Variational formulas for immersions into 3-manifolds

vorgelegt von
Diplom-Mathematiker
David Chubelaschwili
Berlin

Von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation

Promotionsausschuss:

Vorsitzender:	Prof. Dr. Martin Henk
Berichter/Gutachter:	Prof. Dr. Ulrich Pinkall
Berichter/Gutachter:	Prof. Dr. Boris Springborn
Berichter/Gutachter:	Prof. Dr. Christoph Bohle

Tag der wissenschaftlichen Aussprache: 22.01.2016

Berlin 2016

Zusammenfassung

In dieser Arbeit führen wir die S-Theorie ein und wenden diese auf Variationsprobleme für immersierte Riemann Flächen, Elastizitätstheorie und die Spin Theorie an.

Aus der klassischen Mechanik ist bekannt, dass durch Symmetriebetrachtungen geometrische Probleme wesentlich vereinfacht werden. Das Noether-Theorem erklärt dabei, wie man Erhaltungsgrößen aus der invarianten Eigenschaft der betrachteten Energie finden kann. Die zu anfangs bestehende Motivation war es, den Raum der Immersionen

$$\mathcal{M} = \{f\colon M \to \mathbb{R}^3\},$$

für eine orientierte Riemannsche Mannigfaltigkeit M der Dimension kleiner oder gleich drei, eine euklidisch invariante Beschreibung zu verleihen um schliesslich aus geometrische Energien, welche unabhängig von Euklidischen Transformationen sind, automatisch Erhaltungssätze zu den jeweiligen Variationsproblemen herleiten zu können. Im Falle von immersierten Kurven ist bekannt, dass diese durch ihre Krümmungsfunktion $(\kappa_1, \kappa_2, \tau)$ bis auf Euklidische Transformationen eindeutig bestimmt sind. Im Falle von immersierten Flächen und immersierten 3-Mannigfaltigkeiten ist der Sachverhalt komplizierter und führt zu einer nicht konformen Deformationstheorie (S-Theorie).

Die S-Theorie bildet die Grundlage für die folgenden Kapitel dieser Arbeit. Ausgehend von einer Riemannschen Mannigfaltigkeit M und einer Referenzmetrik $\langle\,,\rangle$ lässt sich jede weitere Riemansche Metrik g durch einen positiv definiten Operators S mittels

$$g = \langle S, S\rangle$$

modellieren. Der Operator S ist somit eine Isometrie, d.h. $g = S^*\langle\,,\rangle$. Durch diese Sichtweise lässt sich der Levi-Civita Zusammenhang bezüglich g aus dem Levi-Civita Zusammenhang bezüglich der Referenzmetrik und dem Operator S berechnen.

Weiter werden Spin Bündel über zwei und drei dimensionalen orientierten Riemannschen Mannigfaltigkeiten eingeführt. Dabei haben Spin Bündel, anders als in der Literatur, zusätzlich eine quaternionische Struktur. Viele Formeln werden dadurch übersichtlicher und zugänglicher. Es wird gezeigt, dass jedes Spin Bündel Σ über $(M, \langle\,,\rangle)$ einen eindeutigen Spin Zusammenhang hat und berechnen mittels der S-Theorie den deformierten Spin Zusammenhang bezüglich der Metrik g. Schliesslich betrachten wir das von einer Immersion $f\colon M \to \mathbb{R}^3$ induzierte Spin Bündel und berechnen den dazugehörigen Spin Zusammenhang und finden eine neue Interpretation der Gauss-Codazzi-Gleichung. Wir führen den Dirac Operator ein und berechnen wie sich dieser bezüglich der Metrik g deformiert.

Nun sind wir in der Lage eine geometrische Beschreibung des Raumes $\mathcal{M} = \{f\colon M \to \mathbb{R}^3\}$ der immersierten Flächen im Euklidischen Raum zu geben. Wir berechnen dessen normalen Raum und formulieren das Noether-Theorem. Als Anwendung werden Erhaltungsgrößen des Willmore-Funktionals berechnet.

Schließlich wird eine intrinsische Version der distance-squared Energie auf einer n-dimensionalen Riemanschen Mannigfaltigkeit beschrieben. Dann wird der Spannungstensor eingeführt. Dessen Geschlossenheit liefert eine Charakterisierung für die kritischen Punkte der Energie. Dabei hat die S-Theorie eine fundamentale Bedeutung für das Verständnis des Spannungstensors. Im Falle von Riemannschen Flächen erhalten wir aus der S-Theorie eine Rotationsform, welche für kritischen Punkte der distance-squared Energie, harmonisch ist.

Abstract

In this thesis we introduce the S-theory. We apply the S-theory to variational problems of immersed Riemann surfaces, Elasticity theory and Spin theory.

From classical mechanics it is known that geometric problems can be substantially simplified by symmetry considerations. The Noether theorem describes in this case how to find conservation laws from the invariance property of the observed energy. At first, the motivation was to give the space of immersions

$$\mathcal{M} = \{f\colon M \to \mathbb{R}^3\},$$

for an oriented Riemannian manifold M of dimension less than or equal to three, an Euclidean invariant description. For geometric energies, i.e. energies which are invariant under Euclidean transformations, one would be able to derive conservation laws for the respective variational problems. It is a well known fact that immersed space curves are uniquely determined by their curvature functions $(\kappa_1, \kappa_2, \tau)$ up to Euclidean transformations. In the case of immersed surfaces and immersed 3-manifolds the situation becomes more complicated and leads to a non conformal deformation theory (S-Theorie).

The S-Theory builds the foundations of this thesis. Starting from a Riemannian manifold M and a reference metric $\langle\, , \rangle$ one can modell any other Riemmanian metric g through a positive definite and self adjoint operator S via

$$g = \langle S, S\rangle.$$

The operator S is an isometry, i.e. $g = S^*\langle\, , \rangle$. From this point of view we compute the Levi-Civita connection with respect to g out of Levi-Civita connection of the reference metric and the operator S.

Further we introduce Spin bundles over two and three dimensional oriented Riemannian manifolds. Thereby Spin bundles, in contrast to most of the literature, have additionally a quaternionic structure. Many formulas become clearer and more accessible. We show that any Spin bundle Σ over $(M, \langle\, , \rangle)$ has a unique Spin connection and compute via the S-Theory the deformed Spin connection with respect to the metric g. Eventually we consider the induced Spin bundle of an immersion $f\colon M \to \mathbb{R}^3$ and compute the corresponding Spin connection and find a new interpretation of the Gauss-Codazzi equation. We introduce the Dirac operator and compute its deformation with respect to the metric g.

We are now able to give the space of immersed surfaces $\mathcal{M} = \{f\colon M \to \mathbb{R}^3\}$ the desired geometric description. We compute the normal space of $\mathcal{M}$ and formulate the Noether theorem. As an application we compute conservation laws for the Willmore functional.

Finally we introduce the intrinsic version of the famous distance-squared energy on a n-dimensional Riemannian manifold. We introduce the corresponding stress tensor and show that it's closeness is a characterization for critical points of the energy. Thereby the S-theory provides a fundamental concept for the understanding of the stress tensor. In the case of a Riemann surface, the S-theory provides a canonical harmonic rotation 1-form for critical points of the distance-squared energy.

Contents

Introduction

Elasticity. A fundamental problem in nonlinear elasticity was to understand the relation between the three-dimensional theory and theories of lower dimensional objects (plates, shells, rods, ...). In 2002 Georo Frieseke, Richard D. James and Stefan Müller puplished a striking paper [**8**] where they showed that several lower dimensional theories, like the famous Kármán plate equation, are obtained from the Γ-Limit of the distance-squared Energy

$$E(f) := \frac{1}{2}\int_M \mathrm{dist}(df, \mathrm{SO}(3))^2$$

of immersed maps $f\colon M \to \mathbb{R}^3$ for a three dimensional Riemannian manifold M. Currently Jonas Tervooren and Ulrich Pinkall are writing a paper about elastic deformations of regions in $\mathbb{R}^2$. They found that critical points of the distance-squared energy admit a Weierstrass representation in terms of holomorphic functions. In the fourth chapter we introduce the intrinsic distance-squared energy,

$$E(S) = \frac{1}{2}\int_M \parallel S - I \parallel^2$$

for a given n-dimensional Riemannian manifold $(M, \langle\,,\rangle)$ and a positive definite and self adjoint operator S. Thereby $E(S)$ measures the amount of elastic energy to deform the given metric $\langle\,,\rangle$ to the metric $g := \langle S, S\rangle$. In the case of Riemann surfaces we found that critical points of the above elastic energy are characterized by the following

Theorem 1. *S is a critical point of E if one of the following equivalent conditions are satisfied:*

(1) $(\mathrm{tr}(S) - 1)\tilde{J}S^{-1}$ *is holomorphic.*

(2) The stress tensor $\sigma = (\mathrm{tr}(S) - 1)\tilde{J}S^{-1} - \tilde{J}$ *is closed.*

For critical points S of E, $\hat{\nabla}S^{-1} = -\eta\tilde{J}S^{-1}$, implies η to be co-closed. In particular, if M is embedded in $\mathbb{R}^2$, then η is harmonic.

S-theory. The fundamental concept for deriving such a theorem was the S-theory, which we develop in the first chapter and will be extremely useful for the whole thesis. Starting from a Riemannian manifold M and a reference metric $\langle\,,\rangle$ one can modell any other Riemmanian metric g through a positive definite and self adjoint operator S via

$$g = \langle S, S\rangle.$$

From this point of view we compute the Levi-Civita connection with respect to g out of the Levi-Civita connection of the reference metric and the operator S. The resulting formula for the Levi-Civita connection

$$\tilde{\nabla} := \nabla + S^{-1}(\nabla S) + S\hat{\nabla}S^{-1},$$

is the first step for a non-conformal deformation theory.

The space of Immersions. In the third chapter we are seeking an Euclidean invariant description of the space of immersions

$$\mathcal{M} = \{f\colon M \to \mathbb{R}^3\},$$

for a Riemann surface M. Starting with an isometric immersion $f\colon M \to \mathbb{R}^3$ any other immersion $\tilde{f}\colon M \to \mathbb{R}^3$ can be expressed by

$$d\tilde{f} = \bar{\lambda}(df \circ S)\lambda$$

for a quaternion-valued function $\lambda\colon M \to \mathbb{H}$ with $|\lambda| = 1$ and a positive definite operator $S \in \Gamma(\mathrm{End}(TM)$. This leads to a generalization of Spin transformations **[12]**, **[5]** to non conformal deformations. We derive the corresponding Dirac equation for λ as an integrability condition. We are now able to give the space of immersed surfaces $\mathcal{M} = \{f\colon M \to \mathbb{R}^3\}$ the desired geometric description. It is given by

$$\mathcal{M} := \{\begin{pmatrix} \lambda \\ \rho \\ \tau \end{pmatrix} \in H \mid -\mathrm{Re}((df + \tau) \wedge d\lambda\lambda^{-1}) = \rho|df|^2 \quad \text{and} \quad \bar{\lambda}(df + \tau)\lambda \quad \text{is exact}\}.$$

We compute for any geometric functional

$$\mathcal{F}\colon \mathcal{M} \to \mathbb{R}$$

a conservation law, in particular we derive conservation laws for Willmore surfaces, Minimal surfaces and Constant Mean curvature surfaces.

Spin theory. In the second chapter we discuss the theory of Spin bundles, Spin connections and Dirac operators. The Spin theory, which is necessary for understanding the space of immersed surfaces with all its applications towards variation problems, deserves an interest for its own. Spin theory is a well studied topic in differential geometry and physics. For Riemann surfaces, Spin theory was successfully used in **[6]**, **[1]**, **[13]**, **[10]** for the understanding of global surface theory. There are different approaches of dealing with spin manifolds. Thomas Friedrich **[7]** starts out with the spin cover of $\mathrm{SO}(n)$, connections on principal bundles and associated vector bundles. In contrast, Nicolas Ginoux **[9]** does not apply the principal bundle theory extensively, but his definitions and formulas are formulated in coordinates, which often hides the geometric meaning. Fortunately, in dimensions two and three one circumvents this difficulty by using right from the start that the spin representations of the Clifford algebra are quaternionic in these dimensions. We pay a special attention to the construction of the unique Spin connection (a prerequisite to defining the Dirac operator) for three dimensional oriented Riemannian manifolds. Via the S-theory we compute how the Spin connection transforms under the change of a Riemannian metric $g = \langle S, S\rangle$. Eventually we consider the induced Spin bundle of an immersion $f\colon M \to \mathbb{R}^3$ and compute the corresponding Spin connection and derive integrability conditions for λ and S for finding new immersions. In the special case of a conformal metric deformation ($g = e^{2u}\langle,\rangle$) one obtains

$$d\lambda = -\frac{1}{2}Gdf\lambda,$$

with $G := df(\mathrm{grad}\, u)$ as the integrability condition for the existence of a Spin transformation $\tilde{f}$ of f. In this case the famous Liouville theorem implies that $\tilde{f}$ must be a Möbius transformation of f. The above equation appeared also recently in a paper **[3]**, where the authors found a variational approach to construct deformations which are nearly conformal in a suitable sense. They were seeking for minimizers of

$$E(u, \lambda) := \int_M |d\lambda + \frac{1}{2}Gdf|^2 \quad \text{s.t} \quad |\lambda| = 1.$$

In the language of Spin bundles $d\lambda + \frac{1}{2}Gdf$ turns out to be the deformed Spin connection. In the case of Riemann surfaces the Spin theory yields a new interpretation of the Gauss-Codazzi equation. We prove

Theorem 2. *Let $f\colon M \to \mathbb{R}^3$ be an immersion. We prescribe a new metric $g = \langle S, S\rangle$, a map $\lambda\colon M \to S^3$ and $\tilde{A} \in \Gamma(\mathrm{End}(TM))$. If there exists a $\tilde{f}$ with $d\tilde{f} = \bar{\lambda}df(S)\lambda$ with its shape operator $\tilde{A}$ then*

$$d\lambda\lambda^{-1} = -\frac{1}{2}Ndf(S\tilde{A} - A) + \frac{1}{2}\eta N.$$

Open problem. The above formula is interesting for Discrete Differential Geometry because it can be discretized in a canonical way by taking the piecewise linear limit. A further problem in

Discrete Differential Geometry is to find a notation of discrete holomorphicity that can be derived from a variational problem. In [**2**] the above mentioned distance-squared energy was discretized. For a three dimensional simplicial complex embedded in $\mathbb{R}^3$ via a map $f\colon M \to \mathbb{R}^3$ one can find for any another immersion $\tilde{f}\colon M \to \mathbb{R}^3$ on each tetrahedron $t \in M$ a unique positive definite self adjoint operator S_t and a unique rotation matrix $R_t \in \mathrm{SO}(3)$ such that on the interior of t we have

$$d\tilde{f} = R_t \circ df \circ S_t.$$

Now one can measure the deviation of $\tilde{f}$ from being an isometry (its elastic energy) by the integral

$$\begin{aligned}\int_M |d\tilde{f} - R \circ df|^2 &= \int_M |S - I|^2 \\ &= \sum_t |S_t - I|^2 \mathrm{vol}(t).\end{aligned}$$

In the smooth theory the elastic deformations of regions in $\mathbb{R}^2$ admit a representation of holomorphic functions. We expect that the discretization of two-dimensional elasticity will yield another approach to discrete holomorphicity.

Acknowledgments

I would like to thank my academic advisor, Ulrich Pinkall, for being a great teacher, inspiring personality and giving me all the space and time for writing this thesis. Many of his ideas are written down here. I would like to thank Franz Pedit for helping me during the time i was in Tübingen. I would like to thank my friend Felix Knöppel. He helped me to compute the normal space of immersions and he really loves mathematics. I would like thank Wayne Lam for bringing his creativity and self irony to our group. He made me laugh many times during our daily discussions. I would like to thank Christian Reiher for being my best friend and flatmate. He made me cry many times during our daily living. I would like to thank Boris Springborn and Christoph Bohle for taking their time to support this thesis. I am also indebted to my family, you are simply the best.

1 S-Theory

Let $(M, \langle\,,\rangle)$ be a Riemannian manifold. Any other metric g on M can be expressed with

$$g(X,Y) = \langle BX, Y\rangle,$$

where B is a positive definite and self adjoint operator with respect to $\langle\,,\rangle$. It is a well known fact that B has a unique square root S, i.e.

$$S^2 = B$$

and S is positive definite and self adjoint.

Let $\langle\,,\rangle$ and $g := \langle B, .\rangle = \langle S, S\rangle$ two Riemannian metrics on M. Let $(\nabla, \langle\,,\rangle)$ and $(\tilde{\nabla}, g)$ be the corresponding Levi-Civita connections. For any

$$A \in \Gamma(\mathrm{Hom}((TM, \langle\,,\rangle), (TM, g)))$$

one can define a connection

$$\hat{\nabla}\colon \Gamma(TM) \times \Gamma(\mathrm{Hom}((TM, \langle\,,\rangle), (TM, g))) \to \Gamma(\mathrm{Hom}((TM, \langle\,,\rangle), (TM, g)))$$

$$(\hat{\nabla}_X A)Y := \tilde{\nabla}_X AY - A\nabla_X Y = (\tilde{\nabla}_X A)Y + A(\tilde{\nabla}_X Y - \nabla_X Y).$$

Let

$$R \in \Gamma(\mathrm{Iso}((TM, \langle\,,\rangle), (TM, g))),$$

i.e.

$$g(RX, RY) = \langle X, Y\rangle$$

for any $X, Y \in \Gamma(TM)$. Note that

$$S^{-1} \in \Gamma(\mathrm{Iso}((TM, \langle\,,\rangle), (TM, g))), \tag{0.1}$$

since $g(S^{-1}X, S^{-1}Y) = \langle SS^{-1}X, SS^{-1}Y\rangle = \langle X, Y\rangle$.

Lemma 1. *Let $R \in \Gamma(\mathrm{Iso}((TM, \langle\,,\rangle), (TM, g)))$ then $\hat{\nabla}R = \Omega R$, for some $\Omega \in \Omega^1(M, \mathfrak{so}(TM, g))$. For $\dim M = 2$ we obtain $\hat{\nabla}R = \eta\tilde{J}R$ for some $\eta \in \Omega^1(M)$.*

Proof.

$$\begin{aligned} g((\hat{\nabla}_X R)Y, RY) &= g(\tilde{\nabla}_X RY - R\nabla_X Y, RY) \\ &= \frac{1}{2}Xg(RY, RY) - \frac{1}{2}X\langle Y, Y\rangle \\ &= \frac{1}{2}X\langle RY, RY\rangle - \frac{1}{2}X\langle Y, Y\rangle \\ &= 0. \end{aligned}$$

□

Corollary 1.

$$S\hat{\nabla}S^{-1} \in \Omega^1(M, \mathfrak{so}(TM, \langle\,,\rangle)). \tag{0.2}$$

Proof. From the previous lemma there must exist a $\Omega \in \Omega^1(M, \mathfrak{so}(TM, g))$ with $\hat{\nabla} S^{-1} = \Omega S^{-1}$, hence $S\hat{\nabla} S^{-1} = S\Omega S^{-1} \in \Omega^1(M, \mathfrak{so}(TM, \langle\, , \rangle))$. $\square$

Note that we obtain

$$\nabla_X Y = \tilde{\nabla}_X Y + S(\tilde{\nabla}_X S^{-1})Y - S(\hat{\nabla}_X S^{-1})Y.$$

Analogously one could view $S \in \Gamma(\mathrm{Hom}((TM, g), (TM, \langle\, , \rangle)))$ as an isometry. With

$$(\tilde{\hat{\nabla}}_X S)Y := \nabla_X SY - S\tilde{\nabla}_X Y$$

one obtains

$$S^{-1}\tilde{\hat{\nabla}} S \in \Omega^1(M, \mathfrak{so}(TM, g))$$

and

$$\tilde{\nabla}_X Y = \nabla_X Y + S^{-1}(\nabla_X S)Y - S^{-1}(\tilde{\hat{\nabla}}_X S)Y. \tag{0.3}$$

We have computed the difference of Levi-Civta connections for two Riemannian metrics $\langle\, , \rangle$ and $g = \langle S, S\rangle$. The following theorem examines the question of how to construct the Levi-Civita connection $\tilde{\nabla}$ of a metric $g = \langle S, S\rangle$, while prescribing a positive and self adjont Operator $S \in \Gamma(\mathrm{End}(TM))$.

Theorem 3. *Let $(M, \langle\, , \rangle, \nabla)$ a Riemannian manifold. We prescribe a positive definite and self adjoint operator S and the Riemannian metric $g := \langle S, S\rangle$. Let $\Omega \in \Omega^1(M, \mathfrak{so}(TM, g))$ the unique one form which satisfies*

$$\Omega \wedge I = S^{-1} d^{\nabla} S, \tag{0.4}$$

then the Levi-Civita connection with respect to g is given by

$$\tilde{\nabla} := \nabla + S^{-1}(\nabla S) - \Omega. \tag{0.5}$$

Proof. It is a well known fact that there exist a Levi-Civita connection $\tilde{\nabla}$ of (M, g) and we already know (0.3) that it is of the form

$$\tilde{\nabla} := \nabla + S^{-1}(\nabla S) - \Omega,$$

Let us first check that if $\Omega^1(M, \mathfrak{so}(TM, g))$ then $\tilde{\nabla}$ is compatible with the metric g. Let $X, Y \in \Gamma(TM)$

$$\begin{aligned}
Xg(Y, Y) &= X\langle SY, SY\rangle \\
&= 2\langle \nabla_X SY, SY\rangle \\
&= 2\langle (\nabla_X S)Y + S\nabla_X Y, SY\rangle \\
&= 2g(\nabla_X Y + S^{-1}(\nabla_X S)Y, Y) \\
&= 2g(\nabla_X Y + S^{-1}(\nabla_X S)Y - \Omega(X)Y, Y) \\
&= 2g(\tilde{\nabla}_X Y, Y).
\end{aligned}$$

Further $\tilde{\nabla}$ is torsion free if and only if

$$\begin{aligned}
0 &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\
&= S^{-1}(\nabla_X S)Y - \Omega(X)Y - (S^{-1}(\nabla_Y S)X - \Omega(Y)X) \\
&= S^{-1} d^{\nabla} S(X, Y) - \Omega \wedge I(X, Y).
\end{aligned}$$

This shows that Ω must satisfy

$$\Omega \wedge I = S^{-1} d^{\nabla} S.$$

Since there exist a unique Levi-Civita connection of (M, g) there must exist a unique $\Omega \in \Omega^1(M, \mathfrak{so}(TM, g))$ which solves $\Omega \wedge I = S^{-1} d^{\nabla} S$. $\square$

Knowing ∇ and $\tilde{\nabla}$, then Ω is explicitly given by

$$\Omega = S^{-1}\tilde{\tilde{\nabla}}S.$$

Furthermore,

Corollary 2.

$$\hat{\nabla}S^{-1} = -\Omega S^{-1}. \tag{0.6}$$

Proof.

$$\begin{aligned} S\Omega(X)S^{-1}Y &= S(S^{-1}\tilde{\tilde{\nabla}}_X S)S^{-1}Y \\ &= \nabla_X SS^{-1}Y - S\tilde{\nabla}_X S^{-1}Y \\ &= -S(\tilde{\nabla}_X S^{-1}Y - S^{-1}\nabla_X Y) \\ &= -S(\hat{\nabla}_X S^{-1})Y. \end{aligned}$$

□

Let $(M, \langle\, , \rangle, J)$ be a Riemann surface. By changing the Riemannian metric $g = \langle S, S\rangle$ we obtain a change of the complex structure by

$$\tilde{J} = S^{-1}JS.$$

Corollary 3. *Let $(M, \langle\, , \rangle, J)$ a Riemann surface with the Levi-Civita connection ∇. Then the Levi-Civita connection of (M, g) is given by*

$$\tilde{\nabla}_X Y = \nabla_X Y + S^{-1}(\nabla_X S)Y - \eta(X)\tilde{J}Y, \tag{0.7}$$

where $\eta(X) := \langle JS^{-1}JZ, X\rangle$ and $Z \in \Gamma(TM)$ is the unique vector field defined through $Z\det := d^\nabla S$.

In particular, if $g = e^{2u}\langle\, , \rangle$, then

$$\tilde{\nabla}_X Y = \nabla_X Y + du(X)Y - {*du}(X)JY.$$

Proof. Let $\tilde{\nabla} = \nabla + S^{-1}\nabla S - \Omega$ with $\Omega = \eta\tilde{J}$, $\eta \in \Omega^1(M)$ and $\Omega \wedge I = S^{-1}d^\nabla S$ (0.4) , be the Levi-Civita connection of (M, g). Let (X, JX) an orthonormal basis of eigenvectors of S.

$$\begin{aligned} Z &= S(\Omega(X)JX - \Omega(JX)X) \\ &= \eta(X)S\tilde{J}JX - \eta(JX)S\tilde{J}X \\ &= \eta(X)JSJX - \eta(JX)JSX \\ &= -\eta(X)\langle SJX, JX\rangle X - \eta(JX)\langle SX, X\rangle JX, \end{aligned}$$

therefore

$$\eta(X) = \frac{\langle Z, X\rangle}{\langle SJX, JX\rangle} = \langle JS^{-1}JZ, X\rangle.$$

□

Next we will examine the change of the curvature two form on a Riemann surface M. Let $Y \in \Gamma(TM)$ be direction field, i.e. $\langle Y, Y\rangle = 1$. At least locally such a vector field always exists. The rotation 1-form is

$$\rho(X) = \langle\nabla_X Y, JY\rangle. \tag{0.8}$$

It is a well known fact that

$$d\rho = K\det \tag{0.9}$$

is the curvature 2-form, whereby K denotes the Gaussian curvature. Next we compute the rotation 1-form $\tilde{\rho}$ with respect to g.

Theorem 4. *The rotation 1-form $\tilde{\rho}$ with respect to g is given by*

$$\tilde{\rho} = \rho - \eta. \tag{0.10}$$

In particular one gets $\tilde{\rho} = \rho - *du$ for a conformal change of the metric.

Proof. Let $Y \in \Gamma(TM)$ be a direction field with respect to $\langle\,,\rangle$, then $S^{-1}Y$ is a direction filed with respect to g.

$$\begin{aligned}\tilde{\rho}(X) &= g(\tilde{\nabla}_X S^{-1}Y, \tilde{J}S^{-1}Y)\\ &= \langle S\tilde{\nabla}_X S^{-1}Y, S\tilde{J}S^{-1}Y\rangle\\ &= \langle S\tilde{\nabla}_X S^{-1}Y, JY\rangle.\end{aligned}$$

Applying (0.6) we obtain,

$$-\eta(X)\tilde{J}S^{-1}Y = (\hat{\nabla}_X S^{-1})Y = \tilde{\nabla}_X S^{-1}Y - S^{-1}\nabla_X Y,$$

and therefore

$$S\tilde{\nabla}_X S^{-1}Y = \nabla_X Y - \eta(X)S\tilde{J}S^{-1}Y = \nabla_X Y - \eta(X)JY.$$

This proves the claim. □

Corollary 4. *Let M be an embedded Riemann surface in $\mathbb{R}^2$ then η is closed.*

Proof. Both rotation 1-forms ρ and $\tilde{\rho}$ are closed and therefore η must be closed as well. □

Now we consider the simplest 3 dimensional manifold which is made of a Riemann surface M, namely

$$\tilde{M} := M \times \mathbb{R}$$

with the product metric

$$\langle (X,a),(Y,b)\rangle := \langle X,Y\rangle + ab$$

on $T\tilde{M} = M \times \mathbb{R}$. Now let ∇ be the Levi-Civita connection on M, then the corresponding Levi-Civita connection ∇ of $\tilde{M}$ is given by

$$\nabla_{(X,a)}(Y,b) = (\nabla_X Y, ab').$$

Changing the metric on M by $g = \langle S, S\rangle$ yields a change of the corresponding product metric on $\tilde{M}$. With

$$\tilde{S} := \begin{pmatrix} S & 0\\ 0 & 1\end{pmatrix}$$

the product metric changes with

$$g = \langle \tilde{S}, \tilde{S}\rangle.$$

Applying (0.7), the Levi-Civita connection of $(\tilde{M}, g)$ is given by

$$\tilde{\nabla}_{(X,a)}(Y,b) = (\nabla_X Y + S^{-1}(\nabla_X S)Y - \eta(X)\tilde{J}Y, ab'). \tag{0.11}$$

Alternatively, we can apply (0.3) and get

$$\begin{aligned}\tilde{\nabla}_{(X,a)}(Y,b) &= \nabla_{(X,a)}(Y,b) + \tilde{S}^{-1}(\nabla_{(X,a)}\tilde{S})\begin{pmatrix} Y\\ b\end{pmatrix} - \Omega(\begin{pmatrix} X\\ a\end{pmatrix})\begin{pmatrix} Y\\ b\end{pmatrix}\\ &= \left(\nabla_X Y + S^{-1}(\nabla_X S)Y, ab'\right) - \Omega(\begin{pmatrix} X\\ a\end{pmatrix})\begin{pmatrix} Y\\ b\end{pmatrix}.\end{aligned}$$

With

$$\omega(X) := \eta(X)\partial_t \tag{0.12}$$

one can easily show

$$\Omega(\begin{pmatrix} X\\ a\end{pmatrix})\begin{pmatrix} Y\\ b\end{pmatrix} = \omega(X)\tilde{\times}\begin{pmatrix} Y\\ b\end{pmatrix}, \tag{0.13}$$

where $\tilde{\times}$ is the cross product with respect to g, i.e.

$$X \tilde{\times} Y = S^{-1}(SX \times SY). \tag{0.14}$$

2 | Spin theory

1 Spin bundles

Let M be an oriented Riemannian manifold The Clifford multiplication for any $X, Y \in \Gamma(TM)$ is defined by

$$XY := -\langle X, Y\rangle + X \wedge Y.$$

Let $\mathrm{Cl}_{\langle\,,\rangle}(M)$ be the Clifford bundle over $(M, \langle\,,\rangle)$. The Clifford connection ∇^c on $\mathrm{Cl}_{\langle\,,\rangle}(M)$ is the unique connection, which extends the Levi Civita connection to $\mathrm{Cl}_{\langle\,,\rangle}(M)$ and satisfies the Leibnitz rule with respect to the Clifford multiplication, i.e. $\nabla^c XY = (\nabla X)Y + X\nabla Y$ for all $X, Y \in \Gamma(TM)$. A quaternionic line bundle over M is a real vector bundle Σ over M of rank 4 such that each fibre Σ_p has the structure of a 1-dimensional quaternionic vector space. We require that for a smooth section $\Psi \in \Gamma(\Sigma)$ and smooth function $\lambda\colon M \to \mathbb{H}$ also the section $\Psi\lambda$ is smooth. A hermitian quaternionic line bundle is a quaternionic line bundle Σ together with positive definite quaternionic hermitian forms

$$\langle\,,\rangle_\Sigma \colon \Sigma_p \times \Sigma_p \to \mathbb{H}$$

on each fibre. Quaternionic hermitian means

$$\langle \Psi\lambda, \phi\mu\rangle_\Sigma = \bar{\lambda}\langle \Psi, \phi\rangle_\Sigma \mu$$
$$\langle \Psi, \phi\rangle_\Sigma = \overline{\langle \Psi, \phi\rangle_\Sigma}$$

for all $\Psi, \phi \in \Sigma_p, \lambda, \mu \in \mathbb{H}$. Again we require that for smooth sections Ψ, ϕ of Σ also the function $\langle \Psi, \phi\rangle_\Sigma$ is smooth. Note that

(1) Every quaternionic line bundle can be made into a hermitian quaternionic line bundle (this is proved using a partition of unity).

(2) If $\langle\,,\rangle, \langle \tilde{,} \rangle$ are two positive hermitian forms on the same quaternionic line bundle Σ then there is a function $u \in C^\infty(M)$ such that $\langle \tilde{,} \rangle = e^u \langle\,,\rangle$.

Definition 1. *A Spin bundle Σ over $(M, \langle\,,\rangle)$ is a hermitian quaternionic line bundle such that there exists a non trivial Clifford representation $\widehat{\ }\colon \Gamma(\mathrm{Cl}_{\langle\,,\rangle}(M)) \to \Gamma(\mathrm{End}_{\mathbb{H}}(\Sigma))$ such that $\widehat{qp} = \widehat{q}\widehat{p}$ for all $q, p \in \mathrm{Cl}_{\langle\,,\rangle}(M)$. Further, a Spin connection ∇^Σ is a connection on Σ, such that*

(1) ∇^Σ satisfies the Leibniz rule with respect to the Clifford representation, i.e. for any $q \in \Gamma(\mathrm{Cl}_{\langle\,,\rangle}(M))$ and $\Psi \in \Gamma(\Sigma)$ one gets $\nabla^\Sigma_X \hat{q}\Psi = (\widehat{\nabla^c_X q})\Psi + \widehat{q}\nabla^\Sigma_X \Psi$.

(2) $\nabla^\Sigma_X(\Psi\lambda) = (\nabla^\Sigma_X \Psi)\lambda + \Psi d\lambda$, for any $\lambda \in C^\infty(M, \mathbb{H})$.

(3) $X\langle \Psi, \Phi\rangle_\Sigma = \langle \nabla^\Sigma_X \Psi, \Phi\rangle_\Sigma + \langle \Psi, \nabla^\Sigma_X \Phi\rangle_\Sigma$.

Further we require that $\langle,\rangle_\Sigma$ is compatible with the Clifford multiplication, i.e. for all $q \in \Gamma(\mathrm{Cl}_{\langle,\rangle}(M))$ and $\Psi \in \Gamma(\Sigma)$

$$|\hat{q}\Psi|_\Sigma = |q|_c |\Psi|_\Sigma.$$

Lemma 2. *Let $X \in \Gamma(TM)$ then*

$$\hat{X} \in \Gamma(\mathfrak{so}(\Sigma)).$$

PROOF. Let $\hat{X}\Psi = \Phi\mu$ for $\mu \in C^\infty(M, \mathbb{H})$ and therefore $-\langle X, X\rangle\Psi = (\hat{X}\Phi)\mu$.

$$\langle \hat{X}\Psi, \Phi\rangle_\Sigma = \langle \Phi\mu, \Phi\rangle_\Sigma = \bar{\mu}\langle \Phi, \Phi\rangle_\Sigma = \bar{\mu}|\Phi|^2_\Sigma,$$

$$\begin{aligned}
\langle \Psi, \hat{X}\Phi\rangle_\Sigma &= \langle -\frac{1}{\langle X, X\rangle}(\hat{X}\Phi)\mu, \hat{X}\Phi\rangle_\Sigma \\
&= -\frac{\bar{\mu}}{\langle X, X\rangle}\langle \hat{X}\Phi, \hat{X}\Phi\rangle_\Sigma \\
&= -\frac{\bar{\mu}}{\langle X, X\rangle}|X|^2_c|\Phi|^2_\Sigma = -\frac{\bar{\mu}}{\langle X, X\rangle}|X|^2|\Phi|^2_\Sigma \\
&= -\bar{\mu}|\Phi|^2_\Sigma.
\end{aligned}$$

This shows

$$\langle \hat{X}\Psi, \Phi\rangle_\Sigma = -\langle \Psi, \hat{X}\Phi\rangle_\Sigma.$$

□

2 Spin bundles over 3 dimensional oriented Riemannian manifolds

Since our definition of a spin bundle differs from those in the literature the existence of such a bundle is a priori not clear. It will turn out that only for 2 or 3 dimensional oriented manifolds M such a spin structure exists. The aim of this section is to prove the existence of a Spin bundle over a 3-dimensional oriented Riemannian manifold $(M, \langle\, , \rangle)$. Then we will prove that any such a Spin bundle has a unique Spin connection. But before showing the existence of a Spin bundle and its unique Spin connection we will briefly summarize basic facts about Clifford representations. Let $p \in M$ and X, Y, Z an orthonormal basis of T_pM. The Clifford multiplication for any two vectors $U, V \in T_pM$ is defined by

$$UV := -\langle U, V\rangle + U \wedge W.$$

Let

$$E := -XYZ \tag{2.1}$$

then $X^2 = Y^2 = Z^2 = -1$ and $E^2 = 1$. Further,

$$\begin{aligned}
EXY &= -XYZXY = +Z = X \times Y, \\
EXZ &= -XYZXZ = -Y = X \times Z, \\
EYZ &= -XYZYZ = +X = Y \times Z,
\end{aligned}$$

in particular for any $U, V \in T_pM$ we obtain

$$E(U \wedge V) = U \times V, \tag{2.2}$$

$$E(U \times V) = U \wedge V. \tag{2.3}$$

Now one can easily prove

Lemma 3. *E commutes with vectors and bivectors.*

Let

$$\mathcal{A}_p := \mathbb{R} \oplus ET_pM.$$

We claim that $\mathcal{A}_p$ is an Algebra isomorphic to $\mathbb{H}$. For any $a, b \in \mathbb{R}$ and $U, V \in T_pM$ we get

$$\begin{aligned}
(a + EU)(b + EV) = \\
&= ab + E(bU + aV) + UV \\
&= ab - \langle U, V\rangle + E(bU + aV) + U \wedge V \\
&= ab - \langle U, V\rangle + E(bU + aV + U \times V) \in \mathcal{A}_p.
\end{aligned}$$

The Clifford Algebra of T_pM is

$$\mathrm{Cl}_{\langle\, , \rangle}(T_pM) = \mathcal{A}_p \oplus E\mathcal{A}_p.$$

Note that E commutes with any element of the Clifford Algebra. Let $\mathcal{A} := M \times \mathbb{R} \oplus ETM$ then the Clifford bundle over M is

$$\mathrm{Cl}_{\langle\,,\rangle}(M) = \mathcal{A} \oplus E\mathcal{A}.$$

Lemma 4. *Let $q = g + EY + E(h + EZ) \in \Gamma(\mathrm{Cl}_{\langle\,,\rangle}(M))$, then the Clifford connection is given by*

$$\nabla^c q = dg + E\nabla Y + E(dh + E\nabla Z).$$

Proof. Since $E^2 = 1$ we obtain $0 = \nabla^c_X E^2 = N\nabla^c_X E + E\nabla^c_X E = 2E\nabla^c_X E$, therefor $0 = \nabla^c_X E$. $\square$

The next task will be to show that a representation is completely determined by its values on $\Gamma(\mathcal{A})$.

Lemma 5. *Any representation* $\widehat{\ }: \Gamma(\mathcal{A}) \to \Gamma(\mathrm{End}_{\mathbb{H}}(\Sigma))$ *is irreducible.*

Proof. Let $U \neq 0$ be a real subspace of Σ such that $\widehat{q}U \subset U$ for all $q \in \Gamma(\mathcal{A})$. We show $U = \Sigma$. Let X, Y, Z an orthonormal basis of T_pM and $\Psi \in U$ and $\Psi \neq 0$. We claim that $\Psi, \widehat{(EX)}\Psi, \widehat{(EY)}\Psi, \widehat{(EZ)}\Psi \in U$ are linearly independent. Note that $(\widehat{(EX)})^2 = (\widehat{(EY)})^2 = (\widehat{(EZ)})^2 = -I$ and note that these operators pairwise anti commute. For example we get $\widehat{(EX)}\widehat{(EY)} = \widehat{(EXEY)} = \widehat{(XY)} = \widehat{(X \wedge Y)} = -\widehat{(EY)}\widehat{(EX)}$. Since $\dim_{\mathbb{H}}(\Sigma) = 1$, we can find $a, b, c \in \mathbb{H}$ such that $\widehat{(EX)}\Psi = \Psi a, \widehat{(EY)}\Psi = \Psi b, \widehat{(EZ)}\Psi = \Psi c$, where a, b, c obey $a^2 = b^2 = c^2 = -1$ and pairwise anti commute. Now let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ with

$$\begin{aligned} 0 =& \lambda_1\Psi + \lambda_2\widehat{(EX)}\Psi + \lambda_3\widehat{(EY)}\Psi + \lambda_4\widehat{(EZ)}\Psi \\ =& \lambda_1\Psi + \lambda_2\Psi a + \lambda_3\Psi b + \lambda_4\Psi c \\ =& \Psi(\lambda_1 + \lambda_2 a + \lambda_3 b + \lambda_4 c). \end{aligned}$$

This shows $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Therefore $\dim_{\mathbb{R}} U = \dim_{\mathbb{R}}\Sigma$ and we conclude $U = \Sigma$. $\square$

Lemma 6. *Let* $\widehat{\ }: \Gamma(Cl_{\langle\,,\rangle}(M)) \to \Gamma(\mathrm{End}_{\mathbb{H}}(\Sigma))$ *be a representation, then* $\widehat{E} = \pm I$.

Proof. Let $\widehat{\ }: \Gamma(\mathcal{A}) \to \Gamma(\mathrm{End}_{\mathbb{H}}(\Sigma))$ be the induced representation of $\widehat{\ }$. By the previous lemma this representation is irreducible. Note that $E \notin \mathcal{A}$. Any $\Psi \in \Gamma(\Sigma)$ can be decomposed by

$$\Psi = \frac{1}{2}(\Psi + \widehat{E}\Psi) + \frac{1}{2}(\Psi - \widehat{E}\Psi),$$

thus

$$\Sigma = \Sigma_+ \oplus \Sigma_-,$$

where $\widehat{E}\Psi = \pm\Psi$ for $\Psi \in \Gamma(\Sigma_\pm)$. We claim that Σ_+, Σ_- are invariant subspaces of Σ with respect to the induced representation. Let $q \in \Gamma(\mathcal{A})$ and $\Psi \in \Gamma(\Sigma_\pm)$ then

$$\widehat{E}\widehat{q}\,\Psi = \widehat{E}\widehat{q}\,\Psi = \widehat{Eq}\Psi = \widehat{qE}\Psi = \widehat{q}\widehat{E}\Psi = \pm\widehat{q}\,\Psi,$$

therefore $\widehat{q}\,\Psi \in \Gamma(\Sigma_\pm)$. Since the representation is irreducible we conclude that either $\Sigma = \Sigma_+$ or $\Sigma = \Sigma_-$, which is equivalent to $\widehat{E} = \pm I$. $\square$

Now one can easily prove the following

Theorem 5. Proof. Let $\widehat{\ }$ be a representation for the Spin bundle $(\Sigma, M, \langle\,,\rangle)$, then for any $P, Q \in \Gamma(\mathcal{A})$ one gets $\widehat{P + EQ} = \widehat{P} \pm \widehat{Q}$. Conversely, any representation $\widehat{\ }: \Gamma(\mathcal{A}) \to \Gamma(\mathrm{End}_{\mathbb{H}}(\Sigma))$ can be extended through, $\widehat{P + EQ} := \widehat{P} \pm \widehat{Q}$, to a representation of the Spin bundle Σ.

In particular one can always assume that $\hat{E} = I$, otherwise we define $\widehat{P + EQ} := \widehat{P} - \widehat{Q}$. $\square$

Consequently we obtain

$$\begin{aligned}\widehat{XY} &= -\langle X, Y\rangle + \widehat{X \wedge Y} \\ &= -\langle X, Y\rangle + \widehat{E}\widehat{X \wedge Y} \\ &= -\langle X, Y\rangle + (\widehat{EX \wedge Y}) \\ &= -\langle X, Y\rangle + \widehat{X \times Y}.\end{aligned}$$

Lemma 7. *Let Σ be a Spin bundle over M, $p \in M$ and X, Y, Z a positively oriented orthonormal basis of T_pM. Then there is a $\phi \in \Sigma_p$ such that $|\phi| = 1$ and*

$$\begin{aligned}\hat{X}\phi &= \phi\mathbf{i} \\ \hat{Y}\phi &= \phi\mathbf{j} \\ \hat{Z}\phi &= \phi\mathbf{k}\end{aligned}$$

ϕ is unique up to sign.

Proof. We think of Σ_p as a complex vector space where the complex structure is given as right multiplication by i. $\hat{X}$ is then a complex linear endomorphism of Σ_p with $\hat{X}^2 = -I$. Therefore at least one among the two numbers $\pm\mathbf{i}$ is an eigenvalue of $\hat{X}$. If $\hat{X}\Psi = \Psi(-\mathbf{i})$ then

$$\hat{X}(\Psi\mathbf{j}) = (\hat{X}\Psi)\mathbf{j} = -(\Psi\mathbf{i})\mathbf{j} = (\Psi\mathbf{j})\mathbf{i}.$$

Therefore in any case $\mathbf{i}$ is an eigenvalue and there exist a Ψ with $|\Psi| = 1$ and $\hat{X}\Psi = \Psi\mathbf{i}$. The same will then be true for

$$\begin{aligned}\phi &= \cos\alpha\Psi + \sin\alpha\Psi\mathbf{i} \\ &= \Psi(\cos\alpha + \sin\alpha\mathbf{i})\end{aligned}$$

where $\alpha \in \mathbb{R}$ is arbitrary. Define $\lambda \in \mathbb{H}$ by $\hat{Y}\Phi = \Phi\lambda$. Then

$$-\Psi = \hat{Y}^2\Psi = \Psi\lambda^2$$

implies $|\lambda| = 1$. Furthermore,

$$0 = (\hat{X}\hat{Y} - \hat{Y}\hat{X})\Psi = \Psi(\mathbf{i}\lambda + \lambda\mathbf{i}).$$

Therefore λ anti commutes with $\mathbf{i}$ and therefore there is $\beta \in \mathbb{R}$ such that

$$\lambda = \mathbf{j}(\cos\beta - \sin\beta\mathbf{i}).$$

Now

$$\begin{aligned}\hat{Y}\phi &= \hat{Y}\Psi(\cos\alpha + \sin\alpha\mathbf{i}) \\ \hat{Y}\phi &= \Psi\lambda(\cos\alpha + \sin\alpha\mathbf{i}) \\ &= \Psi\mathbf{j}(\cos\beta - \sin\beta\mathbf{i})(\cos\alpha + \sin\alpha\mathbf{i}) \\ &= \phi(\cos\alpha - \sin\alpha\mathbf{i})\mathbf{j}(\cos\beta - \sin\beta\mathbf{i})(\cos\alpha + \sin\alpha\mathbf{i}) \\ &= \phi\mathbf{j}(\cos(2\alpha - \beta) + \sin(2\alpha - \beta)\mathbf{i}).\end{aligned}$$

This shows that up to sign there is a unique $\phi \in \Sigma_p$ with $|\phi| = 1$ for which the first two equations in the statement of the lemma hold. The third equation then follows automatically. $\square$

Now we will prove the existence of a Spin bundle over M.

Theorem 6. *Every orientable 3-manifold has a Spin bundle Σ. If M is simply connected and $\Sigma, \tilde{\Sigma}$ are two Spin bundles over M then there is a bundle isomorphism $L\colon \Sigma \to \tilde{\Sigma}$ that is quaternionic linear, unitary and compatible with the Clifford multiplication. L is unique up to replacing L by $-L$.*

Proof. Since the tangent bundle of any oriented 3-manifold is trivial, there is an orientation-preserving isometric bundle isomorphism

$$TM \to M \times \mathrm{im}\mathbb{H}$$
$$TM \ni X \mapsto \tilde{X} \in \mathrm{Im}\mathbb{H}$$

Now we can define a Clifford multiplication for $\Sigma := M \times \mathbb{H}$ as

$$\hat{X}(\Phi) = \tilde{X}\Phi.$$

If we have two different Spin bundles Σ and $\tilde{\Sigma}$ on M we can find on small enough open sets U around every sets U around every point $X, Y, Z \in \Gamma(TM_{|U})$, $\phi \in \Gamma(\Sigma_{|U})$ and $\tilde{\phi} \in \Gamma(\tilde{\Sigma}_{|U})$ as in the above Lemma. Then it is easy to see that on U the quaternionic linear bundle map $L\colon \Gamma(\Sigma_{|U}) \to \Gamma(\tilde{\Sigma}_{|U})$ defined by $\tilde{\phi} = L \circ \phi$ is an isomorphism of Spin bundles. On the intersection $U_1 \cap U_2$ of two such open sets we have $L_1 = \epsilon L_2$ where $\epsilon\colon U_1 \cap U_2$ is a locally constant function. If M is simply connected a cover space argument now shows that there is a consistent way to define L globally. ☐

Theorem 7. *Let Σ be a Spin bundle over an oriented Riemannian 3-manifold M. Then there is a unique Spin connection ∇^{Σ} on Σ.*

Proof. Locally we may chose a positively oriented orthonormal frame field X, Y, Z on M and a section $\phi \in \Gamma(\Sigma)$ as in Lemma (7). It is sufficient to prove the theorem locally, so this means that without loss of generality we may assume $TM = M \times \mathrm{Im}\mathbb{H}$, $\Sigma = M \times \mathbb{H}$, the Clifford multiplication is the ordinary multiplication of quaternions and for $\Psi, \phi \in \Gamma(\Sigma)$ we have $\langle \Psi, \phi \rangle_{\Sigma} = \bar{\Psi}\phi$. After these identifications there is an $\mathrm{Im}\mathbb{H}$ valued 1-form ω on M such that for $X, Y \in \Gamma(TM)$ we have

$$\nabla_X Y = d_X Y + \omega(X) \times Y. \tag{2.4}$$

If ∇^{Σ} is a connection for which the last two of the Leibniz rules hold there must be another $\mathrm{Im}\mathbb{H}$ valued 1-form $\tilde{\omega}$ such that

$$\nabla^{\Sigma}_X \Psi = d_X \Psi + \tilde{\omega}(X)\Psi.$$

Now the Leibniz rule with respect to the Clifford representation implies

$$\begin{aligned}(d_X Y)\Psi + Y d_X \Psi + \tilde{\omega}(X) Y \Psi &= d_X(Y\Psi) + \tilde{\omega}(X) Y \Psi \\ &= \nabla^{\Sigma}_X (Y\Psi) \\ &= (d_X Y + \omega(X) \times Y)\Psi + Y(d_X \Psi + \tilde{\omega}(X)\Psi).\end{aligned}$$

Hence

$$\omega(X) \times Y = \tilde{\omega}(X) Y - Y \tilde{\omega}(X) = 2\tilde{\omega}(X) \times Y$$

and therefore

$$\tilde{\omega} = \frac{1}{2}\omega.$$

This shows uniqueness of ∇^{Σ}. Existence amounts to check that $\nabla^{\Sigma} := d + \tilde{\omega}$ satisfies all three Leibniz rules.

☐

We showed the existence and the uniqueness of a Spin connection on Σ. We get the connection ∇^{Σ} in a canonical way from the Levi-Civita connection on M.

Corollary 5. *Let $M \subset \mathbb{R}^3$ an oriented 3-dimensional manifold and $\Sigma := M \times \mathbb{H}$ then*

$$d = \nabla^{\Sigma}.$$

Proof. The Levi-Civita connection coincides with d and therefore the ω from the previous theorem (2.4) vanishes. ☐

3 Spin bundles over Riemann surfaces

In Theorem (6) the existence proof for a Spin bundle Σ over an oriented 3-dimensional manifold relied on the fact that oriented 3-manifolds are parallelizable. Thereof we concluded that any Spin bundle Σ is isomorphic to

$$M \times \mathbb{H}.$$

Clearly this argument does not work for a higher dimensional M. The task of this section will be to show that every Riemann surface M has a Spin bundle Σ with a unique Spin connection. The main idea is to extend M to a 3-dimensional manifold $\tilde{M} := M \times \mathbb{R}$ and apply the results from the previous section.
Let $(M, \langle\,,\rangle, J)$ be a 2 dimensional Riemannian manifold. Let X, JX be an orthonormal basis of T_pM and $N_p := XJX = X \wedge JX$. Note that N_p does not depend on the choice of the orthonormal basis. The Clifford algebra of T_pM is

$$\mathrm{Cl}_{\langle,\rangle}(T_pM) = \mathbb{R} \oplus T_pM \oplus \mathbb{R}N_p.$$

Since $X^2 = (JX)^2 = N_p^2 = -1$ and X, JX, N_p do pairwise commute, the Clifford algebra of T_pM is isomorphic to the quaternions, i.e. $Cl_{\langle,\rangle}(T_pM) = \mathbb{H}$. Therefore the Clifford bundle over $(M, \langle,\rangle)$ is

$$\mathrm{Cl}_{\langle,\rangle}(M) = (M \times \mathbb{R}) \oplus TM \oplus (M \times \mathbb{R})N.$$

isomorphic to the trivial bundle $M \times \mathbb{H}$.

Lemma 8. *For any $U, V \in \Gamma(TM)$ we obtain*

(1) $U \wedge V = \langle JU, V\rangle N$.

(2) $NU = JU = -UN$.

Proof. We decompose $U = \langle U, X\rangle X + \langle U, JX\rangle JX$ and $V = \langle V, X\rangle X + \langle U, JX\rangle JX$. Then

$$\begin{aligned} U \wedge V &= (\langle U, X\rangle\langle V, JX\rangle - \langle U, JX\rangle\langle U, JX\rangle\langle V, X\rangle)N \\ &= \langle JU, V\rangle N. \end{aligned}$$

Further,

$$NU = N(\langle U, X\rangle X + \langle U, JX\rangle JX) = -(\langle U, X\rangle X + \langle U, JX\rangle JX)N = -UN.$$

$\square$

The next task will be to show that every Riemann surface has a Spin bundle Σ with a unique Spin connection ∇^Σ on Σ. We consider the oriented 3-manifold

$$\tilde{M} := M \times \mathbb{R}$$

with the product metric

$$\langle (X, a), (Y, b)\rangle := \langle X, Y\rangle + ab$$

on $T\tilde{M} = M \times \mathbb{R}$. Let X, JX be locally an orthonormal basis for M, then $(X, 0), (JX, 0), \partial_t$ is locally an orthonormal basis for $\tilde{M}$. For $E := -(X, 0)(JX, 0)\partial_t$ we obtain by (2.2)

$$E((U, 0) \wedge (V, 0)) = (U, 0) \times (V, 0) = \langle JU, V\rangle\partial_t.$$

In the previous section (6) we proved that $\tilde{M}$ has a Spin bundle Σ. We claim that Σ is as well a Spin bundle over M. Let $\tilde{\hat{}}$ be the Clifford representation of $(\Sigma, \mathrm{Cl}(\tilde{M}))$. We have to define a representation on $(\Sigma, \mathrm{Cl}(M))$. Let $U, V \in \Gamma(TM)$.

$$\begin{aligned} \widehat{U} &:= \widehat{\widetilde{(U, 0)}} \\ \widehat{U \wedge V} &:= \langle JU, V\rangle\tilde{\hat{\partial}}_t. \end{aligned}$$

Let us check that this really defines a Clifford representation.

$$
\begin{aligned}
\widehat{U}\widehat{V} &= \widehat{\widetilde{(U,0)}}\widehat{\widetilde{(V,0)}} \\
&= \widehat{\widetilde{(U,0)(V,0)}} \\
&= -\langle U,V\rangle I + \widehat{\widetilde{(U,0)\wedge(V,0)}} \\
&= -\langle U,V\rangle I + \widehat{\widetilde{(U,0)\times(V,0)}} \\
&= -\langle U,V\rangle I + \langle JU,V\rangle \widehat{\tilde{\partial}}_t \\
&= -\langle U,V\rangle I + \widehat{U\wedge V} \\
&= \widehat{UV}.
\end{aligned}
$$

Let ∇ be the Levi-Civita connection on M, then the corresponding Levi-Civita connection ∇ of $\tilde{M}$ is given by

$$\nabla_{(X,a)}(Y,b) = (\nabla_X Y, ab').$$

We already know that $(\Sigma,\tilde{M})$ has a unique Spin connection ∇^Σ (7). We claim that

$$\nabla^\Sigma_X \Psi := \nabla^\Sigma_{(X,0)}\Psi \tag{3.1}$$

defines the unique Spin connection on (Σ, M). Indeed, let $Y\in\Gamma(TM)$, then

$$
\begin{aligned}
\nabla^\Sigma_X \widehat{Y}\Psi &= \nabla^\Sigma_{(X,0)}\widehat{\widetilde{(Y,0)}}\Psi \\
&= \nabla_{(X,0)}\widehat{\widetilde{(Y,0)}}\Psi + \widehat{\widetilde{(Y,0)}}\nabla^\Sigma_{(X,0)}\Psi \\
&= \widehat{\nabla_X Y}\Psi + \widehat{Y}\nabla^\Sigma_X\Psi.
\end{aligned}
$$

We haven proven the following theorem.

Theorem 8. *Every Riemann surface has a Spin bundle Σ. If M is simply connected and $\Sigma,\tilde{\Sigma}$ are two Spin bundles over M then there is a bundle isomorphism $L\colon\Sigma\to\tilde{\Sigma}$ that is quaternionic linear, unitary and compatible with the Clifford multiplication. L is unique up to replacing L by $-L$. Furthermore, every Spin bundle has a unique Spin connection.*

Conversely, let M be a Riemann surface and Σ a Spin bundle over M with a representation $\widehat{\ }$. Then Σ is a Spin bundle over $\tilde{M} := M\times\mathbb{R}$, since

$$\widehat{\widetilde{(X,a)}}\Psi := \widehat{X}\Psi + a\widehat{N}\Psi \tag{3.2}$$

defines a Clifford representation. Furthermore, the unique Clifford connection of $(\Sigma,\tilde{M})$ is an extension of the Clifford connection of (Σ, M).

4 Deformation of a spin connection

We constructed (3) the Levi-Civita connection $\tilde{\nabla}$ for a prescribed positive definite and self adjont Operator $S\in\Gamma(\mathrm{End}(TM))$ with respect to $g=\langle S,S\rangle$ out of the Levi-Civita connection ∇ of $(M,\langle\,,\rangle)$. In this section we want to compute the difference between the corresponding Spin connections. As an application, we will be able to formulate elegant compatibility conditions for the existence of immersions.
Let M be an oriented manifold with $n:=\mathrm{dim}M\in\{2,3\}$. Let Σ be a Spin bundle over $(M.\langle\,,\rangle)$. Now we prescribe a new Riemannian metric $g=\langle S,S\rangle$. The corresponding Clifford multiplication is given by

$$X * Y = -g(X,Y) + X\wedge Y.$$

We obtain the corresponding Clifford bundle $\mathrm{Cl}_g(M)$ and its Clifford connection $\tilde{\nabla}^c$.

Lemma 9. *Let Σ be a Spin bundle over $(M, \langle\, ,\rangle, \widehat{\ })$, then* $\tilde{\widehat{\ }} : \Gamma(\mathrm{Cl}_g(M)) \to \Gamma(\mathrm{End}_{\mathbb{H}}(\Sigma))$

$$(X_1 \tilde{\widehat{\wedge \ldots \wedge}} X_m) := (SX_1) \widehat{\wedge \ldots \wedge} (SX_m), \quad \tilde{\hat{1}} := I$$

for any $X_1, .., X_m \in \Gamma(TM)$ with $1 \leq m \leq n$, defines a Clifford representation.

Proof. For any $X, Y \in \Gamma(TM)$ we get

$$\begin{aligned}
\tilde{\hat{X}}\tilde{\hat{Y}} &= \widehat{SX}\widehat{SY} \\
&= (\widehat{SX)(SY}) \\
&= -\langle SX, SY\rangle I + \widehat{SX \wedge SY} \\
&= -g(X,Y)I + \tilde{\widehat{X \wedge Y}} \\
&= \tilde{\widehat{X * Y}}.
\end{aligned}$$

$\square$

Now we will compute the spin connection of (Σ, M, g). Recall (3), the Levi-Civita connection of (M, g) is given by

$$\tilde{\nabla} := \nabla + S^{-1}(\nabla S) - \Omega,$$

where $\Omega \in \Omega^1(M, \mathfrak{so}(TM, g))$ and satisfies $\Omega \wedge I = S^{-1}d^{\nabla}S$. First we will deal with a 3-dimensional manifold M. Since $\Omega \in \Omega^1(M, \mathfrak{so}(TM, g))$ we can find a $\omega \in \Omega^1(M, TM)$ with

$$\Omega(X)Y = \omega(X)\tilde{\times}Y. \tag{4.1}$$

Note that the cross product with respect to g is given by

$$X\tilde{\times}Y = S^{-1}(SX \times SY).$$

Theorem 9. *Let $(M, \langle, \rangle)$ be an oriented 3 dimensional manifold and ∇^{Σ} be the Spin connection of the Spin bundle $(\Sigma, M, \langle, \rangle)$. Then*

$$\tilde{\nabla}^{\Sigma}_X \Psi := \nabla^{\Sigma}_X \Psi - \frac{1}{2}\widehat{S\omega(X)}\Psi \tag{4.2}$$

is the Spin connection of (Σ, M, g). In particular, if $g = e^{2u}\langle, \rangle$, then

$$\tilde{\nabla}^{\Sigma}\Psi := \nabla^{\Sigma}\Psi + \frac{1}{2}(\widehat{\operatorname{grad} u \times})\Psi.$$

Proof. It suffices to check the Leibniz rule with respect to the Clifford representation

$$\begin{aligned}
\tilde{\nabla}^{\Sigma}_X(\tilde{\hat{Y}}\Psi) &= \nabla^{\Sigma}_X(\widehat{SY}\Psi) - \frac{1}{2}\widehat{S\omega(X)}\widehat{SY}\Psi \\
&= \widehat{\nabla_X SY}\Psi + \widehat{SY}\nabla^{\Sigma}_X\Psi - \frac{1}{2}\widehat{S\omega(X)}\widehat{SY}\Psi \\
&= (\widehat{S\tilde{\nabla}_X Y} + \widehat{S\Omega(X)Y} - \frac{1}{2}\widehat{S\omega(X)}\widehat{SY})\Psi + \widehat{SY}\nabla^{\Sigma}_X\Psi \\
&= (\widehat{S\tilde{\nabla}_X Y} + S(\widehat{\omega(X)\tilde{\times}Y}) - \frac{1}{2}\widehat{S\omega(X)}\widehat{SY})\Psi + \widehat{SY}\nabla^{\Sigma}_X\Psi \\
&= \widehat{S\tilde{\nabla}_X Y}\Psi - \frac{1}{2}\widehat{SY}\widehat{S\omega(X)}\Psi + \widehat{SY}\nabla^{\Sigma}_X\Psi \\
&= \tilde{\widehat{\tilde{\nabla}_X Y}}\Psi + \tilde{\hat{Y}}\tilde{\nabla}^{\Sigma}_X\Psi.
\end{aligned}$$

In particular, for

$$S = e^u I,$$

we obtain

$$\Omega(X)Y = \omega(X)\tilde{\times}Y := e^u\omega(X) \times Y.$$

The unique solution of

$$\Omega(X)Y - \Omega(Y)X = S^{-1}d^{\nabla}S(X,Y) = (X \times Y) \times \operatorname{grad} u$$

is given by

$$\omega(X) := -\frac{1}{e^u}\operatorname{grad} u \times X$$

□

Now we will prove the analogous result for Riemann surfaces.

Theorem 10. *Let $(M, \langle\,,\rangle, J)$ be a Riemann surface and let ∇^Σ be the Spin connection on $(\Sigma, M, \langle\,,\rangle)$ then*

$$\tilde{\nabla}^\Sigma_X \Psi = \nabla^\Sigma_X \Psi - \frac{1}{2}\eta(X)\hat{N}\Psi, \tag{4.3}$$

where $\eta(X) := \langle JS^{-1}JZ, X\rangle$ and $Z \in \Gamma(TM)$ is the unique vector field defined through $Z\det := d^\nabla S$ (3). In particular, if $g = e^{2u}\langle,\rangle$, then the Spin connection transforms as

$$\tilde{\nabla}^\Sigma = \nabla^\Sigma + \frac{1}{2} * du\hat{N}.$$

Proof. We extend M to the 3-dimensional manifold $\tilde{M} := M \times \mathbb{R}$. In the previous sections we have seen that Σ is a Spin bundle over $\tilde{M}$. Let $\tilde{\nabla}^\Sigma$ and ∇^Σ be the Spin connections of $(\Sigma, \tilde{M}, g)$ and $(\Sigma, \tilde{M}, \langle\,,\rangle)$. Then the Spin connections of (Σ, M, g) and $(\Sigma, M, \langle\,,\rangle)$ are given by (3.1)

$$\begin{aligned}\nabla^\Sigma_X \Psi &= \nabla^\Sigma_{(X,0)}\Psi,\\ \tilde{\nabla}^\Sigma_X \Psi &= \tilde{\nabla}^\Sigma_{(X,0)}\Psi.\end{aligned}$$

Applying the previous theorem and a result from the first chapter (0.12) we obtain

$$\begin{aligned}\tilde{\nabla}^\Sigma_X \Psi &= \tilde{\nabla}^\Sigma_{(X,0)}\Psi\\ &= \nabla^\Sigma_{(X,0)}\Psi - \frac{1}{2}\widehat{\tilde{S}\omega(X)}\Psi\\ &= \nabla^\Sigma_X \Psi - \frac{1}{2}\eta(X)\hat{\partial}_t\Psi\\ &= \nabla^\Sigma_X \Psi - \frac{1}{2}\eta(X)\hat{N}\Psi.\end{aligned}$$

□

Let Σ be a Spin bundle over a 2 or 3 dimensional oriented manifold $(M, \langle\,,\rangle)$. Let $\tilde{\Sigma}$ be another Spin bundle over M. Then there exists a bundle isomorphism $L\colon \Gamma(\Sigma) \to \Gamma(\tilde{\Sigma})$. For any $X_1, .., X_k \in \Gamma(TM)$ with $1 \leq k \leq 3$

$$X_1 \widetilde{\widehat{\wedge .. \wedge}} X_k := L \circ (SX_1 \widehat{\wedge .. \wedge} SX_k) \circ L^{-1}$$

defines a representation and makes $\tilde{\Sigma}$ a Spin bundle over (M, g). Indeed, for any $X, Y \in \Gamma(TM)$ we get

$$\begin{aligned}\tilde{\hat{X}}\tilde{\hat{Y}} &= L \circ \widehat{SX} \circ L \circ L^{-1} \circ \widehat{SY} \circ L^{-1}\\ &= L \circ \widehat{SXSY} \circ L^{-1}\\ &= -g(X,Y)I + L \circ S\widehat{X \wedge S}YL^{-1}\\ &= -g(X,Y)I + \widetilde{\widehat{X \wedge Y}}\\ &= \widetilde{\widehat{X * Y}}.\end{aligned}$$

Let $\tilde{\nabla}^\Sigma$ be the Spin connection of (Σ, M, g), then

$$\nabla^{\tilde{\Sigma}} L \circ \Psi := L \circ \tilde{\nabla}^\Sigma \Psi \tag{4.4}$$

is the Spin connection of $(\tilde{\Sigma}, M, g)$.

Let $M \subset \mathbb{R}^3$ be an oriented 3 dimensional manifold and $f : M \to \mathbb{R}^3$ be an immersion. We consider the induced metric $\langle X, Y\rangle := \langle df(X), df(Y)\rangle_{\mathbb{R}^3}$ on M. Then

$$\Sigma := f^*(\mathbb{R}^3 \times \mathbb{H}) = M \times \mathbb{H}$$

is a Spin bundle over $(M, \langle,\rangle)$, since for any $X \in \Gamma(TM)$

$$\hat{X}\Psi := df(X)\Psi$$

defines a Clifford representation.

$$\begin{aligned}
\widehat{X}\widehat{Y}\Psi &= df(X)df(Y)\Psi \\
&= -\langle X, Y\rangle\Psi + df(X) \times df(Y)\Psi \\
&= -\langle X, Y\rangle\Psi + df(X \times Y)\Psi \\
&= -\langle X, Y\rangle\Psi + \widehat{X \times Y}\Psi \\
&= -\langle X, Y\rangle\Psi + \widehat{X \wedge Y}\Psi \\
&= \widehat{XY}\Psi.
\end{aligned}$$

Let $g := \langle S, S\rangle$ be another metric on M and $\lambda : M \to \mathbb{S}^3 \subset \mathbb{H}$ a smooth map. We define

$$\begin{aligned}
L\colon \Gamma(\Sigma) &\to \Gamma(\Sigma) \\
\Psi &\mapsto \bar{\lambda}\Psi
\end{aligned}$$

a bundle isomorphism. The inverse is given by $L^{-1}\Psi = \lambda\Psi$. With

$$\tilde{\hat{X}}\Psi := L \circ \widehat{SX} \circ L^{-1}\Psi \tag{4.5}$$

$$= \bar{\lambda}df(SX)\lambda\Psi \tag{4.6}$$

Σ becomes a Spin bundle over (M, g). Since $M \subset \mathbb{R}^3$ both Spin connections, $\tilde{\nabla}^\Sigma$ of $(M \times \mathbb{H}, M, g)$ and ∇^Σ of $(M \times \mathbb{H}, M, \langle\,,\rangle)$, coincide with the trivial connection d (5).

Now let $\Psi \in \Gamma(M \times \mathbb{H})$, applying (4.4) and (4.2) we obtain

$$\begin{aligned}
\bar{\lambda}(d\Psi(X) - \frac{1}{2}\widehat{S\omega(X)})\Psi &= d(\bar{\lambda}\Psi)(X) \\
&= d\bar{\lambda}(X)\Psi + \bar{\lambda}d\Psi(X),
\end{aligned}$$

which is equivalent to

$$-\frac{1}{2}\bar{\lambda}df(S\omega(X))\Psi = d\bar{\lambda}(X)\Psi,$$

or

$$d\lambda\lambda^{-1} = \frac{1}{2}df(S\omega)$$

Any other immersion $\tilde{f}\colon M \to \mathbb{R}^3$ can be expressed by

$$d\tilde{f} = \bar{\lambda}df(S)\lambda,$$

with a smooth map $\lambda : M \to \mathbb{S}^3 \subset \mathbb{H}$ and a positive and self adjoint operator S. The induced metric is

$$\tilde{f}^*\langle X, Y\rangle = \langle d\tilde{f}(X), d\tilde{f}(Y)\rangle = \langle SX, SY\rangle = g(X, Y).$$

Further, the Spin structure induced by $\tilde{f}$ on $\Sigma = M \times \mathbb{H}$ is

$$\begin{aligned}
\tilde{\hat{X}}\Psi &= d\tilde{f}(X)\Psi \\
&= \bar{\lambda}df(SX)\lambda\Psi \\
&= L \circ \widehat{SX} \circ L^{-1}\Psi.
\end{aligned}$$

We haven proven the following

Theorem 11. *Let $M \subset \mathbb{R}^3$ be a three dimensional oriented manifold and $f\colon M \to \mathbb{R}^3$ be an immersion. We prescribe a new metric $g = \langle S, S\rangle$, a map $\lambda\colon M \to S^3$. If there exists a $\tilde{f}$ with $d\tilde{f} = \bar{\lambda} df(S)\lambda$ then*

$$d\lambda\lambda^{-1} = \frac{1}{2} df(S\omega). \tag{4.7}$$

In particular, if $g = e^{2u}\langle\, ,\rangle$ then

$$d\lambda\lambda^{-1} = -\frac{1}{2} df(\operatorname{grad} u \times). \tag{4.8}$$

In this case $\tilde{f}$ is a Möbius transformation of f. □

Let $G := df(\operatorname{grad} u)$, then $df(\operatorname{grad} u \times Z) = df(\operatorname{grad} u) \times df(Z) = G \times df(Z)$. Since $d\lambda\lambda^{-1}$ is an $\mathbb{R}^3$-valued 1-form we obtain

$$d\lambda\lambda^{-1} = -\frac{1}{2} G df.$$

This formula was recently found by Pinkall, Chern and Schroeder [**3**].

5 Gauss-Codazzi-formula

Let us first summarize the idea behind (11). We started with an immersion $f\colon M \to \mathbb{R}^3$. Further we prescribed a new metric $g = \langle S, S\rangle$ and a map $\lambda\colon M \to S^3$ and we were seeking for a new immersion $\tilde{f}$ such that $d\tilde{f} = \bar{\lambda} df(S)\lambda$. A necessary condition for the existence of such a $\tilde{f}$ is given by (4.7)

$$d\lambda\lambda^{-1} = \frac{1}{2} df(S\omega).$$

Since ω is defined through (4.1), ω is determined by S. Thus (4.7) can be interpreted as a compatibility condition for the existence of a new immersion $\tilde{f}$, between the inner geometry described through S and the extrinsic geometry given by λ.

In this section we will derive the corresponding compatibility conditions for immersed surfaces. Let M be a Riemann surface. Let $f : M \to \mathbb{R}^3$ be an immersion and $\langle,\rangle$ the induced metric on M, further let N_f be the Gauss map of f. Let

$$\Sigma := f^*(\mathbb{R}^3 \times \mathbb{H}) = M \times \mathbb{H}.$$

$$\hat{X}\Psi := df(X)\Psi,$$
$$\hat{N}\Psi = N_f\Psi$$

induces a Spin structure on Σ. The shape operator A of f is defined by $dN_f = df(A)$. Note that for any $X, Y \in \Gamma(TM)$ one gets

$$X df(Y) = df(\nabla_X Y) - \langle Y, AX\rangle N_f.$$

Let d be the connection on $\mathbb{R}^3 \times \mathbb{H}$ pulled back to M via f.

Theorem 12.

$$\nabla^{\Sigma}_X \Psi := d\Psi(X) - \frac{1}{2}\widehat{JAX}\Psi. \tag{5.1}$$

is the Spin connection on Σ.

Proof. It suffices to check the Leibniz rule with respect to the Clifford representation. So let $Y \in \Gamma(TM)$.

$$\begin{aligned}\nabla^{\Sigma}_X \hat{Y}\Psi &= X(df(Y)\Psi) - \frac{1}{2}df(JAX)df(Y)\Psi \\ &= (df(\nabla_X Y) - \langle Y, AX\rangle N_f)\Psi + df(Y)d\Psi(X) + \frac{1}{2}\langle AX, Y\rangle N_f\Psi + \frac{1}{2}\langle JAX, Y\rangle\Psi \\ &= df(\nabla_X Y)\Psi + df(Y)(d\Psi(X) - \frac{1}{2}df(JAX)\Psi) \\ &= \widehat{\nabla_X Y}\Psi + \hat{Y}\nabla^{\Sigma}_X\Psi.\end{aligned}$$

□

Let $\tilde{f}\colon M \to \mathbb{R}^3$ be another immersion with

$$d\tilde{f} = \bar{\lambda}df(S)\lambda.$$

The Gauss map of $\tilde{f}$ is given by $\tilde{N} = \lambda^{-1}N\lambda$ and the induced metric is $g = \langle S, S\rangle$. Let $X \in \Gamma(TM)$ then

$$\hat{\tilde{X}}\Psi = d\tilde{f}(X)\Psi = \bar{\lambda}df(SX)\lambda\Psi.$$

is a representation we have already discussed in the previous section (4.6). With (4.3) and (4.4), we computed that the Spin connection on $\tilde{\Sigma} := \tilde{f}^*(\mathbb{R}^3 \times \mathbb{H})$ is given by

$$\begin{aligned}\nabla^{\tilde{\Sigma}}_X \bar{\lambda}\Psi &= \bar{\lambda}\tilde{\nabla}^{\Sigma}_X\Psi \\ &= \bar{\lambda}(\nabla^{\Sigma}_X\Psi - \frac{1}{2}\eta(X)\hat{N}\Psi).\end{aligned}$$

Applying (5.1) we get alternatively

$$\begin{aligned}\nabla^{\tilde{\Sigma}}_X \bar{\lambda}\Psi &= d(\bar{\lambda}\Psi)(X) - \frac{1}{2}\widehat{\tilde{J}\tilde{A}\tilde{X}}\bar{\lambda}\Psi \\ &= d(\bar{\lambda}\Psi)(X) - \frac{1}{2}\bar{\lambda}\widehat{S\tilde{J}\tilde{A}X}\lambda\bar{\lambda}\Psi \\ &= d(\bar{\lambda}\Psi)(X) - \frac{1}{2}\bar{\lambda}\widehat{JS\tilde{A}X}\Psi.\end{aligned}$$

Combining both expressions for $\nabla^{\tilde{\Sigma}}_X\bar{\lambda}\Psi$ we obtain

$$\bar{\lambda}(d\Psi(X) - \frac{1}{2}\widehat{JAX}\Psi - \frac{1}{2}\eta(X)\hat{N}\Psi) = d(\bar{\lambda}\Psi)(X) - \frac{1}{2}\bar{\lambda}\widehat{JS\tilde{A}X}\Psi,$$

which is equivalent to

$$\bar{\lambda}d\Psi(X) - d(\bar{\lambda}\Psi)(X) = \frac{1}{2}\bar{\lambda}(\widehat{JAX} - \widehat{JS\tilde{A}X})\Psi + \frac{1}{2}\eta(X)\hat{N}\Psi,$$

or

$$d\Psi(X) - \lambda d(\bar{\lambda}\Psi)(X) = -\frac{1}{2}Ndf(S\tilde{A} - A)(X)\Psi + \frac{1}{2}\eta(X)N\Psi.$$

Since $d\Psi - \lambda d(\bar{\lambda}\Psi)(X) = -\lambda d\bar{\lambda}(X)\Psi = d\lambda(X)\lambda^{-1}\Psi$, we get

Theorem 13. *Let $f\colon M \to \mathbb{R}^3$ be an immersion. We prescribe a new metric $g = \langle S, S\rangle$, a map $\lambda\colon M \to S^3$ and $\tilde{A} \in \Gamma(\mathrm{End}(TM))$. If there exists a $\tilde{f}$ with $d\tilde{f} = \bar{\lambda}df(S)\lambda$ with its shape operator $\tilde{A}$ then*

$$d\lambda\lambda^{-1} = -\frac{1}{2}Ndf(S\tilde{A} - A) + \frac{1}{2}\eta N. \tag{5.2}$$

□

We want to derive the previous theorem from a more elementary approach. Seeking for an immersion with an induced metric $g = \langle S, S\rangle$ and shape operator $\tilde{A}$ we should at least require

(1) $\bar{\lambda} df(S)\lambda$ is closed.

(2) $d(\lambda^{-1}N\lambda) = \bar{\lambda} df(S\tilde{A})\lambda$.

Since $d(\lambda^{-1}N\lambda) = \lambda^{-1}df(A)\lambda) + \lambda^{-1}(Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N)\lambda)$ the second condition implies

$$Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N = df(S\tilde{A} - A). \tag{5.3}$$

We split

$$d\lambda\lambda^{-1} = (d\lambda\lambda^{-1})_- + (d\lambda\lambda^{-1})_+$$

in its tangential and normal part with respect to f. So for instance

$$(d\lambda\lambda^{-1})_+ = \langle d\lambda\lambda^{-1}, N\rangle N. \tag{5.4}$$

We obtain

$$Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N = 2N(d\lambda\lambda^{-1})_-$$

and therefore

$$(d\lambda\lambda^{-1})_- = -\frac{1}{2}Ndf(S\tilde{A} - A). \tag{5.5}$$

It remains to compute the normal part $(d\lambda\lambda^{-1})_+$. Now we make use of the first condition (1).

$$0 = d(\bar{\lambda}df(S)\lambda) \tag{5.6}$$

$$= -2\mathrm{Im}(\bar{\lambda}df(S) \wedge d\lambda + \bar{\lambda}d(df(S))\lambda. \tag{5.7}$$

(5.7) is equivalent to the existence of a real valued function $\rho \in C^\infty$ such that

$$\rho|df|^2 = -2\bar{\lambda}df(S) \wedge d\lambda\lambda^{-1}\lambda + \bar{\lambda}d(df(S))\lambda \tag{5.8}$$

or

$$\rho|df|^2 = -2df(S) \wedge d\lambda\lambda^{-1} + d(df(S)). \tag{5.9}$$

Note that only $(d\lambda\lambda^{-1})_+$ contributes to the tangential part of $df(S) \wedge d\lambda\lambda^{-1}$. Since $d(df(S)) = df \circ d^\nabla S + \langle S \wedge A\rangle N$ we obtain

$$0 = -2df(S) \wedge (d\lambda\lambda^{-1})_+ + df \circ d^\nabla S.$$

From (3) we know that $d^\nabla S = S(\eta\tilde{J} \wedge I) = \eta \wedge JS$. Therefore

$$0 = -2df(S) \wedge (d\lambda\lambda^{-1})_+ + df \circ d^\nabla S \tag{5.10}$$

$$= -2df(S) \wedge (\langle d\lambda\lambda^{-1}, N\rangle N) + df(\eta \wedge JS) \tag{5.11}$$

$$= -2df(\langle d\lambda\lambda^{-1}, N\rangle \wedge SJ) + df(\eta \wedge JS). \tag{5.12}$$

Thus $(d\lambda\lambda^{-1})_+ = \frac{1}{2}\eta N$ and with (5.5) we obtain again (5.2).

6 Dirac Operator

In (5.1) we could at least locally ensure the existence of an immersion $\tilde{f}$ by a compatibility condition in terms of the geometric data $(\lambda, \tilde{A}, S)$. In this section we want derive the compatibility condition out of the Dirac equation. Let M be a two or three dimensional oriented Riemannian manifold and Σ a Spin bundle over $(M, \langle\, ,\rangle)$. Let $X_1, .., X_n$ be a positively oriented orthonormal basis. For any $\Psi \in \Gamma(\Sigma)$ the Dirac operator is defined by

$$D\Psi = \hat{X}_1\nabla^\Sigma_{X_1}\Psi + .. + \hat{X}_n\nabla^\Sigma_{X_n}\Psi.$$

The Dirac operator is a self adjoint and elliptic differential operator. Let $\Psi, \Phi \in \Gamma(\Sigma)$ then

$$(\Psi, \Phi)(X) := \langle\Psi, \hat{X}\Phi\rangle$$

defines a $\mathbb{H}$-valued 1-form. The square (Ψ, Ψ) is even a $\mathbb{R}^3$-valued 1-form, since $(\Psi, \Psi)(X) := \langle\Psi, \hat{X}\Psi\rangle = -\overline{\langle\Psi, \hat{X}\Psi\rangle}$.

THEOREM 14. *Let M be an oriented 3-manifold and $\Psi \in \Gamma(\Sigma)$. Then*

$$*d(\Psi,\Psi)(Z) = 2\mathrm{Im}\langle \Psi, \nabla^{\Sigma}_{Z}\Psi + \hat{Z}D\Psi\rangle. \tag{6.1}$$

PROOF. Let X, Y, Z be a a positively oriented orthonormal basis, then

$$\begin{aligned}
*d(\Psi,\Psi)(Z) &= d(\Psi,\Psi)(X,Y)\\
&= X(\Psi,\Psi)(Y) - Y(\Psi,\Psi)(X) - (\Psi,\Psi)([X,Y])\\
&= X\langle \Psi, \hat{Y}\Psi\rangle - Y\langle \Psi, \hat{X}\Psi\rangle - \langle \Psi, \widehat{[X,Y]}\Psi\rangle\\
&= \langle \nabla^{\Sigma}_{X}\Psi, \hat{Y}\Psi\rangle + \langle \Psi, \nabla^{\Sigma}_{X}\hat{Y}\Psi\rangle\\
&\quad - (\langle \nabla^{\Sigma}_{Y}\Psi, \hat{X}\Psi\rangle + \langle \Psi, \nabla^{\Sigma}_{Y}\hat{X}\Psi\rangle) - \langle \Psi, \widehat{[X,Y]}\Psi\rangle\\
&= \langle \nabla^{\Sigma}_{X}\Psi, \hat{Y}\Psi\rangle - \overline{\langle \nabla^{\Sigma}_{X}\Psi, \hat{Y}\Psi\rangle}\\
&\quad - ((\langle \nabla^{\Sigma}_{Y}\Psi, \hat{X}\Psi\rangle - \overline{(\langle \nabla^{\Sigma}_{Y}\Psi, \hat{X}\Psi\rangle)})\\
&= 2\mathrm{Im}\langle \nabla^{\Sigma}_{X}\Psi, \hat{Y}\Psi\rangle - 2\mathrm{Im}\langle \nabla^{\Sigma}_{Y}\Psi, \hat{X}\Psi\rangle\\
&= 2\mathrm{Im}\langle \hat{X}\nabla^{\Sigma}_{X}\Psi, \hat{X}\hat{Y}\Psi\rangle - 2\mathrm{Im}\langle \hat{Y}\nabla^{\Sigma}_{Y}\Psi, \hat{Y}\hat{X}\Psi\rangle\\
&= 2\mathrm{Im}\langle D\Psi, \hat{Z}\Psi\rangle - 2\mathrm{Im}\langle \nabla^{\Sigma}_{Z}\Psi, \Psi\rangle\\
&= 2\mathrm{Im}\langle \Psi, \hat{Z}D\Psi\rangle + 2\mathrm{Im}\langle \Psi, \nabla^{\Sigma}_{Z}\Psi\rangle\\
&= 2\mathrm{Im}\langle \Psi, \nabla^{\Sigma}_{Z}\Psi + \hat{Z}D\Psi\rangle.
\end{aligned}$$

□

We are looking for a similar formula in the case of a Riemann surface.

COROLLARY 6. *Let M be a Riemann surface and $\Psi \in \Gamma(\Sigma)$, then*

$$d(\Psi,\Psi) = 2\mathrm{Im}\langle \Psi, \hat{N}D\Psi\rangle \det. \tag{6.2}$$

PROOF. From (3.2) we know that Σ is a spin bundle over $\tilde{M} := M \times \mathbb{R}$ via the representation

$$\widetilde{\widehat{(X,a)}}\Psi := \hat{X}\Psi + a\hat{N}\Psi.$$

For an orthonormal basis $(X,0), (Y,0), \partial_t$ we obtain from (3.1)

$$\begin{aligned}
\tilde{D}\Psi &= \widetilde{\widehat{(X,0)}}\nabla^{\Sigma}_{(X,0)}\Psi + \widetilde{\widehat{(Y,0)}}\nabla^{\Sigma}_{(Y,0)}\Psi + \tilde{\hat{\partial}}_t\nabla^{\Sigma}_{\partial_t}\Psi\\
&= D\Psi + \hat{N}\nabla^{\Sigma}_{\partial_t}\Psi,
\end{aligned}$$

where $\tilde{D}$ is the Dirac operator of $\tilde{M}$ and D the one of M. Applying (6.1) one gets

$$\begin{aligned}
d(\Psi,\Psi)(X,Y) &= d(\Psi,\Psi)((X,0),(Y,0))\\
&= 2\mathrm{Im}\langle \Psi, \nabla^{\Sigma}_{\partial_t}\Psi + \tilde{\hat{\partial}}_t\tilde{D}\Psi\rangle\\
&= 2\mathrm{Im}\langle \Psi, \nabla^{\Sigma}_{\partial_t}\Psi + \hat{N}(D\Psi + \hat{N}\nabla^{\Sigma}_{\partial_t}\Psi)\rangle\\
&= 2\mathrm{Im}\langle \Psi, \hat{N}D\Psi\rangle.
\end{aligned}$$

□

The formula (6.2) suggests to look at the operator

$$\hat{D} = ND.$$

It is not difficult to show that $\hat{D}$ is a self adjoint and an elliptic operator.

COROLLARY 7. *Let $\Psi \in \Gamma(\Sigma)$ then (Ψ,Ψ) is closed if and only if $\hat{D}\Psi = \rho\Psi$ and $\rho \in C^{\infty}(M)$.*

PROOF. $\hat{D}\Psi = \Psi\rho$, where ρ is a quaternion valued function. If (Ψ, Ψ) is closed then we want to show that ρ is a real valued function.

$$\begin{aligned} 0 = d(\Psi, \Psi) &= 2\mathrm{Im}\langle \Psi, \hat{D}\Psi \rangle \\ &= 2\mathrm{Im}\langle \Psi, \Psi\rho \rangle \\ &= 2\mathrm{Im}(\langle \Psi, \Psi \rangle \rho) \\ &= 2|\Psi|^2 \mathrm{Im}\rho. \end{aligned}$$

$\square$

Now let M be a Riemann surface and $f \colon M \to \mathbb{R}^3$ be an immersion. Once again we consider the induced Spin bundle

$$\Sigma = f^*(\mathbb{R}^3 \times \mathbb{H}) = M \times \mathbb{H}.$$

In (5.1) we showed that the corresponding Spin connection was given by

$$\nabla^\Sigma_X \Psi = d\Psi(X) - \frac{1}{2}\widehat{JAX}\Psi.$$

THEOREM 15.

$$\hat{D}\Psi = -\frac{df \wedge d\Psi}{|df|^2} + H\Psi. \tag{6.3}$$

PROOF. Let (X, JX) be a orthonormal basis.

$$\begin{aligned} \hat{D}\Psi &= \hat{N}(\hat{X}\nabla^\Sigma_X \Psi + \hat{JX}\nabla^\Sigma_{JX}\Psi) \\ &= \hat{N}(df(X)(d\Psi(X) - \frac{1}{2}df(JAX)\Psi) + df(JX)(d\Psi(Y) - \frac{1}{2}df(JAJX)\Psi)) \\ &= df(JX)d\Psi(X) - df(X)d\Psi(JX) + \frac{1}{2}(df(X)df(JAJX) - df(JX)df(JAX)\Psi \\ &= -df \wedge d\Psi(X, JX) + H\Psi. \end{aligned}$$

$\square$

Let

$$D_f\Psi := -\frac{df \wedge d\Psi}{|df|^2}$$

then $\hat{D} = D_f + H$ and eigenfunctions of $\hat{D}$ are eigenfunctions of D_f, and vice versa. Let $1_f \in \Gamma(\Sigma)$ with

$$df(X) =: (1_f, \hat{X}1_f).$$

Let $\Psi \in \Gamma(\Sigma)$ and

$$(\Psi, \Psi)\tilde{} := (\Psi, \Psi) \circ S$$

We obtain for a $\lambda \in \Gamma(\Sigma) = C^\infty(M, \mathbb{H})$ with $|\lambda| = 1$

$$\begin{aligned} (\lambda, \lambda)\tilde{}(X) &= \langle \lambda, \tilde{\hat{X}}\lambda \rangle \\ &= \langle \lambda, \widehat{SX}\lambda \rangle \\ &= \bar{\lambda} df(SX)\lambda. \end{aligned}$$

We already computed the Spin connection with respect to the representation $\tilde{\hat{X}} := \widehat{SX}$ in (4.3).

$$\begin{aligned} \tilde{\nabla}^\Sigma \Psi &= \nabla^\Sigma \Psi - \frac{1}{2}\eta\tilde{N}\Psi \\ &= d\Psi - \frac{1}{2}\widehat{JA}\Psi - \frac{1}{2}\eta\tilde{N}\Psi. \end{aligned}$$

Let $\tilde{D}$ be the corresponding Dirac operator, then

$$d(\lambda, \lambda)\tilde{} = 2\mathrm{Im}\langle \lambda, \hat{N}\tilde{D}\lambda\rangle \det S \det .$$

With $\tilde{\tilde{D}} := \hat{N}\tilde{D}$ and (5.7),(5.8) we obtain

Corollary 8. *Let $\lambda \in C^\infty(M.\mathbb{H})$ with $|\lambda| = 1$. Then the following statements are equivalent:*

(1) $(\lambda, \lambda)\tilde{}$ is closed.

(2) There exists a $\rho \in C^\infty(M)$ with $\tilde{\tilde{D}}\lambda = \rho\lambda$ (Dirac equation).

(3) There exists a $\rho \in C^\infty(M)$ with $\rho|df|^2 = -2\bar{\lambda}df(S) \wedge d\lambda + \bar{\lambda}d(df(S))\lambda$.

□

In (8) the compatibility conditions are formulated in terms of (λ, ρ, S). Put simply, the operator $\tilde{\tilde{D}}$ is completely determined by S. Further, if (λ, ρ) satisfy the Dirac equation

$$\tilde{\tilde{D}}\lambda = \rho\lambda,$$

then $(\lambda, \lambda)\tilde{}$ is closed and therefore at least locally we can find an immersion $\tilde{f}\colon M \to \mathbb{R}^3$ with

$$d\tilde{f} = (\lambda, \lambda)\tilde{}.$$

Now we want to drop the restriction $|\lambda| = 1$ in order to work in a vector space. We can decompose $S = \frac{trS}{2}(I + T)$, where $TJ = -JT$. With $\tau := df \circ T$ we can describe any immersion $\tilde{f}$ by

$$d\tilde{f} = \bar{\lambda}(df + \tau)\lambda,$$

for some $\lambda\colon M \to \mathbb{H}^*$. τ describes the change of the conformal structure, i.e. $\tilde{f}$ and f induce the same conformal structure if and only if $\tau = 0$. So instead of looking at (λ, ρ, S) with $|\lambda| = 1$ we will parametrize immersions via $(\lambda, \rho, \tau) \in H$, where H is the Prehilbertspace

$$H := C^\infty(M, \mathbb{H}) \times C^\infty(M) \times \Gamma(\bar{K} \otimes \mathrm{Im}\mathbb{H}).$$

Here $\Gamma(\bar{K} \otimes \mathrm{Im}\mathbb{H}) := \{\tau \in \Omega^1(M, \mathbb{R}^3)| * \tau = -N\tau\}$ is the space of quadratic differentials. (8)(3) translates to

$$-(df + \tau) \wedge d\lambda = (\rho|df|^2 - \frac{1}{2}d\tau)\lambda. \tag{6.4}$$

3 | The space of immersions

1 The space of immersions and its tangent space

Let M be a Riemann surface and $f\colon M \to \mathbb{R}^3$ a reference immersion. By (6.4) the space of immersions up to translations is given by

$$(1.1) \qquad \mathcal{M} := \{ \begin{pmatrix} \lambda \\ \rho \\ \tau \end{pmatrix} \in H \mid -\operatorname{Re}((df+\tau)\wedge d\lambda\lambda^{-1}) = \rho|df|^2 \quad \text{and} \quad \bar{\lambda}(df+\tau)\lambda \quad \text{is exact}\}.$$

The reference immersion f corresponds to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{M}$. The formal tangent space is given by the time derivative at $t=0$ of all smooth curves $\begin{pmatrix} \lambda_t \\ \rho_t \\ \tau_t \end{pmatrix} \in \mathcal{M}$ with

$$(1.2) \qquad \begin{pmatrix} \lambda_0 \\ \rho_0 \\ \tau_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The infinitesimal version of (6.4) is

$$-df \wedge d\dot{\lambda} = \dot{\rho}|df|^2 - \frac{1}{2}d\dot{\tau}.$$

and therefore the tangent space at the base point f is described by

$$(1.3) \qquad T_f\mathcal{M} = \{ \begin{pmatrix} \dot{\lambda} \\ \dot{\rho} \\ \dot{\tau} \end{pmatrix} \in H \mid -\operatorname{Re}(df \wedge d\dot{\lambda}) = \dot{\rho}|df|^2 \quad \text{and} \quad \bar{\dot{\lambda}}df + df\dot{\lambda} + \dot{\tau} \quad \text{is exact}\}.$$

In the following section we will compute the normal space of $T_f\mathcal{M}$ which is a subspace of the dual space of H. First we need to find the dual space of quadratic differentials. Any quaternion valued 1-form α can be decomposed in its normal part and tangential part with respect to an immersion f.

$$\alpha = \alpha_+ + \alpha_-,$$

$$\begin{aligned} \alpha_+ &\in \Omega^1(M) \otimes \Omega^1(M)N, \\ \alpha_- &\in df \circ \Omega^1(M, TM). \end{aligned}$$

Further we can decompose any quaternionic valued 1-form in its conformal and anti conformal part.

$$\begin{aligned} \alpha &= \alpha' + \alpha'' \\ &:= \frac{1}{2}(\alpha - N * \alpha) + \frac{1}{2}(\alpha + N * \alpha), \end{aligned}$$

then

$$\begin{aligned} * \alpha' &= N\alpha' \\ * \alpha'' &= -N\alpha''. \end{aligned}$$

Assume that α' and α'' are both $\mathbb{R}^3$ valued one forms then both are automatically tangential valued 1-forms. In this case we obtain

$$\begin{aligned}\alpha' \wedge \alpha''(X, JX) &= \alpha'(X)\alpha''(JX) - \alpha'(JX)\alpha''(X)\\ &= -\alpha'(X)N\alpha''(JX) - N\alpha'(X)\alpha''(X)\\ &= N\alpha'(X)\alpha''(JX) - N\alpha'(X)\alpha''(X)\\ &= 0.\end{aligned}$$

A quadratic differential τ in this terms is simply $\tau' = 0$ and $\tau_+ = 0$. Let σ be any $\mathbb{R}^3$ valued 1-form and τ a quadratic differential, then

$$\langle \tau \wedge \sigma \rangle = \langle \tau \wedge \sigma_- \rangle \tag{1.4}$$
$$= \langle \tau \wedge (\sigma_-)' + (\sigma_-)'' \rangle \tag{1.5}$$
$$= \langle \tau \wedge (\sigma_-)'' \rangle. \tag{1.6}$$

Note that $(\sigma_-)''$ is a quadratic differential. With

$$\langle \tau, \sigma \rangle := \int_M \langle *\tau \wedge \sigma \rangle$$

we obtain from (1.6) a non degenerated pairing between quadratic differentials with themselves, i.e.

$$\Gamma(\bar{K} \otimes \mathrm{Im}\mathbb{H})^* = \Gamma(\bar{K} \otimes \mathrm{Im}\mathbb{H})$$

and therefore

$$H^* = (\Omega^2(M) \otimes \mathbb{H}) \times \Omega^2(M) \times \Gamma(\bar{K} \otimes \mathrm{Im}\mathbb{H}).$$

Further we consider the non degenerated pairing

$$\langle \begin{pmatrix} \dot\lambda \\ \dot\rho \\ \dot\tau \end{pmatrix}, \begin{pmatrix} \sigma \\ \mu \\ \eta \end{pmatrix} \rangle := \int_M \langle \dot\lambda, \sigma \rangle + \dot\rho\mu + \langle *\dot\tau \wedge \mu \rangle.$$

2 The normal space of immersions

Our next goal is to compute the normal space of immersions at a base point f. Note, $\begin{pmatrix} \dot\lambda \\ \dot\rho \\ \dot\tau \end{pmatrix} \in$ $T_f\mathcal{M}$ if and only if

$$0 = \int_M \mathrm{Re}(df \wedge d\dot\lambda + \dot\rho|df|^2)\Psi + \langle \eta \wedge \bar{\dot\lambda} df + df\dot\lambda + \dot\tau \rangle,$$

for all $\Psi \in C^\infty(M, \mathbb{R})$ with compact support and all closed $\eta \in \Omega^1(M, \mathrm{Im}\mathbb{H})$ with compact support.

Theorem 16. *The normal space of immersions at a base point f is given by*

$$N_f\mathcal{M} = \{ \begin{pmatrix} df \wedge (d\Psi + \eta) \\ \Psi|df|^2 \\ -\frac{1}{2} * \eta_-'' \end{pmatrix} \mid \Psi \in C_0^\infty(M, \mathbb{R}), \eta \in \Omega_0^1(M, \mathrm{Im}\mathbb{H}) \textit{ closed one form}\}. \tag{2.1}$$

PROOF. Let Ψ a smooth function with compact support.

$$\begin{aligned}
&\int_M \mathrm{Re}(df \wedge d\dot{\lambda} + \dot{\rho}|df|^2)\Psi \\
&= \int_M \dot{\rho}\Psi|df|^2 + \langle \Psi, df \wedge d\dot{\lambda}\rangle \\
&= \int_M \dot{\rho}\Psi|df|^2 - \langle \Psi, d(df\dot{\lambda})\rangle \\
&= \int_M \dot{\rho}\Psi|df|^2 + \langle d\Psi \wedge df\dot{\lambda}\rangle \\
&= \int_M \dot{\rho}\Psi|df|^2 + \langle \dot{\lambda}, df \wedge d\Psi\rangle.
\end{aligned}$$

Further, let $\eta \in \Omega^1_0(M, \mathrm{Im}\mathbb{H})$ closed.

$$\begin{aligned}
&\int_M \langle \eta \wedge (\bar{\dot{\lambda}}df + df\dot{\lambda} + \dot{\tau})\rangle \\
&= \int_M 2\langle \dot{\lambda}, df \wedge \eta\rangle + \langle \eta''_- \wedge \dot{\tau}\rangle \\
&= \int_M 2\langle \dot{\lambda}, df \wedge \eta\rangle - \langle *\dot{\tau} \wedge *\eta''_-\rangle.
\end{aligned}$$

Patching all together, we obtain $\begin{pmatrix} \dot{\lambda} \\ \dot{\rho} \\ \dot{\tau} \end{pmatrix} \in T_f\mathcal{M}$ if and only if

$$0 = \int_M \langle \dot{\lambda}, df \wedge (d\Psi + \eta)\rangle + \dot{\rho}\Psi|df|^2 - \frac{1}{2}\langle *\dot{\tau} \wedge *\eta''_-\rangle.$$

□

Consequently the normal space of immersions at a base point f with prescribed conformal structure is given by

$$N_f\mathcal{M} = \{\begin{pmatrix} df \wedge (d\Psi + \eta) \\ \Psi|df|^2 \end{pmatrix} \mid \Psi \in C^\infty_0(M, \mathbb{R}), \eta \in \Omega^1_0(M, \mathrm{Im}\mathbb{H}) \text{ closed one form}\}.$$

3 Noether Theorem on the space of immersions

Let $f\colon M \to \mathbb{R}^3$ be a reference immersion. We consider an arbitrary geometric smooth functional $\mathcal{F}$, i.e.

$$\mathcal{F}\colon \mathcal{M} \to \mathbb{R}, \quad (\lambda, \rho, \tau) \mapsto \int_M \mathcal{L}(\lambda, \rho, \tau)|df|^2,$$

such that for all smooth curves $\begin{pmatrix} \lambda_t \\ \rho_t \\ \tau_t \end{pmatrix} \in \mathcal{M}$ with property (1.2)

$$\frac{d}{dt}_{t=0}\mathcal{F} \circ \begin{pmatrix} \lambda_t \\ \rho_t \\ \tau_t \end{pmatrix}$$

exists. Then

$$\mathrm{grad}\,\mathcal{F} = \begin{pmatrix} \partial_\lambda\mathcal{F} \\ \partial_\rho\mathcal{F} \\ \partial_\tau\mathcal{F} \end{pmatrix} \in H^* = (\Omega^2(M, \mathbb{H}) \times \Omega^2(M) \times \Gamma(\bar{K} \otimes \mathrm{Im}\mathbb{H})$$

such that

$$\frac{d}{dt}_{t=0}\mathcal{F}\circ\begin{pmatrix}\lambda_t\\ \rho_t\\ \tau_t\end{pmatrix} =: \int_M \langle\dot\lambda,\partial_\lambda\mathcal{F}\rangle + \dot\rho\partial_\rho\mathcal{F} + \langle *\dot\tau\wedge\partial_\tau\mathcal{F}\rangle.$$

Let

$$\partial_\lambda\mathcal{F} = (g_1 + df(Y) + g_2N)|df|^2,$$

$$\partial_\rho\mathcal{F} = g_3|df|^2,$$

where $g_1, g_2, g_3 \in C^\infty(M)$ and $Y \in \Gamma(TM)$.

We are interested in critical points of $\mathcal{F}$. Clearly, $\begin{pmatrix}\lambda\\ \rho\\ \tau\end{pmatrix} \in \mathcal{M}$ is a critical point if and only if

$$\operatorname{grad}\mathcal{F} \in N_f\mathcal{M}. \tag{3.1}$$

The following theorem gives a complete characterization for critical points of $\mathcal{F}$.

Theorem 17. $\operatorname{grad}\mathcal{F} \in N_f\mathcal{M}$ *if and only if the* $\operatorname{Im}\mathbb{H}$ *valued one form*

$$\eta := \frac{g_2}{2}df + \frac{g_1}{2} * df + 2 * \partial_\tau\mathcal{F} + (*dg_3 - \langle Y, .\rangle)N \tag{3.2}$$

is closed.

Proof. Assume $\operatorname{grad}\mathcal{F} \in N_f\mathcal{M}$ then there exists a $\Psi \in C^\infty(M)$ and a closed one form $\eta \in \Omega^1(M, \operatorname{Im}\mathbb{H})$ such that $\eta_-^{''} = 2 * \partial_\tau\mathcal{F}$ and

$$\partial_\lambda\mathcal{F} = df \wedge (d\Psi + \eta).$$

We decompose $d\Psi + \eta = (d\Psi + \eta)_+ + \eta_-$. Further we can decompose $(d\Psi + \eta)_+ = (d\Psi + \eta)_+^{'} + (d\Psi+\eta)_+^{''}$ and $\eta_- = \eta_-^{'} + \eta_-^{''}$ in their conformal and anti conformal types. Let $\eta_-^{'} = \alpha_1 df + \alpha_2 * df$ for some functions $\alpha_1, \alpha_2 \in C^\infty(M)$. We obtain

$$df \wedge \eta_-^{'} = 2\alpha_2|df|^2 + 2\alpha_1 N|df|^2.$$

Further $(d\Psi + \eta)_+^{'}$ can be decomposed as

$$(d\Psi + \eta)_+^{'} = \alpha_3 - *\alpha_3 N,$$

where $\alpha_3 = \langle\tilde Y, .\rangle$ fore some $\tilde Y \in \Gamma(TM)$.

$$df \wedge (d\Psi + \eta)_+^{'} = df \wedge \alpha_3 - df \wedge *\alpha_3 N = 2df \wedge \alpha_3$$

and therefore

$$df \wedge (d\Psi + \eta)_+^{'} = -2df(J\tilde Y)|df|^2.$$

Now we put all together.

$$\begin{aligned}(g_1 + df(Y) + g_2N)|df|^2 = \partial_\lambda\mathcal{F} &= df \wedge (d\Psi + \eta)\\ &= df \wedge ((d\Psi + \eta)_+ + h_-)\\ &= df \wedge ((d\Psi + \eta)_+^{'} + h_-^{'})\\ &= -2df(J\tilde Y)|df|^2 + 2\alpha_2|df|^2 + 2\alpha_1 N|df|^2.\end{aligned}$$

We obtain $\alpha_1 = \frac{g_2}{2}$, $\alpha_2 = \frac{g_1}{2}$ and $\tilde Y = \frac{1}{2}JY$. Note that

$$\begin{aligned}&(d\Psi + \eta)_+\\ &= d\Psi + \langle\eta, N\rangle N\\ &= \alpha_3 - *\alpha_3 N + (d\Psi - \alpha_3) + *(d\Psi - \alpha_3)N\\ &= d\Psi + (*d\Psi - 2 * \alpha_3)N\\ &= d\Psi + (*d\psi - \langle Y, \rangle)N.\end{aligned}$$

Finally,

$$\begin{aligned}
&d\Psi + \eta \\
&= (d\Psi + \eta)_+ + \eta_- \\
&= d\Psi + (*d\Psi - \langle Y, \rangle)N + \eta_-^{'} + \eta_-^{''} \\
&= d\Psi + (*d\Psi - \langle Y, \rangle)N + \frac{g_2}{2}df + \frac{g_1}{2} * df + 2 * \partial_\tau \mathcal{F},
\end{aligned}$$

or

$$\eta = (*dg_3 - \langle Y, \rangle)N + \frac{g_2}{2}df + \frac{g_1}{2} * df + 2 * \partial_\tau \mathcal{F}.$$

Conversely, assume η is closed. We want to show that $\operatorname{grad}\mathcal{F} \in N_f\mathcal{M}$. Note that $\partial_\tau\mathcal{F} = -\frac{1}{2} * \eta_-''$. Let $\Psi := g_3$. We have to show

$$\partial_\lambda \mathcal{F} = df \wedge (d\Psi + \eta).$$

$d\Psi + \eta = \frac{g_2}{2}df + \frac{g_1}{2} * df + d\Psi + *d\Psi N + 2 * \partial_\tau\mathcal{F} - \langle Y, \rangle N$. By a type argument we obtain

$$\begin{aligned}
df \wedge (d\Psi + \eta) &= df \wedge (\frac{g_2}{2}df + \frac{g_1}{2} * df - \langle Y, \rangle N) \\
&= g_1|df|^2 + g_2|df|^2 N + df(Y)|df|^2 \\
&= \partial_\lambda \mathcal{F}.
\end{aligned}$$

□

Corollary 9. *For a critical point (λ, ρ, τ) of $\mathcal{F}$ we obtain for two homologous cycles γ_1 and γ_2 in M, $\int_{\gamma_1}\eta = \int_{\gamma_2}\eta$. So on each homology class $[\gamma]$, $\int_\gamma \eta$ is a conserved vector. We call η the momentum flux.*

4 The mean curvature half density

The mean curvature half density of an immersion $f\colon M \to \mathbb{R}^3$ is

$$H|df|. \tag{4.1}$$

Let $\lambda \in C^\infty(M, \mathbb{H})$ with $D_f\lambda = \rho\lambda$. Thus

$$(\lambda, \lambda) = \bar{\lambda} df \lambda$$

is a closed 1-form and therefore we locally can find an immersions $\tilde{f}$ with

$$d\tilde{f} = \bar{\lambda} df \lambda.$$

$\tilde{f}$ and f induce the same conformal structure on M. It is a well known fact **[11]** that the mean curvature half density transforms as

$$\tilde{H}|d\tilde{f}| = (\rho + H)|df|. \tag{4.2}$$

Now we are going to generalize (4.2) for an arbitrary deformation $\tilde{f}$.
Let $\tilde{f}\colon M \to \mathbb{R}^3$ with $d\tilde{f} = \bar{\lambda}(df + \tau)\lambda$ and $\tilde{N} = \lambda^{-1}N\lambda$, $\tilde{\omega} := d\tilde{f} \circ \tilde{Q}$.

Theorem 18. *The mean curvature half-density transforms as*

$$\tilde{H}|d\tilde{f}| = \sqrt{\det(I+T)}\Big((H+\rho)|df| + \frac{1}{|df|}(\langle \tau \wedge (d\lambda\lambda^{-1})_-\rangle - \frac{1}{2}\langle \tau \wedge df \circ J\tilde{Q}\rangle)\Big), \tag{4.3}$$

and the infinitesimal version is

$$(H\dot{|df|}) = \dot{\rho}|df| - \frac{1}{2|df|}\langle *\dot{\tau} \wedge \omega\rangle. \tag{4.4}$$

Proof. $d\tilde{N} = \bar{\lambda}(\tilde{H}(df + \tau) + df \circ U)\lambda$, where $U := \tilde{Q} + T\tilde{Q}$. From $\tilde{N} = \lambda^{-1}N\lambda$ one gets $d\tilde{N} = \lambda^{-1}(Hdf + \omega + 2(Nd\lambda\lambda^{-1})_-)\lambda$, hence

$$|\lambda|^2(\tilde{H}(df + \tau) + df \circ U) = Hdf + \omega + 2(Nd\lambda\lambda^{-1})_-.$$

From $\frac{1}{2} * df \wedge df = -|df|^2$ and wedging the previous equation with $\frac{1}{2} * df$ from the left we obtain by a type argument

$$-|\lambda|^2 \tilde{H}|df|^2 - \frac{1}{2}\mathrm{tr}(U)|df|^2 = -H|df|^2 - \langle (*df \wedge (Nd\lambda\lambda^{-1})_-)\rangle,$$

where we used $\mathrm{Re}(\frac{1}{2} * df \wedge df \circ U) = -\frac{1}{2}\mathrm{tr}(U)|df|^2$ and $\mathrm{Re}(*df \wedge (Nd\lambda\lambda^{-1})_-) = -\langle (*df \wedge (Nd\lambda\lambda^{-1})_-)\rangle$. Taking the real part of the Dirac equation (6.4) we obtain

$$\langle df \wedge (d\lambda\lambda^{-1})_-\rangle + \langle \tau \wedge (d\lambda\lambda^{-1})_-\rangle = \rho|df|^2,$$

or

$$\langle *df \wedge (Nd\lambda\lambda^{-1})_-\rangle = \rho|df|^2 + \langle \tau \wedge (d\lambda\lambda^{-1})_-\rangle,$$

and therefore

$$|\lambda|^2 \tilde{H}|df|^2 = (H + \rho - \frac{1}{2}\mathrm{tr}(U))|df|^2 + \langle \tau \wedge (d\lambda\lambda^{-1})_-\rangle.$$

Furthermore, the area element transforms as $|d\tilde{f}|^2 = |\lambda|^4|df|^2 \det(I+T)$. Note that $\mathrm{tr}(U) = \mathrm{tr}(T\tilde{Q}) = \langle T \wedge J\tilde{Q}\rangle = \langle \tau \wedge df \circ J\tilde{Q}\rangle$, hence

$$\tilde{H}|d\tilde{f}| = \sqrt{\det(I+T)}\Big((H+\rho)|df| + \frac{1}{|df|}(\langle \tau \wedge (d\lambda\lambda^{-1})_-\rangle - \frac{1}{2}\langle \tau \wedge df \circ J\tilde{Q}\rangle)\Big).$$

Let us now compute the infinitesimal version.

$$(H\dot{|df|}) = \dot{\rho}|df| + \frac{1}{2|df|}\langle \dot{\tau} \wedge *\omega\rangle = \dot{\rho}|df| - \frac{1}{2|df|}\langle *\dot{\tau} \wedge \omega\rangle.$$

□

5 The Rivin-Schlenker Schläfli formula

In this section we will give a new proof for a special case of the Rivin-Schlenker Schläfli formula **[15]**.

Lemma 10. *Let* $\begin{pmatrix} \lambda_t \\ \rho_t \\ \tau_t \end{pmatrix} \in \mathcal{M}$ *a deformation of our reference immersion* $f\colon M \to \mathbb{R}^3$.

(1) The infinitesimal change of the metric is

$$\dot{g}(X,Y) = 4\mathrm{Re}\dot{\lambda}\langle X,Y\rangle + 2\langle \dot{T}X, Y\rangle.$$

(2) The infinitesimal change of the area element is

$$\dot{|df|^2} = 4\mathrm{Re}\dot{\lambda}|df|^2.$$

Proof. (1) Let $df_t = \bar{\lambda}_t(df + \tau_t)\lambda_t = \frac{\bar{\lambda}_t}{|\lambda_t|}(|\lambda_t|^2(df+\tau_t)\frac{\lambda_t}{|\lambda_t|}$ and

$$g_t(X,Y) = \langle df_t(X), df_t(Y)\rangle = |\lambda_t|^4\langle (I+T_t)X, (I+T_t)Y\rangle$$

$\Rightarrow$

$$\dot{g}(X,Y) = 4\mathrm{Re}\dot{\lambda}\langle X,Y\rangle + 2\langle \dot{T}X, Y\rangle.$$

(2) $|df_t|^2 = |\lambda_t|^4 \det(I+T_t)|df|^2 \Rightarrow \dot{|df|^2} = 4\mathrm{Re}\dot{\lambda}|df|^2.$

□

THEOREM 19. *Let M be two dimensional Riemannian manifold without boundary and let* $\begin{pmatrix} \dot\lambda \\ \dot\rho \\ \dot\tau \end{pmatrix} \in T_f\mathcal{M}$ *then*

$$\int_M (\dot H + \frac{1}{2}\langle \dot I, II\rangle)|df|^2 = 0.$$

PROOF. Since $\begin{pmatrix} \dot\lambda \\ \dot\rho \\ \dot\tau \end{pmatrix} \in T_f\mathcal{M}$ we get $-df \wedge d\dot\lambda = \dot\rho|df|^2 - \frac{1}{2}d\dot\tau$ and

$$\int_M \dot\rho|df|^2 = 0.$$

In the previous section (4.4) we computed the infinitesimal change of the mean curvature half density $(H\dot{|df|}) = \dot\rho|df| - \frac{1}{2|df|}\langle *\dot\tau \wedge \omega\rangle$.

$$\int_M (H\dot{|df|})|df| = \int_M \dot\rho|df|^2 - \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle = -\int_M \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle,$$

therefore

$$\begin{aligned} 0 &= \int_M (H\dot{|df|})|df| + \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle \\ &= \int_M \dot H|df|^2 + \frac{1}{2}H\dot{|df|^2} + \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle \\ &= \int_M \dot H|df|^2 + 2H\mathrm{Re}\dot\lambda|df|^2 + \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle. \end{aligned}$$

Now let us compute $\frac{1}{2}\langle \dot I, II\rangle|df|^2(X, JX)$ for the eigenbasis (X, JX) of the shape operator.

$$\begin{aligned} \frac{1}{2}\langle \dot I, II\rangle|df|^2(X, JX) &= \frac{1}{4}\mathrm{tr}(\begin{pmatrix} 4\mathrm{Re}\dot\lambda + 2\langle \dot T X, X\rangle & 2\langle \dot T X, JX\rangle \\ 2\langle \dot T X, JX\rangle & 4\mathrm{Re}\dot\lambda - 2\langle \dot T X, X\rangle \end{pmatrix}\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}) \\ &= 2\mathrm{Re}\dot\lambda H + \frac{1}{2}(k_1 - \kappa_2)\langle \dot T X, X\rangle. \end{aligned}$$

Finally, one can easily verify $\frac{1}{2}\langle *\dot\tau \wedge \omega\rangle(X, JX) = \frac{1}{2}(\kappa_1 - \kappa_2)\langle \dot T X, X\rangle$. ☐

COROLLARY 10. *For conformal surface deformations we even obtain*

$$\int_M (H\dot{|df|})|df| = 0.$$

PROOF. $0 = \int_M (\dot H + \frac{1}{2}\langle \dot I, II\rangle)|df|^2 = \int_M (H\dot{|df|})|df| + \int_M \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle = \int_M (H\dot{|df|})|df|$. ☐

6 THE MEAN CURVATURE FUNCTIONAL

Now we will investigate the mean curvature functional

$$\mathcal{H} := \int_M H|df|^2.$$

We obtain

$$\begin{aligned} \dot{\mathcal{H}} &= \int_M (H\dot{|df|})|df| + H|df|\dot{|df|} \\ &= \int_M \dot\rho|df|^2 - \frac{1}{2}\langle *\dot\tau \wedge \omega\rangle + 2H\mathrm{Re}\dot\lambda|df|^2 \\ &= \int_M -\frac{1}{2}\langle *\dot\tau \wedge \omega\rangle + 2H\mathrm{Re}\dot\lambda|df|^2, \end{aligned}$$

and therefore

$$\operatorname{grad}\mathcal{H} = \begin{pmatrix} 2H|df|^2 \\ 0 \\ -\frac{1}{2}\omega \end{pmatrix}.$$

Theorem 20. *$(\lambda, \rho, \tau) \in \mathcal{M}$ is a critical point of $\mathcal{H}$ if and only if the momentum flux NdN is closed. In particular, all critical points have a vanishing Gaussian curvature.*

Proof. We apply the Noether Theorem (3.2) with $g_1 := 2H$, $g_2 = g_3 = 0$, $Y = 0$ and $\partial_\tau\mathcal{H} = -\frac{1}{2}\omega$. The momentum flux $\eta := \frac{g_2}{2}df + \frac{g_1}{2} * df + 2 * \partial_\tau\mathcal{F} + (*dg_3 - \langle Y, .\rangle)N$ simplifies to

$$\eta = H * df - *\omega = HNdf + N\omega = NdN.$$

Further $d(NdN) = dN \wedge dN = 2KN|df|^2$. □

7 Willmore surfaces

In this section we will investigate Willmore surfaces and derive conservation laws.

$$\mathcal{W} = \int_M (H|df|)^2.$$

We compute the gradient of $\mathcal{W}$.

$$\dot{\mathcal{W}} = \int_M 2H|df|(H\dot{|df|}) = \int_M 2H\dot{\rho}|df|^2 - \langle *\dot{\tau} \wedge H\omega\rangle.$$

this shows

$$\operatorname{grad}\mathcal{W} = \begin{pmatrix} 0 \\ 2H|df|^2 \\ -H\omega \end{pmatrix}.$$

Theorem 21. *f is a Willmore surface if and only if the momentum flux*

$$\eta = *dHN - H * \omega \tag{7.1}$$

is closed. Moreover, if η is closed, then the following forms are closed as well:

(1) $f\eta + \omega$ the stretch-torque flux.

(2) $-2\langle f, \eta\rangle f + \langle f, f\rangle\eta - 2f \times \omega$ the Möbius flux.

Proof. Applying the Noether Theorem (3.2), we obtain $0 = \partial_\lambda\mathcal{W}$, hence $g_1 = g_2 = 0$ and $Y = 0$. Further $\partial_\rho\mathcal{W} = 2H|df|^2$ and therefore $g_3 = 2H$. Finally, $\partial_\tau\mathcal{W} = -H\omega$ and one gets $\eta = *dHN - H * \omega$.

(1) $d(f\eta+\omega) = df \wedge \eta + d\omega = df \wedge (*dHN - H*\omega) + d(dN - Hdf) = df \wedge *dHN - dH \wedge df = dH \wedge df - dH \wedge df = 0$.

(2) Since $f\eta + \omega = -\langle f, \eta\rangle + f \times \eta + \omega$, one concludes that the real part $-\langle f, \eta\rangle$ must be closed as well.

$$\begin{aligned} &d(-2\langle f, \eta\rangle f + \langle f, f\rangle\eta - 2f \times \omega) \\ &= 2\langle f, \eta\rangle \wedge df + 2\langle f, df\rangle \wedge \eta - 2f \times d\omega \\ &= 2\langle f, (*dHN - H * \omega)\rangle \wedge df + 2\langle f, df\rangle \wedge (*dHN - H * \omega) + 2f \times dH \wedge df \\ &= 2dH \wedge (\langle f, *df\rangle N - \langle f, N\rangle * df + f \times df) - 2H(\langle f, *\omega\rangle \wedge df + \langle f, df\rangle \wedge *\omega). \end{aligned}$$

Note that $f \times df = -\langle f, *df\rangle N + \langle f, N\rangle * df$ whereby the first term in sum cancels. Let us decompose $f = df(Y) + \langle f, N\rangle N$. Then

$$\begin{aligned}
&\langle f, *\omega\rangle \wedge df(X, JX) \\
&= \langle Y, QJX\rangle df(JX) + \langle Y, QX\rangle df(X) \\
&= df(\langle QY, X\rangle X + \langle QY, JX\rangle JX) \\
&= df(QY).
\end{aligned}$$

Similarly one computes $\langle f, df\rangle \wedge *\omega(X, JX) = -df(QY) \Rightarrow$ the Möbius flux is closed.

□

Note that the momentum flux $\eta = *dHN - H * \omega$ of a Willmore surface already appeared in the Phd thesis of Jörg Richter [**14**]. Soon we will give a really simple prove for the fact that the Möbius flux and the momentum flux are from the Möbius geometric point of view the same. But first we prefer a computational approach.

Lemma 11. *Let $\tilde{f} := f^{-1}$, then the mean curvature and the Hopf differential transform as*

$$\tilde{H} = H|f|^2 - 2\langle f, N\rangle$$

and

$$\tilde{\omega} = f\omega f^{-1}.$$

Proof. Since $d\tilde{f} = -f^{-1}dff^{-1}$ and $\tilde{N} = fNf^{-1}$ one obtains $*d\tilde{f} = \tilde{N}d\tilde{f}$. A straightforward computation shows

$$d\tilde{N} = (H|f|^2 - 2\langle f, N\rangle)d\tilde{f} + f\omega f^{-1}.$$

□

Theorem 22. *The momentum flux of $\tilde{f} = f^{-1}$ is minus the Möbius flux of f. In particular, the Möbius flux is closed if and only if the momentum flux is closed.*

Proof.

$$\begin{aligned}
&* d\tilde{H}\tilde{N} - \tilde{H} * \tilde{\omega} \\
&= (|f|^2 * dH - 2\langle f, *\omega\rangle)fNf^{-1} - (H|f|^2 - 2\langle f, N\rangle)f * \omega f^{-1} \\
&= |f|^2 f(*dHN - H * \omega)f^{-1} + f(2\langle f, N\rangle * \omega - 2\langle f, *\omega\rangle N)f^{-1} \\
&= f(|f|^2\eta - 2f \times \omega)f^{-1} \\
&= -f\eta f + \frac{2}{|f|^2} f(f \times \omega)f \\
&= -(-\langle f, \eta\rangle + f \times \eta)f + \frac{2}{|f|^2}(f \times (f \times \omega))f \\
&= 2\langle f, \eta\rangle - |f|^2\eta + \frac{2}{|f|^2}(f\langle f, \omega\rangle - \omega|f|^2)f \\
&= 2\langle f, \eta\rangle - |f|^2\eta - 2\omega \times f.
\end{aligned}$$

□

Up to now it is not clear how we have found the stretch-torque flux and the Möbius flux. Even the momentum flux, which we obtained from the Noether Theorem, haven't been elaborated in detail yet. Note, each $\begin{pmatrix} \dot{\lambda} \\ \dot{\rho} \\ \dot{\tau} \end{pmatrix} \in T_f\mathcal{M}$ corresponds to an infinitesimal deformation $\dot{f}$, where $\dot{f}$ is defined up to translations. The translation invariant description $\mathcal{M}$ of the space of immersed surfaces is the reason for the existence of the momentum flux. To appreciate our approach and to justify the name Noether Theorem, we will not describe our surfaces with the geometric data (λ, ρ, τ) but with the surface f itself. First we consider the Möbius invariant Willmore functional

$$\mathcal{W}(f) = \int_M (H^2 - K)|df|^2,$$

which has the same critical points as $\mathcal{W}(f) = \int_M H^2|df|^2$. A quite lengthy computation shows, that critical points f of $\mathcal{W}$ are characterized by

$$0 = \int_{\partial N} \langle \dot{f}, *dHN - H * \omega \rangle + \langle \dot{N}, *\omega \rangle - \int_N \langle \dot{f}, (d * dH - H\langle *\omega \wedge \omega \rangle)N \rangle.$$

for any compact 2-dimensional submanifold $N \subset M$ and all variations f_t of f with compact support on N.

So f is a Willmore surface if and only if the 2-form $d * dH - H\langle *\omega \wedge \omega \rangle$ vanishes on M. Let f be a Willmore surface, then for any 2-dimensional compact submanifold $N \subset M$ the restriction f on N is still a Willmore surface. Let $v \in \mathbb{R}^3$ be a constant vector and $f_t = f + tv$ a family of translations of f all having their compact support on N, then obviously f_t are all Willmore surfaces, which all have the same Willmore-energy as f. Therefore

$$0 = \frac{d}{dt}\mathcal{W}_N(f) = \frac{d}{dt}\mathcal{W}_N(f_t) = \int_{\partial N} \langle \dot{f}, *dHN - H * \omega \rangle + \langle \dot{N}, *\omega \rangle.$$

Note that $\dot{f} = v$ and $\dot{N} = 0$. We obtain for all compact 2-dimensional $N \subset M$ and all $v \in \mathbb{R}^3$

$$0 = \int_{\partial N} \langle v, \eta \rangle = \int_N \langle v, d\eta \rangle,$$

which shows that η is closed on M. The Willmore functional is also invariant under rotations. An infinitesimal rotation is given by $\dot{f} = v \times f$ and $\dot{N} = v \times N$ for a constant vector $v \in \mathbb{R}^3$. As before, we get

$$0 = \int_{\partial N} \langle v \times f, \eta \rangle + \langle v \times N, *\omega \rangle = \int_{\partial N} \langle v, f \times \eta + \omega \rangle.$$

This shows that $f \times \eta + \omega$ is closed. The scaling invariance implies that $\langle f, \eta \rangle$ is closed, we will omit the details. Let us check what the inversion invariance will bring us. One can show that the infinitesimal inversions are described by $\dot{f} = faf = -2\langle f, a \rangle f + a\langle f, f \rangle$, for a constant vector $a \in \mathbb{R}^3$. The infinitesimal change of the Gauss map is $\dot{N} = 2N \times (a \times f)$, so

$$\begin{aligned}
0 &= \int_{\partial N} \langle \dot{f}, \eta \rangle + \langle \dot{N}, *\omega \rangle \\
&= \int_{\partial N} \langle -2\langle f, a \rangle f + a\langle f, f \rangle, \eta \rangle + \langle 2N \times (a \times f), *\omega \rangle \\
&= \int_{\partial N} -2\langle f, a \rangle \langle f, \eta \rangle + \langle a, \langle f, f \rangle \eta \rangle - 2\langle a, f \times \omega \rangle \\
&= \int_{\partial N} \langle a, -2\langle f, \eta \rangle f + \langle f, f \rangle \eta - 2f \times \omega \rangle,
\end{aligned}$$

which shows that $-2\langle f, \eta \rangle f + \langle f, f \rangle \eta - 2f \times \omega$ is closed.
This approach shows clearly that the momentum flux is a result of varying the origin, while the Möbius flux results by varying the infinity point. But there is no difference from a Möbius geometric point of view, which implies the equality of the Möbius flux and the momentum flux.

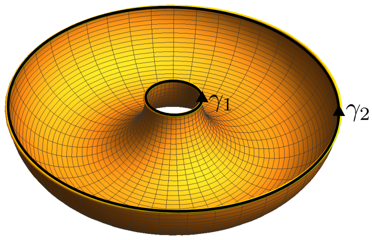

FIGURE 1. Half a torus.

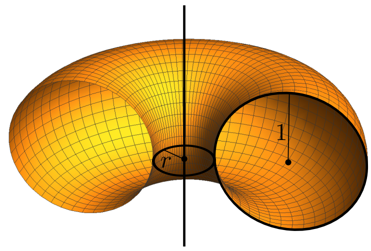

FIGURE 2. The other half.

Now we give an application of our momentum flux.

THEOREM 23. *Among all tori of revolution with a circle of radius 1 as meridian, the Clifford torus is the only Willmore torus.*

PROOF. Let f be such a Willmore torus. We want to show $r = \sqrt{2} - 1$. Let e_3 be a unit vector, which spans the rotation axle. We consider the two homologous cycles γ_1 and γ_2. Since f is Willmore, the momentum flux η is closed. The mean curvature achieves its minimum along γ_1 and its maximum along γ_2 and therefore η simplifies to $\eta = *dHN - H * \omega = -H * \omega$. Further $H \circ \gamma_1 = \frac{-\frac{1}{r}+1}{2}$ and $H \circ \gamma_2 = \frac{\frac{1}{r+2}+1}{2}$. For eigendirections X, JX of the shape operator we get $\omega(X) = \frac{\kappa_1-\kappa_2}{2} df(X)$ and $\omega(JX) = \frac{\kappa_2-\kappa_1}{2} df(JX)$. Hence,

$$\begin{aligned}
\gamma_1^*\eta &= -H(\gamma_1) * \omega(\gamma_1') \\
&= -\frac{-\frac{1}{r}+1}{2} r\omega(JX) \\
&= -\frac{-\frac{1}{r}+1}{2} r \frac{-\frac{1}{r}-1}{2} - e_3 \\
&= -\frac{1}{4}(r - \frac{1}{r})e_3
\end{aligned}$$

and

$$\begin{aligned}
\gamma_2^*\eta &= -H(\gamma_2) * \omega(\gamma_2') \\
&= -\frac{\frac{1}{r+2}+1}{2}(r+2)\omega(JX) \\
&= -\frac{\frac{1}{r+2}+1}{2}(r+2)\frac{\frac{1}{r+2}-1}{2} e_3 \\
&= \frac{1}{4}(r + 2 - \frac{1}{r+2})e_3.
\end{aligned}$$

$\int_{\gamma_1} \eta = \int_{\gamma_2} \eta$ implies $-(r - \frac{1}{r}) = r + 2 - \frac{1}{r+2}$ or

$$r = \sqrt{2} - 1.$$

$\square$

8 MINIMAL SURFACES AND CONSTANT MEAN CURVATURE SURFACES

Now we are going to investigate minimal surfaces. Recall (10), the area element of $\tilde{f}$ changes by

$$|d\tilde{f}|^2 = |\lambda|^4 \det(I + T)|df|^2,$$

and therefore

$$|\dot{df}|^2 = 4\mathrm{Re}\dot{\lambda}|df|^2.$$

The gradient of the area functional is given by

$$\operatorname{grad}\mathcal{A} = \begin{pmatrix} 4|df|^2 \\ 0 \\ 0 \end{pmatrix}.$$

The Noether Theorem immediately implies the following

Theorem 24. *f is a minimal surface if and only if the momentum flux $*df$ is closed. Moreover, if $*df$ is closed, then the torque flux $f \times *df$ is also closed.*

Constant mean curvature surfaces are critical points of the volume functional under all variations which keep the area fixed. The volume functional is only defined for closed surfaces. So let M be a closed surface, then

$$\begin{aligned}\dot{V} &= \int_M \langle \dot{f}, N\rangle \\ &= \frac{1}{2}\int_M \langle \dot{f}, df \wedge df\rangle \\ &= \frac{1}{2}\int_M \langle \dot{f}, d(f df)\rangle \\ &= \frac{1}{2}\int_M -d\langle \dot{f}, f df\rangle + \langle d\dot{f} \wedge f df\rangle \\ &= \frac{1}{2}\int_M \langle d\dot{f} \wedge f df\rangle\end{aligned}$$

Let $d\dot{f} = \bar{\dot{\lambda}} df + df\dot{\lambda} + \dot{\tau}$, then

$$\dot{V} = \int_M \langle (\bar{\dot{\lambda}} df + df\dot{\lambda} + \dot{\tau}) \wedge f df\rangle.$$

$$\langle \bar{\dot{\lambda}} df \wedge f df\rangle = -\mathrm{Re}(df\dot{\lambda} \wedge f df) = \langle \dot{\lambda}, df \wedge df f\rangle = 2\langle \dot{\lambda}, Nf\rangle |df|^2,$$

$$\langle df\dot{\lambda} \wedge f df\rangle = -\mathrm{Re}(\bar{\dot{\lambda}} df \wedge f df) = -\langle \dot{\lambda}, df \wedge f df\rangle = \langle \dot{\lambda}, -2\langle f, N\rangle |df|^2\rangle.$$

Let $f = df(Z) + \langle f, N\rangle N$. Then $f df = df(Z) df + \langle f, N\rangle * df$ and therefore $\langle \dot{\tau} \wedge f df\rangle = 0$.

Finally,

$$\dot{V} = \int_M \langle \dot{\lambda}, Nf - \langle f, N\rangle\rangle |df|^2 = \int_M \langle \dot{\lambda}, -2\langle f, N\rangle + N \times f\rangle |df|^2,$$

$$\operatorname{grad} V = \begin{pmatrix} (-2\langle f, N\rangle + N \times f)|df|^2 \\ 0 \\ 0 \end{pmatrix}$$

Now we want to characterize constant mean curvature surfaces through the corresponding momentum flux. Recall, $f\colon M \to \mathbb{R}^3$ is a CMC surface if and only if $\operatorname{grad} V + a\operatorname{grad}\mathcal{A} \in N_f\mathcal{M}$, where $a \in \mathbb{R}$ is a Lagrangian multiplier.

$$\operatorname{grad} V + a\operatorname{grad}\mathcal{A} = \begin{pmatrix} (-2\langle f, N\rangle + N \times f + 4a)|df|^2 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 25. *f is a CMC surface if and only if*

$$\eta = (2a - \langle f, N\rangle) * df + \langle f, *df\rangle N$$

is closed. Further, η is closed if and only if $H = \frac{1}{2a}$.

Proof. We apply the Noether Theorem with $g_1 = 4a - 2\langle f, N\rangle$, $g_2 = g_3 = 0$, $\partial_\tau \mathcal{F} = 0$ and $df(Y) = N \times f$. For our momentum flux $\eta := \frac{g_2}{2} df + \frac{g_1}{2} * df + 2 * \partial_\tau \mathcal{F} + (*dg_3 - \langle Y, .\rangle)N$ we obtain

$$\eta = (2a - \langle f, N\rangle) * df - \langle Y, \rangle N.$$

Note, for any $X \in \Gamma(TM)$ one gets $\langle Y, X\rangle = \langle df(Y), df(X)\rangle = \langle N \times f, df(X)\rangle = -\langle f, *df(X)\rangle$. Thus,

$$\eta = (2a - \langle f, N\rangle) * df + \langle f, *df\rangle N.$$

Now one can easily compute

$$d\eta = (4aH - 2)|df|^2 N + \langle f, *df\rangle \wedge dN - \langle f, dN\rangle * df = (4aH - 2)|df|^2 N.$$

So η is closed if and only if $H = \frac{1}{2a}$. □

9 The normal space of space curves

Similar to the space of immersions of a Riemann surface M into $\mathbb{R}^3$, which we described by the geometric data (λ, ρ, τ), we will describe the space of space curves in terms of their curvature functions. We investigate space curves with prescribed end points and prescribed frames at the end points and give an Euclidean invariant characterization. Let κ_1, κ_2, τ be smooth functions on the interval $[0, L]$. We consider

$$A = \begin{pmatrix} 0 & -\kappa_1 & -\kappa_2 & 1 \\ \kappa_1 & 0 & -\tau & 0 \\ \kappa_2 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let G be the group of euclidean motions. A framed curve $F \in G$ has the form $F = \begin{pmatrix} T & N & B & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where $(T, N, B)\colon [0.L] \to \mathrm{SO}(3)$ and $\gamma\colon [0, L] \to \mathbb{R}^3$. Then

$$F' = FA$$

is equivalent to

$$\begin{aligned} \gamma' &= T \\ T' &= \quad \kappa_1 N + \kappa_2 B \\ N' &= -\kappa_1 T \qquad + \tau B \\ B' &= -\kappa_2 T - \tau N. \end{aligned}$$

$(\kappa_1, \kappa_2, \tau)$ are the curvature functions of γ with respect to the frame (T, N, B). Thus, the space of space curves with prescribed and points and prescribed frames at the end points is given by

$$\mathcal{M} := \{ \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \tau \end{pmatrix} \in C^\infty([0, L], \mathbb{R}^3) \mid F' = FA \quad \text{with} \quad (F(0), F(L)) \quad \text{are both fixed}\}. \tag{9.1}$$

Let γ be an arc length parametrized curve with curvature functions $(\kappa_1, \kappa_2, \tau)$. Let $(\kappa_{1,t}, \kappa_{2,t}, \tau_t) \in \mathcal{M}$ a variation of γ. Let $F_t' = F_t A_t$ then we obtain

$$(\dot{F}F^{-1})' = \dot{F}' F^{-1} - \dot{F}F^{-1}F'F^{-1} = (\dot{F}A + F\dot{A})F^{-1} - \dot{F}AF^{-1} = F\dot{A}F^{-1}.$$

The condition that the framed curves F_t stay fixed at the end points translates to

$$0 = \dot{F}(L)F^{-1}(L) - \dot{F}(0)F^{-1}(0) = \int_0^L (\dot{F}F^{-1})' = \int_0^L F\dot{A}F^{-1}, \tag{9.2}$$

which is equivalent to

$$\int_0^L \dot{\tau}T - \dot{\kappa}_2 N + \dot{\kappa}_1 B ds = 0$$

and

$$\int_0^L (\dot{\tau}T - \dot{\kappa_2}N + \dot{\kappa_1}B) \times \gamma ds = 0.$$

Here the dot stands for the time derivative at $t = 0$. The tangent space is given by

$$T_{(\kappa_1,\kappa_2,\tau)}\mathcal{M} = \left\{ \begin{pmatrix} \dot{\kappa_1} \\ \dot{\kappa_2} \\ \dot{\tau} \end{pmatrix} \in C^\infty([0,L],\mathbb{R}^3) \;\Big|\; \begin{pmatrix} \int_0^L \dot{\tau}T - \dot{\kappa_2}N + \dot{\kappa_1}B \\ \int_0^L (\dot{\tau}T - \dot{\kappa_2}N + \dot{\kappa_1}B) \times \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Theorem 26. *The normal space of space curves is given by*

$$N_{(\kappa_1,\kappa_2,\tau)}\mathcal{M} = \left\{ \left(\begin{array}{c} \langle B, \mathbf{a} + \gamma \times \mathbf{b} \rangle \\ \langle -N, \mathbf{a} + \gamma \times \mathbf{b} \rangle \\ \langle T, \mathbf{a} + \gamma \times \mathbf{b} \rangle \end{array} \right) \mid \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \right\}. \tag{9.3}$$

Proof. $\begin{pmatrix} \dot{\kappa_1} \\ \dot{\kappa_2} \\ \dot{\tau} \end{pmatrix} \in T_{(\kappa_1,\kappa_2,\tau)}\mathcal{M}$ if and only if for all $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$0 = \int_0^L \langle \mathbf{a}, \dot{\tau}T - \dot{\kappa_2}N + \dot{\kappa_1}B \rangle = \int_0^L \langle \begin{pmatrix} \dot{\kappa_1} \\ \dot{\kappa_2} \\ \dot{\tau} \end{pmatrix}, \begin{pmatrix} \langle B, \mathbf{a} \rangle \\ -\langle N, \mathbf{a} \rangle \\ \langle T, \mathbf{a} \rangle \end{pmatrix} \rangle ds$$

and

$$0 = \int_0^L \langle \mathbf{b}, (\dot{\tau}T - \dot{\kappa_2}N + \dot{\kappa_1}B) \times \gamma \rangle = \int_0^L \langle \begin{pmatrix} \dot{\kappa_1} \\ \dot{\kappa_2} \\ \dot{\tau} \end{pmatrix}, \begin{pmatrix} \langle B, \mathbf{b} \times \gamma \rangle \\ -\langle N, \mathbf{b} \times \gamma \rangle \\ \langle T, \mathbf{b} \times \gamma \rangle \end{pmatrix} \rangle ds.$$

□

The normal space can also be expressed by the linear combination of the six $\mathbb{R}^3$ valued functions

$$\begin{pmatrix} \langle B, e_1 \rangle \\ \langle -N, e_1 \rangle \\ \langle T, e_1 \rangle \end{pmatrix}, \begin{pmatrix} \langle B, e_2 \rangle \\ \langle -N, e_2 \rangle \\ \langle T, e_2 \rangle) \end{pmatrix}, \begin{pmatrix} \langle B, e_3 \rangle \\ \langle -N, e_3 \rangle \\ \langle T, e_3 \rangle \end{pmatrix}, \begin{pmatrix} \langle B \times \gamma, e_1 \rangle \\ \langle -N \times \gamma, e_1 \rangle \\ \langle T \times \gamma, e_1 \rangle \end{pmatrix}, \begin{pmatrix} \langle B \times \gamma, e_2 \rangle \\ \langle -N \times \gamma, e_2 \rangle \\ \langle T \times \gamma, e_2 \rangle \end{pmatrix}, \begin{pmatrix} \langle B \times \gamma, e_3 \rangle \\ \langle -N \times \gamma, e_3 \rangle \\ \langle T \times \gamma, e_3 \rangle \end{pmatrix}.$$

The functions are linear independent unless there exists $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ such that

$$\mathbf{a} + \gamma \times \mathbf{b} = 0.$$

Thus, the normal space has constant rank away from plane curves. By the infinite dimensional version of the implicit function theorem we could say that $\mathcal{M}$ without plane curves defines a manifold.

Let us shortly discuss the space of plane curves with prescribed end points and prescribed frames at the end points. A plane curve γ could be seen as a space curve with $\kappa_2 = \tau = 0$. Therefore we can apply the results which we derived for space curves. Let $\kappa \in \mathcal{M}$ be the curvature function of γ. Then the tangent and normal spaces are

$$T_\kappa\mathcal{M} = \left\{ \dot{\kappa} \;\Big|\; \int_0^L \dot{\kappa} = 0, \int_0^L \dot{\kappa}\gamma = 0 \right\}$$

and

$$N_\kappa\mathcal{M} = \mathrm{span}(1, \gamma_1, \gamma_2).$$

Obviously the three functions $(1, \gamma_1, \gamma_2)$ are linear independent unless the curve γ is a straight line. Consequently the space of all closed regular curves is a manifold. To motivate the following section and to appreciate our approach we discuss the well studied elastic curves. The elastic energy of a plane curve is defined by

$$E(\gamma) = \frac{1}{2}\int_0^L k^2 ds.$$

Elastic curves are critical points of E under all variations with fixed end points, fixed frames at the end points and fixed length. So instead of minimizing the elastic energy of γ we could look at

$$\hat{E}(\kappa) := \frac{1}{2}\int_0^L k^2 ds, \tag{9.4}$$

where κ is the curvature function of γ. We are now looking for a critical points κ of $\hat{E}$ under all variations in $\mathcal{M}$. Obviously

$$\operatorname{grad}\hat{E}(\kappa) = \kappa$$

and κ is a critical point of $\hat{E}$ if and only if $\operatorname{grad}\hat{E}(\kappa) \in N_\kappa\mathcal{M}$. So let $\mathbf{a} \in \mathbb{R}$ and $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ with

$$\kappa = \operatorname{grad}\hat{E}(\kappa) = \mathbf{a} + b_1\gamma_1 + b_2\gamma_2 = \mathbf{a} + \langle \mathbf{b}, \gamma\rangle.$$

This shows that κ is proportional to distance of γ to the axis ib. This is a well known characterization for elastic curves.

10 Noether Theorem on space curves

In this section we want to generalize the idea behind the example (9.4). We consider a geometric functional, i.e.

$$\mathcal{F}\colon \mathcal{M} \to \mathbb{R}, \quad \mathcal{F}(\kappa_1, \kappa_2, \tau) = \int_0^L \mathcal{L}(\kappa_1(s), \kappa_2(s), \tau(s))ds,$$

such that for each smooth variation $\begin{pmatrix}\kappa_{1,t}\\ \kappa_{2,t}\\ \tau_t\end{pmatrix} \in \mathcal{M}$ of $\begin{pmatrix}\kappa_1\\ \kappa_2\\ \tau\end{pmatrix} \in \mathcal{M}$

$$\frac{d}{dt}_{t=0}\mathcal{F} \circ \begin{pmatrix}\kappa_{1,t}\\ \kappa_{2,t}\\ \tau_t\end{pmatrix}$$

exists.

$$\operatorname{grad}\mathcal{F} = \begin{pmatrix}\partial_{\kappa_1}\mathcal{F}\\ \partial_{\kappa_2}\mathcal{F}\\ \partial_\tau\mathcal{F}\end{pmatrix}$$

is defined by

$$\begin{aligned}\frac{d}{dt}_{t=0}\mathcal{F} \circ \begin{pmatrix}\kappa_{1,t}\\ \kappa_{2,t}\\ \tau_t\end{pmatrix} &=: \langle \operatorname{grad}\mathcal{F}(\kappa_1, \kappa_2, \tau), \begin{pmatrix}\dot{\kappa_1}\\ \dot{\kappa_2}\\ \dot{\tau}\end{pmatrix}\rangle_{L^2}\\ &= \int_0^L \langle \begin{pmatrix}\partial_{\kappa_1}\mathcal{F}\\ \partial_{\kappa_2}\mathcal{F}\\ \partial_\tau\mathcal{F}\end{pmatrix}, \begin{pmatrix}\dot{\kappa_1}\\ \dot{\kappa_2}\\ \dot{\tau}\end{pmatrix}\rangle ds.\end{aligned}$$

$\begin{pmatrix}\kappa_1\\ \kappa_2\\ \tau\end{pmatrix} \in \mathcal{M}$ is a critical point of $\mathcal{F}$ if and only if

$$\operatorname{grad}\mathcal{F}(\kappa_1, \kappa_2, \tau) \in N_{(\kappa_1, \kappa_2, \tau)}\mathcal{M}.$$

Let $\begin{pmatrix}\kappa_1\\ \kappa_2\\ \tau\end{pmatrix} \in \mathcal{M}$ be a critical point of $\mathcal{F}$. From

$$\gamma \times \mathbf{b} = \langle \gamma \times \mathbf{b}, T\rangle T + \langle \gamma \times \mathbf{b}, N\rangle N + \langle \gamma \times \mathbf{b}, B\rangle B$$

we obtain

$$\gamma \times \mathbf{b} + \mathbf{a} = (\partial_\tau \mathcal{F})T - (\partial\kappa_2\mathcal{F})N + (\partial\kappa_1\mathcal{F})B.$$

The essential information is in $\mathbf{b}$, so taking the derivative yields

$$T \times \mathbf{b} = ((-\partial_{\kappa_2}\mathcal{F})' + \kappa_1(\partial_\tau\mathcal{F}) - \tau(\partial_{\kappa_1}\mathcal{F}))N + ((\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_\tau\mathcal{F}) - \tau(\partial_{\kappa_2}\mathcal{F}))B$$

Now one can easily compute

$$\langle \mathbf{b}, N\rangle = (\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_\tau\mathcal{F}) - \tau(\partial_{\kappa_2}\mathcal{F}),$$

$$\langle \mathbf{b}, B\rangle = ((\partial_{\kappa_2}\mathcal{F})' - \kappa_1(\partial_\tau\mathcal{F}) + \tau\partial_{\kappa_1}\mathcal{F})$$

and

$$(\partial_\tau\mathcal{F})' = \kappa_2\partial_{\kappa_1}\mathcal{F} - \kappa_1\partial_{\kappa_2}\mathcal{F}. \tag{10.1}$$

The trick now, is to see that

$$\langle \mathbf{b}, T\rangle = \langle \operatorname{grad}\mathcal{F}(\kappa_1,\kappa_2,\tau), (\begin{pmatrix}\kappa_1\\ \kappa_2\\ \tau\end{pmatrix})\rangle - \mathcal{L}(\kappa_1,\kappa_2,\tau) + \mu,$$

for some constant $\mu \in \mathbb{R}$. Indeed,

$$\begin{aligned}
&\langle \mathbf{b}, T\rangle' \\
&\quad = \kappa_1\langle \mathbf{b}, N\rangle + \kappa_2\langle \mathbf{b}, B\rangle \\
&\quad = \kappa_1(\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_{\kappa_2}\mathcal{F})' + \tau(\kappa_2\partial_{\kappa_1}\mathcal{F} - \kappa_1\partial_{\kappa_2}\mathcal{F}) \\
&\quad = \kappa_1(\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_{\kappa_2}\mathcal{F})' + \tau(\partial_\tau\mathcal{F})' \\
&\quad = (\langle \operatorname{grad}\mathcal{F}(\kappa_1,\kappa_2,\tau), (\begin{pmatrix}\kappa_1\\ \kappa_2\\ \tau\end{pmatrix})\rangle - \mathcal{L}(\kappa_1,\kappa_2,\tau))'.
\end{aligned}$$

Summarizing: For critical points of $\mathcal{F}$ we obtain two constant vectors $\mathbf{a}$ and $\mathbf{b}$. We expressed $\mathbf{a}$ and $\mathbf{b}$ in terms of the framed curve F. Of course, one could define $\mathbf{a}$ and $\mathbf{b}$, not only for critical points of $\mathcal{F}$, but in that case $\mathbf{a}$ and $\mathbf{b}$ need not to be constant anymore. So the question which arises: Is the constants of $\mathbf{a}$ and $\mathbf{b}$ a characterization of critical points of $\mathcal{F}$? First, we figure out the relationship between $\mathbf{b}$ and $\mathbf{a}$.

Lemma 12. *Let*

$$\begin{aligned}
\mathbf{b} := {} & (\langle \operatorname{grad}\mathcal{F}(\kappa_1,\kappa_2,\tau), (\begin{pmatrix}\kappa_1\\ \kappa_2\\ \tau\end{pmatrix})\rangle - \mathcal{L}(\kappa_1,\kappa_2,\tau) + \mu)T \\
& + (\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_\tau\mathcal{F}) - \tau(\partial_{\kappa_2}\mathcal{F})N \\
& + ((\partial_{\kappa_2}\mathcal{F})' - \kappa_1(\partial_\tau\mathcal{F}) + \tau\partial_{\kappa_1}\mathcal{F})B
\end{aligned}$$

and

$$\mathbf{a} := (\partial_\tau\mathcal{F})T - (\partial\kappa_2\mathcal{F})N + (\partial\kappa_1\mathcal{F})B - \gamma \times \mathbf{b}.$$

If $\mathbf{b}$ *is a constant vector then* $\mathbf{a}$ *is a constant vector as well.*

Proof. By the definition of $\mathbf{b}$ and $\mathbf{a}$ we obtain

$$\mathbf{a}' = ((\partial_\tau\mathcal{F})' - \kappa_2\partial_{\kappa_1}\mathcal{F} + \kappa_1\partial_{\kappa_2}\mathcal{F})T = 0.$$

For the last equality we used (10.1). This shows that $\mathbf{a}$ is constant. $\square$

Theorem 27. *$(\kappa_1, \kappa_2, \tau)$ is a critical points of $\mathcal{F}$ if and only if*

$$\begin{aligned}\mathbf{b} = (\langle \operatorname{grad} \mathcal{F}(\kappa_1, \kappa_2, \tau), (\begin{pmatrix}\kappa_1 \\ \kappa_2 \\ \tau\end{pmatrix})\rangle - \mathcal{L}(\kappa_1, \kappa_2, \tau) + \mu)T \\ + (\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_\tau \mathcal{F}) - \tau(\partial_{\kappa_2}\mathcal{F})N \\ + ((\partial_{\kappa_2}\mathcal{F})' - \kappa_1(\partial_\tau \mathcal{F}) + \tau\partial_{\kappa_1}\mathcal{F})B\end{aligned}$$

is constant.

Proof. If $(\kappa_1, \kappa_2, \tau)$ is a critical points of $\mathcal{F}$ then we already know that b is conserved. If $\mathbf{b}$ is constant, then $\mathbf{a}$ is constant as well and we obtain

$$T \times \mathbf{b} = ((-\partial_{\kappa_2}\mathcal{F})' + \kappa_1(\partial_\tau \mathcal{F}) - \tau(\partial_{\kappa_1}\mathcal{F}))N + ((\partial_{\kappa_1}\mathcal{F})' + \kappa_2(\partial_\tau \mathcal{F}) - \tau(\partial_{\kappa_2}\mathcal{F}))B,$$

and

$$\gamma \times \mathbf{b} + \mathbf{a} = (\partial_\tau \mathcal{F})T - (\partial \kappa_2 \mathcal{F})N + (\partial \kappa_1 \mathcal{F})B,$$

or

$$\operatorname{grad} \mathcal{F}(\kappa_1, \kappa_2, \tau) = \begin{pmatrix}\langle \mathbf{a}, B\rangle \\ -\langle \mathbf{a}, N\rangle \\ \langle \mathbf{a}, T\rangle\end{pmatrix} + \begin{pmatrix}\langle \gamma \times \mathbf{b}, B\rangle \\ -\langle \gamma \times \mathbf{b}, N\rangle \\ \langle \gamma \times \mathbf{b}, T\rangle\end{pmatrix},$$

which shows that $(\kappa_1, \kappa_2, \tau)$ is a critical points of $\mathcal{F}$. □

An interesting application are Elastic strips [**4**].

$$\mathcal{F} := \int_0^L \kappa^2(1+\lambda^2)^2 ds, \quad \lambda = \frac{\tau}{\kappa}.$$

With our new approach one can easily verify

Theorem 28. *A strip is elastic if and only if the force vector*

$$\begin{aligned}\mathbf{b} :=& \\ &\frac{1}{2}(\kappa^2(1+\lambda^2)^2 + \mu)T \\ &+ (\kappa'(1+\lambda^2)^2 + 2\kappa(1+\lambda^2)\lambda'\lambda)N \\ &- (\kappa^2(1+\lambda^2)^2\lambda + (\frac{\kappa'}{\kappa})(1+\lambda^2)2\lambda)' + ((1+\lambda^2)2\lambda))'')B.\end{aligned}$$

is constant. Further, the torque vector

$$\mathbf{a} = 2\kappa\lambda(1+\lambda^2)T + \frac{1}{\kappa}(2\kappa\lambda(1+\lambda^2))'N + \kappa(1+\lambda^2)(1-\lambda^2)B - b \times \gamma.$$

is constant.

4 | Elastic deformation

1 The intrinsic distance-squared energy

In this chapter we introduce the intrinsic distance-squared energy on a Riemannian manifold M. In contrary to most of the sources in the literature **[2]**, **[8]** it seems to be the first intrinsically defined energy. Ulrich Pinkall and Jonas Tervooren are writing a paper about elastic deformations in the plane. We briefly summarize the main definitions and results.

Let $M \subset \mathbb{R}^n$ be a domain and $\langle\,,\rangle$ the induced metric on M. Let $f\colon M \to \mathbb{R}^n$ be a smooth orientation preserving immersion. The distance-squared energy is

$$E(f) := \frac{1}{2}\int_M \operatorname{dist}(df, \mathrm{SO}(n))^2.$$

Here the distance $\operatorname{dist}(df, \mathrm{SO}(n)) := \min\{\| df - R \| \,|, R \in \mathrm{SO}(n)\}$ and $\| A \|^2 := \frac{1}{n}\operatorname{tr}(AA^*)$ is the euclidean norm on $\mathrm{GL}(n, \mathbb{R})$. Since f is an orientation preserving immersion there exists a unique $R \in \mathrm{SO}(n)$ that minimizes $\| df - R \|$. R is obtained from the polar decomposition $df = RS$ where S is positive definite self adjoint operator. We obtain

$$\operatorname{dist}(df, \mathrm{SO}(n)) = \| S - I \|$$

and therefore

$$E(f) = \frac{1}{2}\int_M \| S - I \|^2 .$$

For a 2 dimensional M the energy can be expressed by

$$E(f) = \int_M (|f_z| - 1)^2 + |f_{\bar z}|^2,$$

where $df = f_z dz + f_{\bar z} d\bar z$ is the usual decomposition of df in its complex linear and complex anti linear parts. They proved, that among all compactly supported variations, the critical points (which are not melting points, i.e. $|f_z| = \frac{1}{2}$) are characterized by the following

Theorem 29. *f is a critical point of E if one of the following equivalent conditions are satisfied.*

(1) $g := (2 - \frac{1}{|f_z|})f_z$ is holomorphic function.

(2) The stress tensor $\sigma := i((1 - \frac{1}{|f_z|})f_z dz - f_{\bar z} d\bar z)$ is closed.

Furthermore, for a critical point f the argument $arg(f_z)$ is harmonic.

Thinking of $M \subset \mathbb{R}^n$ as consisting of a perfectly elastic homogeneous and isotropic material $E(f)$ measures the amount of elastic energy to deform the domain M to the domain $f(M)$. Both on M and $f(M)$ we put the induced metric $\langle\,,\rangle$ of $\mathbb{R}^n$. Since $df = RS$, f induces a metric $g = \langle S, S\rangle$ on M and obviously

$$f\colon (M, g) \to (f(M), \langle\,,\rangle)$$

is an isometry. So instead of measuring the elastic energy to deform M to $f(M)$, we could alternatively set f as the identity map and measure the energy to deform $(M, \langle\,,\rangle)$ to (M, g).

For $f = I$ we obtain the polar decomposition $I = RS$, which leads to $R = S^{-1}$ an orientation preserving isometry in

$$G := \Gamma(\mathrm{Iso}((TM, \langle\, , \rangle), (TM, g))) = S^{-1}(\mathrm{O}(TM, \langle\, , \rangle)).$$

This idea leads us to introduce the following

Definition 2.

$$E(S) := \frac{1}{2}\int_M \mathrm{dist}(I, G)^2 \tag{1.1}$$

is called the intrinsic distance-squared energy. Here the distance is $\mathrm{dist}(I, G) = \min\{\| I - S^{-1}U \|_{\aleph} \,|U \in \mathrm{SO}(TM, \langle\, , \rangle)\}$*, where*

$$\| A \|_{\aleph} := \| SA \| = \sqrt{\frac{1}{n} tr(AA^* S^2)}.$$

The reason for introducing this matrix norm is that we require the distance to remain unchanged under left and right multiplication by matrices $S^{-1}U \in G$*.*

Theorem 30.

$$E(S) = \frac{1}{2}\int_M \| S - I \|^2 .$$

Proof. We have to show

$$\mathrm{dist}(I, G) = \| I - S^{-1} \|_{\aleph} = \| S - I \| .$$

Let $U \in \mathrm{SO}(TM, \langle\, , \rangle)$. Then $\| I - S^{-1}U \|_{\aleph} = \| S - U \|$. So it remains to show $\| S - U \| \geq \| S - I \|$. First we show

$$\mathrm{tr}(SU) \leq \mathrm{tr}(S).$$

Let $X_1 ... X_n$ be an orthonormal basis of eigenvectors of S. Since S is positive definite all eigenvalues λ_i are positive.

$$\begin{aligned} \mathrm{tr}(SU) &= \sum \langle SUU^{-1}X_i, U^{-1}X_i\rangle \\ &= \sum \lambda_i \langle X_i, U^{-1}X_i\rangle \\ &\leq \sum \lambda_i \\ &= \mathrm{tr}(S). \end{aligned}$$

Now

$$\begin{aligned} \| S - U \|^2 &= \frac{1}{n}(\mathrm{tr}((S - U)(S - U^*)) \\ &= \frac{1}{n}(\mathrm{tr}(S^2 + I - SU^* - US)) \\ &= \frac{1}{n}(\mathrm{tr}(S^2 + I) - 2\mathrm{tr}(SU)) \\ &\geq \frac{1}{n}(\mathrm{tr}(S^2 + I) - 2\mathrm{tr}(S)) \\ &= \| S - I \|^2 . \end{aligned}$$

□

2 Euler-Lagrange-equation

Let $(M, \langle\, , \rangle, \nabla)$ be a n-dimensional Riemannian manifold and ∇ the corresponding Levi-Civita connection. We are interested in the critical points of the functional

$$E(S) = \frac{1}{2}\int_M \| S - I \|^2 \, dV,$$

among all positive definite and self adjoint operators $S \in \Gamma(\mathrm{End}(TM))$ with $\mathrm{tr}(S) > 1$. Let $S \in \Gamma(\mathrm{End}(TM))$ then $g = \langle S, S\rangle$ defines a Riemannian metric on M. Let $\tilde{\nabla}$ be the corresponding Levi Civita connection. Note that for any operator $L \in \Gamma(\mathrm{End}(TM))$ the adjont with respect to g is given by $L^{\tilde{*}} := S^{-2}L^*S^2$.

Lemma 13. *Let $X \in \Gamma(TM)$ and $S \in \Gamma(\mathrm{End}(TM))$ then $\dot{g} = L_X g$ implies*

$$S^{-1}\dot{S} + (S^{-1}\dot{S})^{\tilde{*}} = \tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}}.$$

Proof.

$$\begin{aligned} L_X g(Y,Z) &= Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) - g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, Z) - g(Y, \tilde{\nabla}_X Z - \tilde{\nabla}_Z X) \\ &= g(\tilde{\nabla}_Y X, Z) + g(Y, \tilde{\nabla}_Z X) \\ &= g((\tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}})Y, Z) \\ &= \langle S^2(\tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}})Y, Z\rangle, \end{aligned}$$

and

$$\begin{aligned} \dot{g}(Y,Z) &= \langle SY,\dot{\ } SZ\rangle \\ &= \langle \dot{S}Y, SZ\rangle + \langle SY, \dot{S}Z\rangle \\ &= \langle (S\dot{S} + \dot{S}S)Y, Z\rangle. \end{aligned}$$

We obtain $S\dot{S} + \dot{S}S = S^2(\tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}})$ or $S^{-1}\dot{S} + S^{-2}\dot{S}S = \tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}}$. Since $(S^{-1}\dot{S})^{\tilde{*}} = S^{-2}\dot{S}S$ we proved the claim. □

For any $A, L \in \Omega^1(M, TM)$ the Hodge star is defined by

$$\langle *A \wedge L\rangle = \mathrm{tr}(AL^*)\det.$$

We will now compute the Hodge star $\tilde{*}$ with respect to g

Lemma 14.

$$\begin{aligned} *A &= A * I. \\ \tilde{*}A &= (\det S)AS^{-2} * I. \end{aligned}$$

Proof. The first statement follows from

$$\begin{aligned} \langle A * I \wedge L\rangle &= \langle *I \wedge A^*L\rangle \\ &= \mathrm{tr}((A^*L)^*)\det \\ &= \mathrm{tr}(L^*A)\det \\ &= \mathrm{tr}(AL^*)\det \\ &= \langle *A \wedge L\rangle. \end{aligned}$$

The second statement follows from

$$\begin{aligned} \langle S^2\tilde{*}A \wedge L\rangle &= g(\tilde{*}A \wedge L) \\ &= \mathrm{tr}(AL^{\tilde{*}})\tilde{\det} \\ &= \mathrm{tr}(AS^{-2}L^*S^2)\det S\det \\ &= \mathrm{tr}(S^2AS^{-2}L^*)\det S\det \\ &= \det S\langle *S^2AS^{-2} \wedge L\rangle \\ &= \det S\langle S^2AS^{-2} * I \wedge L\rangle. \end{aligned}$$

□

We are now looking for critical points S of E among all compactly supported variations of S.

Theorem 31. *Critical points of E are characterized by*

$$d^{\tilde{\nabla}} * (I - S^{-1}) = 0.$$

We will call

$$\sigma := *(I - S^{-1}) \tag{2.1}$$

the stress tensor.

Proof. Let $\dot{S}$ be a compactly supported variation of S. Thus there exist a $X \in \Gamma(TM)$ with compact support such that $S^{-1}\dot{S} + (S^{-1}\dot{S})^{\tilde{*}} = \tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}}$.

$$\begin{aligned}
\dot{E} :&= \int_M \langle S - I, \dot{S}\rangle dV \\
&= \int_M \operatorname{tr}((S - I)\dot{S})dV \\
&= \int_M \operatorname{tr}(S(S - I)\dot{S}S^{-1})dV \\
&= \int_M \operatorname{tr}((S^2 - S)\dot{S}S^{-1})dV \\
&= \int_M \operatorname{tr}(\dot{S}S^{-1}(S^2 - S))dV \\
&= \int_M \operatorname{tr}(S^{-1}\dot{S}(S^2 - S))dV \\
&= \int_M \frac{1}{2}\operatorname{tr}((S^{-1}\dot{S} + (S^{-1}\dot{S})^{\tilde{*}})(S^2 - S))dV \\
&= \int_M \frac{1}{2}\operatorname{tr}((\tilde{\nabla}X + (\tilde{\nabla}X)^{\tilde{*}})(S^2 - S))dV \\
&= \int_M \operatorname{tr}((\tilde{\nabla}X)(S^2 - S))dV \\
&= \int_M \frac{1}{\det S}\operatorname{tr}((\tilde{\nabla}X)(S^2 - S))dV_g \\
&= \int_M g(\tilde{\nabla}X, \frac{S^2 - S}{\det S})dV_g \\
&= \int_M g(\tilde{*}\frac{S^2 - S}{\det S} \wedge \tilde{\nabla}X) \\
&= \int_M g(d^{\tilde{\nabla}}\tilde{*}\frac{S^2 - S}{\det S}, X).
\end{aligned}$$

Applying the previous lemma we obtain

$$\tilde{*}(\frac{S^2 - S}{\det S}) = *(I - S^{-1}),$$

and therefore

$$d^{\tilde{\nabla}} * (I - S^{-1}) = 0.$$

□

Lemma 15. *For any $A \in \Omega^1(M, TM)$ one obtains $d^{\tilde{\nabla}} * A = \hat{\nabla}A \wedge *I$.*

PROOF. Let $Y_1, ..., Y_n$ be a positively oriented orthonormal basis with respect to $\langle , \rangle$ such that $\nabla_{Y_i} Y_i = 0$ and $[Y_i, Y_j] = 0$.

$$
\begin{aligned}
d^{\tilde{\nabla}} * A(Y_1, ..., Y_n) &= \sum_{j=1}^{n} (-1)^{j-1}(-1)^{n-j} \tilde{\nabla}_{Y_j} AY_j \\
&= (-1)^{n-1} \sum_{j=1}^{n} \tilde{\nabla}_{Y_j} AY_j \\
&= (-1)^{n-1} \sum_{j=1}^{n} \tilde{\nabla}_{Y_j} Y_j - A\nabla_{Y_j} Y_j \\
&= (-1)^{n-1} \sum_{j=1}^{n} (\hat{\nabla}_{Y_j} A) Y_j \\
&= (\hat{\nabla} A \wedge *I)(Y_1, ..., Y_n).
\end{aligned}
$$

□

THEOREM 32. *S is a critical point of E if and only if*

$$(\hat{\nabla} I + \Omega S^{-1}) \wedge *I = 0.$$

PROOF. This follows from $\hat{\nabla}(I - S^{-1}) = \hat{\nabla} I - \hat{\nabla} S^{-1} = \hat{\nabla} I + \Omega S^{-1}$ and the application of the previous lemma. □

3 The two dimensional case

First we want to interpret the results of Pinkall and Tervooren and translate them to our theory. For a 2 dimensional $M \subset \mathbb{R}^2$ Pinkall and Tervooren decomposed the differential of $f \colon M \to \mathbb{R}^2$

$$df = f_z dz + f_{\bar{z}} d\bar{z}$$

in its complex linear and complex anti linear parts. In our case we have to decompose $I \colon (TM, J) \to (TM, \tilde{J})$ in its complex linear and and complex anti linear parts.

$$
\begin{aligned}
I &= I_+ + I_- \\
&= \frac{1}{2}(I - \tilde{J}J) + \frac{1}{2}(I + \tilde{J}J)
\end{aligned}
$$

where $\tilde{J} I_+ = I_+ J$ and $\tilde{J} I_- = -I_- J$. Let $S = \frac{1}{2}\mathrm{tr}(S) I + Q$ with $JQ = -QJ$ then one can easily check

$$I_+ = \frac{1}{2}\mathrm{tr}(S) S^{-1}$$

and

$$I_- = S^{-1} Q.$$

With $f = I$ we obtain $I = f_z dz + f_{\bar{z}} d\bar{z} = \frac{1}{2}\mathrm{tr}(S) S^{-1} + S^{-1} Q$ and therefore the stress tensor found by Pinkall and Tervooren translates to

$$\sigma = i((1 - \frac{1}{|f_z|}) f_z dz - f_{\bar{z}} d\bar{z}) \tag{3.1}$$

$$= \tilde{J}((1 - \frac{2}{\mathrm{tr}(S)}) \frac{1}{2}\mathrm{tr}(S) S^{-1} - S^{-1} Q) \tag{3.2}$$

$$= \tilde{J} S^{-1}((\frac{1}{2}\mathrm{tr}(S) - 1) I - Q) \tag{3.3}$$

$$= \tilde{J} S^{-1}((\mathrm{tr}(S) - 1) I - S) \tag{3.4}$$

$$= (\mathrm{tr}(S) - 1) \tilde{J} S^{-1} - \tilde{J}. \tag{3.5}$$

We now establish the necessary theory to prove the intrinsic counterpart of (29). Let M be a Riemann surface. Any $A \in \Omega^1(M, TM)$ can be decomposed

$$\hat{\nabla} A = \partial A + \bar{\partial} A,$$

with

$$\begin{aligned}(\partial_X A)Y &= \frac{1}{2}((\hat{\nabla}_X A)Y - \tilde{J}(\hat{\nabla}_{JX} A)Y),\\ (\bar{\partial}_X A)Y &= \frac{1}{2}((\hat{\nabla}_X A)Y + \tilde{J}(\hat{\nabla}_{JX} A)Y).\end{aligned}$$

$A \in \Gamma(\mathrm{End}((TM, J), (TM, \tilde{J})))$ is holomorhic if it is complex linear, i.e. $\tilde{J}A = AJ$, and

$$\bar{\partial} A = 0.$$

Let A be complex linear, then for any $X \in \Gamma(TM)$

$$\begin{aligned}(\hat{\nabla}_X A)JY &= \tilde{\nabla}_X AJY - A\nabla_X JY\\ &= \tilde{J}\tilde{\nabla}_X AY - \tilde{J}A\nabla_X Y\\ &= \tilde{J}(\hat{\nabla}_X A)Y.\end{aligned}$$

Therefore $\hat{\nabla}_X A$ is complex linear.

Lemma 16. *Let $A \in \Omega^1(M, TM)$ be complex linear, then*

$$\begin{aligned}d^{\tilde{\nabla}} * A(X, JX) &= -2(\bar{\partial}_X A)X,\\ d^{\tilde{\nabla}} A(X, JX) &= 2\tilde{J}(\bar{\partial}_X A)X.\end{aligned}$$

In particular, a complex linear A is holomorhic if and only if A is closed and co-closed.

Proof.

$$\begin{aligned}d^{\tilde{\nabla}} * A(X, JX) &= \hat{\nabla} A \wedge J(X, JX)\\ &= -(\hat{\nabla}_X A)X - (\hat{\nabla}_{JX} A)JX\\ &= -(\hat{\nabla}_X A)X + \tilde{J}(\hat{\nabla}_{JX} A)X)\\ &= -2(\bar{\partial}_X A)X.\end{aligned}$$

Furthermore,

$$\begin{aligned}d^{\tilde{\nabla}} A(X, JX) &= -d^{\tilde{\nabla}} * *A(X, JX)\\ &= -\hat{\nabla} * A \wedge J(X, JX)\\ &= -\tilde{J}\hat{\nabla} A \wedge J(X, JX)\\ &= 2\tilde{J}(\bar{\partial}_X A)X.\end{aligned}$$

□

Lemma 17. *Let $A = e^u R$ for a $R \in \Gamma(\mathrm{Iso}((TM, J), (TM, \tilde{J})))$ with $\hat{\nabla} R = \eta \tilde{J} R$. If A is holomorphic then is η is co-closed.*

Proof.

$$\begin{aligned}0 =& \frac{1}{2}(\hat{\nabla} A + \tilde{J} * \hat{\nabla} A)\\ &= e^u(duR + \hat{\nabla} R + \tilde{J} * (duR + \hat{\nabla} R))\\ &= e^u(duR + \eta\tilde{J}R + *du\tilde{J}R - *\eta R)\\ &= e^u(du - *\eta)I + (*du + \eta)\tilde{J})R.\end{aligned}$$

Thus A is holomorphic if and only if $du = *\eta$ and $*du = -\eta$. □

Using (0.10) we obtain

Corollary 11. *Let $M \subset \mathbb{R}^2$ and $A = e^u S^{-1}$. Let $\hat{\nabla} S^{-1} = -\eta \tilde{J} S^{-1}$ (0.6). If A is holomorphic then η is harmonic.*

The next lemma is most probably only true for a 2-dimensional M.

Lemma 18.

$$\ast(I - S^{-1}) = (\mathrm{tr}(S) - 1) \ast S^{-1} - \tilde{\ast} I. \tag{3.6}$$

In particular, our stress tensor coincides with (3.5).

Proof. Applying the Cayley-Hamilton theorem we obtain $S^2 - \mathrm{tr}(S)S + (\det S)I = 0$ or $S = (\mathrm{tr}S)I - (\det S)S^{-1} = 0$.

$$\begin{aligned}
\ast(I - S^{-1}) &= J - S^{-1}J \\
&= (S - I)S^{-1}J \\
&= (S - I)\tilde{J}S^{-1} \\
&= (\mathrm{tr}(S) - 1)I - (\det S)S^{-1})\tilde{J}S^{-1} \\
&= (\mathrm{tr}(S) - 1)\tilde{J}S^{-1} - (\det S)S^{-1}\tilde{J}S^{-1} \\
&= (\mathrm{tr}(S) - 1)\tilde{J}S^{-1} - \tilde{J}.
\end{aligned}$$

□

We have proven the intrinsic counterpart of (29).

Theorem 33. *S is a critical point of E if one of the following equivalent conditions are satisfied:*

(1) $(\mathrm{tr}(S) - 1)\tilde{J}S^{-1}$ is holomorphic.

(2) The stress tensor $\sigma = (\mathrm{tr}(S) - 1)\tilde{J}S^{-1} - \tilde{J}$ is closed.

For critical points S of E, $\hat{\nabla} S^{-1} = -\eta \tilde{J} S^{-1}$, implies η to be co-closed. In particular, if M is embedded in $\mathbb{R}^2$, then η is harmonic.

Finally we want to interpret the co-closedness of η. Let us now look at the Dirichlet energy for $R \in \Gamma(\mathrm{Iso}((TM, \langle\,,\rangle), (TM, g)))$

$$\mathcal{D}(R) := \frac{1}{2} \int_M \ast\eta \wedge \eta,$$

where η is defined through $\hat{\nabla} R = \eta \tilde{J} R$.

Theorem 34. *R is a critical point of $\mathcal{D}$ if and only if $d \ast \eta = 0$.*

Proof. Let $R_t \in \Gamma(\mathrm{Iso}((TM, J), (TM, \tilde{J})))$ a variation of R and $\eta_t \in \Omega^1(M)$ such that $\hat{\nabla} R_t = \eta_t \tilde{J} R_t$ and let $\dot{R} = f\tilde{J}R$ for compactly supported function f.

$$\begin{aligned}
\hat{\nabla}\dot{R} &= \dot{\eta}\tilde{J}R + \eta\tilde{J}\dot{R} \\
&= \dot{\eta}\tilde{J}R - f\eta R.
\end{aligned}$$

Further, $\hat{\nabla}(f\tilde{J}R) = df\tilde{J}R - f\eta R$ and therefore $\dot{\eta} = df$.

$$\begin{aligned}
\dot{\mathcal{D}} &= \int_M \ast\eta \wedge \dot{\eta} \\
&= \int_M \ast\eta \wedge df \\
&= \int_M -d(\ast\eta f) + fd(\tilde{\ast}\eta) \\
&= \int_M fd(\ast\eta).
\end{aligned}$$

□

Bibliography

[1] F. E. Burstall, D. Ferus, K. Leschke, F. Pedit, and U. Pinkall, *Conformal geometry of surfaces in S^4 and quaternions*, vol. 1772 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2002.

[2] I. Chao, U. Pinkall, P. Sanan, and P. Schröder, *A simple geometric model for elastic deformations*, ACM Trans. Graph., 29 (2010), pp. 38:1–38:6.

[3] A. Chern, U. Pinkall, and P. Schröder, *Close-to-conformal deformations of volumes*, ACM Trans. Graph., 34 (2015), pp. 56:1–56:13.

[4] D. Chubelaschwili and U. Pinkall, *Elastic strips*, Manuscripta Math., 133 (2010), pp. 307–326.

[5] K. Crane, U. Pinkall, and P. Schröder, *Spin transformations of discrete surfaces*, ACM Trans. Graph., 30 (2011).

[6] D. Ferus, K. Leschke, F. Pedit, and U. Pinkall, *Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic* 2*-tori*, Invent. Math., 146 (2001), pp. 507–593.

[7] T. Friedrich, *Dirac operators in Riemannian geometry*, vol. 25 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[8] G. Friesecke, R. D. James, and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, Comm. Pure Appl. Math., 55 (2002), pp. 1461–1506.

[9] N. Ginoux, *The Dirac spectrum*, vol. 1976 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2009.

[10] N. Hitchin, *Harmonic spinors*, Advances in Math., 14 (1974), pp. 1–55.

[11] G. Kamberov, P. Norman, F. Pedit, and U. Pinkall, *Quaternions, spinors, and surfaces*, vol. 299 of Contemporary Mathematics, American Mathematical Society, Providence, RI, 2002.

[12] G. Kamberov, F. Pedit, and U. Pinkall, *Bonnet pairs and isothermic surfaces*, Duke Math. J., 92 (1998), pp. 637–644.

[13] F. Pedit and U. Pinkall, *Quaternionic analysis on Riemann surfaces and differential geometry*, in Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 389–400 (electronic).

[14] J. Richter, *Conformal Maps of a Riemann Surface into the Space of Quaternions*, PhD thesis, TU Berlin, 99.

[15] I. Rivin and J.-M. Schlenker, *The Schläfli formula in Einstein manifolds with boundary*, Electron. Res. Announc. Amer. Math. Soc., 5 (1999), pp. 18–23 (electronic).