}} \left(\left(D_{v(i)}\psi_{n,\omega} \right) B(0, \sqrt{d}2^{-k}) \right).
$$

Therefore for every positive integers i and j and $\omega \in \hat{\Omega}_k^{\mathbb{N}}$, property

$$
\left(v(j) + [0, 2^{-k})^d \right) \cap \psi_{n,\omega}(v(i) + [0, 2^{-k})^d) \neq \emptyset
$$

implies

$$
B_{\sqrt{d}2^{-k}}(v(j)) \cap \left(\psi_{n,\omega}(v(i)) + B_{\sqrt{d}2^{-k}} \left(\left(D_{v(i)}\psi_{n,\omega} \right) B(0, \sqrt{d}2^{-k}) \right) \right) \neq \emptyset,
$$

and the latter inequality yields

$$
B_{\sqrt{d}2^{-k-1}}(v(j)) \cap \left(\psi_{n,\omega}(v(i)) + B_{\sqrt{d}2^{-k+1}} \left(\left(D_{v(i)}\psi_{n,\omega} \right) B(0, \sqrt{d}2^{-k}) \right) \right) \neq \emptyset.
$$

Therefore we have for $\omega \in \hat{\Omega}_k^{\mathbb{N}}$ by Lemma 4.1.1 applied to $A := D_{v(i)}\psi_{n,\omega}$

$$
\# \left\{ j : (v(j) + [0, 2^{-k})^d) \cap \psi_{n,\omega}(v(i) + [0, 2^{-k})^d) \neq \emptyset \right\} \le K(n, \omega, i),
$$

where

$$
K(n, \omega, i) := C(d) \prod_{u=1}^{d} (\delta_u(D_{v(i)}\psi_{n,\omega}) \vee 1).
$$

the latter inequality implies by Lemma 3.1.2

$$
\begin{aligned}
I_1 &= \limsup_{k\to\infty} -\mathbb{E}1_{\hat{\Omega}_k^{\mathbb{N}}} 2^{-kd} \sum_{i=1}^{2^{kd}} \sum_{j=1}^{2^{kd}} \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \log \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \\
&\leq \limsup_{k\to\infty} \mathbb{E}1_{\hat{\Omega}_k^{\mathbb{N}}} 2^{-kd} \sum_{i=1}^{2^{kd}} \log K(n,\omega,i) \\
&\leq \limsup_{k\to\infty} \mathbb{E}2^{-kd} \sum_{i=1}^{2^{kd}} \log K(n,\omega,i) \\
&= \limsup_{k\to\infty} \mathbb{E}\log K(n,\omega,1),
\end{aligned}
$$

where the last inequality holds because translation invariance of ψ implies the independence of the distribution of $K(n,\omega,i)$ with respect to i. Now by the definition of $K(n,\omega,1)$ we have

$$
\limsup_{k\to\infty} \mathbb{E}\log K(n,\omega,1) = \log C(d) + \mathbb{E}\left[\sum_{u=1}^{d} \log^+(\delta_u(D_0\psi_{n,\omega}))\right].
$$

Thus we have

$$
I_1 \leq \log C(d) + \mathbb{E}\left[\sum_{u=1}^{d} \log^+(\delta_u(D_0\psi_{n,\omega}))\right]. \tag{4.2}
$$

To handle the term I_2 we will again estimate the number of elements $\mathbf{P}_{k,j}$ that can intersect $\psi_{n,\omega}(v(i)+[0,2^{-k})^d)$. To do this we put

$$
L(n,\omega) := \sup_{z\in B(v(i),2\sqrt{d})} \|D_z\psi_{n,\omega}\|
$$

and observe that we have by the mean value theorem for $x,y \in v(i)+[0,2^{-k})^d$ and arbitrary $\omega \in \hat{\Omega}^{\mathbb{N}}$

$$
|\psi_n(x,\omega) - \psi_n(y,\omega)| \leq L(n,\omega)|x-y|,
$$

which implies

$$
\psi_{n,\omega}(v(i)+[0,2^{-k})^d) \subset \psi_{n,\omega}B(v(i),\sqrt{d}2^{-k}) \subset B(\psi_{n,\omega}(v(i)), L(n,\omega)\sqrt{d}2^{-k}).
$$

By Lemma 4.1.1 applied to $A := L(n,\omega)1_{\mathbb{R}^d}$ we conclude

$$
\#\left\{j : v(j)+[0,2^{-k})^d \cap \psi_{n,\omega}(v(i)+[0,2^{-k})^d) \neq \emptyset\right\} \leq C(d)(L(n,\omega)\vee 1)^d
$$

which yields

$$
\begin{aligned}
I_2 =& \limsup_{k\to\infty} \mathbb{E} 1_{\hat{\Omega}^{\mathbb{N}}\setminus\hat{\Omega}_k^{\mathbb{N}}} 2^{-kd} \sum_{i=1}^{2^{kd}} \sum_{j=1}^{2^{kd}} \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \log \mu_{0,1}(\psi_{n,\omega}^{-1}\mathbf{P}_{k,j}|\sigma(\mathbf{P}_{k,i})) \\
\leq& \limsup_{k\to\infty} \mathbb{E}\left[1_{\hat{\Omega}^{\mathbb{N}}\setminus\hat{\Omega}_k^{\mathbb{N}}} 2^{-kd} \sum_{i=1}^{2^{kd}} \log C(d) + d\log^+ L(n,\omega)\right] \\
\leq& \limsup_{k\to\infty} \mathbb{E}\left[1_{\hat{\Omega}^{\mathbb{N}}\setminus\hat{\Omega}_k^{\mathbb{N}}} \log C(d) + d\log^+ L(n,\omega)\right] \\
\leq& \limsup_{k\to\infty} \mathbb{E}\left[1_{\hat{\Omega}^{\mathbb{N}}\setminus\hat{\Omega}_k^{\mathbb{N}}} \log C(d) + d\log^+ L(n,\omega)\right].
\end{aligned}
$$

The latter estimation together with (4.2) provides

$$
\begin{aligned}
nh_\mu(\psi) \leq 2\log C(d) + & \mathbb{E}\left[\sum_{u=1}^{d} \log^+(\delta_u(D_0\psi_{n,\omega}))\right] \\
& + \limsup_{k\to\infty} \mathbb{E}\left[1_{\hat{\Omega}^{\mathbb{N}}\setminus\hat{\Omega}_k^{\mathbb{N}}} d\log^+ L(n,\omega)\right].
\end{aligned}
$$

Since $\mathbb{E}\log^+ L(n,\omega)$ is finite which follows from Assumption 1, the last term vanishes. Hence we have

$$
nh_\mu(\psi) \leq 2\log C(d) + \mathbb{E}\left[\sum_{u=1}^{d} \log^+(\delta_u(D_0\psi_{n,\omega}))\right].
$$

Now we divide by n to obtain

$$
h_\mu(\psi) \leq \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{u=1}^{d} \log^+(\delta_u(D_0\psi_{n,\omega}))\right] \leq \sum_{u=1}^{p} d_u\lambda_u,
$$

where the last inequality holds because of [25], p. 54, inequality (2.6) and [25], Proposition I.3.2. The theorem is proven.

Chapter 5

Pesin's Formula for Translation Invariant Random Dynamical Systems

Pesin's formula is a relation between Kolmogorov-Sinaĭ entropy of a smooth dynamical system and its positive Lyapunov exponents. The formula was first established by Pesin for deterministic DSs on a compact Riemannian manifold, which preserve a smooth invariant probability measure, see [32], [31], and [33]. Later different authors proved Pesin's formula in different settings, see e.g. the discussion of the formula in the introduction. Nevertheless, to the best of our knowledge, this formula has never been established before for systems without invariant probability measure. The problem is that entropy in this case is ill-posed. It is ill-posed for TIRDSs as well, because these systems do not have an invariant probability measure, but the Lebesgue measure, which is an infinite invariant measure.

In this chapter we prove Pesin's formula for volume preserving two-sided TIRDSs with entropy defined as in Definition 3.4.1. To estimate entropy from below in our case, we follow closely Mañé's approach, see [26] and [3]. The idea of Mañé is to estimate entropy from below by the exponential rate of decay of certain numbers. More precisely, these numbers are denoted as measures of Bowen balls with certain state-dependent radii, where Bowen ball (with center x and radius r) is the set of points that stay on the distance at most r from the trajectory of x during the first n iterations of the system. These radii are chosen so that they are not too small, but the Bowen balls with these radii are thin enough if we measure the thickness with respect to the unstable direction. This lets Mañé estimate measures of Bowen balls from above as required.

Our proof is divided into two parts. In the first part (Section 5.2) we consider a two-sided RDS with the fixed origin and bound from above the measures of Bowen ball of the fixed point with certain random radii. This procedure is similar to Mañé's approach, but in this case dynamics on the state space is substituted by dynamics of the shift. It turns out that the triviality of the dynamics on the state space (we always stay at the origin) lets us repeat the ideas of Mañé, even though the probability space, where the shift is defined, is non-compact. In the second part (Section 5.3) we also follow [26] and [3]. Note that in our case we measure the exponential rate of decay in the sense of $\liminf$ and not in the sense of $\limsup$ as Mañé did. This gives us the possibility to simplify the proof for our case.

We formulate the main result for two-sided systems because we use the ideas of Mañé [26] and Bahnmüller [3], which rely on negative times as well. Further, there are two reasons for us to restrict ourselves to the volume preserving case. First of all, the definition of entropy in the two-sided case (see Section 3.4) relies on the preservation of the volume. Another reason is that Mañé in [26] (and then Bahnmüller in [3]) proved Pesin's formula for two-sided systems with an absolutely continuous invariant measure. It turns out that to repeat the ideas of [26] and [3] we have to stick to the volume preserving case.

5.1 Main Result

In this chapter we prove the following theorem

Theorem 5.1.1. *Let ψ be a two-sided TIRDS defined on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$, which satisfies Assumptions 1-3, and has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Further, let the measure $\mathbf{M} = \mu \times \hat{\nu}^{\mathbb{Z}}$ be invariant for the skew product Θ. Then we have*

$$h_\mu(\psi) \geq \sum_{i=1}^{p} d_i \lambda_i^+.$$

Remark 5.1.1. *Because of Ruelle's inequality, see Proposition 4.1.1, the inequality, which appears in Theorem 5.1.1, is equivalent to*

$$h_\mu(\psi) = \sum_{i=1}^{p} d_i \lambda_i^+,$$

which is a usual form of Pesin's formula in the literature.

Remark 5.1.1 together with Theorem 2.3.1 immediately implies Pesin's formula for TIBFs. Let us formulate the result precisely.

Corollary 5.1.1. *Let ϕ be a TIBF and φ be the respective two-sided TIRDS on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$ (i.e. is constructed as in Section 2.2), which has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Further, let $\mu \times \hat{\nu}^{\mathbb{Z}}$ (defined on $\mathbb{R}^d \times \tilde{\Omega}^{\mathbb{Z}}$) be invariant for the skew product Θ. Then we have*

$$h_\mu(\varphi) = \sum_{i=1}^{p} d_i \lambda_i^+.$$

Remark 5.1.2. *The class of TIBFs, for which Corollary 5.1.1 holds, is not empty. For example, it holds for volume preserving isotropic Brownian flows (these flows are discussed for example in [11], [10] Chapter 2 and Chapter 5, and [41]).*

Now let us come back to the main result of the chapter. The proof of the theorem is similar to the proof of Pesin's formula by Mañé, see [26]; see also [3].

Before we start explaining the proof of the theorem, let us define local entropy of ψ with random radius δ, which is the key object in the proof of the theorem.

For a two-sided RDS ψ and a function $\delta : \hat{\Omega}^{\mathbb{Z}} \to (0, \infty)$ define

$$R_n^{\psi,\delta,x} := \bigcap_{j=0}^{n} \psi_{j,\omega}^{-1} B(\psi_{j,\omega}(x), \delta(\theta^j \omega)).$$

Note that $R_n^{\psi,\delta,x}$ in the case of deterministic δ are usually called Bowen balls with center x and radius δ.

Now we define objects, which are similar to local entropy and show that these objects are constants.

Theorem 5.1.2. *Let ψ be a two-sided TIRDS defined on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$ and $\delta : \hat{\Omega}^{\mathbb{Z}} \to (0, 1]$ be a measurable function. Then there exist deterministic numbers $\underline{h}^{loc}(\psi, \delta, x)$ and $\overline{h}^{loc}(\psi, \delta, x)$ (maybe equal to $+\infty$) such that for $\hat{\nu}^{\mathbb{Z}}$-a.a. ω*

$$\overline{h}^{loc}(\psi, \delta, x) = \limsup_{n\to\infty} -\frac{1}{n} \log \mu(R_n^{\psi,\delta,x}) \tag{5.1}$$

and

$$\underline{h}^{loc}(\psi, \delta, x) = \liminf_{n\to\infty} -\frac{1}{n} \log \mu(R_n^{\psi,\delta,x}). \tag{5.2}$$

Remark 5.1.3. *Note that $\overline{h}^{loc}$ and $\underline{h}^{loc}$ are similar to local entropy. However, in the definition of local entropy we additionally pass to the limit in δ, and δ is a (deterministic) number.*

Proof of Theorem 5.1.2. Without loss of generality assume that $x = 0$. Let us formulate a proposition that implies the theorem

Proposition 5.1.1. *Let $\overline{\psi}$ be a two-sided RDS defined on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$ with the fixed origin, i.e. $\overline{\psi}_{1,\omega}(0) = 0$, $\forall \omega \in \overline{\Omega}^{\mathbb{Z}}$, and $\delta : \overline{\Omega}^{\mathbb{Z}} \to (0, 1]$ be a measurable function. Then there exist deterministic numbers $\underline{h}^{loc}(\overline{\psi}, \delta, 0)$ and $\overline{h}^{loc}(\overline{\psi}, \delta, 0)$ (maybe equal to $+\infty$) such that for $\overline{\nu}^{\mathbb{Z}}$-a.a. ω*

$$\overline{h}^{loc}(\overline{\psi}, \delta, 0) = \limsup_{n\to\infty} -\frac{1}{n} \log \mu(R_n^{\overline{\psi},\delta,0}) \tag{5.3}$$

and

$$\underline{h}^{loc}(\overline{\psi}, \delta, 0) = \liminf_{n\to\infty} -\frac{1}{n} \log \mu(R_n^{\overline{\psi},\delta,0}). \tag{5.4}$$

Indeed, we can apply Proposition 5.1.1 to two-sided RDS $\overline{\psi}$ generated by i.i.d. mappings

$$\ldots \psi_{1,\theta^{-1}\omega} - \psi_{1,\theta^{-1}\omega}(0), \psi_{1,\omega} - \psi_{1,\omega}(0), \psi_{1,\theta\omega} - \psi_{1,\theta\omega}(0) \ldots,$$

and so translation invariance of ψ implies the theorem. Now let us prove Proposition 5.1.1. During the proof we abbreviate ψ instead of $\overline{\psi}$. We prove (5.3). The proof of (5.4) is the same.

It suffices to show that

$$\xi := \limsup_{n\to\infty} -\frac{1}{n}\log\mu\left(\bigcap_{i=0}^{n}\psi_{i,\omega}^{-1}(B(0,\delta(\theta^i\omega)))\right) \tag{5.5}$$

is a constant $\overline{\nu}^{\mathbb{Z}}$-almost everywhere. Recall that θ is ergodic, and so in order to prove (5.5) it suffices to check that

$$\xi(\omega) = \xi(\theta\omega), \quad \overline{\nu}^{\mathbb{Z}} - a.e. \tag{5.6}$$

Now let us prove (5.6). Note that

$$\xi = \limsup_{n\to\infty} -\frac{1}{n}\log\mu\left(R_n^{\psi,\delta,0}\right).$$

From now on during the proof of the proposition we abbreviate R_n instead of $R_n^{\psi,\delta,0}$. We start from the following chain of computations

$$\begin{aligned}
\psi_{1,\omega}^{-1}(R_n(\theta\omega)) =& \psi_{1,\omega}^{-1}\left(\bigcap_{i=0}^{n}\psi_{i,\theta\omega}^{-1}(B(0,\delta(\theta^{i+1}\omega)))\right)\\
=& \left(\bigcap_{i=0}^{n}\psi_{i+1,\omega}^{-1}(B(0,\delta(\theta^{i+1}\omega)))\right)\\
\supset& \left(\bigcap_{i=0}^{n+1}\psi_{i,\omega}^{-1}(B(0,\delta(\theta^{i}\omega)))\right) = R_{n+1}(\omega).
\end{aligned}$$

Thus,

$$R_{n+1}(\omega) \subset \psi_{1,\omega}^{-1}(\omega)(R_n(\theta\omega)).$$

Now we have

$$\begin{aligned}
\mu\left(R_{n+1}(\omega)\right) \le& \mu\left(\psi_{1,\omega}^{-1}(\omega)(R_n(\theta\omega))\right)\\
=& \int\limits_{\psi_{1,\omega}^{-1}(\omega)(R_n(\theta\omega))} 1\,\mu(dx)\\
=& \int\limits_{R_n(\theta\omega)} \det[D_y\psi_{1,\omega}^{-1}] \quad \mu(dy)\\
\le& \left(\sup_{y\in B(0,1)}\left\|D_y\psi_{1,\omega}^{-1}\right\|\right)\mu(R_n(\theta\omega)),
\end{aligned}$$

where the last inequality holds because $R_n(\theta\omega) \subset B(0,1)$. Thus

$$\mu(R_n(\omega)) \leq \left(\sup_{y\in B(0,1)} \left\| D_y \psi_{1,\omega}^{-1} \right\| \right) \mu(R_n(\theta\omega)).$$

Then $\sup_{y\in B(0,1)} \left\| D_y \psi_{1,\omega}^{-1} \right\|$ is a random variable, which does not depend on n. Therefore,

$$\limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(R_n(\omega)\right) \geq \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(R_n(\theta\omega)\right). \tag{5.7}$$

Now let us finalize the proof of (5.6). We know that
i) $\xi(\omega) \geq \xi(\theta\omega)$, $\overline{\nu}^{\mathbb{Z}}$-a.e. (due to (5.7));
ii) $\xi(\omega)$ and $\xi(\theta\omega)$ have the same distribution;
The latter two condidions directly imply (5.6). Thus, we have obtained (5.3). The proof of (5.4) is the same. The proposition is proven.

□

The rest of the chapter consists of two sections. In Section 5.2 we prove the following crucial theorem, which estimates $\underline{h}^{loc}$ in terms of the Lyapunov exponents. Note that this is an analogue of a claim in [26], see p. 101; see also [3], inequality (11).

Theorem 5.1.3. *Let ψ be a two-sided TIRDS defined on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$ which satisfies Assumptions 1-3 and also has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Then for all $\epsilon > 0$ there exists $\rho : \hat{\Omega}^{\mathbb{Z}} \to (0, 1]$ with $\log \rho \in L^1(\hat{\nu}^{\mathbb{Z}})$ such that*

$$\underline{h}^{loc}(\psi, \rho, 0) \geq (\sum_{i=1}^{p} d_i \lambda_i^+) - \epsilon. \tag{5.8}$$

Finally, in Section 5.3 we prove Theorem 5.1.1 using Theorem 5.1.3, which is basically estimation of entropy from below by $\underline{h}^{loc}$.

5.2 Proof of Theorem 5.1.3

We mainly follow here [26] and [3], Section 6.

It suffices to prove Theorem 5.1.3 for a two-sided RDS $\overline{\psi}$ on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$ (instead of ψ) generated by i.i.d. mappings

$$\ldots \psi_{1,\theta^{-1}\omega} - \psi_{1,\theta^{-1}\omega}(0), \psi_{1,\omega} - \psi_{1,\omega}(0), \psi_{1,\theta\omega} - \psi_{1,\theta\omega}(0) \ldots$$

Fix $\epsilon \in (0, 1)$. Note that both sides of (5.8) are deterministic, and therefore it suffices to show that that there exist $N \in \mathbb{N}$ and a measurable set K_ϵ with $\overline{\nu}^{\mathbb{Z}}(K_\epsilon) \geq 1 - \epsilon$ such that for all $\omega \in K_\epsilon$ we have

$$\liminf_{n\to\infty} -\frac{1}{nN} \log \mu(R_{nN}^{\overline{\psi},\rho,0}) \geq (\sum_{i=1}^{p} d_i \lambda_i^+) - \epsilon. \tag{5.9}$$

Define

$$C(\omega) := \log^+ \sup_{v \in B(0,1), E \leq \mathbb{R}^d} \left(\left| \det[D_v(\overline{\psi}_{1,\omega})^{-1}|_E] \right| \vee \left| \det[D_v \overline{\psi}_{1,\omega}|_E] \right| \right); \tag{5.10}$$

further, define

$$G(\epsilon) := \frac{1}{2} \sup\{\kappa : 2\kappa + 2 \sup_{L: \overline{\nu}^{\mathbb{Z}}(L) \leq \kappa} \mathbb{E} C^+ 1_L \leq \epsilon\} \tag{5.11}$$

Now let us define N and K_ϵ. Denote $\overline{\psi}^N_{n,\omega} := \overline{\psi}_{Nn,\omega}$ Recall that the subspaces S and U are defined by (2.10) and (2.11).

Fix large enough N so that we can define sets $K_{\epsilon,i} \subset \overline{\Omega}^{\mathbb{Z}}$, $i = \overline{1,5}$ with $\hat{\nu}^{\mathbb{Z}}(K_{\epsilon,i}) > 1 - \frac{\epsilon}{5}$ and numbers $\alpha_1 > \alpha_2 > 1$ in the following way

1. We have $\omega \in K_{\epsilon,1}$ if and only if

$$\left| D_0 \overline{\psi}^N_{n,\omega} v \right| \geq \alpha_1^n |v|, \quad n \geq 1, \quad v \in U(\omega). \tag{5.12}$$

2. We have $\omega \in K_{\epsilon,2}$ if and only if

$$\left| D_0 \overline{\psi}^N_{n,\omega} v \right| \leq \alpha_2^n |v|, \quad n \geq 1, \quad v \in S(\omega). \tag{5.13}$$

3. We have $\omega \in K_{\epsilon,3}$ if and only if

$$\log \left| \det[D_0 \overline{\psi}^N_{n,\omega}|_{U(\omega)}] \right| \geq nN(\sum_{i=1}^p d_i \lambda_i^+ - G(\epsilon)), \quad n \geq 1. \tag{5.14}$$

4. Fix M so large that $\omega \in K_{\epsilon,4}$ if and only if

$$\sup\{\gamma(U(\omega), S(\omega))\} \leq M, \tag{5.15}$$

where γ is defined in the Appendix.

5. Fix $c > 0$ and $a \in (0,1]$ such that $\omega \in K_{\epsilon,5}$ if and only if for every $v \in B(0,a)$, and for every subspace $E \subset \mathbb{R}^d$ which is an $(S(\omega), U(\omega))$-graph with dispersion $\leq c$ (see Definition A.0.1) we have

$$\left| \log \left| \det[D_v \overline{\psi}^N_{1,\omega}|_E] \right| - \log \left| \det[D_0 \overline{\psi}^N_{1,\omega}|_{U(\omega)}] \right| \right| \leq G(\epsilon) \tag{5.16}$$

Note that (5.12) and (5.13) and (5.15) hold because of Theorem 2.3.4; (5.14) holds because of Lemma 2.3.1, Theorem 3.3.3, and (5.16) holds because of the spatial smoothness of $\overline{\psi}$.

Finally, put

$$K_\epsilon := \bigcap_{i=1}^5 K_{\epsilon,i}.$$

Then we have $\overline{\nu}^{\mathbb{Z}}(K_\epsilon) > 1 - \epsilon$.

Lemma 5.2.1. *For all $c > 0$ there exists a random variable $\zeta \in (0,1)$ with $\log \zeta \in L^1(\overline{\nu}^{\mathbb{Z}})$ such that if $\omega \in K_\epsilon$ and $\theta^{Nn}\omega \in K_\epsilon$ for some positive integer n, then every $(S(\omega), U(\omega))$-graph $\mathcal{G}$ with dispersion $\leq c$ and $\mathcal{G} \subset B(0, \prod_{i=0}^{n-1} \zeta(\theta^{Ni}\omega))$ is taken by $\overline{\psi}^N_{n,\omega}$ to an $(S(\theta^{Nn}\omega), U(\theta^{Nn}\omega))$-graph with dispersion $\leq c$.*

Proof. Apply Lemma A.0.1 with $\beta_1 := \alpha_1^n$, $\beta_2 := \alpha_2^n$, $\alpha = M$ and with $r := \prod_{i=0}^{n-1} \zeta(\theta^{Ni}\omega)$. Take in (A.1) β_1 and β_2 which makes the restriction on δ_0 stronger but independent of n. Now fix δ_0 as in Lemma A.0.1 so hat it does not depend on n. To prove Lemma 5.2.1, it suffices to show Assumption (b) in Lemma A.0.1. For this purpose we define

$$C^1_{\sup}(\omega) := \max\left(1, \sup_{v \in B(0,1)} \|D_v \overline{\psi}^N_{1,\omega}\|\right),$$

$$C^2_{\sup}(\omega) := \max\left(1, \sup_{v \in B(0,1)} \|D^2_v \overline{\psi}^N_{1,\omega}\|\right),$$

and

$$C_{\sup}(\omega) := 2((C^1_{\sup}(\omega))^2 + C^2_{\sup}(\omega)).$$

Claim 5.2.1. *For all*

$$v, w \in B\left(0, 1/\prod_{i=0}^{n-1} C^1_{\sup}(\theta^{Ni}\omega)\right)$$

we have

$$\|D_v \overline{\psi}^N_{n,\omega} - D_w \overline{\psi}^N_{n,\omega}\| \leq \prod_{i=0}^{n-1} C_{\sup}(\theta^{Ni}\omega)|v - w|.$$

Proof. By the mean value theorem we have

$$\|D_v \overline{\psi}^N_{1,\omega} - D_w \overline{\psi}^N_{1,\omega}\| \leq C^2_{\sup}(\omega)|v - w|.$$

We prove the claim by induction with respect to n. The case $n = 1$ is clear, since the last inequality holds true and $C_{\sup}(\omega) > C^2_{\sup}(\omega)$. Suppose the claim holds for n. Then by the chain rule

$$\begin{aligned}
&\|D_v \overline{\psi}^N_{n+1,\omega} - D_w \overline{\psi}^N_{n+1,\omega}\| \\
=&\|D_{\overline{\psi}^N_{n,\omega} v} \overline{\psi}^N_{1,\theta^{Nn}\omega} D_v \overline{\psi}^N_{n,\omega} - D_{\overline{\psi}^N_{n,\omega} w} \overline{\psi}^N_{1,\theta^{Nn}\omega} D_w \overline{\psi}^N_{n,\omega} \\
&+ D_{\overline{\psi}^N_{n,\omega} v} \overline{\psi}^N_{1,\theta^{Nn}\omega} D_w \overline{\psi}^N_{n,\omega} - D_{\overline{\psi}^N_{n,\omega} v} \overline{\psi}^N_{1,\theta^{Nn}\omega} D_w \overline{\psi}^N_{n,\omega}\| \\
\leq&\|D_{\overline{\psi}^N_{n,\omega} v} \overline{\psi}^N_{1,\theta^{Nn}\omega} - D_{\overline{\psi}^N_{n,\omega} w} \overline{\psi}^N_{1,\theta^{Nn}\omega}\| \|D_w \overline{\psi}^N_{n,\omega}\| \\
&+ \|D_{\overline{\psi}^N_{n,\omega} v} \overline{\psi}^N_{1,\theta^{Nn}\omega}\| \|D_v \overline{\psi}^N_{n,\omega} - D_w \overline{\psi}^N_{n,\omega}\| := I.
\end{aligned}$$

Note that the mean value theorem implies for $0 \leq m \leq n+1$:

$$|\overline{\psi}_{m,\omega}^{N} v| = |\overline{\psi}_{m,\omega}^{N} v - \overline{\psi}_{m,\omega}^{N} 0| \leq \left(\prod_{i=m}^{n} C_{\sup}^{1}(\theta^{Ni}\omega) \right)^{-1} \leq 1. \tag{5.17}$$

The latter inequality and the mean value theorem implies

$$\|D_{\overline{\psi}_{n,\omega}^{N} v} \overline{\psi}_{1,\theta^{Nn}\omega}^{N} - D_{\overline{\psi}_{n,\omega}^{N} w} \overline{\psi}_{1,\theta^{Nn}\omega}^{N}\| \leq C_{\sup}^{2}(\theta^{Nn}\omega) |\overline{\psi}_{n,\omega}^{N} v - \overline{\psi}_{n,\omega}^{N} w|.$$

Moreover, because of the definition of $C_{\sup}^{1}$ and the chain rule and (5.17) we have

$$\|D_{w} \overline{\psi}_{n,\omega}^{N}\| \leq \prod_{i=0}^{n-1} C_{\sup}^{1}(\theta^{Ni}\omega).$$

Further, because of (5.17) we have

$$\|D_{\overline{\psi}_{n,\omega}^{N} v} \overline{\psi}_{1,\theta^{Nn}\omega}^{N}\| \leq C_{\sup}^{1}(\theta^{Nn}\omega).$$

The latter three inequalities together with the induction assumption imply

$$\begin{aligned} I \leq & C_{\sup}^{2}(\theta^{Nn}\omega) |\overline{\psi}_{n,\omega}^{N} v - \overline{\psi}_{n,\omega}^{N} w| \prod_{i=0}^{n-1} C_{\sup}^{1}(\theta^{Ni}\omega) \\ & + C_{\sup}^{1}(\theta^{Nn}\omega) \prod_{i=0}^{n-1} C_{\sup}(\theta^{Ni}\omega) |v - w| \\ \leq & C_{\sup}^{2}(\theta^{Nn}\omega) (\prod_{i=0}^{n-1} C_{\sup}^{1}(\theta^{Ni}\omega))^{2} \\ & + \leq \prod_{i=0}^{n} C_{\sup}(\theta^{Ni}\omega) |v - w|, \end{aligned}$$

where the latter inequality holds by the definition of $C_{\sup}$. Thus, we have

$$\begin{aligned} |\overline{\psi}_{n,\omega}^{N} v - \overline{\psi}_{n,\omega}^{N} w| \prod_{i=0}^{n-1} C_{\sup}^{1}(\theta^{Ni}\omega) & \leq \prod_{i=0}^{n-1} L^{2}(\theta^{Ni}\omega) |v - w| \\ & \leq \prod_{i=0}^{n-1} C_{\sup}(\theta^{Ni}\omega) |v - w|. \end{aligned}$$

The claim is proven.

□

Denote $\zeta(\omega) := \delta_0 / C_{\sup}(\omega)$ (we can assume $\delta_0 < 1$). Then, according to Claim 5.2.1, for $v \in B\left(0, \prod_{i=0}^{n-1} \zeta(\theta^{Ni}\omega)\right)$ we obtain

$$\begin{aligned}\|D_0\overline{\psi}^N_{n,\omega} - D_v\overline{\psi}^N_{n,\omega}\| &\le \prod_{i=0}^{n-1} C_{\sup}(\theta^{Ni}\omega)|v| \\ &\le \prod_{i=0}^{n-1} C_{\sup}(\theta^{Ni}\omega)\zeta(\theta^{Ni}\omega) = \delta_0^n \le \delta_0.\end{aligned}$$

Finally, $\log\zeta$ is integrable because of Assumptions 1 and 3. The lemma is proven. □

We write

$$D_r(\omega) := \{y_1 + y_2 : y_1 \in S(\omega), y_2 \in U(\omega), |y_1| < r, |y_2| < r\}.$$

Further, let k_1 and k_2 be constants such that for all $\omega \in K_\epsilon$ and for all positive r

$$B(0, k_1 r) \subset D_r(\omega) \subset B(0, k_2 r); \tag{5.18}$$

this is possible because of (5.15). Note that $D_r(\omega)$ is an open subset of $\mathbb{R}^d$.

Now let us define $\rho = \rho_\epsilon(\overline{\psi}, \omega)$. If $\omega \in K_\epsilon$ define $N_K(\omega)$ as the minimum positive integer, such that $\theta^{NN_K}(\omega) \in K_\epsilon$. By Poincaré recurrence theorem N_K is well defined for a.a. $\omega \in K_\epsilon$. We extend N_K to $\overline{\Omega}^{\mathbb{Z}}$ by putting $N_K(\omega) := 0$ for $\omega \in \overline{\Omega}^{\mathbb{Z}} \backslash K_\epsilon$. Recall that the number a is defined in the beginning of the section, see (5.16).

Definition 5.2.1. *Define the random variable $\rho = \rho_\epsilon(\overline{\psi}, \omega)$ by*

$$\rho(\omega) := \frac{k_1}{k_2} \min(a, \prod_{i=0}^{N_K(\omega)-1} \zeta(\theta^{Ni}\omega)),$$

where ζ is defined in Lemma 5.2.1.

We prove Theorem 5.1.3 with ρ defined as in Definition 5.2.1.

Now recall that $\log\zeta$ is integrable. Then $\log\rho$ is integrable as well because of the following lemma

Lemma 5.2.2. *Let $A \in \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}$ and N_A the return function of A If f is a non-negative and integrable random variable, then*

$$\mathbb{E} \sum_{i=0}^{N_A(\omega)-1} f(\theta^{Ni}\omega) 1_A(\omega) \le \mathbb{E} f.$$

Proof. For $j \ge 1$ define $W_j := \{\omega \in A : N_A(\omega) = j\}$. Then up to a set of probability zero we have

$$\bigcup_{n=0}^{\infty} \theta^{Nn}(A) = \bigcup_{j=1}^{\infty} \bigcup_{i=0}^{j-1} \theta^{Ni}(W_j), \tag{5.19}$$

where the latter equality holds because both sides represent the "wandering" of A with respect to θ^N; the sets in the RHS are disjoint, because $\theta^{Ni}(W_j)$ is the set of points, that started from A exactly i steps before and will come back to A in $j-i$ steps. Now we have

$$\begin{aligned}
\mathbb{E}\sum_{i=0}^{N_A(\omega)-1} f(\theta^{Ni}\omega)1_A(\omega) &= \mathbb{E}\sum_{i=0}^{N_A(\omega)-1}\sum_{j=1}^{\infty} f(\theta^{Ni}\omega)1_{W_j}(\omega) \\
&= \sum_{j=1}^{\infty}\mathbb{E}\sum_{i=0}^{N_A(\omega)-1} f(\theta^{Ni}\omega)1_{W_j}(\omega) \\
&= \sum_{j=1}^{\infty}\mathbb{E}\sum_{i=0}^{j-1} f(\theta^{Ni}\omega)1_{W_j}(\omega) \\
&= \sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\mathbb{E}f(\theta^{Ni}\omega)1_{W_j}(\omega) \\
&= \sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\int\limits_{W_j} f\circ\theta^{Ni} d\overline{\nu}^{\mathbb{Z}}.
\end{aligned}$$

Now let us make a change of the measure

$$\overline{\nu}_i^{\mathbb{Z}} = \theta^{Ni}\overline{\nu}^{\mathbb{Z}};$$

Then measures $\overline{\nu}_i^{\mathbb{Z}}$ and $\overline{\nu}^{\mathbb{Z}}$ have the same distributions, and hence

$$\sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\int\limits_{W_j} f\circ\theta^{Ni} d\overline{\nu}^{\mathbb{Z}} = \sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\int\limits_{\theta^{Ni}(W_j)} f\, d\overline{\nu}^{\mathbb{Z}}.$$

Finally, since the sets $\theta^{Ni}(W_j)$ are disjoint and equality (5.19) holds, we have

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\int\limits_{\theta^{Ni}(W_j)} f\, d\overline{\nu}^{\mathbb{Z}} &= \int\limits_{\bigcup_{j=1}^{\infty}\bigcup_{j=1}^{j-1}\theta^{Ni}(W_j)} f\, d\overline{\nu}^{\mathbb{Z}} \\
&\overset{(5.19)}{=} \int\limits_{\bigcup_{n=0}^{\infty}\theta^{Nn}(A)} f\, d\overline{\nu}^{\mathbb{Z}} \leq \mathbb{E}f,
\end{aligned}$$

which completes the proof of the lemma. □

From now on we will abbreviate $R_n^{\overline{\psi},\rho}$ instead of $R_n^{\overline{\psi},\rho,0}$ if there is no risk of ambiguity. Fix any $\omega \in K_\epsilon$. There exists $B(\omega) > 0$, depending only on $(S(\omega), U(\omega))$, such that for all $n \geq 0$

$$\mu(R_{nN}^{\overline{\psi},\rho}) = B(\omega)\int\limits_{S(\omega)} \mu^y(y+U(\omega)) \cap R_{nN}^{\overline{\psi},\rho}\, d\mu^s(y),$$

where μ^s denotes the Lebesgue measure on S, and μ^y the Lebesgue measure on

$y+U$. Define

$$A := \{y \in S : y + U \cap R_{nN}^{\overline{\psi},\rho} \neq \emptyset\}.$$

This is a bounded subset of S. Thus

$$\int_S \mu^y(y+U) \cap R_{nN}^{\overline{\psi},\rho}\, d\mu^s(y) \leq \sup_{y\in S} \mu^y((y+U) \cap R_{nN}^{\overline{\psi},\rho})\mu^s(A).$$

Therefore, to show (5.9) it suffices to prove that

$$\liminf_{n\to\infty} \inf_{y\in S} -\frac{1}{nN} \log \mu^y((y+U) \cap R_{nN}^{\overline{\psi},\rho}) \geq (\sum_{i=1}^p d_i\lambda_i^+) - \epsilon. \tag{5.20}$$

Now define for $y \in S(\omega)$

$$\Lambda_n^y(\omega) := \{v \in y + U(\omega) : \overline{\psi}_j^N(\omega)v \in D_{\rho(\theta^{Nj}\omega)/k_1}(\theta^{Nj}\omega),\, 0 \leq j \leq n\}.$$

By the definition of $R_{nN}^{\overline{\psi},\rho}$ and (5.18) we have

$$\Lambda_n^y(\omega) \supset (y + U(\omega)) \cap R_{nN}^{\overline{\psi},\rho}.$$

Therefore, to show (5.20), it suffices to prove that

$$\liminf_{n\to\infty} \inf_{y\in S} -\frac{1}{nN} \log \mu^y(\Lambda_n^y \cap R_{nN}^{\overline{\psi},\rho}) \geq (\sum_{i=1}^p d_i\lambda_i^+) - \epsilon. \tag{5.21}$$

Lemma 5.2.3. *If $\omega \in K_\epsilon$, $\theta^{Nn}\omega \in K_\epsilon$, $y \in S(\omega)$ and $\Lambda_n^y(\omega) \neq \emptyset$, then $\overline{\psi}_{n,\omega}^N \Lambda_n^y(\omega)$ is an $(S(\theta^{Nn}\omega), U(\theta^{Nn}\omega))$-graph with dispersion $\leq c$.*

Proof. Fix $\omega \in K_\epsilon$. We prove the lemma by induction with respect to n. For $n = 0$, by the definition of Λ, we have

$$\Lambda_0^y(\omega) = (y + U(\omega)) \cap D_{\rho(\omega)/k_1}(\omega).$$

Therefore $\Lambda_0^y(\omega)$ is an open subset of the $(S(\theta^{Nn}\omega), U(\theta^{Nn}\omega))$-graph $y + U(\omega)$ with dispersion 0. We assume that the claim is valid for n. If $\theta^{N(n+m)}\omega \in K_\epsilon$, $\theta^{N(n+j)}\omega \in \overline{\Omega}^{\mathbb{Z}}/K_\epsilon$, $1 \leq j \leq m-1$ and $\Lambda_{n+m}^y(\omega) \neq \emptyset$, then the definition of Λ implies

$$\begin{aligned}\overline{\psi}_{n+m,\omega}^N \Lambda_{n+m}^y(\omega) &= \overline{\psi}_{m,\theta^{Nn}\omega}^N \overline{\psi}_{n,\omega}^N \Lambda_{n+m}^y(\omega) \\ &\subset \overline{\psi}_{m,\theta^{Nn}\omega}^N \overline{\psi}_{n,\omega}^N \Lambda_n^y(\omega) \\ &\quad \cap D_{\rho(\theta^{N(n+m)}\omega)/k_1}(\theta^{N(n+m)}\omega).\end{aligned}$$

By our assumption $\overline{\psi}_{n,\omega}^N \Lambda_n^y(\omega)$ is an $(S(\theta^{Nn}\omega), U(\theta^{Nn}\omega))$-graph with dispersion $\leq c$ and we have

$$
\begin{aligned}
\overline{\psi}^N_{n,\omega}\Lambda^y_n(\omega) &\subset D_{\rho(\theta^{Nn}\omega)/k_1}(\theta^{Nn}\omega) \overset{(5.18)}{\subset} \\
&\subset B(0,\rho(\theta^{Nn}\omega)k_2/k_1) \\
&\subset B\left(0, \prod_{i=0}^{N_K(\theta^{Nn}\omega)-1} \zeta(\theta^{N(n+i)}\omega)\right) \\
&= B(0, \prod_{i=0}^{m-1} \zeta(\theta^{N(n+i)}\omega)).
\end{aligned}
$$

By Lemma 5.2.1 we have $\overline{\psi}^N_{m,\theta^{Nn}\omega}\overline{\psi}^N_{n,\omega}\Lambda^y_n(\omega)$ is an $(S(\theta^{Nn}\omega), U(\theta^{Nn}\omega))$-graph with dispersion $\le c$, and so is its open subset $\overline{\psi}^N_{n+m,\omega}\Lambda^y_{n+m}(\omega)$.

□

Now let us show (5.21). Choose $D > 0$ such that $D > vol^{\dim U}(\mathcal{G})$ for every C^1 $(S(\omega), U(\omega))$-graph with dispersion $\le c$ contained in $D_{\rho(\omega)/k_1}(\omega), \omega \in K_\epsilon$, where $vol^m(S)$ denotes m-dimensional volume of the set $S \subset \mathbb{R}^d$. If $\theta^{Nn}\omega \in K_\epsilon$ and $y \in S(\omega)$, we obtain by Lemma 5.2.3 and the transformation formula for differentiable maps

$$
\begin{aligned}
D > vol^{\dim U}(\overline{\psi}^N_{n,\omega}\Lambda^y_n(\omega)) &= \int_{\Lambda^y_n(\omega)} \left|\det[D_v\overline{\psi}^N_{n,\omega}|_{U(\omega)}]\right| d\mu^y(v) \\
&\ge \int_{\Lambda^y_n(\omega)\cap R^{\overline{\psi},\rho}_{nN}} \left|\det[D_v\overline{\psi}^N_{n,\omega}|_{U(\omega)}]\right| d\mu^y(v).
\end{aligned} \tag{5.22}
$$

We put

$$J_{n,N:} = \{0 \le j \le n : \theta^{Nj}\omega \in K_\epsilon\},$$

and

$$J^c_{n,N} := \{0 \le j \le n : \theta^{Nj}\omega \in \overline{\Omega}^{\mathbb{Z}} \backslash K_\epsilon\}.$$

Recall that $C(\omega)$ is defined via (5.10). Now define

$$C_N(\omega) := \log^+ \sup_{v\in R^{\overline{\psi},\rho}_N, E\le\mathbb{R}^d} \left(\left|\det[D_v(\overline{\psi}^N_{1,\omega})^{-1}|_E]\right| \vee \left|\det[D_v\overline{\psi}^N_{1,\omega}|_E]\right|\right),$$

where $E \le \mathbb{R}^d$ means that E is a subspace of $\mathbb{R}^d$. We have

$$\left|\det[D_v(\overline{\psi}_{1,\omega})^{-1}|_E]\right| \le \left\|D_v(\overline{\psi}_{1,\omega})^{-1}\right\|^{\dim E},$$

and

$$\left|\det[D_v\overline{\psi}_{1,\omega}|_E]\right| \le \left\|D_v\overline{\psi}_{1,\omega}\right\|^{\dim E};$$

hence

$$C^+(\omega) \le d\log^+ \sup_{v\in B(0,1)} \left(\left\|D_v(\overline{\psi}_{1,\omega})^{-1}\right\| \vee \left\|D_v\overline{\psi}_{1,\omega}\right\|\right).$$

Therefore, $C^+ \in L^1(\overline{\nu}^{\mathbb{Z}})$ by Assumptions 1 and 2. Moreover $C_N^+ \in L^1(\overline{\nu}^{\mathbb{Z}})$ because

$$C_N^+(\omega) \leq \sum_{i=0}^{N-1} C^+(\theta^i \omega). \tag{5.23}$$

Now fix $v \in \Lambda_n^y(\omega) \cap R_{Nn}^{\overline{\psi},\rho}$. By the chain rule we have

$$\begin{aligned}
&\log \left|\det[D_v \overline{\psi}_{n,\omega}^N|_{U(\omega)}]\right| \\
=&\sum_{j=0}^{n-1} \log \left|\det[D_{\overline{\psi}_{j,\omega}^N v} \overline{\psi}_{1,\theta^{Nj}\omega}^N|_{\left(D_v \overline{\psi}_{j,\omega}^N\right) U(\omega)}]\right| \\
=&\sum_{j \in J_{n-1,N}} \log \left|\det[D_{\overline{\psi}_{j,\omega}^N v} \overline{\psi}_{1,\theta^{Nj}\omega}^N|_{\left(D_v \overline{\psi}_{j,\omega}^N\right) U(\omega)}]\right| \\
&+ \sum_{j \in J_{n-1,N}^c} \log \left|\det[D_{\overline{\psi}_{j,\omega}^N v} \overline{\psi}_{1,\theta^{Nj}\omega}^N|_{\left(D_v \overline{\psi}_{j,\omega}^N\right) U(\omega)}]\right| \\
=&\sum_{j \in J_{n-1,N}} \log \left|\det[D_{\overline{\psi}_{j,\omega}^N v} \overline{\psi}_{1,\theta^{Nj}\omega}^N|_{\left(D_v \overline{\psi}_{j,\omega}^N\right) U(\omega)}]\right| \\
&- \sum_{j \in J_{n-1,N}^c} \log \left|\det[D_{\overline{\psi}_{j+1,\omega}^N v} (\overline{\psi}_{1,\theta^{Nj}\omega}^N)^{-1}|_{\left(D_v \overline{\psi}_{j+1,\omega}^N\right) U(\omega)}]\right| \\
\geq&\sum_{j \in J_{n-1,N}} \log \left|\det[D_{\overline{\psi}_{j,\omega}^N v} \overline{\psi}_{1,\theta^{Nj}\omega}^N|_{\left(D_v \overline{\psi}_{j,\omega}^N\right) U(\omega)}]\right| - \sum_{j \in J_{n-1,N}^c} C_N^+(\theta^{Nj}\omega),
\end{aligned}$$

where the last inequality holds because $v \in R_{Nn}^{\overline{\psi},\rho}$. Now Lemma 5.2.3 implies that $\left(D_v \overline{\psi}_{j,\omega}^N\right) U(\omega)$ is a $(U(\theta^j\omega), S(\theta^j\omega))$-graph with Lyapunov norm $\leq c$. Further, by the definition of Λ we have

$$\overline{\psi}_{j,\omega}^N v \in D_{\rho(\theta^{nj}\omega)/k_1} \overset{(5.18)}{\subset} B\left(0, k_2 \rho(\theta^{Nj}\omega)/k_1\right) \subset B(0, a).$$

Therefore, (5.16) implies

$$
\begin{aligned}
&\sum_{j\in J_{n-1,N}} \log\left|\det[D_{\overline{\psi}^N_{j,\omega}v}\overline{\psi}^N_{1,\theta^{Nj}\omega}|_{\left(D_v\overline{\psi}^N_{j,\omega}\right)U(\omega)}]\right| - \sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega)\\
&\geq \sum_{j\in J_{n-1,N}} \log\left|\det[D_0\overline{\psi}^N_{1,\theta^{Nj}\omega}|_{U(\theta^{Nj}\omega)}]\right| - G(\epsilon)n - \sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega)\\
&\geq \sum_{j=0}^{n-1} \log\left|\det[D_0\overline{\psi}^N_{1,\theta^{Nj}\omega}|_{U(\theta^{Nj}\omega)}]\right| - G(\epsilon)n - 2\sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega)\\
&= \log\left|\det[D_0\overline{\psi}^N_{n,\omega}|_{U(\omega)}]\right| - G(\epsilon)n - 2\sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega)\\
&\overset{(5.14)}{\geq} Nn(\sum_{i=1}^p d_i\lambda_i^+ - G(\epsilon)) - G(\epsilon)n - 2\sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega).
\end{aligned}
$$

Thus, for all $v \in \Lambda^y_n(\omega)$ we have

$$
\begin{aligned}
\log\left|\det[D_v\overline{\psi}^N_{n,\omega}|_{U(\omega)}]\right| \geq & Nn(\sum_{i=1}^p d_i\lambda_i^+ - G(\epsilon)) - G(\epsilon)n\\
&-2\sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega).
\end{aligned}
\tag{5.24}
$$

Then (5.22) and (5.24) together imply for all $y \in S(\omega)$

$$
\begin{aligned}
D >& \mu^y(\Lambda^y_n \cap R^{\overline{\psi},\rho}_{Nn})\\
&\times \exp\left(nN(\sum_{i=1}^p d_i\lambda_i^+ - G(\epsilon)) - G(\epsilon)n - 2\sum_{j\in J^c_{n-1,N}} C^+_N(\theta^{Nj}\omega)\right)\\
\geq& \mu^y(\Lambda^y_n \cap R^{\overline{\psi},\rho}_{Nn})\\
&\times \exp\left(nN(\sum_{i=1}^p d_i\lambda_i^+ - G(\epsilon)) - G(\epsilon)n - 2\sum_{j\in J^c_{n-1,N}} \sum_{i=0}^{N-1} C^+(\theta^{Nj+i}\omega)\right),
\end{aligned}
$$

where the last inequality holds because of (5.23). By taking logarithms and dividing by n we obtain

$$
\begin{aligned}
-\frac{1}{n}\log\mu^y(\Lambda^y_n(\omega)) >& -\frac{1}{n}\log D + N(\sum_{i=1}^p d_i\lambda_i^+ - G(\epsilon)) - G(\epsilon)\\
&-\frac{2}{n}\sum_{j\in J^c_{n-1,N}} \sum_{i=0}^{N-1} C^+(\theta^{Nj+i}\omega).
\end{aligned}
\tag{5.25}
$$

By Birkhoff's ergodic theorem we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j\in J^c_{n-1,N}}\sum_{i=0}^{N-1}C^+(\theta^{Nj+i}\omega) =& \lim_{n\to\infty}\frac{1}{n}\sum_{j\in J^c_{n-1,N}}\sum_{i=0}^{N-1}C^+(\theta^{Nj+i}\omega)1_{\overline{\Omega}^{\mathbb{Z}}\setminus K_\epsilon}(\theta^{Nj}\omega)\\ =&\mathbb{E}\sum_{i=0}^{N-1}C^+(\theta^i\omega)1_{\overline{\Omega}^{\mathbb{Z}}\setminus K_\epsilon}(\omega).\end{aligned}$$

Now because of the definition of $G(\epsilon)$ we have

$$\mathbb{E}\sum_{i=0}^{N-1}C^+(\theta^i\omega)1_{\overline{\Omega}^{\mathbb{Z}}\setminus K_\epsilon}(\omega)\le N(\epsilon-G(\epsilon)),$$

and therefore

$$NG(\epsilon)+\lim_{n\to\infty}\frac{2}{n}\sum_{j\in J^c_{n-1,N}}C^+_N(\theta^{Nj}\omega)\le N\epsilon-NG(\epsilon).$$

Combining the last inequality with (5.25) and dividing by N, we obtain for n large enough (n does not depend on y)

$$-\frac{1}{nN}\log\mu^y(\Lambda^y_n(\omega)\cap R^{\overline{\psi},\rho}_{Nn})>-\frac{1}{nN}\log D+(\sum_{i=1}^p d_i\lambda_i^+-\epsilon)+\frac{(N-1)G(\epsilon)}{N}.$$

Thus, taking inf over y in the latter inequality first, and then taking $\liminf$ in both sides completes the proof of (5.21). The theorem is proven.

5.3 Proof of Pesin's Formula using Theorem 5.1.3

In this section we prove Theorem 5.1.1. Let us start with the following lemma

Lemma 5.3.1. *If* $x_n\in[0,1)$ *for* $n\ge 1$ *and*

$$\sum_{n=1}^{\infty}nx_n<\infty$$

then (with $0\log 0:=0$*)*

$$-\sum_{n=1}^{\infty}x_n\log x_n<\infty.$$

Proof. See [26], Lemma 1. □

Recall that $\mathbf{M}=\mu\times\hat{\nu}^{\mathbb{Z}}$ is invariant for Θ. Recall that $\mathbf{M}_{0,1}:=\mathbf{M}|_{[0,1)^d\times\hat{\Omega}^{\mathbb{Z}}}$. Further, for a countable measurable partition $\mathcal{Z}$ of $\mathbb{R}^d\times\hat{\Omega}^{\mathbb{Z}}$ denote by $\mathcal{Z}^\omega$ the restriction of $\mathcal{Z}$ to $\mathbb{R}^d\times\{\omega\}$. Now we formulate an analogue of Lemma 2 from [26]; see also Lemma 3.3 from [3].

Lemma 5.3.2. *Let $\delta : \mathbb{R}^d \times \hat{\Omega}^{\mathbb{Z}} \to (0,1]$ be a measurable function with*

$$- \int\limits_{[0,1)^d} \mathbb{E} \log \delta(x, \omega) \, dx < \infty. \tag{5.26}$$

Then there is a countable measurable partition $\mathcal{Z}$ of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{Z}}$ with entropy $H_{\mathbf{M}_{0,1}}(\mathcal{Z}) < \infty$ such that for all $x \in \mathbb{R}^d$ and for all $\omega \in \hat{\Omega}^{\mathbb{Z}}$ we have

$$\mathbf{diam}\mathcal{Z}^{\omega}(x) \leq \delta(x, \omega),$$

where $\mathbf{diam}(\cdot)$ denotes the diameter of a set in $\mathbb{R}^d$, and for a partition $\mathcal{P}$ of $\mathbb{R}^d$ the set $\mathcal{P}(x)$ denotes the element of $\mathcal{P}$ which contains x.

Proof. We mainly follow here Lemma 2 from [26]; see also Lemma 3.3 from [3].

Put $U_n := \{(x, \omega) : 4^{-(n+1)} < \delta(x, \omega) \leq 4^{-n}\}$, $n \geq 0$. This generates a countable measurable partition $\mathcal{U} = \{U_n\}_{n \geq 0}$ of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{Z}}$. The integrability of $\log \delta$ implies for every $m \geq 1$

$$\sum_{n=0}^{m} n\mathbf{M}_{0,1}(U_n) \leq \sum_{n=0}^{m} \int\limits_{U_n} - \log \delta(x, \omega) d\mathbf{M}_{0,1} \leq \int\limits_{\mathbb{R}^d \times \hat{\Omega}^{\mathbb{Z}}} - \log \delta(x, \omega) d\mathbf{M}_{0,1} < \infty.$$

Hence,

$$\sum_{n=0}^{\infty} n\mathbf{M}_{0,1}(U_n) < \infty. \tag{5.27}$$

Recall that $\mathcal{P}_k := \{v + [0, 2^{-k})^d + \mathbb{Z}^d | v \in 2^{-k}\mathbb{Z}^d \cap [0,1)^d\}$. Define a partition $\mathcal{Z}$ in the following way

$$\mathcal{Z} := \{U_n \cap P \times \hat{\Omega}^{\mathbb{Z}} : P \in \mathcal{P}_{2n+d}, n \geq 0\}.$$

Then

$$H_{\mu_{0,1}}(\mathcal{Z}) = \sum_{n=0}^{\infty} (- \sum_{z \in \mathcal{Z}, Z \subset U_n} \mathbf{M}_{0,1}(Z) \log \mathbf{M}_{0,1}(Z)).$$

Therefore

$$\begin{aligned}
& - \sum_{z \in \mathcal{Z}, Z \subset U_n} \mathbf{M}_{0,1}(Z) \log \mathbf{M}_{0,1}(Z) \leq \mathbf{M}_{0,1}(U_n) \\
& \times \left(- \sum_{z \in \mathcal{Z}} \frac{\mathbf{M}_{0,1}(Z \cap U_n)}{\mathbf{M}_{0,1}(U_n)} \log \frac{\mathbf{M}_{0,1}(Z \cap U_n)}{\mathbf{M}_{0,1}(U_n)} - \log \mathbf{M}_{0,1}(U_n) \right) \\
& \leq \mathbf{M}_{0,1}(U_n)(d(n + d/2) \log 4 - \log \mathbf{M}_{0,1}(U_n)),
\end{aligned}$$

where the last inequality holds because of Lemma 3.1.2. Now summing over n we get

$$H_{\mu_{0,1}}(\mathcal{Z}) \leq 2\log 4 \sum_{n=0}^{\infty} d(n+d/2)\mathbf{M}_{0,1}(U_n) - \sum_{n=0}^{\infty} \mathbf{M}_{0,1}(U_n)\log \mathbf{M}_{0,1}(U_n).$$

By (5.27) and Lemma 5.3.1 we obtain

$$H_{\mathbf{M}_{0,1}}(\mathcal{Z}) < \infty.$$

By the definition of $\mathcal{Z}$ it is clear that

$$\mathbf{diam}\mathcal{Z}^{\omega}(x) \leq \delta(x,\omega)$$

which finishes the proof. The lemma is proven.

$\square$

Now fix $\epsilon > 0$. Let us define $\delta = \delta_\epsilon(x,\omega)$. We define it in the same way as ρ, but now with respect to the moving point $\phi_{n,\omega}(x)$ and not with respect to the origin, as in the case of ρ. To avoid ambiguity we provide the strict definition of δ.

Definition 5.3.1. *Recall that $C(\omega)$ and $G(\epsilon)$ are defined via (5.10) and (5.11) respectively. Fix large enough N so that we can define sets $K^x_{\epsilon,i} \subset \hat{\Omega}^{\mathbb{Z}}$, $i = \overline{1,5}$ with $\hat{\nu}^{\mathbb{Z}}\left(K^x_{\epsilon,i}\right) > 1 - \frac{\epsilon}{5}$, $i = \overline{1,5}$ and numbers $\alpha_1 > \alpha_2 > 1$ in the following way (recall that the subspaces S_x and U_x are defined as the subspaces S and U, but with respect to x and not with respect to zero)*

1. We have $\omega \in K^x_{\epsilon,1}$ if and only if

$$\left|D_x\psi^N_{n,\omega}v\right| \geq \alpha_1^n|v|, \quad n \geq 1, \quad v \in U_x(\omega).$$

2. We have $\omega \in K_{\epsilon,2}$ if and only if

$$\left|D_x\psi^N_{n,\omega}v\right| \leq \alpha_2^n|v|, \quad n \geq 1, \quad v \in S_x(\omega).$$

3. We have $\omega \in K_{\epsilon,3}$ if and only if

$$\log\left|\det[D_x\psi^N_{n,\omega}|_{U_x(\omega)}]\right| \geq nN(\sum_{i=1}^{p} d_i\lambda_i^+ - G(\epsilon)).$$

4. Fix M so large that $\omega \in K^x_{\epsilon,4}$ if and only if

$$\sup\{\gamma(U_x(\omega), S_x(\omega))\} < M,$$

where γ is defined in the Appendix.

5. Fix $c > 0$ and $a \in (0,1]$ such that $\omega \in K^x_{\epsilon,5}$ if and only if for every $v \in B(x,a)$, and for every subspace $E \subset \mathbb{R}^d$ which is an $(S_x(\omega), U_x(\omega))$-graph with dispersion $\leq c$ (see Definition A.0.1) we have

$$\left|\log\left|\det[D_v\psi^N_{1,\omega}|_E]\right| - \log\left|\det[D_x\psi^N_{1,\omega}|_{U_x(\omega)}]\right|\right| \leq G(\epsilon).$$

For $\omega \in K_\epsilon$ define $N_K^x(\omega)$ as the minimum positive integer, such that $\theta^{NN_K^x}(\omega) \in K_\epsilon^x$. We extend N_K^x to $\hat{\Omega}^{\mathbb{Z}}$ by putting $N_K^x(\omega) := 0$ for $\omega \in \hat{\Omega}^{\mathbb{Z}} \backslash K_\epsilon^x$. Now define

$$C^1_{\sup}(x,\omega) := \max\left(1, \sup_{v\in B(x,1)} \|D_v \psi^N_{1,\omega}\|\right),$$

$$C^2_{\sup}(x,\omega) := \max\left(1, \sup_{v\in B(x,1)} \|D^2_v \psi^N_{1,\omega}\|\right),$$

and

$$\zeta(x,\omega) := \frac{\delta_0}{2(((C^1_{\sup}(x,\omega))^2 + C^2_{\sup}(x,\omega))},$$

where δ_0 is as in Lemma A.0.1, when $\beta_1 := \alpha_1^n$, $\beta_2 := \alpha_2^n$, $\alpha = M$ and with $r := \prod_{i=0}^{n-1} \zeta(\psi_{Ni,\omega}(x)\theta^{Ni}\omega)$. Additionally take in (A.1) β_1 and β_2 which makes the restriction on δ_0 stronger but independent of n. Now fix δ_0 as in Lemma A.0.1 so hat it does not depend on n (we can assume $\delta_0 < 1$). Then define

$$D^x_r(\omega) := \{y_1 + y_2 : y_1 \in S_x(\omega), y_2 \in U_x(\omega), |y_1| < r, |y_2| < r\}.$$

Further, let k_1 and k_2 be constants such that for all $\omega \in K^x_\epsilon$ and for all positive r

$$B(0, k_1 r) \subset D^x_r(\omega) \subset B(0, k_2 r).$$

Finally, define

$$\delta(x,\omega) := \frac{k_1}{k_2} \min(a, \prod_{i=0}^{N_K^x(\omega)-1} \zeta(\psi^N_{i,\omega}(x), \theta^{Ni}\omega)).$$

Now take the partition $\mathcal{Z}$ given by Lemma 5.3.2, which corresponds to $\delta = \delta_\epsilon$. Note that for such a δ holds (5.26), because

$$\mathbb{E}\log\rho(\omega) < \infty,$$

and ψ is translation invariant. Define

$$\mathcal{Z}^\omega_n := \bigvee_{i=0}^{n-1} \psi^{-1}_{i,\omega} \mathcal{Z}_{\theta^i\omega}.$$

Now let us show that

$$h_\mu(\psi) \geq \int\limits_{\mathbb{R}^d\times\hat{\Omega}^{\mathbb{Z}}} \liminf_{n\to\infty} -\frac{1}{n} \log \mu_{0,1}\left((\mathcal{Z}^\omega_n \cap [0,1)^d)(x)\right) d\mathbf{M}_{0,1}. \tag{5.28}$$

Enlarge the class $\mathcal{B}^{\mathbb{Z}}_{tr}$ to $\mathcal{B}'_{tr}$ in the following way

$$\mathcal{B}'_{tr} := \left\{ \bigcap_{i=1}^m \Theta^{-n_i}\mathbf{A}_i \,\middle|\, \mathbf{A}_i \in \mathcal{Z} \cup \mathcal{B}_{tr}, n_i \in \mathbb{N}_0, m \in \mathbb{N} \right\}.$$

As in the proof of Theorem 3.4.1 we obtain for every element $\mathbf{A} \in \mathcal{B}'_{tr}$

$$\mathbf{M}_{0,1}(\Theta^{-1}\mathbf{A}|\Theta^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}})) = \mathbf{M}_{0,1}(\mathbf{A}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}) \circ \Theta; \tag{5.29}$$

the latter equality directly implies for all countable $\mathcal{B}'_{tr}$-measurable partitions of $\mathbb{R}^d \times \hat{\Omega}^{\mathbb{Z}}$

$$H_{\mathbf{M}_{0,1}}(\Theta^{-1}\mathcal{Z}|(\Theta^{-1}(\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}})) = H_{\mathbf{M}_{0,1}}(\mathcal{Z}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}), \tag{5.30}$$

Therefore, as in Lemma 3.3.1, there exists the following limit

$$h_{\mathbf{M}}(\Theta, \mathcal{Z}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}) := \lim_{n\to\infty} \frac{1}{n} H_{\mathbf{M}_{0,1}}\left(\bigvee_{i=0}^{n-1} \Theta^{-i}\mathcal{Z}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}\right) \tag{5.31}$$

Let now $\mathcal{Z} := \{\mathbf{Z}_1, \mathbf{Z}_2, \ldots\}$ and $\mathcal{Z}^{(k)} := \{\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_k, \cup_{i\geq k+1}\mathbf{Z}_i\}$. Then the number $H_{\mathbf{M}_{0,1}}(\mathcal{Z}^{(k)})$ can be approximated from below by $H_{\mathbf{M}_{0,1}}(\mathcal{Z}^{(k)}) \wedge H_{\mathbf{M}_{0,1}}(\xi \times \eta)$, where $\xi = \{A_1, \ldots, A_k\}$ and $\eta = \{B_1, \ldots, B_m\}$ are finite measurable partitions of $\mathbb{R}^d$ and $\hat{\Omega}^{\mathbb{N}}$ respectively. Further, the number $H_{\mathbf{M}_{0,1}}(\mathcal{Z})$ can be approximated from below by $H_{\mathbf{M}_{0,1}}(\mathcal{Z}^{(k)})$. Therefore, as in Theorem II.1.4 (ii) from [18] we obtain

$$h_{\mathbf{M}}(\Theta, \mathcal{Z}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}) \leq h_{\mathbf{M}}(\Theta|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}) \leq h_{\mu}(\psi). \tag{5.32}$$

Now we have

$$\begin{aligned}
h_{\mathbf{M}}(\Theta, \mathcal{Z}|\mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}) &= \lim_{n\to\infty} \frac{1}{n} \int_{\hat{\Omega}^{\mathbb{Z}}} H_{\mathbf{M}_{0,1}}\left(\mathcal{Z}_n^{\omega}\right) d\hat{\nu}^{\mathbb{Z}} \\
&= \liminf_{n\to\infty} \frac{1}{n} \int_{\hat{\Omega}^{\mathbb{Z}}} H_{\mathbf{M}_{0,1}}\left(\mathcal{Z}_n^{\omega}\right) d\hat{\nu}^{\mathbb{Z}} \\
&\geq \int_{\hat{\Omega}^{\mathbb{Z}}} \liminf_{n\to\infty} \frac{1}{n} H_{\mathbf{M}_{0,1}}\left(\mathcal{Z}_n^{\omega}\right) d\hat{\nu}^{\mathbb{Z}} \\
&\geq \int_{\hat{\Omega}^{\mathbb{Z}}} \liminf_{n\to\infty} \int_{\mathbb{R}^d} -\frac{1}{n} \log \mu_{0,1}\left((\mathcal{Z}_n^{\omega} \cap [0,1)^d)(x)\right) d\mu_{0,1} d\hat{\nu}^{\mathbb{Z}} \\
&\geq \int_{\mathbb{R}^d\times\hat{\Omega}^{\mathbb{Z}}} \liminf_{n\to\infty} -\frac{1}{n} \log \mu_{0,1}\left((\mathcal{Z}_n^{\omega} \cap [0,1)^d)(x)\right) d\mathbf{M}_{0,1},
\end{aligned}$$

which together with (5.32) completes the proof of (5.28).

Note that $\forall x \in [0,1)^d$

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu_{0,1}\left((\mathcal{Z}_n^{\omega} \cap [0,1)^d)(x)\right) \geq \underline{h}^{loc}(\psi, \delta_{\epsilon}(x,\omega), x). \tag{5.33}$$

Indeed, by the construction of $\mathcal{Z}$ we have

$$\mathcal{Z}_n^\omega(x) \subset R_n^{\psi,\delta_\epsilon(x,\omega),x},$$

which yields (5.33). Therefore, by (5.28) we have

$$h_\mu(\psi) \geq \int\limits_{\mathbb{R}^d \times \hat{\Omega}^{\mathbb{Z}}} \underline{h}^{loc}(\psi, \delta_\epsilon(x,\omega), x) d\mathbf{M}_{0,1}. \tag{5.34}$$

Now let us apply Theorem 5.1.3. This theorem implies the following corollary

Corollary 5.3.1. *Let ψ be a two-sided TIRDS defined on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$ which satisfies Assumptions 1-3 and also has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Then for all $x \in \mathbb{R}^d$ and for all $\epsilon > 0$ we have*

$$\underline{h}^{loc}(\psi, \delta_\epsilon(x,\omega), x) \geq (\sum_{i=1}^p d_i \lambda_i^+) - \epsilon,$$

where $\delta_\epsilon(x,\omega)$ is defined in Definition 5.3.1.

Proof. In the proof of Theorem 5.1.3 we consider $\rho(\omega) = \rho_\epsilon(\omega)$, which is defined via Definition 5.2.1. It turns out that by the definition of $\delta_\epsilon(x,\omega)$, see Definition 5.3.1, and the definition of $\rho(\omega)$, these two random variables have the same distribution. Therefore translation invariance of ψ implies the corollary. □

Thus, Corollary 5.3.1 together with (5.34) implies

$$h_\mu(\psi) \geq (\sum_{i=1}^p d_i \lambda_i^+) - \epsilon.$$

Finally, put $\epsilon \to 0$. The theorem is proven.

Chapter 6

Local Ruelle's Inequality for Random Dynamical Systems

In this chapter we define entropy for two-sided RDSs using the idea of Brin and Katok. They define the notion of local entropy, which is similar to the measure-theoretic one, but measures disorder of a system only around the trajectory of a particular point, see [9]. We define entropy similarly but adapt the definition to random dynamical systems case. Further, we prove a local analogue of Ruelle's inequality with the defined entropy. Namely, for the systems with the fixed origin, we prove that the defined entropy is less than or equal to the sum of positive Lyapunov exponents of the system. As a corollary, we also obtain the respective result for two-sided TIRDSs, see Corollary 6.1.1. Note that this result also implies Ruelle's inequality, where entropy for TIBFs is defined as in Brin and Katok's paper, see the remark in the next section.

The chapter is organized as follows. In Section 6.1 we formulate the main result. In Section 6.2 we provide the main idea of the proof. In Section 6.3 we establish the Lyapunov metric for the two-sided RDSs and prove some technical lemmas. In Section 6.4 we prove the main result.

6.1 Main Result

Let ψ be a random dynamical system on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$ with the fixed origin, i.e. $\overline{\psi}_{1,\omega}(0) = 0$, $\forall \omega \in \overline{\Omega}^{\mathbb{Z}}$. First of all, let us define entropy of ψ. Recall that

$$R_n^{\psi,\epsilon,0} := \bigcap_{j=0}^{n} \psi_{j,\omega}^{-1}(B(0,\epsilon)).$$

Further, recall that Proposition 5.1.1 provides the existence of a deterministic value (maybe equal to $+\infty$) $\overline{h}^{loc}(\psi,\epsilon,0)$ such that

$$\overline{h}^{loc}(\psi,\epsilon,0) = \limsup_{n\to\infty} -\frac{1}{n}\log \mu(R_n^{\psi,\epsilon,0}), \quad \text{a.s.}$$

Note that in this chapter ϵ is always a deterministic number; finally, define entropy of ψ in the following way

$$\overline{h}^{loc}(\psi) := \lim_{\epsilon\to 0+} \overline{h}^{loc}(\psi,\epsilon,0).$$

Note that such a limit exists because of the monotonicity arguments. Now we are ready to state the main result of the chapter.

Theorem 6.1.1. *Let ψ be a random dynamical system on a probability space $(\overline{\Omega}^{\mathbb{Z}}, \mathcal{B}(\overline{\Omega})^{\mathbb{Z}}, \overline{\nu}^{\mathbb{Z}})$, which satisfies Assumptions 1-3 and has the fixed origin, i.e. $\psi_{1,\omega}(0) = 0$, $\forall\omega$. Let further ψ have Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Then we have*

$$\overline{h}^{loc}(\psi) \le \sum_{i=1}^{p} d_i \lambda_i^+.$$

This theorem immediately provides the respective result for two-sided TIRDSs. Let us formulate it.

Corollary 6.1.1. *Let ψ be a two-sided translation invariant random dynamical system on a probability space $(\hat{\Omega}^{\mathbb{Z}}, \mathcal{B}(\hat{\Omega})^{\mathbb{Z}}, \hat{\nu}^{\mathbb{Z}})$, which satisfies Assumptions 1-3. Let further ψ has Lyapunov exponents $\lambda_1, \ldots, \lambda_p$ with multiplicities $d_1, \ldots, d_p$. Then we have*

$$\overline{h}^{loc}(\psi) \le \sum_{i=1}^{p} d_i \lambda_i^+.$$

Proof. It suffices to apply Theorem 6.1.1 to a two-sided RDS $\overline{\psi}$ generated by i.i.d. mappings

$$\ldots \psi_{1,\theta^{-1}\omega} - \psi_{1,\theta^{-1}\omega}(0), \psi_{1,\omega} - \psi_{1,\omega}(0), \psi_{1,\theta\omega} - \psi_{1,\theta\omega}(0) \ldots,$$

and so translation invariance of ψ implies the corollary. $\square$

Remark 6.1.1. *Corollary 6.1.1 implies that for every TIBF ϕ we have $\mathbb{P}$-almost surely*

$$\lim_{\epsilon\to 0+} \limsup_{n\to\infty} -\frac{1}{n} \log \mu \left(\bigcap_{j=0}^{n} \phi_{0,j}^{-1}(B(\phi_{0,j}(0,\omega),\epsilon)) \right) \le \sum_{i=1}^{p} d_i \lambda_i^+.$$

This corresponds to the upper bound on local entopy in Brin and Katok's paper, see [9]. They showed that in particular for an ergodic DS f with a finite invariant measure m (under certain assumptions on f and m), for m-almost all x the value

$$\lim_{\epsilon\to 0+} \limsup_{n\to\infty} -\frac{1}{n} \log m \left(\bigcap_{j=0}^{n} f^{-n}(B(f^n(x),\epsilon)) \right)$$

does not exceed metric entropy.

6.2 Main Idea of the Proof

Now we provide the idea of the proof of Theorem 6.1.1. Note that this idea was proposed by A. Blumenthal.

Essentially, we exploit Mañé's and also some Thieullien's ideas, see [26] and [40].

For the sake of simplicity suppose that we have no zero Lyapunov exponents. In a small region around the origin, which we denote by $\mathcal{D}_{n,\epsilon} \subset R_n^{\psi,\epsilon,\omega}$ (which is almost the same as $\mathcal{D}_{0,\epsilon,n}$ in Thieullen's paper, see [40], p. 239), we obtain uniform hyperbolicity in the Lyapunov metric (as Thieullen did; see [40], Proposition II.3.3). Let $y \in \mathcal{D}_{n,\epsilon}$, $D_{n,\epsilon}^y = D_{n,\epsilon}^y(\psi,\epsilon,\omega) := D_{n,\epsilon} \cap (U(\omega)+x)$, and the subspaces S and U are defined as in (2.10) and (2.11). Then we have

$$\begin{aligned} 1 \lesssim \mu(\psi_n(\mathcal{D}_{n,\epsilon}^y)) = & \int\limits_{\mathcal{D}_{n,\epsilon}^y} |\det[D_v\psi_n|_U]|\, d\mu^y(v) \\ \lesssim & \int\limits_{\mathcal{D}_{n,\epsilon}^y} |\det[D_0\psi_n|_U]|\, d\mu^y(v) \\ = & \mu^y(\mathcal{D}_{n,\epsilon})|\det[D_0\psi_n|_U]| \\ \approx & \mu^y(\mathcal{D}_{n,\epsilon}) \exp\left\{ n\sum_{i=1}^p d_i\lambda_i^+ \right\}. \end{aligned}$$

Here $\lesssim$ and $\approx$ means that when we take "$\lim\limits_{n\to\infty} -\frac{1}{n}\log$" of both sides, we might obtain some error, but it is negligible when ϵ goes to zero. The first "approximate" inequality holds because expansion in unstable direction helps us to show that $\mu(\psi_n(R_n^y))$ is not too small (see Lemma 6.3.2). The second "approximate" inequality is rather technical and is formally proven in Lemma 6.3.3 (see also Lemma 6.3.4).

It turns out that $\mathcal{D}_{n,\epsilon}$ is a set such that its projection on S (with respect to U) covers a ball in S centered at zero and with radius $\gtrsim 1$. Intuitively, this happens because stable direction S does not let projections of points on S escape from the unit ball too quickly, see Lemma 6.3.2. Moreover, because of the calculations above, the volume of the "width" with respect to U is of the size approximately at least

$$V_n = \exp\left\{ -n\sum_{i=1}^p d_i\lambda_i^+ \right\}.$$

Therefore, the volume of $\mathcal{D}_{n,\epsilon}$ is approximately at least V_n, and thus the same can be said about $R_n^{\psi,\epsilon,\omega}$, which completes the proof of the theorem.

6.3 Preliminaries Before the Proof of Local Ruelle's Inequality

In this section we provide Lyapunov charts for the two-sided RDS ψ, and also prove some technical lemmas.

6.3.1 Lyapunov Metric

We mainly repeat here the construction of Lyapunov metric, which is done in [40]. Note that the original idea can be found in [33]. However, we adapt it to our case, where we have dynamics generated by θ, whereas in [33] and [40] they define it for the deterministic dynamics on state space.

Denote by $\overline{\Omega}_1^{\mathbb{Z}} \subset \overline{\Omega}^{\mathbb{Z}}$ the set of ω for which Theorem 2.3.4 holds. Recall that $\overline{\nu}^{\mathbb{Z}}(\overline{\Omega}_1^{\mathbb{Z}}) = 1$. We start from the following lemma

Lemma 6.3.1. *For every $\epsilon > 0$ there exists a Borel set (which does not depend on ϵ) $\overline{\Omega}_2^{\mathbb{Z}} \subset \overline{\Omega}^{\mathbb{Z}}$ and a measurable function $l_\epsilon : \overline{\Omega}_2^{\mathbb{Z}} \to (0, \infty)$ such that $\overline{\nu}^{\mathbb{Z}}(\overline{\Omega}_2^{\mathbb{Z}}) = 1$ and for all $\omega \in \overline{\Omega}_2^{\mathbb{Z}}$*

i) $\|D_v\psi_{1,\omega} - D_w\psi_{1,\omega}\| \leq l_\epsilon(\omega)|v - w|, \quad v, w \in B(0, 1)$;

ii) $l_\epsilon(\theta^n\omega) \leq l_\epsilon(\omega)e^{\epsilon n}$, $n \geq 0$.

Proof. See [25], Lemma III.1.4 (see also [8], Lemma 4.4 or [7], Lemma 5.2.4). Note that we can define such an l_ϵ as there because of Assumption 3. □

Recall that $i_0 := \max\{i \in \mathbb{N} : \lambda_i > 0\}$. Define $b := \lambda_{i_0}$. Then $b > 0$. Now fix $\epsilon > 0$ with the following restrictions

Restriction 1: $\epsilon < \frac{b}{2}$. We will use this restriction in Proposition 6.3.1 and also in Restriction 4, see the end of the section.

Restriction 2: $(e^{b-\epsilon} - \epsilon)^{-1} < 1 - \epsilon$. We will use this restriction in Lemma 6.3.2

In the end of the section we provide two additional restrictions, which use the definition of Lyapunov metric defined below.

Now we are ready to define Lyapunov metric, which is treated for example in [33] and [40]. We state the "random version".

As in [33], for $\omega \in \overline{\Omega}_1^{\mathbb{Z}}$ define a norm $\|\cdot\|_{\omega,n}$ on $\mathbb{R}^d$ in the following way

$$\|v\|_{\omega,n} := \sqrt{d}\sum_{m=0}^{+\infty} e^{-\epsilon m}\|D_0\psi_{m,\theta^n\omega}(v)\|, \quad v \in S(\theta^n\omega),$$

$$\|w\|_{\omega,n} := \sqrt{d}\sum_{m=-\infty}^{0} e^{-bm+\epsilon m}\|D_0\psi_{m,\theta^n\omega}(w)\|, \quad w \in U(\theta^n\omega),$$

and

$$\|v + w\|_{\omega,n} := \max\{\|v\|_{\omega,n}, \|w\|_{\omega,n}\}, \quad v \in S(\theta^n\omega), w \in U(\theta^n\omega).$$

The sequence of norms $\|\cdot\|_{\omega,n}$ is usually called *Lyapunov metric* or *Lyapunov norm* at ω (and point 0). Note that for an integer n we have $\|v\|_{0,\theta^n\omega} = \|v\|_{n,\omega}$.

The following lemma repeats Proposition II.2.3 in [40], but is adapted to our settings.

Proposition 6.3.1. *There exists a function $\rho_\epsilon : \overline{\Omega}_1^{\mathbb{Z}} \cap \overline{\Omega}_2^{\mathbb{Z}} \to (0, \epsilon]$ with the following properties*
i) $\rho_\epsilon(\theta^n \omega) \geq e^{-n\epsilon}\rho_\epsilon(\omega)$, $n \geq 0$;
ii) $\|(D_0\psi_{1,\theta^n\omega})v\|_{\omega,n+1} \leq e^{a+\epsilon}\|v\|_{\omega,n}, \quad v \in S(\theta^n\omega), \quad n \in \mathbb{Z}$;
iii) $\|(D_0\psi_{1,\theta^n\omega})w\|_{\omega,n+1} \geq e^{b-\epsilon}\|w\|_{\omega,n}, \quad w \in U(\theta^n\omega), \quad n \in \mathbb{Z}$;
iv) $|v| \leq \|v\|_{\omega,0} \leq |v|/\rho_\epsilon(\omega), \quad v \in \mathbb{R}^d$;
v) For every $v, w \in \mathbb{R}^d$, such that $\|v\|_{\omega,0}, \|w\|_{\omega,0} \leq \rho_\epsilon(\omega)$ we have

$$\|D_v\psi_1 - D_w\psi_1\|_{\omega,1} \leq \epsilon;$$

vi) For every integer n we have

$$\|\mathbf{Pr}_{U(\theta^n\omega),S(\theta^n\omega)}\|_{\omega,n} = \|\mathbf{Pr}_{S(\theta^n\omega),U(\theta^n\omega)}\|_{\omega,n} = 1.$$

Proof. We impose $\rho_\epsilon(\omega) \leq \epsilon \wedge (\epsilon/l_\epsilon(\omega))$. Then Lemma 6.3.1 implies iv). The rest is due to [33], Theorem 1.5.1. □

Define

$$B_{\omega,n}(x, r) := \{v \in \mathbb{R}^d : \|v - x\|_{\omega,n} \leq r\},$$

$$\rho_{\epsilon,n}(\omega) := \rho_\epsilon(\omega)e^{-n\epsilon},$$

and

$$\mathcal{D}_{\epsilon,n}(\omega) := \{v : \forall k = \overline{0,n} : \|\psi_{k,\omega}(v)\|_{\omega,k} < \rho_{\epsilon,k}(\omega)\} = \bigcap_{k=0}^{n} \psi_{k,\omega}^{-1} B_{\omega,k}(0, \rho_{\epsilon,k}(\omega)).$$

Note that

$$\mathcal{D}_{\epsilon,n}(\omega) \subset R_n^{\psi,\epsilon,\omega}. \tag{6.1}$$

Indeed, for every $v \in \mathcal{D}_{\epsilon,n}(\omega)$ we have

$$|v| \leq \|v\|_{\omega,0} \leq \rho_\epsilon(\omega) \leq \epsilon,$$

where the first inequality holds because of Proposition 6.3.1 iv). Thus, (6.1) is proven.

Now we are ready to state two additional restrictions on ϵ.

Restriction 3: Fix $r \in (0, \frac{1}{10})$. Further, let $c = c(r) \leq r$ and $\epsilon = \epsilon(r) \leq r$ (in particular, $\epsilon \vee c < \frac{1}{10}$) be so small, that there exists a measurable set K_r such that $\overline{\nu}^{\mathbb{Z}}(K_r) \geq 1 - r$ and if $\omega \in K_r$ and $v \in B(0, \epsilon)$, then for every subspace $E \subset \mathbb{R}^d$ which is an $(S(\omega), U(\omega))$-graph in Lyapunov norm $\|\cdot\|_{0,\omega}$ with dispersion $\leq c$ (see Definition A.0.1), we have

$$\big|\log|\det[D_v\psi_{1,\omega}|_E]| - \log\big|\det[D_0\psi_{1,\omega}|_{U(\omega)}]\big|\big| \leq r. \tag{6.2}$$

Note that we can find such c and ϵ because of continuity arguments. We will use this restriction in Lemma 6.3.3.

Restriction 4: Let ϵ be so small, that Lemma A.0.1 holds for c, where $\delta_0 = \epsilon$, $F = \psi_{1,\omega}$, $\beta_1 = e^{b-\epsilon}$, $\beta_2 = e^{\epsilon}$, $\alpha = 1$, $r = \rho_\epsilon(\omega)$, $E_1 = S(\omega)$, $E_2 = U(\omega)$, $E = \mathbb{R}^d$ with norm $\|\cdot\|_{\omega,0}$, and $E' = \mathbb{R}^d$ with norm $\|\cdot\|_{\omega,1} \equiv \|\cdot\|_{\theta\omega,0}$. Note that we can make such a substitution because of Restriction 1, and also by Proposition 6.3.1 ii), iii), v) and vi). We will use this restriction in Lemma 6.3.2 and Lemma 6.3.3. Note that this restriction is θ-invariant.

6.3.2 Some Technical Lemmas

Define

$$\rho'_{\epsilon,n}(\omega) := \epsilon\rho_\epsilon(\omega)e^{-4n\epsilon}.$$

Lemma 6.3.2. *For every $\omega \in \overline{\Omega}_1^{\mathbb{Z}} \cap \overline{\Omega}_2^{\mathbb{Z}}$, for every $n \in \mathbb{N}_0$ and for every x with $\|x\|_{\omega,0} < \rho'_{\epsilon,n}(\omega)$ define*

$$G_{x,n}(\omega) := (x + U(\omega)) \cap \mathcal{D}_{\epsilon,n}(\omega).$$

Then

i) $G_{x,n}$ is a $(U(\theta^n\omega), S(\theta^n\omega))$-graph in Lyapunov norm $\|\cdot\|_{\omega,n}$ with dispersion $\le c$;

ii) We have

$$\psi_{n,\omega}(x + U(\omega)) \cap B_{\omega,n}(0, \rho_{\epsilon,n}(\omega)) = \psi_{n,\omega}(G_{x,n}(\omega));$$

iii) The set $\psi_{n,\omega}(G_{x,n}(\omega)) \cap S(\theta^n\omega)$ consists of a single point, which belongs to $B_{\omega,n}(0, e^{3\epsilon n}\|x\|_{\omega,0})$.

Proof. We use induction. For $n = 0$ the statement of the theorem is trivial. Suppose that it holds for $\overline{1, n-1}$. Let us prove the statement of the theorem for n.

First of all, let us prove i) for n. Note that by Proposition 6.3.1 v) and the mean value inequality we have

$$\sup_{v,w \in B_{l,\omega}(0,\rho_\epsilon(\omega)), v \ne w} \frac{\|\psi_{1,\theta^l\omega}(v) - \psi_{1,\theta^l\omega}(w) - (D_0\psi_{1,\theta^l\omega})(v-w)\|_{\omega,l+1}}{\|v-w\|_{\omega,l}} \le \epsilon. \quad (6.3)$$

Restriction 4, Proposition 6.3.1 ii), iii), v), vi), and consecutive application of Lemma A.0.1 to $\omega, \dots, \theta^{j-1}\omega$ together imply that for every $j = \overline{0, n}$

$$\psi_{j,\omega}((U(\omega) + x) \cap \mathcal{D}_{\epsilon,n}(\omega))$$

is a $(U(\theta^j\omega), S(\theta^j\omega))$-graph in Lyapunov norm $\|\cdot\|_{\omega,j}$ with dispersion $\le c$. i) for n is proven.

Now let us prove ii) for n. By induction assumption we have for $i = \overline{1, n-1}$

$$(x + U(\omega)) \cap \psi_{i-1,\omega}^{-1} B_{\omega,i-1}(0, \rho_{\epsilon,i-1}(\omega)) \supset (x + U(\omega)) \cap \psi_{i,\omega}^{-1} B_{\omega,i}(0, \rho_{\epsilon,i}(\omega)). \quad (6.4)$$

It suffices to show that

$$\psi_{n,\omega}(x+U(\omega))\cap B_{\omega,n}(0,\rho_{\epsilon,n}(\omega))\subset\psi_{n,\omega}(G_{x,n}(\omega)). \tag{6.5}$$

and so to prove (6.5) it suffices to show (6.4) for $i=n$. Suppose that it is not true. Then there exists $x'\in(x+U(\omega))\cap\mathcal{D}_{\epsilon,n}(\omega)$, such that $\|\psi_{n-1,\omega}(x')\|_{\omega,n-1}>(1-\epsilon)\rho_{\epsilon,n-1}(\omega)$. To obtain contradiction, we show that

$$\|\psi_{n-1,\omega}(x')\|_{\omega,n-1}<(1-\epsilon)\rho_{\epsilon,n-1}(\omega).$$

Denote by $x_{n-1}:=\psi_{n-1,\omega}(G_{x,n-1})\cap S(\theta^{n-1}\omega)$, which exists by induction assumption (see iii)). Then $\psi^{-1}_{n-1,\omega}(x_{n-1})$ belongs to $\mathcal{D}_{\epsilon,n}(\omega)$, because we have

$$\|\psi_{1,\theta^{n-1}\omega}(x_{n-1})\|_{\omega,n}<\rho_{\epsilon,n}. \tag{6.6}$$

Indeed, by Proposition 6.3.1 ii) and by induction assumption (see iii))

$$\begin{aligned}\|\mathbf{Pr}_{S(\theta^n\omega),U(\theta^n\omega)}((\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}&\overset{(6.3)}{\leq}(e^\epsilon+\epsilon)e^{3(n-1)\epsilon}\|x\|_{\omega,0}\\&\leq(e^\epsilon+\epsilon)e^{3(n-1)\epsilon}\rho'_{\epsilon,n-1}<\frac{\rho_{\epsilon,n}}{2}<\rho_{\epsilon,n}\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\|\mathbf{Pr}_{U(\theta^n\omega),S(\theta^n\omega)}(\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}&\overset{(6.3)}{\leq}\epsilon e^{3(n-1)\epsilon}\|x\|_{\omega,0}\\&\leq\epsilon e^{3(n-1)\epsilon}\rho'_{\epsilon,n-1}<\frac{\rho_{\epsilon,n}}{2}<\rho_{\epsilon,n},\end{aligned} \tag{6.8}$$

which completes the proof of (6.6). Denote by $x^*:=\psi_{n-1,\omega}(x')$. By Proposition 6.3.1 iii) and induction assumption (see i)) we have

$$\begin{aligned}\|\mathbf{Pr}_{U(\theta^{n-1}\omega),S(\theta^{n-1}\omega)}(x^*)\|_{\omega,n-1}\leq&\|\mathbf{Pr}_{U(\theta^{n-1}\omega),S(\theta^{n-1}\omega)}(x^*-x_{n-1})\|_{\omega,n-1}\\&+\|\mathbf{Pr}_{U(\theta^{n-1}\omega),S(\theta^{n-1}\omega)}(x_{n-1})\|_{\omega,n-1}=:I;\end{aligned}$$

now inequality (6.3) implies

$$\begin{aligned}I\leq&\|\mathbf{Pr}_{U(\theta^n\omega),S(\theta^n\omega)}(\psi_{1,\theta^{n-1}\omega}(x^*)-\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}(e^{b-\epsilon}-\epsilon)^{-1}+0\\\overset{(6.8)}{\leq}&(\rho_{\epsilon,n}+\epsilon e^{3(n-1)\epsilon}\rho'_{\epsilon,n-1})(e^{b-\epsilon}-\epsilon)^{-1}\\=&\rho_{\epsilon,n-1}e^{-\epsilon}(1+\epsilon)(e^{b-\epsilon}-\epsilon)^{-1}<\rho_{\epsilon,n-1}(e^{b-\epsilon}-\epsilon)^{-1}<(1-\epsilon)\rho_{\epsilon,n-1},\end{aligned}$$

where the last inequality holds because of Restriction 2. Recall that $\epsilon\vee c<\frac{1}{10}$, see Restriction 3. Then we have

$$\begin{aligned}\|\mathbf{Pr}_{S(\theta^{n-1}\omega),U(\theta^{n-1}\omega)}(x^*)\|_{\omega,n-1} \leq &\|\mathbf{Pr}_{S(\theta^{n-1}\omega),U(\theta^{n-1}\omega)}(x^*-x_{n-1})\|_{\omega,n-1}\\ &+\|\mathbf{Pr}_{S(\theta^{n-1}\omega),U(\theta^{n-1}\omega)}(x_{n-1})\|_{\omega,n-1}\\ \leq& 2c\rho_{\epsilon,n-1}+e^{\epsilon}e^{3(n-1)\epsilon}\|x\|_{\omega,0},\end{aligned}$$

where the last inequality holds by induction assumption (see iii)). It is easy to check that

$$2c\rho_{\epsilon,n-1}+e^{\epsilon}e^{3(n-1)\epsilon}\|x\|_{\omega,0}<(1-\epsilon)\rho_{\epsilon,n-1},$$

and therefore

$$\begin{aligned}\|x^*\|_{\omega,n-1} \leq& \max\{\|\mathbf{Pr}_{U(\theta^{n-1}\omega),S(\theta^{n-1}\omega)}(x^*)\|_{\omega,n-1},\|\mathbf{Pr}_{S(\theta^{n-1}\omega),U(\theta^{n-1}\omega)}(x^*)\|_{\omega,n-1}\}\\ <&(1-\epsilon)\rho_{\epsilon,n-1}.\end{aligned}$$

The contradiction is obtained. ii) for n is proven.

Note that (6.7) and (6.8) show that

$$\|\mathbf{Pr}_{S(\theta^n\omega),U(\theta^n\omega)}((\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}<\frac{\rho_{\epsilon,n}}{2}$$

and

$$\|\mathbf{Pr}_{U(\theta^n\omega),S(\theta^n\omega)}(\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}<\frac{\rho_{\epsilon,n}}{2},$$

and therefore i) and ii) imply that the set $\psi_{n,\omega}(G_{x,n})\cap U(\theta^n\omega)$ consists of a single point. Denote it by x_n. Finally, by induction assumption (see iii)) and (6.8), we have

$$\begin{aligned}\|\mathbf{Pr}_{S(\theta^n\omega),U(\theta^n\omega)}(x_n)\|_{\omega,n} \leq&\|\mathbf{Pr}_{S(\theta^n\omega),U(\theta^n\omega)}(x_n-\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}\\ &+\|\mathbf{Pr}_{S(\theta^n\omega),U(\theta^n\omega)}(\psi_{1,\theta^{n-1}\omega}(x_{n-1}))\|_{\omega,n}\\ \leq& c\epsilon e^{3(n-1)\epsilon}\|x\|_{\omega,0}+(e^{\epsilon}+\epsilon)e^{3(n-1)\epsilon}\|x\|_{\omega,0}\\ <&(e^{\epsilon}+2\epsilon)e^{3(n-1)\epsilon}\|x\|_{\omega,0}\leq(e^{\epsilon}+e^{2\epsilon}-1)e^{3(n-1)\epsilon}\|x\|_{\omega,0}\\ \leq& e^{3\epsilon n}\|x\|_{\omega,0},\end{aligned}$$

which proves iii) for n. The lemma is proven.

□

Recall that r and K_r are defined in Restriction 3, see (6.2). Define

$$J_n:=\{0\leq j\leq n:\theta^j\omega\in K_r\},$$

and

$$J_n^c:=\{0\leq j\leq n:\theta^j\omega\in\overline{\Omega}^{\mathbb{Z}}\backslash K_r\}.$$

Further, define

$$C(\omega):=\log^+\left(\sup_{v\in B(0,1),E\leq\mathbb{R}^d}|\det[D_v\psi_{1,\omega}|_E]|\vee\sup_{v\in B(0,1),E\leq\mathbb{R}^d}\left|\det[D_v(\psi_{1,\omega}^{-1})|_E]\right|\right).$$

where $E \leq \mathbb{R}^d$ means that E is a subspace of $\mathbb{R}^d$.

Lemma 6.3.3. *For every* $\omega \in \overline{\Omega}_1^{\mathbb{Z}} \cap \overline{\Omega}_2^{\mathbb{Z}}$ *and for every* $v \in \mathcal{D}_{\epsilon,n}(\omega)$ *such that* $\|\mathbf{Pr}_{S(\omega),U(\omega)}(v)\|_{\omega,0} \leq \rho'_{\epsilon,n}$ *we have*

$$\log \left|\det[D_v\psi_{n,\omega}|_{U(\omega)}]\right| \leq \log \left|\det[D_0\psi_{n,\omega}|_{U(\omega)}]\right| + F_{n,r}(\omega),$$

where $F_{n,r}(\omega) := rn + 2 \sum_{j\in J_{n-1}^c} C^+(\theta^j\omega)$.

Proof. By the chain rule we have

$$\begin{aligned}\log \left|\det[D_v\psi_{n,\omega}|_{U(\omega)}]\right| &= \sum_{j=0}^{n-1} \log \left|\det[D_{\psi_{j,\omega}(v)}\psi_{1,\theta^j\omega}|_{(D_v\psi_{j,\omega})U(\omega)}]\right| \\ &= \sum_{j\in J_{n-1}} \log \left|\det[D_{\psi_{j,\omega}(v)}\psi_{1,\theta^j\omega}|_{(D_v\psi_{j,\omega})U(\omega)}]\right| \\ &+ \sum_{j\in J_{n-1}^c} \log \left|\det[D_{\psi_{j,\omega}(v)}\psi_{1,\theta^j\omega}|_{(D_v\psi_{j,\omega})U(\omega)}]\right| := I;\end{aligned}$$

now let us bound the latter expression from above

$$I \leq \sum_{j\in J_{n-1}} \log \left|\det[D_{\psi_{j,\omega}(v)}\psi_{1,\theta^j\omega}|_{(D_v\psi_{j,\omega})U(\omega)}]\right| + \sum_{j\in J_{n-1}^c} C^+(\theta^j\omega).$$

Let us show that

$$\sum_{j\in J_{n-1}} \log \left|\det[D_{\psi_{j,\omega}(v)}\psi_{1,\theta^j\omega}|_{(D_v\psi_{j,\omega})U(\omega)}]\right| \leq \sum_{j\in J_{n-1}} \log \left|\det[D_0\psi_{1,\theta^j\omega}|_{U(\theta^j\omega)}]\right| + rn.$$

Indeed, $\psi_{j,\omega}(v) \in B(0,\epsilon)$, because $v \in \mathcal{D}_{\epsilon,n}(\omega)$. Moreover, by Lemma 6.3.2 i), for every $j = \overline{0,n}$

$$\psi_{j,\omega}((U(\omega) + v) \cap \mathcal{D}_{\epsilon,n}(\omega))$$

is a $(U(\theta^j\omega), S(\theta^j\omega))$-graph in Lyapunov norm $\|\cdot\|_{\omega,j}$ with dispersion $\leq c$. Therefore, $(D_v\psi_{j,\omega})\,U(\omega)$ is also a $(U(\theta^j\omega), S(\theta^j\omega))$-graph with Lyapunov norm $\leq c$. Thus, Restriction 3 completes the proof of the inequality. Thus, we have

$$\begin{aligned}\log \left|\det[D_v\psi_{n,\omega}|_{U(\omega)}]\right| &\leq \sum_{j\in J_{n-1}} \log \left|\det[D_0\psi_{1,\theta^j\omega}|_{U(\theta^j\omega)}]\right| + rn + \sum_{j\in J_{n-1}^c} C^+(\theta^j\omega) \\ = \log \left|\det[D_0\psi_{n,\omega}|_{U(\omega)}]\right| &- \sum_{j\in J_{n-1}^c} \log \left|\det[D_0\psi_{1,\theta^j\omega}|_{U(\theta^j\omega)}]\right| + rn + \sum_{j\in J_{n-1}^c} C^+(\theta^j\omega),\end{aligned}$$

where the last equality holds because of the chain rule. Finally, by the definition of C, for every $j \in J_n^c$ we have

$$-\log \left|\det[D_0\psi_{1,\theta^j\omega}|_{U(\theta^j\omega)}]\right| = \log \left|\det[D_0\psi^{-1}_{1,\theta^j\omega}|_{U(\theta^{j+1}\omega)}]\right| \leq C^+(\theta^j\omega),$$

which completes the proof of the lemma. $\square$

Lemma 6.3.4. *There exists a positive number G_r with $G_r \to 0$, $r \to 0+$, such that for $\overline{\nu}^{\mathbb{Z}}$-a.a. ω we have*

$$\limsup_{n\to\infty} \frac{1}{n} F_{n,r} \le G_r.$$

Proof. We have

$$|\det[D_v\psi_{1,\omega}|_E]| \le \|D_v\psi_{1,\omega}\|^{\dim E}$$

and

$$\left|\det[D_v\psi_{1,\omega}^{-1}|_E]\right| \le \left\|D_v\psi_{1,\omega}^{-1}\right\|^{\dim E}.$$

Hence

$$C^+(\omega) \le d\log^+\left(\sup_{v\in B(0,1)} \|D_v\psi_{1,\omega}\| \vee d\log^+ \sup_{v\in B(0,1)} \left\|D_v\psi_{1,\omega}^{-1}\right\|\right),$$

and therefore, $C^+ \in L^1(\overline{\nu}^{\mathbb{Z}})$ by Assumptions 1 and 2. Now define

$$G_r := r + 2 \sup_{L:\overline{\nu}^{\mathbb{Z}}(L)\le r} \mathbb{E}C^+1_L.$$

It is easy to see that indeed $G_r \to 0$ when $r \to 0+$. Further, by Birkhoff's ergodic theorem for $\overline{\nu}^{\mathbb{Z}}$-a.a. ω we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j\in J_{n-1}} C^+(\theta^j\omega) = \mathbb{E}C1_{K_r}+ \le (G_r/2) - r,$$

and therefore

$$r + \lim_{n\to\infty} \frac{2}{n}\sum_{j=0}^{n-1} C^+(\theta^j\omega) \le G_r.$$

The lemma is proven.

□

6.4 Proof of Local Ruelle's Inequality

In this Section we prove Theorem 6.1.1.

Recall that r, ϵ and c are fixed (see Section 6.3.1). Fix $\omega \in \overline{\Omega}_1^{\mathbb{Z}} \cap \overline{\Omega}_2^{\mathbb{Z}}$. There exists $B = B(\omega) > 0$, which depends only on $(S(\omega), U(\omega))$ such that for all $n \in \mathbb{N}_0$

$$
\begin{aligned}
\mu(D_{\epsilon,n}) =& B\int\limits_{S} \mu^y((y+U)\cap D_{\epsilon,n})d\mu^s(y)\\
\geq& B \int\limits_{B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S} \mu^y((y+U)\cap D_{\epsilon,n})d\mu^s(y)\\
\geq& B\mu^s(B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S) \inf_{y\in B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S} \mu^y((y+U)\cap D_{\epsilon,n}),
\end{aligned}
$$

where μ^s denotes the Lebesgue measure on S, and μ^y the Lebesgue measure on $y+U$. Thus, we have

$$
\mu(D_{\epsilon,n}) \geq e^{-5dn\epsilon} B_1(\omega) \inf_{B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S} \mu^y(\Lambda^y_{n,\epsilon}), \tag{6.9}
$$

where $B_1(\omega) := B\mu^s(B(0,\rho'_{\epsilon,0}\rho_{\epsilon,0})\cap S)$, and $\Lambda^y_{n,\epsilon} := (y+U)\cap D_{\epsilon,n}$. Fix $y\in B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S$. Then by Proposition 6.3.1 iii) we have

$$
B(y,(\rho_{\epsilon,n})^2/2) \subset B_{\omega,n}(y,\rho_{\epsilon,n}/2) \subset B_{\omega,n}(0,\rho_{\epsilon,n}),
$$

where the last implication holds because $y\in B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S$. Note that by Lemma 6.3.2 i) the set $\psi_{n,\omega}\Lambda^y_{n,\epsilon}$ is a $(U(\theta^n\omega),S(\theta^n\omega))$-graph with dispersion $\leq c$. Recall that for a set $S\subset\mathbb{R}^d$ the value $vol^m(S)$ denotes the m-dimensional volume of S. Then by Lemma 6.3.2 we have

$$
\epsilon_1((\rho_{\epsilon,n})^2/2)^d < vol^{\dim U}(\psi_{n,\omega}\Lambda^y_{n,\epsilon}),
$$

where $\epsilon_1>0$ is a deterministic lower bound on possible $\dim U$-dimensional volume of a $(U(\theta^n\omega),S(\theta^n\omega))$-graph of class C^1 in a ball of radius 1, passing through 0. Further, by transformation formula we have

$$
\begin{aligned}
vol^{\dim U}(\psi_{n,\omega}\Lambda^y_{n,\epsilon}) =& \int\limits_{\Lambda^y_{n,\epsilon}} |\det[D_v\psi_{n,\omega}|_U]|\, d\mu^y(v)\\
&\leq \left(|\det[D_0\psi_{n,\omega}|_U]|\, e^{F_{n,r}}\right)\mu^y(\Lambda^y_{n,\epsilon}),
\end{aligned}
$$

where the last inequality holds by Lemma 6.3.3. Therefore,

$$
\begin{aligned}
\epsilon_1((\rho_{\epsilon,n})^2/2)^d \leq& \left(|\det[D_0\psi_{n,\omega}|_U]|\, e^{F_{n,r}}\right) \inf_{y\in B(0,\rho'_{n,\epsilon}\rho_{n,\epsilon})\cap S} \mu^y(\Lambda^y_{n,\epsilon})\\
\overset{(6.9)}{\leq}& e^{5dn\epsilon}B_1^{-1}\left(|\det[D_0\psi_{n,\omega}|_U]|\, e^{F_{n,r}}\right)\mu(D_{\epsilon,n}).
\end{aligned}
$$

By taking logarithms and dividing by n we obtain

$$
\begin{aligned}
&-\frac{1}{n}\log\mu(D_{\epsilon,n}) \\
\leq & 5d\epsilon - \frac{1}{n}\log\left(\epsilon_1(\rho_\epsilon^2/2)^d B_1\left(\left|\det[D_0\psi_{n,\omega}|_U]\right| e^{F_{n,r}}\right)^{-1}\right) \qquad (6.10)\\
= & 5d\epsilon - \frac{1}{n}\log\left(\epsilon_1(\rho_\epsilon^2/2)^d B_1\right) + \frac{1}{n}\log\left(\left|\det[D_0\psi_{n,\omega}|_U]\right| e^{F_{n,r}}\right).
\end{aligned}
$$

Now put $n \to \infty$. We have

$$
\begin{aligned}
\overline{h}^{loc}(\psi,\epsilon,\omega) &\overset{\overline{\nu}^{\mathbb{Z}}\text{-a.a.}}{\leq} \limsup_{n\to\infty} -\frac{1}{n}\log\mu(R_n^{\psi,\epsilon,0}) \leq \limsup_{n\to\infty} -\frac{1}{n}\log\mu(D_{\epsilon,n}) \\
&\leq 5d\epsilon + \limsup_{n\to\infty}\frac{1}{n}\log\left(\left|\det[D_0\psi_{n,\omega}|_U]\right| e^{F_{n,r}}\right) \\
&\overset{\overline{\nu}^{\mathbb{Z}}\text{-a.a.}}{\leq} 5d\epsilon + G_r + \limsup_{n\to\infty}\frac{1}{n}\log\left(\left|\det[D_0\psi_{n,\omega}|_U]\right|\right),
\end{aligned}
$$

where the last inequality holds by Lemma 6.3.4. Further, by Lemma 2.3.1

$$
\limsup_{n\to\infty}\frac{1}{n}\log\left(\left|\det[D_0\psi_{n,\omega}|_U]\right|\right) \overset{\overline{\nu}^{\mathbb{Z}}\text{-a.a.}}{=} \sum_{i=1}^{p} d_i\lambda_i^+.
$$

Thus,

$$
\overline{h}^{loc}(\psi,\epsilon,\omega) \overset{\overline{\nu}^{\mathbb{Z}}\text{-a.a.}}{\leq} 5d\epsilon + G_r + \sum_{i=1}^{p} d_i\lambda_i^+.
$$

Finally, put $r \to 0+$ (recall that ϵ depends on r, see Resriction 3). The theorem is proven.

Chapter 7

Open Problems

The first question which arises from the thesis is whether we can obtain Pesin's formula for two-sided TIRDSs, that are not volume preserving. The problem is that the proof relies on the existence and invariance of the unstable direction of two-sided RDS with the fixed origin. That forced us to use negative times and also to define entropy for partitions that depend on negative times. But then it is unclear why Θ should have a smooth invariant measure. An alternative approach could be to find a substitution for the unstable direction, which does not depend on negative times. More precisely, we can consider a randomly distributed direction, where the randomness does not depend on the randomness of the RDS. Further, the distribution should coincide with a stationary with respect to θ distribution for the Markov chain on the space of $(\dim U)$-fold exterior power of $\mathbb{R}^d$, which corresponds to the space of possible directions, transversal to $S(\omega)$. Then we do not use negative times anymore, but still enjoy stationarity of the unstable direction. That could lead to a proof of an analogue of Theorem 5.1.3. Then one might hope to deduce Pesin's formula from the analogue.

Another open question is whether we can obtain a local Pesin's formula, i.e. is it true that local entropy of two-sided RDSs with the fixed origin (or of two-sided TIRDSs) is equal to the sum of positive Lyapunov exponents? In this case we have to estimate the volume of the respective Bowen balls from above. However, we can not use the approach from Chapter 6, where we use Lyapunov charts, because we meet the following problem: the Lyapunov charts are basically charts that have a random volume which is small for some ω, so the Bowen balls can't be covered by these charts. That means that we have to find an approach to estimate the volume of the Bowen balls beyond the Lyapunov charts, which seems to be a challenging problem. We propose two approaches which may resolve the problem for TIBFs (maybe in some partial cases only).

The first approach is to add linear drift towards the origin, obtaining another stochastic flow with an invariant probability measure. For example, in the case of isotropic Brownian flows, we obtain isotropic Ornstein-Uhlenbeck flows, that are described, for example, in [42]. In this case we expect that the Bowen balls of the obtained flows have larger volume in distribution because of the definition of Bowen balls and the added drift. However, it turns out to be a non-trivial statement and one has to check it. Furthermore, even for the new flow, the question of the existence of a local Pesin's formula is not trivial because of lack of compactness.

Another approach could be to periodize the initial TIBF, i.e. to obtain another flow so that its local behaviour around the trajectory of a fixed point (say zero) is the same in distribution, but the spatial behaviour is spatially periodic ω-wise.

To periodize the flow, one should change the covariance tensor of the system so that the obtained one coincides with the covariance tensor of the initial flow in some neighbourhood of zero, but also becomes periodic. The problem is that the changed covariance tensor should be positive-definite. Therefore, for the tensors with high smoothness, such a task seems to be more challenging. Perhaps one should find a way to periodize tensors with a singularity at zero and then to extend the result to the flows with a smooth tensor.

In the thesis we try to use the definition of Brin and Katok, but one can also think about some other "local" definitions of entropy. A possible approach is to consider information function of the respective RDS (say at zero). Shannon-McMillan-Breiman Theorem asserts that for ergodic dynamical systems information function often coincides a.e. with Kolmogorov-Sinaĭ entropy of the system. To approach the problem, one can try to prove the analogue of Shannon-McMillan-Breiman Theorem for TIRDSs, where instead of Kolmogorov-Sinaĭ entropy we consider entropy defined in Chapter 3.

Now let us discuss a possibility to establish Pesin's formula in the case of Kunita-type SDEs (on $\mathbb{R}^d$; for the sake of simplicity let $d = 1$) with delay. Let us be more precise. Denote by

$$x_t(s) := x(t+s), \qquad s \in [-1, 0], \qquad t \geq 0;$$

now consider the following delay equations

$$\begin{cases} dx(t) = F(x_t)dt + M(dt, x(t)), \quad t \geq 0, \\ x_0 \in C([-1, 0], \mathbb{R}), \end{cases}$$

i.e. Kunita-type delay equations, where M is a translation invariant martingale field, F is a translation invariant with respect to constants drift term, i.e. $F(x_t) \equiv F(x_t - c)$, $c \in \mathbb{R}$. From [29] and [28] we know that, under certain mild assumptions, the equation above generates a stochastic flow which even has Lyapunov spectrum. Respectively, a discretized flow can be seen as a one-sided RDS on the state space $C([0, 1])$. Hence, one might think of defining entropy for such RDSs (perhaps similar to the definition in Chapter 3) and prove an analogue of Pesin's formula for such systems.

Appendix A

Transformations of graphs

Here we, following [26], pp. 98–99 with [27], introduce graphs with bounded dispersion and also state a lemma about transformation of such graphs.

Note that all vector spaces mentioned in the Appendix are finite-dimensional.

Definition A.0.1. *Let $(E, \|\cdot\|)$ be a normed vector space with splitting $E = E_1 \oplus E_2$. We call a subset $\mathcal{G}$ of E an (E_1, E_2)-graph if there is an open set $U \subset E_2$ and a C^1 map $f : U \to E_1$ such that*

$$\mathcal{G} = \{f(x) + x : x \in U\}.$$

The dispersion of $\mathcal{G}$ is defined by

$$\sup_{x,y \in U, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

Hence we can conclude that dispersion of a graph is like Lipschitz constant in the coordinate system, generated by E_1 and E_2

Let $\pi_1 : E \to E_2$ be the projection onto E_1 with kernel E_2 and let $\pi_2 : E \to E_1$ be the projection onto E_2 with kernel E_1. Then we define

$$\gamma(E_1, E_2) := \max(\|\pi_1\|, \|\pi_2\|).$$

Now we are ready to formulate the main result of Appendix.

Lemma A.0.1. *Given $\beta_1 > \beta_2 > 1$, $\alpha > 0$, and $c > 0$, then for every*

$$\delta_0 \in (0, \min\{\beta_1 \alpha^{-1}(1 + c)^{-1}, (\beta_1 - \beta_2)c\alpha^{-1}(1 + c)^{-2}\}) \tag{A.1}$$

the following property holds. If $E = E_1 \oplus E_2$ with $\gamma(E_1, E_2) \leq \alpha$, and F is a C^1 embedding of a ball $B(0, r) \subset E$ into another Banach space E' satisfying

(a) D_0F is an isomorphism and $\gamma((D_0F)E_1, (D_0F)E_2) \leq \alpha$;

(b) $\|D_0F - D_xF\| \leq \delta_0$ for all $x \in B(0, r)$;

(c) $\|(D_0F)v\| \geq \beta_1\|v\|$ for all $v \in E_2$;

(d) $\|(D_0F)v\| \leq \beta_2\|v\|$ for all $v \in E_1$;

then for every (E_1, E_2)-graph $\mathcal{G}$ with dispersion $\leq c$ contained in the ball $B(0, r)$, its image $F(\mathcal{G})$ is a $((D_0F)E_1, (D_0F)E_2)$-graph $\mathcal{G}$ with dispersion $\leq c$.

Proof. See [26], Lemma 3 with [27]. □

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Index of Notation and Abbreviations

$(\cdot)^+$	positive part of $(\cdot)$
$\lvert \cdot \rvert$	Euclidean norm
$\lVert \cdot \rVert$	operator norm coming from $\lvert \cdot \rvert$
a.a.	almost all
a.e.	almost everywhere
a.s.	almost surely
$\mathcal{B}(\cdot)$	Borel σ-algebra
$B_r(A)$	set of points that are on the distance at most r from set A
$B(x, r)$	closed r-ball, centered at $x \in \mathbb{R}^d$
$\mathcal{B}_{tr}$	class of 1-periodic in distribution sets in one-sided case
$\mathcal{B}_{tr}^{\mathbb{Z}}$	class of 1-periodic in distribution sets in two-sided case
$\mathcal{B}'_{tr}$	class of sets, see p. 64
d	natural number, dimension of the state space
DS	dynamical system
$\mathbf{diam}(A)$	diameter of set A
$\mathbf{diam}(\mathcal{P})$	$\sup_{C \in \mathcal{P}} \mathbf{diam}(C)$, diameter of partition $\mathcal{P}$
D_v	spatial derivative
e.g.	exempli gratia, for example
$H_{\mathbb{P}}(\xi \mid \mathcal{G})$	conditional entropy of partition ξ given σ-algebra $\mathcal{G}$
$h_{\mathbf{M}}(\Theta)$	entropy of skew product Θ
$h_{\mathbf{M}}(\Theta \mid \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}})$	entropy of skew product Θ given randomness
$h_{\mathbf{M}}(\Theta, \mathcal{P} \mid \mathbb{R}^d \times \mathcal{B}(\hat{\Omega})^{\mathbb{Z}})$	entropy of skew product Θ given randomness with respect to partition $\mathcal{P}$
$h_\mu(\psi)$	entropy of ψ
$h_\mu(\psi, \mathcal{P})$	entropy of ψ with respect to partition $\mathcal{P}$
$\mathbf{id}_{\mathbb{R}^d}$	identity map on $\mathbb{R}^d$
i.i.d.	independent, identically distributed
i.e.	id est, this is
$\mathbb{N}$	set of positive integers
$\mathbb{N}_0$	set of non-negative integers
$\mathbf{M}$	$:= \mu \times \nu^{\mathbb{N}}$
$\mathbf{M}_{0,1}$	$\mathbf{M}\vert_{[0,1)^d \times \hat{\Omega}}$
$\mathbf{M}^+$	$:= \mu \times \nu^{\mathbb{Z}}$

$\mathbf{M}^+_{0,1}$	$\mathbf{M}^+\|_{[0,1)^d \times \hat{\Omega}}$
$\mathbf{Pr}_{S_1,S_2}(A)$	projection of set A on subspace S_1, which is parallel to subspace S_2
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of non-negative real numbers
RDS	random dynamical system
S	stable direction
SDE	stochastic differential equation
TIBF	translation invariant Brownian flow
TIRDS	translation invariant random dynamical system
$vol^m(S)$	m-dimensional volume of S
U	unstable direction
Θ	skew product of two-sided RDS
Θ_+	skew product of one-sided RDS
θ	left shift operator of two-sided RDS
θ_+	left shift operator of one-sided RDS
λ_i	Lyapunov exponent
μ	Lebesgue measure on $\mathbb{R}^d$
$\mu_{m,n} := \mu\|_{[m,n)^d}$	restriction of Lebesgue measure μ to cube $[m,n)^d$.