Renormalized solutions for nonlinear partial differential equations with variable exponents and L^1-data

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Notation

Ω	open domain in $\mathbb{R}^N$ for $N \geq 1$
$\partial\Omega$	topological boundary of Ω
$x = (x_1, \ldots, x_n)$	point in $\mathbb{R}^N$ for $N \geq 1$
$dx = dx_1 \ldots dx_N$	Lebesgue measure in Ω
Q_T	$(0,T) \times \Omega$ for $T > 0$
t	time variable
Σ_T	$(0,T) \times \partial\Omega$
Du	gradient of a function u
$\operatorname{supp} f$	support of a function f
f^+, f^-	$\max(f,0)$, $\min(f,0)$
$\mathcal{D}(\Omega)$, $\mathcal{D}(Q_T)$,...	test functions in Ω, Q_T,...
$\mathcal{D}^+(\Omega)$, $\mathcal{D}^+(Q_T)$,...	positive test functions in Ω, Q_T,...
$\mathcal{C}(\Omega)$,$\mathcal{C}(Q_T)$, ...	the function space of continuous functions in Ω, Q_T,...
$\mathcal{C}^k(\Omega)$,$\mathcal{C}^k(Q_T)$, ...	for $1 \leq k \leq \infty$ the function space of k times continuously differentiable functions in Ω, Q_T,...
$L^p(\Omega)$	$\{f : \Omega \to \mathbb{R} \mid f \text{ measurable}, \int_\Omega \|f(x)\|^p dx < \infty\}$ for $1 \leq p < \infty$
$L^\infty(\Omega)$	$\{f : \Omega \to \mathbb{R} \mid f \text{ measurable}, \operatorname{ess\,sup}_{x\in\Omega} \|f(x)\| < \infty\}$
$W^{1,p}(\Omega)$	$\{f : \Omega \to \mathbb{R} \mid f \in L^p(\Omega),\ Df \in (L^p(\Omega))^N\}$ for $1 \leq p \leq +\infty$
$W_0^{1,p}(\Omega)$	Closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ for $1 \leq p < +\infty$
$H^1(\Omega)$	$\{f : \Omega \to \mathbb{R} \mid f \in L^2(\Omega),\ Df \in (L^2(\Omega))^N\}$
$H^{-1}(\Omega)$	dual space of $H^1(\Omega)$
X	arbitrary Banach space
X'	dual space of the Banach space X
$L^p(0,T;X)$	for $1 \leq p < \infty$, $\{f : \Omega \to X \mid \int_0^T \\|f(t)\\|_X^p dt < \infty\}$
$L^\infty(0,T;X)$	$\{f : (0,T) \to X \mid \text{ess sup} \\|f(t)\\|_X < \infty\}$
$\mathcal{C}([0,T];X)$	function space of continuous functions defined on $[0,T]$ with values in X
$\mathcal{D}((0,T);X)$	test functions with values in X
I	$I : X \to X$ identity mapping on X

Chapter 1

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with Lipschitz boundary $\partial\Omega$ if $N \geq 2$. Our aim is to prove existence and uniqueness of renormalized solutions to the nonlinear elliptic equation

$$(E, f) \begin{cases} \beta(u) - \operatorname{div}(a(x, Du) + F(u)) \ni f & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

and to the corresponding parabolic problem

$$(P, f, b_0) \begin{cases} \beta(u)_t - \operatorname{div}(a(x, Du) + F(u)) \ni f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ \beta(u(0, \cdot)) \ni b_0 & \text{in } \Omega \end{cases}$$

with right-hand side $f \in L^1(\Omega)$ for (E, f) and $f \in L^1(Q_T)$ for (P, f, b_0). Furthermore, $F : \mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz continuous and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set-valued, maximal monotone mapping such that $0 \in \beta(0)$.
$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

$(A1)$ There exist a continuous function $p : \overline{\Omega} \to (1, \infty)$, $1 < \min_{x \in \overline{\Omega}} p(x) \leq N$ (the case $\min_{x \in \overline{\Omega}} p(x) > N$ is easy and can be solved by variational methods) and a positive constant γ such that $a(x, \xi) \cdot \xi \geq \gamma |\xi|^{p(x)}$ holds for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

$(A2)$ $|a(x, \xi)| \leq d(x) + |\xi|^{p(x)-1}$ for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where d is a nonnegative function in $L^{p'(\cdot)}(\Omega)$ and $p'(x) := p(x)/(p(x) - 1)$ for a.e. $x \in \Omega$.

$(A3)$ $(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq 0$ for almost every $x \in \Omega$ and for every ξ, $\eta \in \mathbb{R}^N$.

Due to the assumptions $(A1)$, $(A2)$ and $(A3)$, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. The theory of Lebesgue and Sobolev spaces with variable exponent has experienced a revival of interest, shown in a substantial amount of publications over the past few years. An extensive list of references concerning the recent advances and open problems can be found in Diening et al. [38].

The equation (E,f) can be viewed as generalization of the $p(x)$-Laplacian equation

$$(L,f)\begin{cases} -\operatorname{div}(|Du|^{p(x)-2}Du) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

In case of a constant exponent $p(\cdot) \equiv p$, $1 < p < \infty$, and $f \in W^{-1,p'}(\Omega)$ it follows from Minty-Browder Theorem that there exists a unique solution $u \in W_0^{1,p}(\Omega)$ to (L,f) in the sense of distributions. For $1 < p < N$ and right-hand side $f \in L^1(\Omega)$ we can not expect solutions $u \in W_0^{1,p}(\Omega)$. Indeed, supposing that for each $f \in L^1(\Omega)$ there exists a solution $u \in W_0^{1,p}(\Omega)$ of (L,f), as $-\operatorname{div}(|Du|^{p-2}Du) \in W^{-1,p'}(\Omega)$, it follows that $L^1(\Omega) \subset W^{-1,p'}(\Omega)$. By duality, this implies $W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$. From Sobolev embedding theorems we know that this is not true in general if $1 < p < N$. Supposing $1 < p < 2$ and that there exists a solution $u \in W_0^{1,1}(\Omega)$ of (L,f), as $|Du|^{p-2}Du \in (L^{1/(p-1)}(\Omega))^N$ we get $-\operatorname{div}(|Du|^{p-2}Du) \in W^{-1,1/(p-1)}(\Omega)$ and therefore $L^1(\Omega) \subset W^{-1,1/(p-1)}(\Omega)$. By duality, this implies $W_0^{1,1/(2-p)}(\Omega) \subset L^\infty(\Omega)$. By Sobolev embedding theorems, this is true if $p > 2-\frac{1}{N}$. Therefore we can not even expect solutions $u \in W_0^{1,1}(\Omega)$ of (L,f) for $1 < p < 2-\frac{1}{N}$ and $f \in L^1(\Omega)$. In the case of constant $p > 2-\frac{1}{N}$ the existence of a distributional solution u of (L,f) in the space

$$\bigcap_{q<\frac{N(p-1)}{N-1}} W_0^{1,q}(\Omega)$$

has been shown in [25]. As it has been shown in [70] and [62], the distributional solution u is in general not unique. In order to get well-posedness for $L^1(\Omega)$-data, the notion of an entropy solution for problem (L,f) was introduced by Bénilan et al. in [15] in the framework of a constant $p(\cdot) \equiv p$. Moreover, existence and uniqueness of an entropy solution of (L,f) has been established for $1 < p < N$. In [67] the result of [15] has been extended to nonconstant $p \in W^{1,\infty}(\Omega)$. An equivalent notion of solution for problem (L,f) is called renormalized solution. The concept of renormalized solutions was introduced by DiPerna and Lions in [39]. This notion was then extended to the study of various problems of partial differential equations of parabolic, elliptic-parabolic and hyperbolic type, we refer to [20], [23], [56], [32], [2] and the references therein for more details. In [10], existence of renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations has been shown and in [11] the notion of renormalized solution was adapted to an anisotropic reaction-diffusion-advection system. In [12], existence and uniqueness of renormalized solutions to (L, f) has been shown for continuous functions $p : \overline{\Omega} \to (1, \infty)$, such that $2 - \frac{1}{N} < \min_{x \in \overline{\Omega}} p(x)$.

(P, f, b_0) can be viewed as a generalisation of the parabolic $p(x)$-Laplacian equation

$$(L, f, u_0) \begin{cases} u_t - \operatorname{div}(|Du|^{p(x)-2} Du) = f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(0, \cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases}$$

One of the motivations for studying (E, F) and (P, f, b_0) comes from applications to electro-rheological fluids (see [64], [49] for more details) as an important class of non-Newtonian fluids. Other important applications are related to image processing and elasticity (see [33], [75]). Note that (L, f, u_0) has a more complicated nonlinearity than the classical p-Laplacian since it is nonhomogenous. In [13] existence and uniqueness of renormalized solutions to (L, f, u_0) was shown for arbitrary L^1-data.

This thesis is organized as follows: In the next chapter we will present some general definitions and results concerning the necessary function spaces. Moreover, we will introduce some notation and functions which will be used frequently. In the third chapter we will study existence and uniqueness of weak and renormalized solutions to the elliptic problem (E, f). These results will serve us as a basis for the study of the evolution problem associated with the same convection-diffusion operator: More precisely, from these results we deduce that there exists a mild solution of the abstract Cauchy problem corresponding to (P, f, b_0) in the sense of nonlinear semigroup theory. The nonlinear semigroup theory gives a general notion of solution, called 'mild' solution for abstract Cauchy problems of the form

$$\frac{du}{dt} + Au \ni f$$

where A is (a possibly multivalued) operator in a Banach space X and $f \in L^1(0, T; X)$. The mild solution is, roughly speaking, the uniform limit of piecewise constant approximate solutions of time-discretized equations given by an implicit Euler scheme. This result will lead us to the appropriate energy space for weak and renormalized solutions of (P, f, b_0) with variable exponent.

At the end of the last chapter we will show existence and uniqueness of renormalized solutions using the ideas developed in [18], [2], [68] and [22] for the case of a constant exponent.

For all the basic definitions and results from nonlinear semigroup theory we mostly refer to the unpublished book of Bénilan, Crandall and Pazy (see [17]). Other references are [8], [72], [59], [58] [14], [36], [30]. Consider also [16], [34], [9], [35] and [19] for furhter reading and applications to partial differential equations.

For all basic definitions and results concerning (linear and nonlinear) functional analysis and classical Lebesgue and Sobolev spaces we refer to, e.g., [29], [66]. For the theory of vector-valued integration and Sobolev spaces see, e.g., [40], [48].

Chapter 2

Function spaces and notation

2.1 Lebesgue and Sobolev spaces with variable exponent

We recall in what follows some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponent (see for example [52], [44], [43], [37], [38] for proofs and details and [60] for general theory of Orlicz spaces). For an open set $\Omega \subset \mathbb{R}^N$, let $p : \Omega \to [1, \infty)$ be a measurable function, which is called the variable exponent, such that

$$1 \leq p^- := \text{ess}\ \inf_{x\in\Omega} p(x) \leq p^+ := \text{ess}\ \sup_{x\in\Omega} p(x) < \infty.$$

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of classes of almost everywhere equal measurable functions $f : \Omega \to \mathbb{R}$, such that the modular $\rho_p(f) := \int_\Omega |f(x)|^{p(x)}dx$ is finite. On $L^{p(\cdot)}(\Omega)$, $f \to \|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \rho_p(\frac{f}{\lambda}) < 1\}$ defines a norm, $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is a Banach space and $\mathcal{D}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$. If $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega fg\ dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)} \tag{2.1.1}$$

holds true. Convergence with respect to the modular is equivalent to convergence with respect to the norm. We have the following relation between the modular and the norm:

$$\min\left\{\|f\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|f\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\} \leq \int_\Omega |f(x)|^{p(x)}dx \leq \max\left\{\|f\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|f\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}. \tag{2.1.2}$$

Since we always assume Ω to be bounded, we have

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \tag{2.1.3}$$

with continuous embedding for all variable exponents $q \in L^\infty(\Omega)$ such that $p(x) \leq q(x)$ almost everywhere in Ω. We define the Sobolev space with variable exponent

$$W^{1,p(\cdot)}(\Omega) = \{f \in L^{p(\cdot)}(\Omega) : |Df| \in L^{p(\cdot)}(\Omega)\}.$$

For $f \in W^{1,p(\cdot)}(\Omega)$, $f \to \|f\|_{W^{1,p(\cdot)}(\Omega)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \||Df|\|_{L^{p(\cdot)}(\Omega)}$ defines a norm such that, for $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{W^{1,p(\cdot)}})$ is a Banach space and we have a continuous embedding

$$W^{1,q(\cdot)}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega) \tag{2.1.4}$$

for all variable exponents $q \in L^\infty(\Omega)$ such that $p(x) \leq q(x)$ almost everywhere in Ω. Moreover, if $p^- > 1$, then $W^{1,p(\cdot)}(\Omega)$ is reflexive. For $N = 1$ and $\Omega = (a, b)$, $a, b \in \mathbb{R}$, $a < b$, it is an immediate consequence of (2.1.4) that

$$W^{1,p(\cdot)}(a, b) \hookrightarrow W^{1,p^-}(a, b) \hookrightarrow \mathcal{C}([a, b]) \tag{2.1.5}$$

with continuous and dense embedding. We define also

$$W_0^{1,p(\cdot)}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}}.$$

For exponents $p \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^+)$, $p^- \geq 1$ and $f \in W_0^{1,p(\cdot)}(\Omega)$ the Poincaré inequality

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq C\||Df|\|_{L^{p(\cdot)}(\Omega)} \tag{2.1.6}$$

holds true and the embedding $W_0^{1,p(\cdot)}(\Omega)$ into $L^{p(\cdot)}(\Omega)$ is compact (see [52], [44]). In particular, $W_0^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space if $p^- > 1$. Its dual space will be denoted by $W^{-1,p'(\cdot)}(\Omega)$. According to [44] and [37], for a bounded domain Ω with Lipschitz boundary and $p^+ < N$, we have a compact embedding

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

for all measurable exponents $q : \Omega \to [1, \infty)$ such that $q(x) < p^*(x) - \varepsilon$ almost everywhere in Ω for some $\varepsilon > 0$, where $p^*(x) := Np(x)/(N - p(x))$ almost everywhere in Ω. Some more general Sobolev embedding results for variable exponents $p \in L^\infty(\Omega)$ such that $p^- > 1$ can be found in [52]. To the best of our knowledge, no general necessary and sufficient conditions for the Poincaré inequality (2.1.6) are known beyond continuity of the variable exponent. This is different in the particular case of one space dimension: If $\Omega = (a,b)$ for $a,b \in \mathbb{R}$, $a < b$, from Poincaré inequality in $W_0^{1,1}(a,b)$ it follows that $W_0^{1,1}(a,b)$ is continuously embedded into $\mathcal{C}([a,b])$. Since $W_0^{1,p(\cdot)}(a,b) \subset W_0^{1,1}(a,b)$ there exists $C_1 > 0$ such that

$$\|f\|_{L^\infty(a,b)} \leq C_1 \||Df|\|_{L^1(a,b)} \tag{2.1.7}$$

for all $f \in W_0^{1,p(\cdot)}(a,b)$. Now, from (2.1.7) and the continuous embedding of $L^{p(\cdot)}(a,b)$ into $L^1(a,b)$ it follows that there exists $C_2 > 0$ such that

$$\|f\|_{L^\infty(a,b)} \leq C_2 \||Df|\|_{L^{p(\cdot)}(a,b)} \tag{2.1.8}$$

for all $f \in W_0^{1,p(\cdot)}(a,b)$. Since $L^\infty(a,b) \hookrightarrow L^{p(\cdot)}(a,b)$, from (2.1.8) it follows that

$$\|f\|_{L^{p(\cdot)}(a,b)} \leq C_3 \||Df|\|_{L^{p(\cdot)}(a,b)} \tag{2.1.9}$$

holds for all $f \in W_0^{1,p(\cdot)}(a,b)$, where $C_3 > 0$ does not depend on f. Hence, the Poincaré inequality (2.1.6) holds for any $p \in L^\infty(a,b)$ such that $p^- \geq 1$. However, we will exclusively work with Lebesgue and Sobolev spaces with continuous variable exponent $p : \overline{\Omega} \to [1,\infty)$ such that $1 < p^-$. We do not assume $p(\cdot)$ to be log-Hölder continuous:

Definition 2.1.1. The continuous function $p : \overline{\Omega} \to [1,\infty)$ satisfies the log-Hölder continuity condition iff there exists a non-decreasing function $\omega : (0,\infty) \to \mathbb{R}$ such that $\limsup_{t\to 0^+} \omega(t)\ln(1/t) < +\infty$ and

$$|p(x) - p(y)| < \omega(|x-y|) \tag{2.1.10}$$

holds for all $x, y \in \overline{\Omega}$, $|x-y| < 1$.

If log-Hölder continuity condition (2.1.10) holds, $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and

$$W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Moreover, if $1 < p^- < p^+ < N$, then the Sobolev embedding holds also for $q(\cdot) = p^*(\cdot)$ (see [37] for more details). An additional difficulty to our setting arises from the fact that $W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are in general not equal, hence different duality frameworks for (E,f) are possible and lead to different notions of solution (see [74], [73], [3] for more details). We will restrict ourselves to the $W_0^{1,p(\cdot)}(\Omega)/W^{-1,p'(\cdot)}(\Omega)$ duality. We refer to [3] for some existence and uniqueness results to (E,f) in the case $F \equiv 0$ and $p \in L^\infty(\Omega)$ such that $p^- > 1$ where different duality frameworks and notions of solution have been considered. Note that $W_0^{1,p(\cdot)}(\Omega)$ is stable by composition with Lipschitz functions, even if for a function $w \in W^{1,p(\cdot)}(\Omega)$ having trace zero does not guarantee that $w \in W_0^{1,p(\cdot)}(\Omega)$. Indeed, let $L : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous such that $L(0) = 0$ and $u \in W_0^{1,p(\cdot)}(\Omega)$. Then there exists a sequence $(u_n)_n \subset \mathcal{D}(\Omega)$, such that $u_n \to u$ in $W^{1,p(\cdot)}(\Omega)$ as $n \to \infty$. From the Lipschitz continuity of L it follows immediately that $L(u_n) \to L(u)$ as $n \to \infty$ in $L^{p(\cdot)}(\Omega)$. Since L' is essentially bounded and $D(L \circ u_n) = L'(u_n)D(u_n)$ almost everywhere in Ω and in $\mathcal{D}'(\Omega)$ for each $n \in \mathbb{N}$, we have $L(u_n) \in W_0^{1,p^+}(\Omega)$, hence in $W_0^{1,p(\cdot)}(\Omega)$ (by continuous embedding of $W_0^{1,p^+}(\Omega)$ into $W_0^{1,p(\cdot)}(\Omega)$). Moreover, there exists a constant $C > 0$ not depending on $n \in \mathbb{N}$ such that

$$\| |D(L(u_n))| \|_{p(\cdot)} \leq C.$$

By reflexivity of $W_0^{1,p(\cdot)}(\Omega)$ it follows that there exists a (not relabeled) subsequence of $(L(u_n))_n$ converging to $L(u)$ weak in $W_0^{1,p(\cdot)}(\Omega)$. Therefore, $L(u) \in W_0^{1,p(\cdot)}(\Omega)$.

2.2 Function spaces for the evolution problem

If X is a Banach space, $1 \leq q \leq \infty$ and $T > 0$, then $L^q(0,T;X)$ denotes the space of strongly measurable functions $u : (0,T) \to X$ such that $t \to \|u(t)\|_X \in L^q(0,T)$. Moreover, $\mathcal{C}([0,T];X)$ denotes the space of continuous functions $u : [0,T] \to X$ endowed with the norm

$$\|u\|_{\mathcal{C}([0,T];X)} = \max_{t \in [0,T]} \|u(t)\|_X.$$

The following density result will be used in the study of the evolution problem:

Proposition 2.2.1. *Let $X = L^p(\Omega)$ or $X = W^{1,p}(\Omega)$ and $1 \leq p < \infty$. Then, $\mathcal{D}((0,T) \times \Omega)$ is dense in $L^q(0,T;X)$ for any $1 \leq q < \infty$.*

Proof: From [40], Cor. 1.3.1, p. 13 it follows that

$$Z := \left\{ \sum_{i=1}^{n} \phi_i(x)\psi_i(t),\ n \geq 1, \quad \phi_i \in \mathcal{D}(\Omega), \psi_i \in \mathcal{D}(0,T) \right\} \subset \mathcal{D}((0,T) \times \Omega)$$

is dense in $L^q(0,T;X)$ for any Banach space X such that $\mathcal{D}(\Omega)$ is dense in X and $1 \leq q < \infty$. $\square$

For $T > 0$ let $Q_T := (0,T) \times \Omega$. Extending a variable exponent $p : \overline{\Omega} \to [1,\infty)$ to $\overline{Q_T}$ by setting $p(t,x) := p(x)$ for all $(t,x) \in \overline{Q_T}$, we may also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q_T) := \left\{ u : Q_T \to \mathbb{R};\ u \text{ is measurable, } \int_{Q_T} |u(t,x)|^{p(x)} d(t,x) < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q_T)} := \inf_{\mu>0} \left\{ \int_{Q_T} \left| \frac{u(t,x)}{\mu} \right|^{p(x)} d(t,x) \leq 1 \right\},$$

which, of course shares the same properties as $L^{p(\cdot)}(\Omega)$. Moreover, if $p(\cdot)$ is log-Hölder continuous in Ω, so it is in Q_T. Indeed, if $p(\cdot)$ satisfies the log-Hölder continuity condition in Ω, according to Definition 2.1.1, there exists a non-decreasing function $\omega : (0,\infty) \to \mathbb{R}$ such that $\limsup_{t\to 0^+} \omega(t)\ln(1/t) < +\infty$ and

$$|p(t,x) - p(s,y)| = |p(x) - p(y)| < \omega(|x-y|) \leq \omega(|(t,x) - (s,y)|) \quad (2.2.1)$$

holds for all (t,x), $(s,y) \in \overline{Q_T}$ such that $|(t,x) - (s,y)| < 1$.

Let $p : \overline{\Omega} \to [1,\infty)$ be a continuous variable exponent and $T > 0$. The abstract Bochner spaces $L^{p^+}(0,T;L^{p(\cdot)}(\Omega))$ and $L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ will be important in the study of renormalized solutions to (P,f,b_0). In the following we identify an abstract function like $v \in L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ with the real-valued function v defined by $v(t,x) = v(t)(x)$ for almost all $t \in (0,T)$ and almost all $x \in \Omega$. In the same way we associate to any function $v \in L^{p(\cdot)}(Q_T)$ an abstract function $v : (0,T) \to L^{p(\cdot)}(\Omega)$ by setting $v(t) := v(t,\cdot)$ for almost every $t \in (0,T)$.

Lemma 2.2.2. *We have the following continuous dense embeddings:*

$$L^{p^+}(0,T;L^{p(\cdot)}(\Omega)) \overset{d}{\hookrightarrow} L^{p(\cdot)}(Q_T) \overset{d}{\hookrightarrow} L^{p^-}(0,T;L^{p(\cdot)}(\Omega)). \quad (2.2.2)$$

Proof: For $v \in L^{p(\cdot)}(Q_T)$, the corresponding abstract function $v : (0,T) \to L^{p(\cdot)}(\Omega)$ is strongly Bochner measurable (by the Dunford-Pettis Theorem, since it is weakly measurable and $L^{p(\cdot)}(\Omega)$ is seperable). Moreover, using

(2.1.2) and the Jensen inequality, we find the estimate

$$\begin{aligned}
&\int_0^T \|v(t)\|^{p^-}_{L^{p(\cdot)}(\Omega)} dt \\
&\leq \int_0^T \max\left[\int_\Omega |v(t,x)|^{p(x)}dx, \left(\int_\Omega |v(t,x)|^{p(x)}dx\right)^{p^-/p^+}\right] dt \\
&\leq \int_0^T \int_\Omega |v(t,x)|^{p(x)}dxdt + T^{1-p^-/p^+}\left(\int_0^T\int_\Omega |v(t,x)|^{p(x)}dxdt\right)^{p^-/p^+} \\
&\leq \max\left[\|v\|^{p^-}_{L^{p(\cdot)}(Q_T)}, \|v\|^{p^+}_{L^{p(\cdot)}(Q_T)}\right] \\
&+ T^{1-p^-/p^+}\max\left[\|v\|^{(p^-)^2/p^+}_{L^{p(\cdot)}(Q_T)}, \|v\|^{p^-}_{L^{p(\cdot)}(Q_T)}\right].
\end{aligned} \tag{2.2.3}$$

Therefore, the embedding of $L^{p(\cdot)}(Q_T)$ into $L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ is continuous. If $u \in L^{p^+}(0,T;L^{p(\cdot)}(\Omega))$, from $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ it follows that $u \in L^{p^+}(0,T;L^1(\Omega))$, hence, according to [40], Prop. 1.8.1, p. 28, the corresponding real-valued function $u : (0,T)\times\Omega \to \mathbb{R}$ is measurable and using the same arguments as above we find the continuous embedding of $L^{p^+}(0,T;L^{p(\cdot)}(\Omega))$ into $L^{p(\cdot)}(Q_T)$. It is left to prove that both embeddings are dense. We consider the first embedding and fix $u \in L^{p(\cdot)}(Q_T)$. Since $\mathcal{D}(Q_T)$ is dense $L^{p(\cdot)}(Q_T)$, we find a sequence $(u_n)_n \subset \mathcal{D}(Q_T)$ converging to u in $L^{p(\cdot)}(Q_T)$ as $n \to \infty$. According to Proposition 2.2.1, $\mathcal{D}(Q_T)$ is densely embedded into $L^{p^+}(0,T;L^{p^+}(\Omega))$, therefore $u_n \in L^{p^+}(0,T;L^{p(\cdot)}(\Omega))$ for all $n \in \mathbb{N}$. To prove the denseness of the second embedding, we fix $v \in L^{p-}(0,T;L^{p(\cdot)}(\Omega))$. Taking a standard sequence of mollifiers $(\rho_n)_n \subset \mathcal{D}(\mathbb{R})$ and extending v by zero onto $\mathbb{R}$, from [40], Proposition 1.7.1, p. 25, it follows that the regularized (in time) function

$$(\rho_n * v)(\cdot) := \int_{\mathbb{R}} \rho_n(\cdot - s)v(s)ds \tag{2.2.4}$$

is in $L^{p^+}(\mathbb{R};L^{p(\cdot)}(\Omega))$ for each $n \in \mathbb{N}$, hence in $L^{p(\cdot)}(Q_T)$ and converges to v in $L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ (see [40], Théorème 1.7.1, p. 27). □

2.3 Notation and functions

Let us introduce some notation and functions that will be frequently used. If $A \subset \Omega$ is a Lebesgue measurable set, we will denote its Lebesgue measure by $|A|$ and by χ_A its characteristic function. For any $u : \Omega \to \mathbb{R}$ and $k \geq 0$, we write $\{|u| \leq (<,>,\geq,=)k\}$ for the set $\{x \in \Omega : |u(x)| \leq (<,>,\geq,=)k\}$. For

$r \in \mathbb{R}$, let $r \to r^+ := \max(r, 0)$, $r \to \operatorname{sign}_0(r)$ the usual sign function which is equal to -1 on $]-\infty, 0[$, to 1 on $]0, \infty[$ and to 0 for $r = 0$. $r \to \operatorname{sign}_0^+(r)$ is the function defined by $\operatorname{sign}_0^+(r) = 1$ if $r > 0$ and $\operatorname{sign}_0^+(r) = 0$ if $r \leq 0$. Let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by $h_l(r) := \min((l + 1 - |r|)^+, 1)$ for each $r \in \mathbb{R}$. For any given $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(r) := \begin{cases} -k, & \text{if } r \leq -k, \\ r, & \text{if } |r| < k, \\ k, & \text{if } r \geq k. \end{cases} .$$

For $\delta > 0$ we define $H_\delta^+ : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta^+(r) = \begin{cases} 0, & r < 0 \\ \frac{1}{\delta} r, & 0 \leq r \leq \delta \\ 1, & r > \delta \end{cases}$$

and $H_\delta : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta(r) = \begin{cases} -1, & r < -\delta \\ \frac{1}{\delta} r, & -\delta \leq r \leq \delta \\ 1, & r > \delta. \end{cases}$$

Clearly, H_δ^+ is an approximation of sign_0^+ and H_δ is an approximation of sign_0.

Remark 2.3.1. The following argument will be frequently used to treat the „convection“ term $F(u) \cdot Du$ in (E, f) and (P, f, b_0): Observe that for $F = (F_1, \ldots, F_N) \in L^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $F(0) = 0$, $u \in W_0^{1,p(\cdot)}(\Omega)$ we have

$$\int_\Omega F(u) \cdot Du = 0. \tag{2.3.1}$$

Proof: Let us define $\int_0^s F(\sigma) d\sigma := \left(\int_0^s F_1(\sigma) d\sigma, \ldots, \int_0^s F_N(\sigma) d\sigma \right)$ and $\phi : \mathbb{R} \to \mathbb{R}^N$, $\phi(s) := \int_0^s F(\sigma) d\sigma$ for $s \in \mathbb{R}$. Observe that ϕ is Lipschitz continuous such that $\phi(0) = 0$ and therefore $\phi \circ u$ is in $(W_0^{1,p(\cdot)}(\Omega))^N$. Hence,

$$\frac{\partial}{\partial x_i} (\phi \circ u) = F(u) \frac{\partial}{\partial x_i} u \tag{2.3.2}$$

in $\mathcal{D}'(\Omega)$ for any $i = 1, \ldots, N$. Consequently,

$$-\operatorname{div} \left(\int_0^u F(\sigma) d\sigma \right) = F(u) \cdot Du \tag{2.3.3}$$

in $\mathcal{D}'(\Omega)$ and (2.3.1) follows using (2.3.3) and the Gauss-Green Theorem for Sobolev functions from $u = 0$ almost everywhere on $\partial\Omega$.

Chapter 3

The elliptic case

3.1 Renormalized solutions

Definition 3.1.1. A renormalized solution to (E, f) is a pair of functions (u, b) satisfying the following conditions:

$(R1)$ $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega)$, $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

$(R2)$ For each $k > 0$, $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega bh(u)\varphi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\varphi) = \int_\Omega fh(u)\varphi \qquad (3.1.1)$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

$(R3)$ $\int_{\{k<|u|<k+1\}} a(x, Du) \cdot Du \to 0$ as $k \to \infty$.

Remark 3.1.1. We can easily check that all the terms in $(R2)$ make sense. We recall that a function u such that $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$, for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, it is possible to define its generalized gradient (still denoted by Du) as the unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$DT_k(u) = v\chi_{\{|u|<k\}}$$

for a.e. $x \in \Omega$ and for all $k > 0$, where χ_E denotes the characteristic function of a measurable set E. Moreover, if $u \in W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u. See [15], [67] for more details.

The main existence result of this chapter is the following theorem:

Theorem 3.1.2. *For $f \in L^1(\Omega)$ there exists at least one renormalized solution (u, b) to (E, f).*

3.2 Existence for L^∞-data

To prove Theorem 3.1.2, we will introduce and solve approximating problems. To this end, for $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ we define $f_{m,n} : \Omega \to \mathbb{R}$ by

$$f_{m,n}(x) = \max(\min(f(x), m), -n)$$

for almost every $x \in \Omega$. Clearly, $f_{m,n} \in L^\infty(\Omega)$ for each $m, n \in \mathbb{N}$, $|f_{m,n}(x)| \leq |f(x)|$ a.e. in Ω, hence $\lim_{n\to\infty} \lim_{m\to\infty} f_{m,n} = f$ in $L^1(\Omega)$ and almost everywhere in Ω. The next proposition will give us existence of renormalized solutions $(u_{m,n}, b_{m,n})$ of $(E, f_{m,n})$ for each $m, n \in \mathbb{N}$:

Proposition 3.2.1. *For $f \in L^\infty(\Omega)$ there exists at least one renormalized solution (u, b) to (E, f).*

The proof of Proposition 3.2.1 will be divided into several steps.

3.2.1 Approximate solutions for L^∞-data

At first we approximate (E, f) for $f \in L^\infty(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0 < \varepsilon \leq 1$, let $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the Yosida approximation (see [28]) of β. We introduce the operators

$$\mathcal{A}_{1,\varepsilon} : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))',$$
$$u \to \beta_\varepsilon(T_{1/\varepsilon}(u)) + \varepsilon \arctan(u) - \operatorname{div}\, a(x, Du)$$

and

$$\mathcal{A}_{2,\varepsilon} : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))',$$
$$u \to -\operatorname{div}\, F(T_{1/\varepsilon}(u)).$$

Because of $(A2)$ and $(A3)$, $\mathcal{A}_{1,\varepsilon}$ is well-defined and monotone (see [55], p. 157). Since $\beta_\varepsilon \circ T_{1/\varepsilon}$ and arctan are bounded and continuous and thanks to the growth condition $(A2)$ on a, it follows that $\mathcal{A}_{1,\varepsilon}$ is hemicontinuous (see [55], p.157). From the continuity and boundedness of $F \circ T_{1/\varepsilon}$ it follows that $\mathcal{A}_{2,\varepsilon}$ is strongly continuous. Therefore the operator $\mathcal{A}_\varepsilon := \mathcal{A}_{1,\varepsilon} + \mathcal{A}_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the „convection" term $\int_\Omega F(T_{1/\varepsilon}(u)) \cdot Du$, we show with similar arguments as in [12] that $\mathcal{A}_\varepsilon$ is coercive and bounded. Then it follows from [55], Theorem 2.7, that $\mathcal{A}_\varepsilon$ is surjective, i.e., for each $0 < \varepsilon \leq 1$ and $f \in (W_0^{1,p(\cdot)}(\Omega))'$ there exists at least one solution $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ to the problem

$$(E_\varepsilon, f) \begin{cases} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon) - \operatorname{div}(a(x, Du_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

such that

$$\int_\Omega (\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon))\varphi + \int_\Omega (a(x, Du_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon)) \cdot D\varphi = \langle f, \varphi\rangle \tag{3.2.1}$$

holds for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,p(\cdot)}(\Omega)$ and $(W_0^{1,p(\cdot)}(\Omega))'$.

In the next proposition, we establish uniqueness of solutions u_ε of (E_ε, f) with right-hand sides $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of renormalized solutions to (E, f) with $f \in L^1(\Omega)$.

Proposition 3.2.2. *For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^\infty(\Omega)$ let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ be solutions of (E_ε, f) and $(E_\varepsilon, \tilde{f})$, respectively. Then, the following comparison principle holds:*

$$\varepsilon \int_\Omega (\arctan(u_\varepsilon) - \arctan(\tilde{u}_\varepsilon))^+ \leq \int_\Omega (f - \tilde{f}) \operatorname{sign}_0^+(u_\varepsilon - \tilde{u}_\varepsilon). \tag{3.2.2}$$

Proof: We use the test function $\varphi = H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ in the weak formulation (3.2.1) for u_ε and $\tilde{u}_\varepsilon$. Subtracting the resulting inequalities, we obtain

$$I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta} + I^3_{\varepsilon,\delta} + I^4_{\varepsilon,\delta} = I^5_{\varepsilon,\delta}$$

where

$$\begin{aligned}
I^1_{\varepsilon,\delta} &= \int_\Omega (\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - \beta_\varepsilon(T_{1/\varepsilon}(\tilde{u}_\varepsilon)))H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^2_{\varepsilon,\delta} &= \int_\Omega (\varepsilon \arctan(u_\varepsilon) - \varepsilon \arctan(\tilde{u}_\varepsilon))H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),
\end{aligned}$$

$$\begin{aligned}
I^3_{\varepsilon,\delta} &= \int_\Omega (a(x, Du_\varepsilon) - a(x, D\tilde{u}_\varepsilon)) \cdot DH_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^4_{\varepsilon,\delta} &= \int_\Omega (F(T_{1/\varepsilon}(u_\varepsilon)) - F(T_{1/\varepsilon}(\tilde{u}_\varepsilon))) \cdot DH_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^5_{\varepsilon,\delta} &= \int_\Omega (f - \tilde{f})H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon).
\end{aligned}$$

Passing to the limit with $\delta \downarrow 0$, (3.2.2) follows. □

Remark 3.2.3. Let $f, \tilde{f} \in L^\infty(\Omega)$ be such that $f \leq \tilde{f}$ almost everywhere in Ω, $\varepsilon > 0$ and $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ solutions to (S_ε, f), $(S_\varepsilon, \tilde{f})$ respectively. Then it is an immediate consequence of Propsition 3.2.2 that $u_\varepsilon \leq \tilde{u}_\varepsilon$ almost everywhere in Ω. Furthermore, from the monotonicity of $\beta_\varepsilon \circ T_{1/\varepsilon}$ it follows that also $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \leq \beta_\varepsilon(T_{1/\varepsilon}(\tilde{u}_\varepsilon))$ almost everywhere in Ω.

3.2.2 A priori estimates

Lemma 3.2.4. *For* $0 < \varepsilon \leq 1$ *and* $f \in L^\infty(\Omega)$ *let* $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ *be a solution of* (E_ε, f)*. Then,*

i.) there exists a constant $C_1 = C_1(\|f\|_\infty, \gamma, p(\cdot), N) > 0$*, not depending on* ε*, such that*

$$\||Du_\varepsilon|\|_{L^{p(\cdot)}(\Omega)} \leq C_1. \tag{3.2.3}$$

Moreover,

ii.)

$$\|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\|_\infty \leq \|f\|_{L^\infty(\Omega)} \tag{3.2.4}$$

holds for all $0 < \varepsilon \leq 1$ *and*

iii.)

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \leq k \int_{\{|u_\varepsilon|>l\}} |f| \tag{3.2.5}$$

holds for all $0 < \varepsilon \leq 1$ *and all* $l, k > 0$*.*

Proof: Taking u_ε as a test function in (3.2.1), by $(A1)$ we obtain

$$\gamma \int_\Omega |Du_\varepsilon|^{p(x)} \leq C(p(\cdot), N)\|f\|_{L^\infty(\Omega)}\||Du_\varepsilon|\|_{L^{p(\cdot)}(\Omega)}, \tag{3.2.6}$$

where $C(p(\cdot), N) > 0$ is a constant coming from the Hölder and Poincaré inequalities. From (2.1.2) and (3.2.6) it follows that either

$$\||Du_\varepsilon|\|_{L^{p(\cdot)}(\Omega)} \leq \left(\frac{1}{\gamma}\|f\|_{L^\infty(\Omega)} C(p(\cdot), N)\right)^{\frac{1}{p^- - 1}}$$

or

$$\||Du_\varepsilon|\|_{L^{p(\cdot)}(\Omega)} \leq \left(\frac{1}{\gamma}\|f\|_{L^\infty(\Omega)} C(p(\cdot), N)\right)^{\frac{1}{p^+ - 1}}.$$

Setting

$$C_1 := \max\left(\left(\frac{1}{\gamma}\|f\|_{L^\infty(\Omega)} C(p(\cdot), N)\right)^{\frac{1}{p^+ - 1}}, \left(\frac{1}{\gamma}\|f\|_{L^\infty(\Omega)} C(p(\cdot), N)\right)^{\frac{1}{p^- - 1}}\right),$$

we get *i.)*. Taking $\frac{1}{\delta}(T_{k+\delta}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))$ as a test function in (3.2.1), passing to the limit with $\delta \downarrow 0$ and choosing $k > \|f\|_\infty$ we obtain *ii.)*. For $k, l > 0$ fixed we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (3.2.1) to obtain *iii.)*. □

Remark 3.2.5. For $k > 0$, from Lemma 3.2.4, *iii.*), we deduce

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \le k\|f\|_\infty |\{|u_\varepsilon| > l\}| \le C_2(k) l^{-p^-} \tag{3.2.7}$$

for any $0 < \varepsilon \le 1$ and a constant $C_2(k) > 0$ not depending on ε. Indeed,

$$\begin{aligned} |\{|u_\varepsilon| \ge l\}| &\le \int_{\{|u_\varepsilon| \ge l\}} \frac{|T_l(u_\varepsilon)|^{p^-}}{l^{p^-}} \\ &\le C(p^-, N) l^{-p^-} \left(\int_{\{|Du_\varepsilon| \ge 1\}} |Du_\varepsilon|^{p(x)} + \int_{\{|Du_\varepsilon| < 1\}} |Du_\varepsilon|^{p^-} \right) \\ &\le C(p^-, N) l^{-p^-} \left(\int_\Omega |Du_\varepsilon|^{p(x)} + |\Omega| \right), \end{aligned} \tag{3.2.8}$$

where $C(p^-, N) > 0$ is a constant from the Poincaré inequality in $W_0^{1,p^-}(\Omega)$. Combining (3.2.6), (3.2.3) and (3.2.8), setting

$$C(p(\cdot), p^-, \gamma, C_1) := C(p^-, N) \left(\frac{C(p(\cdot), N)\|f\|_\infty}{\gamma} C_1 + |\Omega| \right) > 0,$$

we obtain

$$|\{|u_\varepsilon| \ge l\}| \le l^{-p^-} C(p(\cdot), p^-, \gamma, C_1). \tag{3.2.9}$$

Now, (3.2.7) follows from (3.2.9) with $C_2(k) := C(p(\cdot), p^-, \gamma, C_1) k \|f\|_\infty > 0$.

3.2.3 Basic convergence results

In the following it is always understood that ε takes values in a sequence in $(0, 1)$ tending to zero. The a priori estimates in Lemma 3.2.4 and Remark 3.2.5 imply the following basic convergences:

Lemma 3.2.6. *For $0 < \varepsilon \le 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$ be the solution of (E_ε, f). There exist $u \in W_0^{1,p(\cdot)}(\Omega)$, $b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\le 1}$ as $\varepsilon \downarrow 0$:*

$$u_\varepsilon \to u \text{ in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega \tag{3.2.10}$$

$$Du_\varepsilon \rightharpoonup Du \text{ in } (L^{p(\cdot)}(\Omega))^N \tag{3.2.11}$$

and

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{3.2.12}$$

Moreover, for any $k > 0$,

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \text{ in } (L^{p(\cdot)}(\Omega))^N \tag{3.2.13}$$

and

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \text{ in } (L^{p'(\cdot)}(\Omega))^N. \tag{3.2.14}$$

Proof: Since (3.2.10) - (3.2.13) follow directly from Lemma 3.2.4 and Remark 3.2.5, (3.2.14) is left to prove. To this end, we fix $k > 0$ and take $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.2.1). Using Gauss-Green Theorem for Sobolev functions and passing to the limit with $\varepsilon \downarrow 0$ and then with $l \to \infty$ we obtain

$$\limsup_{\varepsilon \downarrow 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D(T_k(u_\varepsilon) - T_k(u)) \le 0. \tag{3.2.15}$$

By $(A2)$ and (3.2.3) it follows that given any subsequence of $a(x, DT_k(u_\varepsilon))_\varepsilon$, there exists a subsequence, still denoted by $a(x, DT_k(u_\varepsilon))_\varepsilon$ such that

$$a(x, DT_k(u_\varepsilon))_\varepsilon \rightharpoonup \Phi_k \ \text{ weakly in } (L^{p'(\cdot)}(\Omega))^N.$$

We show that

$$\Phi_k(x) = a(x, DT_k(u)) \text{ for almost every } x \in \Omega, \tag{3.2.16}$$

which allows us to conclude that the whole sequence $a(x, DT_k(u_\varepsilon))_\varepsilon$ converges to $a(x, DT_k(u))$. To this end, we define the variational operator

$$\mathcal{A} : (L^{p(\cdot)}(\Omega))^N \to (L^{p'(\cdot)}(\Omega))^N$$

for $G \in (L^{p(\cdot)}(\Omega))^N$ by

$$(\mathcal{A}G)(H) = \int_\Omega a(x, G) \cdot H, \ H \in (L^{p(\cdot)}(\Omega))^N.$$

By $(A2)$, $\mathcal{A}$ ist well-defined and hemicontinuous, by $(A3)$ it is monotone, hence $\mathcal{A}$ is maximal monotone (see [65], Lemma 3.4, p. 88). Using $(A3)$ and (3.2.15), we calculate

$$\int_\Omega (\Phi_k - a(x, H)) \cdot (DT_k(u) - H) \ge 0 \text{ for all } H \in (L^{p(\cdot)}(\Omega))^N. \tag{3.2.17}$$

Since $\mathcal{A}$ is maximal monotone, (3.2.16) follows from (3.2.17). □

Remark 3.2.7. As an immediate consequence of (3.2.15) and $(A3)$ we obtain

$$\lim_{\varepsilon \downarrow 0} \int_\Omega a(x, DT_k(u_\varepsilon) - a(x, DT_k(u)) \cdot D(T_k u_\varepsilon - T_k(u)) = 0. \tag{3.2.18}$$

Combining (3.2.7) and (3.2.18), using the same arguments as in [6] it follows that

$$\lim_{l \to \infty} \int_{\{l < |u| < l+1\}} a(x, Du) \cdot Du = 0. \tag{3.2.19}$$

3.2.4 Proof of existence

Now, we are able to conclude the proof of Proposition 3.2.1:
Proof: Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be arbitrary. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (3.2.1), we obtain

$$I^1_{\varepsilon,l} + I^2_{\varepsilon,l} + I^3_{\varepsilon,l} + I^4_{\varepsilon,l} = I^5_{\varepsilon,l} \tag{3.2.20}$$

where

$$\begin{aligned}
I^1_{\varepsilon,l} &= \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h_l(u_\varepsilon)h(u)\phi,\\
I^2_{\varepsilon,l} &= \varepsilon \int_\Omega \arctan(u_\varepsilon)h_l(u_\varepsilon)h(u)\phi,\\
I^3_{\varepsilon,l} &= \int_\Omega a(x, Du_\varepsilon) \cdot D(h_l(u_\varepsilon)h(u)\phi),\\
I^4_{\varepsilon,l} &= \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)) \cdot D(h_l(u_\varepsilon)h(u)\phi),\\
I^5_{\varepsilon,l} &= \int_\Omega f h_l(u_\varepsilon)h(u)\phi.
\end{aligned}$$

Step 1: Passing to the limit with $\varepsilon \downarrow 0$
Obviously,

$$\lim_{\varepsilon \downarrow 0} I^2_{\varepsilon,l} = 0. \tag{3.2.21}$$

Using the convergence results (3.2.10), (3.2.12) from Lemma 3.2.6 we can immediately calculate the following limits:

$$\lim_{\varepsilon \downarrow 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\phi, \tag{3.2.22}$$

$$\lim_{\varepsilon \downarrow 0} I^5_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\phi. \tag{3.2.23}$$

We write

$$I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l},$$

where

$$\begin{aligned}
I^{3,1}_{\varepsilon,l} &= \int_\Omega h_l'(u_\varepsilon)a(x, Du_\varepsilon) \cdot Du_\varepsilon h(u)\phi,\\
I^{3,2}_{\varepsilon,l} &= \int_\Omega h_l(u_\varepsilon)a(x, Du_\varepsilon) \cdot D(h(u)\phi).
\end{aligned}$$

Using (3.2.7) we get the estimate

$$|\lim_{\varepsilon\downarrow 0} I^{3,1}_{\varepsilon,l}| \leq \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)} C_2(1) l^{-p^-}. \tag{3.2.24}$$

Since modular convergence is equivalent to norm convergence in $L^{p(\cdot)}(\Omega)$, by Lebesgue Dominated Convergence Theorem it follows that

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\phi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\phi)$$

for any $i \in \{1,\dots,N\}$ in $L^{p(\cdot)}(\Omega)$ as $\varepsilon \downarrow 0$. Keeping in mind that

$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon) a(x, DT_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi),$$

by (3.2.14), we get

$$\lim_{\varepsilon\downarrow 0} I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u) a(x, DT_{l+1}(u)) \cdot D(h(u)\phi). \tag{3.2.25}$$

Let us write

$$I^4_{\varepsilon,l} = I^{4,1}_{\varepsilon,l} + I^{4,2}_{\varepsilon,l},$$

where

$$\begin{aligned} I^{4,1}_{\varepsilon,l} &= \int_\Omega h'_l(u_\varepsilon) F(T_{1/\varepsilon}(u_\varepsilon)) \cdot Du_\varepsilon h(u)\phi, \\ I^{4,2}_{\varepsilon,l} &= \int_\Omega h_l(u_\varepsilon) F(T_{1/\varepsilon}(u_\varepsilon)) \cdot D(h(u)\phi). \end{aligned}$$

For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon)) F(T_{l+1}(u_\varepsilon)) \cdot DT_{l+1}(u_\varepsilon) h(u)\phi. \tag{3.2.26}$$

Using Gauss-Green Theorem for Sobolev functions in (3.2.26) we get

$$I^{4,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r) F(r) dr \cdot D(h(u)\phi). \tag{3.2.27}$$

Now, using (3.2.10) and the Gauss-Green Theorem, after the passage to the limit with $\varepsilon \downarrow 0$ we get

$$\lim_{\varepsilon\downarrow 0} I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(u) F(u) \cdot Du\; h(u)\phi. \tag{3.2.28}$$

Choosing ε small enough, we can write

$$I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi) \tag{3.2.29}$$

and conclude

$$\lim_{\varepsilon\downarrow 0} I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi). \tag{3.2.30}$$

Step 2: Passage to the limit with $l \to \infty$.
Combining (3.2.20) with (3.2.21) - (3.2.30) we find

$$I^1_l + I^2_l + I^3_l + I^4_l + I^5_l = I^6_l, \tag{3.2.31}$$

where

$$\begin{aligned} I^1_l &= \int_\Omega bh_l(u)h(u)\phi, \\ I^2_l &= \int_\Omega h_l(u)a(x, DT_{l+1}(u)) \cdot D(h(u)\phi), \\ |I^3_l| &\leq l^{-p^-} C_2(1)\|h\|_\infty \|\phi\|_\infty, \\ I^4_l &= \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi), \\ I^5_l &= \int_\Omega h'_l(u)F(u) \cdot Du\ h(u)\phi, \end{aligned}$$

and

$$I^6_l = \int_\Omega fh_l(u)h(u)\phi.$$

Obviously, we have

$$\lim_{l\to\infty} I^3_l = 0. \tag{3.2.32}$$

Choosing $m > 0$ such that $\operatorname{supp} h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I^1_l, \ldots, I^6_l$. Therefore, it follows that

$$\lim_{l\to\infty} I^1_l = \int_\Omega bh(u)\phi,, \tag{3.2.33}$$

$$\lim_{l\to\infty} I^2_l = \int_\Omega a(x, Du) \cdot D(h(u)\phi), \tag{3.2.34}$$

$$\lim_{l\to\infty} I^4_l = \int_\Omega F(u) \cdot D(h(u)\phi), \tag{3.2.35}$$

$$\lim_{l\to\infty} I^5_l = 0, \tag{3.2.36}$$

$$\lim_{l\to\infty} I^6_l = \int_\Omega fh(u)\phi. \tag{3.2.37}$$

Combining (3.2.31) with (3.2.32) - (3.2.37) we obtain

$$\int_\Omega bh(u)\phi + \int_\Omega (a(x,Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega fh(u)\phi \tag{3.2.38}$$

for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

3. Step: Subdifferential argument
It is left to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β a is maximal monotone graph, there exists a convex, l.s.c. and proper function $j : \mathbb{R} \to [0,\infty]$, such that

$$\beta(r) = \partial j(r) \text{ for all } r \in \mathbb{R}.$$

According to [28], for $0 < \varepsilon \leq 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the following properties:

i.) For any $0 < \varepsilon \leq 1$, j_ε is convex and differentiable for all $r \in \mathbb{R}$, such that

$$j_\varepsilon'(r) = \beta_\varepsilon(r) \text{ for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \leq 1$$

ii.) $j_\varepsilon(r) \uparrow j(r)$ for all $r \in \mathbb{R}$ as $\varepsilon \downarrow 0$.

From *i.)* it follows that for any $0 < \varepsilon \leq 1$

$$j_\varepsilon(r) \geq j_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{3.2.39}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω. Let $E \subset \Omega$ be an arbitrary measurable set and χ_E its characteristic function. We fix $\varepsilon_0 > 0$. Multiplying (3.2.39) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using *ii.)*, we obtain

$$j(r)\int_E h_l(u_\varepsilon) \geq \int_E j_{\varepsilon_0}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{3.2.40}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < \min(\varepsilon_0, \frac{1}{l})$. As $\varepsilon \downarrow 0$, taking into account that E is arbitrary we obtain from (3.2.40)

$$j(r)h_l(u) \geq j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{3.2.41}$$

for all $r \in \mathbb{R}$ almost everywhere in Ω. Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \downarrow 0$ in (3.2.41) finally yields

$$j(r) \geq j(u(x)) + b(x)(r - u(x)) \tag{3.2.42}$$

for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, hence $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω. With this last step the proof of Proposition 3.2.1 is concluded. $\square$

3.3 Weak solutions for L^∞-data

Definition 3.3.1. A weak solution to (S, f) is a pair of functions $(u, b) \in W_0^{1,p(\cdot)}(\Omega) \times L^1_{\mathrm{loc}}(\Omega)$ satisfying $F(u) \in (L^1_{\mathrm{loc}}(\Omega))^N$, $b \in \beta(u)$ almost everywhere in Ω and

$$b - \operatorname{div}(a(x, Du) + F(u)) = f \tag{3.3.1}$$

in $\mathcal{D}'(\Omega)$.

Remark 3.3.1. Note that if (u, b) is a renormalized solution to (E, f) such that $u \in W_0^{1,p(\cdot)}(\Omega)$, then (u, b) in general is not a weak solution in the sense of Definition 3.3.1, since we did not assume a growth condition on F and therefore $F(u)$ in general may fail to be locally integrable. If (u, b) is a renormalized solution of (E, f) such that $u \in L^\infty(\Omega)$, it is a direct consequence of Definition 3.1.1 that u is in $W_0^{1,p(\cdot)}(\Omega)$ and since (3.1.1) holds with the formal choice $h \equiv 1$, (u, b) is a weak solution. Indeed, let us choose $\varphi \in \mathcal{D}(\Omega)$ and plug $h_l(u)\varphi$ as a test function in (3.1.1). Since $u \in L^\infty(\Omega)$, we can pass to the limit with $l \to \infty$ and find that u solves (E, f) in the sense of distributions.

In the next proposition we will show that renormalized solutions to (E, f) for right-hand side $f \in L^\infty(\Omega)$ are weak solutions. In one space dimension this follows immediately from Remark 3.3.1 since $u \in W_0^{1,p(\cdot)}(\Omega)$ implies $u \in \mathcal{C}(\overline{\Omega})$ (see Proposition 3.4.6). Therefore, for the rest of this section we may assume $N \geq 2$.

Proposition 3.3.2. *Let (u, b) be a renormalized solution to (E, f) for $f \in L^\infty(\Omega)$. Then $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and thus, in particular, u is a weak solution to (E, f).*

Proof: From Lemma 3.2.6 it follows that $u \in W^{1,p(\cdot)}(\Omega)$. It suffices to prove that $u \in L^\infty(\Omega)$. For $\varepsilon, k > 0$, we take $h_l(u)\frac{1}{\varepsilon}T_\varepsilon(u - T_k(u))$ as a test function in (3.1.1). Neglecting positive terms and passing to the limit with $l \to \infty$, we obtain

$$\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |Du|^{p(x)} \leq \|f\|_N \left(\phi(k)\right)^{(N-1)/N}, \tag{3.3.2}$$

where $\phi(k) := |\{|u| > k\}|$ for $k > 0$. Now we use similar arguments as in [18]. We apply the continuous embedding of $W_0^{1,1}(\Omega)$ into $L^{N/(N-1)}(\Omega)$ and the Hölder inequality to get

$$\frac{1}{\varepsilon C_N}\|T_\varepsilon(u-T_k(u))\|_{\frac{N}{N-1}} \leq \left(\frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon}\right)^{1/(p^-)'} \left(\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |Du|^{p^-}\right)^{1/p^-}, \tag{3.3.3}$$

where $C_N > 0$ is the constant coming from the Sobolev embedding. Notice that

$$\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |Du|^{p^-} \leq \frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |Du|^{p(x)}, \quad (3.3.4)$$

hence from (3.3.2), (3.3.3) and (3.3.4) we deduce

$$\frac{1}{\varepsilon C_N}\|T_\varepsilon(u-T_k(u))\|_{\frac{N}{N-1}} \leq$$
$$\left(\frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon}\right)^{1/(p^-)'}\left(\frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon} + \|f\|_N\,(\phi(k))^{(N-1)/N}\right)^{1/p^-}. \quad (3.3.5)$$

From (3.3.5) and Young's inequality with $\alpha > 0$ it follows that

$$\frac{1}{C_N C}(\phi(k+\varepsilon))^{(N-1)/N} - \frac{\alpha^{p^-}}{p^- C}\|f\|_N\,(\phi(k))^{(N-1)/N} - \frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon} \leq 0, \quad (3.3.6)$$

where

$$C := \left(\frac{1}{\alpha^{(p^-)'}(p^-)'} + \frac{\alpha^{p^-}}{p^-}\right) > 0.$$

The mapping $(0,\infty) \ni k \to \phi(k)$ is non-increasing and therefore of bounded variation, hence it is differentiable almost everywhere on $(0,\infty)$ with $\phi' \in L^1_{\text{loc}}(0,\infty)$. Since it is also continuous from the right, we can pass to the limit with $\varepsilon \downarrow 0$ in (3.3.6) to find

$$C''(\phi(k))^{(N-1)/N} + \phi'(k) \leq 0 \quad (3.3.7)$$

for almost every $k > 0$ and $\alpha > 0$ chosen small enough such that

$$C'' := \left(\frac{C_N}{C} - \frac{\alpha^{p^-}}{p^- C}\|f\|_N\right) > 0.$$

Now, the conclusion of the proof follows by contradiction. We assume that $\phi(k) > 0$ for each $k > 0$. For $k > 0$ fixed, we choose $k_0 < k$. From (3.3.7) it follows that

$$\frac{1}{N}C'' + \frac{d}{ds}\left((\phi(s))^{(1/N)}\right) \leq 0 \quad (3.3.8)$$

for almost all $s \in (k_0, k)$. The left hand side of (3.3.8) is in $L^1(k_0,k)$, hence we integrate (3.3.8) over $[k_0,k]$. Moreover, since ϕ is non-increasing, integrating (3.3.8) over (k_0,k) we get

$$(\phi(k))^{1/N} \leq \phi(k_0)^{1/N} + \frac{1}{N}C''(k_0 - k) \quad (3.3.9)$$

and from (3.3.9) the contradiction follows. □

3.4 Proof of Theorem 3.1.2

3.4.1 Approximate solutions for L^1-data

The comparison principle from Proposition 3.2.2 will be the main tool in the second approximation procedure. For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ let $f_{m,n} \in L^\infty(\Omega)$ be defined as in the beginning of Section 4. From Proposition 3.2.1 it follows that for any $m, n \in \mathbb{N}$ there exists $u_{m,n} \in W_0^{1,p(\cdot)}(\Omega)$, $b_{m,n} \in L^\infty(\Omega)$, such that $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$. Therefore

$$\int_\Omega b_{m,n} h(u_{m,n})\phi + \int_\Omega (a(x, Du_{m,n}) + F(u_{m,n})) \cdot D(h(u_{m,n})\phi) = \int_\Omega f_{m,n} h(u_{m,n})\phi \tag{3.4.1}$$

holds for all $m, n \in \mathbb{N}$, $h \in \mathcal{C}_c^1(\mathbb{R})$, $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. In the next Lemma, we give a priori estimates that will be important in the following:

Lemma 3.4.1. *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then,*

i.) For any $k > 0$ we have

$$\int_\Omega |DT_k(u_{m,n})|^{p(x)} \leq \frac{k}{\gamma} \|f\|_1. \tag{3.4.2}$$

ii.) For $k > 0$, there exists a constant $C_3(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that

$$\| |DT_k(u_{m,n})| \|_{p(\cdot)} \leq C_3(k). \tag{3.4.3}$$

iii.)

$$\|b_{m,n}\|_1 \leq \|f\|_1 \tag{3.4.4}$$

holds for all $m, n \in \mathbb{N}$.

Proof: Proof: For $l, k > 0$, we plug $h_l(u_{m,n})T_k(u_{m,n})$ as a test function in (3.4.1). Then *i.)* and *ii.)* follow with similar arguments as used in the proof of Lemma 3.2.4. To prove *iii.)*, we neglect the positive term

$$\int_\Omega a(x, DT_k(u_{m,n}))DT_k(u_{m,n})$$

and keep

$$\int_\Omega b_{m,n} T_k(u_{m,n}) \leq \int_\Omega f_{m,n} u_{m,n}. \tag{3.4.5}$$

Since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, from (3.4.5) it follows that

$$\int_{\{|u_{m,n}|>k\}} |b_{m,n}| \leq \int_\Omega |f|, \tag{3.4.6}$$

and we find *iii.)* by passing to the limit with $k \downarrow 0$. □

By definition we have

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n} \tag{3.4.7}$$

From Proposition 3.2.2 it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \text{ and } u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n}, \tag{3.4.8}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$, hence passing to the limit with $\varepsilon \downarrow 0$ in (3.4.8) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n}, \tag{3.4.9}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. Setting $b_\varepsilon := \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$, using (3.4.8), Remark 3.2.3 and the fact that $b^\varepsilon_{m,n} \overset{*}{\rightharpoonup} b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order we get

$$b_{m,n} \leq b_{m+1,n} \text{ and } b_{m,n+1} \leq b_{m,n} \tag{3.4.10}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. By (3.4.10) and (3.4.4), for any $n \in \mathbb{N}$ there exists $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ as $m \to \infty$ in $L^1(\Omega)$ and almost everywhere and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. By (3.4.9), the sequence $(u_{m,n})_m$ is monotone increasing, hence, for any $n \in \mathbb{N}$, $u_{m,n} \to u^n$ almost everywhere in Ω, where $u^n : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Using (3.4.9) again, we conclude that the sequence $(u^n)_n$ is monotone decreasing, hence $u^n \to u$ almost everywhere in Ω, where $u : \Omega \to \overline{\mathbb{R}}$ is a measurable function. In order to show that that u is finite almost everywhere we will give an estimate on the level sets of $u_{m,n}$ in the next lemma:

Lemma 3.4.2. *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then, there exist a constant $C_4 > 0$, not depending on $m, n \in \mathbb{N}$, such that*

$$|\{|u_{m,n}| \geq l\}| \leq C_4 \, l^{-(p^- -1)} \tag{3.4.11}$$

for all $l \geq 1$.

Proof: With the same arguments as in Remark 3.2.5 we obtain

$$|\{|u_{m,n}| \geq l\}| \leq C(p^-, N) l^{-p^-} \left(\int_\Omega |DT_l(u_{m,n})|^{p(x)} + |\Omega| \right), \tag{3.4.12}$$

for all $m, n \in \mathbb{N}$ where $C(p^-, N)$ is the constant from Sobolev embedding in $L^{p^-}(\Omega)$. Now, we plug (3.4.2) into (3.4.12) to obtain (3.4.11). $\square$

Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m,

$$\lim_{m\to\infty} |\{u_{m,n} > l\}| = |\{u^n > l\}| \tag{3.4.13}$$

and

$$\lim_{m\to\infty} |\{u_{m,n} \leq -l\}| = |\{u^n \leq -l\}|. \tag{3.4.14}$$

Combining (3.4.11) with (3.4.13) and (3.4.14) we get

$$|\{u^n \leq -l\}| + |\{u^n > l\}| \leq C_4 \; l^{-(p^- - 1)}, \tag{3.4.15}$$

for any $l \geq 1$, hence u^n is finite almost everywhere for any $n \in \mathbb{N}$. By the same arguments we get

$$|\{u < -l\}| + |\{u > l\}| \leq C_4 \; l^{-(p^- - 1)} \tag{3.4.16}$$

from (3.4.15), hence u is finite almost everywhere. Now, since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Ω it follows by a subdifferential argument that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ almost everywhere in Ω.

Remark 3.4.3. If $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$, using $h_\nu(u_{m,n}) T_k(u_{m,n} - T_l(u_{m,n}))$ as a test function in (3.4.1), neglecting positive terms and passing to the limit with $\nu \to \infty$ we obtain

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}) \cdot Du_{m,n} \leq k \left(\int_{\{|u_{m,n}|>l\}\cap\{|f|<\sigma\}} |f| + \int_{\{|f|>\sigma\}} |f| \right) \tag{3.4.17}$$

for any $k, l, \sigma > 0$. Now, applying (3.4.11) to (3.4.17), we find that

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}) \cdot Du_{m,n} \leq \sigma k C_4 \; l^{-(p^- - 1)} + k \int_{\{|f|>\sigma\}} |f| \tag{3.4.18}$$

holds for any $k, \sigma > 0$, $l \geq 1$ uniformly in $m, n \in \mathbb{N}$.

Lemma 3.4.4. *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting $f_n := f_{m(n),n}, b_n := b_{m(n),n}, u_n := u_{m(n),n}$ we have*

$$u_n \to u \text{ almost everywhere in } \Omega. \tag{3.4.19}$$

Moreover, for any $k > 0$,

$$T_k(u_n) \to T_k(u) \text{ in } L^{p(\cdot)}(\Omega) \text{ and almost everywhere in } \Omega, \tag{3.4.20}$$

$$DT_k(u_n) \rightharpoonup DT_k(u) \text{ in } (L^{p(\cdot)}(\Omega))^N, \tag{3.4.21}$$

$$a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u)) \text{ in } (L^{p'(\cdot)}(\Omega))^N. \tag{3.4.22}$$

as $n \to \infty$.

Proof: Applying the diagonal principle in $L^1(\Omega)$, we construct a subsequence $(m(n))_n$, such that

$$\arctan(u_{m(n),n}) \to \arctan(u),$$

$$b_n := b_{m(n),n} \to b,$$
$$f_n := f_{m(n),n} \to f$$

as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. It follows that (3.4.19) and (3.4.20) hold. Combining (3.4.20) with (3.4.3) we get $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$, $T_k(u_n) \to T_k(u)$ in $L^{p(\cdot)}(\Omega)$ and (3.4.21) holds for any $k > 0$. From (3.4.2) and $(A2)$ it follows, that, for fixed $k > 0$, given any subsequence of $a(x, DT_k(u_n))_n$, there exists a subsequence, still denoted by $a(x, DT_k(u_n))_n$, such that

$$a(x, DT_k(u_n))_n \rightharpoonup \Phi_k \text{ in } (L^{p'(\cdot)}(\Omega))^N$$

as $n \to \infty$. Since $h_l(u_n)(T_k(u_n) - T_k(u))$ is an admissible test function in (3.4.1),

$$\limsup_{n\to\infty} \int_\Omega a(x, DT_k(u_n)) \cdot D(T_k(u_n) - T_k(u)) \leq 0 \tag{3.4.23}$$

holds. Then, (3.4.22) follows with the same arguments as in the proof of Lemma 3.2.6. □

Remark 3.4.5. With the same arguments as in Remark 3.2.7, we have

$$\lim_{n\to\infty} \int_\Omega a(x, DT_k(u_n) - a(x, DT_k(u))) \cdot D(T_k(u_n) - T_k(u)) = 0, \tag{3.4.24}$$

$$\lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Du = 0. \tag{3.4.25}$$

3.4.2 Conclusion of the proof of Theorem 3.1.2

It is left to prove that (u, b) satisfies

$$\int_\Omega bh(u)\phi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega fh(u)\phi. \tag{3.4.26}$$

for all $h \in \mathcal{C}_c^1(\mathbb{R})$, $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. To this end, we take $h \in \mathcal{C}_c^1(\mathbb{R})$, $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ arbitrary and plug $h_l(u_n)h(u)\phi$ into (3.4.1) to obtain

$$I_{n,l}^1 + I_{n,l}^2 + I_{n,l}^3 = I_{n,l}^4, \tag{3.4.27}$$

where

$$\begin{aligned} I_{n,l}^1 &= \int_\Omega b_n h_l(u_n)h(u)\phi \\ I_{n,l}^2 &= \int_\Omega a(x, Du_n) \cdot D(h_l(u_n)h(u)\phi) \\ I_{n,l}^3 &= \int_\Omega F(u_n) \cdot D(h_l(u_n)h(u)\phi) \\ I_{n,l}^4 &= \int_\Omega f_n h_l(u_n)h(u)\phi. \end{aligned}$$

Step 1: Passing to the limit with $n \to \infty$
Applying the convergence results from Lemma 3.4.4 we get

$$\lim_{n\to\infty} I_{n,l}^1 = \int_\Omega bh_l(u)h(u)\phi, \tag{3.4.28}$$

$$\lim_{n\to\infty} I_{n,l}^2 = \int_\Omega fh_l(u)h(u)\phi. \tag{3.4.29}$$

Let us write

$$I_{n,l}^2 = I_{n,l}^{2,1} + I_{n,l}^{2,2},$$

where

$$\begin{aligned} I_{n,l}^{2,1} &= \int_\Omega h_l(u_n)a(x, Du_n) \cdot D(h(u)\phi), \\ I_{n,l}^{2,2} &= \int_\Omega h_l'(u_n)a(x, Du_n) \cdot Du_n \ h(u)\phi. \end{aligned}$$

With similar arguments as in the proof of (3.2.25) it follows that

$$\lim_{n\to\infty} I_{n,l}^{2,1} = \int_\Omega h_l(u)a(x, Du) \cdot D(h(u)\phi). \tag{3.4.30}$$

By (3.4.18), we get the estimate

$$|\lim_{n\to\infty} I^{2,2}_{n,l}| \leq \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}\left(\sigma C_4\ l^{-(p^- -1)} + \int_{\{|f|>\sigma\}}|f|\right) \tag{3.4.31}$$

for all $n \in \mathbb{N}$ and all $l \geq 1$, $\sigma > 0$. Next, we write

$$I^3_{n,l} = I^{3,1}_{n,l} + I^{3,2}_{n,l},$$

where

$$\lim_{n\to\infty} I^{3,1}_{n,l} = \int_\Omega h_l(u)F(u)\cdot D(h(u)\phi), \tag{3.4.32}$$

$$\lim_{n\to\infty} I^{3,2}_{n,l} = \int_\Omega h_l'(u)F(u)\cdot Du\ h(u)\phi \tag{3.4.33}$$

follows with the same arguments as in (3.2.26) - (3.2.30).

Step 2: Passage to the limit with $l \to \infty$.
Combining (3.4.27) with (3.4.28) - (3.4.33) we get for all $\sigma > 0$ and all $l \geq 1$

$$I^1_l + I^2_l + I^3_l + I^4_l + I^5_l = I^6_l \tag{3.4.34}$$

where

$$\begin{aligned}
I^1_l &= \int_\Omega bh_l(u)h(u)\phi, \\
I^2_l &= \int_\Omega h_l(u)a(x, DT_{l+1}(u))\cdot D(h(u)\phi), \\
|I^3_l| &\leq \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}\left(\sigma C_4\ l^{-(p^- -1)} + \int_{\{|f|>\sigma\}}|f|\right),
\end{aligned}$$

for any $\sigma > 0$ and

$$\begin{aligned}
I^4_l &= \int_\Omega h_l'(u)F(u)\cdot h(u)\phi Du, \\
I^5_l &= \int_\Omega h_l(u)F(u)\cdot D(h(u)\phi), \\
I^5_l &= \int_\Omega fh_l(u)h(u)\phi.
\end{aligned}$$

Choosing $m > 0$ such that $\operatorname{supp} h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, \ldots, I_l^6$, hence

$$\begin{aligned}
\lim_{l\to\infty} I_l^1 &= \int_\Omega bh(u)\phi, && (3.4.35)\\
\lim_{l\to\infty} I_l^2 &= \int_\Omega a(x, Du) \cdot D(h(u)\phi), && (3.4.36)\\
\lim_{l\to\infty} |I_l^3| &\leq \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)} \int_{\{|f|>\sigma\}} |f|, && (3.4.37)
\end{aligned}$$

$$\begin{aligned}
\lim_{l\to\infty} I_l^4 &= 0, && (3.4.38)\\
\lim_{l\to\infty} I_l^5 &= \int_\Omega F(u) \cdot D(h(u)\phi), && (3.4.39)\\
\lim_{l\to\infty} I_l^6 &= \int_\Omega fh(u)\phi && (3.4.40)
\end{aligned}$$

for all $\sigma > 0$. Combining (3.4.34) with (3.4.35) - (3.4.40) we finally obtain that (3.4.1) holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\phi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. □

3.4.3 Existence in one space dimension

For $N = 1$, $\Omega = (a, b)$ with $a, b \in \mathbb{R}$, $a < b$ the following improved existence result holds:

Proposition 3.4.6. *For $f \in L^1(a, b)$ there exists at least one weak solution (u, b) to (E, f) in the sense of Defintion 3.3.1.*

Proof: We fix $f \in L^1(a, b)$. By the continuous embedding of $W_0^{1,p(\cdot)}(a, b)$ into $\mathcal{C}([a, b])$ we have $L^1(a, b) \subset (W_0^{1,p(\cdot)}(a, b))'$. Now it follows from [55], Theorem 2.7 that for any $\varepsilon > 0$ there exists $u^\varepsilon \in W_0^{1,p(\cdot)}(a, b)$ such that

$$\beta_\varepsilon(u^\varepsilon) - (a(x, (u^\varepsilon)_x) + F(u^\varepsilon))_x = f \tag{3.4.41}$$

holds in $\mathcal{D}'(a, b)$. For right hand sides $f_{m,n} \in L^\infty(a, b)$ as defined in Section 3.2, all a priori estimates stated in Lemma 3.2.4 hold uniformly in $\varepsilon > 0$. Moreover, the sequence $(u_{m,n}^\varepsilon)$ is uniformly bounded in $L^\infty(a, b)$ for $\varepsilon > 0$ and $m, n \in \mathbb{N}$. Therefore, using similar arguments as in the conclusion of the proof of Theorem 3.1.2, we find that (u, b) is a weak solution to (E, f).

3.5 Uniqueness of renormalized solutions

In this section, we prove a uniqeness result for renormalized solutions to the problem (E, f) with $f \in L^1(\Omega)$.

Theorem 3.5.1. *For $f, \tilde{f} \in L^1(\Omega)$ let (u, b) be a renormalized solution of (S, f) and $(\tilde{u}, \tilde{b})$ be a renormalized solution of $(S, \tilde{f})$. Then, the following comparison principle holds:*

$$\int_\Omega (b-\tilde{b})^+ \leq \int_\Omega (f-\tilde{f})\,\mathrm{sign}_0^+(u-\tilde{u}) + \int_{\{u=\tilde{u}\}} (f-\tilde{f})\,\mathrm{sign}_0^+(b-\tilde{b}). \quad (3.5.1)$$

Proof: We choose $\pi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ such that $0 \leq \pi \leq 1$ almost everywhere in Ω. For $l > 0$ arbitrary, we use $h_l(u)H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi)$ as a test function in the renormalized formulation for (u, b) and $h_l(\tilde{u})H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi)$ as a test function in the renormalized formulation for $(\tilde{u}, \tilde{b})$. Subtracting the resulting equalities, we obtain

$$I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 + I_{l,\delta}^5 + I_{l,\delta}^6 + I_{l,\delta}^7 = I_{l,\delta}^8, \quad (3.5.2)$$

where $M := \{0 < T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi < \delta\}$ and

$$\begin{aligned}
I_{l,\delta}^1 &= \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi),\\
I_{l,\delta}^2 &= \int_\Omega (h_l'(u)a(x,Du)\cdot Du - h_l'(\tilde{u})a(x,D\tilde{u})\cdot D\tilde{u})H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi),\\
I_{l,\delta}^3 &= \frac{1}{\delta}\int_M (h_l(u)a(x,Du) - h_l(\tilde{u})a(x,D\tilde{u}))\cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u})),\\
I_{l,\delta}^4 &= \int_M (h_l(u)a(x,Du) - h_l(\tilde{u})a(x,D\tilde{u}))\cdot D\pi,\\
I_{l,\delta}^5 &= \int_\Omega (h_l'(u)F(u)\cdot Du - h_l'(\tilde{u})F(\tilde{u})\cdot D\tilde{u})H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi),\\
I_{l,\delta}^6 &= \frac{1}{\delta}\int_M (h_l(u)F(u) - h_l(\tilde{u})F(\tilde{u}))\cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u})),\\
I_{l,\delta}^7 &= \int_M (h_l(u)F(u) - h_l(\tilde{u})F(\tilde{u}))\cdot D\pi,\\
I_{l,\delta}^8 &= \int_\Omega (fh_l(u) - \tilde{f}h_l(\tilde{u}))H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi).
\end{aligned}$$

1. Step: Passage to the limit as $\delta \downarrow 0$
Since $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi) \to \mathrm{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) + \chi_{\{T_{l+1}(u)=T_{l+1}(\tilde{u})\}}\pi$

as $\delta \downarrow 0$ almost everywhere in Ω it follows that

$$\lim_{\delta\downarrow 0} I^1_{l,\delta} = \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))(\mathrm{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) + \chi_{\{T_{l+1}(u)=T_{l+1}(\tilde{u})\}}\pi), \tag{3.5.3}$$

$$\lim_{\delta\downarrow 0} I^7_{l,\delta} = 0, \tag{3.5.4}$$

$$\lim_{\delta\downarrow 0} I^8_{l,\delta} = \int_\Omega (fh_l(u) - \tilde{f}h_l(\tilde{u}))(\mathrm{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) + \chi_{\{T_{l+1}(u)=T_{l+1}(\tilde{u})\}}\pi). \tag{3.5.5}$$

Let us recall that, because of the definition of h_l, we can replace u by $T_{l+1}(u)$ and $\tilde{u}$ by $T_{l+1}(\tilde{u})$ which belong to $W_0^{1,p(\cdot)}(\Omega)$ in $I^1_{l,\delta}, \ldots, I^8_{l,\delta}$ and so $DT_{l+1}(u) = DT_{l+1}(\tilde{u})$ almost everywhere in $\{T_{l+1}(u) = T_{l+1}(\tilde{u})\}$. Therefore,

$$\lim_{\delta\downarrow 0} I^2_{l,\delta} = \int_\Omega (h_l'(u)a(x, Du)\cdot Du - h_l'(\tilde{u})a(x, D\tilde{u})\cdot D\tilde{u})(\mathrm{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) \tag{3.5.6}$$

and

$$\lim_{\delta\downarrow 0} I^4_{l,\delta} = 0. \tag{3.5.7}$$

Let us write

$$I^3_{l,\delta} = I^{3,1}_{l,\delta} + I^{3,2}_{l,\delta},$$

where

$$\begin{aligned} I^{3,1}_{l,\delta} &= \frac{1}{\delta}\int_M (h_l(u) - h_l(\tilde{u}))a(x, DT_{l+1}(u)) \cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u})),\\ I^{3,2}_{l,\delta} &= \frac{1}{\delta}\int_M h_l(\tilde{u})(a(x, DT_{l+1}(u)) - a(x, DT_{l+1}(\tilde{u}))) \cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u})). \end{aligned}$$

By $(A3)$, $I^{3,2}_{l,\delta}$ is nonnegative. As $\|h_l'\|_\infty \leq 1$ for all $l > 0$, we have the estimate

$$|I^{3,1}_{l,\delta}| \leq \int_{\{0<T_{l+1}(u)-T_{l+1}(\tilde{u})<\delta\}} |a(x, DT_{l+1}(u)) \cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u}))| \tag{3.5.8}$$

and from (3.5.8) it follows that

$$\limsup_{\delta\downarrow 0} I^3_{l,\delta} \geq 0. \tag{3.5.9}$$

Now, we write

$$\begin{aligned} I^5_{l,\delta} &= \int_\Omega \mathrm{div}\left(\int_{T_{l+1}(\tilde{u})}^{T_{l+1}(u)} h_l'(r)F(r)dr\right) H_\delta(T_{l+1}(u) - T_{l+1}(\tilde{u}) + \delta\pi)\\ &= I^{5,1}_{l,\delta} + I^{5,2}_{l,\delta}, \end{aligned}$$

where

$$
\begin{aligned}
I_{l,\delta}^{5,1} &= -\int_{\{0<T_{l+1}(u)-T_{l+1}(\tilde{u})+\delta\pi<\delta\}} \int_{T_{l+1}(u)}^{T_{l+1}(\tilde{u})} h_l'(r)F(r)dr \cdot D\pi, \\
I_{l,\delta}^{5,2} &= -\frac{1}{\delta}\int_{\{0<T_{l+1}(u)-T_{l+1}(\tilde{u})+\delta\pi<\delta\}} \int_{T_{l+1}(\tilde{u})}^{T_{l+1}(u)} h_l'(r)F(r)dr \cdot D(T_{l+1}(u)-T_{l+1}(\tilde{u})).
\end{aligned}
$$

It is easy to calculate that

$$
\lim_{\delta\downarrow 0} I_{l,\delta}^{5,1} = 0 \tag{3.5.10}
$$

and from

$$
|I_{l,\delta}^{5,2}| \le \max_{s\in[-l-1,l+1]} |F(s)| \int_{\{0<T_{l+1}(u)-T_{l+1}(\tilde{u})<\delta\}} |D(T_{l+1}(u)-T_{l+1}(\tilde{u}))| \tag{3.5.11}
$$

it follows that

$$
\lim_{\delta\downarrow 0} I_{l,\delta}^{5} = 0. \tag{3.5.12}
$$

Let us write

$$
I_{l,\delta}^{6} = I_{l,\delta}^{6,1} + I_{l,\delta}^{6,2},
$$

where

$$
\begin{aligned}
I_{l,\delta}^{6,1} &= \frac{1}{\delta}\int_M h_l(u)(F(T_{l+1}(u))-F(T_{l+1}(\tilde{u}))) \cdot D(T_{l+1}(u)-T_{l+1}(\tilde{u})), \\
I_{l,\delta}^{6,2} &= \frac{1}{\delta}\int_M (h_l(u)-h_l(\tilde{u}))F(T_{l+1}(\tilde{u})) \cdot D(T_{l+1}(u)-T_{l+1}(\tilde{u})).
\end{aligned}
$$

Let $L_F > 0$ be the Lipschitz constant of F. Then we find

$$
|I_{l,\delta}^{6,1}| \le L_F \int_{\{0<T_{l+1}(u)-T_{l+1}(\tilde{u})<\delta\}} |D(T_{l+1}(u)-T_{l+1}(\tilde{u}))|, \tag{3.5.13}
$$

$$
|I_{l,\delta}^{6,2}| \le \max_{s\in[-l-1,l+1]} |F(s)| \int_{\{0<T_{l+1}(u)-T_{l+1}(\tilde{u})<\delta\}} |D(T_{l+1}(u)-T_{l+1}(\tilde{u}))|, \tag{3.5.14}
$$

hence, from (3.5.13) and (3.5.14) it follows that

$$
\lim_{\delta\downarrow 0} I_{l,\delta}^{6} = 0. \tag{3.5.15}
$$

Combining (3.5.2) with (3.5.3) - (3.5.15) we obtain

$$
I_l^1 + I_l^2 \le I_l^3, \tag{3.5.16}
$$

where

$$
\begin{aligned}
I_l^1 &= \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))(\operatorname{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) + \chi_{\{T_{l+1}(u)=T_{l+1}(\tilde{u})\}}\pi),\\
I_l^2 &= \int_\Omega (h_l'(u)a(x,Du)\cdot Du - h_l'(\tilde{u})a(x,D\tilde{u})\cdot D\tilde{u})(\operatorname{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})),\\
I_l^3 &= \int_\Omega (fh_l(u) - \tilde{f}h_l(\tilde{u}))(\operatorname{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) + \chi_{\{T_{l+1}(u)=T_{l+1}(\tilde{u})\}}\pi).
\end{aligned}
$$

2. Step: Passage to the limit with $l \to \infty$
Thanks to (3.4.25) it follows that

$$\lim_{l\to\infty} I_l^2 = 0. \tag{3.5.17}$$

Since we have $h_l(u) = 0$ almost everywhere on $\{|u| \geq l+1\}$ and $h_l(\tilde{u}) = 0$ almost everywhere on $\{|\tilde{u}| \geq l+1\}$ it follows that

$$I_l^1 = I_l^{1,1} + I_l^{1,2},$$

where

$$
\begin{aligned}
I_l^{1,1} &= \int_{\{|u|<l+1\}\cap\{|\tilde{u}|<l+1\}} (bh_l(u) - \tilde{b}h_l(\tilde{u}))\chi_{\{u=\tilde{u}\}}\pi,\\
I_l^{1,2} &= \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))\operatorname{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})).
\end{aligned}
$$

Using that $u, \tilde{u}$ are almost everywhere finite, we have

$$\lim_{l\to\infty} I_l^{1,1} = \int_\Omega (b - \tilde{b})\chi_{\{u=\tilde{u}\}}\pi. \tag{3.5.18}$$

and

$$\lim_{l\to\infty} \operatorname{sign}_0^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) = \operatorname{sign}_0^+(u - \tilde{u}) \tag{3.5.19}$$

almost everywhere in Ω and weak-$*$ in $L^\infty(\Omega)$. Therefore,

$$\lim_{l\to\infty} I_l^{1,2} = \int_\Omega (b - \tilde{b})\operatorname{sign}_0^+(u - \tilde{u}). \tag{3.5.20}$$

With similar arguments we conclude

$$\lim_{l\to\infty} I_l^3 = \int_\Omega (f - \tilde{f})(\operatorname{sign}_0^+(u - \tilde{u}) + \chi_{\{u=\tilde{u}\}}\pi). \tag{3.5.21}$$

Combining (3.5.16) with (3.5.17) - (3.5.21) we get

$$\int_\Omega (b-\tilde{b})(\mathrm{sign}_0^+(u-\tilde{u})+\chi_{\{u=\tilde{u}\}}\pi)) \leq \int_\Omega (f-\tilde{f})(\mathrm{sign}_0^+(u-\tilde{u})+\chi_{\{u=\tilde{u}\}}\pi) \tag{3.5.22}$$

for any $\pi \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ such that $0 \leq \pi \leq 1$ almost everywhere in Ω.

3. Step:
Following the idea of [22], for $n \in \mathbb{N}$ we choose $\pi = T_1(\pi_n)$ in (3.5.22), where $(\pi_n)_n \subset \mathcal{C}_c^\infty(\Omega)$ is an approximation of $\mathrm{sign}_0^+(b-\tilde{b})$ in $L^1(\Omega)$. Passing to the limit with $n \to \infty$, we finally obtain (3.5.1). □

Remark 3.5.2. Let $f \in L^1(\Omega)$. Given any two renormalized solutions (u,b), $(\tilde{u},\tilde{b})$ of (E,f), it is an immediate consequence of Theorem 3.5.1 that $b=\tilde{b}$ almost everywhere in Ω. If β is a strictly increasing, continuous function then it follows immediately that $u=\tilde{u}$. If β is a monotone graph, $F \equiv 0$ and $a(x,Du)$ is strictly monotone, using the fact that (u,b), $(\tilde{u},\tilde{b})$ are renormalized solutions to (S,f) iff they are entropy solutions (see [67] for a definition) we obtain $u=\tilde{u}$ almost everywhere in Ω by similar arguments as in [67]. In the general case with convection, we cannot get $u=\tilde{u}$ almost everywhere in Ω from Theorem 3.5.1 even if $p(\cdot)$ is assumed to be constant. A one-dimensional counterexample can be found in [26].

3.6 Extensions and remarks

The condition that F is locally Lipschitz continuous is not crucial for the existence of renormalized solutions to (E,f) with $f \in L^1(\Omega)$. Indeed, any continuous function $F:\mathbb{R}\to\mathbb{R}^N$ can be approximated uniformly on compact sets by a sequence $(F_k)_k$ of Lipschitz continuous functions $F_k:\mathbb{R}\to\mathbb{R}^N$. Given $f_{m,n} \in L^\infty(\Omega)$ as in the beginning of Section 3.2, the weak solutions $u_\varepsilon^{k,m,n}$ of the approximate problems $(E_\varepsilon^k, f_{m,n})$ with locally Lipschitz continuous flux function F_k will converge to a weak solution $u_\varepsilon^{m,n}$ of $(E_\varepsilon, f_{m,n})$ with continuous flux function F and the comparison principle of Proposition 3.2.2 still holds for $u_\varepsilon^{m,n}$. Therefore, we are able to construct a sequence of approximate solutions $(u_n,b_n)_n$ as in Lemma 3.4.4. Using the same arguments as in the conclusion of the proof of Theorem 3.1.2, we obtain a renormalized solution (u,b) of (E,f) with continuous flux function F. The uniqueness of renormalized solutions of (E,f), however, is an open problem if F is only continuous. If $a=a(Du)$ does not depend on the space variable x (by $(A1)$ and $(A2)$ this implies $p(\cdot)\equiv p$ is constant), according to [32], uniqueness can be proved by the method of doubling variables.

Even if we assume $a(x,\xi) = A(x)\xi$ for all $\xi \in \mathbb{R}^N$ where $A(x) = (A_{i,j}(x))_{i,j} \in \mathbb{R}^{N\times N}$ for any $x \in \Omega$, $A_{i,j} \in W^{1,\infty}(\Omega)$ for all $i, j = 1, \dots N$, $\gamma > 0$ such that $\xi^T A(x)\xi \geq \gamma|\xi|^2$ holds for all $\xi \in \mathbb{R}^N$, $x \in \Omega$, β to be the identity mapping and L^∞-data, the method of doubling variables does not apply. Note, however, the following uniqueness result which we have been able to establish in the particular case of linear diffusion in one space dimension:

Proposition 3.6.1. *For $a, b \in \mathbb{R}$, $a < b$, let $F : \mathbb{R} \to \mathbb{R}$ be continuous, $A \in L^\infty(a,b)$ such that there exists $\gamma > 0$ with $A(x) \geq \gamma$ for almost all $x \in (a,b)$ and let f be in $L^1(a,b)$. Then the weak solution $u \in H_0^1(a,b)$ of*

$$(PB)\begin{cases} u - (A(x)u_x)_x - (F(u))_x = f \text{ in } (a,b), \\ u(a) = 0, u(b) = 0, \end{cases}$$

is unique.

Proof: Let f be in $L^1(\Omega)$. From Proposition 3.4.6 it follows that there exists at least one weak solution to (PB). The proof is based on the continuity of weak solutions of (PB). If $u, v \in H^1(a,b)$ are two weak solutions of (PB) with right-hand side f, we will identify $u, v \in H^1(a,b)$ with their continuous representatives $u, v \in \mathcal{C}([a,b])$ without changing notation. Assuming that there exists $x_0 \in]a,b[$ such that $u(x_0) \neq v(x_0)$, from the continuity of u, v it follows that there exist $c, d \in \mathbb{R}$, $c < d$, an interval $(c,d) \subset (a,b)$, such that $u(c) = v(c)$, $u(d) = v(d)$, $x_0 \in (c,d)$ and $u > v$ or $u < v$ on (c,d). Now, we will show that our assumption leads to a contradiction. The proof will be divided into several steps.

Step 1: First, we construct a family of test functions $(\xi_h)_{h>0}$, satisfying the following conditions:

i.) $0 \leq \xi_h \leq 1$ holds for all $0 < h < (d-c)/2$,

ii.) $\xi_h \in H_0^1(c,d)$ for all $0 < h < (d-c)/2$,

iii.) $\xi_h \to \chi_{(c,d)}$ almost everywhere in (c,d) as $h \downarrow 0$,

iv.) There exist constants $C_1, C_2 > 0$ not depending on $0 < h < (d-c)/2$, such that

$$\int_c^d |(\xi_h)_x|^2 \leq \frac{C_1}{h}, \tag{3.6.1}$$

$$\int_c^d |(\xi_h)_x| \leq C_2 \tag{3.6.2}$$

holds for all $0 < h < (d-c)/2$.

v.) If $W \in H_0^1(c,d)$ is nonnegative, then

$$\int_c^d A(x)(\xi_h)_x W_x dx \geq 0 \tag{3.6.3}$$

holds for all $0 < h < (d-c)/2$.

The construction of the test functions ξ_h for $h > 0$ has been introduced in [5] for $A(x) \equiv 1$. It is well-known that for any $0 < h < (d-c)/2$, the problems

$$(PB_h^1) \begin{cases} (A(x)u_x)_x = 0 \text{ on } (c, c+h) \\ u(c) = 0, u(c+h) = h, \end{cases}$$

$$(PB_h^2) \begin{cases} (A(x)u_x)_x = 0 \text{ on } (d-h, d) \\ u(d-h) = h, u(d) = 0. \end{cases}$$

have solutions $u_h^1 \in H_0^1(c, c+h)$, $u_h^2 \in H_0^1(d-h, d)$ respectively, such that $(A(x)(u_h^1)_x)_x = 0$ holds in $H^{-1}(c, c+h)$ and $(A(x)(u_h^2)_x)_x = 0$ holds in $H^{-1}(d-h, d)$. Now, we define $\xi_h : (c,d) \to \mathbb{R}$ by

$$\xi_h(x) = \begin{cases} \frac{2}{h}\min(u_h^1(x), \frac{h}{2}), & \text{for } x \in [c, c+h), \\ 1, & \text{for } x \in [c+h, d-h], \\ \frac{2}{h}\min(u_h^2(x), \frac{h}{2}), & \text{for } x \in (d-h, d]. \end{cases} \tag{3.6.4}$$

Therefore, *i.)* follows directly from (3.6.4). A short calculation gives

$$(\xi_h)_x = \frac{2}{h}\left((u_h^1)_x \chi_{\{\{u_h^1 < h/2\} \cap (c,c+h)\}} + (u_h^2)_x \chi_{\{\{u_h^2 < h/2\} \cap (d-h,d)\}}\right) \tag{3.6.5}$$

in $\mathcal{D}'(c,d)$, hence $\xi_h \in H_0^1(c,d)$ and $\operatorname{supp}(\xi_h)_x \subset \{x \in (c, c+h) : u_h^1 < h/2\} \cap \{x \in (d-h, d) : u_h^2 < h/2\}$ for all $0 < h < (d-c)/2$. Choosing any $x \in (c,d)$, there exists $h_0 > 0$ such that $x \in [c+h, d-h]$ for all $0 < h < h_0$, hence $\xi_h(x) = 1$ for all $h < h_0$ and *iii.)* follows. Note that u_h^1 is the unique solution of the minimization problem

$$\min_{u \in K} \left\{ \frac{1}{2} \int_c^{c+h} A(x)(u_x)^2 \right\}$$

where $K := (x-c) + H_0^1(c, c+h)$ and u_h^2 is the unique solution of the minimization problem

$$\min_{u \in \tilde{K}} \left\{ \frac{1}{2} \int_{d-h}^{d} A(x)(u_x)^2 \right\}$$

where $\tilde{K} := (d-x) + H_0^1(d-h,d)$ (see [7], Theorem 6.5.1, p. 246). Using (3.6.5) it follows that

$$\begin{aligned}\gamma\int_c^d |(\xi_h)_x|^2 &\leq \frac{4}{h^2}\left(\int_c^{c+h} A(x)(u_h^1)_x^2 + \int_{d-h}^d A(x)(u_h^2)_x^2\right)\\ &\leq \frac{8}{h^2}\left(\frac{1}{2}\int_c^{c+h} A(x) + \frac{1}{2}\int_{d-h}^d A(x)\right)\\ &\leq \frac{8\|A\|_{L^\infty(a,b)}}{h} \qquad (3.6.6)\end{aligned}$$

hence (3.6.1) holds. Applying the Hölder inequality to $\int_c^d |(\xi_h)_x|$ and using (3.6.1), we get (3.6.2). To prove $v.$), we choose $W \in H_0^1(c,d)$ such that $W \geq 0$ in (c,d) and write

$$\int_c^d A(x)(\xi_h)_x W_x = I_1 + I_2, \qquad (3.6.7)$$

where

$$\begin{aligned}I_1 &= \int_c^{c+h} A(x)(\xi_h)_x W_x,\\ I_2 &= \int_{d-h}^d A(x)(\xi_h)_x W_x.\end{aligned}$$

In the next steps, we will show $I_1 \geq 0$ and $I_2 \geq 0$. Since $W(\cdot)H_\delta^+(h/2-u_h^1) \in H_0^1(c,c+h)$ is an admissible test function in (PB_h^1), we have

$$\begin{aligned}0 &= -\int_c^{c+h} A(x)(u_h^1)_x(W(x)H_\delta^+(h/2-u_h^1))_x\\ &= -\int_c^{c+h} A(x)(u_h^1)_x H_\delta^+(h/2-u_h^1)W_x + \frac{1}{\delta}\int_{\{0<h/2-u_h^1<\delta\}} A(x)(u_h^1)_x^2 W(x)\end{aligned} \qquad (3.6.8)$$

neglecting the positive term and passing to the limit with $\delta \downarrow 0$ from (3.6.8) it follows that

$$0 \leq \frac{2}{h}\int_c^{c+h} A(x)(u_h^1)_x \chi_{\{u_h^1<h/2\}} W_x = I_1. \qquad (3.6.9)$$

Since Since $W(\cdot)H_\delta^+(h/2-u_h^2) \in H_0^1(d-h,d)$ is an admissible test function in (PB_h^2), we have

$$0 = -\int_{d-h}^d A(x)(u_h^2)_x(W(x)H_\delta^+(h/2-u_h^2))_x, \qquad (3.6.10)$$

therefore

$$0 \leq \frac{2}{h} \int_{d-h}^{d} A(x)(u_h^2)_x \chi_{\{u_h^2 < h/2\}} W_x = I_2 \tag{3.6.11}$$

and $v.$) holds.

Step 2: Conclusion. Let us assume $u > v$ on (c, d). As $\xi_h \in H_0^1(c, d)$ for all $0 < h < (d-c)/2$, we will identify ξ_h with its extension by zero on (a, b) that is in $H_0^1(a, b)$. Using ξ_h as a test function in the weak formulations for u and v respectively we find

$$\int_c^d (u-v)\xi_h + \int_c^d (F(u) - F(v))(\xi_h)_x + \int_c^d A(x)(u_x - v_x)(\xi_h)_x = 0. \tag{3.6.12}$$

Now, the function $u - v$ is nonnegative and in $H_0^1(c, d)$, since by assumption we have $u > v$ on (c, d) and $u(c) = v(c)$, $u(d) = v(d)$. Hence the last integral on the left-hand side in (3.6.13) is nonnegative by $v.$). Now we write

$$\int_c^d (F(u) - F(v))(\xi_h)_x = I_1^h + I_2^h, \tag{3.6.13}$$

where

$$\begin{aligned} I_1^h &= \int_c^{c+h} (F(u) - F(v))(\xi_h)_x \leq I_{1,1}^h + I_{1,2}^h \\ I_2^h &= \int_{d-h}^{d} (F(u) - F(v))(\xi_h)_x \leq I_{2,1}^h + I_{2,2}^h \end{aligned} \tag{3.6.14}$$

and

$$\begin{aligned} I_{1,1}^h &= \int_c^{c+h} |F(u) - F(u(c))| \, |(\xi_h)_x|, \\ I_{1,2}^h &= \int_c^{c+h} |F(v) - F(v(c))| \, |(\xi_h)_x|, \\ I_{2,1}^h &= \int_{d-h}^{d} |F(u) - F(u(c))| \, |(\xi_h)_x|, \\ I_{2,2}^h &= \int_{d-h}^{d} |F(v) - F(v(c))| \, |(\xi_h)_x|. \end{aligned} \tag{3.6.15}$$

By continuity of $F \circ u$, (3.6.2) and $iv.$), for any $\varepsilon > 0$ we find $h_0 > 0$ such that

$$I_{1,1}^h + I_{1,2}^h + I_{2,1}^h + I_{2,2}^h \leq 4\varepsilon C_2 \tag{3.6.16}$$

for all $0 < h \le h_0$. Therefore, using (3.6.14), (3.6.15), (3.6.16), *iii.*) and neglecting the nonnegative term we can pass to the limit with $h \downarrow 0$ in (3.6.12) to obtain

$$\int_c^d |u-v| = 0, \tag{3.6.17}$$

hence $u = v$ on (c, d) and we have a contradiction. Assuming $u < v$ on (c, d) and using the same arguments leads to the same contradiction, hence the proof is completed. □

It is an open problem whether this result can be generalized to problems with linear diffusion in several space dimensions replacing the continuity of solutions by their cap-p quasicontinuity and using capacity theory (see [45], [57]). Another possible object of future work is the generalisation of Proposition 3.6.1 to nonlinear problems, i.e. replacing $(A(x)u_x)$ in (PB) by, e.g., $(|u_x|^{p(x)-2}u_x)$. This question is closely related to the question of existence of solutions to the problems

$$(P\tilde{B}_h^1)\begin{cases}(|u_x|^{p(x)-2}u_x)_x = 0 \text{ on } (c, c+h)\\ u(c) = 0, u(c+h) = h,\end{cases}$$

$$(P\tilde{B}_h^2)\begin{cases}(|u_x|^{p(x)-2}u_x)_x = 0 \text{ on } (d-h, d)\\ u(d-h) = h, u(d) = 0,\end{cases}$$

for $h > 0$ and $c, d \in \mathbb{R}$.

Chapter 4

The parabolic case

4.1 Mild solutions of the abstract Cauchy problem

4.1.1 Existence of mild solutions

It follows from Theorem 3.1.2 of the previous chapter that for all $f \in L^1(\Omega)$, $\lambda > 0$ there exists a renormalized solution (u, b) to

$$(S, \lambda, f) \begin{cases} \beta(u) - \lambda \operatorname{div}(a(x, Du) + F(u)) \ni f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

For $f, \tilde{f} \in L^1(\Omega)$ let (u, b) and $(\tilde{u}, \tilde{b})$ be renormalized solutions of (S, λ, f), $(S, \lambda, \tilde{f})$ respectively. Writing $|b - \tilde{b}| = (b - \tilde{b})^+ + (\tilde{b} - b)^+$ and applying the comparison principle from Theorem 3.5.1, we find that

$$\|b - \tilde{b}\|_{L^1(\Omega)} \leq \|f - \tilde{f}\|_{L^1(\Omega)}. \tag{4.1.1}$$

In terms of nonlinear operators the preceeding results read as follows: If A_β is the nonlinear operator defined in $L^1(\Omega)$ by

$$\begin{aligned} A_\beta := \big\{ &(b, w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R} \text{ measurable, } b \in \beta(u) \\ &\text{almost everywhere in } \Omega \text{ and } u \text{ is a renormalized solution of} \\ &- \operatorname{div}(a(x, Du) + F(u)) = w \big\} \end{aligned} \tag{4.1.2}$$

then A_β is m-accretive in $L^1(\Omega)$, i.e., the resolvent mapping

$$f \in L^1(\Omega) \to (I + \lambda A_\beta)^{-1} f =: J^\lambda_{A_\beta}(f) \in L^1(\Omega)$$

is a contraction in the L^1-norm (because of (4.1.1)) and the range condition

$$R(I + \lambda A_\beta) = L^1(\Omega) \tag{4.1.3}$$

holds for all $\lambda > 0$. Indeed, for any $f \in L^1(\Omega)$, $\lambda > 0$ there exists $(b, w) \in A_\beta$ such that

$$b + \lambda w = f \tag{4.1.4}$$

almost everywhere in Ω: If (u, b) is the renormalized solution to (S, λ, f), then we have $b \in \beta(u)$ almost everywhere in Ω and u is the renormalized solution to

$$-\lambda \operatorname{div}(a(x, Du) + F(u)) = f - b.$$

Therefore $(b, \frac{f-b}{\lambda}) \in A_\beta$ and (4.1.4) holds with $w = \frac{f-b}{\lambda}$. By the general theory of nonlinear semigroups (see [17], [8]) we conclude that the abstract Cauchy problem corresponding to (P, f, b_0)

$$(ACP)(f, b_0) \begin{cases} \dfrac{db}{dt} + A_\beta b \ni f \text{ in } (0, T), \\ b(0) = b_0 \end{cases}$$

admits a unique mild solution $b \in \mathcal{C}([0, T]; L^1(\Omega))$ for any initial datum $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ and any right-hand side $f \in L^1(0, T; L^1(\Omega)) \cong L^1(Q_T)$. As we will see in the next subsection,

$$\overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} = \left\{ b \in L^1(\Omega) : b \in \overline{R(\beta)} \text{ a.e. in } \Omega \right\}.$$

Roughly speaking, a mild solution is a continuous abstract function $b \in \mathcal{C}([0, T]; L^1(\Omega))$ which is the uniform limit of piecewise constant functions $b_\varepsilon : (0, T) \to L^1(\Omega)$ defined by $b_\varepsilon(0) = b_0^\varepsilon$, $b_\varepsilon(t) = b_i^\varepsilon$ on $]t_{i-1}^\varepsilon, t_i^\varepsilon]$ for $i = 1, \ldots, N(\varepsilon)$ where $(b_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ are solutions of time-discretized problems given by an implicit Euler scheme of the form

$$\frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{t_i^\varepsilon - t_{i-1}^\varepsilon} + A_\beta b_i^\varepsilon \ni f_i^\varepsilon,\ i = 1, \ldots, N(\varepsilon), \tag{4.1.5}$$

where $\varepsilon > 0$, $N(\varepsilon) \in \mathbb{N}$, $0 = t_0^\varepsilon < t_1^\varepsilon < \ldots < t_{N(\varepsilon)}^\varepsilon \leq T$ and $f_i^\varepsilon \in L^1(\Omega)$, $i = 1, \ldots, N(\varepsilon)$ such that, as $\varepsilon \to 0$,

$$\sum_{i=1}^{N(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \|f(t) - f_i^\varepsilon\|_{L^1(\Omega)} dt \to 0,$$

$\max_{1,\dots,N(\varepsilon)}(t_i^\varepsilon - t_{i-1}^\varepsilon) \to 0$, $T - t_{N(\varepsilon)}^\varepsilon \to 0$ and $\|b_0 - b_0^\varepsilon\|_{L^1(\Omega)} \to 0$.

Let us recall that the mild solution of $(ACP)(f, b_0)$ depends continuously on the data, more precisely, if $b, v \in \mathcal{C}([0,T]; L^1(\Omega))$ are mild solutions of $(ACP)(f, b_0)$, $(ACP)(g, v_0)$ respectively, then

$$\|b(t) - v(t)\|_{L^1(\Omega)} \leq \|b_0 - v_0\|_{L^1(\Omega)} + \int_0^t \|f(s) - g(s)\|_{L^1(\Omega)} ds \tag{4.1.6}$$

holds for any $0 \leq t \leq T$. Moreover, a function $b \in \mathcal{C}([0,T]; L^1(\Omega))$ is the unique mild solution of $(ACP)(f, b_0)$, if and only if b is the unique integral solution of $(ACP)(f, b_0)$ in the sense of Bénilan ([14], [17], [8]), i.e. if b satisfies the following family of integral inequalities: For any $(v, w) \in A_\beta$, for any $0 \leq s \leq t \leq T$, we have

$$\|b(t) - v\|_{L^1(\Omega)} \leq \|b(s) - v\|_{L^1(\Omega)} + \int_s^t [u(\tau) - v, f(\tau) - w] d\tau, \tag{4.1.7}$$

where, for $g, h \in L^1(\Omega)$, the bracket $[g, h]$ denotes the right-hand side Gâteaux derivative of the L^1-norm at g in the direction of h, i.e.,

$$\begin{aligned} [g, h] &= \lim_{\lambda \to 0} \frac{\|g - \lambda h\|_{L^1(\Omega)} - \|g\|_{L^1(\Omega)}}{\lambda} \\ &= \int_\Omega \mathrm{sign}_0(g) h \, dx + \int_{g=0} |h| \, dx. \end{aligned}$$

4.1.2 The closure of $D(A_\beta)$

The following proposition gives us a description of $\overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$:

Proposition 4.1.1. *Let A_β be the operator defined in* (4.1.2). *Then*

$$\overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} = \left\{ b \in L^1(\Omega) : b \in \overline{R(\beta)} \text{ a.e. in } \Omega \right\}. \tag{4.1.8}$$

Proof: If we define

$$M := \left\{ b \in L^1(\Omega) : b \in \overline{R(\beta)} \text{ a.e. in } \Omega \right\},$$

then obviously $\overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \subset M$. Therefore we will prove $M \subset \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. The proof will be divided into several steps.

Step 1: We choose an appropriate dense subset of M:

Lemma 4.1.2. *The set*

$$D := \{b \in L^\infty(\Omega) : \exists u \in L^\infty(\Omega) : b \in \beta(u) \text{ a.e. in } \Omega\}$$

is dense in M.

Proof: We choose $b \in M$ and set

$$\Omega_1 := \{x \in \Omega : b(x) \in R(\beta)\},$$
$$\Omega_2^* := \{x \in \Omega : b(x) \in \overline{R(\beta)} \setminus R(\beta)\}.$$

Let $b^+ := \sup R(\beta)$, $b^- := \inf R(\beta)$. Assuming Ω_2^* is nonempty implies that at most one among b^+, b^- does not belong to $R(\beta)$, so $b^- < b^+$ and therefore, since β is a maximal monotone graph,

$$\Omega_2^* = \Omega_2 \cup \Omega_3,$$

where

$$\Omega_2 := \{x \in \Omega : b(x) = b^+\}$$

and

$$\Omega_3 := \{x \in \Omega : b(x) = b^-\}.$$

Clearly, Ω_1, Ω_2 and Ω_3 are measurable subsets of Ω and $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup N$, where $N \subset \Omega$ is measurable with $|N| = 0$. If $r \in D(\beta)$, let $v^0 \in \beta(r)$ be the element of $\beta(r)$ with minimal norm. For $r \in \mathbb{R}$ we define

$$\beta^0(r) := \begin{cases} v^0 \in \beta(r), & \text{if } r \in D(\beta), \\ +\infty, & \text{if } [r, +\infty[\cap D(\beta) = \emptyset, \\ -\infty, & \text{if }]-\infty, r] \cap D(\beta) = \emptyset, \end{cases}$$

$$\beta^{\max}(r) := \begin{cases} \sup \beta(r), & \text{if } r \in D(\beta), \\ +\infty, & \text{if } [r, +\infty[\cap D(\beta) = \emptyset, \\ -\infty, & \text{if }]-\infty, r] \cap D(\beta) = \emptyset, \end{cases}$$

$$\beta^{\min}(r) := \begin{cases} \inf \beta(r), & \text{if } r \in D(\beta), \\ +\infty, & \text{if } [r, +\infty[\cap D(\beta) = \emptyset, \\ -\infty, & \text{if }]-\infty, r] \cap D(\beta) = \emptyset, \end{cases}$$

By the definition of b^+ and b^-, there exist sequences $(b_n^+)_n$, $(b_n^-)_n \subset R(\beta)$, $(r_n^-)_n$, $(r_n^+)_n \subset D(\beta)$, such that $b_n^+ \in \beta(r_n^+)$, $b_n^- \in \beta(r_n^-)$ for all $n \in \mathbb{N}$ and $b_n^+ \uparrow b^+$, $b_n^- \downarrow b^-$ as $n \to \infty$.

We fix $b \in M$. Now, we are ready to construct sequences $(b_n)_n$, $(u_n)_n \subset L^\infty(\Omega)$ such that $u_n \in D(\beta)$, $b_n \in \beta(u_n)$ almost everywhere in Ω for all $n \in \mathbb{N}$ and $b_n \to b$ in $L^1(\Omega)$ for $n \to \infty$. The construction depends on $D(\beta)$. Keeping in mind that $0 \in D(\beta)$ by assumption, we have the following cases:

i.) $\overline{D(\beta)} = [c, d]$ for $c, d \in \mathbb{R}$, $c < 0 < d$. Since β is maximal monotone it follows that $R(\beta) = \mathbb{R}$ and we have:

$$u_n(x) := ((\beta^{-1})^0(b(x)) \wedge (d - \frac{1}{n})) \vee (c + \frac{1}{n}),\ x \in \Omega,$$

$$b_n(x) := (b(x) \wedge \beta^{\max}(d - \frac{1}{n})) \vee \beta^{\min}(c + \frac{1}{n}),\ x \in \Omega.$$

ii.) $D(\beta) = \mathbb{R}$:

$$u_n(x) := \begin{cases} ((\beta^{-1})^0(b(x)) \wedge n) \vee -n, & \text{if } x \in \Omega_1, \\ r_n^+, & \text{if } x \in \Omega_2, \\ r_n^-, & \text{if } x \in \Omega_3. \end{cases}$$

$$b_n(x) := \begin{cases} (b(x) \wedge \beta^{\max}(n)) \vee \beta^{\min}(-n), & \text{if } x \in \Omega_1, \\ b_n^+, & \text{if } x \in \Omega_2, \\ b_n^-, & \text{if } x \in \Omega_3. \end{cases}$$

iii.) $\overline{D(\beta)} = [a, \infty]$ for $a \in \mathbb{R}$, $a < 0$. Since β is maximal monotone it follows that Ω_3 is empty and we have:

$$u_n(x) := \begin{cases} ((\beta^{-1})^0(b(x)) \wedge n) \vee (a + \frac{1}{n}), & \text{if } x \in \Omega_1, \\ r_n^+, & \text{if } x \in \Omega_2. \end{cases}$$

$$b_n(x) := \begin{cases} (b(x) \wedge \beta^{\max}(n)) \vee \beta^{\min}(a + \frac{1}{n}), & \text{if } x \in \Omega_1, \\ b_n^+, & \text{if } x \in \Omega_2. \end{cases}$$

iv.) $\overline{D(\beta)} = [-\infty, a]$ for $a \in \mathbb{R}$, $a > 0$. Since β is maximal monotone it follows that Ω_2 is empty and we have:

$$u_n(x) := \begin{cases} ((\beta^{-1})^0(b(x)) \wedge (a - \frac{1}{n})) \vee -n, & \text{if } x \in \Omega_1, \\ r_n^-, & \text{if } x \in \Omega_3. \end{cases}$$

$$b_n(x) := \begin{cases} (b(x) \wedge \beta^{\max}(a - \frac{1}{n})) \vee \beta^{\min}(-n), & \text{if } x \in \Omega_1, \\ b_n^-, & \text{if } x \in \Omega_3. \end{cases}$$

$\square$

Step 2: According to Lemma 4.1.2, it is left to prove that $D \subset \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. By general nonlinear semigroup theory (see [17]), since A_β is m-accretive in $L^1(\Omega)$, this is true if, for each $b \in D$ and $\lambda > 0$,

$$\lim_{\lambda \downarrow 0} J^\lambda_{A_\beta}(b) = b \tag{4.1.9}$$

holds in $L^1(\Omega)$. To this end, we fix $b \in D$ and choose $u \in L^\infty(\Omega)$ such that $b \in \beta(u)$ almost everywhere in Ω. For $\widehat{\beta} := \beta + I$ the operator $A_{\widehat{\beta}}$ is m-accretive in $L^1(\Omega)$, hence for each $\lambda > 0$ there exist $(\widehat{b}_\lambda, \widehat{w}_\lambda) \in A_{\widehat{\beta}}$, $\widehat{u}_\lambda \in L^1(\Omega)$ such that

$$\widehat{b}_\lambda + \lambda \widehat{w}_\lambda = b + u \tag{4.1.10}$$

holds in $L^1(\Omega)$,

$$\widehat{b}_\lambda \in \widehat{\beta}(\widehat{u}_\lambda) \tag{4.1.11}$$

almost everywhere in Ω and $\widehat{u}_\lambda$ is a renormalized solution to

$$-\operatorname{div}(a(x, D\widehat{u}_\lambda) + F(\widehat{u}_\lambda)) = \widehat{w}_\lambda. \tag{4.1.12}$$

Step 3: A priori estimates and convergence.

Lemma 4.1.3. *For all $\lambda > 0$, $b \in D$, $u \in L^\infty(\Omega)$ such that $b \in \beta(u)$ almost everywhere in Ω, $(\widehat{b}_\lambda, \widehat{w}_\lambda) \in A_{\widehat{\beta}}$, $\widehat{u}_\lambda \in L^1(\Omega)$ satisfying* (4.1.10), (4.1.12) *and* (4.1.11)*, the following holds true:*

i.)

$$\|\widehat{u}_\lambda\|_{L^\infty(\Omega)} \le \|b + u\|_{L^\infty(\Omega)}, \tag{4.1.13}$$

ii.) $\widehat{u}_\lambda \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *and there exists a constant $C > 0$ not depending on $\lambda > 0$ such that*

$$\lambda \int_\Omega |D\widehat{u}_\lambda|^{p(x)} \le C\|b + u\|_{L^\infty(\Omega)}, \tag{4.1.14}$$

iii.)

$$\lim_{\lambda \downarrow 0} \lambda \int_\Omega |D\widehat{u}_\lambda|^{p(x)-1} = 0, \tag{4.1.15}$$

iv.)

$$\|\widehat{b}_\lambda\|_{L^2(\Omega)} \le \|b + u\|_{L^2(\Omega)} \tag{4.1.16}$$

and $\widehat{b}_\lambda \to b + u$ in $L^2(\Omega)$.

Proof: We fix $\lambda > 0$. By the definition of $\widehat{\beta}$ and (4.1.11), there exists $d_\lambda \in L^1(\Omega)$ satisfying $d_\lambda \in \beta(\widehat{u}_\lambda)$ for almost all $x \in \Omega$ and

$$\widehat{b}_\lambda = d_\lambda + \widehat{u}_\lambda \tag{4.1.17}$$

almost everywhere in Ω. By (4.1.10), (4.1.12) and (4.1.17),

$$\int_\Omega (d_\lambda + \widehat{u}_\lambda)h(\widehat{u}_\lambda)\phi + \lambda(a(x, D\widehat{u}_\lambda) + F(\widehat{u}_\lambda)) \cdot D(h(\widehat{u}_\lambda)\phi) = \int_\Omega (b+u)h(\widehat{u}_\lambda)\phi \tag{4.1.18}$$

holds for all $h \in \mathcal{C}^1_c(\mathbb{R})$ and all $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. We choose $h_l(\widehat{u}_\lambda)\frac{1}{\delta}(T_{k+\delta}(\widehat{u}_\lambda) - T_k(\widehat{u}_\lambda))$ as a test function in (4.1.18). Neglecting positive terms we pass to the limit with $\delta \downarrow 0$ and then with $l \to \infty$. Setting $k = \|b+u\|_{L^\infty(\Omega)}$, we find (4.1.13) and *i.*) holds. Since $\widehat{u}_\lambda$ is a renormalized solution to (4.1.12) in follows from (4.1.13) that $\widehat{u}_\lambda \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Therefore, using the same arguments as in the proof of Proposition 3.3.2, we can use $\widehat{u}_\lambda$ as a test function and find

$$\int_\Omega d_\lambda \widehat{u}_\lambda + (\widehat{u}_\lambda)^2 + \lambda \int_\Omega (a(x, D\widehat{u}_\lambda) + F(\widehat{u}_\lambda)) \cdot D\widehat{u}_\lambda = \int_\Omega (b+u)\widehat{u}_\lambda. \tag{4.1.19}$$

Now, (4.1.14) follows from (4.1.19) and $(A1)$. To prove (4.1.15), we choose $0 < \lambda < 1$. Then, using the Young inequality with $p(x)$ and $p'(x) = p(x)/(p(x)-1)$ almost everywhere in Ω and (4.1.14) it follows that

$$\begin{aligned} \lambda \int_\Omega |D\widehat{u}_\lambda|^{p(x)-1} &\leq \lambda^{1-\frac{1}{(p')^-}} \int_\Omega \lambda^{\frac{1}{p'(x)}} |D\widehat{u}_\lambda|^{p(x)-1} \\ &\leq \lambda^{1-\frac{1}{(p')^-}} \lambda \frac{1}{(p')^-} \int_\Omega |D\widehat{u}_\lambda|^{p(x)} + \lambda^{1-\frac{1}{(p')^-}} \frac{1}{p^-}|\Omega| \\ &\leq \lambda^{1-\frac{1}{(p')^-}} \left(\frac{1}{(p')^-} C\|b+u\|_{L^\infty(\Omega)} + \frac{1}{p^-}|\Omega| \right) \end{aligned}$$

where $C > 0$ does not depend on λ. Now, *iii.*) follows since $1 - \frac{1}{(p')^-} > 0$. To prove *iv.*), for $\varepsilon > 0$, let β_ε be the Yosida approximation of β and set $\widehat{\beta}_\varepsilon := I + \beta_\varepsilon$. Then for each $\varepsilon, \lambda > 0$, there exists a weak solution $\widehat{u}_\varepsilon^\lambda \in W_0^{1,p(\cdot)}(\Omega)$ to

$$\widehat{\beta}_\varepsilon(T_{1/\varepsilon}(\widehat{u}_\varepsilon^\lambda)) - \lambda \operatorname{div}(a(x, D\widehat{u}_\varepsilon^\lambda) + F(T_{1/\varepsilon}(\widehat{u}_\varepsilon^\lambda)) = b + u. \tag{4.1.20}$$

Choosing $\widehat{\beta}_\varepsilon(T_{1/\varepsilon}(\widehat{u}_\varepsilon^\lambda))$ as a test function in (4.1.20), we find that

$$\|\widehat{\beta}_\varepsilon(T_{1/\varepsilon}(\widehat{u}_\varepsilon^\lambda))\|_{L^2(\Omega)} \leq \|b+u\|_{L^2(\Omega)} \tag{4.1.21}$$

holds for all $\varepsilon, \lambda > 0$. Using the same arguments as in the proof of Proposition 3.2.1, it follows that $\widehat{\beta}_\varepsilon(T_{1/\varepsilon}(\widehat{u}_\varepsilon^\lambda)) \rightharpoonup \widehat{b}_\lambda$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$. Therefore, (4.1.16) holds for all $\lambda > 0$. It is an immediate consequence of (4.1.15), (4.1.10) and (4.1.12) that $\widehat{b}_\lambda \to b + u$ in $\mathcal{D}'(\Omega)$ and now it follows by (4.1.16) that $\widehat{b}_\lambda \to b + u$ in $L^2(\Omega)$ as $\lambda \downarrow 0$, hence the proof of *iv.*) is complete. $\square$

Step 4: Conclusion: From the proof of Lemma 4.1.3 we recall that by (4.1.11), for each $\lambda > 0$ we can write

$$\widehat{b}_\lambda = d_\lambda + \widehat{u}_\lambda, \tag{4.1.22}$$

where $d_\lambda \in L^1(\Omega)$ satisfies $d_\lambda \in \beta(\widehat{u}_\lambda)$ almost everywhere in Ω. From (4.1.10) and (4.1.12) it follows that $(d_\lambda, \widehat{w}_\lambda) \in A_\beta$ and

$$d_\lambda = J^\lambda_{A_\beta}(b + u - \widehat{u}_\lambda) \tag{4.1.23}$$

for all $\lambda > 0$. Now, using the contractivity property of the resolvent mapping $J^\lambda_{A_\beta}$ and the contractivity of $(\beta + I)^{-1}$, we get the estimate

$$\begin{aligned}
\|J^\lambda_{A_\beta}(b) - b\|_{L^1(\Omega)} &\leq \|J^\lambda_{A_\beta}(b) - J^\lambda_{A_\beta}(b + u - \widehat{u}_\lambda)\|_{L^1(\Omega)} + \|\widehat{b}_\lambda - \widehat{u}_\lambda - b\|_{L^1(\Omega)} \\
&\leq 2\|\widehat{u}_\lambda - u\|_{L^1(\Omega)} + \|\widehat{b}_\lambda - (b + u)\|_{L^1(\Omega)} \\
&\leq 3\|\widehat{b}_\lambda - (b + u)\|_{L^1(\Omega)}.
\end{aligned} \tag{4.1.24}$$

Applying *iv.*) from Lemma 4.1.3, (4.1.9) follows and the proof of Proposition 4.1.1 is complete.

4.2 Solutions and function spaces for the evolution problem

For a constant exponent $p(\cdot) \equiv p$, the notion of renormalized solution to (P, f, b_0) is well known (see [2], [68], [22]) and (P, f, b_0) is well-posed in the space $L^p(0, T; W^{1,p}_0(\Omega))$. We remind that (u, b) is a renormalized solution to (P, f, b_0) for $f \in L^1(Q_T)$ in this particular case, iff (u, b) is satisfying the following conditions:

(1) $u : Q_T \to \mathbb{R}$ is measurable, $b \in L^1(Q_T)$, $u(t, x) \in D(\beta(t, x))$ and $b(t, x) \in \beta(u(t, x))$ for a.e. $(t, x) \in Q_T$,

(2) $b(0, x) = b_0(x)$ a.e. in Ω,

(3) For each $k > 0$, $T_k(u) \in L^p(0, T; W^{1,p}_0(\Omega))$,

(4)

$$
\begin{aligned}
- & \int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h \circ (\beta^{-1})^0(r)drd(t,x) \\
+ & \int_{Q_T} (a(x, Du) + F(u)) \cdot D(h(u)\xi)d(t,x) = \int_{Q_T} fh(u)\xi d(t,x)
\end{aligned}
$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\xi \in \mathcal{D}([0,T) \times \Omega)$.

(5) $\int_{Q_T \cap \{k<|u|<k+1\}} a(x, Du) \cdot Du \; d(t,x) \to 0$ as $k \to \infty$.

In the general case, the situation is more delicate. We will use the above definition as a starting point in our study because we already know from the previous section that there exists a mild solution to the corresponding abstract Cauchy problem (ACP, f, b_0) for any $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ and any $f \in L^1(Q_T)$. This kind of data (f, b_0) will be called L^1-data in the following. In the next lemma we give a priori estimates on the solutions of the discretized problem (DP_ε) associated to (P, f, b_0) for arbitary L^1-data. From these results we will deduce the generalised notion of renormalized solution to (P, f, b_0) for variable exponents. Then we will define the appropriate functional setting and the notion of weak solution to (P, f, b_0). In this approach we tacticly assume an integration-by-parts Lemma for variable exponents that will be proved later (see Lemma 4.2.11).

Another possible approach would be to start with data that allow weak solutions in the discretized problems (DP_ε) and to obtain energy estimates and function spaces for weak solutions. Then, the integration-by-parts Lemma would naturally lead us to the notion of renormalized solution for variable exponents.

Lemma 4.2.1. *For $f \in L^1(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$, $0 < \varepsilon \leq 1$, $N(\varepsilon) \in \mathbb{N}$ and*

$$
(D_\varepsilon) \begin{cases}
t_0^\varepsilon = 0 < t_1^\varepsilon < \ldots < t_{N(\varepsilon)}^\varepsilon \leq T, \\
t_i^\varepsilon - t_{i-1}^\varepsilon \leq \varepsilon, \; T - t_{N(\varepsilon)}^\varepsilon \leq \varepsilon \; \forall i = 1, \ldots, N(\varepsilon), \\
f_i^\varepsilon \in L^1(\Omega), i = 1, \ldots, N(\varepsilon), : \sum_{i=1}^{N(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \|f(t) - f_i^\varepsilon\|_{L^1(\Omega)} dt \leq \varepsilon \\
b_0^\varepsilon \in L^1(\Omega) : \|b_0^\varepsilon - b_0\|_{L^1(\Omega)} \leq \varepsilon
\end{cases}
$$

let $(b_i^\varepsilon, u_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ *be a solution of the discretized problem*

$$(DP_\varepsilon)\begin{cases} b_i^\varepsilon \in L^1(\Omega), u_i^\varepsilon : \Omega \to \mathbb{R} \text{ measurable, } T_k(u_i^\varepsilon) \in W_0^{1,p(\cdot)}(\Omega)\ \forall k>0, \\ \int_{\{n<|u_i^\varepsilon|<n+1\}} a(x, Du_i^\varepsilon)\cdot Du_i^\varepsilon \to 0 \text{ as } n\to\infty, \\ \int_\Omega \frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{t_i^\varepsilon - t_{i-1}^\varepsilon} h(u_i^\varepsilon)\varphi + \int_\Omega (a(x,Du_i^\varepsilon)+F(u_i^\varepsilon))\cdot D(h(u_i^\varepsilon)\varphi) = \int_\Omega f_i^\varepsilon h(u_i^\varepsilon)\varphi \\ \forall \varphi \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega), h\in \mathcal{C}_c^1(\mathbb{R}), \\ b_i^\varepsilon \in \beta(u_i^\varepsilon) \text{ a.e. in } \Omega \text{ for all } i=1,\dots,N(\varepsilon). \end{cases}$$

For $k>0$*, we define the piecewise constant functions* $f_\varepsilon : (0,T] \to L^1(\Omega)$*,* $b_\varepsilon : [0,T]\to L^1(\Omega)$ *and* $T_k(u_\varepsilon) : (0,T] \to W_0^{1,p(\cdot)}(\Omega)$ *as follows:* $f_\varepsilon(t) = f_i^\varepsilon$*,* $b_\varepsilon(0) = b_0^\varepsilon$*,* $b_\varepsilon(t) = b_i^\varepsilon$ *and* $T_k(u_\varepsilon(t)) = T_k(u_i^\varepsilon)$ *for* $t\in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ *and* $i = 1,\dots,N(\varepsilon)$*. If* $t_{N(\varepsilon)}^\varepsilon < T$*,* f_ε*,* b_ε *and* u_ε *are extended by setting* $f_\varepsilon(t) = f_{N(\varepsilon)}^\varepsilon$*,* $T_k(u_\varepsilon(t)) = T_k(u_{N(\varepsilon)}^\varepsilon)$ *and* $b_\varepsilon(t) = b_{N(\varepsilon)}^\varepsilon$ *for all* $t \in (t_{N(\varepsilon)}^\varepsilon, T]$*.*

Then the following estimates hold true for all $k>0$ *and* $0<\varepsilon\le 1$*:*

i.) There exists a constant $C_1(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k) > 0$ *not depending on* $\varepsilon > 0$*, such that*

$$\int_0^T \int_\Omega |DT_k(u_\varepsilon)|^{p(x)} dxdt \le C_1(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k). \tag{4.2.1}$$

ii.) There exists a constant $C_2(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, T, p(\cdot), \Omega) > 0$ *not depending on* $\varepsilon>0$*, such that*

$$\|T_k(u_\varepsilon)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} \le C_2(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, T, p(\cdot), \Omega). \tag{4.2.2}$$

iii.) There exist constants $C_3(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, p(\cdot), \Omega) > 0$*,* $C_4(C_1(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k)) > 0$ *not depending on* $\varepsilon>0$*, such that*

$$\|a(x, DT_k(u_\varepsilon))\|_{L^{(p')^-}(0,T;L^{p'(\cdot)}(\Omega))} \le C_3(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, p(\cdot), \Omega) \tag{4.2.3}$$

and

$$\int_0^T \int_\Omega |a(x, DT_k(u_\varepsilon))|^{p'(x)} \le C_4(C_1(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k)). \tag{4.2.4}$$

Remark 4.2.2. Since A_β is m-accretive, by nonlinear semigroup theory (see [17]), (DP_ε) has a solution $(b_i^\varepsilon, u_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ for every discretisation (D_ε), $\varepsilon > 0$, $f \in$

$L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. Moreover, the piecewise constant function b_ε defined in Lemma 4.2.1 converges in $L^\infty(0,T;L^1(\Omega))$ as $\varepsilon \downarrow 0$ to the mild solution $b \in \mathcal{C}([0,T];L^1(\Omega))$ of $(ACP)(f, b_0)$.

Proof of Lemma (4.2.1): For $i \in \{1, \dots, N(\varepsilon)\}$ we take $T_k(u_i^\varepsilon)h_l(u_i^\varepsilon)$, $k, l > 0$, as a test function in (DP_ε) to obtain

$$I_1 + I_2 + I_3 = I_4,$$

where

$$\begin{aligned}
I_1 &= \int_\Omega \frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{t_i^\varepsilon - t_{i-1}^\varepsilon} h_l(u_i^\varepsilon) T_k(u_i^\varepsilon), \\
I_2 &= \int_\Omega a(x, Du_i^\varepsilon) \cdot D(h_l(u_i^\varepsilon) T_k(u_i^\varepsilon)), \\
I_3 &= \int_\Omega F(u_i^\varepsilon) \cdot D(h_l(u_i^\varepsilon) T_k(u_i^\varepsilon)), \\
I_4 &= \int_\Omega f_i^\varepsilon h_l(u_i^\varepsilon) T_k(u_i^\varepsilon).
\end{aligned}$$

By Gauss-Green Theorem, it follows that $I_3 = 0$ for all $l > k$. Applying $(A1)$ in I_2, we can pass to the limit with $l \to \infty$ and find

$$\int_\Omega \frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{t_i^\varepsilon - t_{i-1}^\varepsilon} T_k(u_i^\varepsilon) + \gamma \int_\Omega |DT_k(u_i^\varepsilon)|^{p(x)} \le k \int_\Omega |f_i^\varepsilon|. \tag{4.2.5}$$

If we define the convex, l.s.c., proper function $\phi_{T_k} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi_{T_k}(r) := \begin{cases} \int_0^r T_k((\beta^{-1})^0(\sigma)) d\sigma, & \text{if } r \in \overline{R(\beta)}, \\ +\infty, & \text{otherwise}, \end{cases}$$

then $T_k(u_i^\varepsilon) \subset \partial \phi_{T_k}(b_i^\varepsilon)$ for all $i = 1, \dots, N(\varepsilon)$ and

$$\phi_{T_k}(b_i^\varepsilon) - \phi_{T_k}(b_{i-1}^\varepsilon) \le (b_i^\varepsilon - b_{i-1}^\varepsilon) T_k(u_i^\varepsilon) \tag{4.2.6}$$

holds almost everywhere in Ω. Therefore from (4.2.5) and (4.2.6) it follows that

$$\int_\Omega \frac{\phi_{T_k}(b_i^\varepsilon) - \phi_{T_k}(b_{i-1}^\varepsilon)}{t_i^\varepsilon - t_{i-1}^\varepsilon} + \gamma \int_\Omega |DT_k(u_i^\varepsilon)|^{p(x)} \le k \int_\Omega |f_i^\varepsilon|. \tag{4.2.7}$$

Integrating (4.2.7) over $(t_{i-1}^\varepsilon, t_i^\varepsilon]$, and taking the sum over $i = 1, \dots, N(\varepsilon)$ yields

$$\int_\Omega \phi_{T_k}(b_\varepsilon(T)) dx + \gamma \int_0^T \int_\Omega |DT_k(u_\varepsilon)|^{p(x)} dx dt \le \int_\Omega \phi_{T_k}(b_\varepsilon(0)) + k \int_0^T \int_\Omega |f_\varepsilon| dx dt. \tag{4.2.8}$$

According to (D_ε), $b_\varepsilon(0) = b_0^\varepsilon$ converges to b_0 in $L^1(\Omega)$ and f_ε converges to f in $L^1(0,T;L^1(\Omega))$ as $\varepsilon \downarrow 0$. Therefore, the right-hand side of (4.2.8) is bounded by a constant $C_1(\|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k) > 0$ that does not depend on ε. Now, (4.2.1) follows from (4.2.8) if we neglect the positive term and use $(A1)$. To prove *ii.*), we apply (2.1.2) and find

$$\begin{aligned} &\int_0^T \|DT_k(u_\varepsilon)\|^{p^-}_{L^{p(\cdot)}(\Omega)} dt \\ &\le \int_0^T \max\left[\int_\Omega |DT_k(u_\varepsilon)|^{p(x)}, \left(\int_\Omega |DT_k(u_\varepsilon)|^{p(x)}\right)^{p^-/p^+}\right] dt, \end{aligned} \tag{4.2.9}$$

hence (4.2.2) follows from the Poincaré inequality in $W_0^{1,p(\cdot)}(\Omega)$ and (4.2.1). To prove *iii.*), we use $(A2)$ and the same arguments as above. $\square$

Remark 4.2.3. As we will see in the following, estimate (4.2.1) plays a crucial role in order to get a well-posed problem. Note that, if $p(\cdot) = p$ is constant, then, of course, (4.2.2) implies (4.2.1) and the problem can be settled within the classical functional setting of the Bochner-Lebesgue spaces $L^p(0,T;W_0^{1,p}(\Omega))$. In the general case, a function $v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ does not automatically satisfy (4.2.1). As an example, consider $N = 2$, $\Omega = (-1,1)^2$, $p(x,y) = 3/2 - |x|/4$, $(x,y) \in \overline{\Omega}$. Then $p^+ = 5/4$, $p^- = 3/2$ and the function $v : [0,T]\times\overline{\Omega} \to \mathbb{R}$ defined by $v(t,x,y) = t^{-2/3}(1-|x|)(1-|y|)$ is an element of $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, but $v_x, v_y \notin L^{p(\cdot)}(Q_T)$ (see [13] for more details).

4.2.1 Renormalized solution

In view of the results in [2], [22], [68] and [13], the a priori estimates in Lemma 4.2.1 naturally lead to an appropriate notion of a renormalized solution to (P, f, b_0):

Definition 4.2.1. For $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ a renormalized solution to (P, f, b_0) is a pair of functions (u, b) satisfying the following conditions:

$(P1)$ $u : Q_T \to \mathbb{R}$ is measurable, $b \in L^1(Q_T)$, $u(t,x) \in D(\beta(t,x))$ and $b(t,x) \in \beta(u(t,x))$ for a.e. $(t,x) \in Q_T$,

$(P2)$ $b(0,x) = b_0(x)$ a.e. in Ω,

$(P3)$ For each $k > 0$, $T_k(u) \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and $DT_k(u) \in (L^{p(\cdot)}(Q_T))^N$,

$(P4)$

$$
\begin{aligned}
- & \int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h \circ (\beta^{-1})^0(r) dr d(t,x) \\
+ & \int_{Q_T} (a(x, Du) + F(u)) \cdot D(h(u)\xi) d(t,x) = \int_{Q_T} f h(u) \xi d(t,x)
\end{aligned}
$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\xi \in \mathcal{D}([0,T) \times \Omega)$.

$(P5)$ $\int_{Q_T \cap \{k<|u|<k+1\}} a(x, Du) \cdot Du \; d(t,x) \to 0$ as $k \to \infty$.

Remark 4.2.4. Using the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$, we can associate to every measurable function $u : Q_T \to \mathbb{R}$ satisfying

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)), \;\; DT_k(u) \in (L^{p(\cdot)}(Q_T))^N$$

for all $k > 0$, a generalized gradient (still denoted by Du), defined as the unique measurable function satisfying $Du = DT_k(u)$ a.e. on $\{|u| < k\}$ for all $k > 0$ (see, e.g., [15]). It follows that all the terms in $(P4)$ are well-defined. In particular, the first member of $(P4)$ makes sense as

$$\int_{b_0}^{b(t,x)} h \circ (\beta^{-1})^0(r) dr \leq \|h\|_{L^\infty(\mathbb{R})} |b(t,x) - b_0|$$

almost everywhere in Q_T and $b \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$.

Remark 4.2.5. A definiton of renormalized solutions involving the spaces $L^{p(\cdot)}(Q_T)$ and $L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$ has been recently proposed in [71] for the special case when β is the identity mapping $F \equiv 0$ and $a(x, Du) = \operatorname{div}(|Du|^{p(x)-2} Du)$. But just a few comments about those spaces have been made. To the best of our knowledge, the study of properties of the spaces $L^{p(\cdot)}(Q_T)$ and $L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$ in Lemma 2.2.2, their introduction in Definition 4.2.1 as a natural consequence of the a priori estimates on the solutions of the time-discretized problems in Lemma 4.2.1 and the following study of the energy space V for weak solutions is new. Note that the proof of the main Theorem 1.1 in [71] is false. Nevertheless, the existence result holds true as a special case of (P, f, b_0). A detailed proof and additional regularity results can found in [13].

4.2.2 Functional setting and weak solutions

At the end of this subsection we will give a definition of weak solution (u, b) to (P, f, b_0). The main problem is to find an appropriate energy space for u. As we will see in the next remark, it is not enough to claim $u \in L^{p-}(0,T; W_0^{1,p(\cdot)}(\Omega))$.

Remark 4.2.6. According to the a priori estimates in Lemma 4.2.1 let us assume that for given $f \in L^\infty(Q_T)$, $b_0 \in L^1(\Omega)$ there exists $(u, b) \in L^{p-}(0, T; W_0^{1,p(\cdot)}(\Omega)) \times L^1(Q_T)$ such that $Du \in (L^{p(\cdot)}(Q_T))^N$, $b(0, x) = b_0(x)$ almost everywhere in Ω, $F(u) \in (L^{p'(\cdot)}(Q_T))^N$. Furthermore we assume

$$-\int_{Q_T}(b - b_0)\xi_t + \int_{Q_T}(a(x, Du) + F(u)) \cdot D\xi = \int_{Q_T} f\xi \tag{4.2.10}$$

to hold for all $\xi \in \mathcal{D}([0, T) \times \Omega)$. From the embeddings of Lemma 2.2.2 and $(A2)$ we have $a(x, Du) + F(u) \in L^{(p')^-}(0, T; L^{p(\cdot)}(\Omega))$ and therefore it follows that $(b - b_0)_t \in L^{(p')^-}(0, T; W^{-1,p'(\cdot)}(\Omega))$ in the sense of distributions. But since

$$\begin{aligned}
(p')^- &= \operatorname*{ess\,inf}_{x\in\Omega}\left(\frac{p(x)}{p(x)-1}\right) \\
&= 1 + \frac{1}{p^+ - 1} = (p^+)' \\
&\leq 1 + \frac{1}{p^- - 1} = (p^-)' \qquad (4.2.11)
\end{aligned}$$

and equality holds in (4.2.11) if and only if $p(\cdot) \equiv p$ is constant, in general

$$L^{(p^-)'}(0, T; W^{-1,p'(\cdot)}(\Omega)) \subsetneqq L^{(p')^-}(0, T; W^{-1,p'(\cdot)}(\Omega)).$$

Hence we can not use test functions $\xi \in L^{p^-}(0, T; W_0^{1,p(\cdot)}(\Omega))$ in (4.2.10) and (P, f, b_0) is not well-posed in $L^{p^-}(0, T; W_0^{1,p(\cdot)}(\Omega))$.

In fact, the function u from the preceeding remark is indeed more regular than $L^{p^-}(0, T; W_0^{1,p(\cdot)}(\Omega))$. Since we claimed $Du \in (L^{p(\cdot)}(Q_T))^N$, u is an element of the functional space

$$V := \{f \in L^{p^-}(0, T; W_0^{1,p(\cdot)}(\Omega)) : |Df| \in L^{p(\cdot)}(Q_T)\} \tag{4.2.12}$$

which, endowed with the norm

$$\|f\|_V := \|f\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \||Df|\|_{L^{p(\cdot)}(Q_T)}$$

is a separable and reflexive Banach space. We state some further properties of V in the following lemma:

Lemma 4.2.7. *Let V be defined as in* (4.2.12) *and V' denote the dual space of V. Then,* *i.)* *we have the following continuous dense embeddings:*

$$L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega)) \overset{d}{\hookrightarrow} V \overset{d}{\hookrightarrow} L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)). \tag{4.2.13}$$

In particular, since $\mathcal{D}(Q_T)$ *is dense in* $L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega))$*, it is dense in* V *and for the corresponding dual spaces we have*

$$L^{(p^-)'}(0,T;W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow V' \hookrightarrow L^{(p^+)'}(0,T;W^{-1,p'(\cdot)}(\Omega)). \tag{4.2.14}$$

ii.) $\|f\|_V := \||Df|\|_{L^{p(\cdot)}(Q_T)}$ *is an equivalent norm on* V*,*

iii.) *one can represent the elements of* V' *as follows: If* $T \in V'$*, then there exists* $F = (f_1, \dots, f_N) \in (L^{p'(\cdot)}(Q_T))^N$ *such that* $T = \operatorname{div}_x F$ *in the sense that*

$$\langle T, \xi \rangle_{V',V} = \int_0^T \int_\Omega F \cdot D\xi dxdt$$

for any $\xi \in V$*. Moreover, an equivalent norm on* V' *is given by*

$$\|T\|_{V'} = \max\{\|f_i\|_{L^{p(\cdot)}(Q_T)},\ i = 1, \dots, n\}.$$

Proof: *i.)*: The continuous embeddings in (4.2.13) follow immediately from Lemma 2.2.2, (2.2.2). To prove the density of the first embedding in *i.)*, we fix $v \in V$. Let $(\rho_n)_n$ be a standard sequence of mollifiers in $\mathbb{R}$ and $\overline{v}$ the abstract function v extended by zero onto $\mathbb{R}$. It follows from [40], Proposition 1.7.1, p. 25 and Théorème 1.7.1, p. 27, that the convolution (only) in t of ρ_n and $\overline{v}$ defined as $v_n := \rho_n * \overline{v} \in L^{p^+}(\mathbb{R}; W_0^{1,p(\cdot)}(\Omega))$ for all $n \in \mathbb{N}$ converges to v in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ as $n \to \infty$. Since $\frac{\partial}{\partial x_i}(\rho_n * v)(t) = (\rho_n * \frac{\partial v}{\partial x_i})(t)$ for all $t \in \mathbb{R}$ and $i = 1, \dots, n$, it is left to prove that

$$\frac{\partial v_n}{\partial x_i} = \rho_n * \frac{\partial v}{\partial x_i} \to \frac{\partial v}{\partial x_i}$$

for each $i = 1, \dots, n$ in $L^{p(\cdot)}(Q_T)$ as $n \to \infty$. To this end, we fix $\varepsilon > 0$, $i \in \{1, \dots, n\}$ and choose (by Lemma 2.2.2) a function $u_\varepsilon^i \in L^{p^+}(0,T;L^{p(\cdot)}(\Omega))$ such that

$$\int_{Q_T} \left| u_\varepsilon^i - \frac{\partial v}{\partial x_i} \right|^{p(x)} d(t,x) < \frac{\varepsilon}{3 \cdot 4^{2p^+}} \tag{4.2.15}$$

Then

$$\int_0^T \int_\Omega \left| \frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^{p(x)} dxdt \le 4^{2p^+}(I_1 + I_2 + I_3), \tag{4.2.16}$$

where

$$\begin{aligned}
I_1 &= \int_0^T \int_\Omega \left| \int_{\mathbb{R}} \rho_n(t-s) \left(\frac{\partial v}{\partial x_i}(s,x) - u_\varepsilon^i(s,x) \right) ds \right|^{p(x)} dxdt, \\
I_2 &= \int_0^T \int_\Omega \left| \int_{\mathbb{R}} \rho_n(t-s) u_\varepsilon^i(s,x) ds - u_\varepsilon^i(t,x) \right|^{p(x)} dxdt, \\
I_3 &= \int_0^T \int_\Omega \left| u_\varepsilon^i - \frac{\partial v}{\partial x_i} \right|^{p(x)} dxdt. \qquad (4.2.17)
\end{aligned}$$

Substituting, applying the Jensen inequality and Fubini Theorem we get the estimate

$$\begin{aligned}
I_1 &\le \int_{-1}^1 \rho(\sigma) \int_0^T \int_\Omega \left| \frac{\partial v}{\partial x_i}(t+\sigma/n, x) - u_\varepsilon^i(t+\sigma/n, x) \right|^{p(x)} dxdtd\sigma \\
&\le \int_0^T \int_\Omega \left| \frac{\partial v}{\partial x_i}(t,x) - u_\varepsilon^i(t,x) \right|^{p(x)} dxdt. \qquad (4.2.18)
\end{aligned}$$

Since $I_2 \to 0$ as $n \to \infty$, choosing n_0 large enough and plugging (4.2.15), (4.2.17) and (4.2.18) into (4.2.16) we have shown that the first embedding in (4.2.13) is dense. To prove that the second embedding is dense, we fix $u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$. Using the same arguments as in the proof of Lemma 2.2.2, $u_n := \rho_n * u$ is in $L^{p^+}(0,T; W_0^{1,p(\cdot)}(\Omega))$, hence in $L^{p(\cdot)}(Q_T)$ for all $n \in \mathbb{N}$ and converges to u in $L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$ as $n \to \infty$. *ii.*) follows directly from Poincaré inequality and Lemma 2.2.2. To prove *iii.*), note that the mapping $i : V \to (L^{p(\cdot)}(Q_T))^N$ defined by

$$i(u) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right)$$

for $u \in V$ is linear, continuous and, by *ii.*), norm preserving. Identifying each $T \in V'$ with $T \circ i^{-1} \in (i(V))'$ and using Hahn-Banach Theorem, we can extend T to a continuous linear functional on $(L^{p(\cdot)}(Q_T))^N$ and the assertion follows by the $L^{p(\cdot)}(Q_T)$-$L^{p'(\cdot)}(Q_T)$ duality. □

Remark 4.2.8. Note that, if u is a renormalized solution of (P, f, b_0), then $T_k(u), h(u) \in V \cap L^\infty(Q_T)$ for all $h \in \mathcal{C}_c^1(\mathbb{R})$. Let us also remark that $V \cap L^\infty(Q_T)$ endowed with the norm

$$\|v\|_{V \cap L^\infty(Q_T)} := \max \left\{ \|v\|_V, \|v\|_{L^\infty(Q_T)} \right\}, \ v \in V \cap L^\infty(Q_T)$$

is a Banach space. In fact it is the dual space of the Banach space $V'+L^1(Q_T)$ endowed with the norm

$$\|v\|_{V'+L^1(Q_T)} = \inf\left\{\|v_1\|_{V'} + \|v_2\|_{L^1(Q_T)};\ v = v_1 + v_2,\ v_1 \in V',\ v_2 \in L^1(Q_T)\right\}.$$

Moreover,

$$\frac{\partial}{\partial t}\int_{b_0}^{b(t,x)} h\circ(\beta^{-1})^0(r)dr \in V' + L^1(Q_T). \tag{4.2.19}$$

Indeed, for any $h \in \mathcal{C}_c^1(\mathbb{R})$, $k > 0$ such that $\operatorname{supp} h \subset [-k,k]$ and $\xi \in \mathcal{D}(Q_T)$ we have

$$\int_{Q_T}(a(x,Du) + F(u))\cdot D(h(u)\xi) + \int_{Q_T} fh(u)\xi = I_1 + I_2 \tag{4.2.20}$$

where, by Remark 4.2.4, there exist constants $K_1, K_2 > 0$ not depending on ξ such that

$$\begin{aligned}
|I_1| &= \left|\int_{Q_T} h(u)(a(x,DT_k(u)) + F(T_k(u)))\cdot D\xi\right| \le K_1\||D\xi|\|_{L^{p(\cdot)}(Q_T)},\\
|I_2| &= \left|\int_{Q_T}(h'(u)(a(x,DT_k(u)) + F(T_k(u)))\cdot DT_k(u) + fh(u))\xi\right| \le K_2\|\xi\|_{L^\infty(Q_T)}
\end{aligned}$$

hence, by Lemma 4.2.7, *iii.)* there exists $G_1 \in V'$ such that

$$I_1 = \langle G_1, \xi\rangle_{V',V} \tag{4.2.21}$$

and

$$I_2 = \langle h'(u)(a(x,DT_k(u)) + F(T_k(u)))\cdot DT_k(u) + fh(u), \xi\rangle_{L^1(Q_T),L^\infty(Q_T)}. \tag{4.2.22}$$

Now, by $(P4)$, (4.2.21) and (4.2.22) it follows that

$$\frac{\partial}{\partial t}\int_{b_0}^{b(t,x)} h\circ(\beta^{-1})^0(r)dr = G_1 + h'(u)(a(x,DT_k(u)) + F(T_k(u)))\cdot DT_k(u) + fh(u) \tag{4.2.23}$$

in $\mathcal{D}'(Q_T)$ and (4.2.19) holds. Since $\mathcal{D}(Q_T)$ is dense in V, for any $\phi \in V \cap L^\infty$ there exists a sequence $(\phi_n) \subset \mathcal{D}(Q_T)$ such that $\phi_n \to \phi$ as $n \to \infty$ in V and weak-$*$ in $L^\infty(Q_T)$. Therefore, replacing

$$-\int_{Q_T}\xi_t\int_{b_0}^{b(t,x)} h\circ(\beta^{-1})^0(r)dr$$

by

$$\left\langle \frac{\partial}{\partial t} \int_{b_0}^{b(t,x)} h \circ (\beta^{-1})^0(r)dr, \xi \right\rangle \tag{4.2.24}$$

in the left-hand side of $(P4)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V' + L^1(Q_T)$ and $V \cap L^\infty(Q_T)$, we can also use test functions $\xi \in V \cap L^\infty(Q_T)$.

Now, we are finally in the position to give a generalisation to the notion of weak solutions to (P, f, b_0) for variable exponents:

Definition 4.2.2. For $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ a weak solution to (P, f, b_0) is a pair of functions $(u, b) \in V \times L^1(Q_T)$ satisfying $F(u) \in (L^{p'(\cdot)}(Q_T))^N$, $b \in \beta(u)$ almost everywhere in Q_T, $b(0, x) = b_0$ almost everywhere in Ω such that

$$-\int_{Q_T} (b - b_0)\xi_t + \int_{Q_T} (a(x, Du) + F(u)) \cdot D\xi = \int_{Q_T} f\xi \tag{4.2.25}$$

holds for all $\xi \in \mathcal{D}([0, T) \times \Omega)$.

Remark 4.2.9. Note that if (u, b) is a renormalized solution to (P, f, b_0) such that $u \in L^\infty(Q_T)$ then (u, b) is a weak solution to (P, f, b_0). Indeed, as an immediate consequence from $(P1)$, and $(P3)$ we get $u \in V$. Now we fix $\xi \in \mathcal{D}([0, T) \times \Omega)$ and choose $h_l(u)\xi$ as a test function in $(P4)$. As usual, we apply the Gauss-Green Theorem and the boundary condition on the „convection“ term $\int_{Q_T} h_l'(u)\xi \ F(u) \cdot Du$ and $(P5)$ to estimate $\int_{Q_T} h_l'(u)\xi \ a(x, Du) \cdot Du$. Passing to the limit with $l \to \infty$, we find (4.2.2). The remaining conditions for being a weak solution follow from $(P1)$ and $(P2)$.

Remark 4.2.10. For $f \in L^\infty(Q_T)$, from (4.2.25) it follows that $(b - b_0)_t \in V'$. If we replace

$$-\int_{Q_T} (b - b_0)\xi_t$$

by

$$\langle (b - b_0)_t, \xi \rangle_{V', V}$$

in (4.2.25), by density of $\mathcal{D}(Q_T)$ in V we can use test functions in V. Therefore, the problem (P, f, b_0) is well-posed in V.

4.2.3 Integration-by-parts-formula

In the next Lemma, we prove an integration-by-parts-formula that will be crucial in the following. The idea of the proof is the same as in [1] and the generalisations considered in [31] and [61]. The essential point is that the Stekhlov average of a function $v \in V \cap L^\infty(Q_T)$ defined by $v_\eta(\cdot) = \frac{1}{\eta}\int_\cdot^{\cdot+\eta} v(\sigma)d\sigma$, $\eta > 0$, (appropriately prolongated outside $(0,T)$) still belongs to $V \cap L^\infty(Q_T)$ and converges to v in V and weak-$*$ in $L^\infty(Q_T)$ as $\eta \downarrow 0$.

Lemma 4.2.11. *Let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph, $u \in V$, $b \in L^1(Q_T)$ such that $b \in \beta(u)$ almost everywhere in Q_T, $b_0 \in L^1(\Omega)$ with $b(0,x) = b_0$ almost everywhere in Ω and $u_0 : \Omega \to \overline{\mathbb{R}}$ be a measurable function such that $b_0 \in \beta(u_0)$ almost everywhere in Ω. Furthermore, we assume that there exists $G \in V' + L^1(Q_T)$ satisfying*

$$\int_{Q_T} (b - b_0)\xi_t = \langle G, \xi \rangle, \tag{4.2.26}$$

for all $\xi \in \mathcal{D}([0,T) \times \Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V' + L^1(Q_T)$ and $V \cap L^\infty(Q_T)$. Then,

$$\int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h((\beta^{-1})^0(\sigma))d\sigma = \langle G, h(u)\xi \rangle \tag{4.2.27}$$

for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$.

Remark 4.2.12. Note that $(b - b_0)_t \in \mathcal{D}'(Q_T)$ is identified to $G \in V' + L^1(Q_T) \subset \mathcal{D}'(Q_T)$ by formula (4.2.26).

Proof of Lemma 4.2.11: First note that there exist Lipschitz continuous functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that h_1 is non-decreasing, h_2 is non-increasing and $h = h_1 + h_2$. Furhermore, there exists $k > 0$ such that $\operatorname{supp} h \subset [-k, k]$, hence $h(u) = h(T_k(u)) = h_1(T_k(u)) + h_2(T_k(u))$ and obviously $h_1 \circ T_k(u), h_2 \circ T_k(u) \in V \cap L^\infty(Q_T)$. For $\xi \in \mathcal{D}^+([0,T) \times \overline{\Omega})$, $(t,x) \in Q_T$ and $\eta > 0$ we set $\zeta := h_1(T_k(u))\xi$ and

$$\xi_\eta(t,x) := \frac{1}{\eta}\int_t^{t+\eta} \zeta(\sigma, x)d\sigma. \tag{4.2.28}$$

Note that $\zeta \in V \cap L^\infty(Q_T)$ and the function $\xi_\eta : Q_T \to \mathbb{R}$ is in $V \cap W^{1,\infty}(Q_T)$ with $\xi_\eta(T,x) = 0$ for all $x \in \Omega$ and $\eta > 0$. Using similar arguments as in Remark 4.2.8, it follows that ξ_η is an admissible test function in (4.2.26), hence

$$\begin{aligned}
\langle G, \xi_\eta \rangle &= \int_{Q_T} (b - b_0)(\xi_\eta)_t \\
&= \int_{Q_T} \frac{1}{\eta}(\zeta(t+\eta, x) - \zeta(t,x))(b(t,x) - b(0,x)) \\
&= \frac{1}{\eta}(I_1 + I_2 + I_3),
\end{aligned} \tag{4.2.29}$$

where $\zeta(t,x)=0$ for $t>T$, $b(t,x)=b_0$ for $t<0$, and

$$\begin{aligned} I_1 &= \int_0^T\int_\Omega \zeta(t+\eta,x)b(t,x) = \int_\eta^T\int_\Omega \zeta(t,x)b(t-\eta,x) \\ I_2 &= -\int_0^T\int_\Omega \zeta(t,x)b(t,x), \end{aligned} \tag{4.2.30}$$

$$\begin{aligned} I_3 &= -\int_0^T\int_\Omega \zeta(t+\eta,x)-\zeta(t,x))b(0,x) \\ &= \int_0^T\int_\Omega\int_t^{t+\eta}\zeta(\sigma,x)d\sigma\cdot 0 - \int_\Omega\int_t^{t+\eta}\zeta(\sigma,x)d\sigma\cdot b(0,x)\Big|_0^T \\ &= \int_0^\eta\int_\Omega \zeta(t,x)b(t-\eta,x). \end{aligned} \tag{4.2.31}$$

From (4.2.30) and (4.2.31) we get

$$\langle G,\xi_\eta\rangle = \frac{1}{\eta}\int_{Q_T}\zeta(t,x)(b(t-\eta,x)-b(t,x)). \tag{4.2.32}$$

Now, since $b(t,x)\in\beta(u(t,x))$ almost everywhere in Q_T, $h_1\circ T_k$ nondecreasing and $\xi\geq 0$ it follows that

$$\frac{1}{\eta}\int_{Q_T}\zeta(t,x)(b(t-\eta,x)-b(t,x)) \;\leq\; \frac{1}{\eta}\int_{Q_T}\xi(t,x)\int_{b(t,x)}^{b(t-\eta,x)} h_1\circ T_k\circ(\beta^{-1})^0(\sigma)d\sigma \tag{4.2.33}$$

If we define $\phi_{h_1}:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ by

$$\phi_{h_1}(r) := \begin{cases} \int_0^r h_1\circ T_k\circ(\beta^{-1})^0(\sigma)d\sigma, & r\in\overline{R(\beta)}, \\ +\infty, & \text{otherwise}, \end{cases}$$

from (4.2.32) and (4.2.33) it follows with the same arguments as in (4.2.31) that

$$\begin{aligned} \langle G,\xi_\eta\rangle &\leq \frac{1}{\eta}\int_{Q_T}\xi(t,x)(\phi_{h_1}(b(t-\eta,x))-\phi_{h_1}(b(t,x))) \\ &= \frac{1}{\eta}\int_{Q_T}(\xi(t+\eta,x)-\xi(t,x))(\phi_{h_1}(b(t,x))-\phi_{h_1}(b(0,x)) \\ &= \frac{1}{\eta}\int_{Q_T}(\xi(t+\eta,x)-\xi(t,x))\int_{b_0}^{b(t,x)} h_1\circ T_k\circ(\beta^{-1})^0(\sigma)d\sigma. \end{aligned} \tag{4.2.34}$$

Passing to a subsequence if necessary, we have $\xi_\eta \to \xi h_1(T_k(u))$ as $\eta \downarrow 0$ in V and weak-$*$ in $L^\infty(Q_T)$, hence passing to the limit in (4.2.34) yields

$$\langle G, h_1(T_k(u))\xi\rangle \le \int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma)d\sigma. \tag{4.2.35}$$

Note that since $T_k(u_0) \in L^\infty(\Omega)$, there exists a sequence $(u_{0,n})_n \subset \mathcal{D}(\Omega)$ such that $T_k(u_{0,n}) \to T_k(u_0)$ in $L^1(\Omega)$ and almost everywhere in Ω as $n \to \infty$. For $t < 0$ and all $x \in \Omega$ we write $u(t,x) = u_{0,n}$ and $b(t,x) = b_0$. For $\xi \in \mathcal{D}^+([0,T) \times \overline{\Omega})$ we define $\xi(t,x) := \xi(-t,x)$ for $t < 0$ and all $x \in \Omega$. If $\zeta := h_1(T_k(u))\xi$, for $(t,x) \in Q_T$ and $\eta > 0$ we define

$$\tilde{\xi}_\eta(t,x) := \frac{1}{\eta}\int_{t-\eta}^{t} \zeta(\sigma,x)d\sigma, \tag{4.2.36}$$

then $\tilde{\xi}_\eta : Q_T \to \mathbb{R}$ is in $V \cap W^{1,\infty}(Q_T)$ such that $\xi_\eta(T,x) = 0$ for all $x \in \Omega$ and $\eta > 0$ sufficiently small. Using similar arguments as in Remark 4.2.8, it follows that $\tilde{\xi}_\eta$ is an admissible test function in (4.2.26), hence

$$\begin{aligned}
\langle G, \tilde{\xi}_\eta\rangle &= \int_{Q_T} (b - b_0)(\tilde{\xi}_\eta)_t \\
&= \int_{Q_T} \frac{1}{\eta}(\zeta(t,x) - \zeta(t-\eta,x))(b(t,x) - b(0,x)) \\
&= \frac{1}{\eta}(J_1 + J_2 + J_3),
\end{aligned} \tag{4.2.37}$$

where

$$\begin{aligned}
J_1 &= \int_0^T \int_\Omega \zeta(t,x)b(t,x) = \int_\eta^{T+\eta} \int_\Omega \zeta(t-\eta,x)b(t-h,x) \\
J_2 &= -\int_0^T \int_\Omega \zeta(t-\eta,x)b(t,x),
\end{aligned} \tag{4.2.38}$$

and, for $\eta > 0$ sufficiently small,

$$\begin{aligned}
J_3 &= -\int_0^T \int_\Omega \zeta(t,x) - \zeta(t-\eta,x))b(0,x) \\
&= \int_0^T \int_\Omega \int_{t-\eta}^{t} \zeta(\sigma,x)d\sigma \cdot 0 - \int_\Omega \int_{t-\eta}^{t} \zeta(\sigma,x)d\sigma \cdot b(0,x)\Big|_0^T \\
&= \int_{-\eta}^{0} \int_\Omega \zeta(t,x)b(t,x) \\
&= \int_0^\eta \int_\Omega \zeta(t-\eta,x)b(t-\eta,x).
\end{aligned} \tag{4.2.39}$$

From (4.2.38) and (4.2.39) we get

$$\langle G, \tilde{\xi}_\eta \rangle = \frac{1}{\eta}(I_1 + I_2), \tag{4.2.40}$$

where

$$\begin{aligned} I_1 &= \int_\eta^T \int_\Omega \zeta(t-\eta,x)(b(t-\eta,x) - b(t,x)), \\ I_2 &= \int_0^\eta \int_\Omega h_1(T_k(u_0))\xi(b(t-\eta,x) - b(t,x)) \\ &+ \int_0^\eta \int_\Omega (h_1(T_k(u_{0,n})) - h_1(T_k(u_0)))\xi(b(t-\eta,x) - b(t,x)) \end{aligned}$$

Since $-(h_1 \circ T_k \circ (\beta^{-1})^0)$ is nonincreasing, for any $\eta > 0$ we have

$$\int_{b(t-\eta,x)}^{b(t,x)} -(h_1 \circ T_k \circ (\beta^{-1})^0)(\sigma)d\sigma \leq -(b(t,x) - b(t-\eta,x))h_1(T_k(u(t-\eta,x))) \tag{4.2.41}$$

almost everywhere in $(\eta, T) \times \Omega$ and

$$\int_{b(t-\eta,x)}^{b(t,x)} -(h_1 \circ T_k \circ (\beta^{-1})^0)(\sigma)d\sigma \leq -(b(t,x) - b_0)h_1(T_k(u_0)) \tag{4.2.42}$$

almost everywhere in $(0, \eta) \times \Omega$. Now, putting together (4.2.40), (4.2.41) and (4.2.42) yields

$$\begin{aligned} \langle G, \tilde{\xi}_\eta \rangle &\geq \frac{1}{\eta} \int_{Q_T} \xi(t-\eta,x)(\phi_{h_1}(b(t,x)) - \phi_{h_1}(b(t-\eta,x))) \\ &+ \int_0^\eta \int_\Omega (h_1(T_k(u_{0,n})) - h_1(T_k(u_0)))\xi(b_0 - b(t,x)) \\ &\geq \frac{1}{\eta} \int_{Q_T} (\xi(t-\eta,x) - \xi(t,x))(\phi_{h_1}(b(t,x)) - \phi_{h_1}(b(0,x))) \\ &- \int_{Q_T} |h_1(T_k(u_{0,n})) - h_1(T_k(u_0))||\xi|(|b_0| + |b(t,x)|) \\ &= \frac{1}{\eta} \int_{Q_T} (\xi(t-\eta,x) - \xi(t,x)) \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma)d\sigma \\ &- \int_{Q_T} |h_1(T_k(u_{0,n})) - h_1(T_k(u_0))||\xi|(|b_0| + |b(t,x)|). \end{aligned} \tag{4.2.43}$$

Passing to the limit with $\eta \downarrow 0$ an then with $n \to \infty$ in (4.2.43) by Lebesgue Dominated Convergence Theorem we get

$$\langle G, h_1(T_k(u))\xi\rangle \geq \int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma)d\sigma. \tag{4.2.44}$$

Combining (4.2.35) and (4.2.44) finally we get

$$\langle G, h_1(T_k(u))\xi\rangle = \int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma)d\sigma. \tag{4.2.45}$$

for all $h_1 : \mathbb{R} \to \mathbb{R}$ non-decreasing and Lipschitz continuous and all $\xi \in \mathcal{D}^+([0,T) \times \overline{\Omega})$. Replacing $h_1(T_k(u))$ by $-h_2(T_k(u))$ in (4.2.45) it follows that (4.2.45) also holds for $-h_2(T_k(u))$ and $h_2(T_k(u))$, hence we can also replace $h_1(T_k(u))$ by $h(T_k(u)) = h(u)$ in (4.2.45). For $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$ we have $\xi = \xi^+ + \xi^-$ where $\xi^+ := \max(0,\xi)$, $\xi^- := \min(0,\xi)$ are in $W^{1,\infty}(Q_T)$. By density, we can plug ξ^+, ξ^- in (4.2.45) to finally obtain (4.2.27) for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$. □

Remark 4.2.13. The integration-by-parts-formula of Lemma 4.2.26 still holds for any $h \in W^{1,\infty}(\mathbb{R})$. Indeed, there exists a sequence $(h_n)_n \subset \mathcal{C}_c^1(\mathbb{R})$ converging to h in $W^{1,p}(\mathbb{R})$ for any $1 \leq p < \infty$ and in $L^\infty(\mathbb{R})$ as $n \to \infty$. Hence, for $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$ and $u \in V$, $h_n(u)\xi$ converges to $h(u)\xi$ in V and in $L^\infty(Q_T)$ as $n \to \infty$. Therefore we can pass to the limit in (4.2.27).

Proposition 4.2.14. *For $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ such that there exists a measurable function $u_0 : \Omega \to \overline{\mathbb{R}}$ with $b_0 \in \beta(u_0)$ almost everywhere in Ω let (u,b) be a weak solution to (P,f,b_0). Then (u,b) is a renormalized solution to (P,f,b_0).*

Proof: Clearly, (u,b) satisfies $(P1)$, $(P2)$ and $(P5)$.
Since $u \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, we have $u \in L^1(0,T;W_0^{1,1}(\Omega))$ and $T_k(u) \in L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ for any $k > 0$. Moreover, the truncation function T_k is Lipschitz continuous for any $k > 0$, hence $T_k(u(t)) \in W_0^{1,p(\cdot)}(\Omega)$ for almost all $t \in (0,T)$. According to the chain rule in $W_0^{1,1}(\Omega)$ (see [76], Theorem 2.1.11, p. 48-49, [51], Corollary A.6, p. 54), it follows that

$$DT_k(u(t)) = D(T_k \circ u(t)) = \chi_{\{|u(t)|<k\}} Du(t)$$

in $\mathcal{D}'(\Omega)$ for almost all $t \in (0,T)$, hence $DT_k(u) = \chi_{\{|u|<k\}} Du$ almost everywhere in Q_T. By assumption we have $u \in V$, hence u is finite almost everywhere in Q_T and it follows that $|\{k < |u| < k+1\}| \to 0$ as $k \to \infty$. In particular, $|Du|^{p(\cdot)} \in L^1(Q_T)$ and therefore $(P3)$ holds. From (4.2.2) we get that $(b-b_0)_t \in V' + L^1(Q_T)$. Now, $(P4)$ follows from the integration-by-parts-formula in Lemma 4.2.11. □

4.3 Existence of renormalized solutions

The main result of this section is the following theorem:

Theorem 4.3.1. *For each $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ there exists a renormalized solution to (P, f, b_0).*

To prove Theorem 4.3.1, we will use several approximation procedures. First, we prove existence of weak solutions for L^∞-data.

4.3.1 Existence for L^∞-data

In a first step, for bounded data $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, we prove existence of a weak solution to our elliptic-parabolic problem with an additional strictly monotone and continuous perturbation $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(0) = 0$, i.e.

$$(P, \psi, f, b_0) \begin{cases} \beta(u)_t + \psi(u) - \operatorname{div}(a(x, Du) + F(u)) \ni f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ \beta(u(0, \cdot)) \ni b_0 & \text{in } \Omega \end{cases}$$

To this end, we define the nonlinear operator

$$\begin{aligned} A_{\beta,\psi} := \big\{ &(b, w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R} \text{ measurable}, \\ &b \in \beta(u) \text{ a.e. in } \Omega,\ u \text{ is a renormalized solution of} \\ &- \operatorname{div}(a(x, Du) + F(u)) + \psi(u) = w \big\}, \end{aligned} \tag{4.3.1}$$

where a definition of a renormalized solution to the above problem is obtained from Definition 3.1.1 upon setting $f = w - \psi(u) - b_0$. Using the same arguments as in Subsection 4.1.1 it follows that $A_{\beta,\psi}$ is m-accretive in $L^1(\Omega)$ and $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} = \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$, i.e. to each $(f, b_0) \in L^1(Q_T) \times \overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ there exists a unique mild solution $b \in \mathcal{C}([0, T]; L^1(\Omega))$ of the abstract Cauchy problem

$$(ACP)(f, \psi, b_0) \begin{cases} \dfrac{db}{dt} + A_{\beta,\psi} b \ni f \text{ in } (0, T), \\ b(0) = b_0 \end{cases}$$

corresponding to (P, ψ, f, b_0). Moreover, for $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, b is the uniform limit of piecewise constant functions $b_\varepsilon : (0, T) \to$

$L^1(\Omega)$ defined by $b_\varepsilon = b_i^\varepsilon$ on $]t_{i-1}^\varepsilon, t_i^\varepsilon]$, $i = 1, \ldots, N(\varepsilon)$, $b_\varepsilon(0) = b_0^\varepsilon$ where $(u_i^\varepsilon, b_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ is a solution of the discretized problem (see Proposition 3.3.2)

$$(DP_{\varepsilon,\psi}) \begin{cases} b_i^\varepsilon \in L^1(\Omega), u_i^\varepsilon \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \\ \int_{\{n<|u_i^\varepsilon|<n+1\}} a(x, Du_i^\varepsilon) \cdot Du_i^\varepsilon \to 0 \text{ as } n \to \infty, \\ \int_\Omega \frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{\varepsilon}\varphi + \int_\Omega (a(x, Du_i^\varepsilon) + F(u_i^\varepsilon)) \cdot D\varphi + \int_\Omega \psi(u_i^\varepsilon)\varphi = \int_\Omega f_i^\varepsilon \varphi \\ \forall \varphi \in W_0^{1,p(\cdot)}(\Omega) \\ b_i^\varepsilon \in \beta(u_i^\varepsilon) \text{ a.e. in } \Omega, \\ i = 1, \ldots, N(\varepsilon). \end{cases}$$

given by an equidistant time discretisation of the form

$$(D_\varepsilon) \begin{cases} t_0^\varepsilon = 0 < t_1^\varepsilon < \ldots < t_{N(\varepsilon)}^\varepsilon = T, \\ t_i^\varepsilon - t_{i-1}^\varepsilon = \varepsilon, \ \forall i = 1, \ldots, N(\varepsilon), \\ f_i^\varepsilon \in L^\infty(\Omega), i = 1, \ldots, N(\varepsilon) : \sum_{i=1}^{N(\varepsilon)} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \|f(t) - f_i^\varepsilon\|_{L^1(\Omega)} dt \le \varepsilon \\ b_0^\varepsilon \in L^\infty(\Omega) : \|b_0^\varepsilon - b_0\|_{L^1(\Omega)} \le \varepsilon. \end{cases}$$

If we define the piecewise constant function $u_\varepsilon : (0, T) \to W_0^{1,p(\cdot)}(\Omega)$ by $u_\varepsilon(t) = u_i^\varepsilon$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $i = 1, \ldots, N(\varepsilon)$, the following a priori estimates hold:

Lemma 4.3.2. *Let u_ε be defined as above. Then, the following results hold for all $\varepsilon > 0$:*

i.) There exists a constant $C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}) > 0$ not depending on $\varepsilon > 0$, such that

$$\|\psi(u_\varepsilon)\|_{L^\infty(Q_T)} \le C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}). \tag{4.3.2}$$

ii.) There exists a constant $C_2(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}, \psi) > 0$ not depending on $\varepsilon > 0$, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le C_2(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}, \psi). \tag{4.3.3}$$

iii.) There exists a constant $C_3(\gamma, C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)})) > 0$ not depending on $\varepsilon > 0$, such that

$$\int_0^T \int_\Omega |Du_\varepsilon|^{p(x)} \le C_3(\gamma, C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)})). \tag{4.3.4}$$

Proof: As in [18], [2], for $i = 1, \dots, N(\varepsilon)$ we choose $p(u_i^\varepsilon)$ as a test function in $(DP_{\varepsilon,\psi})$ where $p \in \mathcal{P}_0 = \{p \in \mathcal{C}^\infty(\mathbb{R});\ 0 \leq p' \leq 1,\ \text{supp } p' \text{ compact},\ 0 \notin \text{supp } p\}$. Upon integrating over $(t_{i-1}^\varepsilon, t_i^\varepsilon)$ and summing over $i = 1, \dots N(\varepsilon)$ we obtain *i.)* and from *i.)* we deduce *ii.)* since ψ is strictly increasing and continuous. To prove *iii.)*, for $i = 1, \dots, N(\varepsilon)$ we plug u_i^ε a test function in $(DP_{\varepsilon,\psi})$ and use analogous arguments as in the proof of Lemma 4.2.1. $\square$

In the next steps we will show that a subsequence $(u_\varepsilon, b_\varepsilon)_\varepsilon$, converges to a weak solution (u, b) of (P, ψ, f, b_0) as $\varepsilon \downarrow 0$. Since the (uniform) convergence of $(b_\varepsilon)_\varepsilon$ is a straightforward consequence from nonlinear semigroup theory, the main difficulty is to obtain almost everywhere convergence (up to a subsequence) of $(u_\varepsilon)_\varepsilon$. In general (see [2], [68] for the case of a constant exponent), we have to solve approximate problems (P_k, ψ, f, b_0) where we replace β by $\beta + \frac{1}{k}I$, for $k > 0$. The result of the following lemma allows us to skip this approximation step in the particular case when β is a continuous, non-decreasing function.

A technical lemma

Lemma 4.3.3. *Let β be a continuous and non-decreasing function, $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$. For ε, $\delta > 0$ let (D_ε), (D_δ) be equidistant time discretisations and $(u_i^\varepsilon, b_i^\varepsilon)_{i=1}^{N(\varepsilon)}$, $(u_j^\delta, b_j^\delta)_{j=1}^{M(\delta)}$ solutions of the corresponding discretised problems $(DP_{\varepsilon,\psi})$ and $(DP_{\delta,\psi})$. Assume that the piecewise constant functions $b_\varepsilon, b_\delta : [0, T] \to L^1(\Omega)$ defined by $b_\varepsilon(0) = b_0^\varepsilon$, $b_\delta(0) = b_0^\delta$, $b_\varepsilon(t) = b_i^\varepsilon$, $b_\delta(t) = b_j^\delta$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $t \in (t_{j-1}^\delta, t_j^\delta]$ respectively, $i = 1, \dots, N(\varepsilon)$, $j = 1, \dots, M(\delta)$ converge to a function $b \in \mathcal{C}([0, T]; L^1(\Omega))$ as $\varepsilon, \delta \downarrow 0$ in $L^\infty(0, T; L^1(\Omega))$. Then,*

$$\lim_{\varepsilon,\delta \downarrow 0} \int_0^T \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)| = 0 \tag{4.3.5}$$

holds for the piecewise constant functions $u_\varepsilon, u_\delta : (0, T) \to W_0^{1,p(\cdot)}(\Omega)$ defined by $u_\varepsilon(t) = u_i^\varepsilon$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $i = 1, \dots, N(\varepsilon)$, $u_\delta(t) = u_j^\delta$ for $t \in (t_{j-1}^\delta, t_j^\delta]$ and $j = 1, \dots, M(\delta)$.

Proof: Let t, s denote two variables in $[0, T]$, $\xi \in \mathcal{D}(0, T)$, $\xi \geq 0$ and $(\rho_n)_n$ be a sequence of mollifiers. We write the variable t in (D_ε) and the variable s in (D_δ). For $j \in \{1, \dots, M(\delta)\}$ fixed, we plug $\frac{1}{k} T_k(u_i^\varepsilon - u_j^\delta) \rho_n(t_i^\varepsilon - s_j^\delta) \xi(t_i^\varepsilon)$

as a test function into $(DP_{\varepsilon,\psi})$ to obtain

$$
\begin{aligned}
&\frac{1}{k}\int_\Omega \frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{\varepsilon} T_k(u_i^\varepsilon - u_j^\delta)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon) \\
+&\frac{1}{k}\int_\Omega \rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon)(a(x, Du_i^\varepsilon) + F(u_i^\varepsilon)) \cdot DT_k(u_i^\varepsilon - u_j^\delta) \\
+&\frac{1}{k}\int_\Omega \psi(u_i^\varepsilon) T_k(u_i^\varepsilon - u_j^\delta)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon) \\
=&\frac{1}{k}\int_\Omega f_i^\varepsilon T_k(u_i^\varepsilon - u_j^\delta)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon)
\end{aligned}
\tag{4.3.6}
$$

for all $i = 1, \ldots, N(\varepsilon)$. For $i \in \{1, \ldots, N(\varepsilon)\}$, fixed, we plug $T_k(u_j^\delta - u_i^\varepsilon)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon)$ as a test function into $(DP_{\delta,\psi})$ to obtain

$$
\begin{aligned}
&\frac{1}{k}\int_\Omega \frac{b_j^\delta - b_{j-1}^\delta}{\delta} T_k(u_j^\delta - u_i^\varepsilon)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon) \\
+&\frac{1}{k}\int_\Omega \rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon)(a(x, Du_j^\delta) + F(u_j^\delta)) \cdot DT_k(u_j^\delta - u_i^\varepsilon) \\
+&\frac{1}{k}\int_\Omega \psi(u_j^\delta) T_k(u_j^\delta - u_i^\varepsilon)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon) \\
=&\frac{1}{k}\int_\Omega f_j^\delta T_k(u_j^\delta - u_i^\varepsilon)\rho_n(t_i^\varepsilon - s_j^\delta)\xi(t_i^\varepsilon)
\end{aligned}
\tag{4.3.7}
$$

for all $j = 1, \ldots, M(\delta)$. If we define the piecewise constant functions $f_\varepsilon : (0,T) \to L^\infty(\Omega)$, $\xi_\varepsilon : (0,T) \to \mathbb{R}$ and $\rho_n^{\varepsilon,\delta} : (0,T)^2 \to \mathbb{R}$ by $f_\varepsilon(t) = f_i^\varepsilon$, $\xi_\varepsilon(t) = \xi(t_i^\varepsilon)$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $\rho_n^{\varepsilon,\delta}(t,s) = \rho_n(t_i^\varepsilon - s_j^\delta)$ for $(t,s) \in (t_{i-1}^\varepsilon, t_i^\varepsilon] \times (s_{j-1}^\delta, s_j^\delta]$, integrating (4.3.6) over $(t_{i-1}^\varepsilon, t_i^\varepsilon)$ and summing over $i = 1, \ldots, N(\varepsilon)$ yields

$$I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon = I_4^\varepsilon, \tag{4.3.8}$$

where

$$I_4^\varepsilon = \frac{1}{k}\int_0^T \int_\Omega f_\varepsilon(t) T_k(u_\varepsilon(t,x) - u_j^\delta)\rho_n^{\varepsilon,\delta}(t, s_j^\delta)\xi_\varepsilon(t)\,dxdt, \tag{4.3.9}$$

$$I_3^\varepsilon = \frac{1}{k}\int_0^T \int_\Omega \psi(u_\varepsilon(t,x)) T_k(u_\varepsilon(t,x) - u_j^\delta)\rho_n^{\varepsilon,\delta}(t, s_j^\delta)\xi_\varepsilon(t)\,dxdt, \tag{4.3.10}$$

$$I_2^\varepsilon = \frac{1}{k}\int_0^T \int_\Omega \rho_n^{\varepsilon,\delta}(t, s_j^\delta)\xi_\varepsilon(t)(a(x, Du_\varepsilon) + F(u_\varepsilon)) \cdot DT_k(u_\varepsilon(t,x) - u_j^\delta)\,dxdt. \tag{4.3.11}$$

Furthermore, for $r \in \mathbb{R}$ and $j = 1, \ldots, M(\delta)$, we define

$$\phi_{T_k(\cdot - u_j^\delta)}(r) := \begin{cases} \int_{b_j^\delta}^r T_k((\beta^{-1})^0(\sigma) - u_j^\delta) d\sigma, & \text{if } r \in \overline{R(\beta)} \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $T_k(u_i^\varepsilon - u_j^\delta) \subset \partial\phi_{T_k(\cdot - u_j^\delta)}(b_i^\varepsilon)$ for all $i = 1, \ldots, N(\varepsilon)$ and almost everywhere in Ω, we have

$$I_1^\varepsilon \geq \frac{1}{k} \int_\Omega \sum_{i=1}^{N(\varepsilon)} (\phi_{T_k(\cdot - u_j^\delta)}(b_i^\varepsilon) - \phi_{T_k(\cdot - u_j^\delta)}(b_{i-1}^\varepsilon)) \rho_n(t_i^\varepsilon - s_j^\delta) \xi(t_i^\varepsilon) dx. \quad (4.3.12)$$

Now, observe that

$$\begin{aligned} & \sum_{i=1}^{N(\varepsilon)} (\phi_{T_k(\cdot - u_j^\delta)}(b_i^\varepsilon) - \phi_{T_k(\cdot - u_j^\delta)}(b_{i-1}^\varepsilon)) \rho_n(t_i^\varepsilon - s_j^\delta) \xi(t_i^\varepsilon) \\ = & - \sum_{i=0}^{N(\varepsilon)-1} \phi_{T_k(\cdot - u_j^\delta)}(b_i^\varepsilon) (\rho_n(t_{i+1}^\varepsilon - s_j^\delta) \xi(t_{i+1}^\varepsilon) - \rho_n(t_i^\varepsilon - s_j^\delta) \xi(t_i^\varepsilon)) \quad (4.3.13) \\ = & - \sum_{i=0}^{N(\varepsilon)-1} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} \phi_{T_k(\cdot - u_j^\delta)}(b_i^\varepsilon) \frac{\partial}{\partial t} (\rho_n(t - s_j^\delta) \xi(t)) dt, \end{aligned}$$

hence setting $b_\varepsilon(t) = b_0^\varepsilon$ for all $t \in (-\varepsilon, 0]$ from (4.3.12) and (4.3.13) it follows that

$$I_1^\varepsilon \geq -\frac{1}{k} \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega \phi_{T_k(\cdot - u_j^\delta)}(b_\varepsilon)(\rho_n(t+\varepsilon-s_j^\delta)\xi'(t+\varepsilon) + \rho_n'(t+\varepsilon-s_j^\delta)\xi(t+\varepsilon)) dx dt. \quad (4.3.14)$$

If we define the piecewise constant functions $u_\delta : (0,T) \to W_0^{1,p(\cdot)}(\Omega)$, $\rho_n^\delta, \rho_{n,\delta}' : (0,T)^2 \to \mathbb{R}$ by $u_\delta(s) = u_j^\delta$, $\rho_n^\delta(t,s) = \rho_n(t - s_j^\delta)$, $\rho_{n,\delta}'(t,s) = \rho_n'(t - s_j^\delta)$ for $t \in [0,T]$ and $s \in (s_{j-1}^\delta, s_j^\delta]$, $j = 1, \ldots, M(\delta)$, integrating (4.3.8) over $(s_{j-1}^\delta, s_j^\delta)$ and summing over $j = 1, \ldots, M(\delta)$ from (4.3.9) - (4.3.14) it follows that

$$\begin{aligned} & -\frac{1}{k} \int_0^T \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega \phi_{T_k(\cdot - u_\delta(s))}(b_\varepsilon)(\rho_n^\delta(t+\varepsilon, s)\xi'(t+\varepsilon) + \rho_{n,\delta}'(t+\varepsilon, s)\xi(t+\varepsilon)) dx dt ds \\ & + \frac{1}{k} \int_{[0,T]^2} \int_\Omega \rho_n^{\varepsilon,\delta}(t,s) \xi_\varepsilon(t) (a(x, Du_\varepsilon) + F(u_\varepsilon)) \cdot DT_k(u_\varepsilon(t,x) - u_\delta(s,x)) dx dt ds \\ & + \frac{1}{k} \int_{[0,T]^2} \int_\Omega \psi(u_\varepsilon) T_k(u_\varepsilon(t,x) - u_\delta(s,x)) \rho_n^{\varepsilon,\delta}(t,s) \xi_\varepsilon(t) dx dt ds \\ \leq & \frac{1}{k} \int_{[0,T]^2} \int_\Omega f_\varepsilon(t) T_k(u_\varepsilon(t,x) - u_\delta(s,x)) \rho_n^{\varepsilon,\delta}(t,s) \xi_\varepsilon(t) dx dt ds. \end{aligned} \quad (4.3.15)$$

Now, we define the piecewise constant function $f_\delta : (0,T) \to L^\infty(\Omega)$ by $f_\delta(s) = f_j^\delta$ for $s \in (s_{j-1}^\delta, s_j^\delta]$, and $j = 1,\dots,M(\delta)$, integrate (4.3.7) over $(s_{j-1}^\delta, s_j^\delta)$ and sum over $j = 1,\dots,M(\delta)$ to obtain

$$I_1^\delta + I_2^\delta + I_3^\delta = I_4^\delta, \tag{4.3.16}$$

where

$$I_4^\delta = \frac{1}{k}\int_0^T \int_\Omega f_\delta T_k(u_\delta(s,x) - u_i^\varepsilon)\rho_n^{\varepsilon,\delta}(t_i^\varepsilon, s)\xi(t_i^\varepsilon)dxds, \tag{4.3.17}$$

$$I_3^\delta = \frac{1}{k}\int_0^T \int_\Omega \psi(u_\delta) T_k(u_\delta(s,x) - u_i^\varepsilon)\rho_n^{\varepsilon,\delta}(t_i^\varepsilon, s)\xi(t_i^\varepsilon)dxds, \tag{4.3.18}$$

and

$$I_2^\delta = \frac{1}{k}\int_0^T \int_\Omega \rho_n^{\varepsilon,\delta}(t_i^\varepsilon, s)\xi(t_i^\varepsilon)(a(x, Du_\delta) + F(u_\delta)) \cdot DT_k(u_\delta(s,x) - u_i^\varepsilon)dxds. \tag{4.3.19}$$

Setting $b_\delta(s) = b_0^\delta$ for all $s \in (-\delta, 0]$, with similar arguments as in (4.3.14) it follows that

$$\begin{aligned} I_1^\delta \;&\geq\; \frac{1}{k}\int_{-\delta}^{T-\delta} \int_\Omega \phi_{T_k(\cdot - u_i^\varepsilon)}(b_\delta)\rho_n'(t_i^\varepsilon - (s+\delta))\xi(t_i^\varepsilon)dxds \\ &-\; \frac{1}{k}\int_\Omega \phi_{T_k(\cdot - u_i^\varepsilon)}(b_\delta(0))\rho_n(t_i^\varepsilon)\xi(t_i^\varepsilon)dx, \end{aligned} \tag{4.3.20}$$

where, for $r \in \mathbb{R}$ and $i = 1,\dots,N(\varepsilon)$,

$$\phi_{T_k(\cdot - u_i^\varepsilon)}(r) := \begin{cases} \int_{b_i^\varepsilon}^r T_k((\beta^{-1})^0(\sigma) - u_i^\varepsilon)d\sigma, & \text{if } r \in \overline{R(\beta)} \\ +\infty, & \text{otherwise.} \end{cases}$$

Next, we define the piecewise constant functions $\rho_n^\varepsilon, \rho_{n,\varepsilon}' : (0,T)^2 \to \mathbb{R}$ by $\rho_n^\varepsilon(t,s) := \rho_n(t_i^\varepsilon - s)$ and $\rho_{n,\varepsilon}'(t,s) := \rho_n'(t_i^\varepsilon - s)$ for all $s \in [0,T]$, $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$, $i = 1,\dots,N(\varepsilon)$. Integrating (4.3.16) over $(t_{i-1}^\varepsilon, t_i^\varepsilon)$ and summing over $i =$

$1, \ldots, N(\varepsilon)$ from (4.3.17) - (4.3.20) it follows that

$$
\begin{aligned}
&\frac{1}{k}\int_0^T \int_{-\delta}^{T-\delta} \int_\Omega \phi_{T_k(\cdot - u_\varepsilon)}(b_\delta)\rho'_{n,\varepsilon}(t, s+\delta)\xi_\varepsilon(t)dxdsdt \\
&-\frac{1}{k}\int_{[0,T]\times\Omega} \phi_{T_k(\cdot - u_\varepsilon)}(b_\delta(0))\rho_n^\varepsilon(t,0)\xi_\varepsilon(t)dxdt \\
&+\frac{1}{k}\int_{[0,T]^2}\int_\Omega \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)(a(x,Du_\delta)+F(u_\delta))\cdot DT_k(u_\delta(s,x)-u_\varepsilon(t,x))dxdtds \\
&+\frac{1}{k}\int_{[0,T]^2}\int_\Omega \psi(u_\delta)T_k(u_\delta(s,x)-u_\varepsilon(t,x))\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds \\
&\leq \frac{1}{k}\int_{[0,T]^2}\int_\Omega f_\delta T_k(u_\delta(s,x)-u_\varepsilon(t,x))\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds.
\end{aligned}
\tag{4.3.21}
$$

Taking the sum of (4.3.15) and (4.3.21) we find

$$
I_1^{k,\varepsilon,\delta,n} + I_2^{k,\varepsilon,\delta,n} + I_3^{k,\varepsilon,\delta,n} + I_4^{k,\varepsilon,\delta,n} + I_5^{k,\varepsilon,\delta,n} + I_6^{k,\varepsilon,\delta,n} \leq I_7^{k,\varepsilon,\delta,n} \tag{4.3.22}
$$

where

$$
\begin{aligned}
I_1^{k,\varepsilon,\delta,n} =& -\frac{1}{k}\int_0^T \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega \phi_{T_k(\cdot - u_\delta)}(b_\varepsilon) \\
&\cdot(\rho_n^\delta(t+\varepsilon,s)\xi'(t+\varepsilon)+\rho'_{n,\delta}(t+\varepsilon,s)\xi(t+\varepsilon))dxdtds, \qquad (4.3.23)\\
I_2^{k,\varepsilon,\delta,n} =& \frac{1}{k}\int_0^T \int_{-\delta}^{T-\delta} \int_\Omega \phi_{T_k(\cdot - u_\varepsilon)}(b_\delta)\rho'_{n,\varepsilon}(t,s+\delta)\xi_\varepsilon(t)dxdsdt,
\end{aligned}
$$

$$
\begin{aligned}
I_3^{k,\varepsilon,\delta,n} =& -\frac{1}{k}\int_0^T \int_\Omega \phi_{T_k(\cdot - u_\varepsilon)}(b_\delta(0))\rho_n^\varepsilon(t,0)\xi_\varepsilon(t)dxdt, \\
I_4^{k,\varepsilon,\delta,n} =& \frac{1}{k}\int_{[0,T]^2}\int_\Omega \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)(a(x,Du_\varepsilon)-a(x,Du_\delta)) \\
&\cdot DT_k(u_\varepsilon - u_\delta)dxdtds,
\end{aligned}
\tag{4.3.24}
$$

$$
\begin{aligned}
I_5^{k,\varepsilon,\delta,n} =& \frac{1}{k}\int_{[0,T]^2}\int_\Omega \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)(F(u_\varepsilon)-F(u_\delta))\cdot \\
& DT_k(u_\varepsilon - u_\delta)\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds, \\
I_6^{k,\varepsilon,\delta,n} =& \frac{1}{k}\int_{[0,T]^2}\int_\Omega (\psi(u_\varepsilon)-\psi(u_\delta))T_k(u_\varepsilon - u_\delta)\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds, \\
I_7^{k,\varepsilon,\delta,n} =& \frac{1}{k}\int_{[0,T]^2}\int_\Omega (f_\varepsilon - f_\delta)T_k(u_\varepsilon - u_\delta)\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds.
\end{aligned}
\tag{4.3.25}
$$

Now, we will pass to the limit with $k \downarrow 0$ in (4.3.22). Note that by $(DP_{\varepsilon,\psi})$, $(DP_{\delta,\psi})$ and the continuity of β we have $b_\varepsilon = \beta(u_\varepsilon)$ and $b_\delta = \beta(u_\delta)$ almost everywhere in Q_T. Since $\frac{1}{k}T_k(u_\varepsilon - u_\delta) \to \operatorname{sign}_0(u_\varepsilon - u_\delta)$ almost everywhere in Q_T as $k \downarrow 0$, by Lebesgues Dominated Convergence Theorem it follows that

$$\begin{aligned}
\lim_{k\downarrow 0} I_1^{k,\varepsilon,\delta,n} &= -\int_0^T \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega |b_\delta(s) - b_\varepsilon(t)|(\rho_n^\delta(t+\varepsilon,s)\xi'(t+\varepsilon) \\
&+ \rho'_{n,\delta}(t+\varepsilon,s)\xi(t+\varepsilon))dxdtds, \\
\lim_{k\downarrow 0} I_2^{k,\varepsilon,\delta,n} &= \int_0^T \int_{-\delta}^{T-\delta} \int_\Omega |b_\delta(s) - b_\varepsilon(t)|\rho'_{n,\varepsilon}(t,s+\delta)\xi_\varepsilon(t)dxdsdt,
\end{aligned} \tag{4.3.26}$$

$$\begin{aligned}
\lim_{k\downarrow 0} I_3^{k,\varepsilon,\delta,n} &= -\int_0^T \int_\Omega |b_\delta(0) - b_\varepsilon(t)|\rho_n^\varepsilon(t,0)\xi_\varepsilon(t)dxdt, \\
\lim_{k\downarrow 0} I_6^{k,\varepsilon,\delta,n} &= \int_{[0,T]^2} \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)|\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds, \\
\lim_{k\downarrow 0} I_7^{k,\varepsilon,\delta,n} &= \int_{[0,T]^2} \int_\Omega (f_\varepsilon - f_\delta)\operatorname{sign}_0(u_\varepsilon - u_\delta)\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds.
\end{aligned} \tag{4.3.27}$$

Since F is locally Lipschitz continuous, it follows that

$$\lim_{k\downarrow 0} I_5^{k,\varepsilon,\delta,n} = 0 \tag{4.3.28}$$

and, by $(A3)$ we have

$$\limsup_{k\downarrow 0} I_4^{k,\varepsilon,\delta,n} \geq 0, \tag{4.3.29}$$

hence, using (4.3.26), (4.3.28) and (4.3.29) from (4.3.22) we get

$$\begin{aligned}
&-\int_0^T \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega |b_\delta(s) - b_\varepsilon(t)| \cdot \\
&\quad (\rho_n^\delta(t+\varepsilon,s)\xi'(t+\varepsilon) + \rho'_{n,\delta}(t+\varepsilon,s)\xi(t+\varepsilon))dxdtds \\
&+\int_0^T \int_{-\delta}^{T-\delta} \int_\Omega |b_\delta(s) - b_\varepsilon(t)|\rho'_{n,\varepsilon}(t,s+\delta)\xi_\varepsilon(t)dxdtds
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \int_\Omega |b_\delta(0) - b_\varepsilon(t)| \rho_n^\varepsilon(t,0)\xi_\varepsilon(t)dxdt \\
&+\int_{[0,T]^2} \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)| \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds \qquad (4.3.30)\\
&\leq \int_{[0,T]^2} \int_\Omega |f_\varepsilon - f_\delta| \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds.
\end{aligned}
$$

By assumption, $b_\delta, b_\varepsilon \to b$ as $\varepsilon, \delta \downarrow 0$ in $L^\infty(0,T;L^1(\Omega))$ and $b \in \mathcal{C}([0,T];L^1(\Omega))$. Moreover, as $\varepsilon, \delta \downarrow 0$, we have $\xi_\varepsilon(t) \to \xi(t)$, $\rho_n^\varepsilon(t,0) \to \rho_n(t)$ for all $t \in (0,T)$, $\rho_n^{\varepsilon,\delta}(t,s)$, $\rho_n^\varepsilon(t,s)$ and $\rho_n^\delta(t,s) \to \rho_n(t-s)$, $\rho'_{n,\varepsilon}(t,s)$ and $\rho'_{n,\delta}(t,s) \to \rho'_n(t-s)$ for all $(t,s) \in (0,T)^2$ and all these function sequences are uniformly bounded in (ε,δ). Therefore, we can pass to the limit with $\varepsilon, \delta \downarrow 0$ in (4.3.30) and obtain

$$I_1^n + I_2^n + I_3^n \leq 0, \qquad (4.3.31)$$

where

$$
\begin{aligned}
I_1^n &= -\int_{[0,T]^2} \int_\Omega |b(t) - b(s)| \rho_n(t-s)\xi'(t)dxdtds,\\
I_2^n &= \int_0^T \int_\Omega |b(0) - b(t)| \rho_n(t)\xi(t)dxdt,\\
I_3^n &= \limsup_{\varepsilon,\delta\downarrow 0} \int_{[0,T]^2} \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)| \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds. \qquad (4.3.32)
\end{aligned}
$$

Finally, $n \to \infty$ in (4.3.31) yields

$$\limsup_{n\to\infty} \limsup_{\varepsilon,\delta\downarrow 0} \int_{[0,T]^2} \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)| \rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds \leq 0. \qquad (4.3.33)$$

Now we are in the position to conclude the proof: Choosing an arbitrary (not relabeled) subsequence of

$$\left(\int_0^T \int_\Omega \psi(u_\varepsilon(t)) - \psi(u_\delta(t))dxdt\right)_{\varepsilon,\delta},$$

by Lemma 4.3.2, (4.3.2) there exists $\alpha \in L^\infty(Q_T)$, such that, extracting another (not relabeled) subsequence if necessary,

$$(|\psi(u_\varepsilon) - \psi(u_\delta)|)_{\varepsilon,\delta} \rightharpoonup \alpha \qquad (4.3.34)$$

weak-$*$ in $L^\infty(Q_T)$ as $\varepsilon, \delta \downarrow 0$. Since $\int_0^T \rho_n^{\varepsilon,\delta}(\cdot,s)ds\, \xi_\varepsilon(\cdot)$ converges to $\int_0^T \rho_n(\cdot -$

$s)ds\ \xi(\cdot)$ as $\varepsilon, \delta \downarrow 0$ in $L^1(0,T)$, it follows from (4.3.34) that

$$
\begin{aligned}
&\lim_{n\to\infty} \lim_{\varepsilon,\delta\downarrow 0} \int_0^T \int_\Omega \int_0^T \rho_n^{\varepsilon,\delta}(t,s)ds\ \xi_\varepsilon(t)|\psi(u_\varepsilon(t)) - \psi(u_\delta(t))|dxdt \\
&= \lim_{n\to\infty} \int_0^T \int_\Omega \int_0^T \rho_n(t-s)ds\ \xi(t)\alpha(t,x)dxdt \\
&= \int_0^T \int_\Omega \xi(t)\alpha(t,x)dxdt \\
&= \lim_{\varepsilon,\delta\downarrow 0} \int_0^T \int_\Omega \xi(t)|\psi(u_\varepsilon(t)) - \psi(u_\delta(t))|dxdt.
\end{aligned}
\tag{4.3.35}
$$

For $0 < \theta < \tau < T$, we choose $\xi \in \mathcal{D}(0,T)$ such that $0 \le \xi \le 1$ in $[0,T]$ and $\xi = 1$ in $[\theta, \tau]$. Now, using (4.3.2), (4.3.35) and (4.3.33) we get

$$
\begin{aligned}
&\lim_{\varepsilon,\delta\downarrow 0} \int_\theta^\tau \int_\Omega |\psi(u_\varepsilon(t)) - \psi(u_\delta(t))|dxdt \\
&\le \lim_{\varepsilon,\delta\downarrow 0} \int_0^T \int_\Omega |\psi(u_\varepsilon(t)) - \psi(u_\delta(t))|\xi(t)dxdt \\
&= \lim_{n\to\infty} \lim_{\varepsilon,\delta\downarrow 0} \int_0^T \int_\Omega \int_0^T \rho_n^{\varepsilon,\delta}(t,s)ds\ \xi_\varepsilon(t)|\psi(u_\varepsilon(t)) - \psi(u_\delta(t))|dxdt \\
&\le \limsup_{n\to\infty} \limsup_{\varepsilon,\delta\downarrow 0} \int_0^T \int_0^T \int_\Omega |\psi(u_\varepsilon(t)) - \psi(u_\delta(s))|\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t)dxdtds \\
&+ \limsup_{n\to\infty} \limsup_{\varepsilon,\delta\downarrow 0} \int_0^T \int_0^T \int_\Omega |\psi(u_\delta(t)) - \psi(u_\delta(s))|\rho_n^{\delta,\delta}(t,s)\xi_\delta(t)dxdtds \\
&+ \limsup_{n\to\infty} \limsup_{\varepsilon,\delta\downarrow 0} \int_0^T \int_0^T \int_\Omega |\psi(u_\delta(t)) - \psi(u_\delta(s))| \\
&\cdot (\rho_n^{\varepsilon,\delta}(t,s)\xi_\varepsilon(t) - \rho_n^{\delta,\delta}(t,s)\xi_\delta(t))dxdtds \\
&\le 0,
\end{aligned}
\tag{4.3.36}
$$

hence

$$
\lim_{\varepsilon,\delta\downarrow 0} \int_0^T \int_\Omega |\psi(u_\varepsilon(t)) - \psi(u_\delta(t))|dxdt = 0. \tag{4.3.37}
$$

Since we started with an arbitrary subsequence, (4.3.37) holds for the whole sequence

$$
\left(\int_0^T \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)|dxdt \right)_{\varepsilon,\delta}
$$

and the proof of the lemma is completed. $\square$

Remark 4.3.4. In the general case when β may be set-valued, we could follow the idea of [22] to choose $T_k(u_i^\varepsilon - u_j^\delta + k\pi)$ as a test function in (4.3.6) and $T_k(u_j^\delta - u_i^\varepsilon - k\pi)$ as a test function in (4.3.7), where $\pi \in \mathcal{D}(\Omega)$. Then,

$$\begin{aligned}\lim_{k\downarrow 0} I_1^{k,\varepsilon,\delta,n} = &-\int_0^T \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega \int_{b_\delta}^{b_\varepsilon} \operatorname{sign}_0((\beta^{-1})^0(\sigma) - u_\delta) + \chi_{\{(\beta^{-1})^0(\sigma)=u_\delta\}}\pi d\sigma \\ &\cdot ((\rho_n^\delta(t+\varepsilon,s)\xi'(t+\varepsilon) + \rho'_{n,\delta}(t+\varepsilon,s)\xi(t+\varepsilon))dxdtds\end{aligned} \tag{4.3.38}$$

and

$$\begin{aligned}\lim_{k\downarrow 0} I_2^{k,\varepsilon,\delta,n} = &-\int_0^T \int_{-\delta}^{T-\delta} \int_\Omega \int_{b_\varepsilon}^{b_\delta} \operatorname{sign}_0((\beta^{-1})^0(\sigma) - u_\varepsilon) - \chi_{\{(\beta^{-1})^0(\sigma)=u_\varepsilon\}}\pi d\sigma \\ &\cdot \rho'_{n,\varepsilon}(t,s+\delta)\xi_\varepsilon(t)dxdsdt.\end{aligned} \tag{4.3.39}$$

To continue in the proof of Lemma 4.3.3, it would be necessary to choose $\pi = \pi_l$ such that $\pi_l \to \operatorname{sign}_0(u_\varepsilon(t) - u_\delta(s))$ as $l \to \infty$ in $L^1([0,T]^2 \times \Omega)$. Unfortunately, it is an open problem how to perform the former calculation of the proof if π is assumed to be time dependent.

β continuous, non-decreasing

The a priori estimates of Lemma 4.3.2 and Lemma 4.3.3 imply the following convergence results for the approximate solutions of $(DP_{\varepsilon,\psi})$ for $\varepsilon \downarrow 0$:

Lemma 4.3.5. *Let $\varepsilon > 0$ take values in a sequence in $(0,1)$ tending to 0. For $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ let u_ε, b_ε be the piecewise constant functions defined by $(DP_{\varepsilon,\psi})$. Then there exist functions $b \in C([0,T];L^1(\Omega))$, $u \in V \cap L^\infty(Q_T)$ and $\Phi \in (L^{p'(\cdot)}(Q_T))^N$ such that for a not relabeled subsequence of $(u_\varepsilon)_\varepsilon$ we have the following convergence results for $\varepsilon \downarrow 0$:*

i.) $u_\varepsilon \to u$ *almost everywhere in* Q_T, *weak-$*$ in* $L^\infty(Q_T)$ *and weak in* $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$,

ii.) $b_\varepsilon \to b$ *in* $L^\infty(0,T;L^1(\Omega))$ *and* $b = \beta(u)$ *almost everywhere in* Q_T.

iii.) $Du_\varepsilon \rightharpoonup Du$ *in* $(L^{p(\cdot)}(Q_T))^N$,

iv.) $a(x,Du_\varepsilon) \rightharpoonup \Phi$ *in* $(L^{p'(\cdot)}(Q_T))^N$.

Proof: If $(\varepsilon_n)_n \subset (0,1)$ is a sequence tending to 0 as $n \to \infty$, applying Lemma 4.3.3 with $u_\varepsilon = u_{\varepsilon_n}$, $u_\delta = u_{\varepsilon_m}$ for $m, n \in \mathbb{N}$ yields that (passing to a subsequence if necessary)

$$|\psi(u_{\varepsilon_n}) - \psi(u_{\varepsilon_m})| \to 0$$

almost everywhere in Q_T as $m, n \to \infty$. Hence, $(u_{\varepsilon_n})_n$ is a Cauchy sequence almost everywhere in Q_T and there exists a measurable function $u : Q_T \to \mathbb{R}$ such that $u_{\varepsilon_n} \to u$ almost everywhere in Q_T as $n \to \infty$. By (4.3.3) and (4.3.4) it follows that $u \in V \cap L^\infty(Q_T)$ and *i.*), *iii.*) hold. By definition of the operator $A_{\beta,\psi}$, assuming β to be a continuous, non-decreasing function it follows that $b_\varepsilon(t) = \beta(u_\varepsilon(t))$ a.e. in $(0,T)$ for all $\varepsilon > 0$. Keeping in mind that by nonlinear semigroup theory, $(b_\varepsilon)_\varepsilon$ converges to the mild solution $b \in C([0,T]; L^1(\Omega))$ of $(ACP)(f, \psi, b_0)$ as $\varepsilon \downarrow 0$ and using *i.*) and the continuity of β, *ii.*) holds. Applying (4.3.4) and $(A2)$ (and passing to a subsequence if necessary), we find that $a(x, Du_\varepsilon) \rightharpoonup \Phi$ in $(L^{p'(\cdot)}(Q_T))^N$ for some $\Phi \in (L^{p'(\cdot)}(Q_T))^N$. □

Using the convergence results of the preceeding Lemma, we have the following existence result:

Proposition 4.3.6. *If $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function, then there exists a weak solution (in the sense of Definition 4.2.2) $(u, b = \beta(u))$ to (P, ψ, f, b_0) for any $f \in L^\infty(Q_T)$ and any $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$.*

Proof: Let $\widetilde{b}_\varepsilon : [0,T] \to L^1(\Omega)$ be the piecewise linear function defined by $\widetilde{b}_\varepsilon(t) = b^\varepsilon_{i-1} + \frac{t - t^\varepsilon_{i-1}}{t^\varepsilon_i - t^\varepsilon_{i-1}}(b^\varepsilon_i - b^\varepsilon_{i-1})$ for $t \in [t^\varepsilon_{i-1}, t^\varepsilon_i]$, $i = 1, \ldots, N(\varepsilon)$. For arbitrary $\xi \in \mathcal{D}([0,T) \times \Omega)$ and $t \in [0,T)$ the function $\Omega \ni x \to \xi(t,x)$ is in $\mathcal{D}(\Omega)$, hence we can use it as a test function each equation of $(DP_{\varepsilon,\psi})$. Integrating over $(t^\varepsilon_{i-1}, t^\varepsilon_i)$ and summing over $i = 1, \ldots, N(\varepsilon)$ we find

$$\begin{aligned} &-\int_0^T \int_\Omega \widetilde{b}_\varepsilon \xi_t + (a(x, Du_\varepsilon) + F(u_\varepsilon)) \cdot D\xi + \psi(u_\varepsilon)\xi - \int_\Omega \widetilde{b}_\varepsilon(0)\xi(0,\cdot) \\ &= \int_0^T \int_\Omega f_\varepsilon \xi. \end{aligned} \tag{4.3.40}$$

Since $\widetilde{b}_\varepsilon \to b$ in $C([0,T]; L^1(\Omega))$ as $\varepsilon \downarrow 0$, using the convergence results of Lemma 4.3.5 we can pass to the limit in (4.3.40) to obtain

$$-\int_0^T \int_\Omega (b - b_0)\xi_t + (\Phi + F(u)) \cdot D\xi + \psi(u)\xi = \int_0^T \int_\Omega f\xi. \tag{4.3.41}$$

for all $\xi \in \mathcal{D}([0,T)\times\Omega)$, where $b = \beta(u)$. It is left to prove that $\Phi = a(x, Du)$. To this end let κ be a non-negative function in $\mathcal{C}_c^\infty([0,T))$. We discretise κ with respect to $(DP_{\varepsilon,\psi})$ by setting $\kappa_\varepsilon(0) = \kappa(0)$ and $\kappa_\varepsilon(t) = \kappa(t_i^\varepsilon)$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $i = 1, \dots, N(\varepsilon)$. Taking $\kappa(t_i^\varepsilon)u_i^\varepsilon$ as a test function in $(DP_{\varepsilon,\psi})$ yields:

$$\kappa(t_i^\varepsilon)\int_\Omega \frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{\varepsilon}u_i^\varepsilon + (a(x, Du_i^\varepsilon) + F(u_i^\varepsilon)) \cdot Du_i^\varepsilon + \psi(u_i^\varepsilon)u_i^\varepsilon = \int_\Omega f_i^\varepsilon \kappa(t_i^\varepsilon)u_i^\varepsilon \tag{4.3.42}$$

for all $i = 1, \dots, N(\varepsilon)$. If we define $\phi_{id} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi_{id}(r) = \begin{cases} \int_0^r (\beta^{-1})^0(\sigma)d\sigma, & \text{if } r \in \overline{R(\beta)}, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.3.43}$$

since $b_i^\varepsilon = \beta(u_i^\varepsilon)$ for all $i = 1, \dots, N(\varepsilon)$ it follows that

$$\frac{b_i^\varepsilon - b_{i-1}^\varepsilon}{\varepsilon}u_i^\varepsilon \geq \frac{1}{\varepsilon}\int_{b_{i-1}^\varepsilon}^{b_i^\varepsilon} (\beta^{-1})^0(\sigma)d\sigma = \frac{1}{\varepsilon}(\phi_{id}(b_i^\varepsilon) - \phi_{id}(b_{i-1}^\varepsilon)) \tag{4.3.44}$$

a.e. in Ω. Now, integration over $(t_{i-1}^\varepsilon, t_i^\varepsilon)$ in (4.3.42) and summation over $i = 1, \dots, N(\varepsilon)$ yields:

$$\begin{aligned} &\sum_{i=1}^{N(\varepsilon)} \int_\Omega (\phi_{id}(b_i^\varepsilon) - \phi_{id}(b_{i-1}^\varepsilon))\kappa(t_i^\varepsilon) \\ &+ \int_{Q_T} \kappa_\varepsilon((a(x, Du_\varepsilon) + F(u_\varepsilon)) \cdot Du_\varepsilon + \psi(u_\varepsilon)u_\varepsilon) \leq \int_{Q_T} f_\varepsilon u_\varepsilon \kappa_\varepsilon, \end{aligned} \tag{4.3.45}$$

where $u_\varepsilon : (0,T) \to W_0^{1,p(\cdot)}(\Omega)$ is the piecewise constant function defined by $u_\varepsilon(t) = u_i^\varepsilon$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$, $i = 1, \dots, N(\varepsilon)$ and $f_\varepsilon : (0,T) \to L^\infty(\Omega)$ is the piecewise constant function defined by $f_\varepsilon(t) = f_i^\varepsilon$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$, $i = 1, \dots, N(\varepsilon)$. Using summation by parts in the first term of (4.3.45) and setting $b_\varepsilon(t) = b_i^\varepsilon$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$, $i = 1, \dots, N(\varepsilon)$, $b_\varepsilon(t) = b_0^\varepsilon$ for $t \in (-\varepsilon, 0]$ it follows that

$$\begin{aligned} &\int_{Q_T} \kappa_\varepsilon a(x, Du_\varepsilon) \cdot Du_\varepsilon \\ &\leq \int_{-\varepsilon}^{T-\varepsilon} \int_\Omega \kappa_t(t+\varepsilon)\phi_{id}(b_\varepsilon) + \int_\Omega \phi_{id}(b_0^\varepsilon)\kappa_\varepsilon(0) \\ &- \int_{Q_T} \kappa_\varepsilon(F(u_\varepsilon) \cdot Du_\varepsilon + (\psi(u_\varepsilon) - f_\varepsilon)u_\varepsilon). \end{aligned} \tag{4.3.46}$$

Using the convergence results of Lemma 4.3.5, there is no problem to pass to the limit with $\varepsilon \downarrow 0$ on the right-hand side of (4.3.46). Moreover,

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa_\varepsilon a(x,Du_\varepsilon)\cdot Du_\varepsilon\\ &\geq \limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa a(x,Du_\varepsilon)\cdot Du_\varepsilon+\liminf_{\varepsilon\downarrow 0}\int_{Q_T}(\kappa_\varepsilon-\kappa)a(x,Du_\varepsilon)\cdot Du_\varepsilon\end{aligned}\tag{4.3.47}$$

where the second term on the right hand side of (4.3.47) is 0 by $(A2)$, (4.3.4) and since $\|\kappa_\varepsilon-\kappa\|_{L^\infty(0,T)}\to 0$ as $\varepsilon\downarrow 0$. Combining (4.3.46) with (4.3.47) and passing to the limit with $\varepsilon\downarrow 0$ we find

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa a(x,Du_\varepsilon)\cdot Du_\varepsilon\\ &\leq \int_{Q_T}\kappa_t(t)\phi_{id}(b)+\int_\Omega \phi_{id}(b_0)\kappa(0)-\int_{Q_T}\kappa(F(u)\cdot Du+\psi(u)u-fu).\end{aligned}\tag{4.3.48}$$

Since (4.3.41) holds, we can apply the integration-by-parts formula of Lemma 4.2.26 with $h(u)=u$ and $\xi=\kappa\chi_\Omega$ to obtain

$$\begin{aligned}&\int_{Q_T}\kappa_t\int_{b_0}^{b(t,x)}(\beta^{-1})^0(\sigma)d\sigma\\ &=\int_{Q_T}\kappa(F(u)\cdot Du+\Phi\cdot Du+\psi(u)u-fu)\end{aligned}\tag{4.3.49}$$

for all $\kappa\in\mathcal{C}_c^\infty([0,T))$, $\kappa\geq 0$. Combining (4.3.48) and (4.3.49) we finally get

$$\limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa a(x,Du_\varepsilon)\cdot Du_\varepsilon\leq\int_{Q_T}\kappa\Phi\cdot Du.\tag{4.3.50}$$

Therefore it follows that

$$\limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa(a(x,Du_\varepsilon)-a(x,Du))\cdot(Du_\varepsilon-Du)\leq 0\tag{4.3.51}$$

for all $\kappa\in\mathcal{C}_c^\infty([0,T))$, $\kappa\geq 0$. Using $(A3)$ and Minty's monotonicity argument we get $\Phi=a(x,Du)$ from (4.3.50) and (4.3.51). Moreover, choosing $\kappa=\chi_{(0,\tau)}$ for $0<\tau<T$, from (4.3.51) we obtain $a(x,Du_\varepsilon)\cdot Du_\varepsilon\rightharpoonup a(x,Du)\cdot Du$ weak in $L^1((0,\tau)\times\Omega)$. $\square$

The general case of multivalued β

Now, let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph. To continue the proof of Theorem 4.3.1, we proceed as in [68] (see also [69]) in the case of a constant exponent and combine the techniques developed in [22] with the approach from [2] so that we do not need the additional assumption that β^{-1} is continuous and defined on $\mathbb{R}$ if we accept one more approximation procedure. Once we have found the appriopriate energy space V, there will be no great difficulties arising from the variable exponent. However, the following technical assumption on β will be needed later: To prove a comparison principle that will be used instead of Lemma 4.3.3, we assume from now on that β is such that for any $n \in \mathbb{N}$, the number of points in $[-n, n] \cap D(\beta)$ where the image of β in an interval is $M_n \in \mathbb{N}$. Since β is maximal monotone in $\mathbb{R} \times \mathbb{R}$, it follows immediately that the set of points in $D(\beta)$ where β is set-valued must by at most countable. Due to our technical assumption we exclude the case that this set has an accumulation point in a finite interval of $D(\beta)$.

For the first approximation procedure let us regularize β by $\beta_k := \beta + \frac{1}{k} I$, $k > 0$. Clearly the results of Subsection 4.1.1 still apply to the nonlinear operator

$$\begin{aligned} A_{\beta_k,\psi} := \big\{ &(b_k, w_k) \in L^1(\Omega) \times L^1(\Omega) : \exists u_k : \Omega \to \mathbb{R} \text{ measurable}, \\ &b_k \in \beta_k(u_k) \text{ a.e. in } \Omega \text{ and } u_k \text{ is a renormalized solution of} \\ &-\operatorname{div}(a(x, Du_k) + F(u_k)) + \psi(u_k) = w_k \big\} \end{aligned}$$

and therefore there exists a unique mild solution $b^k \in \mathcal{C}([0,T]; L^1(\Omega))$ of the abstract Cauchy problem

$$(ACP)_k(\psi, f, b_0^k) \begin{cases} \dfrac{db^k}{dt} + A_{\beta_k,\psi} b^k \ni f \text{ in } (0,T), \\ b^k(0) = b_0^k \end{cases}$$

corresponding to (P_k, ψ, f, b_0^k) for any given $f \in L^1(Q_T)$, $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ and $k > 0$. Moreover, it follows from the results for the elliptic case (see Theorem 3.1.2) that for any $f \in L^1(\Omega)$

$$\lim_{k \to \infty} \|(I + A_{\beta_k,\psi})^{-1} f - (I + A_{\beta,\psi})^{-1} f\|_{L^1(\Omega)} = 0. \tag{4.3.52}$$

Applying the a priori estimates of Lemma 4.3.2 we get the following convergence results for the solutions of the discretized problems $(DP_{\varepsilon,\psi}^k)$:

Lemma 4.3.7. *For f in $L^\infty(Q_T)$, $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ and $\varepsilon, k > 0$ let $(b_i^{\varepsilon,k}, u_i^{\varepsilon,k})_{i=1}^{N(\varepsilon)}$ be a solution of the discretized problem $(DP^k_{\varepsilon,\psi})$. For $k > 0$, let $b^k \in \mathcal{C}([0,T]; L^1(\Omega))$ be the $L^\infty(0,T;L^1(\Omega))$-limit of the sequence of piecewise constant functions $(b_\varepsilon^k)_\varepsilon$ defined by $b_\varepsilon^k : (0,T) \to L^1(\Omega))$, $b_\varepsilon^k(0) = b_0^{\varepsilon,k}$, $b_\varepsilon^k(t) = b_i^{\varepsilon,k}$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $i = 1, \dots, N(\varepsilon)$. If we define u_ε^k : $(0,T) \to W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ by $u_\varepsilon^k(t) = u_i^{\varepsilon,k}$ for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$, $i = 1, \dots, N(\varepsilon)$, there exists $u^k \in V \cap L^\infty(Q_T)$ and a subsequence of $(u_\varepsilon^k)_\varepsilon$ such that, as $\varepsilon \downarrow 0$,*

i.) $u_\varepsilon^k \to u^k$ *almost everywhere in Q_T, weak-$*$ in $L^\infty(Q_T)$ and weak in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$,*

ii.) $b_\varepsilon^k \to b^k$ *in $L^\infty(0,T;L^1(\Omega))$, in $L^1(Q_T)$ and almost everywhere in Q_T. Moreover, $b^k \in \beta_k(u^k)$ almost everywhere in Q_T,*

iii.) $Du_\varepsilon^k \rightharpoonup Du^k$ *in $(L^{p(\cdot)}(Q_T))^N$,*

iv.) $a(x, Du_\varepsilon^k) \rightharpoonup a(x, Du^k)$ *in $(L^{p'(\cdot)}(Q_T))^N$.*

Proof: Using the a priori estimates (4.3.2), (4.3.3) and (4.3.4), it follows immediately that there exists $u^k \in V \cap L^\infty(Q_T)$ such that, passing to a subsequence if necessary, *iii.)* holds and $u_\varepsilon^k \rightharpoonup u^k$ weak-$*$ in $L^\infty(Q_T)$ and weak in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$. The convergence of b_ε^k to b^k in $L^\infty(0,T;L^1(\Omega))$ follows immediately from nonlinear semigroup theory and this implies the other convergence results for a subsequence of $(b_\varepsilon^k)_\varepsilon$. By $(DP^k_{\varepsilon,\psi})$, we have $b_\varepsilon^k \in \beta_k(u_\varepsilon^k)$ almost everywhere in Q_T. Since β is a maximal monotone graph, $(\beta + \frac{1}{k}I)^{-1} = k(k\beta + I)^{-1}$ is single-valued and Lipschitz continuous in $\mathbb{R}$, hence $u_\varepsilon^k := (\beta + \frac{1}{k}I)^{-1} b_\varepsilon^k$ converges to $u^k = (\beta + \frac{1}{k}I)^{-1} b^k$ almost everywhere in Q_T. Therefore *i.)* and *ii.)* hold. Finally, *iv.)* follows with the same arguments as in the proof of Lemma 4.3.5 and Proposition 4.3.6. □

Using the convergence results of Lemma 4.3.7 we can prove the following result:

Proposition 4.3.8. *For any $k > 0$, $f \in L^\infty(Q_T)$ and $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ there exists a weak solution (u^k, b^k) to (P_k, ψ, f, b_0^k). In particular, b^k is the mild solution of $(ACP)_k(\psi, f, b_0^k)$.*

Proof: The assertion follows according to the convergence results of Lemma 4.3.7 and by similar arguments as in the proof of Proposition 4.3.6. □

Next we want to obtain a weak solution (u, b) of (P, ψ, f, b_0) for $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ by passing to the limit with $k \to \infty$

in the approximate equations (P_k, ψ, f, b_0). The convergence of the sequence $(b^k)_k$ is an immediate consequence of nonlinear semigroup theory:

Lemma 4.3.9. *If b_0 is in $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ such that there exists $(b_0^k)_k \subset L^1(\Omega)$ with $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ for all $k > 0$ and $b_0^k \to b_0$ in $L^1(\Omega)$ as $k \to \infty$, then b^k converges in $\mathcal{C}([0,T]; L^1(\Omega))$ to the mild solution b of $(ACP)(\psi, f, b_0)$ as $k \to \infty$.*

Proof: From (4.3.52) it follows that $A_{\beta,\psi} \subset \liminf_{k\to\infty} A_{\beta_k,\psi}$ and therefore the assertion follows according to nonlinear semigroup theory (see, e.g. [17]). $\square$

The main difficulty of this step is to show almost everywhere convergence of a subsequence of $(u^k)_k$. As we have already mentioned in Remark 4.3.4, it is not possible to genereralize the result of Lemma 4.3.3 directly. The following comparison principle is a corresponding result for multivalued β and was proved in [22] and [68] in the constant exponent case:

Lemma 4.3.10. *For $f \in L^1(Q_T)$, $l, k > 0$, b_0^k, $b_0^l \in L^1(\Omega)$ such that*

$$\lim_{k,l\to\infty} \|b_0^k - b_0^l\|_{L^1(\Omega)} = 0$$

let (u^k, b^k), (u^l, b^l) be the weak solution of (P_k, ψ, f, b_0^k), (P_l, ψ, f, b_0^l) respectively. Then,

$$\lim_{k,l\to\infty} \int_\tau^\theta \int_\Omega |\psi(u_k) - \psi(u_l)| = 0 \tag{4.3.53}$$

holds for all $0 < \tau < \theta < T$.

Proof: The proof of this lemma follows the same lines as the proof of the corresponding result in the case of a constant exponent p as stated in [68], Proposition 4.2.2., p. 104-114 and p.116 (see also [22], Proposition 4.2, p. 402-416) and is omitted here in detail. It is based on Kruzhkov's doubling of variable technique (see e.g. [53]) that has been adapted by other authors (see [46], [47], [61], [31], [32], [2], [22], [68]) to prove uniqueness results and comparison principles for elliptic-parabolic problems. In our particular case, we only have to double the time variables: Let t, s denote two variables in $[0, T]$. We write the t variable in the weak formulation of (P_k, ψ, f, b_0^k) and the s variable in the weak formulation of (P_l, ψ, f, b_0^l). For $\delta > 0$, and $r \in \mathbb{R}$ we define the function $r \to \eta_\delta(r)$ by $\eta_\delta(r) := \frac{1}{\delta}T_\delta(r)$. According to the integration-by-parts formula of Lemma 4.2.11 we choose $\eta_\delta(u_k(t,x) - u_l(s,x) + \delta\pi(x))\phi(t)\rho_n(t-s)$ as a test function in (P_k, ψ, f, b_0^k) and (P_l, ψ, f, b_0^l)

where $\pi \in \mathcal{D}(\Omega)$ such that $0 \leq \pi \leq 1$, $\phi \in \mathcal{D}([0,T))$ such that $\phi \geq 0$ and $(\rho_n)_n$ is a sequence of mollifiers in $\mathbb{R}$. There is no problem to pass to the limit with $\delta \downarrow 0$ in the diffusion and the convection term because F is assumed to be locally Lipschitz continuous. In order to pass to the limit in the parabolic term, we have to distinguish between the points where β is single-valued and where it is set-valued and want the sums over the number of points in $[-n,n] \cap D(\beta)$ where the image of β is an interval to be finite for fixed $n \in \mathbb{N}$. Here we need the technical assuption on β that these numbers have to be $M_n \in \mathbb{N} < +\infty$ for $[-n,n] \cap D(\beta)$ and $n \in \mathbb{N}$. Further we proceed with similar arguments as in Lemma 4.3.3. □

Using this result, we can prove the following

Proposition 4.3.11. *For any $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ there exists $u \in V \cap L^\infty(Q_T)$ and $b \in \mathcal{C}([0,T];L^1(\Omega))$ such that (u,b) is a weak solution to (P,ψ,f,b_0). In particular, b is the mild solution of $(ACP)(\psi,f,b_0)$.*

Proof: According to Proposition 4.1.1, we have

$$\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \subset \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$$

for all $k > 0$. Therefore, if b_0 is in $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ it is in $\overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ for all $k > 0$ and by Proposition 4.3.8, (P_k,ψ,b_0,f) has a weak solution $(u^k,b^k) \subset (V \cap L^\infty(Q_T)) \times \mathcal{C}([0,T];L^1(\Omega))$ for all $k > 0$. In particular,

$$\int_{Q_T} -(b^k - b_0)\xi_t + (a(x,Du^k) + F(u^k)) \cdot D\xi + \psi(u^k)\xi = \int_{Q_T} f\xi \quad (4.3.54)$$

holds for any $\xi \in \mathcal{D}([0,T) \times \Omega)$. By Lemma 4.3.9, $b^k \to b$ as $k \to \infty$ in $\mathcal{C}([0,T];L^1(\Omega))$, where $b \in \mathcal{C}([0,T];L^1(\Omega))$ is the mild solution of $(ACP)(\psi,f,b_0)$ and therefore $b(0) = b_0$ almost everywhere in Ω. From Lemma 4.3.10 it follows with the same arguments as in the proof of Lemma 4.3.5 that there exists a measurable function $u : Q_T \to \mathbb{R}$ and a (not relabeled) subsequence of $(u^k)_k$ such that $u^k \to u$ almost everywhere as $k \to \infty$. Since the a priori estimates of Lemma 4.3.2 still hold for u^k, independently of $k > 0$, it follows that, as $k \to \infty$ and up to a non-relabeled subsequence, u^k converges to u weak-$*$ in $L^\infty(Q_T)$ and weak in $L^{p-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, $Du^k \rightharpoonup Du$ in $(L^{p(\cdot)}(Q_T))^N$, hence u is in $V \cap L^\infty(Q_T)$. Moreover, there exists $\Phi \in (L^{p'(\cdot)}(Q_T))^N$ such that $a(x,Du^k) \rightharpoonup \Phi$ weak in $(L^{p'(\cdot)}(Q_T))^N$ as

$k \to \infty$. Using these convergence results we can pass to the limit in (4.3.54) and find that

$$-\int_{Q_T}(b-b_0)\xi_t + \int_{Q_T}(\Phi + F(u))\cdot D\xi + \int_{Q_T}\psi(u)\xi = \int_{Q_T} f\xi \tag{4.3.55}$$

holds for all $\xi \in \mathcal{D}([0,T)\times\Omega)$. Next, we prove $a(x,Du) = \Phi$. To this end, we fix $\sigma \in \mathcal{D}([0,T))$, $\sigma \geq 0$ and $l > 0$. Since (4.3.54) holds, by Lemma 4.2.11 we can use $\sigma T_l(u^k)$ as a test function and obtain

$$\begin{aligned}
&-\int_{Q_T}\sigma_t \int_{b_0}^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds + \int_{Q_T}\sigma a(x,Du^k)\cdot DT_l(u^k) \\
&= -\int_{Q_T}\sigma(F(u^k)\cdot DT_l(u^k) + (\psi(u^k)-f)T_l(u^k)).
\end{aligned} \tag{4.3.56}$$

There is no problem to pass to the limit with $k \to \infty$ on the right-hand side of (4.3.56). To pass to the limit in the first term on the left-hand side, we write

$$\int_{b_0}^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds = I_1 + I_2 \tag{4.3.57}$$

where, since $b^k \in (\beta + \frac{1}{k}I)u^k$ almost everywhere in Q_T,

$$I_1 = \int_0^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds = b^k T_l(u^k) - \int_0^{T_l(u^k)}(\beta^0 + \frac{1}{k}I)(s)ds \tag{4.3.58}$$

(see [1], [6], [61]) almost everywhere in Q_T and

$$I_2 = -\int_0^{b_0} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds. \tag{4.3.59}$$

Now, setting $u_0 := (\beta^{-1})^0(b_0)$ we have $(b_0 + \frac{1}{k}u_0) \in (\beta + \frac{1}{k}I)u_0$, hence

$$I_2 = -b_0T_l(u_0) + \int_0^{T_l(u_0)}(\beta^0 + \frac{1}{k}I)(s)ds - \int_{b_0}^{b_0+\frac{1}{k}u_0} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds \tag{4.3.60}$$

almost everywhere in Ω. Passing to the limit with $k \to \infty$ we find:

$$\begin{aligned}
\lim_{k\to\infty} I_1 &= -bT_l(u) + \int_0^{T_l(u)}(\beta^0)(s)ds \\
&= -\int_0^b T_l \circ (\beta^{-1})^0(s)ds
\end{aligned} \tag{4.3.61}$$

almost everywhere in Q_T and

$$\begin{aligned}\lim_{k\to\infty} I_2 &= -b_0 T_l(u_0) + \int_0^{T_l(u_0)} (\beta^0)(s)ds \\ &= -\int_0^{b_0} T_l \circ (\beta^{-1})^0(s)ds\end{aligned} \tag{4.3.62}$$

almost everywhere in Ω. Now, thanks to Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{k\to\infty} -\int_{Q_T} \sigma_t \int_{b_0^k}^{b^k} T_l \circ (\beta + \frac{1}{k} I)^{-1}(s)ds = -\int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ (\beta^{-1})^0(s)ds \tag{4.3.63}$$

and therefore

$$\begin{aligned}&\limsup_{k\to\infty} \int_{Q_T} \sigma a(x, Du^k) \cdot DT_l(u^k) \\ &= \int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ (\beta^{-1})^0(s)ds - \int_{Q_T} \sigma(F(u) \cdot DT_l(u) + (\psi(u) - f)T_l(u)).\end{aligned} \tag{4.3.64}$$

Now we use $\sigma T_l(u)$ as a test function in (4.3.55). By Lemma 4.2.11 we get

$$\begin{aligned}&\int_{Q_T} \sigma \Phi \cdot DT_l(u) \\ &= \int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ (\beta^{-1})^0(s)ds - \int_{Q_T} \sigma(F(u) \cdot DT_l(u) + (\psi(u) - f)T_l(u)).\end{aligned} \tag{4.3.65}$$

Subtracting (4.3.65) from (4.3.64) and choosing $l = \|u\|_{L^\infty(Q_T)}$ it follows that

$$\limsup_{k\to\infty} \int_{Q_T} \sigma a(x, Du^k) \cdot Du^k \le \int_{Q_T} \sigma \Phi \cdot Du \tag{4.3.66}$$

for all $\sigma \in \mathcal{D}([0,T))$, $\sigma \ge 0$. Furthermore, using (4.3.66) we have

$$\lim_{k\to\infty} \int_{Q_T} \sigma(a(x, Du^k) - a(x, Du)) \cdot (Du^k - Du) = 0 \tag{4.3.67}$$

for all $\sigma \in \mathcal{D}([0,T))$, $\sigma \ge 0$. Now, $\Phi = a(x, Du)$ follows from (4.3.67) by the Minty monotonicity argument. It is left to prove that $b \in \beta(u)$ almost everywhere in Q_T. Since we have $b^k \in \beta_k(u^k)$ almost everywhere in Q_T, for any $k>0$ there exists $B^k \in \beta(u^k)$ such that $b^k = B^k + \frac{1}{k}u^k$ and since $B^k \to b$ for $k \to \infty$ almost everywhere in Q_T. If we define $j : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by $j(r) = \int_0^r \beta^0(\sigma)d\sigma$ if $r \in \overline{D(\beta)}$ and $j(r) = +\infty$ otherwise, it is easy to see that j is a convex, l.s.c, proper function such that $\beta = \partial j$. Therefore,

$$j(r) \geq j(u^k) + B^k(r - u^k) \tag{4.3.68}$$

holds for any $r \in \mathbb{R}$ and almost everywhere in Q_T. Now, by the almost everywhere convergence of B^k to b and u^k to u, from (4.3.68) it follows that $b \in \beta(u)$ almost everywhere in Q_T. □

Remark 4.3.12. The weak solution (u, b) to (P, ψ, f, b_0) for $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ obtained in Proposition 4.3.11 is also a renormalized solution. Note that there exists a measurable function $u_0 : \Omega \to \overline{\mathbb{R}}$ such that $b_0 \in \beta(u_0)$ almost everywhere in Ω and we can apply Proposition 4.2.14.

4.3.2 L^1-contraction and uniqueness of renormalized solutions

The following L^1-contraction result still holds true in the case of a variable exponent:

Proposition 4.3.13. *For $f_1, f_2 \in L^1(Q_T)$, $b_0^1, b_0^2 \in L^1(\Omega)$ let (u_1, b_1), (u_2, b_2) be renormalized solutions of (P, f_1, b_0^1), (P, f_2, b_0^2) respectively. Then*

$$\int_\Omega (b_1(t) - b_2(t))^+ \leq \int_0^t \int_\Omega (f_1 - f_2)^+ + \int_\Omega (b_0^1 - b_0^2)^+ \tag{4.3.69}$$

holds for almost all $t \in (0, T)$.

Proof: We can copy the proof in [22], Theorem 4.1, p. 401-416 for the case of a constant exponent with slight modifications such as exchanging the space $L^p(0, T; W_0^{1,p(\cdot)}(\Omega))$ by V. □

Remark 4.3.14. The result of Proposition 4.3.13 still holds if we replace (P, f_i, b_0^i) by (P, ψ_i, f_i, b_0^i), where $\psi_i : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function for $i = 1, 2$.

Remark 4.3.15. Uniqueness of renormalized solutions is a direct consequence of Proposition 4.3.13: If (u, b) is a renormalized solution to (P, f, b_0) for $f \in L^1(\Omega)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$, then b is unique. We cannot expect uniqueness of the function u without additional assumptions on β.

4.3.3 A comparison principle and weak solutions for L^∞-data

The proof following Lemma is a straightforward generalisation of the case of a constant exponent and will be not given in detail:

Lemma 4.3.16. *Let $b_0, \tilde{b}_0$ be in $L^1(\Omega)$, $f, \tilde{f} \in L^1(Q_T)$, $\psi, \tilde{\psi} : \mathbb{R} \to \mathbb{R}$ be strictly increasing, continuous functions and (u, b), $(\tilde{u}, \tilde{b})$ be weak solutions to (P, ψ, f, b_0) and $(P, \tilde{\psi}, \tilde{f}, \tilde{b}_0)$ respectively. If we have $b_0 \leq \tilde{b}_0$ almost everywhere in Ω, $f \leq \tilde{f}$ almost everywhere in Q_T and $\tilde{\psi}(r) \leq \psi(r)$ for all $r \in \mathbb{R}$, then*

$$u \leq \tilde{u}$$

holds almost everywhere in Q_T.

Proof: As in the proof of the corresponding result in the case of a constant exponent ([68], Lemme 4.3.1., p. 120 and [22], Proposition 4.2., p. 402-416) we use the doubling of (time) variables: Let t, s denote two variables in $[0, T]$. As in the proof of Lemma 4.3.10 and thanks to the integration-by-parts formula of Lemma 4.2.11 we choose $H_\delta^+(u(t, x) - \tilde{u}(s, x) + \delta\zeta(x))\phi(t)\rho_n(t-s)$ as a test function, where η_δ^+ is an approximation of the sign_0^+ function, $\zeta \in \mathcal{D}(\Omega)$ such that $0 \leq \zeta \leq 1$, $\phi \in \mathcal{D}([0, T))$ such that $\phi \geq 0$ and $(\rho_n)_n$ is a sequence of mollifiers. Proceeding with similar arguments as in Lemma 4.3.10 we show that

$$\limsup_{n\to\infty} \int_0^T \int_{Q_T} (\psi(u) - \tilde{\psi}(\tilde{u}))^+ \rho_n(t-s)\,dxdtds \leq 0. \tag{4.3.70}$$

The conclusion of the proof is similar to the end of the proof of Lemma 4.3.3. □

Thanks to Lemma 4.3.16 there is existence of a weak solution to (P, f, b_0) for L^∞-data:

Proposition 4.3.17. *For $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ there exists a weak solution (u, b) to (P, f, b_0). In particular, b is the mild solution of $(ACP)(f, b_0)$.*

Proof: For $n \in \mathbb{N}$, we define the continuous, strictly increasing function $\psi_n : \mathbb{R} \to \mathbb{R}$ by

$$\psi_n(r) := \frac{1}{n}(\arctan(r) + \frac{\pi}{2}),\ r \in \mathbb{R}.$$

Then, by Proposition 4.3.11, there exists a weak solution $(u_n, b_n) \in (V \cap L^\infty(Q_T)) \times \mathcal{C}([0, T]; L^1(\Omega))$ to (P, ψ_n, f, b_0) for any $n \in \mathbb{N}$. Since $A_\beta \subset$

$\liminf_{n\to\infty} A_{\beta,\psi_n}$ and b_n is the mild solution of $(ACP)(\psi_n, f, b_0)$, it follows that b_n converges in $\mathcal{C}([0,T];L^1(\Omega))$ to the mild solution b of $(ACP)(f,b_0)$ as $n\to\infty$. Since $\psi_n \geq \psi_{n+1}$ in $\mathbb{R}$ for all $n\in\mathbb{N}$, from Lemma 4.3.16 it follows that $u_n \leq u_{n+1}$ almost everywhere in Q_T for all $n\in\mathbb{N}$, hence there exists a measurable function $u: Q_T \to \overline{\mathbb{R}}$ such that $u_n \uparrow u$ almost everywhere in Q_T. Moreover, $\arctan(r) - \pi/2 \leq \psi_n \leq \arctan(r) + \pi/2$ for all $r\in\mathbb{R}$ and all $n\in\mathbb{N}$ and from Lemma 4.3.16 it follows that $u_{\pi/2} \leq u_n \leq u_{-\pi/2}$ almost everywhere in Q_T for all $n\in\mathbb{N}$ where $(u_{\pi/2}, b_{\pi/2}), (u_{-\pi/2}, b_{-\pi/2}) \in (V\cap L^\infty(Q_T))\times\mathcal{C}([0,T];L^1(\Omega))$ are the weak solutions to $(P, \arctan+\pi/2, f, b_0)$ and $(P, \arctan-\pi/2, f, b_0)$ respectively. Therefore $u\in L^\infty(Q_T)$ and $u_n \to u$ weak-$*$ in $L^\infty(Q_T)$ for a not relabeled subsequence of $(u_n)_n$. For $\delta > 0$ we define $\phi_\delta : [0,T]\to\mathbb{R}$ by $\phi_\delta(t) := \min(\frac{1}{\delta}\max(T-\delta-t,0),1)$. Thanks to the integration-by-parts formula of Lemma 4.2.11, can use $\phi_\delta T_k(u_n)$ as a test function and obtain for $k = \|u\|_{L^\infty(Q_T)}$:

$$\begin{aligned}
&\frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega\int_0^{b_n(t,x)} T_{\|u\|_{L^\infty(Q_T)}}\circ(\beta^{-1})^0(\sigma)d\sigma\\
&-\int_\Omega\int_0^{b_0} T_{\|u\|_{L^\infty(Q_T)}}\circ(\beta^{-1})^0(\sigma)d\sigma\\
&+\int_{Q_T}\phi_\delta(((a(x,Du_n)+F(u_n))\cdot Du_n)+\frac{1}{n}\arctan(u_n)u_n)\\
&=\int_{Q_T}\phi_\delta u_n(f-\frac{\pi}{2}).
\end{aligned}\tag{4.3.71}$$

We neglect positive terms and use $(A1)$, pass to the limit with $\delta\downarrow 0$ and obtain

$$\int_{Q_T}|Du_n|^{p(x)} \leq C\|u\|_{L^\infty(Q_T)}\left(\left\||f|+\frac{\pi}{2}\right\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)}\right), \tag{4.3.72}$$

where $C>0$ is a positive constant not depending on $n\in\mathbb{N}$. From (4.3.72) we get $u\in V$ and there exists a (not relabeled) subsequence of $(u_n)_n$, such that $Du_n \rightharpoonup Du$ weak in $(L^{p(\cdot)}(Q_T))^N$ and $a(x,Du_n)\rightharpoonup\Phi$ weak in $(L^{p'(\cdot)}(Q_T))^N$ for a function $\Phi\in(L^{p'(\cdot)}(Q_T))^N$. Now we can pass to the limit with $n\to\infty$ in the weak formulation for (P,f,ψ_n,b_0) and obtain

$$-\int_{Q_T}(b-b_0)\xi_t + (\Phi+F(u))\cdot D\xi = \int_{Q_T} f\xi \tag{4.3.73}$$

for all $\xi\in\mathcal{D}([0,T)\times\Omega)$. With the same arguments as in the proof of Proposition 4.3.11 it follows that $\Phi = a(x,Du)$ (by Minty monotonicity argument) and $b\in\beta(u)$ (by a subdifferential argument).

4.3.4 Proof of Theorem 4.3.1

To conclude the proof of Theorem 4.3.1, we use the ideas developed in [2] and [68] for the case of a constant exponent. As in the proof of Theorem 3.1.2 for the elliptic case, we will construct monotone sequences of weak solutions for L^∞-data and show convergence (up to a subsequence) to a renormalized solution. The comparison principles from Lemma 4.3.16 and Lemma 4.3.13 will be a main tool in this approximation procedure.

Approximate solutions and a priori estimates

Let f be in $L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. For $m, n \in \mathbb{N}$, let $f_{m,n} := \max(\min(f, m)), -n)$, $b_0^{m,n} := \max(\min(b_0, m)), -n)$. Furthermore, we define $\psi_{m,n} : \mathbb{R} \to \mathbb{R}$ by $\psi_{m,n}(r) := \frac{1}{m}\max(r, 0) - \frac{1}{n}\max(-r, 0)$ for $r \in \mathbb{R}$. By Proposition 4.3.11, $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$ has a weak solution $(u_{m,n}, b_{m,n})$ for all $m, n \in \mathbb{N}$. We have $\psi_{m,n} \geq \psi_{m+1,n}$ for all $n \in \mathbb{N}$ and $\psi_{m,n} \leq \psi_{m,n+1}$ on $\mathbb{R}$. By Lemma 4.3.16 it follows that $u_{m,n} \leq u_{m+1,n}$ and $u_{m,n+1} \leq u_{m,n}$ almost everywhere in Q_T for any $m, n \in \mathbb{N}$. Hence, there exist measurable functions $u^n : Q_T \to \mathbb{R} \cup \{+\infty\}$, $u : Q_T \to \overline{\mathbb{R}}$ such that $u_{m,n} \uparrow u^n$ as $m \to \infty$ and $u^n \downarrow u$ as $n \to \infty$ almost everywhere in Q_T. By Lemma 4.3.13, it follows that $b_{m,n} \leq b_{m+1,n}$ and $b_{m,n+1} \leq b_{m,n}$ almost everywhere in Q_T for any $m, n \in \mathbb{N}$. Note that we have also $\psi_{m,n}(r) \downarrow \psi^n(r) := -\frac{1}{n}\max(-r, 0)$ as $m \to \infty$ and $\psi^n(r) \uparrow 0$ as $n \to \infty$ for all $r \in \mathbb{R}$. Therefore, $A_{\psi_n,\beta} \subset \liminf_{m\to 0} A_{\psi_{m,n},\beta}$ and $A_\beta \subset \liminf_{n\to 0} A_{\psi_n,\beta}$. By nonlinear semigroup theory it follows that $b_{m,n} \uparrow b^n$ as $m \to \infty$ in $\mathcal{C}([0, T]; L^1(\Omega))$ where b^n is the mild solution of $(ACP)(\psi^n, f^n, b_0^n)$ with $\psi^n(r) := -\frac{1}{n}\max(-r, 0)$, $f^n := \max(f, -n)$, and $b_0^n := \max(b_0, -n)$ for $n \in \mathbb{N}$. Moreover, $b^n \downarrow b$ as $n \to \infty$ in $\mathcal{C}([0, T]; L^1(\Omega))$ where b is the mild solution of $(ACP)(f, b_0)$. In the next steps, we will prove that (u, b) is a renormalized solution to (P, f, b_0). Therefore we need the following a priori estimate:

Lemma 4.3.18. *For $m, n \in \mathbb{N}$, $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ let $(u_{m,n}, b_{m,n})$ be the weak solution to $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$. Then there exists a constant $C > 0$ not depending on $m, n \in \mathbb{N}$ such that*

$$\int_0^T \int_\Omega |DT_k(u_{m,n})|^{p(x)} dxdt \leq Ck(\|f\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)}) \tag{4.3.74}$$

holds for any $k > 0$ and all $m, n \in \mathbb{N}$.

Proof: We fix $k > 0$. For $\delta > 0$ we define $\phi_\delta : [0, T] \to \mathbb{R}$ by $\phi_\delta(t) := \min(\frac{1}{\delta}\max(T - \delta - t, 0), 1)$. Using the integration-by-parts formula of Lemma

4.2.11 and density arguments, we plug $\phi_\delta T_k(u_{m,n})$ as a test function into $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$. Then, for $\delta > 0$ small enough, we find

$$I_1 + I_2 + I_3 + I_4 = I_5, \tag{4.3.75}$$

where

$$\begin{aligned} I_1 &= -\int_{T-2\delta}^{T-\delta}\int_\Omega (\phi_\delta)_t \int_{b_0^{m,n}}^{b_{m,n}(t,x)} T_k \circ (\beta^{-1})^0(\sigma)d\sigma \\ &= \frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega \int_0^{b_{m,n}(t,x)} T_k \circ (\beta^{-1})^0(\sigma)d\sigma \\ &- \int_\Omega \int_0^{b_0^{m,n}} T_k \circ (\beta^{-1})^0(\sigma)d\sigma. \end{aligned} \tag{4.3.76}$$

By $(A1)$ we get

$$\begin{aligned} I_2 &= \int_{Q_T} \phi_\delta a(x, DT_k(u_{m,n})) \cdot DT_k(u_{m,n}) \\ &\geq \gamma \int_0^{T-2\delta}\int_\Omega |DT_k(u_{m,n})|^{p(x)} dxdt. \end{aligned} \tag{4.3.77}$$

Note that applying Gauss-Green Theorem and the boundary condition

$$\int_\Omega F(T_k(u_{m,n}(t))) \cdot DT_k(u_{m,n}(t)) = 0$$

for almost all $t \in (0, T)$. Hence,

$$I_3 = \int_0^T \phi_\delta \int_\Omega F(T_k(u_{m,n})) \cdot DT_k(u_{m,n}) = 0. \tag{4.3.78}$$

Moreover, by the monotonicity of $\psi_{m,n}$ we have

$$\begin{aligned} I_4 &= \int_{Q_T} \psi_{m,n}(u_{m,n})T_k(u_{m,n})\phi_\delta \geq 0, \\ I_5 &= \int_{Q_T} f_{m,n}T_k(u_{m,n})\phi_\delta \leq \|f\|_{L^1(Q_T)}k. \end{aligned} \tag{4.3.79}$$

Now, plugging (4.3.76) - (4.3.79) into (4.3.75) and neglecting non-negative terms we arrive at

$$\begin{aligned} \gamma\int_0^{T-2\delta}\int_\Omega |DT_k(u_{m,n})|^{p(x)} &\leq k\|f\|_{L^1(Q_T)} + \int_\Omega \int_0^{b_0^{m,n}} T_k \circ (\beta^{-1})^0(\sigma)d\sigma \\ &\leq (\|f\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)})k. \end{aligned} \tag{4.3.80}$$

for all $k > 0$ and all $m, n \in \mathbb{N}$. For $\delta \downarrow 0$, the assertion follows. □

Remark 4.3.19. Using Lemma 4.3.18, we show with analogous arguments as in the elliptic case (see Remark 3.2.5, Lemma 3.4.11 and the proof of Theorem 3.1.2) that u^n and u are finite almost everywhere in Q_T (see also [6], p.2777, Step 4 of the proof of Theorem 2.6 for the parabolic case with constant exponent), more precisely there exists a constant $C > 0$, not depending on $n, l \in \mathbb{R}$ such that

$$|\{|u_{m,n}| \geq l\}| \leq Cl^{-(p^- -1)}, \tag{4.3.81}$$

and from (4.3.81) it follows that

$$\lim_{l\to\infty} |\{|u| > l\}| = 0. \tag{4.3.82}$$

Since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Q_T, it follows with subdifferential arguments as in the proof of Proposition 4.3.11 that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ almost everywhere in Q_T.

Convergence results

Now, applying the diagonal principle and Lemma 4.3.18 we get the following convergence results:

Lemma 4.3.20. *For $m, n \in \mathbb{N}$, $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ let $(u_{m,n}, b_{m,n})$ be the weak solution to $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$. Then, there exists a subsequence $(m(n))_n$ such that setting $\psi^n := \psi_{m(n),n}$, $f_n := f_{m(n),n}$, $b_{0,n} := b_0^{m(n),n}$, $b_n := b_{m(n),n}$, $u_n := u_{m(n),n}$ we have the following convergence results for $n \to \infty$:*

i.) $f_n \to f$ *in* $L^1(Q_T)$,

ii.) $u_n \to u$ *almost everywhere in* Q_T,

iii.) $b_{0,n} \to b_0$ *in* $L^1(\Omega)$, $b_n \to b$ *in* $\mathcal{C}([0,T]; L^1(\Omega))$ *and* $b \in \beta(u)$ *almost everywhere in* Q_T

and the uniform renormalized condition

$$\lim_{l\to\infty} \sup_n \int_{\{l<|u_n|<l+1\}} a(x, Du_n) \cdot Du_n = 0 \tag{4.3.83}$$

holds true. Moreover, for any $k > 0$, we have

iv.) $T_k(u_n) \rightharpoonup T_k(u)$ *in* $L^{p^-}(0, T, W_0^{1,p(\cdot)}(\Omega))$,

v.) $DT_k(u_n) \rightharpoonup DT_k(u)$ *in* $(L^{p(\cdot)}(Q_T))^N$, *vi.)* $a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u))$ *in* $(L^{p'(\cdot)}(Q_T))^N$,

vii.) $a(x, DT_k(u_n)) \cdot DT_k(u_n) \rightharpoonup a(x, DT_k(u)) \cdot DT_k(u)$ *weak in* $L^1((0,\tau)\times\Omega)$ *for any* $0 < \tau < T$.

Proof: *i.)* - *v.)* are direct consequences of the approximation procedure, Lemma 4.3.18 and Remark 4.3.19. To prove the uniform renormalized condition, we take $T_k(u_n)\phi_\delta$, $\phi_\delta(t) := \min(\frac{1}{\delta}\max(T-\delta-t, 0), 1)$ as a test function and apply Lemma 4.2.11 in the weak formulation for (P, ψ^n, f_n, b_0^n). By Gauss-Green Theorem for Sobolev functions and the boundary condition, we have

$$\int_\Omega F(u_n(t)) \cdot DT_k(u_n(t))dx = 0$$

almost everywhere in $(0, T)$, hence the „convection“ term vanishes. Then we set $k = l + 1$ and after $k = l$. Subtracting the corresponding equalities and neglecting positive terms we obtain

$$\int_{\{l<|u_n|<l+1\}} \phi_\delta a(x, Du_n) \cdot Du_n \leq \int_{\{|u_n|>l\}} |f| + \int_\Omega \int_0^{|b_0|} G_l((\beta^{-1})^0)(s)ds \tag{4.3.84}$$

where $G_l = T_{l+1} - T_l$ and the uniform renormalized condition follows applying (4.3.81) in (4.3.84). It is left to show that *vi.)* and *vii.)* hold. From Lemma 4.3.18 and $(A2)$ it follows that for any $k > 0$ there exists $\Phi_k \in (L^{p'(\cdot)}(Q_T))^N$ such that $a(x, DT_k(u_n)) \rightharpoonup \Phi_k$ weak in $(L^{p'(\cdot)}(Q_T))^N$ as $n \to \infty$. To prove that $\Phi_k = a(x, DT_k(u))$, we proceed as in [68], p. 122-127 for the case of a constant exponent (see also [13] for variable exponent and [22], Theorem 3.6., p. 394-398, [2], [6], [21] for constant exponent) and define a special time regularisation of $T_k(u)$ by the regularisation method of Landes ([54]). For $\mu > 0$, we denote this time regularized function by $(T_k(u))_\mu : Q_T \to \mathbb{R}$ defined by

$$(T_k(u))_\mu(t, x) := \mu \int_{-\infty}^t e^{\mu(s-t)} T_k(u(s, x))ds \tag{4.3.85}$$

where, for $s \leq 0$ and $\mu > 0$ we extend $u(s, x)$ by a function $\omega_0^\mu \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ such that $(\omega_0^\mu)_\mu$ is a sequence of functions with $\|\omega_0^\mu\|_{L^\infty(\Omega)} \leq k$ for all $\mu > 0$, $\frac{1}{\mu}\|\omega_0^\mu\|_{W_0^{1,p(\cdot)}(\Omega)} \to 0$ as $\mu \to \infty$, $\omega_0^\mu \to T_k(u_0)$ almost everywhere in Ω as $\mu \to \infty$ and $u_0 : \Omega \to \overline{\mathbb{R}}$ is a measurable function such that $b_0 \in \beta(u_0)$ almost everywhere in Ω. We can easily check that $(T_k(u))_\mu \in V \cap L^\infty(Q_T)$ is the unique solution of the equation

$$\partial_t(T_k(u))_\mu + \mu((T_k(u)) - T_k(u)_\mu) = 0$$

in $\mathcal{D}'(Q_T)$ satisfying the initial condition $(T_k(u))_\mu(0,x) = \omega_0^\mu$ almost everywhere in Ω. In particular, $(T_k(u))_\mu$ is differentiable for almost every $t \in (0,T)$ with $\partial_t(T_k(u))_\mu = \mu(T_k(u) - (T_k(u))_\mu) \in V \cap L^\infty(Q_T)$. Moreover, we have $D(T_k(u))_\mu = (DT_k(u))_\mu$ in $\mathcal{D}'([0,T)\times\Omega)$, $\|(T_k(u))_\mu\|_{L^\infty(Q_T)} \leq k$ for all $\mu > 0$, $(T_k(u))_\mu \to T_k(u)$ almost everywhere in Q_T, weak-$*$ in $L^\infty(Q_T)$ and strongly in V as $\mu \to \infty$. Now, for $\kappa \in \mathcal{D}^+([0,T))$ we show that

$$\limsup_{\mu\to\infty} \limsup_{n\to\infty} \int_{Q_T} \kappa a(x, DT_k(u_n)) \cdot D(T_k(u_n) - (T_k(u))_\mu) \leq 0. \qquad (4.3.86)$$

Therefore, we choose $\kappa h_l(u_n)(T_k(u_n) - (T_k(u))_\mu$ as a test function in (P, ψ^n, f_n, b_0^n) and obtain

$$I_1 + I_2 + I_3 + I_4 + I_5 = I_6, \qquad (4.3.87)$$

where

$$I_1 = \langle (b_n - b_0^n)_t, \kappa h_l(u_n)(T_k(u_n) - (T_k(u))_\mu) \rangle \qquad (4.3.88)$$

and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V' + L^1(Q_T)$ and $V \cap L^\infty(Q_T)$,

$$\begin{aligned} I_2 &= \int_{Q_T} \kappa h_l(u_n) a(x, Du_n) \cdot D(T_k(u_n) - (T_k(u))_\mu), \\ I_3 &= \int_{Q_T} \kappa h_l'(u_n)(T_k(u_n) - (T_k(u))_\mu) a(x, Du_n) \cdot Du_n, \\ I_4 &= \int_{Q_T} \kappa F(u_n) \cdot D(h_l(u_n)(T_k(u_n) - (T_k(u))_\mu)), \end{aligned}$$

$$\begin{aligned} I_5 &= \int_{Q_T} \kappa h_l(u_n) \psi^n(u_n)(T_k(u_n) - (T_k(u))_\mu), \\ I_6 &= \int_{Q_T} \kappa f_n h_l(u_n)(T_k(u_n) - (T_k(u))_\mu). \end{aligned}$$

Now, we want to pass to the limit with $n \to \infty$ and then with $\mu \to \infty$. To handle I_2, we choose $l > k$ and apply the uniform renormalized condition. It is easy to see that I_6, I_5 and I_4 tend to 0 as $n \to \infty$, $\mu \to \infty$. Using the uniform renormalized condition (4.3.83), it follows that $I_3 \leq \omega(l,k)$ and $\omega(l,k) \to 0$ as $l \to \infty$. To handle the parabolic term I_1, we need the following

Lemma 4.3.21.

$$\liminf_{n\to\infty} \liminf_{\mu\to\infty} \langle (b_n - b_0^n)_t, \kappa h_l(u_n)(T_k(u_n) - (T_k(u))_\mu) \rangle \geq 0 \qquad (4.3.89)$$

Proof: The proof is similar to the proof of the corresponding result in the case of a constant exponent (see [13] for the case of a variable exponent and [68], [22], [2], [6], [21], [11] for the case of a constant exponent). □

If (4.3.86) holds, then, by $(A3)$ it follows that

$$\limsup_{n\to\infty}\int_{Q_T}\kappa a(x, DT_k(u_n))\cdot D(T_k(u_n)-(T_k(u)))\leq 0 \tag{4.3.90}$$

and $vi.$) follows from (4.3.90) by the standard Minty-Browder argument. $vii.$) follows from (4.3.90) by choosing κ to be a smooth approximation of $\chi_{(0,\tau)}$ for $0<\tau<T$. □

Conclusion of the proof of Theorem 4.3.1

Now, we are able to conclude the proof of Theorem 4.3.1: By Remark 4.3.19 and Lemma 4.3.20 it follows immediately that $(P1)$, $(P2)$ and $(P3)$ hold for all $k>0$. For $h\in\mathcal{C}_c^1(\mathbb{R})$ and $\xi\in\mathcal{D}([0,T)\times\Omega)$ we can plug $h(u_n)\xi$ into (P,ψ^n,f_n,b_0^n) by the integration-by-parts formula of Lemma 4.2.11 and obtain

$$I_1+I_2+I_3+I_4=I_5, \tag{4.3.91}$$

where

$$\begin{aligned}
I_1 &= \int_{Q_T}\xi_t\int_{b_0^n}^{b_n}h\circ(\beta^{-1})^0(s)ds,\\
I_2 &= \int_{Q_T}a(x,Du_n)\cdot D(h(u_n)\xi),\\
I_3 &= \int_{Q_T}F(u_n)\cdot D(h(u_n)\xi),
\end{aligned}$$

$$\begin{aligned}
I_4 &= \int_{Q_T}\psi^n(u_n)h(u_n)\xi\\
I_5 &= \int_{Q_T}f_nh(u_n)\xi.
\end{aligned}$$

Thanks to the convergence results of Lemma 4.3.20, we can pass to the limit with $n\to\infty$ in $I_1,\ldots,I_5$: It follows immediately that

$$\lim_{n\to\infty}I_1=\int_{Q_T}\xi_t\int_{b_0}^{b}h\circ(\beta^{-1})^0(s)ds. \tag{4.3.92}$$

Now we choose $M > 0$ such that $\operatorname{supp} h \subset [-M, M]$. Next, we write

$$I_2 = I_{2,1} + I_{2,2}, \tag{4.3.93}$$

where

$$I_{2,1} = \int_{(0,\tau)\times\Omega} h'(T_M(u_n))\xi a(x, DT_M(u_n)) \cdot DT_M(u_n),$$

for $0 < \tau < T$ is such that $\operatorname{supp} \xi \subset [0, \tau) \times \Omega$. By Lemma 4.3.20, *vii.*), $a(x, DT_M(u_n)) \cdot DT_M(u_n) \rightharpoonup a(x, DT_M(u)) \cdot DT_M(u)$ weak in $L^1([0, \tau) \times \Omega)$. Since $h'(u_n)\xi \to h(u)\xi$ almost everywhere in Q_T and $\|h(u_n)\xi\|_{L^\infty((0,\tau)\times\Omega} \leq \|h\|_{L^\infty(Q_T)}\|\xi\|_{L^\infty(Q_T)}$, we may pass to the limit in $I_{2,1}$ and obtain

$$\lim_{n\to\infty} I_{2,1} = \int_{Q_T} h'(u)\xi a(x, Du) \cdot Du. \tag{4.3.94}$$

Where Du denotes the generalized gradient in the sense of Remark 4.2.4. By Lebesgue Dominated Convergence Theorem it follows that $h(T_M(u_n)) \to h(T_M(u))$ as $n \to \infty$ in $L^{p(\cdot)}(Q_T)$. Using Lemma 4.3.20, *vi.*), we can pass to the limit in

$$I_{2,2} = \int_{Q_T} h(T_M(u_n))a(x, DT_M(u_n)) \cdot D\xi, \tag{4.3.95}$$

and find

$$\lim_{n\to\infty} I_{2,2} = \int_{Q_T} h(u)a(x, Du) \cdot D\xi. \tag{4.3.96}$$

Next we write $I_3 = I_{3,1} + I_{3,2}$ where

$$\begin{aligned} I_{3,1} &= \int_{Q_T} h'(T_M(u_n))\xi F(T_M(u_n)) \cdot DT_M(u_n), \\ I_{3,2} &= \int_{Q_T} h(T_M(u_n))F(T_M(u_n)) \cdot D\xi \end{aligned}$$

Since $h'(T_M(u_n))F(T_M(u_n))$ converges to $h'(T_M(u))F(T_M(u))$ in $(L^{p'(\cdot)}(Q_T))^N$ as $n \to \infty$ using Lemma 4.3.18, *v.*) we have

$$\lim_{n\to\infty} I_{3,1} = \int_{Q_T} h'(u)\xi F(u) \cdot Du \tag{4.3.97}$$

and moreover

$$\lim_{n\to\infty} I_{3,2} = \int_{Q_T} h(u)F(u) \cdot D\xi. \tag{4.3.98}$$

Note that

$$|I_4| \leq \frac{M}{n} \|h\|_{L^\infty(Q_T)} \|\xi\|_{L^\infty(Q_T)} \to 0 \tag{4.3.99}$$

for $n \to \infty$. Finally, we have

$$\lim_{n\to\infty} I_5 = \int_{Q_T} f h(u)\xi \tag{4.3.100}$$

and from (4.3.91)-(4.3.100) it follows that (u, b) satisfies the renormalized formulation $(P4)$. $(P5)$ follows from the uniform renormalized condition (4.3.83) and Lemma 4.3.20, *vii.*). □

4.4 Extensions and open problems

4.4.1 Strong L^1-convergence

We can improve the convergence result from Lemma 4.3.20, *vii.*). For $0 < \tau < T$ and for any $k > 0$ from (4.3.90) it follows that

$$\lim_{n\to\infty} \int_0^\tau \int_\Omega (a(x, DT_k(u_n)) - a(x, DT_k(u))) \cdot (DT_k(u_n) - DT_k(u)) = 0 \tag{4.4.1}$$

and since the integrand in (4.4.1) is nonnegative, this implies

$$\lim_{n\to\infty} (a(x, DT_k(u_n)) - a(x, DT_k(u))) \cdot (DT_k(u_n) - DT_k(u)) = 0 \tag{4.4.2}$$

almost everywhere in Q_T. If the function a is strictly monotone (this is true in the case of the $p(x)$-Laplacian), it is well known that $DT_k(u_n) \to DT_k(u)$ almost everywhere in Q_T. For the proof we refer to [13] in the case of the $p(x)$-Laplacian, [27], [4] in the case of a constant exponent and [50] for a proof using Young measures in the case of a constant exponent. If the function a is only monotone and we have the convergence $a(x, DT_k(u_n)) \cdot a(x, DT_k(u_n)) \to a(x, DT_k(u)) \cdot a(x, DT_k(u))$ as $n \to \infty$ almost everywhere in Q_T, from Lemma 4.3.20, *vii.*) and [41], Lemma 8.4, p. 1099 it follows that $a(x, DT_k(u_n)) \cdot a(x, DT_k(u_n)) \to a(x, DT_k(u)) \cdot a(x, DT_k(u))$ strong in $L^1((0, \tau) \times \Omega)$ for $0 < \tau < T$.

4.4.2 A regularity result

We have an additional regularity result for the special case $\beta = I$ if we assume $2 - \frac{1}{N+1} < p^- \leq p^+ < N$:

Lemma 4.4.1. *Let $p \in C(\overline{\Omega})$ with $2 - \frac{1}{N+1} < p^- \le p^+ < N$ and let $\gamma > 0$. Then there exists a constant $c > 0$, depending on γ, such that, for any function $g \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ with*

$$\|g\|_{L^\infty(0,T;L^1(\Omega))} = \operatorname{ess}\sup_{t\in(0,T)} \int_\Omega |g(t,x)|dx \le \gamma, \tag{4.4.3}$$

and

$$\sup_{l\ge 0} \int\int_{B_l} |Dg|^{p(x)}\, dxdt \le \gamma, \tag{4.4.4}$$

where, for $l \ge 0$, $B_l = \{l \le |u| \le l+1\}$, it follows that

$$\|g\|_{L^{q^-}(0,T;W_0^{1,q(\cdot)}(\Omega))} \le c, \tag{4.4.5}$$

for all continuous functions $q(\cdot)$ on Ω satisfying

$$1 \le q(x) < \frac{N(p(x)-1)+p(x)}{N+1} \text{ for all } x \in \Omega. \tag{4.4.6}$$

Proof: See [13] for the case of a variable exponent and also [24] for the case of a constant exponent. □

From Lemma 4.4.1 it follows that if u is a renormalized solution to (P,f,u_0) with a variable exponent such that $2 - \frac{1}{N+1} < p^- \le p^+ < N$ and $\beta = I$, then u is in $L^{q^-}(0,T;W_0^{1,q(\cdot)}(\Omega))$ for all continuous functions $q(\cdot)$ on Ω satisfying $1 < q(x) < \frac{N(p(x)-1)+p(x)}{N+1}$ for all $x \in \Omega$.

4.4.3 Entropy solutions

In the case of a constant exponent, a notion of entropy solution for the classical p-Laplacian problem

$$(Lc,f,u_0)\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) = f & \text{in } Q_T,\\ u = 0 & \text{on } \Sigma_T,\\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega \end{cases}$$

has been introduced in [63]. The next definition will be a straightforward generalisation for the parabolic $p(x)$-Laplacian equation

$$(L,f,u_0)\begin{cases} u_t - \operatorname{div}(|Du|^{p(x)-2}Du) = f & \text{in } Q_T,\\ u = 0 & \text{on } \Sigma_T,\\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases}$$

Definition 4.4.1. For $k > 0$, let us define $\theta_k : \mathbb{R} \to \mathbb{R}$ by $\theta_k(r) := \int_0^r T_k(\sigma)d\sigma$, $r \in \mathbb{R}$ and

$$E := \{\phi \in V \cap L^\infty(Q_T); \phi_t \in V' + L^1(Q_T)\}.$$

An entropy solution to (L, f, u_0) is a measurable function $u : Q_T \to \mathbb{R}$ such that

i.) $T_k(u) \in V$ for all $k > 0$,

ii.) The mapping

$$[0,T] \ni t \to \int_\Omega \theta_k(u-\phi)(t,x)dx$$

is a.e. equal to a continuous function for all $k > 0$ and all $\phi \in E$,

iii.) If $\langle \cdot, \cdot \rangle$ is the duality pairing between $V' + L^1(Q_T)$ and $V \cap L^\infty$,

$$\begin{aligned} &\int_\Omega \theta_k(u-\phi)(T) - \int_\Omega \theta_k(u_0-\phi)(0) + \langle \phi_t, T_k(u-\phi)\rangle \\ &+ \int_{Q_T} |Du|^{p(x)-2}Du \cdot DT_k(u-\phi) \leq \int_{Q_T} fT_k(u-\phi) \end{aligned} \tag{4.4.7}$$

holds for all $k > 0$ and all $\phi \in E$.

The following result is known for (Lc, f, u_0) and can be easily generalized:

Proposition 4.4.2. *A function u is a renormalized solution to (L, f, u_0) iff it is an entropy solution.*

The proof of Proposition 4.4.2 follows the same lines as the proof of the corresponding result in the case of a constant exponent (see [42]). In particular, we use the same arguments as in the proof of Lemma 7.1 in [42] to show that the following integration-by-parts-formula holds true:

Lemma 4.4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is zero outside a compact set of $\mathbb{R}$. Let us denote $F(s) = \int_0^s f(r)dr$. If $u \in V$ is such that $u_t \in V^* + L^1(Q_T)$ and if $\psi \in \mathcal{C}^\infty(\overline{Q_T})$, then we have*

$$\langle u_t, f(u)\psi\rangle = \int_\Omega F(u(T))\psi(T)dx - \int_\Omega F(u(0))\psi(0)dx - \int_{Q_T} \psi_t F(u), \tag{4.4.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V^* + L^1(Q_T)$ and $V \cap L^\infty(Q_T)$.

A natural way to generalize the notion of entropy solution of Definition 4.4.1 to the problem (P, f, b_0) is to exchange θ_k by the function $\theta_{k,\beta}$ defined by $\theta_{k,\beta}(r) := \int_0^r T_k \circ (\beta^{-1})^0(\sigma)d\sigma$ for $r \in \overline{R(\beta)}$. More precisely, we write $\theta_{k,\beta}(b-\phi)$ instead of $\theta_k(u-\phi)$ with $b \in L^1(Q_T)$ such that $b \in \beta(u)$ almost everywhere in Definition 4.4.1. It is an open problem to show equivalence between renormalized and entropy solutions to the general problem (P, f, b_0).

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