DE GRUYTER Advances in Nonlinear Analysis 2023; 12: 1–22

Research Article

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Identification of discontinuous parameters in double phase obstacle problems

https://doi.org/10.1515/anona-2022-0223
received November 4, 2021; accepted December 30, 2021

Abstract: In this article, we investigate the inverse problem of identification of a discontinuous parameter and a discontinuous boundary datum to an elliptic inclusion problem involving a double phase differential operator, a multivalued convection term (a multivalued reaction term depending on the gradient), a multivalued boundary condition and an obstacle constraint. First, we apply a surjectivity theorem for multivalued mappings, which is formulated by the sum of a maximal monotone multivalued operator and a multivalued pseudomonotone mapping to examine the existence of a nontrivial solution to the double phase obstacle problem, which exactly relies on the first eigenvalue of the Steklov eigenvalue problem for the p-Laplacian. Then, a nonlinear inverse problem driven by the double phase obstacle equation is considered. Finally, by introducing the parameter-to-solution-map, we establish a continuous result of Kuratowski type and prove the solvability of the inverse problem.

Keywords: discontinuous parameter, double phase operator, elliptic obstacle problem, inverse problem, mixed boundary condition, multivalued convection, Steklov eigenvalue problem

MSC 2020: 35J20, 35J25, 35J60, 35R30, 49N45, 65J20

Special Issue – Nonlinear analysis: perspectives and synergies

1 Introduction

The aim of this article is to study an inverse problem to an elliptic differential inclusion problem involving a double phase differential operator, a multivalued convection term (dependence on the gradient of the solution), a multivalued boundary condition and an obstacle constraint. To this end, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\Gamma \coloneqq \partial\Omega$ such that Γ is divided into three mutually disjoint parts Γ_1, Γ_2 and Γ_3 with Γ_1 having positive Lebesgue measure. We study the problem

$$\begin{aligned}
-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) + g(x,u) + \mu(x)|u|^{q-2}u \in f(x,u,\nabla u) \quad & \text{in } \Omega,\\
u = 0 \quad & \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} = h(x) \quad & \text{on } \Gamma_2,\\
\frac{\partial u}{\partial \nu_a} \in U(x,u) \quad & \text{on } \Gamma_3,\\
u(x) \leq \Phi(x) \quad & \text{in } \Omega,
\end{aligned} \tag{1.1}$$

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where $1 < p < N$, $p < q$, $\mu : \overline{\Omega} \to [0, \infty)$ is a bounded function,

$$\frac{\partial u}{\partial \nu_a} := (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot \nu,$$

with ν being the outward unit normal vector on Γ, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ and $U : \Gamma_3 \times \mathbb{R} \to 2^{\mathbb{R}}$ are two given multivalued functions, $\Phi : \Omega \to \mathbb{R}$ is an obstacle function and $a : \Omega \to (0, +\infty)$, $h : \Gamma_2 \to \mathbb{R}$ are two possibly discontinuous parameters.

The main contribution of this article is twofold. The first intention of the article is to establish the nonemptiness, boundedness and closedness of the solution set to problem (1.1) (in the weak sense), in which our main methods are based on a surjectivity theorem for multivalued mappings, which is formulated by the sum of a maximal monotone multivalued operator and a multivalued pseudomonotone mapping, the theory of nonsmooth analysis and the properties of the Steklov eigenvalue problem for the p-Laplacian. The second contribution of the article is to develop a general framework for studying the inverse problem under consideration and to establish the solvability for such inverse problems. To the best of our knowledge, this is the first work studying the identification of discontinuous parameters for such general nonlinear elliptic equations. The problem under consideration combines several interesting phenomena such as double phase operators, multivalued right-hand sides, mixed boundary conditions and obstacle constraints.

First we point out that, motivated by several applications, the inverse problem of parameter identification in partial differential equations is an important field in mathematics and even though such problems in form of equations and inequalities have been studied a lot, there are still several open problems to be solved. Our work is motivated by the article of Migórski et al. [39], who studied the inverse problem of mixed quasi-variational inequalities of the form

$$\langle T(a, u), v - u\rangle + \varphi(v) - \varphi(u) \geq \langle m, v - u\rangle \quad \text{for all } v \in K(u),$$

where $K : C \to 2^C$ is a set-valued map, $T : B \times V \to V^*$ is a nonlinear map, $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ is a functional and $m \in V^*$, while V is a real reflexive Banach space, B is another Banach space and C is a nonempty, closed, convex subset of V. Their abstract result applies to p-Laplacian inequalities, see also [38] for hemivariational inequalities. We also mention the works of Clason et al. [10] for noncoercive variational problems, Gwinner [27] for variational inequalities of second kind, Gwinner et al. [28] for an optimization setting and Migórski and Ochal [40] for nonlinear parabolic problems, see also references therein. In addition, we refer to the recent work of Papageorgiou and Vetro [45] about existence and relaxation theorems for different types of problems which can be applied to variational inequalities and control systems.

A second interesting phenomenon is the occurrence of the weighted double phase operator, namely,

$$\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) \quad \text{for } u \in W^{1,\mathcal{H}}(\Omega). \tag{1.2}$$

For $a \equiv 1$, this operator corresponds to the energy functional given by

$$\int_\Omega (|\nabla u|^p + \mu(x)|\nabla u|^q)\mathrm{d}x. \tag{1.3}$$

Functionals of the form (1.3) have been initially introduced by Zhikov [56] in 1986 in order to describe models for strongly anisotropic materials and it also turned out its relevance in the study of duality theory as well as in the context of the Lavrentiev phenomenon, see Zhikov [57]. Observe that the energy density in (1.3) changes its ellipticity and growth properties according to the point in the domain. In general, double phase differential operators and corresponding energy functionals interpret various comprehensive natural phenomena and model several problems in Mechanics, Physics and Engineering Sciences. For example, in the elasticity theory, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents p and q, see Zhikov [58]. Functionals given in (1.3) have been intensively studied in the last few years concerning regularity of local minimizers. We mention the famous works of Baroni et al. [2,3], Byun and Oh [7], Colombo and Mingione [12,13], Marcellini [36,37] and Ragusa and Tachikawa [49].

A third interesting phenomenon is not only the multivalued right-hand side, which is motivated by several physical applications (see, e.g., Panagiotopoulos [42,43], Carl and Le [8] and references therein) but also its dependence on the gradient of the solutions often called convection term. The main difficulty in the study of gradient dependent right-hand sides is their nonvariational character, that is, the standard variational tools to the corresponding energy functionals are not applicable. In the past few years, several interesting works have been published with convection terms, we refer to the papers of El Manouni et al. [16], Faraci et al. [17], Faraci and Puglisi [18], Figueiredo and Madeira [20], Gasiński and Papageorgiou [23], Liu et al. [32], Liu and Papageorgiou [33], Marano and Winkert [35], Papageorgiou et al. [44] and Zeng and Papageorgiou [55].

Finally, we mention some existence results on the recent topic of double phase operators published within the last few years. We refer to Bahrouni et al. [1], Benslimane et al. [4], Biagi et al. [5], Colasuonno and Squassina [11], Fiscella [21], Farkas and Winkert [19], Gasiński and Papageorgiou [22], Gasiński and Winkert [24–26], Liu and Dai [31], Liu and Winkert [34], Papageorgiou et al. [46], Perera and Squassina [48], Stegliński [51] and Zeng et al. [52–54].

This article is organized as follows. Section 2 recalls preliminary material including Musielak-Orlicz Lebesgue and Musielak-Orlicz Sobolev spaces, the p-Laplacian eigenvalue problem with Steklov boundary condition, pseudomonotone operators and a surjectivity theorem for multivalued mappings. Under very general assumptions on the data, Section 3 proves the nonemptiness and compactness of the solution set to problem (1.1). In Section 4, we present a new existence result to the inverse problem of (1.1).

2 Preliminaries

This section is devoted to recall some basic definitions and preliminaries, which will be used in the next sections to derive the main results of the article. To this end, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ such that Γ is decomposed into three mutually disjoint parts Γ_1, Γ_2 and Γ_3 with Γ_1 having positive Lebesgue measure. In what follows, we denote by $M(\Omega)$ the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual, we identify two functions which differ on a Lebesgue-null set. Let $r \in [1, \infty)$ and D be a nonempty subset of $\overline{\Omega}$. We denote the usual Lebesgue spaces by $L^r(D) := L^r(D;\mathbb{R})$ and $L^r(D;\mathbb{R}^N)$ equipped with the standard r-norm $\|\cdot\|_{r,D}$ and $L^r(\Gamma)$ stands for the boundary Lebesgue spaces with norm $\|\cdot\|_{r,\Gamma}$.

Let $L^r(D)_+ := \{u \in L^r(D) : u(x) \geq 0 \text{ for a.a. } x \in \Omega\}$. By $W^{1,r}(\Omega)$ we define the corresponding Sobolev space endowed with the norm $\|\cdot\|_{1,r,\Omega}$ given by

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

For any fixed $s > 1$, the conjugate of s is defined by $s' > 1$ such that $\frac{1}{s} + \frac{1}{s'} = 1$. Moreover, we use the symbols s^* and s_* to represent the critical exponents to s in the domain and on the boundary, respectively, given by

$$s^* = \begin{cases} \dfrac{Ns}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \geq N, \end{cases} \quad \text{and} \quad s_* = \begin{cases} \dfrac{(N-1)s}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \geq N. \end{cases} \tag{2.1}$$

Let us comment on the r-Laplacian eigenvalue problem with Steklov boundary condition given by

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u \ \text{ in } \Omega, \\ |u|^{r-2}u \cdot \nu &= \lambda |u|^{r-2}u \ \text{ on } \Gamma, \end{aligned} \tag{2.2}$$

for $1 < r < \infty$. From Lê [30] we know that (2.2) has a smallest eigenvalue $\lambda_{1,r}^S > 0$, which is isolated and simple. Besides, we know that $\lambda_{1,r}^S > 0$ can be characterized by

$$\lambda_{1,r}^S = \inf_{u \in W^{1,r}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{r,\Omega}^r + \|u\|_{r,\Omega}^r}{\|u\|_{r,\Gamma}^r}. \tag{2.3}$$

The following assumptions are supposed in the entire article:

$$1 < p < N, \quad p < q < p^* \quad \text{and} \quad 0 \le \mu(\cdot) \in L^\infty(\Omega). \tag{2.4}$$

Now we define the nonlinear function $\mathcal{H} : \Omega \times [0, \infty) \to [0, \infty)$ given by

$$\mathcal{H}(x, t) = t^p + \mu(x)t^q \quad \text{for all } (x, t) \in \Omega \times [0, \infty).$$

Then, the Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ driven by the function $\mathcal{H}$ is given by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) : \rho_{\mathcal{H}}(u) < +\infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Here, the modular function is given by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|)\mathrm{d}x = \int_\Omega (|u|^p + \mu(x)|u|^q)\mathrm{d}x.$$

We know that $L^{\mathcal{H}}(\Omega)$ is uniformly convex and so a reflexive Banach space. Moreover, we introduce the seminormed space $L^q_\mu(\Omega)$

$$L^q_\mu(\Omega) = \left\{u \in M(\Omega) : \int_\Omega \mu(x)|u|^q \,\mathrm{d}x < +\infty\right\}$$

endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q \mathrm{d}x\right)^{\frac{1}{q}}.$$

The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is given by

$$W^{1,\mathcal{H}}(\Omega) = \{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. As before, it is known that $W^{1,\mathcal{H}}(\Omega)$ is a reflexive Banach space.

Next, we introduce a closed subspace V of $W^{1,\mathcal{H}}(\Omega)$ given by

$$V := \{u \in W^{1,\mathcal{H}}(\Omega) : u = 0 \quad \text{on } \Gamma_1\}$$

endowed with the norm $\|u\|_V = \|u\|_{1,\mathcal{H}}$ for all $u \in V$. Of course, V is also a reflexive Banach space. In the following, we denote the norm of the dual space V^* of V by $\|\cdot\|_{V^*}$.

Let us recall some embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, see Gasiński and Winkert [26] or Crespo-Blanco et al. [14].

Proposition 2.1. *Let* (2.4) *be satisfied and denoted by p^*, p_* the critical exponents to p as given in* (2.1) *for $s = p$.*

(*i*) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ *are continuous for all* $r \in [1, p]$;

(*ii*) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1, p^*]$ *and compact for all* $r \in [1, p^*)$;

(*iii*) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Gamma)$ *is continuous for all* $r \in [1, p_*]$ *and compact for all* $r \in [1, p_*)$;

(*iv*) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_\mu(\Omega)$ *is continuous*;

(*v*) $L^q(\Gamma) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

We point out that if we replace the space $W^{1,\mathcal{H}}(\Omega)$ by V in Proposition 2.1, then the embeddings (ii) and (iii) remain valid.

The following proposition is due to Liu and Dai [31, Proposition 2.1].

Proposition 2.2. *Let* (2.4) *be satisfied and let* $y \in L^{\mathcal{H}}(\Omega)$. *Then the following hold*:

(i) *if* $y \neq 0$, *then* $\|y\|_{\mathcal{H}} = \lambda$ *if and only if* $\rho_{\mathcal{H}}\left(\frac{y}{\lambda}\right) = 1$;
(ii) $\|y\|_{\mathcal{H}} < 1$ (*resp.* >1 *and* $=1$) *if and only if* $\rho_{\mathcal{H}}(y) < 1$ (*resp.* >1 *and* $=1$);
(iii) *if* $\|y\|_{\mathcal{H}} < 1$, *then* $\|y\|_{\mathcal{H}}^q \leq \rho_{\mathcal{H}}(y) \leq \|y\|_{\mathcal{H}}^p$;
(iv) *if* $\|y\|_{\mathcal{H}} > 1$, *then* $\|y\|_{\mathcal{H}}^p \leq \rho_{\mathcal{H}}(y) \leq \|y\|_{\mathcal{H}}^q$;
(v) $\|y\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(y) \to 0$;
(vi) $\|y\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(y) \to +\infty$.

We suppose that

$$a \in L^\infty(\Omega) \quad \text{such that } \inf_{x\in\Omega} a(x) > 0. \tag{2.5}$$

Next, we introduce the nonlinear operator $F : V \to V^*$ given by

$$\langle F(u), v\rangle := \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla v\,\mathrm{d}x + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v\,\mathrm{d}x, \tag{2.6}$$

for $u, v \in V$ with $\langle\cdot,\cdot\rangle$ being the duality pairing between V and its dual space V^*. The following proposition states the main properties of $F : V \to V^*$. We refer to Liu and Dai [31, Proposition 3.1] or Crespo-Blanco et al. [14, Proposition 3.4] for its proof.

Proposition 2.3. *Let hypotheses* (2.4) *and* (2.5) *be satisfied. Then, the operator F defined by* (2.6) *is bounded, continuous, monotone* (*hence maximal monotone*) *and of type* (S_+), *that is*,

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{and} \quad \limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0,$$

imply $u_n \to u$ *in* V.

We now recall some notions and results concerning nonsmooth analysis and multivalued analysis. Throughout the article the symbols "$\xrightarrow{w}$" and "$\to$" stand for the weak and the strong convergence, respectively, in various spaces. Moreover, let us recall the notions of pseudomonotonicity and generalized pseudomonotonicity in the sense of Brézis for multivalued operators (see, e.g., Migórski et al. [41, Definition 3.57]), which will be useful in the sequel.

Definition 2.4. Let X be a reflexive real Banach space. The operator $A : X \to 2^{X^*}$ is called

(a) pseudomonotone (in the sense of Brézis) if the following conditions hold:
  (i) the set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$;
  (ii) A is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*;
  (iii) if $\{u_n\} \subset X$ with $u_n \xrightarrow{w} u$ in X and $u_n^* \in A(u_n)$ are such that

$$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0,$$

then to each element $v \in X$, there exists $u^*(v) \in A(u)$ with

$$\langle u^*(v), u - v\rangle_{X^*\times X} \leq \liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle_{X^*\times X};$$

(b) generalized pseudomonotone (in the sense of Brézis) if the following holds: Let $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with $u_n^* \in A(u_n)$. If $u_n \xrightarrow{w} u$ in X and $u_n^* \xrightarrow{w} u^*$ in X^* and

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$

then the element u^* lies in $A(u)$ and

$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}.$$

It is not difficult to see that every pseudomonotone operator is generalized pseudomonotone, see, e.g., Carl et al. [9, Proposition 2.122]. Also, under an additional assumption of boundedness, we obtain the converse statement, see, e.g., Carl et al. [9, Proposition 2.123].

Proposition 2.5. *Let X be a reflexive real Banach space and assume that $A : X \to 2^{X^*}$ satisfies the following conditions*:

(*i*) *for each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^*.*

(*ii*) *$A : X \to 2^{X^*}$ is bounded.*

(*iii*) *A is generalized pseudomonotone, i.e., if $u_n \xrightarrow{w} u$ in X and $u_n^* \xrightarrow{w} u^*$ in X^* with $u_n^* \in A(u_n)$ and*

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$

then $u^* \in A(u)$ and

$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}.$$

Then the operator $A : X \to 2^{X^*}$ is pseudomonotone.

Let us now recall the definition of Kuratowski limits, see, e.g., Papageorgiou and Winkert [47, Definition 6.7.4].

Definition 2.6. Let (X, τ) be a Hausdorff topological space and let $\{A_n\}_{n\in\mathbb{N}} \subset 2^X$ be a sequence of sets. We define the τ-Kuratowski lower limit of the sets A_n by

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x \in X : x = \tau\text{-}\lim_{n\to\infty} x_n,\, x_n \in A_n \text{ for all } n \ge 1 \right\},$$

and the τ-Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x \in X : x = \tau\text{-}\lim_{k\to\infty} x_{n_k},\, x_{n_k} \in A_{n_k},\, n_1 < n_2 < \ldots < n_k < \ldots \right\}.$$

If

$$A = \tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n,$$

then A is called τ-Kuratowski limit of the sets A_n.

Finally, we recall the following surjectivity theorem for multivalued mappings, which is formulated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping, see Le [29, Theorem 2.2].

Theorem 2.7. *Let X be a real reflexive Banach space, let $\mathcal{G} : D(\mathcal{G}) \subset X \to 2^{X^*}$ be a maximal monotone operator, let $\mathcal{F} : D(\mathcal{F}) = X \to 2^{X^*}$ be a bounded multivalued pseudomonotone operator, let $\mathcal{L} \in X^*$ and let $B_R(0) := \{u \in X : \|u\|_X < R\}$. Assume that there exist $u_0 \in X$ and $R \ge \|u_0\|_X$ such that $D(\mathcal{G}) \cap B_R(0) \neq \varnothing$ and*

$$\langle \xi + \eta - \mathcal{L}, u - u_0\rangle_{X^*\times X} > 0 \tag{2.7}$$

for all $u \in D(\mathcal{G})$ with $\|u\|_X = R$, for all $\xi \in \mathcal{G}(u)$ and for all $\eta \in \mathcal{F}(u)$. Then the inclusion

$$\mathcal{F}(u) + \mathcal{G}(u) \ni \mathcal{L}$$

has a solution in $D(\mathcal{G})$.

Obviously, if

$$\lim_{\substack{\|u\|_X \to +\infty \\ u \in D(\mathcal{G})}} \frac{\langle \xi + \eta, u - u_0 \rangle_{X^* \times X}}{\|u\|_X} = +\infty \tag{2.8}$$

is satisfied, then the estimate in (2.7) holds automatically for some R large enough.

3 Double phase elliptic obstacle inclusion problem

In this section, we are interested in the study of the existence of a solution to the double phase elliptic obstacle inclusion problem (1.1) and in deriving some relevant properties of the solution set to problem (1.1). More precisely, we are going to apply a surjectivity theorem for multivalued mappings, which is formulated by the sum of a maximal monotone multivalued operator and a multivalued pseudomonotone operator, to examine the solvability of problem (1.1).

First, we formulate the hypotheses on the data of problem (1.1).

H(f): The multivalued convection mapping $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ has nonempty, bounded, closed and convex values and

(i) the multivalued mapping $x \mapsto f(x, s, \xi)$ is measurable in Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

(ii) the multivalued mapping $(s, \xi) \mapsto f(x, s, \xi)$ is upper semicontinuous for a.a. $x \in \Omega$;

(iii) there exist $\alpha_f \in L^{\frac{r}{r-1}}(\Omega)_+$ and $a_f, b_f \geq 0$ such that

$$|\eta| \leq a_f |\xi|^{\frac{p(r-1)}{r}} + b_f |s|^{r-1} + \alpha_f(x),$$

for all $\eta \in f(x, s, \xi)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$ and for a.a. $x \in \Omega$, where $1 < r < p^*$ with the critical exponent p^* in the domain Ω given in (2.1) for $s = p$;

(iv) there exist $\beta_f \in L^1(\Omega)_+$ and constants $c_f, d_f \geq 0$ such that

$$\eta s \leq c_f |\xi|^p + d_f |s|^p + \beta_f(x),$$

for all $\eta \in f(x, s, \xi)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$ and for a.a. $x \in \Omega$.

H(g): The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) for all $s \in \mathbb{R}$, the function $x \mapsto g(x, s)$ is measurable;

(ii) for a.a. $x \in \Omega$, the function $s \mapsto g(x, s)$ is continuous;

(iii) there exist $a_g > 0$ and $b_g \in L^1(\Omega)$ such that

$$g(x, s)s \geq a_g |s|^{\varsigma} - b_g(x),$$

for all $s \in \mathbb{R}$ and for a.a. $x \in \Omega$, where $p < \varsigma < p^*$;

(iv) for any $u, v \in L^{p^*}(\Omega)$, the function $x \mapsto g(x, u(x))v(x)$ belongs to $L^1(\Omega)$.

H(Φ): The function $\Phi : \Omega \to [0, \infty)$ is measurable, that is, $\Phi \in M(\Omega)$.

H(U): $U : \Gamma_3 \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies the following conditions:

(i) $U(x, s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a.a. $x \in \Gamma_3$ and for all $s \in \mathbb{R}$;

(ii) $x \mapsto U(x, s)$ is measurable on Γ_3 for all $s \in \mathbb{R}$;

(iii) $s \mapsto U(x, s)$ is u.s.c.;

(iv) there exist $\alpha_U \in L^{\delta'}(\Gamma_3)_+$ and $a_U \geq 0$ such that

$$|U(x,s)| \leq \alpha_U(x) + a_U|s|^{\delta-1}$$

for a.a. $x \in \Gamma_3$ and for all $s \in \mathbb{R}$, where $1 < \delta < p_*$ with the critical exponent p_* on the boundary Γ given in (2.1);

(v) there exist $\beta_U \in L^1(\Gamma_3)_+$ and $b_U \geq 0$ such that

$$\xi s \leq b_U|s|^p + \beta_U(x)$$

for all $\xi \in U(x,s)$, for all $s \in \mathbb{R}$ and for a.a. $x \in \Gamma_3$.

H(0): $a \in L^\infty(\Omega)$ is such that $\inf_{x\in\Omega} a(x) \geq c_\Lambda > 0$ and $h \in L^{p'}(\Gamma_2)$.

H(1): The inequality holds

$$c_\Lambda - c_f - b_U(\lambda_{1,p}^S)^{-1} > 0,$$

where $\lambda_{1,p}^S$ is the first eigenvalue of the p-Laplacian with Steklov boundary condition, see (2.2) and (2.3).

Remark 3.1. It should be mentioned that if hypotheses H(f)(iv) and H(U)(v) are replaced by the following conditions:

H(f)(iv)': there exist $\beta_f \in L^1(\Omega)_+$ and constants $c_f, d_f \geq 0$ such that

$$\eta s \leq c_f|\xi|^{\varrho_1} + d_f|s|^p + \beta_f(x)$$

for all $\eta \in f(x,s,\xi)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$ and for a.a. $x \in \Omega$, where $1 < \varrho_1 < p$;

H(U)(v)': there exist $\beta_U \in L^1(\Gamma_3)_+$ and $b_U \geq 0$ such that

$$\xi s \leq b_U|s|^{\varrho_2} + \beta_U(x)$$

for all $\xi \in U(x,s)$, for all $s \in \mathbb{R}$ and for a.a. $x \in \Gamma_3$, where $1 < \varrho_2 < p$,

then hypothesis H(1) can be removed. Indeed, it follows from Young's inequality with $\varepsilon > 0$ that

$$\begin{aligned}\eta s &\leq c_f|\xi|^{\varrho_1} + d_f|s|^p + \beta_f(y) \leq \varepsilon|\xi|^p + c_1(\varepsilon) + d_f|s|^p + \beta_f(y)\\ \xi s &\leq b_U|s|^{\varrho_2} + \beta_U(x) \leq \varepsilon|s|^p + c_2(\varepsilon) + \beta_U(x)\end{aligned}$$

for all $\eta \in f(y,s,\xi)$, for all $\xi \in U(x,s)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, for a.a. $y \in \Omega$ and for a.a. $x \in \Gamma_3$ with some $c_1(\varepsilon), c_2(\varepsilon) > 0$. If we choose $\varepsilon \in \left(0, \frac{c_\Lambda}{1+(\lambda_{1,p}^S)^{-1}}\right)$, then the inequality in H(1) holds automatically.

Let K be a subset of V given by

$$K := \{v \in V : v \leq \Phi \text{ in } \Omega\}. \tag{3.1}$$

Under H(Φ) we see that the set K is a nonempty, closed and convex subset of V. In fact, from H(Φ) (i.e., $\Phi(x) \geq 0$ for a.a. $x \in \Omega$), we know that $0 \in K$, i.e., $K \neq \emptyset$. Furthermore, it is clear that K is convex. For the closedness, let $\{u_n\}_{n\in\mathbb{N}} \subset K$ be a sequence such that $u_n \to u$ in V for some $u \in V$. The continuity of V into $L^p(\Omega)$ implies that $u_n \to u$ in $L^p(\Omega)$. Passing to a subsequence if necessary, we may suppose that $u_n(x) \to u(x)$ for a.a. $x \in \Omega$. Therefore,

$$\Phi(x) \geq \lim_{n\to\infty} u_n(x) = u(x) \quad \text{for a.a. } x \in \Omega.$$

Hence, $u \in K$ and so K is closed.

Next, we state the definition of a weak solution to problem (1.1).

Definition 3.2. A function $u \in K$ is said to be a weak solution of problem (1.1), if there exist functions $\eta \in L^{r'}(\Omega)$ and $\xi \in L^{\delta'}(\Gamma_3)$ with $\eta(x) \in f(x, u(x), \nabla u(x))$ for a.a. $x \in \Omega$, $\xi(x) \in U(x, u(x))$ for a.a. $x \in \Gamma_3$ and the equality

$$\begin{aligned}&\int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla(v-u)\mathrm{d}x + \int_\Omega g(x,u)(v-u)\mathrm{d}x + \int_\Omega \mu(x)|u|^{q-2}u(v-u)\mathrm{d}x\\ &\quad \geq \int_\Omega \eta(x)(v-u)\mathrm{d}x + \int_{\Gamma_2} h(x)(v-u)\mathrm{d}\Gamma + \int_{\Gamma_3}\xi(x)(v-u)\mathrm{d}\Gamma\end{aligned}$$

is satisfied for all $v \in K$, where the set K is defined by (3.1).

The following theorem which is the main result in this section shows that for each pair $(a, h) \in L^\infty(\Omega)_+ \times L^{p'}(\Gamma_2)$ satisfying H(0), the solution set to problem (1.1), denoted by $\mathcal{S}(a, h)$, is nonempty, bounded and weakly closed.

Theorem 3.3. *Let hypotheses* (2.4), $H(f)$, $H(g)$, $H(\Phi)$, $H(U)$, $H(0)$ and $H(1)$ *be satisfied. Then, the solution set of problem* (1.1) *is nonempty, bounded and weakly closed* (*hence, weakly compact*).

Proof. We divide the proof into three parts.
I Existence:

First, we consider the following nonlinear functions $F : V \to V^*$, $G : V \subset L^\varsigma(\Omega) \to L^{\varsigma'}(\Omega)$ and $L : L^p(\Omega) \to L^{p'}(\Omega)$ defined by

$$\begin{aligned}\langle Fu, v\rangle &:= \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla v\mathrm{d}x + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v\mathrm{d}x,\\ \langle Gu, w\rangle_{L^{\varsigma'}(\Omega)\times L^\varsigma(\Omega)} &:= \int_\Omega g(x,u)w\mathrm{d}x\\ \langle Ly, z\rangle_{L^{p'}(\Omega)\times L^p(\Omega)} &:= \int_\Omega |y|^{p-2}yz\mathrm{d}x\end{aligned}$$

for all $u, v \in V$, for all $w \in L^\varsigma(\Omega)$ and for all $y, z \in L^p(\Omega)$.

Let $u \in V$ be fixed. By the Yankov-von Neumann-Aumann selection theorem (see e.g., Papageorgiou and Winkert [47, Theorem 2.7.25]) and assumptions H(f)(i) and (ii), we know that the multivalued function $x \mapsto f(x, u(x), \nabla u(x))$ admits a measurable selection. Let $\eta : \Omega \to \mathbb{R}$ be a measurable selection of $x \mapsto f(x, u(x), \nabla u(x))$, that is, $\eta(x) \in f(x, u(x), \nabla u(x))$ for a.a. $x \in \Omega$. From H(f)(iii) and the inequality

$$(|r_1| + |r_2|)^s \leq 2^{s-1}(|r_1|^s + |r_2|^s) \quad \text{for all } r_1, r_2 \in \mathbb{R} \quad \text{with } s \geq 1,$$

it follows that there exist constants $M_1, M_2 > 0$ satisfying

$$\begin{aligned}\int_\Omega |\eta(x)|^{r'}\mathrm{d}x &\leq \int_\Omega \left(a_f|\nabla u|^{\frac{p}{r'}} + b_f|u|^{r-1} + \alpha_f(x)\right)^{r'}\mathrm{d}x\\ &\leq M_1\int_\Omega (|\nabla u|^p + |u|^r + \alpha_f(x)^{r'})\mathrm{d}x\\ &= M_1(\|\nabla u\|_{p,\Omega}^p + \|u\|_{r,\Omega}^r + \|\alpha_f\|_{r',\Omega}^{r'})\\ &\leq M_2(\|u\|_V^p + \|u\|_V^r + \|\alpha_f\|_{r',\Omega}^{r'}),\end{aligned} \tag{3.2}$$

where we have used the fact that the embeddings of V into $W^{1,p}(\Omega)$ and of V into $L^r(\Omega)$ are continuous. Hence, $\eta \in L^{r'}(\Omega)$. This permits us to consider the Nemytskij operator $N_f : V \subset L^r(\Omega) \to 2^{L^{r'}(\Omega)}$ associated with the multivalued mapping f defined by

$$N_f(u) := \{\eta \in L^{r'}(\Omega) : \eta(x) \in f(x, u(x), \nabla u(x)) \quad \text{for a.a. } x \in \Omega\}$$

for all $u \in V$. Similarly, because of hypotheses H(U)(i), (ii) and (iii), for each $u \in L^\delta(\Gamma_3)$ fixed, we are able to find a measurable function $\xi : \Gamma_3 \to \mathbb{R}$ satisfying $\xi(x) \in U(x, u(x))$ for a.a. $x \in \Gamma_3$ and

$$\begin{aligned}\|\xi\|_{\delta',\Gamma_3}^{\delta'} &= \int_{\Gamma_3} |\xi(x)|^{\delta'} \mathrm{d}\Gamma \\ &\le \int_{\Gamma_3} (\alpha_U(x) + a_U|u|^{\delta-1})^{\delta'} \mathrm{d}\Gamma \\ &\le M_3 \int_{\Gamma_3} (\alpha_U(x)^{\delta'} + |u|^\delta) \mathrm{d}\Gamma \\ &= M_3(\|\alpha_U\|_{\delta',\Gamma_3}^{\delta'} + \|u\|_{\delta,\Gamma_3}^{\delta})\end{aligned} \tag{3.3}$$

for some $M_3 > 0$. Therefore, in what follows, we denote by $N_U : L^\delta(\Gamma_3) \to 2^{L^{\delta'}(\Gamma_3)}$ the Nemytskij operator corresponding to the multivalued mapping U defined by

$$N_U(u) := \{\eta \in L^{\delta'}(\Gamma_3) : \eta(x) \in U(x, u(x)) \text{ for a.a. } x \in \Gamma_3\}$$

for all $u \in L^\delta(\Gamma_3)$.

Let $\iota : V \to L^r(\Omega)$, $\omega : V \to L^\varsigma(\Omega)$ and $\beta : V \to L^p(\Omega)$ be the embedding operators of V to $L^r(\Omega)$, V to $L^\varsigma(\Omega)$ and V to $L^p(\Omega)$, respectively, with its adjoint operators $\iota^* : L^{r'}(\Omega) \to V^*$, $\omega^* : L^{\varsigma'}(\Omega) \to V^*$ and $\beta^* : L^{p'}(\Omega) \to V^*$, respectively. Also, we denote by $\gamma : V \to L^\delta(\Gamma_3)$ the trace operator of V into $L^\delta(\Gamma_3)$ with its adjoint operator $\gamma^* : L^{\delta'}(\Gamma_3) \to V^*$. Consider the indicator function of the set K formulated as

$$I_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Under the aforementioned definitions, we could use a standard procedure for variation calculus to obtain that $u \in K$ is a weak solution of problem (1.1) if and only if it solves the following nonlinear inclusion problem:

$$Fu + \omega^* Gu - \beta^* Lu - \iota^* N_f(u) - \gamma^* N_U(u) + \partial_c I_K(u) \ni h \quad \text{in } V^*,$$

where $\partial_c I_K$ is the convex subdifferential operator of I_K.

Observe that the functions F, G and L are bounded. The latter combined with (3.2), (3.3) and hypotheses $H(f)$ and $H(U)$ implies that for each $u \in V$ the set

$$H(u) := Fu + \omega^* Gu - \beta^* Lu - \iota^* N_f(u) - \gamma^* N_U(u)$$

is nonempty, bounded, closed and convex. We show that H is a pseudomonotone operator. Let $\{u_n\}_{n\in\mathbb{N}} \subset V$, $\{\zeta_n\}_{n\in\mathbb{N}} \subset V^*$ be sequences and let $(u, \zeta) \in V \times V^*$ be such that

$$\zeta_n \in H(u_n) \quad \text{for each } n \in \mathbb{N}, \quad \zeta_n \xrightarrow{w} \zeta \quad \text{and} \quad \limsup_{n\to\infty} \langle \zeta_n, u_n - u\rangle \le 0. \tag{3.4}$$

Then, for every $n \in \mathbb{N}$, there are $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ such that

$$\zeta_n = Fu_n + \omega^* Gu_n - \beta^* Lu_n - \iota^* \eta_n - \gamma^* \xi_n \quad \text{for all } n \in \mathbb{N}.$$

Taking (3.2) and (3.3) into account, we can see that the sequences $\{\eta_n\}_{n\in\mathbb{N}} \subset L^{r'}(\Omega)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset L^{\delta'}(\Gamma_3)$ are both bounded. Without any loss of generality, we may assume that there exist functions $(\eta, \xi) \in L^{r'}(\Omega) \times L^{\delta'}(\Gamma_3)$ such that

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r'}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{\delta'}(\Gamma_3).$$

Recall that V is embedded compactly into $L^\varsigma(\Omega)$, $L^r(\Omega)$ and $L^p(\Omega)$, respectively, and $\gamma : V \to L^\delta(\Gamma_3)$ is compact. Using this we have

$$\begin{aligned}
\lim_{n\to\infty} \langle \omega^* G u_n, u_n - u\rangle &= \lim_{n\to\infty} \langle G u_n, \omega(u_n - u)\rangle_{L^{\varsigma'}(\Omega)\times L^{\varsigma}(\Omega)} = 0,\\
\lim_{n\to\infty} \langle \beta^* L u_n, u_n - u\rangle &= \lim_{n\to\infty} \langle L u_n, \beta(u_n - u)\rangle_{L^{p'}(\Omega)\times L^{p}(\Omega)} = 0,\\
\lim_{n\to\infty} \langle \iota^* \eta_n, u_n - u\rangle &= \lim_{n\to\infty} \langle \eta_n, \iota(u_n - u)\rangle_{L^{r'}(\Omega)\times L^{r}(\Omega)} = 0,\\
\lim_{n\to\infty} \langle \gamma^* \xi_n, u_n - u\rangle &= \lim_{n\to\infty} \langle \xi_n, \gamma(u_n - u)\rangle_{L^{\delta'}(\Gamma_3)\times L^{\delta}(\Gamma_3)} = 0.
\end{aligned} \tag{3.5}$$

Inserting (3.5) into the inequality in (3.4) yields

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty} \langle \zeta_n, u_n - u\rangle\\
&\geq \limsup_{n\to\infty} \langle F u_n, u_n - u\rangle + \liminf_{n\to\infty} \langle \omega^* G u_n, u_n - u\rangle - \limsup_{n\to\infty} \langle \beta^* L u_n, u - u_n\rangle + \liminf_{n\to\infty} \langle \iota^* \eta_n, u - u_n\rangle\\
&\quad + \liminf_{n\to\infty} \langle \gamma^* \xi_n, u - u_n\rangle\\
&\geq \limsup_{n\to\infty} \langle F u_n, u_n - u\rangle.
\end{aligned}$$

From Proposition 2.3 we know that F is of type (S_+). Therefore,

$$u_n \to u \quad \text{in } V.$$

Passing to a subsequence if necessary, we may assume that

$$u_n(x) \to u(x) \quad \text{and} \quad \nabla u_n(x) \to \nabla u(x) \quad \text{for a.a. } x \in \Omega. \tag{3.6}$$

Applying Mazur's theorem there exists a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ of convex combinations to $\{\eta_n\}_{n\in\mathbb{N}}$ satisfying

$$\chi_n \to \eta \quad \text{in } L^{r'}(\Omega).$$

Therefore, we can suppose that $\chi_n(x) \to \eta(x)$ for a.a. $x \in \Omega$. Due to the convexity of f we see that

$$\chi_n(x) \in f(x, u_n(x), \nabla u_n(x)) \quad \text{for a.a. } x \in \Omega.$$

Recall that f is u.s.c. and has nonempty, bounded, closed and convex values (see hypotheses $H(f)$(i) and (ii)). So, we can use Proposition 4.1.9 of Denkowski et al. [15] to infer that the graph of $(s, \xi) \mapsto f(x, s, \xi)$ is closed for a.a. $x \in \Omega$. Taking the convergence properties in (3.6) and $\chi_n(x) \to \eta(x)$ for a.a. $x \in \Omega$ into account, we obtain

$$\eta(x) \in f(x, u(x), \nabla u(x)) \quad \text{for a.a. } x \in \Omega.$$

This shows that $\eta \in N_f(u)$. Applying the same arguments as we did before, we conclude that $\xi \in N_U(u)$. Recall that F, G and L are continuous. So we can use the convergence (3.4) in order to obtain

$$\zeta_n = F u_n + \omega^* G u_n - \beta^* L u_n - \iota^* \eta_n - \gamma^* \xi_n \xrightarrow{w} F u + \omega^* G u - \beta^* L u - \iota^* \eta - \gamma^* \xi = \zeta \quad \text{in } V^*.$$

This implies that $\zeta \in H(u)$. Hence, we have

$$\begin{aligned}
\lim_{n\to\infty} \langle \zeta_n, u_n\rangle &= \lim_{n\to\infty} \langle F u_n + \omega^* G u_n - \beta^* L u_n - \iota^* \eta_n - \gamma^* \xi_n, u_n\rangle\\
&= \lim_{n\to\infty} \langle F u_n + \omega^* G u_n - \beta^* L u_n, u_n\rangle - \lim_{n\to\infty} \langle \eta_n, \iota u_n\rangle_{L^{r'}(\Omega)\times L^{r}(\Omega)} - \lim_{n\to\infty} \langle \xi_n, \gamma u_n\rangle_{L^{\delta'}(\Gamma_3)\times L^{\delta}(\Gamma_3)}\\
&= \langle F u + \omega^* G u - \beta^* L u - \iota^* \eta - \gamma^* \xi, u\rangle = \langle \zeta, u\rangle.
\end{aligned}$$

This shows that H is a generalized pseudomonotone operator. Employing Proposition 2.5, we conclude that H is pseudomonotone.

Next, we show the coercivity of H. To this end, we introduce a subspace W of $W^{1,p}(\Omega)$ defined by

$$W := \{u \in W^{1,p}(\Omega) : u = 0 \quad \text{on } \Gamma_1\}. \tag{3.7}$$

Because Γ_1 has positive measure, it is not difficult to prove that W endowed with the norm

$$\|u\|_W := \|\nabla u\|_{p,\Omega} \quad \text{for all } u \in W$$

is a reflexive and separable Banach space. Moreover, since the embedding of V into W is continuous, there exists a constant $C_{VW} > 0$ such that

$$\|u\|_W \le C_{VW}\|u\|_V \quad \text{for all } u \in V.$$

Let $u \in V$ and $\zeta \in H(u)$ be arbitrary. Then, we can find functions $\eta \in N_f(u)$ and $\xi \in N_U(u)$ such that $\zeta = Fu + \omega^* Gu - \beta^* Lu - \iota^*\eta - \gamma^*\xi$ and

$$\begin{aligned}
\langle \zeta, u\rangle &= \langle Fu, u\rangle + \langle \omega^* Gu - \beta^* Lu, u\rangle - \langle \eta, u\rangle_{L^{r'}(\Omega)\times L^{r}(\Omega)} - \langle \xi, u\rangle_{L^{\delta'}(\Gamma_3)\times L^{\delta}(\Gamma_3)} \\
&\ge c_\Lambda\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{q,\mu}^q - \int_\Omega c_f|\nabla u|^p + d_f|u|^p + \beta_f(x)\mathrm{d}x \\
&\quad - \int_{\Gamma_3} b_U|u|^p + \beta_U(x)\mathrm{d}\Gamma + \int_\Omega a_g|u|^\varsigma - b_g(x)\mathrm{d}x \\
&\ge (c_\Lambda - c_f)\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + a_g\|u\|_{\varsigma,\Omega}^\varsigma + \|u\|_{q,\mu}^q - \|b_g\|_{1,\Omega} - d_f\|u\|_{p,\Omega}^p - \|\beta_f\|_{1,\Omega} - b_U\|u\|_{p,\Gamma_3}^p \\
&\quad - \|\beta_U\|_{1,\Gamma_3}.
\end{aligned} \tag{3.8}$$

We set

$$\varepsilon = \frac{a_g}{2((\lambda_{1,p}^S)^{-1}b_U + d_f + 1)}.$$

Keeping in mind that $\varsigma > p$, it follows from Young's inequality and the eigenvalue problem of the p-Laplacian with Steklov boundary condition (see (2.2) and (2.3)) that the following inequalities hold

$$b_U\|u\|_{p,\Gamma_3}^p \le b_U(\lambda_{1,p}^S)^{-1}(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p) \tag{3.9}$$

and

$$\|u\|_{p,\Omega}^p = \int_\Omega |u|^p\mathrm{d}x \le \varepsilon\int_\Omega |u|^\varsigma\mathrm{d}x + c(\varepsilon) = \varepsilon\|u\|_{\varsigma,\Omega}^\varsigma + c(\varepsilon) \tag{3.10}$$

with some $c(\varepsilon) > 0$. Using (3.9) and (3.10) in (3.8), we obtain

$$\begin{aligned}
\langle \zeta, u\rangle &\ge (c_\Lambda - c_f - b_U(\lambda_{1,p}^S)^{-1})\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu}^q - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} \\
&\quad - \|\beta_U\|_{1,\Gamma_3} - c(\varepsilon) \\
&\ge \hat{M}_0(\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu}^q) + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c(\varepsilon) \\
&= \hat{M}_0\varrho_{\mathcal{H}}(u) + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c(\varepsilon) \\
&\ge \hat{M}_0\min\{\|u\|_V^p, \|u\|_V^q\} + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c(\varepsilon),
\end{aligned} \tag{3.11}$$

where $\hat{M}_0 > 0$ is defined by

$$\hat{M}_0 := \min\{c_\Lambda - c_f - b_U(\lambda_{1,p}^S)^{-1}, 1\}.$$

Since $c_\Lambda - c_f - b_U(\lambda_{1,p}^S)^{-1} > 0$, we deduce that H is coercive.

It is well-known that I_K is a proper, convex and l.s.c. function. Note that (see, e.g., Proposition 1.10 of Brézis [6])

$$I_K(u) \ge \alpha_K\|u\|_V \quad \text{for all } u \in V \quad \text{with some } \alpha_K < 0.$$

So, we have

$$\langle \kappa, u\rangle \ge I_K(u) - I_K(0) \ge \alpha_K\|u\|_V \quad \text{for all } \kappa \in \partial_c I_K(u) \text{ and for all } u \in K,$$

where we have used the fact that $0 \in K$. Combining the inequality above and (3.11) gives

$$\langle \zeta + \kappa - h, u\rangle \geq \hat{M}_0 \min\{\|u\|_V^p, \|u\|_V^q\} + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^{\varsigma} - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c(\varepsilon) - |\alpha_K|\|u\|_V$$
$$- M_4\|h\|_{p',\Gamma_2}\|u\|_V$$

for all $\zeta \in H(u)$ and for all $\kappa \in \partial_c I_K(u)$ with some $M_4 > 0$. Therefore, we infer that (2.8) is satisfied with $u_0 = 0$, $\mathcal{G} = \partial_c I_K$ and $\mathcal{F} = H$. Thus, all conditions of Theorem 2.7 are verified. Using this theorem, we conclude that problem (1.1) has at least one weak solution $u \in K$. Recalling that $f(x,0,0) \neq \{0\}$, u turns out to be a nontrivial weak solution of problem (1.1).

II Boundedness:

Suppose that the solution set $\mathcal{S}(a,h)$ is unbounded. Then, without loss of generality, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(a,h)$ such that

$$\|u_n\|_V \to +\infty \quad \text{as } n \to \infty.$$

Employing the same arguments as in the proof of the first part, we obtain the estimate

$$0 \geq \hat{M}_0 \min(\|u_n\|_V^p, \|u_n\|_V^q) + \frac{a_g}{2}\|u_n\|_{\varsigma,\Omega}^{\varsigma} - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - M_5\|h\|_{p',\Gamma_2}\|u_n\|_V - M_5 \tag{3.12}$$

for all $n \in \mathbb{N}$ and for some $M_5 > 0$. Letting $n \to \infty$ in the inequality above, we obtain a contradiction. Therefore, the solution set $\mathcal{S}(a,h)$ is bounded in V.

III Closedness:

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(a,h)$ be a sequence such that

$$u_n \xrightarrow{w} u \quad \text{in } V$$

for some $u \in K$. Then, there exist functions $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ such that

$$\langle Fu_n + \omega^* Gu_n - \beta^* Lu_n, v - u_n\rangle \geq \int_\Omega \eta_n(v-u_n)\mathrm{d}x + \int_{\Gamma_2} h(x)(v-u_n)\mathrm{d}\Gamma + \int_{\Gamma_3} \xi_n(v-u_n)\mathrm{d}\Gamma \tag{3.13}$$

for all $v \in K$. Due to the boundedness of the operators N_f and N_U we may suppose that there are functions $\eta \in L^{r'}(\Omega)$ and $\xi \in L^{\delta'}(\Gamma_3)$ satisfying

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r'}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{\delta'}(\Gamma_3).$$

Taking $v = u$ in (3.13) and passing to the upper limit as $n \to \infty$ in the resulting inequality, we obtain

$$\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq \lim_{n\to\infty}\int_\Omega (\eta_n + |u_n|^{p-2}u_n + g(x,u_n))(u-u_n)\mathrm{d}x + \lim_{n\to\infty}\int_{\Gamma_2} h(x)(u-u_n)\mathrm{d}\Gamma$$
$$+ \lim_{n\to\infty}\int_{\Gamma_3}\xi_n(u-u_n)\mathrm{d}\Gamma \leq 0.$$

Applying Proposition 2.3 we obtain that $u_n \to u$ in V. Using the upper semicontinuity of f and U, one has $\eta \in N_f(u)$ and $\xi \in N_U(u)$. Passing to the upper limit as $n \to \infty$ in equality (3.13), we derive that $u \in \mathcal{S}(a,h)$ and so, $\mathcal{S}(a,h)$ is weakly closed. This completes the proof. ☐

4 An inverse problem for double phase elliptic obstacle inclusion systems

This section is concerned with the study of an inverse problem to identify a discontinuous parameter in the domain and a discontinuous boundary datum for the double phase elliptic obstacle problem given in (1.1).

For any $g \in L^1(\Omega)$ fixed, in what follows, we denote by $\mathrm{TV}(g)$ the total variation of the function g given by

$$\mathrm{TV}(g) := \sup_{\varphi \in C^1(\Omega;\, \mathbb{R}^N)} \left\{ \int_\Omega g(x)\, \mathrm{div}\varphi(x) \mathrm{d}x : |\varphi(x)| \le 1 \quad \text{for all } x \in \Omega \right\}.$$

By $\mathrm{BV}(\Omega)$, we denote the function space of all integrable functions with bounded variation, namely,

$$\mathrm{BV}(\Omega) := \{g \in L^1(\Omega) : \mathrm{TV}(g) < +\infty\}.$$

It is well-known that $\mathrm{BV}(\Omega)$ endowed with the norm

$$\|g\|_{\mathrm{BV}(\Omega)} := \|g\|_{1,\Omega} + \mathrm{TV}(g) \quad \text{for all } g \in \mathrm{BV}(\Omega)$$

is a Banach space.

In the sequel, let H be a nonempty, closed and convex subset of $L^{p'}(\Gamma_3)$. Given positive constants c_Λ and d_Λ, we denote by Λ the set of all admissible parameters for the double phase differential operator given in (1.2) defined by

$$\Lambda := \{a \in \mathrm{BV}(\Omega) : 0 < c_\Lambda \le a(x) \le d_\Lambda \quad \text{for a.a. } x \in \Omega\}.$$

Obviously, we see that the admissible set Λ is a closed and convex subset of both $\mathrm{BV}(\Omega)$ and $L^\infty(\Omega)$.

Given two regularization parameters $\kappa > 0$ and $\tau > 0$ and the known observed or measured datum $z \in L^p(\Omega; \mathbb{R}^N)$, we consider the inverse problem formulated in the following regularized optimal control framework:

Problem 4.1. Find $a^* \in \Lambda$ and $h^* \in H$ such that

$$\inf_{a \in \Lambda \text{ and } h \in H} C(a, h) = C(a^*, h^*), \tag{4.1}$$

where the cost functional $C : \Lambda \times H \to \mathbb{R}$ is given by

$$C(a, h) := \min_{u \in \mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)} + \kappa \mathrm{TV}(a) + \tau \|h\|_{p',\Gamma_2}, \tag{4.2}$$

and $\mathcal{S}(a, h)$ stands for the solution set of the double phase elliptic obstacle problem (1.1) with respect to $a \in L^\infty(\Omega)$ and $h \in L^{p'}(\Gamma_2)$.

The main result in this section is the following existence result for the regularized optimal control problem given in Problem 4.1.

Theorem 4.2. *Assume that all conditions of* Theorem 3.3 *are satisfied. Then the solution set of* Problem 4.1 *is nonempty and weakly compact.*

Proof. The proof of this theorem is divided into four steps.
Step 1: The functional C defined in (4.2) is well-defined.

We only need to verify that for $(a, h) \in \Lambda \times H$ fixed, the optimal problem

$$\min_{u \in \mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)}$$

is solvable. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(a, h)$ is a minimizing sequence of the problem $\inf_{u\in\mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)}$, that is,

$$\inf_{u \in \mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)} = \lim_{n\to\infty} \|\nabla u_n - z\|_{L^p(\Omega;\, \mathbb{R}^N)}.$$

From Theorem 3.3, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V. Passing to a subsequence if necessary, we can assume that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in V$. This fact along with the weak closedness of $\mathcal{S}(a, h)$ ensures that $u^* \in \mathcal{S}(a, h)$. On the other hand, the weak lower semicontinuity of the norm $\|\cdot\|_{L^p(\Omega;\, \mathbb{R}^N)}$ implies that

$$\inf_{u \in \mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)} = \liminf_{n\to\infty} \|\nabla u_n - z\|_{L^p(\Omega;\, \mathbb{R}^N)} \ge \|\nabla u^* - z\|_{L^p(\Omega;\, \mathbb{R}^N)} \ge \inf_{u \in \mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)}.$$

This means that for each $(a, h) \in \Lambda \times H$ there exists $u^* \in \mathcal{S}(a, h)$ such that

$$\inf_{u \in \mathcal{S}(a,h)} \|\nabla u - z\|_{L^p(\Omega;\, \mathbb{R}^N)} = \|\nabla u^* - z\|_{L^p(\Omega;\, \mathbb{R}^N)}.$$

Hence, C is well-defined.

For any $(a, h) \in \Lambda \times H$ and $u \in \mathcal{S}(a, h)$ fixed, it follows from (3.12) that

$$0 \geq \hat{M}_0 \min\{\|u\|_V^p, \|u\|_V^q\} + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^{\varsigma} - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - M_6\|h\|_{p',\Gamma_2}\|u\|_V - M_6$$

for some $M_6 > 0$. Therefore, we conclude that $\mathcal{S}$ maps bounded sets of $\Lambda \times H \subset \mathrm{BV}(\Omega) \times L^{p'}(\Gamma_2)$ into bounded sets of K.

Step 2: If $\{(a_n, h_n)\}_{n\in\mathbb{N}} \subset \Lambda \times H$ is a sequence such that $\{a_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $a_n \to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in H for some $(a, h) \in L^1(\Omega) \times H$, then $a \in \Lambda$ and one has

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}(a_n, h_n) \subset \mathcal{S}(a, h). \tag{4.3}$$

Let $\{(a_n, h_n)\}_{n\in\mathbb{N}} \subset \Lambda \times H$ be a sequence such that $a_n \to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in H for some $(a, h) \in L^1(\Omega) \times H$. By the properties of Λ (i.e., Λ is nonempty, closed and convex in $\mathrm{BV}(\Omega)$ and $L^1(\Omega)$), one has $(a, h) \in \Lambda \times H$. Moreover, the boundedness of $\{a_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and the map $\mathcal{S}$ implies that $\cup_{n\geq 1}\mathcal{S}(a_n, h_n)$ is bounded in K. Also, the reflexivity of V guarantees that the set $w\text{-}\limsup_{n\to\infty}\mathcal{S}(a_n, h_n)$ is nonempty.

For any $u \in w\text{-}\limsup_{n\to\infty}\mathcal{S}(a_n, h_n)$, passing to a subsequence if necessary, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset K$ such that

$$u_n \in \mathcal{S}(a_n, h_n) \quad \text{and} \quad u_n \xrightarrow{w} u \quad \text{in } V.$$

Hence, for every $n \in \mathbb{N}$, we are able to find functions $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ such that

$$\begin{aligned} &\int_\Omega (a_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot\nabla(v - u_n)\mathrm{d}x \\ &\quad + \int_\Omega \mu(x)|u_n|^{q-2}u_n(v - u_n)\mathrm{d}x + \int_\Omega g(x, u_n)(v - u_n)\mathrm{d}x \\ &\quad \geq \int_\Omega \eta_n(x)(v - u_n)\mathrm{d}x + \int_{\Gamma_2} h_n(x)(v - u_n)\mathrm{d}\Gamma + \int_{\Gamma_3} \xi_n(x)(v - u_n)\mathrm{d}\Gamma \end{aligned} \tag{4.4}$$

for all $v \in K$. Taking $v = u$ in (4.4) gives

$$\begin{aligned} &\int_\Omega (a_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot\nabla(u - u_n)\mathrm{d}x + \int_\Omega \mu(x)|u_n|^{q-2}u_n(u - u_n)\mathrm{d}x \\ &\quad + \int_\Omega g(x, u_n)(u - u_n)\mathrm{d}x \\ &\quad \geq \int_\Omega \eta_n(x)(u - u_n)\mathrm{d}x + \int_{\Gamma_2} h_n(x)(u - u_n)\mathrm{d}\Gamma + \int_{\Gamma_3} \xi_n(x)(u - u_n)\mathrm{d}\Gamma. \end{aligned} \tag{4.5}$$

Hypotheses $H(f)$(iii) and $H(U)$(iv) imply that the sequences $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are bounded in $L^{r'}(\Omega)$ and $L^{\delta'}(\Gamma_3)$, respectively. Since the embeddings of V to $L^\varsigma(\Omega)$ and $L^r(\Omega)$ are compact, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \mu(x)|u_n|^{q-2}u_n(u_n-u)\mathrm{d}x=0,\\
&\lim_{n\to\infty}\int_\Omega g(x,u_n)(u-u_n)\mathrm{d}x=0,\\
&\lim_{n\to\infty}\int_\Omega \eta_n(x)(u-u_n)\mathrm{d}x=0,\\
&\lim_{n\to\infty}\int_{\Gamma_2} h_n(x)(u-u_n)\mathrm{d}\Gamma=0,\\
&\lim_{n\to\infty}\int_{\Gamma_3} \xi_n(x)(u-u_n)\mathrm{d}\Gamma=0,
\end{aligned}\tag{4.6}$$

where we have also used the compactness of $V \hookrightarrow L^p(\Gamma_2)$ and $V \hookrightarrow L^\delta(\Gamma_3)$.

From Simon [50, formula (2.2)] we have the well-known inequalities

$$M_s|\xi-\eta|^s \le (|\xi|^{s-2}\xi-|\eta|^{s-2}\eta)\cdot(\xi-\eta), \quad \text{if } s\ge 2, \tag{4.7}$$

$$\mathcal{M}_s|\xi-\eta|^2 \le (|\xi|^{s-2}\xi-|\eta|^{s-2}\eta)\cdot(\xi-\eta)(|\xi|^s+|\eta|^s)^{\frac{2-s}{s}}, \quad \text{if } 1\le s\le 2, \tag{4.8}$$

for all $\xi,\eta\in\mathbb{R}^N$ with some constants M_s, $\mathcal{M}_s>0$ independent of $\xi,\eta\in\mathbb{R}^N$.

Next, we consider the following cases: $1<p<2$ and $p\ge 2$. If $p\ge 2$, then we use (4.7) in order to obtain

$$\int_\Omega a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x \ge c_\Lambda M_p\|u_n-u\|_W^p,$$

where the function space W is given in (3.7). Consider the sets

$$\begin{aligned}
\Omega_n&=\{x\in\Omega:\nabla u_n\neq 0\}\cup\{x\in\Omega:\nabla u\neq 0\},\\
\Sigma_n&=\{x\in\Omega:\nabla u=\nabla u_n=0\}.
\end{aligned}$$

We observe that $\Omega=\Omega_n\cup\Sigma_n$ and $\Omega_n\cap\Sigma_n=\emptyset$.

By the absolute continuity of the Lebesgue integral, one has

$$\int_{\Sigma_n} a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x=0.$$

Hence, we have

$$\begin{aligned}
&\int_\Omega a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x\\
&\quad=\int_{\Omega_n} a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x+\int_{\Sigma_n} a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x\\
&\quad=\int_{\Omega_n} a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x.
\end{aligned}$$

When $1<p<2$, we can apply (4.8) and obtain

$$\begin{aligned}
&\int_\Omega a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\mathrm{d}x\\
&\quad=\int_{\Omega_n} a_n(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n-u)\frac{(|\nabla u_n|^p+|\nabla u_n|^p)^{\frac{2-p}{p}}}{(|\nabla u_n|^p+|\nabla u_n|^p)^{\frac{2-p}{p}}}\mathrm{d}x\\
&\quad\ge \mathcal{M}_p\int_{\Omega_n} a_n(x)|\nabla u_n-\nabla u|^2(|\nabla u_n|^p+|\nabla u_n|^p)^{\frac{p-2}{p}}\mathrm{d}x\\
&\quad\ge c_\Lambda\mathcal{M}_p\int_{\Omega_n} |\nabla u_n-\nabla u|^2(|\nabla u_n|^p+|\nabla u_n|^p)^{\frac{p-2}{p}}\mathrm{d}x.
\end{aligned}\tag{4.9}$$

Due to $1<p<2$, one has $\frac{2}{p}>1$. Using this and Hölder's inequality yields

$$
\begin{aligned}
\int_\Omega |\nabla u_n - \nabla u|^p \mathrm{d}x &= \int_\Omega |\nabla u_n - \nabla u|^{2\cdot\frac{p}{2}} \mathrm{d}x \\
&= \int_\Omega \left(|\nabla u_n - \nabla u|^2 (|\nabla u_n|^p + |\nabla u|^p)^{\frac{p-2}{p}} \right)^{\frac{p}{2}} (|\nabla u_n|^p + |\nabla u|^p)^{\frac{2-p}{2}} \mathrm{d}x \\
&\le \left(\int_\Omega |\nabla u_n - \nabla u|^2 (|\nabla u_n|^p + |\nabla u|^p)^{\frac{p-2}{p}} \mathrm{d}x \right)^{\frac{p}{2}} \times \left(\int_\Omega (|\nabla u_n|^p + |\nabla u|^p) \mathrm{d}x \right)^{\frac{2-p}{2}}.
\end{aligned}
$$

This means that

$$
\int_\Omega |\nabla u_n - \nabla u|^2 (|\nabla u_n|^p + |\nabla u_n|^p)^{\frac{p-2}{p}} \mathrm{d}x \ge \left(\int_\Omega |\nabla u_n - \nabla u|^p \mathrm{d}x \right)^{\frac{2}{p}} \left(\int_\Omega (|\nabla u_n|^p + |\nabla u_n|^p) \mathrm{d}x \right)^{-\frac{2-p}{p}}.
$$

Combining the inequality above and (4.9), we obtain

$$
\begin{aligned}
&\int_\Omega a_n(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u) \mathrm{d}x \\
&\ge c_\Lambda \mathcal{M}_p \left(\int_\Omega |\nabla u_n - \nabla u|^p \mathrm{d}x \right)^{\frac{2}{p}} \left(\int_\Omega (|\nabla u_n|^p + |\nabla u|^p) \mathrm{d}x \right)^{-\frac{2-p}{p}} \\
&\ge M_7 c_\Lambda \mathcal{M}_p \left(\int_\Omega |\nabla u_n - \nabla u|^p \mathrm{d}x \right)^{\frac{2}{p}},
\end{aligned}
\tag{4.10}
$$

where $M_7 > 0$ is such that $(\|u_n\|_W^p + \|u\|_W^p)^{-\frac{2-p}{p}} \ge M_7$ owing to the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in V and the continuity of embedding from V to W.

Next, we apply Hölder's inequality to obtain

$$
\begin{aligned}
&\int_\Omega ((a_n(x) - a(x))|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u) \mathrm{d}x \\
&\ge -\int_\Omega |a_n(x) - a(x)||\nabla u|^{p-1}|\nabla(u_n - u)| \mathrm{d}x \\
&= -\int_\Omega |a_n(x) - a(x)|^{\frac{p-1}{p}} |\nabla u|^{p-1} |a_n(x) - a(x)|^{\frac{1}{p}} |\nabla(u_n - u)| \mathrm{d}x \\
&\ge -\left(\int_\Omega |a_n(x) - a(x)||\nabla u|^p \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_\Omega |a_n(x) - a(x)||\nabla(u_n - u)|^p \mathrm{d}x \right)^{\frac{1}{p}} \\
&\ge -(2c_\Lambda)^{\frac{1}{p}} \|u_n - u\|_W \left(\int_\Omega |a_n(x) - a(x)||\nabla u|^p \mathrm{d}x \right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Since $a_n \to a$ in $L^1(\Omega)$, without loss of generality, we may assume that $a_n(x) \to a(x)$ for a.a. $x \in \Omega$. Passing to the limit as $n \to \infty$ in the last estimate and using Lebesgue's dominated convergence theorem as well as the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in W yields

$$
\lim_{n\to\infty} \int_\Omega ((a_n(x) - a(x))|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u) \mathrm{d}x \ge \lim_{n\to\infty} \left[-2c_\Lambda \|u_n - u\|_W \left(\int_\Omega |a_n(x) - a(x)||\nabla u|^p \mathrm{d}x \right)^{\frac{p-1}{p}} \right] = 0. \tag{4.11}
$$

Therefore, we have

$$\begin{aligned}
&\int_\Omega (a_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot\nabla(u_n - u)\mathrm{d}x \\
&\quad= \int_\Omega a_n(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u)\mathrm{d}x + \int_\Omega ((a_n(x) - a(x))|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u)\mathrm{d}x \\
&\qquad+ \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u)\mathrm{d}x + \int_\Omega (\mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot\nabla(u_n - u)\mathrm{d}x.
\end{aligned}$$

Passing to the upper limit as $n \to \infty$ in (4.5) and using (4.6), (4.10), (4.11) as well as

$$\lim_{n\to\infty} \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla(u_n - u)\mathrm{d}x = 0,$$

$$\int_\Omega (\mu(x)(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u))\cdot\nabla(u_n - u)\mathrm{d}x \geq 0,$$

we obtain for $p \geq 2$

$$\limsup_{n\to\infty} c_\Lambda M_p \|u_n - u\|_W^p \leq 0$$

and for $1 < p < 2$

$$\limsup_{n\to\infty} \mathcal{M}_p \|u - u_n\|_W^2 \leq 0.$$

We conclude that $u_n \to u$ in W.

Moreover, the boundedness of $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$, as well as the reflexivity of $L^{r'}(\Omega)$ and $L^{\delta'}(\Gamma_3)$ permit us to find functions $\eta \in L^{r'}(\Omega)$ and $\xi \in L^{\delta'}(\Gamma_3)$ such that, by passing to a subsequence if necessary,

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r'}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{\delta'}(\Gamma_3).$$

Arguing as in the proof of Theorem 3.3, we obtain $\eta \in N_f(u)$ and $\xi \in N_U(u)$. Without loss of generality, we may assume that $\nabla u_n(x) \to \nabla u(x)$ for a.a. $x \in \Omega$. Applying Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega (a_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot\nabla(v - u_n)\mathrm{d}x \\
&\quad= \int_\Omega \lim_{n\to\infty}(a_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot\nabla(v - u_n)\mathrm{d}x \\
&\quad= \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla(v - u)\mathrm{d}x.
\end{aligned}$$

Letting $n \to \infty$ in equality (4.4) and using the aforementioned convergence properties we obtain

$$\begin{aligned}
&\int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla(v - u)\mathrm{d}x + \int_\Omega \mu(x)|u|^{q-2}u(v - u)\mathrm{d}x + \int_\Omega g(x, u)(v - u)\mathrm{d}x \\
&\quad\geq \int_\Omega \eta(x)(v - u)\mathrm{d}x + \int_{\Gamma_2} h(x)(v - u)\mathrm{d}\Gamma + \int_{\Gamma_3} \xi(x)(v - u)\mathrm{d}\Gamma
\end{aligned}$$

for all $v \in K$. Therefore, we can observe that $u \in K$ is a solution of problem (1.1) corresponding to $(a, h) \in \Lambda \times H$, that is, $u \in \mathcal{S}(a, h)$. Hence, w-$\limsup_{n\to\infty}\mathcal{S}(a_n, h_n) \subset \mathcal{S}(a, h)$ and so we have proved (4.3).

Step 3: If $\{(a_n, h_n)\}_{n\in\mathbb{N}} \subset \Lambda \times H$ is such that $\{a_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $a_n \to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in $L^{p'}(\Gamma_2)$ for some $(a, h) \in L^1(\Omega) \times H$, then the inequality

$$C(a,h) \le \liminf_{n\to\infty} C(a_n,h_n) \tag{4.12}$$

holds.

Let $\{(a_n,h_n)\}_{n\in\mathbb{N}} \subset \Lambda\times H$ be such that $a_n \to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in $L^{p'}(\Gamma_2)$ for some $(a,h)\in L^1(\Omega)\times H$. From Step 2 one has $a\in\Lambda$. Let $\{u_n\}_{n\in\mathbb{N}}\subset K$ be a sequence such that

$$u_n \in \mathcal{S}(a_n,h_n) \quad \text{and} \quad \inf_{u\in\mathcal{S}(a_n,h_n)} \|\nabla u - z\|_{L^p(\Omega;\,\mathbb{R}^N)} = \|\nabla u_n - z\|_{L^p(\Omega;\,\mathbb{R}^N)} \tag{4.13}$$

for each $n\in\mathbb{N}$.

Recalling that $\cup_{n\ge1}\mathcal{S}(a_n,h_n)$ is bounded, passing to a subsequence if necessary, we have $u_n \xrightarrow{w} u^*$ in V for some $u^*\in K$, that is, $u^* \in w\text{-}\limsup_{n\to\infty}\mathcal{S}(a_n,h_n)$. Applying again Step 2, we conclude that $u^*\in\mathcal{S}(a,h)$. Therefore, from the lower semicontinuity of the function $L^1(\Omega)\ni a\mapsto \mathrm{TV}(a)\in\mathbb{R}$ and the weak lower semicontinuity of $W\ni u\mapsto \|\nabla u - z\|_{L^p(\Omega;\,\mathbb{R}^N)}\in\mathbb{R}$ and $L^{p'}(\Gamma_2)\ni h\mapsto \|h\|_{p',\Gamma_2}\in\mathbb{R}$, it follows that

$$\begin{aligned}
\liminf_{n\to\infty} C(a_n,h_n) &= \liminf_{n\to\infty}[\|\nabla u_n - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa \mathrm{TV}(a_n) + \tau\|h_n\|_{p',\Gamma_2}]\\
&\ge \liminf_{n\to\infty}\|\nabla u_n - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \liminf_{n\to\infty}\kappa\mathrm{TV}(a_n) + \liminf_{n\to\infty}\tau\|h_n\|_{p',\Gamma_2}\\
&\ge \|\nabla u^* - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa\mathrm{TV}(a) + \tau\|h\|_{p',\Gamma_2}\\
&\ge \inf_{u\in\mathcal{S}(a,h)}\|\nabla u - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa\mathrm{TV}(a) + \tau\|h\|_{p',\Gamma_2}\\
&= C(a,h).
\end{aligned}$$

Hence (4.12) follows.

Step 4: The solution set of Problem 4.1 is nonempty and weakly compact.

By the definition of C, we see that C is bounded from below. Let $\{(a_n,h_n)\}_{n\in\mathbb{N}}\subset\Lambda\times H$ be a minimizing sequence of 4.1, namely,

$$\inf_{a\in\Lambda \text{ and } h\in H} C(a,h) = \lim_{n\to\infty} C(a_n,h_n). \tag{4.14}$$

This indicates that the sequences $\{a_n\}_{n\in\mathbb{N}}\subset\Lambda$ and $\{h_n\}_{n\in\mathbb{N}}\subset L^{p'}(\Gamma_2)$ are bounded in $\mathrm{BV}(\Omega)$ and $L^{p'}(\Gamma_2)$, respectively. Passing to a subsequence if necessary we have

$$a_n \to a^* \quad \text{in } L^1(\Omega) \quad \text{and} \quad h_n \xrightarrow{w} h^* \quad \text{in } L^{p'}(\Gamma_2) \tag{4.15}$$

for some $(a^*,h^*)\in\Lambda\times L^{p'}(\Gamma_2)$, where we have used the closedness of Λ in $L^1(\Omega)$ and the compactness of the embedding $\mathrm{BV}(\Omega)$ to $L^1(\Omega)$.

Let us consider a sequence $\{u_n\}_{n\in\mathbb{N}}\subset K$ satisfying (4.13). Employing the convergence (4.15) and the boundedness of $\mathcal{S}$, it implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V. So, we are able to select a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that $u_n \xrightarrow{w} u^*$ in V for some $u^*\in K$. From Step 2 it is clear that $u^*\in\mathcal{S}(a^*,h^*)$. Therefore, we have

$$\begin{aligned}
\liminf_{n\to\infty} C(a_n,h_n) &= \liminf_{n\to\infty}[\|\nabla u_n - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa\mathrm{TV}(a_n) + \tau\|h_n\|_{p',\Gamma_2}]\\
&\ge \liminf_{n\to\infty}\|\nabla u_n - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa\liminf_{n\to\infty}\mathrm{TV}(a_n) + \tau\liminf_{n\to\infty}\|h_n\|_{p',\Gamma_2}\\
&\ge \|\nabla u^* - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa\mathrm{TV}(a^*) + \tau\|h^*\|_{p',\Gamma_2}\\
&\ge \inf_{u\in\mathcal{S}(a^*,h^*)}\|\nabla u - z\|_{L^p(\Omega;\,\mathbb{R}^N)} + \kappa\mathrm{TV}(a^*) + \tau\|h^*\|_{p',\Gamma_2}\\
&= C(a^*,h^*)\\
&\ge \inf_{a\in\Lambda \text{ and } h\in H} C(a,h).
\end{aligned} \tag{4.16}$$

The latter combined with (4.14) implies that $(a^*,h^*)\in\Lambda\times H$ is a solution of Problem 4.1.

Finally, we prove the weak compactness of the solution set to Problem 4.1. To this end, let $\{(a_n,h_n)\}_{n\in\mathbb{N}}$ be a sequence of solutions to Problem 4.1. It is obvious that $\{a_n\}_{n\in\mathbb{N}}\subset\Lambda$ is bounded in $\mathrm{BV}(\Omega)$ and $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\Gamma_2)$. Using the same arguments, we may assume that (4.15) holds with some $(a^*,h^*)\in\Lambda\times L^{p'}(\Gamma_2)$. Similarly, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that (4.13) is fulfilled and $u_n \xrightarrow{w} u^*$

in V for some $u^* \in S(a^*, h^*)$. As done before, we can prove the validity of (4.16). This means that $(a^*, h^*) \in \Lambda \times H$ is a solution of Problem 4.1. Therefore, the solution set of Problem 4.1 is weakly compact. This completes the proof. □

Remark 4.3. The results of this section remain valid if the functional (4.2) is replaced by the following regularized cost functional:

$$C(a, h) = \min_{u \in S(a,h)} \|\nabla u - z\|^{\omega_1}_{L^p(\Omega;\, \mathbb{R}^N)} + \kappa \mathrm{TV}(a) + \tau \|h\|^{\omega_2}_{p',\Gamma_2},$$

where $1 < \omega_1 \le p$ and $1 < \omega_2 \le p'$. The latter for $\omega_1 = \omega_2 = 2$ is the most popular and commonly used the output least-squares objective functional utilized in the numerical approaches.

Acknowledgments: The authors acknowledge support by the German Research Foundation and the Open Access Publication Fund of TU Berlin.

Funding information: This project has received funding from the NNSF of China Grant Nos. 12001478 and 12101143, the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Preludium Project No. 2017/25/N/ST1/00611 and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It is also supported by Natural Science Foundation of Guangxi Grants Nos. 2021GXNSFFA196004, 2020GXNSFBA297137 and GKAD21220144 and the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019.

Conflict of interest: The authors declare that they have no conflict of interest.

References

[1] A. Bahrouni, V. D. Rădulescu, and P. Winkert, *Double phase problems with variable growth and convection for the Baouendi-Grushin operator*, Z. Angew. Math. Phys. **71** (2020), no. 6, Paper no. 183, 15 pp.

[2] P. Baroni, M. Colombo, and G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121** (2015), 206–222.

[3] P. Baroni, M. Colombo, and G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differ. Equ. **57** (2018), no. 2, 62, 48 pp.

[4] O, Benslimane, A. Aberqi, and J. Bennouna, *Existence and uniqueness of entropy solution of a nonlinear elliptic equation in anisotropic Sobolev-Orlicz space*, Rend. Circ. Mat. Palermo (2) **70** (2021), no. 3, 1579–1608.

[5] S. Biagi, F. Esposito, and E. Vecchi, *Symmetry and monotonicity of singular solutions of double phase problems*, J. Differ. Equ. **280** (2021), 435–463.

[6] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[7] S.-S. Byun and J. Oh, *Regularity results for generalized double phase functionals*, Anal. PDE **13** (2020), no. 5, 1269–1300.

[8] S. Carl and V. K. Le, *Multi-valued Variational Inequalities and Inclusions*, Springer, Cham, 2021.

[9] S. Carl, V. K. Le, and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities*, Springer, New York, 2007.

[10] C. Clason, A. A. Khan, M. Sama, and C. Tammer, *Contingent derivatives and regularization for noncoercive inverse problems*, Optimization **68** (2019), no. 7, 1337–1364.

[11] F. Colasuonno and M. Squassina, *Eigenvalues for double phase variational integrals*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 6, 1917–1959.

[12] M. Colombo and G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218** (2015), no. 1, 219–273.

[13] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.

[14] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, and P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*. J. Differ. Equ. **322** (2022), 182–228.

[15] Z. Denkowski, S. Migórski, and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic Publishers, Boston, MA, 2003.
[16] S. El Manouni, G. Marino, and P. Winkert, *Existence results for double phase problems depending on Robin and Steklov eigenvalues for the p-Laplacian*, Adv. Nonlinear Anal. **11** (2022), no. 1, 304–320.
[17] F. Faraci, D. Motreanu, and D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var. Partial Differ. Equ. **54** (2015), no. 1, 525–538.
[18] F. Faraci and D. Puglisi, *A singular semilinear problem with dependence on the gradient*, J. Differ. Equ. **260** (2016), no. 4, 3327–3349.
[19] C. Farkas and P. Winkert, *An existence result for singular Finsler double phase problems*, J. Differ. Equ. **286** (2021), 455–473.
[20] G. M. Figueiredo and G. F. Madeira, *Positive maximal and minimal solutions for non-homogeneous elliptic equations depending on the gradient*, J. Differ. Equ. **274** (2021), 857–875.
[21] A. Fiscella, *A double phase problem involving Hardy potentials*, Appl. Math. Optim. **85** (2022), 45.
[22] L. Gasiński and N. S. Papageorgiou, *Constant sign and nodal solutions for superlinear double phase problems*, Adv. Calc. Var. **14** (2021), no. 4, 613–626.
[23] L. Gasiński and N. S. Papageorgiou, *Positive solutions for nonlinear elliptic problems with dependence on the gradient*, J. Differ. Equ. **263** (2017), 1451–1476.
[24] L. Gasiński and P. Winkert, *Constant sign solutions for double phase problems with superlinear nonlinearity*, Nonlinear Anal. **195** (2020), 111739.
[25] L. Gasiński and P. Winkert, *Existence and uniqueness results for double phase problems with convection term*, J. Differ. Equ. **268** (2020), no. 8, 4183–4193.
[26] L. Gasiński and P. Winkert, *Sign changing solution for a double phase problem with nonlinear boundary condition via the Nehari manifold*, J. Differ. Equ. **274** (2021), 1037–1066.
[27] J. Gwinner, *An optimization approach to parameter identification in variational inequalities of second kind*, Optim. Lett. **12** (2018), no. 5, 1141–1154.
[28] J. Gwinner, B. Jadamba, A. A. Khan and M. Sama, *Identification in variational and quasi-variational inequalities*, J. Convex Anal. **25** (2018), no. 2, 545–569.
[29] V. K. Le, *A range and existence theorem for pseudomonotone perturbations of maximal monotone operators*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1645–1658.
[30] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. **64** (2006), no. 5, 1057–1099.
[31] W. Liu and G. Dai, *Existence and multiplicity results for double phase problem*, J. Differ. Equ. **265** (2018), no. 9, 4311–4334.
[32] Z. Liu, D. Motreanu, and S. Zeng, *Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient*, Calc. Var. Partial Differ. Equ. **58** (2019), no. 1, Paper no. 28, 22 pp.
[33] Z. Liu and N. S Papageorgiou, *Positive solutions for resonant (p, q)-equations with convection*, Adv. Nonlinear Anal. **10** (2021), no. 1, 217–232.
[34] W. Liu and P. Winkert, *Combined effects of singular and superlinear nonlinearities in singular double phase problems in* $\mathbb{R}^N$, J. Math. Anal. Appl. **507** (2022), no. 2, 125762, 19 pp.
[35] S. A. Marano and P. Winkert, *On a quasilinear elliptic problem with convection term and nonlinear boundary condition*, Nonlinear Anal. **187** (2019), 159–169.
[36] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Differ. Equ. **90** (1991), no. 1, 1–30.
[37] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284.
[38] S. Migórski, A. A. Khan, and S. Zeng, *Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems*, Inverse Problems **36** (2020), no. 2, 024006, 20 pp.
[39] S. Migórski, A. A. Khan, and S. Zeng, *Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of p-Laplacian type*, Inverse Problems **35** (2019), no. 3, 035004, 14 pp.
[40] S. Migórski and A. Ochal, *An inverse coefficient problem for a parabolic hemivariational inequality*, Appl. Anal. **89** (2010), no. 2, 243–256.
[41] S. Migórski, A. Ochal, and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities*, Springer, New York, 2013.
[42] P. D. Panagiotopoulos, *Nonconvex problems of semipermeable media and related topics*, Z. Angew. Math. Mech. **65** (1985), no. 1, 29–36.
[43] P. D. Panagiotopoulos, *Hemivariational Inequalities*, Springer-Verlag, Berlin, 1993.
[44] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, *Positive solutions for nonlinear Neumann problems with singular terms and convection*, J. Math. Pures Appl. (9) **136** (2020), 1–21.
[45] N. S. Papageorgiou and C. Vetro, *Existence and relaxation results for second order multivalued systems*, Acta Appl. Math. **173** (2021), Paper No. 5, 36 pp.
[46] N. S. Papageorgiou, C. Vetro, and F. Vetro, *Solutions for parametric double phase Robin problems*, Asymptot. Anal. **121** (2021), no. 2, 159–170.
[47] N. S. Papageorgiou and P. Winkert, *Applied Nonlinear Functional Analysis: An Introduction*, De Gruyter, Berlin, 2018.

[48] K. Perera and M. Squassina, *Existence results for double-phase problems via Morse theory*, Commun. Contemp. Math. **20** (2018), no. 2, 1750023, 14 pp.

[49] M. A. Ragusa and A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.

[50] J. Simon, *Régularité de la solution d'une équation non linéaire dans* $\mathbb{R}^N$, Journées d'Analyse Non Linéaire **665** (1978), 205–227, (Proc. Conf. Besançon, 1977, Springer, Berlin).

[51] R. Stegliński, *Infinitely many solutions for double phase problem with unbounded potential in* $\mathbb{R}^N$, Nonlinear Anal. **214** (2022), Paper No. 112580.

[52] S. Zeng, Y. Bai, L. Gasiński, and P. Winkert, *Convergence analysis for double phase obstacle problems with multivalued convection term*, Adv. Nonlinear Anal. **10** (2021), no. 1, 659–672.

[53] S. Zeng, Y. Bai, L. Gasiński, and P. Winkert, *Existence results for double phase implicit obstacle problems involving multivalued operators*, Calc. Var. Partial Differ. Equ. **59** (2020), no. 5, 176, 18pp.

[54] S. Zeng, L. Gasiński, P. Winkert, and Y. Bai, *Existence of solutions for double phase obstacle problems with multivalued convection term*, J. Math. Anal. Appl. **501** (2021), no. 1, 123997.

[55] S. Zeng and N. S. Papageorgiou, *Positive solutions for* (p, q)*-equations with convection and a sign-changing reaction*, Adv. Nonlinear Anal. **11** (2022), no. 1, 40–57.

[56] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.

[57] V. V. Zhikov, *On Lavrentiev's phenomenon*, Russian J. Math. Phys. **3** (1995), no. 2, 249–269.

[58] V. V. Zhikov, *On variational problems and nonlinear elliptic equations with nonstandard growth conditions*, J. Math. Sci. **173** (2011), no. 5, 463–570.