R. Altmann, T. Levajković, H. Mena

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OPERATOR DIFFERENTIAL ALGEBRAIC EQUATIONS WITH NOISE ARISING IN FLUID DYNAMICS

R. ALTMANN*, T. LEVAJKOVIĆ†,‡ AND H. MENA‡

Abstract. We study linear semi-explicit stochastic operator differential-algebraic equations (DAEs) for which the constraint equation is given in an explicit form. In particular, this includes the Stokes equations arising in fluid dynamics. We combine a white noise polynomial chaos expansion approach to include stochastic perturbations with deterministic regularization techniques. With this, we are able to include Gaussian noise and stochastic convolution terms as perturbations in the differential as well as in the constraint equation. By the application of the polynomial chaos expansion method, we reduce the stochastic operator DAE to an infinite system of deterministic operator DAEs for the stochastic coefficients. Since the obtained system is very sensitive to perturbations in the constraint equation, we analyze a regularized version of the system. This then allows to prove the existence and uniqueness of the solution of the initial stochastic operator DAE in a certain weighted space of stochastic processes.

Key words. operators DAE, noise disturbances, chaos expansions, Itô-Skorokhod integral stochastic convolution, regularization

AMS subject classifications. 65J10, **60H40**, **60H30**, **35R60**

1. Introduction

The governing equations of an incompressible flow of a Newtonian fluid are described by the Navier-Stokes equations [Tem77]. Therein, one searches for the evolution of a velocity field u and the pressure p to given initial data, a volume force, and boundary conditions. For results on the existence of a (unique) solution, we refer to [Tem77, Ch. III], [Tar06, Ch. 25], and [HR90].

In this paper, we consider the linear case but allow a more general constraint, namely that the divergence of the velocity does not vanish. Note that this changes the analysis and numerics since the state-of-the-art methods are often tailored for the particular case of a vanishing divergence. An application with non-vanishing divergence is given by the optimal control problem constrained by the Navier-Stokes equations where cost functional includes the pressure [Hin00].

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The Navier-Stokes equations, as well as the linearized Stokes equations, can be formulated as differential-algebraic equations (DAEs) in an abstract setting [AH15, Alt15]. These so-called *operator DAEs* correspond to the weak formulation of the partial differential equations (PDEs). As generalization of finite-dimensional DAEs, see [GM86, KM06, LMT13] for an introduction, also here considered constrained PDEs suffer from instabilities and ill-posedness. This is the reason why the stable approximation of the pressure (which is nothing else than a Lagrange multiplier to enforce the incompressibility) is a great challenge.

One solution strategy is to perform a regularization which corresponds to an *index reduction* in the finite-dimensional setting. With this, the issue of instabilities with respect to perturbations is removed. In the case of fluid dynamics, this has been shown in [AH15].

In this paper, we study the stochastic version of operator DAEs considered in the framework of the *polynomial chaos expansion* method on white noise spaces [GS91, HKPS93]. More precisely, we consider semi-explicit operator DAEs with perturbations of stochastic type. Particularly, in the fluid flow case, the stochastic equations are of the form

$$\begin{array}{ccccccccc} \dot{u}(t) & - & \Delta u(t) & + & \nabla p(t) & = & \mathcal{F}(t) & + & \text{"noise"}, \\ & & \operatorname{div} u(t) & & & = & \mathcal{G}(t) & + & \text{"noise"} \end{array}$$

with an initial value for $u(0)$. In order to preserve the mean dynamics, we deal with stochastic perturbations of zero mean. This implies that the expected value of the stochastic solution equals the solution of the corresponding deterministic operator DAE. For the "noise" processes we consider either a general Gaussian white noise process or perturbations which can be expressed in the form of a stochastic convolution.

With the application of the polynomial chaos expansion method, also known as the *propagator method*, the problem of solving the initial stochastic equations is reduced to the problem of solving an infinite triangular systems of deterministic operator DAEs, which can be solved recursively. Summing up all coefficients of the expansion and proving convergence in an appropriate space of stochastic processes, one obtains the stochastic solution of the initial problem.

The chaos expansion methodology is a very useful technique for solving many types of stochastic differential equations, see e.g. [GS91, XK02, BTZ04, LR06, LR10, LPS11a, LPS11b, LPSŽ15]. The main statistical properties of the solution, its mean, variance, and higher moments, can be calculated from the formulas involving only the coefficients of the chaos expansion representation [MR12, EMSU12].

The proposed method allows to apply regularization techniques from the theory of deterministic operator DAEs to the related stochastic system. The main applications arise in fluid dynamics, but it is not restricted to this case. The same procedure can be used to regularize other classes of equations that fulfill our setting. A specific example involving equations with the operators of the Malliavin calculus is described in Section 5. For this reason, in the present paper, we developed a general abstract setting based on white noise analysis and chaos expansions. Numerical experiments with truncated chaos expansions (stochastic Galerkin methods) are not included in this paper. However, once we regularize each system, it becomes numerically well-posed [Alt15] and then the stochastic equation is well-posed. We intent to provide numerical experiments with detailed error analysis in our future work.

The paper is organized as follows. In Section 2 we introduce the concept of (deterministic) operator DAEs with special emphasis on applications in fluid dynamics. Considering perturbation results for such systems, we detect the necessity of a regularization in order to allow stochastic perturbations. The stochastic setting for the chaos expansion is then given in Section 3. Furthermore, we discuss stochastic noise terms in the differential as well as in the constraint equation and the systems which result from the chaos expansion. The extension to more general cases is then subject of Section 4. Therein, we consider more general operators and stochastic convolution terms. Finally, in Section 5 we consider shortly a specific example of DAEs that involve stochastic operators arising in Malliavin calculus.

2. Operator DAEs

2.1. **Abstract Setting.** First we consider operator DAEs (also called PDAEs) which equal constrained PDEs in the weak setting or DAEs in an abstract framework [Alt15, EM13]. Thus, we work with generalized derivatives in time and space. In particular, we consider semi-explicit operator DAEs for which the constraint equation is explicitly stated.

We consider real, separable, and reflexive Banach spaces $\mathcal{V}$ and $\mathcal{Q}$ and a real Hilbert space $\mathcal{H}$. Furthermore, we assume that we have a Gel'fand triple of the form

$$\mathcal{V} \subseteq \mathcal{H} \subseteq \mathcal{V}^*$$

which means that $\mathcal{V}$ is continuously and densely embedded in $\mathcal{H}$ [Zei90, Ch. 23]. As a consequence, well-known embedding theorems yield the continuous embedding

$$\big\{v \in L^2(T;\mathcal{V}) : \ \dot{v} \in L^2(T;\mathcal{V}^*)\big\} \hookrightarrow C(T;\mathcal{H}).$$

Note that $L^2(T;\mathcal{V})$ denotes the Bochner space of abstract functions on a time interval T with values in $\mathcal{V}$, see [Emm04, Ch. 7.1] for an introduction. The corresponding norm of $L^2(T;\mathcal{V})$, which we denote by $\|\cdot\|_{L^2(\mathcal{V})}$, is given by

$$\|u\|^2_{L^2(\mathcal{V})} := \|u\|^2_{L^2(T;\mathcal{V})} := \int_T \|u(t)\|^2_{\mathcal{V}} \,\mathrm{d}t.$$

The (deterministic) problem of interest has the form

$$\dot{u}(t) + Ku(t) + B^*\lambda(t) = F(t) \quad \text{in } \mathcal{V}^*, \tag{1a}$$

$$Bu(t) = G(t) \quad \text{in } \mathcal{Q}^* \tag{1b}$$

with (consistent) initial condition $u(0) = u^0 \in \mathcal{H}$. The need of consistent initial values is one characteristic of DAEs in the finite dimensional setting [BCP96, KM06]. The condition in the infinite-dimensional case is discussed in Remark 2.1 below.

For the right-hand sides we assume

$$F \in L^2(T; \mathcal{V}^*) \quad \text{and} \quad G \in H^1(T; \mathcal{Q}^*) \hookrightarrow C(T; \mathcal{Q}^*).$$

The involved operators should satisfy

$$B \colon \mathcal{V} \to \mathcal{Q}^* \qquad \text{and} \qquad K \colon \mathcal{V} \to \mathcal{V}^*$$

and can be extended to Nemytskii mappings of the form $B \colon L^2(T; \mathcal{V}) \to L^2(T; \mathcal{Q}^*)$ as well as $K \colon L^2(T; \mathcal{V}) \to L^2(T; \mathcal{V}^*)$, see [Rou05, Ch. 1.3]. Furthermore, we restrict ourselves to the linear case, i.e., the constraint operator B is linear and there exists a right-inverse which is denoted by B^-. Furthermore, we assume K to be linear, positive on the kernel of B, and continuous. As search space for the solution (u, λ) we consider

$$u \in L^2(T; \mathcal{V}) \text{ with } \dot{u} \in L^2(T; \mathcal{V}^*) \quad \text{and} \quad \lambda \in L^2(T; \mathcal{Q}).$$

Note that the actual meaning of equation (1a) is that for all test functions $v \in \mathcal{V}$ and $\Phi \in C^\infty(T)$ it holds that

$$\int_T \big\langle \dot{u}(t) + Ku(t) + B^*\lambda(t), v \big\rangle \Phi(t) \,\mathrm{d}t = \int_T \big\langle F(t), v \big\rangle \Phi(t) \,\mathrm{d}t.$$

Remark 2.1 (Consistent initial values). DAEs require consistent initial data because of the given constraints which also apply for the initial data. This remains valid for the operator case. However, since we allow $u_0 \in \mathcal{H}$, the constraint operator B is not applicable to u^0. In this case, the condition has the form

$$u^0 = u^0_B + B^- G(0)$$

where u^0_B is an arbitrary element from the closure of the kernel of B in $\mathcal{H}$ [EM13, AH15]. If $u_0 \in \mathcal{V}$ is given, then we get the same decomposition but with $u^0_B \in \ker B$.

Theorem 2.2 (Stability estimate)**.** *Under the given assumptions on the operators K and B, the right-hand sides F and G, and the initial condition u^0, the solution of the operator DAE* (1) *satisfies for some positive constant c the estimate*

$$\|u\|^2_{L^2(\mathcal{V})} \leq\; c\, \Big[\|u^0_B\|^2_{\mathcal{H}} + \|F\|^2_{L^2(\mathcal{V}^*)} + \|G\|^2_{H^1(\mathcal{Q}^*)} \Big]. \tag{2}$$

Proof. This estimate can be found in [Alt15, Sect. 6.1.3]. We emphasize that the bound on the right-hand side includes the derivative of G. □

Since this paper focuses on fluid flows, we show that the linearized Stokes equations fit into the given framework. Note that also the Navier-Stokes equations may be considered in the given setting if we allow the operator K in (1) to be nonlinear.

Example 2.3 (Stokes equations)**.** The Stokes equations are a linearized version of the Navier-Stokes equations and describe the incompressible flow of a Newtonian fluid in a bounded domain D, cf. [Tem77]. We consider homogeneous Dirichlet boundary conditions and set

$$\mathcal{V} = [H^1_0(D)]^d, \qquad \mathcal{H} = [L^2(D)]^d, \qquad \mathcal{Q} = L^2(D)/\mathbb{R}.$$

Furthermore, we define $G \equiv 0$, $B = \operatorname{div}$ with dual operator $B^* = -\nabla$, and K which equals the weak form of the Laplace operator, i.e.,

$$\langle Ku, v\rangle := \int_D \nabla u \cdot \nabla v \,\mathrm{d}x.$$

The solution u describes the velocity of the fluid whereas λ measures the pressure. The operator equations (1) then equal the weak formulation of the Stokes equations

$$\dot{u} - \Delta u + \nabla\lambda = f, \qquad \nabla\cdot u = 0, \qquad u(0) = u_0.$$

2.2. **Influence of Perturbations.** DAEs are known for its high sensitivity to perturbations. The reason for this is that derivatives of the right-hand sides appear in the solution. In particular, this implies that a certain smoothness of the right-hand sides is necessary for the existence of solutions. Furthermore, the numerical approximation is much harder than for ODEs since small perturbations - such as round-off errors or errors within iterative methods - may have a large influence [Pet82].

The level of difficulty in the numerical approximation or the needed smoothness of the right-hand sides in order to guarantee the existence of a classical solution is described by the *index* of a DAE. There exist several index concepts [Meh13] and we focus on the *differentiation index*, see [BCP96, Def. 2.2.2] for a precise definition.

Although there exists no general index concept for operator DAEs, the influence of perturbations should be analyzed as well. A spatial discretization of system (1) by finite elements (under some basic assumptions) leads to a DAE of index 2. Thus, it seems likely that the operator case includes similar stability issues.

We consider system (1) with additional perturbations $\delta \in L^2(T;\mathcal{V}^*)$ and $\theta \in H^1(T;\mathcal{Q}^*)$. The perturbed solution $(\hat{u},\hat{\lambda})$ then satisfies the system

$$\begin{aligned} \dot{\hat{u}} \;+\; K\hat{u} \;+\; B^*\hat{\lambda} \;&=\; F + \delta \quad \text{in } \mathcal{V}^*,\\ B\hat{u} \qquad\qquad &=\; G + \theta \quad \text{in } \mathcal{Q}^*. \end{aligned}$$

Let e_1 denote the difference of u and $\hat{u}$ projected to the kernel of the constraint operator B. Accordingly, we denote the projected initial error by $e_{1,0}$. In [Alt15] it is shown that with the given assumptions on the operators K and B, we have

$$\|e_1\|^2_{C(T;\mathcal{H})} + \|e_1\|^2_{L^2(\mathcal{V})} \lesssim |e_{1,0}|^2 + \|\delta\|^2_{L^2(\mathcal{V}^*)} + \|\theta\|^2_{L^2(\mathcal{Q}^*)} + \|\dot{\theta}\|^2_{L^2(\mathcal{Q}^*)}. \tag{3}$$

Therein, $a \lesssim b$ means that there exists a positive constant c with $a \leq cb$. This estimate shows that the error depends on the derivative of the perturbation θ. Note that this is crucial if we consider stochastic perturbations in Section 3 where we apply the Wiener-Itô chaos expansion to reduce the given problem to an infinite number of deterministic systems. Similar to index reduction procedures for DAEs, cf. [BCP96, KM06], the operator DAE can be regularized in view of an improved behaviour with respect to perturbations.

2.3. **Regularization of Operator DAEs.** In this subsection, we introduce an operator DAE which is equivalent to (1) but where the solution of the perturbed system does not depend on derivatives of the perturbations. Furthermore, a semi-discretization in space of the regularized system directly leads to a DAE of index 1 and thus, is better suited for numerical integration [KM06].

In the case of the Stokes equations, the right-hand side G vanishes since we search for divergence-free velocities. In this case, the constrained system is often reduced to the kernel of the constraint operator B which leads to an operator ODE, i.e., a time-dependent PDE. However, with the stochastic noise term in the constraint, we cannot ignore the inhomogeneity anymore. In addition, the inclusion of G enlarges the class of possible applications. Thus, we propose to apply a regularization of the operator DAE.

For the regularization we follow the procedure introduced first in [Alt13] for second-order systems. The idea is to add the derivative of the constraint, the so-called *hidden constraint*, to the system. In order to balance the number of equations and variables, we add a so-called *dummy variable* v_2 to the system. The assumptions are as before but we split the space $\mathcal{V}$ into $\mathcal{V} = \mathcal{V}_\mathcal{B} \oplus \mathcal{V}^c$ were

$$\mathcal{V}_\mathcal{B} := \ker B$$

and $\mathcal{V}^c$ is any complementary space on which B is invertible, i.e., there exists a right-inverse of B, namely $B^-\colon \mathcal{Q}^* \to \mathcal{V}^c$ with $BB^-q = q$ for all $q \in \mathcal{Q}^*$. In the example of the Stokes equations, cf. Example 2.3, $\mathcal{V}_\mathcal{B}$ is the space of divergence-free functions which build a proper subspace of $\mathcal{V}$. We then search for a solution (u_1, u_2, v_2, λ) where u_1 takes values in $\mathcal{V}_\mathcal{B}$ and u_2, v_2 in the complement $\mathcal{V}^c$. The extended (but equivalent) system then reads

$$\dot u_1(t) + v_2(t) + K\big(u_1(t) + u_2(t)\big) + B^*\lambda(t) = F(t) \quad \text{in } \mathcal{V}^*, \tag{4a}$$

$$Bu_2(t) = G(t) \quad \text{in } \mathcal{Q}^*, \tag{4b}$$

$$Bv_2(t) = \dot G(t) \quad \text{in } \mathcal{Q}^* \tag{4c}$$

with initial condition

$$u_1(0) = u_B^0 - B^-G(0) \in \mathcal{H}. \tag{4d}$$

Recall that u_B^0 is an element of the closure of $\mathcal{V}_\mathcal{B}$ in $\mathcal{H}$, cf. Remark 2.1. The connection of system (1) and (4) is given by $u = u_1 + u_2$ and $v_2 = \dot u_2$. Note, however, that in system (4) u_2 is no differential variable anymore and corresponds to an algebraic variable in the finite-dimensional case.

For the regularized formulation in (4) we obtain the following stability result. Consider perturbations $\delta \in L^2(T;\mathcal{V}^*)$ and $\theta, \xi \in L^2(T;\mathcal{Q}^*)$ of the right-hand sides and the corresponding perturbed solution $(\hat{u}_1, \hat{u}_2, \hat{v}_2, \hat{\lambda})$. Then, the error in u_1, namely $e_1 = \hat{u}_1 - u_1$ satisfies the estimate

$$\|e_1\|^2_{C(T;\mathcal{H})} + \|e_1\|^2_{L^2(\mathcal{V})} \lesssim |e_{1,0}|^2 + \|\delta\|^2_{L^2(\mathcal{V}^*)} + \|\theta\|^2_{L^2(\mathcal{Q}^*)} + \|\xi\|^2_{L^2(\mathcal{Q}^*)}. \tag{5}$$

Thus, in contrast to the result of the previous subsection, the difference does not depend on derivatives of perturbations. This is a crucial result if we consider stochastic perturbations.

3. Inclusion of Stochastic Perturbations

In this section, we consider the DAE (1) with additional stochastic perturbation terms, also called noise terms. Clearly, we perturb the deterministic system with zero mean disturbances. First, we consider the noise, only in the differential equation, i.e., we study

$$\dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) + \text{"noise"}, \tag{6a}$$

$$\mathcal{B}u(t) = \mathcal{G}(t). \tag{6b}$$

Afterwards, we also add a noise term in the constraint equation,

$$\dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) + \text{"noise"}, \tag{7a}$$

$$\mathcal{B}u(t) = \mathcal{G}(t) + \text{"noise"}. \tag{7b}$$

As discussed in Section 2.2, perturbations in the second equation, i.e., in the constraint equation, lead to instabilities. Thus, we also consider the regularized operator equations (4) with stochastic perturbations. In any case, we assume a consistent initial condition of the form $u(0) = u^0$. Note that with the inclusion of stochastic perturbations, we also allow the initial data u^0 to be random.

3.1. Preliminaries.

We consider stochastic DAEs in the white noise framework. For this, the spaces of stochastic test and generalized functions are built by use of series decompositions via orthogonal functions as a basis with certain weight sequences. The classical Hida approach [HKPS93] suggests to start with a Gel'fand triple

$$\mathcal{E} \subseteq L^2(\mathbb{R}) \subseteq \mathcal{E}',$$

with continuous inclusions, formed by a nuclear space $\mathcal{E}$ and its dual $\mathcal{E}'$. As basic probability space we set $\Omega = \mathcal{E}'$ endowed with the Borel sigma algebra of the weak topology and an appropriate probability measure, see [HKPS93, HØUZ10]. Without loss of generality, in this paper we assume that the underlying probability space is the Gaussian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$. Therefore, we take $\mathcal{E}$ and $\mathcal{E}'$ to be the *Schwartz spaces* of rapidly decreasing test functions $S(\mathbb{R})$ and tempered distributions $S'(\mathbb{R})$, respectively, and $\mathcal{B}$ the Borel sigma algebra generated by the weak topology on $S'(\mathbb{R})$. By the Bochner-Minlos theorem, there exists a unique measure μ on $(S'(\mathbb{R}),\mathcal{B})$ such that for each $\phi\in S(\mathbb{R})$ the relation

$$\int_{S'(\mathbb{R})} e^{\langle\omega,\phi\rangle}\, d\mu(\omega) \;=\; e^{-\frac{1}{2}\,\|\phi\|^2_{L^2(\mathbb{R})}}$$

holds, where $\langle\omega,\phi\rangle$ denotes the action of a tempered distribution $\omega\in S'(\mathbb{R})$ on a test function $\phi\in S(\mathbb{R})$. We denote by $L^2(\Omega,\mu)$, or in short $L^2(\Omega)$, the space of square integrable random variables $L^2(\Omega)=L^2(\Omega,\mathcal{B},\mu)$. It is the Hilbert space of random variables which have finite second moments. Here, the scalar product is $(F,G)_{L^2(\Omega)}=\mathbb{E}_\mu(F\cdot G)$, where $\mathbb{E}_\mu$ denotes the expectation with respect to the measure μ. In the sequel, we omit μ and simply write $\mathbb{E}$.

In the case of a Gaussian measure, the orthogonal polynomial basis of $L^2(\Omega)$ can be represented as a family of orthogonal *Fourier-Hermite polynomials* defined by use of the Hermite functions and the Hermite polynomials. We denote by $\{h_n(x)\}_{n\in\mathbb{N}_0}$ the family of *Hermite polynomials* and $\{\xi_n(x)\}_{n\in\mathbb{N}}$ the family of *Hermite functions*, where

$$\begin{aligned} h_n(x) &= (-1)^n\, e^{\frac{x^2}{2}}\; \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), && n\in\mathbb{N}_0,\\ \xi_n(x) &= \frac{1}{\sqrt[4]{\pi}\sqrt{(n-1)!}}\; e^{-\frac{x^2}{2}}\; h_{n-1}(\sqrt{2}x), && n\in\mathbb{N}, \end{aligned}$$

for $x\in\mathbb{R}$. The family of Hermite polynomials forms an orthogonal basis of the space $L^2(\mathbb{R})$ with respect to the Gaussian measure $d\mu=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$, while the family of Hermite functions forms a complete orthonormal system in $L^2(\mathbb{R})$ with respect to the Lebesque measure. We follow the characterization of the Schwartz spaces in terms of the Hermite basis [HØUZ10]. Clearly, the Schwartz space of rapidly decreasing functions can be constructed as the projective limit of the family of spaces

$$S_l(\mathbb{R})=\Big\{f(t)=\sum_{k\in\mathbb{N}} a_k\,\xi_k(t)\in L^2(\mathbb{R})\colon\ \|f\|_l^2=\sum_{k\in\mathbb{N}} a_k^2\,(2k)^l<\infty\Big\},\ l\in\mathbb{N}_0.$$

The Schwartz space of tempered distributions is isomorphic to the inductive limit of the family of spaces

$$S_{-l}(\mathbb{R})=\Big\{F(t)=\sum_{k\in\mathbb{N}} b_k\,\xi_k(t)\colon\ \|F\|_{-l}^2=\sum_{k\in\mathbb{N}} b_k^2\,(2k)^{-l}<\infty\Big\},\ l\in\mathbb{N}_0.$$

It holds $S(\mathbb{R})=\bigcap_{l\in\mathbb{N}_0} S_l(\mathbb{R})$ and $S'(\mathbb{R})=\bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$. The action of a generalized function $F=\sum_{k\in\mathbb{N}} b_k\,\xi_k\in S'(\mathbb{R})$ on a test function $f=\sum_{k\in\mathbb{N}} a_k\,\xi_k\in S(\mathbb{R})$ is given by $\langle F,f\rangle=\sum_{k\in\mathbb{N}} a_k\,b_k$.

3.1.1. *Spaces of random variables.* Let $\mathcal{I}=(\mathbb{N}_0^{\mathbb{N}})_c$ be the set of sequences of non-negative integers which have only finitely many nonzero components $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_m,0,0,\ldots)$, $\alpha_i\in\mathbb{N}_0$, $i=1,2,\ldots,m$, $m\in\mathbb{N}$. The k-th unit vector $\varepsilon^{(k)}=(0,\ldots,0,1,0,\ldots)$, $k\in\mathbb{N}$, is the sequence of zeros with the entry 1 as the k-th component and $\mathbf{0}$ is the multi-index with only zero components. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha| := \sum_{k=1}^{\infty} \alpha_k$. Note that with $(2\mathbb{N})^{\alpha} = \prod_{k=1}^{\infty}(2k)^{\alpha_k}$, it holds that $\sum_{\alpha\in\mathcal{I}}(2\mathbb{N})^{-p\alpha} < \infty$ for $p > 1$.

We define by

$$H_{\alpha}(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, \xi_k \rangle), \quad \alpha \in \mathcal{I},$$

the Fourier-Hermite orthogonal basis of $L^2(\Omega)$ such that $\|H_{\alpha}\|^2_{L^2(\Omega)} = \mathbb{E}(H_{\alpha})^2 = \alpha!$. In particular, $H_{\mathbf{0}} = 1$, and for the k-th unit vector $H_{\varepsilon^{(k)}}(\omega) = h_1(\langle \omega, \xi_k \rangle) = \langle \omega, \xi_k \rangle$, $k \in \mathbb{N}$. The Wiener-Itô chaos expansion theorem [HØUZ10] states that each element $f \in L^2(\Omega)$ has a unique representation of the form

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} a_{\alpha}\, H_{\alpha}(\omega), \quad a_{\alpha} \in \mathbb{R},\ \omega \in \Omega$$

such that it holds

$$\|f\|^2_{L^2(\Omega)} = \sum_{\alpha\in\mathcal{I}} a_{\alpha}^2\, \alpha! < \infty.$$

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions. They have no point value for $\omega \in \Omega$ but an average value with respect to a test random variable. Following the idea of the construction of $S'(\mathbb{R})$ as an inductive limit space over $L^2(\Omega)$ with appropriate weights [Zem87], one can define stochastic generalized random variable spaces over $L^2(\Omega)$ by adding certain weights in the convergence condition of the series expansion. Several spaces of this type, weighted by a sequence $q = (q_{\alpha})_{\alpha\in\mathcal{I}}$, denoted by $(Q)_{-\rho}$, for $\rho \in [0, 1]$ were described in [LS11]. Thus a Gelfand triple

$$(Q)_{\rho} \subseteq L^2(\Omega) \subseteq (Q)_{-\rho}$$

is obtained, where the inclusions are again continuous. The most common weights and spaces appearing in applications are $q_{\alpha} = (2\mathbb{N})^{\alpha}$ which correspond to the Kondratiev spaces of stochastic test functions $(S)_{\rho}$ and stochastic generalized functions $(S)_{-\rho}$, for $\rho \in [0, 1]$. Exponential weights $q_{\alpha} = e^{(2\mathbb{N})^{\alpha}}$ are linked with the *exponential growth spaces* of stochastic test functions $\exp(S)_{\rho}$ and stochastic generalized functions $\exp(S)_{-\rho}$ [HKPS93, HØUZ10, PS07]. In this paper, we consider the largest Kondratiev space of stochastic distributions, i.e., $\rho = 1$.

The space of the *Kondratiev test random variables* $(S)_1$ can be constructed as the projective limit of the family of spaces, $p \in \mathbb{N}_0$,

$$(S)_{1,p} = \Big\{ f(\omega) = \sum_{\alpha\in\mathcal{I}} a_{\alpha} H_{\alpha}(\omega) \in L^2(\Omega) \colon \|f\|^2_{1,p} = \sum_{\alpha\in\mathcal{I}} a_{\alpha}^2 (\alpha!)^2 (2\mathbb{N})^{p\alpha} < \infty \Big\}.$$

The space of the *Kondratiev generalized random variables* $(S)_{-1}$ can be constructed as the inductive limit of the family of spaces, $p \in \mathbb{N}_0$,

$$(S)_{-1,-p} = \Big\{ F(\omega) = \sum_{\alpha \in \mathcal{I}} b_\alpha \, H_\alpha(\omega) \colon \ \|f\|^2_{-1,-p} = \sum_{\alpha \in \mathcal{I}} b_\alpha^2 \, (2\mathbb{N})^{-p\alpha} < \infty \Big\}.$$

It holds $(S)_1 = \bigcap_{p \in \mathbb{N}_0} (S)_{1,p}$ and $(S)_{-1} = \bigcup_{p \in \mathbb{N}_0} (S)_{-1,p}$. The action of a generalized random variable $F = \sum_{\alpha \in \mathcal{I}} b_\alpha \, H_\alpha(\omega) \in (S)_{-1}$ on a test random variable $f = \sum_{\alpha \in \mathcal{I}} b_\alpha \, H_\alpha(\omega) \in (S)_1$ is given by $\langle F, f \rangle = \sum_{\alpha \in \mathcal{I}} \alpha! \, a_\alpha \, b_\alpha$. It holds that $(S)_1$ is a nuclear space with the Gel'fand triple structure

$$(S)_1 \subseteq L^2(\Omega) \subseteq (S)_{-1},$$

with continuous inclusions.

The problem of pointwise multiplications of generalized stochastic functions in the white noise analysis is overcome by introducing the Wick product, which represents the stochastic convolution. The fundamental theorem of stochastic calculus states the important property of the Wick multiplication, namely its relation to the Itô-Skorokhod integration [HØUZ10], which will be the subject of Section 4.2.

Let L and S be random variables given in their chaos expansions $L = \sum_{\alpha \in \mathcal{I}} \ell_\alpha H_\alpha$ and $S = \sum_{\alpha \in \mathcal{I}} s_\alpha \, H_\alpha$, $\ell_\alpha, s_\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{I}$. Then, the *Wick product* $L \Diamond S$ is defined by

$$L \Diamond S = \sum_{\gamma \in \mathcal{I}} \Big(\sum_{\alpha + \beta = \gamma} \ell_\alpha s_\beta \Big) H_\gamma(\omega). \tag{8}$$

Note here that the space $L^2(\Omega)$ is not closed under the Wick multiplication, while the Kondratiev spaces $(S)_1$ and $(S)_{-1}$ are.

3.1.2. *Stochastic processes.* Classical stochastic process can be defined as a family of functions $v \colon T \times \Omega \to \mathbb{R}$ such that for each fixed $t \in T$, $v(t, \cdot)$ is an $\mathbb{R}$-valued random variable and for each fixed $\omega \in \Omega$, $v(\cdot, \omega)$ is an $\mathbb{R}$-valued deterministic function, called a *trajectory.* Here, following [PS07], we generalize the definition of a classical stochastic process and define generalized stochastic processes. By replacing the space of trajectories with some space of deterministic generalized functions, or by replacing the space of random variables with some space of generalized random variables, different types of generalized stochastic processes can be obtained. In this manner, we obtain processes generalized with respect to the t argument, the ω argument, or even with respect to both arguments [PS07, HØUZ10].

A very general concept of generalized stochastic processes, based on chaos expansions was developed in [PS07, HØUZ10, LS11]. In [HØUZ10] generalized stochastic processes are defined as measurable mappings $T \to (S)_{-1}$. Thus, they are defined pointwise with respect to the parameter $t \in T$ and generalized with respect to $\omega \in \Omega$. We define such processes by their chaos expansion representations in terms of an orthogonal polynomial basis.

Let X be a Banach space. We consider a *generalized stochastic process* u which belongs to $X \otimes (S)_{-1}$ and is given by the chaos expansion form

$$u = \sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha = u_{\mathbf{0}}(t) + \sum_{k \in \mathbb{N}} u_{\varepsilon^{(k)}} \otimes H_{\varepsilon^{(k)}} + \sum_{|\alpha|>1} u_\alpha \otimes H_\alpha. \tag{9}$$

Therein, the coefficients $u_\alpha \in X$ satisfy for some $p \in \mathbb{N}_0$ the convergence condition

$$\|u\|^2_{X \otimes (S)_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \|u_\alpha\|^2_X \, (2\mathbb{N})^{-p\alpha} < \infty.$$

Value p corresponds to the *level of singularity* of the process u. Note that the deterministic part of u in (9) is the coefficient $u_{\mathbf{0}}$, which represents the generalized expectation of u. In the applications of fluid flows, the space X will equal one of the Sobolev-Bochner spaces $L^2(T;\mathcal{V})$ or $L^2(T;\mathcal{Q})$.

If we take, as an example, $X = L^2(\mathbb{R})$, then $u \in L^2(\mathbb{R}) \otimes L^2(\Omega)$ is given in the form

$$u(t,\omega) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t)\, H_\alpha(\omega), \qquad t \in \mathbb{R},\ \omega \in \Omega$$

such that it holds

$$\|u\|^2_{L^2(\mathbb{R})\otimes L^2(\Omega)} = \sum_{\alpha \in \mathcal{I}} \alpha!\, \|u_\alpha\|^2_{L^2(\mathbb{R})} = \sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}} \alpha!\, |u_\alpha(t)|^2 \,\mathrm{d}t < \infty.$$

Stochastic processes which are elements of $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ are defined similarly, cf. [LPS11a, LS11, LS15, LPSŽ15]. More precisely, $F \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ has a chaos expansion representation

$$F = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} a_{\alpha,k} \otimes \xi_k \otimes H_\alpha = \sum_{\alpha \in \mathcal{I}} b_\alpha \otimes H_\alpha = \sum_{k \in \mathbb{N}} c_k \otimes \xi_k, \tag{10}$$

where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \in X \otimes S'(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes H_\alpha \in X \otimes (S)_{-1}$, and $a_{\alpha,k} \in X$. Thus, for some $p, l \in \mathbb{N}_0$, it holds

$$\|F\|^2_{X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \|a_{\alpha,k}\|^2_X \, (2k)^{-l} (2\mathbb{N})^{-p\alpha} < \infty.$$

The generalized expectation of F is the zero-th coefficient in the expansion representation (10), i.e., it is given by $\sum_{k\in\mathbb{N}} a_{\mathbf{0},k} \otimes \xi_k = b_{\mathbf{0}}$.

As an example, consider $X = C^k(T)$, $k \in \mathbb{N}$, where T denotes again a time interval. It is well-known that differentiation of a stochastic process can be carried out componentwise in the chaos expansion, i.e., due to the fact that $(S)_{-1}$ is a nuclear space it holds that $C^k(T;(S)_{-1}) = C^k(T) \otimes (S)_{-1}$, cf. [LPSŽ15, LS15]. This means that a stochastic process $u(t,\omega)$ is k times continuously differentiable if and only if all of its coefficients u_α, $\alpha \in \mathcal{I}$ are in $C^k(T)$. The same holds for Banach space valued stochastic processes, i.e., for elements of $C^k(T;X) \otimes (S)_{-1}$, where X is an arbitrary Banach space.

By the nuclearity of $(S)_{-1}$, these processes can be regarded as elements of the tensor product space

$$C^k(T;X\otimes(S)_{-1}) = C^k(T;X)\otimes(S)_{-1} = \bigcup_{p=0}^{\infty} C^k(T;X)\otimes(S)_{-1,-p}.$$

Since we consider weak solutions, i.e., solutions in Sobolev-Bochner spaces such as $L^2(T;X)$, we emphasize that the nuclearity of $(S)_{-1}$ also implies

$$L^2(T;X\otimes(S)_{-1}) = L^2(T;X)\otimes(S)_{-1},$$

as well as

$$H^1(T;X\otimes(S)_{-1}) = H^1(T;X)\otimes(S)_{-1}.$$

In this way, by representing stochastic processes in their polynomial chaos expansion form, we are able to separate the deterministic component from the randomness of the process.

Example 3.1. The *Brownian motion* $B_t(\omega) := \langle\omega, \chi_{[0,t]}\rangle$, $\omega\in S'(\mathbb{R})$, $t\geq 0$ is defined by passing though the limit in $L^2(\mathbb{R})$, where $\chi_{[0,t]}$ is the characteristic function on $[0,t]$. The chaos expansion representation has the form

$$B_t(\omega) = \sum_{k\in\mathbb{N}} \int_0^t \xi_k(s)\ \mathrm{d}s\ H_{\varepsilon^{(k)}}(\omega).$$

Note that for fixed t, B_t is an element of $L^2(\Omega)$. Brownian motion is a Gaussian process with zero expectation and the covariance function $E(B_t(\omega)B_s(\omega)) = \min\{t,s\}$. Furthermore, almost all trajectories are continuous, but nowhere differentiable functions.

Singular white noise is defined by the formal chaos expansion

$$W_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t) H_{\varepsilon^{(k)}}(\omega), \tag{11}$$

and is an element of the space $C^\infty(\mathbb{R})\otimes(S)_{-1,-p}$ for $p>1$, cf. [HØUZ10]. With weak derivatives in the $(S)_{-1}$ sense, it holds that $\frac{\mathrm{d}}{\mathrm{d}t}B_t = W_t$. Both, Brownian motion and singular white noise, are Gaussian processes and have chaos expansion representations via Fourier-Hermite polynomials with multi-indeces of length one, i.e., belong to the Wiener chaos space of order one.

In general, the chaos expansion of a Gaussian process G_t in $S'(\mathbb{R})\otimes(S)_{-1}$, which belongs to the Wiener chaos space of order one, is given by

$$G_t(\omega) = \sum_{k=1}^{\infty} m_k(t) H_{\varepsilon^{(k)}}(\omega) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} m_{kn}\,\xi_n(t) H_{\varepsilon^{(k)}}(\omega), \tag{12}$$

with coefficients m_k being deterministic generalized functions and $m_{kn}\in\mathbb{R}$ such that the condition

$$\sum_{k=1}^{\infty} \|m_k\|_{-l}^2 (2k)^{-p} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} m_{kn}^2\,(2n)^{-l}\,(2k)^{-p} < \infty$$

holds for some $l, p \in \mathbb{N}_0$. In an analogue way one can consider a generalized Gaussian process $G \in X \otimes (S)_{-1}$ with a Banach space X of the form

$$G = \sum_{k=1}^{\infty} m_k \, H_{\varepsilon^{(k)}},$$

with coefficients $m_k \in X$, e.g. $X = L^2(T; \mathcal{V}^*)$ in Section 3.3, that satisfy

$$\sum_{k=1}^{\infty} \|m_k\|_X^2 (2k)^{-p} < \infty. \tag{13}$$

The Wick product of two stochastic processes is defined in an analogue way as it was defined for random variables in (8) and generalized random variables [LPS11b]. Let F and G be stochastic processes given in their chaos expansion forms $F = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha$ and $G = \sum_{\alpha \in \mathcal{I}} g_\alpha \otimes H_\alpha$, $f_\alpha, g_\alpha \in X$ for all $\alpha \in \mathcal{I}$. Assuming that $f_\alpha \, g_\beta \in X$, for all $\alpha, \beta \in \mathcal{I}$, the *Wick product* $F \lozenge G$ is defined by

$$F \lozenge G = \sum_{\gamma \in \mathcal{I}} \Big(\sum_{\alpha + \beta = \gamma} f_\alpha g_\beta \Big) \otimes H_\gamma. \tag{14}$$

This definition generalizes the one for random variables (8).

3.1.3. *Coordinatewise Operators.* We follow the classification of stochastic operators given in [LPSŽ15] and consider two classes. We say that an operator $\mathcal{A}$ defined on $X \otimes (S)_{-1}$ is a *coordinatewise operator* if it is composed of a family of operators $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ such that for a process

$$u = \sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha \in X \otimes (S)_{-1}, \quad u_\alpha \in X, \ \alpha \in \mathcal{I}$$

it holds

$$\mathcal{A} u = \sum_{\alpha \in \mathcal{I}} A_\alpha u_\alpha \otimes H_\alpha \, .$$

If $A_\alpha = A$ for all $\alpha \in \mathcal{I}$, then the operator $\mathcal{A}$ is called a *simple coordinatewise operator*.

3.2. **Chaos Expansion Approach.** We return to the stochastic equations (6) and (7) where the noise terms are generalized Gaussian stochastic processes as given in (12). Within the next two subsections, we consider the influence of these perturbations. Applying the chaos expansion method, we transform the stochastic systems into deterministic problems, which we solve by induction over the length of the multi-index α. Clearly, we represent all the processes appearing in the stochastic equation by their chaos expansion forms and, since the representation in the Fourier-Hermite polynomial basis is unique, equalize the coefficients. In this section, we assume $\mathcal{K}$ and $\mathcal{B}$ to be simple coordinatewise operators, i.e., for $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha$ we have

$$\mathcal{K}u = \sum_{\alpha\in\mathcal{I}} Ku_\alpha \otimes H_\alpha \quad \text{and} \quad \mathcal{B}u = \sum_{\alpha\in\mathcal{I}} Bu_\alpha \otimes H_\alpha. \tag{15}$$

Note that this implies that $\mathcal{B}^*$ is a simple coordinatewise operator as well. The more general case of coordinatewise operators is considered in Section 4. In the following, we assume that K and B are linear and that they satisfy the assumptions made in Section 2.1. For the right-hand side of the differential equation (6a), namely stochastic process $\mathcal{F}$, and the constraint (6b), namely stochastic process $\mathcal{G}$, we assume that they are given in the chaos expansion forms

$$\mathcal{F} = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha \quad \text{and} \quad \mathcal{G} = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes H_\alpha. \tag{16}$$

Therein, corresponding to the deterministic setting of Section 2.1, the deterministic coefficients satisfy $f_\alpha \in L^2(T;\mathcal{V})$ and $g_\alpha \in H^1(T;\mathcal{Q})$. Furthermore, we assume that for some positive p it holds that

$$\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^2(\mathcal{V}^*)}(2\mathbb{N})^{-p\alpha} < \infty \quad \text{and} \quad \sum_{\alpha\in\mathcal{I}} \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}(2\mathbb{N})^{-p\alpha} < \infty. \tag{17}$$

Remark 3.2. Since the family of spaces $(S)_{-1,-p}$ is monotone, i.e., it holds $(S)_{-1,-p_1} \subset (S)_{-1,-p}$ for $p_1 < p$, we may assume in (17) that all the convergence conditions hold for the same level of singularity p. Clearly, for two different p_1 and p_2 we can take p to be $p = \max\{p_1, p_2\}$ and thus, obtain that generalized stochastic processes satisfies (17) in the biggest space $(S)_{-1,-p}$. In that sense, we use in the sequel always the same level of singularity p.

We seek for solutions u and λ which are *stochastic processes* belonging to $\mathcal{V}\otimes(S)_{-1}$ and $\mathcal{Q}\otimes(S)_{-1}$, respectively. Their chaos expansions are given by

$$u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \quad \text{and} \quad \lambda = \sum_{\alpha\in\mathcal{I}} \lambda_\alpha \otimes H_\alpha.$$

The aim is to calculate the unknown coefficients u_α and λ_α for all $\alpha \in \mathcal{I}$, which then give the overall solutions u and λ. Furthermore, we prove bounds on the solutions, provided that the stated assumptions on the right-hand sides, the initial condition, and the noise terms are fulfilled.

Considering the stochastic DAE equations, we apply at first the chaos expansion method to the initial condition $u(0) = u^0$ and obtain

$$u^0 = \sum_{\alpha\in\mathcal{I}} u_\alpha(0)\, H_\alpha = \sum_{\alpha\in\mathcal{I}} u^0_\alpha\, H_\alpha.$$

Thus, the initial condition reduces to the family of conditions $u_\alpha(0) = u^0_\alpha \in \mathcal{H}$ for every $\alpha \in \mathcal{I}$. In order to achieve consistency (in the case of a sufficiently smooth solution this means $\mathcal{B}u^0 = \mathcal{G}(0)$ the initial data have to be of the form

$$u^0_\alpha = u^0_{B,\alpha} + B^- g_\alpha(0), \qquad \alpha \in \mathcal{I} \tag{18}$$

with an arbitrary $u^0_{B,\alpha}$ from the closure of the kernel of B in $\mathcal{H}$ and B^- denoting the right-inverse of the operator B, cf. Remark 2.1. Finally, we assume for the initial data that

$$\sum_{\alpha\in\mathcal{I}} \|u^0_{B,\alpha}\|^2_{\mathcal{H}}(2\mathbb{N})^{-p\alpha} < \infty. \tag{19}$$

3.3. **Noise in the Differential Equation.** Consider system (6) with a stochastic perturbation given in the form of a generalized Gaussian stochastic process in the Wiener chaos space of order one as in (12), i.e., we consider the initial value problem

$$\begin{array}{rcrcrcrcl} \dot u(t) & + & \mathcal{K}u(t) & + & \mathcal{B}^*\lambda(t) & = & \mathcal{F}(t) & + & G_t, \\ & & \mathcal{B}u(t) & & & = & \mathcal{G}(t), & & \end{array} \qquad u(0) = u^0_{\mathcal{B}} + \mathcal{B}^-\mathcal{G}(0). \tag{20}$$

Processes $\mathcal{F}$ and $\mathcal{G}$ are given by (16) such that they satisfy (17). Gaussian process G_t is of the form (12) and its variance may be infinite. We represent all the processes in (20) in their chaos expansion forms, apply (15) and thus reduce it to an infinite triangular system of deterministic initial value problems, which can be solved recursively over the length of multi-index α. We obtain the system

$$\begin{aligned} \sum_{\alpha\in\mathcal{I}} \big(\dot u_\alpha(t) + Ku_\alpha(t) + B^*\lambda_\alpha(t)\big)\, H_\alpha(\omega) &= \sum_{\alpha\in\mathcal{I}} f_\alpha(t)\, H_\alpha(\omega) + \sum_{k\in\mathbb{N}} m_k(t)\, H_\alpha(\omega), \\ \sum_{\alpha\in\mathcal{I}} Bu_\alpha(t)\, H_\alpha(\omega) &= \sum_{\alpha\in\mathcal{I}} g_\alpha(t)\, H_\alpha(\omega) \end{aligned}$$

with $u(0) = u^0$, i.e., initial conditions (18) that satisfy (19). Thus,

- for $|\alpha| = 0$, i.e., for $\alpha = \mathbf{0} = (0, 0, ...)$, we have to solve

$$\begin{array}{rcrcrcl} \dot u_{\mathbf{0}}(t) & + & Ku_{\mathbf{0}}(t) & + & B^*\lambda_{\mathbf{0}}(t) & = & f_{\mathbf{0}}(t), \\ & & Bu_{\mathbf{0}}(t) & & & = & g_{\mathbf{0}}(t), \end{array} \qquad u_{\mathbf{0}} = u^0_{B,\mathbf{0}} + B^- g_{\mathbf{0}}(0). \tag{21}$$

  Note that system (21) represents a deterministic problem of the form (1), where F and G from (1) are equal to the zero-th components $f_{\mathbf{0}}$ and $g_{\mathbf{0}}$, respectively. Moreover, (21) is a system obtained by taking the expectation of system (20). The assumptions on the operators and right-hand sides $f_{\mathbf{0}} \in L^2(T;\mathcal{V}^*)$, $g_{\mathbf{0}} \in H^1(T;\mathcal{Q}^*)$ imply the existence of a solution $u_{\mathbf{0}} \in L^2(T;\mathcal{V})$, $\lambda_{\mathbf{0}} \in L^2(T;\mathcal{Q})$.

- for $|\alpha| = 1$, i.e., for $\alpha = \varepsilon^{(k)}$, $k \in \mathbb{N}$, we obtain the system

$$\begin{array}{rcrcrcl} \dot u_{\varepsilon^{(k)}}(t) & + & Ku_{\varepsilon^{(k)}}(t) & + & B^*\lambda_{\varepsilon^{(k)}}(t) & = & f_{\varepsilon^{(k)}}(t) + m_k(t), \\ & & Bu_{\varepsilon^{(k)}}(t) & & & = & g_{\varepsilon^{(k)}}(t) \end{array} \tag{22}$$

  with initial condition $u_{\varepsilon^{(k)}}(0) = u^0_{B,\varepsilon^{(k)}} + B^- g_{\varepsilon^{(k)}}(0)$. Also for each $k \in \mathbb{N}$ the system (22) is a deterministic initial value problem of the form (1), with the choice $F = f_{\varepsilon^{(k)}} + m_k$ and $G = g_{\varepsilon^{(k)}}$.

- for $|\alpha| > 1$, we finally solve

$$\begin{array}{ccccccc} \dot{u}_\alpha(t) & + & Ku_\alpha(t) & + & B^*\lambda_\alpha(t) & = & f_\alpha(t), \\ & & Bu_\alpha(t) & & & = & g_\alpha(t), \end{array} \quad u_\alpha(0) = u^0_{B,\alpha} + B^- g_\alpha(0). \tag{23}$$

  Again, system (23) is a deterministic operator DAE, which can be solved in the same manner as the system (21).

From system (21) we obtain $u_{\mathbf{0}}$ and $\lambda_{\mathbf{0}}$. Further, we obtain from (22) the coefficients u_α and λ_α for $|\alpha| = 1$ and from (23) the remaining coefficients. Note that all these systems may be solved in parallel.

Remark 3.3. If $\mathcal{F}$ in (20) is a deterministic function, it can be represented as $\mathcal{F} = f_{\mathbf{0}}$, since the coefficients $f_\alpha = 0$ for all $|\alpha| > 0$. In this case, systems (22) and (23) further simplify.

As the last step of the analysis, we prove the convergence of the obtained solution in the space of Kondratiev generalized stochastic processes, i.e., we prove that $\|u\|^2_{\mathcal{V}\otimes(S)_{-1}} < \infty$, for $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha$. For this, we need the following estimate which shows that the chaos expansion approach leads to a bounded solution even with noise terms.

Theorem 3.4. *Let $\mathcal{K}$ and $\mathcal{B}$ be simple coordinatewise operators with corresponding deterministic operators K and B which satisfy the assumptions stated in Section 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be the stochastic processes defined in* (16) *which satisfy the bounds* (17) *and let G_t be the Gaussian noise term such that the estimate* (13) *holds. Then, for any consistent initial data that satisfies* (19) *there exists a unique solution $u \in \mathcal{V}\otimes(S)_{-1}$ of the stochastic DAE* (20) *such that*

$$\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_{L^2(\mathcal{V})} (2\mathbb{N})^{-p\alpha} < \infty.$$

Proof. The estimate from Theorem 2.2 can be applied to the deterministic operator DAEs (21)-(23) for the coefficients u_α. For $u_{\mathbf{0}}$ we obtain by system (21) and by (2) we obtain the estimate

$$\|u_{\mathbf{0}}\|^2_{L^2(\mathcal{V})} \lesssim \|u^0_{B,\mathbf{0}}\|^2_{\mathcal{H}} + \|f_{\mathbf{0}}\|^2_{L^2(\mathcal{V}^*)} + \|g_{\mathbf{0}}\|^2_{H^1(\mathcal{Q}^*)}.$$

Similarly, for $|\alpha| = 1$ and $|\alpha| > 1$, by the systems (22) and (23) respectively, we obtain the estimates

$$\begin{aligned} \|u_{\varepsilon^{(k)}}\|^2_{L^2(\mathcal{V})} &\lesssim \|u^0_{B,\varepsilon^{(k)}}\|^2_{\mathcal{H}} + \|f_{\varepsilon^{(k)}} + m_k\|^2_{L^2(\mathcal{V}^*)} + \|g_{\varepsilon^{(k)}}\|^2_{H^1(\mathcal{Q}^*)}, \quad k \in \mathbb{N} \\ \|u_\alpha\|^2_{L^2(\mathcal{V})} &\lesssim \|u^0_{B,\alpha}\|^2_{\mathcal{H}} + \|f_\alpha\|^2_{L^2(\mathcal{V}^*)} + \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}, \quad |\alpha| > 1. \end{aligned}$$

Note that the involved constants are equal for all estimates because of the assumed simple coordinatewise operators. Summarizing the results, we obtain for the solution u,

$$\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_{L^2(\mathcal{V})}(2\mathbb{N})^{-p\alpha} \lesssim \sum_{\alpha\in\mathcal{I}} \|u^0_{B,\alpha}\|^2_{\mathcal{H}}(2\mathbb{N})^{-p\alpha} + \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^2(\mathcal{V}^*)}(2\mathbb{N})^{-p\alpha} + \sum_{k=1}^{\infty} \|m_k\|^2_{L^2(\mathcal{V}^*)}(2k)^{-p} + \sum_{\alpha\in\mathcal{I}} \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}(2\mathbb{N})^{-p\alpha} < \infty,$$

where we have used the linearity, the triangular inequality, and the relation $(2\mathbb{N})^{\varepsilon^{(k)}} = (2k)$, $k \in \mathbb{N}$. The assumptions (13), (17), and (19) show that the right-hand side is bounded for p and thus, completes the proof the theorem. □

Remark 3.5. A similar result may be given for the Lagrange multiplier λ. However, this requires more smoothness of the data in the form of $f_\alpha \in L^2(T;\mathcal{H}^*)$ and $u^0_{B,\alpha} \in \mathcal{V}$, cf. [Alt15].

Remark 3.6. One may also consider a more general form of the Gaussian noise G_t, i.e., $G_t(\omega) = \sum_{|\alpha|>0} m_\alpha(t)\, H_\alpha(\omega)$, where G_t has also non-zero coefficients of order greater than one. The solution for this case can be provided in the same manner as in the presented case for a Gaussian noise in the Wiener chaos space of order one.

3.4. **Noise in the Constraint Equation.** Consider system (7) with noise given in the form of two Gaussian white noise processes $G^{(1)}$ and $G^{(2)}$. More precisely, we consider the initial value problem

$$\begin{array}{ccccccc} \dot u(t) & + & \mathcal{K}u(t) & + & \mathcal{B}^*\lambda(t) & = & \mathcal{F}(t) + G^{(1)}_t, \\ & & \mathcal{B}u(t) & & & = & \mathcal{G}(t) + G^{(2)}_t \end{array} \tag{24}$$

with the initial condition $u(0) = u^0$. Note that the initial data u^0 has to be consistent again. Here, the consistency condition includes the perturbation $G^{(2)}_t$ such that the (consistent) initial data of the unperturbed problem may not be consistent in this case. As before, we assume

$$G^{(1)}_t(\omega) = \sum_{k=1}^{\infty} m^{(1)}_k(t)\, H_{\varepsilon^{(k)}}(\omega) \quad \text{and} \quad G^{(2)}_t(\omega) = \sum_{k=1}^{\infty} m^{(2)}_k(t)\, H_{\varepsilon^{(k)}}(\omega).$$

Note that $m^{(1)}_k \in L^2(T;\mathcal{V}^*)$, whereas $m^{(2)}_k \in H^1(T;\mathcal{Q}^*)$. We still assume the operators $\mathcal{K}$ and $\mathcal{B}$ to be simple coordinatewise operators and consider right-hand sides $\mathcal{F}$ and $\mathcal{G}$ as in (16). Then, system (24) reduces to the following deterministic systems:

- for $|\alpha| = 0$, i.e., for $\alpha = \mathbf{0} = (0,0,...)$, we obtain

$$\begin{array}{ccccccc} \dot u_{\mathbf{0}}(t) & + & Ku_{\mathbf{0}}(t) & + & B^*\lambda_{\mathbf{0}}(t) & = & f_{\mathbf{0}}(t), \\ & & Bu_{\mathbf{0}}(t) & & & = & g_{\mathbf{0}}(t), \end{array} \qquad u_{\mathbf{0}} = u^0_{\mathbf{0}}. \tag{25}$$

- for $|\alpha| = 1$, i.e., for $\alpha = \varepsilon^{(k)}$, $k \in \mathbb{N}$, we have

$$\begin{aligned} \dot{u}_{\varepsilon^{(k)}}(t) \;+\; Ku_{\varepsilon^{(k)}}(t) \;+\; B^*\lambda_{\varepsilon^{(k)}}(t) &= f_{\varepsilon^{(k)}}(t) + m_k^{(1)}(t), \\ Bu_{\varepsilon^{(k)}}(t) &= g_{\varepsilon^{(k)}}(t) + m_k^{(2)}(t), \end{aligned} \tag{26}$$

  with initial condition $u_{\varepsilon^{(k)}}(0) = u^0_{\varepsilon^{(k)}}$.
- for $|\alpha| > 1$, we solve

$$\begin{aligned} \dot{u}_\alpha(t) \;+\; Ku_\alpha(t) \;+\; B^*\lambda_\alpha(t) &= f_\alpha(t), \\ Bu_\alpha(t) &= g_\alpha(t), \end{aligned} \qquad u_\alpha(0) = u^0_\alpha. \tag{27}$$

We emphasize that the DAEs (25)-(27) can be solved in parallel again.

However, equation (26) is a deterministic DAE with a perturbation in the constraint, cf. Section 2.2 with $\theta = m_k^{(2)}$. Estimate (3) shows that this results in instabilities such that the stochastic truncation cannot converge. Thus, we have to consider the regularized formulation.

3.5. **Regularization.** We have seen that the solution behaves very sensitive to perturbations in the constraint equation. As in the deterministic case in Section 2.3, we need a regularization. The extended system with stochastic noise terms has the form

$$\dot{u}_1(t) + v_2(t) + \mathcal{K}\big(u_1(t) + u_2(t)\big) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) + G_t^{(1)} \quad \text{in } \mathcal{V}^*, \tag{28a}$$

$$\mathcal{B}u_2(t) = \mathcal{G}(t) + G_t^{(2)} \quad \text{in } \mathcal{Q}^*, \tag{28b}$$

$$\mathcal{B}v_2(t) = \dot{\mathcal{G}}(t) + G_t^{(3)} \quad \text{in } \mathcal{Q}^* \tag{28c}$$

Note that, because of the extension of the system, we consider a third perturbation $G_t^{(3)}$ in (28). The chaos expansion approach leads again to a system of deterministic operator DAEs. Since the perturbations have zero mean and are of order one only, we only consider the case with $\alpha = \varepsilon^{(k)}$ which leads to

$$\begin{aligned} \dot{u}_{1,\varepsilon^{(k)}} + v_{2,\varepsilon^{(k)}} + K\big(u_{1,\varepsilon^{(k)}} + u_{2,\varepsilon^{(k)}}\big) + B^*\lambda_{\varepsilon^{(k)}} &= f_{\varepsilon^{(k)}} + m_k^{(1)}, \\ Bu_{2,\varepsilon^{(k)}} &= g_{\varepsilon^{(k)}} + m_k^{(2)}, \\ Bv_{2,\varepsilon^{(k)}} &= \dot{g}_{\varepsilon^{(k)}} + m_k^{(3)}. \end{aligned}$$

Recall that this formulation allows an estimate of the coefficients $u_{1,\varepsilon^{(k)}}$ without the derivatives of the perturbations, cf. equation (5). This then leads to a uniform bound of the solution u_1, u_2, similarly as in Theorem 3.4. Furthermore, the regularization solves the problem of finding consistent initial data. Here, the condition reads $u_{1,\varepsilon^{(k)}}(0) = u^0_{1,\varepsilon^{(k)}}$ and thus, does not depend on the perturbations.

3.6. **Convergence of the Truncated Expansion.** In practice, only the coefficients u_α, λ_α for multi-indices of a maximal length P, i.e., up to a certain order P, can be computed. Thus, the infinite sum has to be truncated such that a given tolerance is achieved. Clearly, denoting by $\tilde{u}$ the approximated (truncated) solution and u_r the truncation error, i.e.,

$$\tilde{u} = \sum_{|\alpha|\leq P} u_\alpha \otimes H_\alpha \qquad \text{and} \qquad u_r = \sum_{|\alpha|>P} u_\alpha \otimes H_\alpha,$$

we can represented the process $u = \tilde{u}+u_r$. In applications, one computes u_α for $|\alpha| < P$ such that the desired bound $\|u_r\|_{\mathcal{V}\otimes L^2(\Omega)} = \|u-\tilde{u}\|_{\mathcal{V}\otimes L^2(\Omega)} \leq \epsilon$ is carried out. Convergence in L^2 is attained if the sum is truncated properly [XK02, LR06, KFY11]. The truncation procedure relies on the regularity of the solution, the type of noise, and the discretization method for solving the deterministic equations involved, see e.g. [BL12] for finite element methods.

Similar results for specific equations can be found, e.g., in [ANZ98, BSDDM05, BBP13]. A general truncation method is stated in [KFY11]. Although the same ideas can be applied to our equations once we have performed the regularization to the deterministic system (such that operator DAE is well-posed in each level), the convergence of the truncated expansion is, in general, guaranteed by the stability result of Theorem 3.4. A study on the analytical dependence of the solution with respect to the noise has to be done in order to test this approach numerically. However, this is not the main focus of this work.

The main steps of the numerical approach are sketched in Algorithm 3.1.

Algorithm 3.1 Main steps of the numerical approximation

1: Define a finite dimensional approximations of the infinite dimensional Gaussian processes.
2: Choose a finite set of polynomials H_α and truncate the random series.
3: Regularize the operator DAEs if necessary.
4: Compute/approximate the solutions of the resulting systems.
5: Generate H_α to compute the approximate solution.
6: Compute the approximate statistics of the solution from the obtained coefficients.

4. More General Cases

This section is devoted to two generalizations of the previous models. First, we consider general coordinatewise operators instead of simple coordinatewise operators as in the previous section. Thus, following the definition from Section 3.1.3, we allow the operators $\mathcal{K}$ and $\mathcal{B}$ to be composed out from families of deterministic operators $\{K_\alpha\}_{\alpha\in\mathcal{I}}$ and $\{B_\alpha\}_{\alpha\in\mathcal{I}}$, respectively, which may not be the same for all multi-indices. Second, we replace the Gaussian noise term by a stochastic integral term. The mean dynamics will remain unchanged, while the perturbation in the differential equation will be given in the form of a stochastic convolution.

4.1. **Coordinatewise Operators.** In the given application, we consider the coordinatewise operators $\mathcal{K}$, $\mathcal{B}$ with

$$\mathcal{K}u = \sum_{\alpha\in\mathcal{I}} K_\alpha u_\alpha \, H_\alpha \qquad \text{and} \qquad \mathcal{B}u = \sum_{\alpha\in\mathcal{I}} B_\alpha u_\alpha \, H_\alpha.$$

Therein, the deterministic operators satisfy $K_\alpha \colon \mathcal{V} \to \mathcal{V}^*$ and $B_\alpha \colon \mathcal{V} \to \mathcal{Q}^*$ for all $\alpha \in \mathcal{I}$. This also implies that $\mathcal{B}^*$ is a coordinatewise operator, which corresponds to the family of operators $\{B_\alpha^*\}_{\alpha\in\mathcal{I}}$ such that for $\lambda = \sum_{\alpha\in\mathcal{I}} \lambda_\alpha H_\alpha$ it holds

$$\mathcal{B}^*\lambda = \sum_{\alpha\in\mathcal{I}} B_\alpha^* \lambda_\alpha \, H_\alpha.$$

The chaos expansion method applied to the system with the Gaussian noise in the constraint equation (24) then leads to the following deterministic systems:

- for $|\alpha| = 0$, i.e., for $\alpha = \mathbf{0}$,

$$\begin{array}{rcl} \dot u_{\mathbf{0}}(t) + K_{\mathbf{0}} u_{\mathbf{0}}(t) + B_{\mathbf{0}}^* \lambda_{\mathbf{0}}(t) &=& f_{\mathbf{0}}(t), \\ B_{\mathbf{0}} u_{\mathbf{0}}(t) &=& g_{\mathbf{0}}(t), \end{array} \qquad u_{\mathbf{0}} = u_{\mathbf{0}}^0.$$

- for $|\alpha| = 1$, i.e., for $\alpha = \varepsilon^{(k)}$, $k \in \mathbb{N}$,

$$\begin{array}{rcl} \dot u_{\varepsilon^{(k)}}(t) + K_{\varepsilon^{(k)}} u_{\varepsilon^{(k)}}(t) + B_{\varepsilon^{(k)}}^* \lambda_{\varepsilon^{(k)}}(t) &=& f_{\varepsilon^{(k)}}(t) + m_k^{(1)}(t), \\ B_{\varepsilon^{(k)}} u_{\varepsilon^{(k)}}(t) &=& g_{\varepsilon^{(k)}}(t) + m_k^{(2)}(t), \end{array}$$

  with $u_{\varepsilon^{(k)}}(0) = u_{\varepsilon^{(k)}}^0$.

- for the remaining $|\alpha| > 1$,

$$\begin{array}{rcl} \dot u_\alpha(t) + K_\alpha u_\alpha(t) + B_\alpha^* \lambda_\alpha(t) &=& f_\alpha(t), \\ B_\alpha u_\alpha(t) &=& g_\alpha(t), \end{array} \qquad u_\alpha(0) = u_\alpha^0.$$

As before, these systems may be solved in parallel.

4.2. **Stochastic Convolution.** Consider problem (7), where the stochastic disturbance is given in terms of a stochastic convolution term. More precisely, we are dealing with the problem of the form

$$\begin{array}{rcl} \dot u(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) &=& \mathcal{F}(t) + \delta(\mathcal{C}u), \\ \mathcal{B}u(t) &=& \mathcal{G}(t) + G_t \end{array} \tag{29}$$

with a consistent initial condition $u(0) = u^0$. Operators $\mathcal{K}$ and $\mathcal{B}$ are coordinatewise operators, processes $\mathcal{F}$ and $\mathcal{G}$ are stochastic processes as before, and G_t is a Gaussian noise as in (12). The term $\delta(\mathcal{C}u)$ stays for an *Itô-Skorokhod stochastic integral* δ. The Skorokhod integral is a generalization of the Itô integral for processes which are not necessarily adapted. The fundamental theorem of stochastic calculus connects the Itô-Skorokhod integral with the Wick product by

$$\delta(\mathcal{C}u) = \int_{\mathbb{R}} \mathcal{C}u \, \mathrm{d}B_t = \int_{\mathbb{R}} \mathcal{C}u \Diamond W_t \; \mathrm{d}t, \tag{30}$$

where the integral on the right-hand side of the relation is the Riemann integral and the derivative is taken in sense of distributions [HØUZ10]. We assume that the operator $\mathcal{C}$ is a linear coordinatewise operator composed of a family of bounded operators $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ such that $\mathcal{C}u$ is integrable in the Skorokhod sense [HØUZ10]. The stochastic integral is the Itô-Skorokhod integral and it exists not only for processes adapted to the filtration but also for non-adapted ones. It is equal to the Riemann integral of a process $\mathcal{C}u$, stochastically convoluted with a singular white noise.

Operator δ is the adjoint operator of the Malliavin derivative $\mathbb{D}$. Their composition is known as the Ornstein-Uhlenbeck operator $\mathcal{R}$ which is a self-adjoint operator. These operators are the main operators of an infinite dimensional stochastic calculus of variations called the *Malliavin calculus* [Nua06]. We consider these operators in Section 5.

For adapted processes v the Itô integral and the Skorokhod integral coincide, i.e., $I(v) = \delta(v)$ and because of this fact we are referring to the stochastic integral as the Itô-Skorokhod integral. Applying the definition of the Wick product (14) to the chaos expansion representation (9) of a process v and the representation (11) of a singular white noise in the definition (30) of $\delta(v)$, we obtain a chaos expansion representation of the Skorokhod integral. Clearly, for $v = \sum_{\alpha\in\mathcal{I}} v_\alpha(t) H_\alpha$ we have

$$v \Diamond W_t = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v_\alpha(t) \; \xi_k(t) \, H_{\alpha+\varepsilon^{(k)}}(\omega),$$

and thus, it holds that

$$\delta(v) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v_{\alpha,k} \, H_{\alpha+\varepsilon^{(k)}}(\omega). \tag{31}$$

Therein, we have used that $v_\alpha(t) = \sum_{k\in\mathbb{N}} v_{\alpha,k} \, \xi_k(t) \in L^2(\mathbb{R})$ is the chaos expansion representation of v_α in the orthonormal Hermite functions basis with coefficients $v_{\alpha,k}$. Furthermore, we are able to represent stochastic perturbations appearing in the stochastic equation (29) explicitly. Note that $\delta(v)$ belongs to the Wiener chaos space of higher order than v, see also [HØUZ10, LS15].

We say that a $L^2(\mathbb{R})$-valued stochastic process $v = \sum_{\alpha\in\mathcal{I}} v_\alpha \, H_\alpha$, with coefficients $v_\alpha(t) = \sum_{k\in\mathbb{N}} v_{\alpha,k} \, \xi_k(t)$, $v_{\alpha,k} \in \mathbb{R}$, for all $\alpha \in \mathcal{I}$ is integrable in the Itô-Skorokhod sense if it holds

$$\sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v_{\alpha,k}^2 \; |\alpha| \; \alpha! \, < \, \infty. \tag{32}$$

Then, the Itô-Skorokhod integral of v is of the form (31) and we write $v \in Dom(\delta)$.

Theorem 4.1. *The Skorokhod integral δ of an $L^2(\mathbb{R})$-valued stochastic process is a linear and continuous mapping*

$$\delta \colon Dom(\delta) \;\to\; L^2(\Omega).$$

Proof. Let v satisfy condition (32). Then we have

$$\begin{aligned}\|\delta(v)\|^2_{L^2(\Omega)} &= \Big\| \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v_{\alpha,k}\; H_{\alpha+\varepsilon^{(k)}} \Big\|^2_{L^2(\Omega)} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v^2_{\alpha,k}\,(\alpha+\varepsilon^{(k)})! \\ &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v^2_{\alpha,k}\;(\alpha_k+1)\;\alpha! \le c \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v^2_{\alpha,k}\,|\alpha|\;\alpha! < \infty,\end{aligned}$$

where we used $(\alpha+\varepsilon^{(k)})! = (\alpha_k+1)\,\alpha!$, for $\alpha\in\mathcal{I}$, $k\in\mathbb{N}$. □

A detailed analysis of the domain and the range of operators of the Malliavin calculus in spaces of stochastic distributions can be found in [LS15].

Applying the polynomial chaos expansion method to problem (29), we obtain the following systems of deterministic operator DAEs:

- for $|\alpha| = 0$, i.e., for $\alpha = \mathbf{0}$

$$\begin{array}{rcccccl}\dot u_{\mathbf{0}}(t) &+& K_{\mathbf{0}} u_{\mathbf{0}}(t) &+& B^*_{\mathbf{0}}\lambda_{\mathbf{0}}(t) &=& f_0(t),\\ && B_{\mathbf{0}} u_{\mathbf{0}}(t) &&&=& g_0(t).\end{array} \tag{33}$$

- for $|\alpha| = 1$, i.e., for $\alpha = \varepsilon^{(k)}$, $k\in\mathbb{N}$

$$\begin{array}{rcccccl}\dot u_{\varepsilon^{(k)}}(t) &+& K_{\varepsilon^{(k)}}\,u_{\varepsilon^{(k)}}(t) &+& B^*_{\varepsilon^{(k)}}\,\lambda_{\varepsilon^{(k)}}(t) &=& f_{\varepsilon^{(k)}} + (C_{\mathbf{0}}\,u_{\mathbf{0}})_k,\\ && B_{\varepsilon^{(k)}}\,u_{\varepsilon^{(k)}}(t) &&&=& g_{\varepsilon^{(k)}}(t) + m_k(t).\end{array} \tag{34}$$

- for $|\alpha| > 1$

$$\begin{array}{rcccccl}\dot u_{\alpha}(t) &+& K_{\alpha}\,u_{\alpha}(t) &+& B^*_{\alpha}\,\lambda_{\alpha}(t) &=& f_{\alpha}(t) + \sum\limits_{k\in\mathbb{N}} (C_{\alpha-\varepsilon^{(k)}}\,u_{\alpha-\varepsilon^{(k)}})_k,\\ && B_{\alpha}\,u_{\alpha}(t) &&&=& g_{\alpha}(t).\end{array} \tag{35}$$

Note that the corresponding initial conditions are given as in systems (25)-(27). The term $(Cu_{\mathbf{0}})_k$ appearing in (34) represents the kth component of the action of the operator $C_{\mathbf{0}}$ on the solution $u_{\mathbf{0}}$, obtained in the previous step, i.e. on the solution of the system (33). Similarly, the term $(C_{\alpha-\varepsilon^{(k)}}\,u_{\alpha-\varepsilon^{(k)}})_k$ from (35) represents the kth coefficient obtained by the action of the operator $C_{\alpha-\varepsilon^{(k)}}$ on $u_{\alpha-\varepsilon^{(k)}}$ calculated in the previous steps. We use the convention that $C_{\alpha-\varepsilon^{(k)}}$ exists only for those $\alpha\in\mathcal{I}$ for which $\alpha_k \ge 1$. Therefore, the sum $\sum_{k\in\mathbb{N}}(C_{\alpha-\varepsilon^{(k)}}\,u_{\alpha-\varepsilon^{(k)}})_k$ has as many summands as the multi-index α has non-zero components. For example, for $\alpha = (2,0,1,0,0,...)$ with two non-zero components $\alpha_1 = 2$ and $\alpha_3 = 1$, the sum has two terms $(C_{(1,0,1,0,0,...)}u_{(1,0,1,0,0,...)})_1$ and $(C_{(2,0,0,0,0,...)}u_{(2,0,0,0,0,...)})_3$.

We point out that, in contrast to the previous cases, in (33)-(35) the unknown coefficients are obtained by recursion. Thus, in order to calculate u_α, we need the solutions u_β for $\beta<\alpha$ from the previous steps. Also this case can be found in applications, see for example [LR06, LPS11a, KFY11, LPS15].

The convergence of the obtained solution in the Kondratiev space of generalized processes can be proven in the similar manner as in Theorem 3.4. We prove the theorem for the case of the stochastic operator DAE (29) with the stochastic perturbations given in the terms of stochastic convolution and with no disturbance in the constrained equation. We need to assume additionally the uniform boundness of the family of operators C_α, $\alpha \in \mathcal{I}$.

Theorem 4.2. *Let $\mathcal{K}$ and $\mathcal{B}$ be coordinatewise operators with corresponding families of deterministic operators $\{K_\alpha\}_{\alpha\in\mathcal{I}}$ and $\{B_\alpha\}_{\alpha\in\mathcal{I}}$ which satisfy the assumptions stated in Section 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be the stochastic processes defined by* (16)*, which satisfy the bounds* (17) *and let $\mathcal{C}$ be a coordinatewise operator that corresponds to a family of deterministic operators $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, $C_\alpha : \mathcal{V} \to \mathcal{V}^*$ for $\alpha \in \mathcal{I}$ that satisfy*

$$\|C_\alpha\| \leq d < 1, \qquad \text{for all } \alpha \in \mathcal{I}. \tag{36}$$

Then, for any consistent initial data that satisfies (19) *there exists a unique solution $u \in \mathcal{V} \otimes (S)_{-1}$ of the stochastic operator DAE*

$$\begin{array}{rcccccccc} \dot{u}(t) & + & \mathcal{K}u(t) & + & \mathcal{B}^*\lambda(t) & = & \mathcal{F}(t) & + & \delta(\mathcal{C}u), \\ & & \mathcal{B}u(t) & & & = & \mathcal{G}(t) & & \end{array} \tag{37}$$

such that it holds

$$\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_{L^2(\mathcal{V})} (2\mathbb{N})^{-p\alpha} < \infty.$$

Proof. We are looking for the solution in the chaos expansion form (9). By applying the estimate (2) from Theorem 2.2 to the deterministic operator DAEs (33)-(35) for the coefficients u_α in each step recursively, we prove the convergence estimate.

For $|\alpha| = 0$, from the system (33) and by (2) we estimate the coefficient $u_{\mathbf{0}}$, i.e.,

$$\|u_{\mathbf{0}}\|^2_{L^2(\mathcal{V})} \lesssim \|u^0_{B,\mathbf{0}}\|^2_{\mathcal{H}} + \|f_{\mathbf{0}}\|^2_{L^2(\mathcal{V}^*)} + \|g_{\mathbf{0}}\|^2_{H^1(\mathcal{Q}^*)}.$$

For $|\alpha| = 1$, i.e. for $\alpha = \varepsilon^{(k)}$, $k \in \mathbb{N}$ by the systems

$$\begin{array}{rcccccl} \dot{u}_{\varepsilon^{(k)}}(t) & + & K_{\varepsilon^{(k)}}\, u_{\varepsilon^{(k)}}(t) & + & B^*_{\varepsilon^{(k)}}\, \lambda_{\varepsilon^{(k)}}(t) & = & f_{\varepsilon^{(k)}} + (C_{\mathbf{0}}\, u_{\mathbf{0}})_k, \\ & & B_{\varepsilon^{(k)}}\, u_{\varepsilon^{(k)}}(t) & & & = & g_{\varepsilon^{(k)}}(t) \end{array}$$

we obtain the estimate

$$\|u_{\varepsilon^{(k)}}\|^2_{L^2(\mathcal{V})} \lesssim \|u^0_{B,\varepsilon^{(k)}}\|^2_{\mathcal{H}} + \|f_{\varepsilon^{(k)}} + (C_{\mathbf{0}}u_{\mathbf{0}})_k\|^2_{L^2(\mathcal{V}^*)} + \|g_{\varepsilon^{(k)}}\|^2_{H^1(\mathcal{Q}^*)}, \quad k \in \mathbb{N},$$

while for $|\alpha| > 1$ from (35) we obtain

$$\|u_\alpha\|^2_{L^2(\mathcal{V})} \lesssim \|u^0_{B,\alpha}\|^2_{\mathcal{H}} + \|f_\alpha + \sum_{k\in\mathbb{N}} (C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}})_k\|^2_{L^2(\mathcal{V}^*)} + \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}, \ \ |\alpha| > 1.$$

We sum up all the coefficients and apply the obtained estimates. Thus, we get

(38)
$$\begin{aligned}\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_{L^2(\mathcal{V})}(2\mathbb{N})^{-p\alpha} \lesssim \sum_{\alpha\in\mathcal{I}} \|u^0_{B,\alpha}\|^2_{\mathcal{H}}(2\mathbb{N})^{-p\alpha} + \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^2(\mathcal{V}^*)}(2\mathbb{N})^{-p\alpha} \\ + \sum_{\alpha\in\mathcal{I}} \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}(2\mathbb{N})^{-p\alpha} \\ + \sum_{\alpha\in\mathcal{I},|\alpha|>0} \Big(\sum_{k\in\mathbb{N}} (C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}})_k\Big)^2 (2\mathbb{N})^{-p\alpha}.\end{aligned}$$

From the assumptions (17) and (19) it follows that the first three summands on the right hand side of (38) are finite. The last term can be estimated in the following way

$$\begin{aligned}&\sum_{\alpha\in\mathcal{I},|\alpha|>0} \Big(\sum_{k\in\mathbb{N}} (C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}})_k\Big)^2 (2\mathbb{N})^{-p\alpha} \\ &\quad\le \sum_{\alpha\in\mathcal{I},|\alpha|>0} \sum_{k\in\mathbb{N}} \|C_{\alpha-\varepsilon^{(k)}} u_{\alpha-\varepsilon^{(k)}}\|^2 (2\mathbb{N})^{-p\alpha} \\ &\quad= \sum_{\beta\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|C_\beta u_\beta\|^2 (2\mathbb{N})^{-p(\beta+\varepsilon^{(k)})} \\ &\quad\le M\cdot \sum_{\beta\in\mathcal{I}} \|C_\beta u_\beta\|^2 (2\mathbb{N})^{-p\beta} = M\cdot d\cdot \|u\|^2_{L^2(\mathcal{V})\otimes(S)_{-1,-p}},\end{aligned}$$

where we have first used the triangular inequality, then the substitution $\alpha = \beta + \varepsilon^{(k)}$, the property

$$(2\mathbb{N})^{\beta+\varepsilon^{(k)}} = (2\mathbb{N})^{\beta}\cdot(2\mathbb{N})^{\varepsilon^{(k)}} = (2\mathbb{N})^{\beta}\cdot(2k),$$

then the uniformly boundness (36) of the family $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, and at last the sum $M = \sum_{k\in\mathbb{N}}(2k)^{-p} < \infty$ for $p>1$. Finally, putting everything together in (38), we obtain

$$\begin{aligned}\|u\|^2_{L^2(\mathcal{V})\otimes(S)_{-1}} \le c\Big(\sum_{\alpha\in\mathcal{I}} \|u^0_{B,\alpha}\|^2_{\mathcal{H}}(2\mathbb{N})^{-p\alpha} + \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^2(\mathcal{V}^*)}(2\mathbb{N})^{-p\alpha} \\ + \sum_{\alpha\in\mathcal{I}} \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}(2\mathbb{N})^{-p\alpha}\Big) + Md\|u\|^2_{L^2(\mathcal{V})\otimes(S)_{-1}}.\end{aligned}$$

We group two summands with the term $\|u\|^2_{L^2(\mathcal{V})\otimes(S)_{-1}}$ on the left hand side of the inequality and obtain

$$\begin{aligned}\|u\|^2_{L^2(\mathcal{V})\otimes(S)_{-1}}(1-Md) \lesssim \sum_{\alpha\in\mathcal{I}} \|u^0_{B,\alpha}\|^2_{\mathcal{H}}(2\mathbb{N})^{-p\alpha} + \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^2(\mathcal{V}^*)}(2\mathbb{N})^{-p\alpha} \\ + \sum_{k=1}^{\infty} \|m_k\|^2_{L^2(\mathcal{V}^*)}(2k)^{-p} + \sum_{\alpha\in\mathcal{I}} \|g_\alpha\|^2_{H^1(\mathcal{Q}^*)}(2\mathbb{N})^{-p\alpha}.\end{aligned}$$

Since (36) holds, one can choose p large enough so that $1 - Md > 0$. With this we prove that $\|u\|^2_{L^2(\mathcal{V})\otimes(S)_{-1}}$ is finite and thus, complete the proof the theorem. $\square$

If the disturbance in the constrained equation of the stochastic operator DAE (29) is considered, an analogous to Theorem 4.2 can be proved. Note that following the results from Section 3.4 and Section 3.5, one can formulate the regularized stochastic operator DAE, equivalent to the initial one (29), and therefore prove the convergence of the solution.

5. The Fully Stochastic Case

The deterministic problem (1) which involves deterministic functions and operators can be generalized for stochastic functions and specific stochastic operators in the following way. We focus on semi-explicit systems that include the stochastic operators from the Malliavin calculus and use their duality relations. Denote by $\mathbb{D}$ and δ the Malliavin derivative operator and the Itô-Skorokhod integral, respectively. As mentioned above, the Itô-Skorokhod integral is the adjoint operator of the Malliavin derivative, i.e., the duality relationship between the operators $\mathbb{D}$ and δ

$$E\left(F \cdot \delta(u)\right) \;=\; E\left(\langle \mathbb{D}F,\, u\rangle\right),$$

holds for stochastic functions u and F belonging to appropriate spaces [Nua06].

Assume that the stochastic operator $\mathcal{K}$ is a coordinatewise operator such that the corresponding deterministic operators $\{K_\alpha\}_{\alpha\in\mathcal{I}}$ are densely defined on a given Banach space X and are generating C_0-semigroups. Taking $\mathcal{B} = \mathbb{D}$ and thus $\mathcal{B}^* = \delta$ we can consider the stochastic operator DAE of the form

$$\begin{aligned} \dot{u} + \mathcal{K}\, u + \,\delta\, \lambda &= v \\ \mathbb{D}\, u &= y \end{aligned} \tag{39}$$

such that the initial condition $u(0) = u^0$ holds and given stochastic processes v and y.

The results concerning the generalized Malliavin calculus and the equations involving these operators can be found in [LPS11a, LPSŽ15, LPS15, LS15]. The chaos expansion method combined with the regularization techniques presented in the previous sections can be applied also in this case. Here we present the direct chaos expansion approach and prove the convergence of the obtained solution.

In the generalized $S'(\mathbb{R})$ setting, the operators of the Malliavin calculus are defined as follows:

(1) The *Malliavin derivative*, namely $\mathbb{D}$, as a stochastic gradient in the direction of white noise, is a linear and continuous mapping $\mathbb{D}\colon X \otimes (S)_{-1} \to X \otimes S'(\mathbb{R} \otimes (S)_{-1}$ given by

$$\mathbb{D}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k\, u_\alpha \,\otimes\, \xi_k \,\otimes H_{\alpha-\varepsilon_k} \quad \text{for } u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha.$$

The operator $\mathbb{D}$ reduces the order of the Wiener chaos space and it holds that the kernel $\ker(\mathbb{D})$ consists of constant random variables, i.e., random variables having the chaos expansion in the Wiener chaos space of order zero. In terms of quantum theory, this corresponds to the annihilation operator.

(2) The *Itô-Skorokhod integral*, namely δ, is a linear and continuous mapping $\delta\colon X \otimes S'(\mathbb{R}) \otimes (S)_{-1} \to X \otimes (S)_{-1}$ given by

$$\delta(F) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} f_\alpha \otimes v_{\alpha,k} \otimes H_{\alpha+\varepsilon_k}$$

for

$$F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes \Big(\sum_{k\in\mathbb{N}} v_{\alpha,k}\, \xi_k\Big) \otimes H_\alpha.$$

It is the adjoint operator of the Malliavin derivative. The operator δ increases the order of the Wiener chaos space and in terms of quantum theory δ corresponds to the creation operator.

(3) The *Ornstein-Uhlenbeck operator*, namely $\mathcal{R}$, as the composition $\delta \circ \mathbb{D}$, is the stochastic analogue of the Laplacian. It is a linear and continuous mapping $\mathcal{R}\colon X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ given by

$$\mathcal{R}(u) = \sum_{\alpha\in\mathcal{I}} |\alpha| u_\alpha \otimes H_\alpha \quad \text{for } u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha.$$

In terms of quantum theory, the operator $\mathcal{R}$ corresponds to the number operator. It is a self-adjoint operator with eigenvectors equal to the basis elements H_α, $\alpha \in \mathcal{I}$, i.e., $\mathcal{R}(H_\alpha) = |\alpha| H_\alpha$, $\alpha \in \mathcal{I}$. Thus, Gaussian processes from the Wiener chaos space of order one with zero expectation are the only fixed points for the Ornstein-Uhlenbeck operator.

Note that the operator $\mathcal{R}$ is a coordinatewise operator, while $\mathbb{D}$ and δ are not coordinatewise operators. More details on the generalized operators of the Malliavin calculus can be found in [LPS11b, LS15, LPSŽ15, LPS15].

The direct method of solving system (39) relies on the results from [LPSŽ15, LPS15, LPS11a]. First, we solve the second equation with the initial condition in (39) and obtain the solution λ in the space of $S'(\mathbb{R})$-stochastic distributions. Then by subtracting the obtained solution λ in the first equation of (39) and by applying the semigroup theory of stochastic processes and the result from [LPSŽ15], we obtain the explicit form of u in the space of generalized stochastic processes.

Assume that $h \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$, where X is the space of continuously differentiable functions, the initial value problem involving the Malliavin derivative operator

$$\mathbb{D}u = y, \qquad Eu = u^0 \in X \tag{39}$$

can be solved by applying the integral operator on both sides of the equation. For a given process $y \in X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-q}$, $p \in \mathbb{N}_0$, $q > p+1$, represented in its chaos expansion form $y = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} y_{\alpha,k} \otimes \xi_k \otimes H_\alpha$ with differentiable coefficients, the equation (5) has a unique solution in $Dom(\mathbb{D})$ given by

$$u = u^0 + \sum_{\alpha \in \mathcal{I}, |\alpha|>0} \frac{1}{|\alpha|} \sum_{k \in \mathbb{N}} y_{\alpha-\varepsilon^{(k)},k} \otimes H_\alpha. \tag{39}$$

Remark 5.1. The initial value problem (5) can be solved by the superposition $u = u_1 + u_2$, where u_1 solves the corresponding homogeneous equation, and u_2 corresponds to the inhomogeneous part. Thus, the initial problem may be reduced to the two problems

$$\mathbb{D}\, u_1 = 0, \qquad E u_1 = u^0$$

and

$$\mathbb{D}\, u_2 = v, \qquad E u_2 = 0.$$

Moreover, the integral equation

$$\delta \lambda = v_1\,, \tag{39}$$

where $v_1 = v - \frac{\mathrm{d}}{\mathrm{d}t} u - \mathcal{K} u$ can be solved in the space of generalized processes. Let $f \in X \otimes (S)_{-1,-p}$, $p \in \mathbb{N}_0$ with zero expectation have the chaos expansion representation of the form

$$v = \sum_{\alpha \in \mathcal{I}, |\alpha| \geq 1} v_\alpha^{(1)} \otimes H_\alpha, \quad f_\alpha \in X.$$

Then the integral equation (5) has a unique solution λ in $X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p}$, for $l > p+1$, given by

$$\lambda = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (\alpha_k + 1) \, \frac{f^{(1)}_{\alpha+\varepsilon^{(k)}}}{|\alpha + \varepsilon^{(k)}|} \otimes \xi_k \otimes H_\alpha. \tag{38}$$

Therefore, the solution (u, λ) of (39) is given by (5) and (5).

6. Conclusion

We have analysed the influence of stochastic perturbations to linear operator DAEs of semi-explicit structure. With the application of the polynomial chaos expansion, we have reduced the problem to a system of deterministic operator DAEs for which regularization techniques are known. With this, we could prove the existence and uniqueness of a solution of the stochastic operator DAE in a weighted space of generalized stochastic processes. A study on the analytical dependence of the solution with respect to the noise have to be done in order to apply this approach numerically. We intent to investigate this in a future work.

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* Institut für Mathematik, Technische Universität Berlin, Germany
E-mail address: raltmann@math.tu-berlin.de

† Institut für Mathematik, Universität Innsbruck, Austria
E-mail address: hermann.mena@uibk.ac.at, tijana.levajkovic@uibk.ac.at

‡ Faculty of Traffic and Transport Engineering, University of Belgrade, Serbia