POLYHEDRAL ASPECTS OF CARDINALITY CONSTRAINED COMBINATORIAL OPTIMIZATION PROBLEMS

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Zusammenfassung

Diese Dissertation befasst sich mit polyedrischen Strukturen von kardinalitätsbeschränkten kombinatorischen Optimierungsproblemen. Aus einem kombinatorischen Optimierungsproblem erhält man ein kardinalitätsbeschränktes kombinatorisches Optimierungsproblem, indem man nur solche Lösungen erlaubt, deren Kardinalitäten Elemente einer festgelegten Menge von nichtnegativen ganzen Zahlen sind.

Wir beschäftigen uns sowohl mit der polyedrischen Analyse ausgewählter kombinatorischer Optimierungsprobleme als auch mit allgemeinen Methoden, um starke gültige Ungleichungen herzuleiten, die einen Bezug zu Kardinalitätsbeschränkungen haben.

Im Mittelpunkt der Arbeit steht die Untersuchung der Facettialstrukturen der kardinalitätsbeschränkten Matroid-, Wege-, und Kreis-Polytope. Wie es exemplarisch für Matroid-, Wege-, und Kreis-Polytope gezeigt wird, ist eine facettendefinierende Ungleichung für ein nicht-kardinalitätsbeschränktes Polytop gewöhnlich auch für die kardinalitätsbeschränkte Version facettendefinierend. Insbesondere interessieren wir uns aber für Ungleichungen, die solche Lösungen abschneiden, die für das Basisproblem zulässig sind, aber nicht für dessen kardinalitätsbeschränkte Version. Die wichtigste Klasse von Ungleichungen sind in diesem Zusammenhang die sogenannten *forbidden cardinality inequalities*. Das sind Ungleichungen, die für ein mit einem kardinalitätsbeschränkten kombinatorischen Optimierungsproblem assoziiertem Polytop gültig sind, unabhängig von dessen kombinatorischer Struktur. Diese Ungleichungen verwenden wir als Prototyp für Ungleichungen,die kombinatorische Strukturen eines gegebenen Problems einbinden. Auf diese Weise gelingt es uns, für verschiedene kardinalitätsbeschränkte Probleme facettendefinierende Ungleichungen herzuleiten, insbesondere für die oben namentlich genannten Polytope. Außerdem präsentieren wir weitere Klassen facettendefinierender Ungleichungen, die einen Bezug zu Kardinalitätsbeschränkungen haben, für kardinalitätsbeschränkte Wege- und Kreis-Polytope. Insbesondere befassen wir uns auch mit solchen Ungleichungen, die spezifisch für gerade/ungerade Kreise/Wege oder Wege mit höchstens k Kanten sind.

Die Arbeit präsentiert und benutzt verschiedene Methoden und Ideen, um starke gültige Ungleichungen, die einen Bezug zu Kardinalitätsbeschränkungen haben, herzuleiten: matroidale Relaxierungen, Lifting, Projektion oder auch algorithmische Aspekte. Es wird beispielsweise gezeigt, dass die dem Moore-Bellman-Ford Algorithmus innewohnende Struktur dazu verwendet werden kann, um facettendefinierende Ungleichungen für das Polytop der gerichteten (s, t)-Wege mit höchstens k Kanten, herzuleiten. Für zwei Relaxierungen dieses Polytops liefert unser Ansatz eine Klassifizierung aller facettendefinierenden Ungleichungen mit Koeffizienten in $\{0, 1\}$ bzw. $\{-1, 0, 1\}$.

Schlüsselworte: Kardinalitätsbeschränkungen, Matroid-Polytop, Polymatroid, Kreis- und Wege-Polyeder, Dynamische Programmierung, Projektion

Mathematics Subject Classification (2000): 05C38, 52B40, 90C27, 90C39

Abstract

This thesis deals with polyhedral structures of cardinality constrained combinatorial optimization problems. Given a combinatorial optimization problem, we obtain a cardinality constrained version of this problem by permitting only those solutions whose cardinalities are elements of a specified set of nonnegative integral numbers.

We study both the polyhedral analysis of selected cardinality constrained combinatorial optimization problems and general methods for deriving strong valid inequalities that bear relations to cardinality restrictions.

The focus of this thesis is the investigation of the facial structure of the cardinality constrained matroid, path and cycle polytopes. As it might be expected and is exemplarily shown for path, cycle, and matroid polytopes, an inequality that induces a facet of the polytope associated with the ordinary problem usually induces a facet of the polytope associated with the cardinality restricted version. However, we are in particular interested in inequalities that cut off the incidence vectors of solutions that are feasible for the ordinary problem but infeasible for its cardinality restricted version. In this context, the most important class of inequalities for this thesis are the so-called *forbidden cardinality inequalities.* These inequalities are valid for a polytope associated with a cardinality constrained combinatorial optimization problem independent of its specific combinatorial structure. Using these inequalities as prototype for inequalities incorporating combinatorial structures of a problem, we derive facet defining inequalities for polytopes associated with several cardinality constrained combinatorial optimization problems, in particular, for the above mentioned polytopes. Moreover, for cardinality constrained path and cycle polytopes we derive further classes of facet defining inequalities related to cardinality restrictions, also those inequalities specific to odd/even path/cycles and hop constrained paths.

The thesis presents and uses different methods and ideas for deriving strong inequalities related to cardinality restrictions: matroidal relaxations, lifting, projection, and also algorithmic ingredients. For example, it will be shown that the inherent structure of the Moore-Bellman-Ford algorithm can be used to find facet defining inequalities for the hop constrained path polytope, that is, the convex hull of the incidence vectors of paths having at most k arcs. For two relaxations of this polytope, our approach yields a classifications of all facet defining inequalities with coefficients in $\{0, 1\}$ and $\{-1, 0, 1\}$, respectively.

Keywords: cardinality constraints, matroid polytope, polymatroid, path and cycle polyhedra, dynamic programming, projection

Mathematics Subject Classification (2000): 05C38, 52B40, 90C27, 90C39

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I also want to thank Akin Tanatmis from TU Kaiserslautern for his help to detect an application for cardinality constrained matroids and polymatroids in the field of linear block coding theory. I have decided, however, not to address it in my thesis, as it is work in progress. Instead of that I am glad that my student Ingo Spiegelberg is engaged with the investigation of these subjects.

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Contents

Introduction

Given a combinatorial optimization problem and a finite subset N of the nonnegative integral numbers $\mathbb{Z}_+$, we obtain a cardinality constrained version of this problem by permitting only those feasible solutions whose cardinalities are elements of N.

Well-known examples of cardinality constrained combinatorial optimization problems are the traveling salesman problem and the minimum odd cycle problem. Both problems are for themselves combinatorial optimization problems, but in the line of sight of the minimum cycle problem, they are cardinality restricted versions of the latter problem.

We give now formal definitions of combintorial optimization problems and their cardinality constrained versions.

Definition 0.1. Let E be a finite set, $\mathcal{I}$ a subset of the power set 2^E of E, and $w : E \rightarrow \mathbb{R},\ e \mapsto w(e)$ a weight function. For any $F \subseteq E$ and any $y \in \mathbb{R}^E$, we set $y(F) := \sum_{e \in F} y_e$. The mathematical program

$$\max\{w(F) : F \in \mathcal{I}\}$$

is called a *combinatorial optimization problem* (COP). We also refer to it as the triple $\Pi = (E, \mathcal{I}, w)$. Elements of $\mathcal{I}$ are called *feasible solutions.*

As usual, we apprehend a combinatorial optimization problem also as the collection of all its problem instances. But we do not distinguish between the problem $\Pi^\star$ and its instances $\Pi = (E, \mathcal{I}, w)$. Moreover, if we say that an algorithm A solves the COP $\Pi = (E, \mathcal{I}, w)$ in polynomial time, then we mean, strictly speaking, that there is an integer r such that A runs in time $O(n^r)$, where n is the input size of the given instance, and all numbers in intermediate computations can be stored with $O(n^r)$ bits.

By setting cardinality constraints on the set of feasible solutions, we obtain a cardinality constrained version of a COP. The *cardinality* of any finite set M, denoted by $|M|$, is the number of its elements.

Definition 0.2. Let $\Pi = (E, \mathcal{I}, w)$ be a COP and $N \subseteq \mathbb{Z}_+$ a finite set of nonnegative integral numbers. Then, the mathematical program

$$\max\{w(F) : F \in \mathcal{I},\ |F| \in N\},$$

also denoted by $\Pi_N = (E, \mathcal{I}, w, N)$, is said to be the *cardinality constrained version of* Π. It is also called a *cardinality constrained combinatorial optimization problem* (CCCOP). W.l.o.g. we assume that $\max N \leq |E|$.

Throughout this thesis, N will be represented by a so-called *cardinality sequence*, which is a sequence $c = (c_1, c_2, \dots, c_m)$ of integers such that

$$0 \leq c_1 < c_2 < \dots c_m \leq |E|$$

and $N = \{c_1, \dots, c_m\}$. Moreover, Π_N will be identified with Π_c. The set of feasible solutions with respect to Π_c will also be denoted by $\mathcal{I}_c$, that is, $\mathcal{I}_c := \{I \in \mathcal{I} : |I| = c_p \text{ for some } p\}$. Π_c is, considered for itself, again a COP. If $c = (k)$ for some $k \in \mathbb{Z}_+$, we speak of a k-COP and write Π_k instead of $\Pi_{(k)}$ provided that it is clear from the context that c refers to a sequence and k to an integer. An overview on k-COPs is given by Bruglieri et al. [16].

This thesis focuses on polyhedral aspects of cardinality constrained combinatorial optimization problems. Many combinatorial optimization problems are polyhedrally well studied. Given a COP $\Pi = (E, \mathcal{I}, w)$, the polyhedral investigation usually refers to the associated polytope $P_{\mathcal{I}}(E)$ defined as the convex hull of the incidence vectors χ^I of the feasible solutions $I \in \mathcal{I}$. Here, we study the polytope

$$P^c_{\mathcal{I}}(E) := \text{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I}_c\},$$

that is, the convex hull of the incidence vectors of feasible solutions with respect to Π_c. Since $\mathcal{I}_c \subseteq \mathcal{I}$, it follows that $P^c_{\mathcal{I}}(E) \subseteq P_{\mathcal{I}}(E)$. Thus, any valid inequality for $P_{\mathcal{I}}(E)$ is also valid for $P^c_{\mathcal{I}}(E)$. It stands to reason that many facet defining inequalities for $P_{\mathcal{I}}(E)$ are also facet defining or at least strong inequalities for $P^c_{\mathcal{I}}(E)$. Although the manifestation of this conjecture might be interesting in its own right and gets some place in this thesis, we are more interested in strong valid inequalities that cut off solutions that are feasible for Π but forbidden for Π_c.

Throughout this thesis, we assume the reader to be familiar with basic concepts of complexity theory, polyhedral combinatorics, linear programming, and combinatorial optimization. For complexity theory, we refer to the book of Garey and Johnson [39]. For polyhedral combinatorics, linear programming, and combinatorial optimization we refer to the books of Grötschel, Lovász, and Schrijver [46], Nemhauser and Wolsey [67], and Schrijver [74, 76].

Nevertheless, since some notions are used in the entire thesis, we explicitely introduce some basic definitions. Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $a^T x \leq a_0$ a valid inequality for P. Any point $x' \in P$ satisfying $a^T x' = a_0$ is said to be *tight* (with respect to the inequality $a^T x \leq a_0$). In our context,

P is usually $P_{\mathcal{I}}(E)$ or $P^c_{\mathcal{I}}(E)$ and x' is the incidence vector of some feasible solution I. In such a case, we also refer to I as tight. The inequality $a^T x \leq a_0$ is called an *implicit equation* if every $x' \in P$ is tight. Let $a^T x \leq a_0$ and $b^T x \leq b_0$ be two valid inequalities for P. We say that $a^T x \leq a_0$ is (*strictly*) *dominated* by $b^T x \leq b_0$, if the face $\{x \in P : a^T x = a_0\}$ is (strictly) contained in $\{x \in P : b^T x = b_0\}$. They are said to be *equivalent* if one can be obtained from the other by multiplication with a positive scalar and adding appropriate multiples of implicit equations for P. Clearly, equivalent inequalities induce the same face of P.

Finally, we repeat the definitions of the membership and the separation problem. Given a polyhedron $P \subseteq \mathbb{R}^n$ and a point $x^* \in \mathbb{R}^n$, the *membership problem* consists of deciding whether $x^\star \in P$ or not. The *separation problem* consists of the membership problem and the additional task to find a valid inequality $b^T x \leq b_0$ for P that is violated by $x^\star$ in case of $x^\star \notin P$. For practical reasons, the separation problem is often considered for families of valid inequalities for P. This is the case, for instance, if one has only a partial description of P by valid inequalities. The separation problem for a family $\mathcal{F}$ of valid inequalities for P consists of finding a violated member of $\mathcal{F}$, i.e., an inequality $b^T x \leq b_0$ belonging to $\mathcal{F}$ such that $b^T x^* > b_0$, or to assert that $x^\star$ satisfies all inequalities of $\mathcal{F}$.

Sometimes it is profitable to find a most violated member, i.e., an inequality $b^T x \leq b_0$ belonging to $\mathcal{F}$ and maximizing the *degree of violation* $b^T x^* - b_0$ (optimization version), because sometimes a maximally violated inequality exhibit a strong combinatorial structure, which can be exploited for separation. Furthermore, note that the complexity of the separation problem depends strongly on the exact definition of the family $\mathcal{F}$. For example, it can arise the case that the separation problem for $\mathcal{F}$ is solvable in polynomial time but for a subclass $\mathcal{F}' \subseteq \mathcal{F}$ it could be NP-hard.

The thesis is organized as follows.

CHAPTER 1: In this chapter, we first briefly touch the question under which conditions a COP Π and its cardinality constrained version Π_c belong to the same complexity class. We conjecture that they usually belong to the same complexity class. The aim of this chapter is not to give a concluding answer to this question, but to mark the challenges to answer this question if we do not study a specific COP. The arising difficulties result, among other things, from missing information about the structure of the set of feasible solutions.

Next, we present an important family of linear inequalities for the polyhedral investigation of CCCOPs, which we call *forbidden cardinality inequalities*, FC-inequality for short. These inequalities were independently introduced by Maurras [62] and Grötschel [45]. Together with the *cardinality* *bounds*

$$c_1 \leq \sum_{e \in E} x_e \leq c_m$$

they cut off solutions that are feasible for Π, but forbidden for Π_c. This results into an integer programming formulation for Π_c provided we know one for Π.

CHAPTER 2: This chapter is dedicated to cardinality constrained matroids and polymatroids. It serves, among other things, as an example for the polyhedral analysis of the cardinality constrained version of a polynomial time solvable combinatorial optimization problem. Maurras [61] has given a complete linear description of the cardinality constrained matroid polytope. We give an elementary proof of this result. Moreover, we characterize the facets of this polytope and state a polynomial time separation procedure. Based on the results for the cardinality constrained matroid, we give a complete linear description of the cardinality constrained polymatroid and present a polynomial time algorithm that solves the associated separation problem.

CHAPTER 3: As an example of NP-hard cardinality constrained combinatorial optimization problems, we extensively study polyhedra associated with cardinality constrained versions of path and cycle problems defined on directed and undirected graphs. We show that a modification of forbidden cardinality inequalities leads to strong inequalities related to cardinality constraints. Moreover, as one would expect, inequalities that define facets of the polytope associated with the ordinary problem usually define facets of the polytope associated with the cardinality constrained version.

CHAPTER 4: In their original form, forbidden cardinality inequalities are mostly not facet defining for the polyhedra associated with a given CCCOP. Based on the polyhedral insights of Chapter 2 and 3, we give three recommendations for deriving strong valid inequalities that are related to cardinality constraints.

CHAPTER 5: In this chapter, we again study polyhedra associated with cardinality constrained paths, namely, those connected to *hop constrained paths*, that is, paths that use at most k arcs. The polyhedra are the hop constrained path polytope, which is the convex hull of the incidence vectors of hop constrained paths, and two of its relaxations: its dominant and the hop constrained walk polytope. To differentiate between general cardinality constrained paths (walks) and those that use at most k arcs (or edges), we decided to declare them as hop constrained paths (walks) as it is done, for instance, in [22, 23].

The aim of this chapter is to show how the inherent structure of the Moore-Bellman-Ford algorithm, which can be essentially expressed by the well known Bellman equations, helps to derive facet defining inequalities for the hop constrained path polytope via projection. Our findings result into characterizations of all facet defining 0/1-inequalities for the dominant and all facet defining $0/\pm 1$-inequalities for the hop constrained walk polytope. Although the derived inequalities are already known, such classifications were not previously given by the best knowledge of the author. Moreover, we derive a further class of facet defining inequalities for the hop constrained walk polytope, with coefficients not all in $\{-1, 0, 1\}$.

Chapter 1

General aspects of cardinality constrained combinatorial optimization problems

This chapter summarizes some observations about cardinality constrained combinatorial optimization problems that can be made independently of the special structure of a given COP Π and its cardinality constrained version Π_c. As one would expect, to make a general statement seems to be difficult as long as we do not have additional information about the combinatorial structure of Π.

Section 1.1 raises the question under which conditions Π and Π_c are in the same complexity class. We give no concluding answer, but mark the arising difficulties to classify those COPs.

In Section 1.2, we present the most important family of linear inequalities for the polyhedral investigation of CCCOPs, which we call *forbidden cardinality inequalities*. These inequalities were independently introduced by Maurras [62] and Grötschel [45]. Together with the *cardinality bounds* $c_1 \leq x(E) \leq c_m$ they cut off solutions that are feasible for Π, but forbidden for Π_c. This results into an integer programming formulation for Π_c provided we have one for Π. These are the good news. The bad news are that the forbidden cardinality inequalities are usually quite weak for the cardinality constrained version of a given COP Π. This means that these inequalities usually induce low dimensional faces of the polytope associated with Π_c.

We will illustrate some aspects of this chapter by examples from matroid and graph theory. The definitions of the notions used can be found in Chapters 2 and 3.

1.1 Complexity

We briefly study conditions under which a COP and its cardinality constrained version belong to the same complexity class. For simplicity, we restrict our considerations to the classes of polynomial time solvable problems (P) and nondeterministic polynomial time solvable problems (NP).

Let $\Pi = (E, \mathcal{I}, w)$ be a COP. For fixed k, the k-COP Π_k can be solved in polynomial time by enumeration on all $\binom{n}{k}$ subsets I of E of cardinality k. Thus, independent of the complexity of the ordinary problem Π, Π_c can be solved in polynomial time if the cardinality sequence c is fixed.

Bruglieri et al. [16] argue that several polynomial time solvable COPs of the form

$$\min\{w(F) : F \in \mathcal{I}\}$$

with a nonnegative weight function $w : E \to \mathbb{R}_+$ become NP-hard as soon as one requires that the feasible solutions have cardinality k. For example, one can find in polynomial time a shortest simple cycle, that is, a simple cycle of minimum weight, if the weight function is nonnegative, but the TSP, which arises by taking k as the number of nodes, is NP-hard. Or, in order to give another example, the min-cut problem can be solved in polynomial time for a nonnegative weight function, but most of the cardinality constrained cut problems are NP-hard (e.g. the equicut problem).

The observation of Bruglieri et al. is of course correct. We believe, however, that less the cardinality restriction itself is responsible for the NP-hardness of some k-COP, but rather the fact that the original problem is also NP-hard as soon as one admits negative weights, or, in order to formulate it more tentative, both ingredients "arbitrary weights" and "cardinality restriction" for a COP seem to be equivalent in many cases with respect to the complexity of a COP. One argument for this hypothesis is that the restriction to nonnegative weights is irrelevant for k-COPs: For any $M \in \mathbb{R}$ and any two feasible solutions F_1, F_2 of a k-COP Π_k we have $w(F_1) \leq w(F_2)$ if and only if $w'(F_1) \leq w'(F_2)$, where $w'_e := w_e + M$. That means, Π_k is invariant under shifting of the weights by a constant. By setting $M := 0$ if all weights are already nonnegative, and $M := |\min\{w_e : e \in E\}|$ otherwise, the weights w'_e are nonnegative. Moreover, the input size of w' is polynomial in w.

We would like to support our hypothesis by two examples. The above mentioned polynomial-time solvable shortest cycle problem becomes NP-hard if arbitrary weights are admitted or the set of feasible solutions is limited to Hamiltonian cycles, see Garey and Johnson [39]. Denoting by CYCLE the shortest cycle problem, the latter fact implies that CYCLE$_k$ is NP-hard for arbitrary k belonging to the input. For instance, let $G = (V, E)$ with $n := |V|$ be an instance of the TSP. Adding to G a set V' of n isolated nodes, we obtain a graph $G' = (V \cup V', E)$ of order $m := 2n$. Every Hamiltonian cycle in G is obviously a cycle of cardinality $\lfloor m/2 \rfloor$ in G', and vice versa. Thus, the TSP can be polynomially reduced to the problem of finding in a graph on n nodes a shortest cycle of cardinality $\lfloor n/2 \rfloor$. This implies that CYCLE$_{\lfloor n/2 \rfloor}$ is NP-hard. Even more, if CYCLE$_k$ is NP-hard for arbitrary

k, then it is very unlikely to find a polynomial time algorithm that solves the general cardinality constrained shortest cycle problem CYCLE_c for an arbitrary cardinality sequence c.

Another example is the maximum weight independent set problem over a matroid, where the independence system is given by an independence oracle. We denote the problem by IND. It can be solved in polynomial time with the greedy algorithm for arbitrary weights. In accordance with our hypothesis, also the cardinality constrained version of this problem IND_k can be solved in polynomial time for each k and hence, also for each c, see Chapter 2.

To our knowledge there is no polynomial time solvable combinatorial optimization problem discussed in the literature for which the cardinality constrained version is NP-hard (supposed that arbitrary weights are admitted). However, classes of polynomial time solvable combinatorial optimization problems that are completely irrelevant but whose cardinality constrained versions are NP-hard, can be constructed quite easily. For instance, let $G = (V, E, w)$ be a weighted graph on $n = |V|$ nodes and consider the *embedded traveling salesman problem* (ETSP) defined as follows:

$$\min\{w(T) : T \subseteq E, \text{ if } |T| = n, \text{ then } T \text{ is a tour}\}.$$

It can obviously be solved in polynomial time. To this end, let $T^\star := \{e \in E : w_e < 0\}$. If $|T^\star| \neq n$ or $|T^\star| = n$ and $T^\star$ is a Hamiltonian cycle, then $T^\star$ is optimal. Otherwise, that is, in the case $|T^\star| = n$ and $T^\star$ is not a Hamiltonian cycle, let $e^- \in T^\star$, $e^+ \in E \setminus T^\star$ (if $E \setminus T^\star \neq \varnothing$) be edges of maximal and minimal weight, respectively. By construction, $w(e^-) < 0$ and $w(e^+) \geq 0$. Now it follows immediately that $w(F) \geq \min\{w(T^\star \setminus \{e^-\}), w(T^\star \cup \{e^+\})\}$ for all $F \subseteq E$ with $|F| \neq n$. Moreover, for any Hamiltonian cycle T we have $w(T) \geq w(T^\star \setminus \{e^-\})$. Hence, $T^\star \setminus \{e^-\}$ or $T^\star \cup \{e^+\}$ is the optimal solution. So, the ETSP can indeed be solved in polynomial time. However, the cardinality constrained version ETSP_n of ETSP is the TSP which is known to be NP-hard. Of course, such a construction – namely the embedding of an NP-hard combinatorial optimization problem into a trivial setting – can be done not only for the TSP but also for other NP-hard COPs like the linear ordering problem.

The discussion in the previous paragraph shows that, in general, we are not able to extrapolate from the polynomial time solvability of a COP to the polynomial time solvability of its cardinality constrained version. We can only give a much weaker result. Let $\Pi = (E, \mathcal{I}, w)$ be a COP, let $n_{\max} = \max\{|I| : I \in \mathcal{I}\}$, and $n_{\min} = \min\{|I| : I \in \mathcal{I}\}$. The *maximum cardinality COP* $\Pi_{\max}$ is the optimization problem

$$\max\{w(I) : I \in \mathcal{I},\ |I| = n_{\max}\}.$$

Similarly, the *minimum cardinality COP* $\Pi_{\min}$ is the optimization problem

$$\max\{w(I) : I \in \mathcal{I},\ |I| = n_{\min}\}.$$

Theorem 1.1. If a COP $\Pi = (E, \mathcal{I}, w)$ can be solved in polynomial time for all weightings $w : E \to \mathbb{R}$, the same holds for $\Pi_{\max}$ and $\Pi_{\min}$. Hence, if $\Pi_{\min}$ or $\Pi_{\max}$ is NP-hard, then Π is too.

Proof. The optimal solutions of $\Pi_{\max}$ and $\Pi_{\min}$ are invariant under shifting of the weights by a constant. For any instance $\Pi = (E, \mathcal{I}, w)$, set $M := |E| \cdot W + 1$, where $W = \max\{|w_e| : e \in E\}$. Then, an optimal solution X^1 of Π with weights $w_e^1 := w_e + M$ is of maximum cardinality and an optimal solution X^2 of Π with weights $w_e^2 := w_e - M$ is of minimum cardinality. In particular, X^1 and X^2 are optimal solutions for $\Pi_{\max}$ and $\Pi_{\min}$, respectively. Since the transformations are polynomial, the claim follows. $\square$

Maybe better results are obtainable if one excludes such artificial COPs as the ETSP. This can be perhaps achieved when adding requirements on the homogeneity of the feasible solutions. For instance, one could require that all feasible solutions of Π have some common property P independent of the cardinality of the solutions. The ETSP could easily be excluded by adding the constraint that each feasible solution has to be a cycle.

During this section we collected three conditions for the comparison of the complexity of an arbitrary COP Π and its cardinality constrained version Π_c to be meaningful – provided P$\neq$NP (otherwise the distinction makes no sense):

- c may not to be fixed;
- w is an arbitrary objective function;
- the set $\mathcal{I}$ of feasible solutions is in some sense homogeneous.

1.2 Integer programming formulations

In this section, we present an important class of inequalities for describing cardinality constrained combinatorial optimization problems polyhedrally. Let $\Pi = (E, \mathcal{I}, w)$ be any COP, Π_c its cardinality constrained version, and $P_{\mathcal{I}}(E)$ and $P_{\mathcal{I}}^c(E)$ the polytopes associated with Π and Π_c, respectively. Recall that $\mathcal{I}_c$ denotes the set of feasible solutions $I \in \mathcal{I}$ with $|I| = c_p$ for some p. The facial structure of $P_{\mathcal{I}}^c(E)$ is essentially determined by the combinatorial structures coming from both the combinatorial optimization problem and the cardinality conditions.

Uncoupled from the combinatorial structure of the optimization problem, the polyhedral structure induced by the cardinality constraints can be described as follows. The set

$$\mathrm{CHS}^c(E) := \{F \subseteq E : |F| = c_p \text{ for some } p\}$$

is called a *cardinality homogeneous set system*. It follows immediately that $\mathcal{I}_c = \mathcal{I} \cap \mathrm{CHS}^c(E)$ and hence,

$$P^c_{\mathcal{I}}(E) = \mathrm{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I} \cap \mathrm{CHS}^c(E)\}.$$

The polytope associated with $\mathrm{CHS}^c(E)$, that is, the convex hull of the incidence vectors of $I \in \mathrm{CHS}^c(E)$, is completely described by the *trivial inequalities* $0 \leq x_e \leq 1$, $e \in E$, the *cardinality bounds*

$$c_1 \leq x(E) \leq c_m, \tag{1.1}$$

and the *forbidden cardinality inequalities*

$$\begin{array}{c}(c_{p+1} - |F|)x(F) \; - \; (|F| - c_p)x(E \setminus F) \leq c_p(c_{p+1} - |F|) \\ \text{for all } F \subseteq E \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \in \{1, \dots, m-1\},\end{array} \tag{1.2}$$

see Maurras [62] and Grötschel [45]. Here, for any subset F of E, $x(F) := \sum_{e \in F} x_e$. Maurras and Grötschel investigated the polytope associated with $\mathrm{CHS}^c(E)$ independent of each other. Maurras calls inequalities (1.2) *cropped inequalities*, while Grötschel calls them *cardinality forcing inequalities*. We have chosen the name *forbidden cardinality inequalities*, since the inequality class consists of exactly one inequality for each subset F of E of forbidden cardinality between the lower bound c_1 and the upper bound c_m.

For an illustration of the forbidden cardinality inequalities, see Figure 1.1. There we show the support graph of a forbidden cardinality inequality associated with a subset F of cardinality 5. Since $c = (2, 3, 8, 9)$, it follows immediately $p = 2$, that is $c_p = c_2 = 3$ and $c_{p+1} = c_3 = 8$. Thus, the coefficients associated with the elements in F are $c_3 - |F| = 3$, those associated with the elements which are not in F are $-(|F| - c_2) = -2$, and the right and side of the inequality is $c_2(c_3 - |F|) = 9$.

The cardinality bounds (1.1) exclude all subsets of E whose cardinalities are less than the lower bound c_1 or greater than the upper bound c_m, while the forbidden cardinality inequalities do this for all subsets of E with forbidden cardinality between the lower and the upper bound. To see this, let $F \in E$ with $c_p < |F| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Then the forbidden

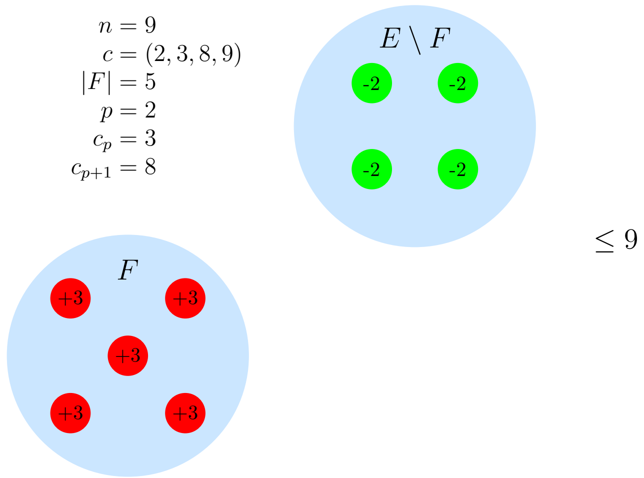


Figure 1.1: Support graph of a forbidden cardinality inequality.

cardinality inequality associated with F is violated by the incidence vector χ^F of F:

$$(c_{p+1}-|F|)\chi^F(F) - (|F|-c_p)\overbrace{\chi^F(E \setminus F)}^{=0} = |F|(c_{p+1}-|F|) > c_p(c_{p+1}-|F|).$$

However, every $I \in \mathrm{CHS}^c(E)$ satisfies the inequality associated with F. If $|I| \leq c_p$, then

$$\begin{aligned}(c_{p+1} - |F|)\chi^I(F) \; - \; &\overbrace{(|F| - c_p)}^{>0}\overbrace{\chi^I(E \setminus F)}^{\geq 0} \\ &\leq (c_{p+1} - |F|)\chi^I(I \cap F) \leq c_p(c_{p+1} - |F|),\end{aligned}$$

and equality holds if $|I| = c_p$ and $I \subseteq F$. If $|I| \geq c_{p+1}$, then

$$\begin{aligned}(c_{p+1} - |F|)\,&\overbrace{\chi^I(F)}^{\leq|F|} - (|F| - c_p)\overbrace{\chi^I(E \setminus F)}^{\geq c_{p+1}-|F|} \\ &\leq (c_{p+1} - |F|)|F| - (|F| - c_p)(c_{p+1} - |F|) = c_p(c_{p+1} - |F|),\end{aligned}$$

and equality holds for $|I| = c_{p+1}$ and $I \cap F = F$.

Although the class of forbidden cardinality inequalities consists of exponentially many members, Grötschel [45] showed that the associated separation problem is solvable in polynomial time by the greedy algorithm. Let $x^\star \in \mathbb{R}^E$ be any nonnegative vector satisfying the cardinality bounds (1.1). Sort the components of $x^\star$ such that $x^\star_{e_1} \geq x^\star_{e_2} \geq \cdots \geq x^\star_{e_{|E|}}$. Then, for each integer q with $c_p < q < c_{p+1}$, $x^\star$ satisfies the forbidden cardinality inequality associated with $F^q := \{e_1, \dots, e_q\}$ if and only if $x^\star$ satisfies all forbidden cardinality inequalities associated with sets $F \subseteq E$ of cardinality q. In other words, the separation problem can be solved by checking the forbidden cardinality inequality associated with F^q for each forbidden integer q between c_1 and c_m.

Inequalities (1.2) and the cardinality bounds (1.1) can be straightforwardly included in an integer programming formulation for Π to derive one for Π_c.

Theorem 1.2. Let

$$\begin{array}{ll} \max & w^T x \\ \text{s.t.} & Ax \leq b \\ & x_e \in \{0,1\} \quad \text{for all } e \in E \end{array} \tag{1.3}$$

be an integer programming formulation for a COP $\Pi = (E, \mathcal{I}, w)$. Furthermore, let $c = (c_1, \dots, c_m)$ be any cardinality sequence. Then, system (1.3) together with the cardinality bounds (1.1) and the forbidden cardinality inequalities (1.2) provide an integer programming formulation for Π_c.

Proof. First, note that inequalities (1.2) are valid for $P^c_{\mathcal{I}}(E)$, since $\mathcal{I}_c \subseteq \text{CHS}^c(E)$. Next, the inequality system $Ax \leq b$ together with the *integrality constraints* $x_e \in \{0,1\}$ for $e \in E$ ensure that x is the incidence vector of some $I \in \mathcal{I}$. Finally, the cardinality bounds (1.1) and the forbidden cardinality inequalities (1.2) exclude all incidence vectors of $I \in \mathcal{I}$ of forbidden cardinality. □

Clearly, if Π incorporates cardinality restrictions a priori as for perfect matchings, minimal spanning trees, or the TSP, the approach is nonsense.

So far, we have seen that the polyhedral structure induced by $\text{CHS}^c(E)$ can easily be gotten under control. Moreover, we immediately obtain an integer programming formulation for Π_c provided we have one for Π. Adding the forbidden cardinality inequalities (1.2) to an integer programming formulation, however, does not necessarily result in facet defining inequalities of the associated polytope. For instance, consider cardinality constrained matroids.

The linear program

$$\begin{array}{lll} \max \sum_{e \in E} w_e \, x_e & & \\ \text{s.t. } x(F) \leq r(F) & \text{for all } \varnothing \neq F \subseteq E, \\ \qquad x_e \geq 0 & \text{for all } e \in E \end{array} \tag{1.4}$$

is a well-known formulation for finding a maximum weight independent set in a matroid $M = (E, \mathcal{I})$, see Edmonds [29]. Here, for any $F \subseteq E$, $r(F)$ denotes the *rank* of F, which is defined to be the maximum size of an independent set $I \subseteq F$. Given a cardinality sequence $c = (c_1, \dots, c_m)$ with $0 \leq c_1 < \cdots < c_m \leq |E|$, the cardinality restricted version of this problem can be formulated as follows:

$$
\begin{aligned}
\max \sum_{e \in E} & w_e x_e \\
\text{s.t. } & x(E) \geq c_1, \\
& x(E) \leq c_m, \\
(c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) & \leq c_p(c_{p+1} - |F|) \\
& \text{for all } F \subseteq E \text{ with } vc_p < |F| < c_{p+1} \text{ for some } p, \\
x(F) \leq r(F) & \qquad \text{for all } \varnothing \neq F \subseteq E, \\
x_e \in \{0,1\} & \qquad \text{for all } e \in E.
\end{aligned}
\tag{1.5}
$$

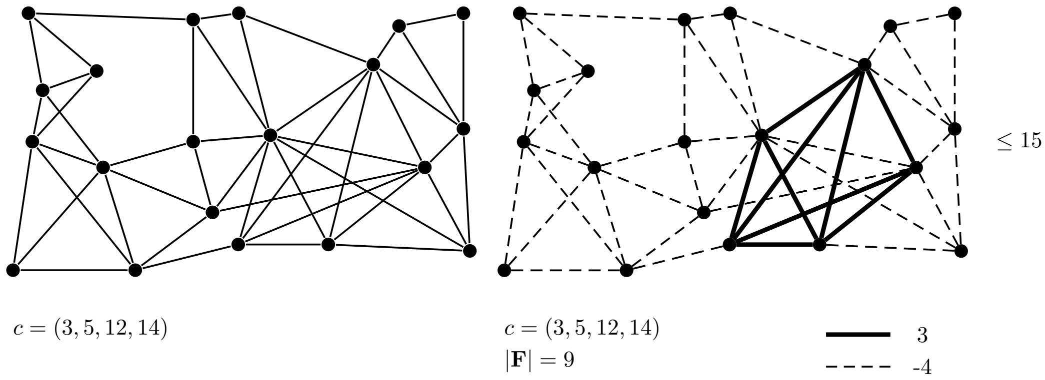
By Theorem 1.2, the integer points of the associated *cardinality constrained matroid polytope*

$$P^c_{\mathrm{M}}(E) := \mathrm{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I} \cap \mathrm{CHS}^c(E)\}$$

are described by the constraints of the integer program (1.5). However, the above IP-formulation for finding a maximum weight independent set $I \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ is quite weak, since, in general, none of the forbidden cardinality inequalities is facet defining for the cardinality constrained matroid polytope.

For instance, let $M = (E, \mathcal{I})$ be the graphic matroid defined on the graph $G = (V, E)$ shown in Figure 1.2, that is, $\mathcal{I}$ is the collection of all forests of G. Figure 1.2 illustrates the support graph of the forbidden cardinality inequality associated with the set F of bold edges with respect to the cardinality sequence $c = (3, 5, 12, 14)$. Denote the inequality by $a^T x \leq 15$. If x is the incidence vector of a forest with exactly three edges, which are all in F, then the left hand side of the inequality evaluates to 9. As is easily seen, 9 is the maximum that can be obtained by the incidence vector of a forest that satisfies the cardinality restrictions. In other words, the depicted inequality is completely irrelevant, since no feasible point of the associated cardinality constrained matroid polytope satisfies the inequality at equality.

We can make a similar observation when $\mathcal{I}$ represents the collection of all simple cycles of G. Now we can find cycles of cardinality 5 whose incidence vectors satisfy the inequality $a^T x \leq 15$ at equality, namely the 5-cycles whose edges are all in F. However, it is not hard to see that there is no cycle of feasible cardinality that contains a dashed edge and satisfies $a^T x \leq 15$ at



(a) Graph $G = (V, E)$ (b) Support graph of an FC-inequality

Figure 1.2: Illustration of a forbidden cardinality inequality.

equality. This immediately implies that the inequality $a^T x \leq 15$ does not induce a facet of the polytope associated with cardinality constrained cycles.

The problem how one can derive strong inequalities related to cardinality conditions is the topic of Chapters 2-4.

Chapter 2

Cardinality constrained matroids and polymatroids

This chapter is dedicated to cardinality constrained matroids and polymatroids. The part on matroids in Section 2.1 is joint work with Jean François Maurras and appears in [65].

After a short introduction to the necessary foundations of matroid theory, we give a complete linear description of the cardinality constrained matroid polytope according to [61] and present an elementary proof of this result. Moreover, we characterize the facet defining inequalities and give a polynomial time procedure to solve the separation problem. Based on the results for the cardinality constrained matroid, we give in Section 2.2 a complete linear description of the cardinality constrained polymatroid and present a polynomial time algorithm that solves the associated separation problem.

2.1 The cardinality constrained matroid polytope

Let E be a finite set and $\mathcal{I}$ a subset of the power set of E. The pair $(E, \mathcal{I})$ is called an *independence system* if (i) $\varnothing \in \mathcal{I}$ and (ii) whenever $I \in \mathcal{I}$ then $J \in \mathcal{I}$ for all $J \subset I$. If $I \subseteq E$ is in $\mathcal{I}$, then I is called an *independent set*, otherwise it is called a *dependent set*. Dependent sets $\{e\}$ with $e \in E$ are called *loops*. For any set $F \subseteq E$, $B \subseteq F$ is called a *basis* of F if $B \in \mathcal{I}$ and $B \cup \{e\}$ is dependent for all $e \in F \setminus B$. The *rank* of F is defined by $r_{\mathcal{I}}(F) := \max\{|B| : B \text{ basis of } F\}$. The set of all bases B of E is called a *basis system*. There are many different ways to characterize when an independence system is a matroid. Fur our purposes the following definition will be most comfortable. $(E, \mathcal{I})$ is called a *matroid*, and then it will be denoted by $M = (E, \mathcal{I})$, if

(iii) $\qquad I, J \in \mathcal{I}, |I| < |J| \;\Rightarrow\; \exists\, K \subseteq J \setminus I : |I \cup K| = |J|,\; K \cup I \in \mathcal{I}.$

Equivalent to (iii) is the requirement that for each $F \subseteq E$ all its bases have the same cardinality. Throughout this chapter we deal only with loopless matroids. The results of the chapter can be easily brought forward to matroids containing loops.

Let $M = (E, \mathcal{I})$ be a matroid. A set $F \subseteq E$ is said to be *closed* if $r_{\mathcal{I}}(F) < r_{\mathcal{I}}(F \cup \{e\})$ for all $e \in E \setminus F$. It is called *inseparable* if there are no sets $F_1 \neq \varnothing \neq F_2$ with $F_1 \dot\cup F_2 = F$ such that $r_{\mathcal{I}}(F_1) + r_{\mathcal{I}}(F_2) \leq r_{\mathcal{I}}(F)$; otherwise it is *separable*.

Given any independence system $(E, \mathcal{I})$ and any weights $w_e \in \mathbb{R}$ on the elements $e \in E$, the combinatorial optimization problem $\max w(I), I \in \mathcal{I}$ is called the *maximum weight independent set problem*. The convex hull of the incidence vectors of the feasible solutions $I \in \mathcal{I}$ is called the *independent set polytope* and will be denoted by $P_{\text{IND}}(E)$. If $(E, \mathcal{I})$ is a matroid, then $P_{\text{IND}}(E)$ is also called the *matroid polytope* and will be always denoted by $P_{\text{M}}(E)$. The cardinality constrained versions of $P_{\text{IND}}(E)$ and $P_{\text{M}}(E)$ are the *cardinality constrained independent set polytope* $P^c_{\text{IND}}(E)$ and the *cardinality constrained matroid polytope* $P^c_{\text{M}}(E)$, respectively. Throughout the chapter we assume that the cardinality sequence $c = (c_1, \dots, c_m)$ consists of at least two members.

As it is well known, the maximum weight independent set problem on a matroid can be solved to optimality with the greedy algorithm. Moreover, the matroid polytope $P_{\text{M}}(E)$ is determined by the *rank inequalities* and the *nonnegativity constraints* (see Edmonds [29]), i.e., $P_{\text{M}}(E)$ is the set of all points $x \in \mathbb{R}^E$ satisfying

$$\begin{aligned} x(F) &\leq r_{\mathcal{I}}(F) && \text{for all } \varnothing \neq F \subseteq E, \\ x_e &\geq 0 && \text{for all } e \in E. \end{aligned} \tag{2.1}$$

The rank inequality associated with F is facet defining for $P_{\text{M}}(E)$ if and only if F is closed and inseparable (see Edmonds [29]).

Let $(E, \mathcal{I})$ be an independence system, c a cardinality sequence, and $w : E \to \mathbb{R}$ an objective function. The combinatorial optimization problem $\max\{w(I) : I \in \mathcal{I} \cap \text{CHS}^c(E)\}$ is called the *cardinality constrained maximum weight independent set problem*. Provided that $(E, \mathcal{I})$ is a matroid, it can be solved in polynomial time with the greedy algorithm (see Algorithm 1), since each intermediate greedy-solution $I_k \in \mathcal{I}$ of cardinality k is an optimal solution over all feasible solutions $I \in \mathcal{I}$ of cardinality k. Thus, choosing a best solution among $\{I_{c_p} : p = 1, \dots, m\}$ yields an optimal solution.

Turning to the polyhedral aspects, we have already shown in Chapter 1 that the system

$$\begin{aligned} x(F) &\leq r_{\mathcal{I}}(F) && \text{for all } \varnothing \neq F \subseteq E, \\ (c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) &\leq c_p(c_{p+1} - |F|) && \text{for all } F \subseteq E \\ & && \text{with } c_p < |F| < c_{p+1} \text{ for some } p, \\ x(E) &\geq c_1, \\ x(E) &\leq c_m, \\ x_e &\in \{0, 1\} && \text{for all } e \in E \end{aligned}$$

Algorithm 1: Greedy algorithm for cardinality constrained matroids.

Input: Matroid $M = (E, \mathcal{I})$ by an independence testing oracle, a cardinality sequence $c = (c_1, \ldots, c_m)$ such that $c_m \leq r_{\mathcal{I}}(E)$, and weights $w_e \in \mathbb{R}$ for all $e \in E$.
Output: $I \in \mathcal{I}$ with $|I| = c_p$ for some $p \in \{1, \ldots, m\}$ that maximizes w.

```
(Sort) Number the elements e ∈ E such that w_{e_1} ≥ w_{e_2} ≥ ... ≥ w_{e_{|E|}}.
Set I_0 := ∅, p := 0, i := 1.
for p := 1 to m do
    Set I_p := I_{p-1}.
    while |I_p| < c_p do
        if I_p ∪ {e_i} ∈ 𝓘 then
            I_p := I_p ∪ {e_i}.
        i := i + 1.
        end
    end
    if p ≥ 2 and w(I_p) ≤ w(I_{p-1}) then
        return I_{p-1}.
    end
end
return I_m.
```

characterizes the integer points of the cardinality constrained matroid polytope $P^c_{\mathrm{M}}(E)$, since $P^c_{\mathrm{M}}(E)$ is contained in the intersection of both polytopes $P_{\mathrm{M}}(E)$ and $P^c(E)$, where $P^c(E)$ denotes the polytope associated with cardinality homogeneous set system $\mathrm{CHS}^c(E)$, that is,

$$P^c(E) := \mathrm{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathrm{CHS}^c(E)\}.$$

However, usually $P^c_{\mathrm{M}}(E) = P_{\mathrm{M}}(E) \cap P^c(E)$ does not hold. As a counterexample, consider the graphic matroid defined on the complete graph $K_4 = (V, E)$ on four nodes together with the cardinality sequence $c = (1, 3)$. $\mathcal{I}$ is the collection of all forests of K_4. The point $x^\star := [1, 0.5, 0.5, 0, 0, 0]$, whose support graph is depicted in Figure 2.1, is the sum of the incidence vectors of the forests $\{e_1, e_2\}$ and $\{e_1, e_3\}$ divided by two. Hence, $x^\star \in P_{\mathrm{M}}(E)$. Moreover, $x^\star \in P^{(1,3)}(E)$, which can be easily verified by application of Grötschel's separation routine for $F_2 := \{e_1, e_2\}$, which has been explained in Section 1.2 of Chapter 1. The forbidden cardinality inequality associated with F_2 is the inequality $x_{e_1} + x_{e_2} - x_{e_3} - x_{e_4} - x_{e_5} - x_{e_6} \leq 1$. The point $x^\star$ satisfies this inequality, and hence it satisfies all forbidden cardinality inequalities. However, $x^\star \notin P^{(1,3)}_{\mathrm{M}}(E)$. Suppose, for the sake of contradiction, that

$x^\star \in P_{\mathrm{M}}^{(1,3)}(E)$. Then exist $\lambda_I \geq 0$ for $I \in \mathcal{I}_{(1,3)} = \mathcal{I} \cap \mathrm{CHS}^{(1,3)}(E)$ such that $x^\star = \sum_{I \in \mathcal{I}_{(1,3)}} \lambda_I \chi^I$ and $\sum_{I \in \mathcal{I}_{(1,3)}} \lambda_I = 1$. Since $x_i^\star = 0$ for $i = 4, 5, 6$, it follows immediately that $\lambda_I = 0$ for all $I \in \mathcal{I}_{(1,3)}$ with $I \cap \{e_4, e_5, e_6\} \neq \varnothing$. Thus, $x^\star = \lambda_{I_1}\chi^{I_1} + \lambda_{I_2}\chi^{I_2} + \lambda_{I_3}\chi^{I_3}$, where $I_j := \{e_j\}$ for $j = 1, 2, 3$. This in turn implies $\lambda_{I_1} = 1$ and $\lambda_{I_2} = \lambda_{I_3} = 0.5$, a contradiction, since $\lambda_{I_1} + \lambda_{I_2} + \lambda_{I_3} = 2$.

Consequently, the forbidden cardinality inequalities (1.2) together with the other valid inequalities are usually not sufficient to provide a complete linear description of $P_{\mathrm{M}}^{c}(E)$. The reason is that the inequalities (1.2) usually define very low dimensional faces of P_{M} as we have already observed at the end of Chapter 1. Consider again the example there depicted in Figure 2.2(a). None of the forests of cardinality 3, 5, 12, or 14 is tight with respect to the illustrated inequality. However, if we fill up F with further edges such that we obtain an edge set, say F', of rank 9, then the resulting inequality, which is illustrated in Figure 2.2(b), is still valid, and in addition, there shows up tight forests of cardinality 5 and 12.

We call an inequality as shown in Figure 2.2(b) *rank induced forbidden cardinality inequality.* Maurras [61] proved in his Phd Thesis that the following system provides a complete linear description of $P_{\mathrm{M}}^{c}(E)$:

$$
\begin{aligned}
\mathrm{FC}_F(x) := (c_{p+1} - r_{\mathcal{I}}(F))x(F) - (r_{\mathcal{I}}(F) - c_p)x(E \setminus F) & \\
\leq c_p(c_{p+1} - r_{\mathcal{I}}(F)) & \\
\text{for all } F \subseteq E \text{ with } c_p < r_{\mathcal{I}}(F) < c_{p+1}, & \\
p = 1, \dots, m-1, & \qquad (2.2)
\end{aligned}
$$

$$x(E) \geq c_1, \qquad (2.3)$$

$$x(E) \leq c_m, \qquad (2.4)$$

$$x(F) \leq r_{\mathcal{I}}(F) \qquad \text{for all } \varnothing \neq F \subseteq E, \qquad (2.5)$$

$$x_e \geq 0 \qquad \text{for all } e \in E. \qquad (2.6)$$

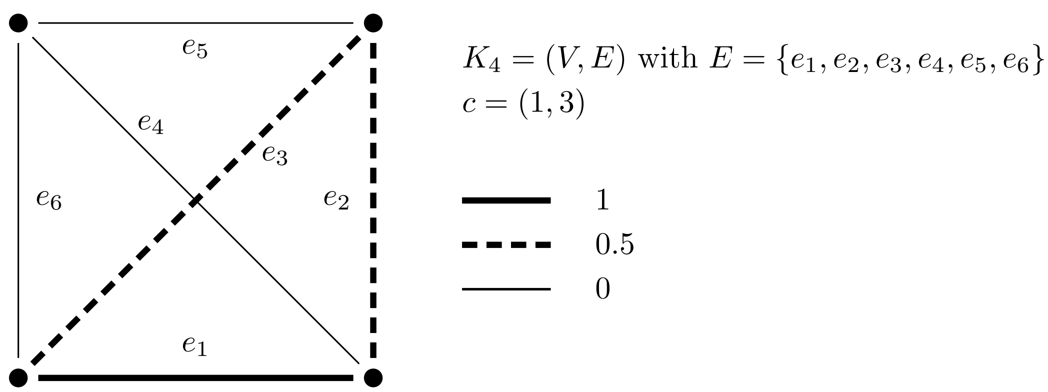
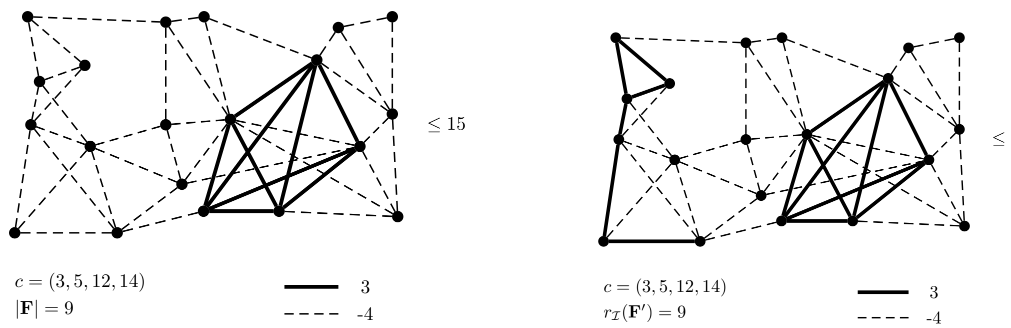


Figure 2.1: Support graph of $x^\star = [1, 0.5, 0.5, 0, 0, 0]$.



(a) Support graph of an ordinary FC-inequality

(b) Support graph of a rank induced FC-inequality

Figure 2.2: Ordinary FC-inequality and rank induced FC-inequality.

In the following subsection, we give an elementary proof of this result.

The rank induced forbidden cardinality inequality $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r_\mathcal{I}(F))$ associated with F, where $c_p < r_\mathcal{I}(F) < c_{p+1}$, is valid as can be seen as follows. The incidence vector of any $I \in \mathcal{I}$ of cardinality at most c_p satisfies the inequality, since $|I \cap F| = r_\mathcal{I}(I \cap F) \leq c_p$:

$$\begin{aligned}(c_{p+1} - r_\mathcal{I}(F))\chi^I(F) - (r_\mathcal{I}(F) - c_p)\chi^I(E \setminus F) &\leq (c_{p+1} - r_\mathcal{I}(F))\chi^I(F) \\ &\leq (c_{p+1} - r_\mathcal{I}(F))c_p.\end{aligned}$$

The incidence vector of any $I \in \mathcal{I}$ of cardinality at least c_{p+1} satisfies also the inequality, since $r_\mathcal{I}(I \cap F) \leq r_\mathcal{I}(F)$ and thus $r_\mathcal{I}(I \cap (E \setminus F)) \geq c_{p+1} - r_\mathcal{I}(F)$:

$$\begin{aligned}&(c_{p+1} - r_\mathcal{I}(F))\chi^I(F) - (r_\mathcal{I}(F) - c_p)\chi^I(E \setminus F) \\ \leq\ &(c_{p+1} - r_\mathcal{I}(F))r_\mathcal{I}(F) - (r_\mathcal{I}(F) - c_p)\chi^I(E \setminus F) \\ \leq\ &(c_{p+1} - r_\mathcal{I}(F))r_\mathcal{I}(F) - (r_\mathcal{I}(F) - c_p)(c_{p+1} - r_\mathcal{I}(F)) \\ =\ &c_p(c_{p+1} - r_\mathcal{I}(F)).\end{aligned}$$

However, it is not hard to see that some incidence vectors of independent sets I with $c_p < |I| < c_{p+1}$ violate the inequality.

When $M = (E, \mathcal{I})$ is the trivial matroid, i.e., all $F \subseteq E$ are independent sets, then $\mathcal{I} \cap \mathrm{CHS}^c(E) = \mathrm{CHS}^c(E)$. Thus, cardinality constrained matroids are a generalization of cardinality homogenous set systems.

2.1.1 A complete linear description

Let $M = (E, \mathcal{I})$ be a matroid. As already mentioned, $P_{\mathrm{M}}(E)$ is determined by (2.1). For any natural number $k \leq r_\mathcal{I}(E)$, the independence system

$M_k := (E, \mathcal{I}_k)$ defined by $\mathcal{I}_k := \{I \in \mathcal{I} : |I| \leq k\}$ is again a matroid and is called the *k-truncation* of M. Therefore, the matroid polytope $P_{\mathrm{M}_k}(E)$ associated with the k-truncation of M is defined by system (2.1), where the rank inequalities are indexed with $\mathcal{I}_k$ instead of $\mathcal{I}$. Following an argument of Gamble and Pulleyblank [38], the only set of the k-truncation which might be closed and inseparable with respect to the truncation, but not with respect to the original matroid M is E itself, and the rank inequality associated with E is the cardinality bound $x(E) \leq k$. Hence, we have shown

Theorem 2.1. Let $M = (E, \mathcal{I})$ be a matroid and $M_k := (E, \mathcal{I}_k)$ its k-truncation. Then, $P_{\mathrm{M}_k}(E)$ is determined by

$$\begin{array}{ll} x(F) \leq r_{\mathcal{I}}(F) & \text{for all } \varnothing \neq F \subseteq E, \\ x(E) \leq k, & \\ \quad x_e \geq 0 & \text{for all } e \in E. \end{array} \tag{2.7}$$

$\square$

Of course, the connection to cardinality constraints is obvious, since $P_{\mathrm{M}_k}(E) = P_{\mathrm{M}}^{(0,\dots,k)}(E)$. The basis system of M_k is the set of all bases B of E with respect to M_k, and all bases have cardinality k. The associated polytope

$$\mathrm{conv}\{\chi^B \in \mathbb{R}^E : B \text{ basis of } E \text{ with respect to } M_k\}$$

is determined by

$$\begin{array}{ll} x(F) \leq r_{\mathcal{I}}(F) & \text{for all } \varnothing \neq F \subseteq E, \\ x(E) = k, & \\ \quad x_e \geq 0 & \text{for all } e \in E. \end{array} \tag{2.8}$$

These preliminary remarks are sufficient to present the complete linear description. In what follows, we denote the rank function by r instead of r_I.

Theorem 2.2 (Maurras [61])**.** The cardinality constrained matroid polytope $P_{\mathrm{M}}^{c}(E)$ is completely described by system (2.2)-(2.6).

We give an elementary proof of this theorem.

Proof. Since all inequalities of system (2.2)-(2.6) are valid, $P_{\mathrm{M}}^{c}(E)$ is contained in the polyhedron defined by (2.2)-(2.6). To show the converse, we consider any valid inequality $b^T x \leq b_0$ for $P_{\text{M}}^c(E)$ and associate with the inequality the following subsets of E:

$$\begin{aligned} P &:= \{e \in E : b_e > 0\}, \\ Z &:= \{e \in E : b_e = 0\}, \\ N &:= \{e \in E : b_e < 0\}. \end{aligned}$$

We will show by case by case enumeration that the inequality $b^T x \leq b_0$ is dominated by some inequality of the system (2.2)-(2.6). By definition, $E = P \dot\cup Z \dot\cup N$, and hence, if $P = Z = N = \varnothing$, then $E = \varnothing$, and it is nothing to show. By a scaling argument we may assume that either $b_0 = 1$, $b_0 = 0$, or $b_0 = -1$.

(1) $b_0 = -1$.

(1.1) $c_1 = 0$. Then $0 \in P_{\text{M}}^c(E)$, and hence $0 = b \cdot 0 \leq -1$, a contradiction.

(1.2) $c_1 > 0$.

(1.2.1) $P = Z = \varnothing$, $N \neq \varnothing$. Assume that there is some tight $I \in \mathcal{I}$ with $|I| = c_p$, $p \geq 2$. Then, for any $J \subset I$ with $|J| = c_1$ holds: $\chi^J \in P_{\text{M}}^c(E)$ and $b^T \chi^J > b^T \chi^I = -1$, a contradiction. Therefore, if any $I \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|I| = c_1$. Thus, $b^T x \leq -1$ is dominated by the cardinality bound $x(E) \geq c_1$.

(1.2.2) $P \cup Z \neq \varnothing$, $N = \varnothing$. Then, $b^T y \geq 0$ for all $y \in P_{\text{M}}^c(E)$, a contradiction.

(1.2.3) $P \cup Z \neq \varnothing$, $N \neq \varnothing$. If $c_1 \leq r(P \cup Z)$, then there is some independent set $I \subseteq P \cup Z$ of cardinality c_1, and hence, $b^T \chi^I \geq 0$, a contradiction. Thus, $c_1 > r(P \cup Z)$. Assume, for the sake of contradiction, that there is some tight independent set J of cardinality c_p with $p \geq 2$. If $J \subseteq N$, then the incidence vector of any $K \subset J$ with $|K| = c_1$ violates $b^T x \leq -1$. Hence, $J \cap (P \cup Z) \neq \varnothing$. On the other hand, $J \cap N \neq \varnothing$ due to $c_p > c_1 > r(P \cup Z)$. However, by removing any $(c_p - c_1)$ elements in $N \cap J$, we obtain some independent set K of cardinality c_1 whose incidence vector violates the inequality $b^T x \leq -1$, a contradiction. Therefore, if any $T \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|T| = c_1$. Thus, $b^T x \leq -1$ is dominated by the bound $x(E) \geq c_1$.

(2) $b_0 = 0$.

(2.1) $P \cup Z \neq \varnothing$, $N = \varnothing$. Then, either $b^T x \leq 0$ is not valid or $b = 0$.

(2.2) $P = \varnothing$, $Z \cup N \neq \varnothing$. Then, $b^T x \leq 0$ is dominated by the nonnegativity constraints $x_e \geq 0$ for $e \in N$ or $b = 0$.

(2.3) $P \neq \varnothing$, $N \neq \varnothing$.

(2.3.1) $c_1 > 0$. If $c_1 \leq r(P \cup Z)$, then there is some independent set $I \subseteq P \cup Z$ with $I \cap P \neq \varnothing$ of cardinality c_1, and hence, $b^T \chi^I > 0$, a contradiction. Thus, $c_1 > r(P \cup Z)$. Assume, for the sake of contradiction, that there is some tight independent set J of cardinality c_p with $p \geq 2$. Since $c_p > c_1 > r(P \cup Z)$ and J is tight, $J \cap (P \cup Z) \neq \varnothing \neq J \cap N$. From here, the proof for this case can be finished as the proof for the case (1.2.3) with $b_0 = 0$ instead of $b_0 = -1$ in order to show that $b^T x \leq 0$ is dominated by the cardinality bound $x(E) \geq c_1$.

(2.3.2) $c_1 = 0$. As in case (2.3.1), it follows immediately that $c_2 > r(P \cup Z)$, and if $I \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ is tight, then $|I| = c_1 = 0$, that is, $I = \varnothing$, or $|I| = c_2$. Moreover, if $I \in \mathcal{I}$ with $|I| = c_2$ is tight, then follows $|I \cap (P \cup Z)| = r(P \cup Z)$. Hence, $b^T x \leq b_0$ is dominated by the rank induced forbidden cardinality inequality $\mathrm{FC}_F(x) \leq 0$ with $F = P \cup Z$.

(3) $b_0 = 1$.

(3.1) $P = \varnothing$, $Z \cup N \neq \varnothing$. Then, $b \leq 0$, and hence $b^T x \leq 1$ is dominated by any nonnegativity constraint $x_e \geq 0$, $e \in E$.

(3.2) $P \cup Z \neq \varnothing$, $N = \varnothing$. Assume that there is some $I \in \mathcal{I}, I \notin \mathrm{CHS}^c(E)$ with $|I| < c_m$ that violates $b^T x \leq 1$. Then, of course, all independent sets $J \supset I$ violate $b^T x \leq 1$, in particular, those J with $|J| = c_m$, a contradiction. Hence, $b^T x \leq 1$ is not only a valid inequality for $P^c_{\mathrm{M}}(E)$ but also for $P^{(0,1,\ldots,c_m)}_{\mathrm{M}}(E)$, that is, $b^T x \leq 1$ is dominated by some inequality of the system (2.7) with $k = c_m$.

(3.3) $P \neq \varnothing$, $N \neq \varnothing$. Let $p \in \{1, \ldots, m\}$ be minimal such that there is a tight independent set $I^\star$ of cardinality c_p. Of course, $c_p > 0$, because otherwise $I^\star$ could not be tight. If $p = m$, then $b^T x \leq 1$ is dominated by the cardinality bound $x(E) \leq c_m$, because then all tight $J \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ have to be of cardinality $c_p = c_m$. So, let $0 < c_p < c_m$. We distinguish 2 subcases.

(3.3.1) $c_p \geq r(P \cup Z)$. Suppose, for the sake of contradiction, that there is some tight independent set I of cardinality c_p such that $|I \cap (P \cup Z)| < r(P \cup Z)$. Then, $I \cap (P \cup Z)$ can be completed to a basis B of $P \cup Z$, and since $|B| \leq |I|$, there is some $K \subseteq I \setminus B$ such that $I' := B \cup K \in \mathcal{I}$ and $|I'| = |I|$. K is maybe the empty set. Anyway, by construction, I' is of cardinality c_p and violates the inequality $b^T x \leq 1$. Thus, $|I \cap (P \cup Z)| = r(P \cup Z)$. For the same reason, any tight $J \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ satisfies $|J \cap (P \cup Z)| = r(P \cup Z)$, and since p is minimal, $|J| \geq c_p$. Now, with similar arguments as in case (1.2.3) one can show that if $T \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ is tight, then $|T| = c_p$. Thus, $c_p = c_1 > 0$ and $b^T x \leq 1$ is dominated by the cardinality bound $x(E) \geq c_1$.

(3.3.2) $c_p < r(P \cup Z)$. Following the argumentation line in (3.3.1), we see that $I \subseteq P \cup Z$ and $|I \cap P|$ has to be maximal for any tight independent set I of cardinality c_p. Assume that $c_{p+1} \leq r(P \cup Z)$. Then, from any tight independent set I with $|I| = c_p$ we can construct a tight independent set J with $|J| = c_{p+1}$ by adding some elements $e \in Z$. However, it is not hard to see that there is no tight $K \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ that contains some $e \in N$. Thus, when $c_{p+1} \leq r(P \cup Z)$, $b^T x \leq 1$ is dominated by the nonnegativity constraints $y_e \geq 0$, $e \in N$. Therefore, $c_{p+1} > r(P \cup Z)$. The following is now immediate: If $I \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ is tight, then $|I| = c_p$ or $|I| = c_{p+1}$; if $|I| = c_p$, then $I \subset P \cup Z$, and if $|I| = c_{p+1}$, then $|I \cap (P \cup Z)| = r(P \cup Z)$ and $c_{p+1} > r(P \cup Z)$. Thus, $b^T x \leq 1$ is dominated by the rank induced forbidden cardinality inequality $\mathrm{FC}_{P \cup Z}(x) \leq c_p(c_{p+1} - r(P \cup Z))$.

□

2.1.2 Facets

We first study the facial structure of a single cardinality constrained matroid polytope $P_{\mathrm{M}}^{(k)}(E)$. All points of $P_{\mathrm{M}}^{(k)}(E)$ satisfy the equation $x(E) = k$, and hence, any inequality $x(F) \leq r(F)$ is equivalent to the inequality $x(E \setminus F) \geq k - r(F)$. Motivated by this observation, we introduce the following definitions. For any $F \subseteq E$, the number $r^k(F) := k - r(E \setminus F)$ is called the *k-rank* of F. Due to the submodularity of r we have $r^k(F_1) + r^k(F_2) \leq r^k(F)$ for all F_1, F_2 with $F = F_1 \dot\cup F_2$, and F is said to be *k-separable* if equality holds for some $F_1 \neq \varnothing \neq F_2$, otherwise *k-inseparable*. Due to the equation $x(E) = k$, $\dim P_{\mathrm{M}}^{(k)}(E) \leq |E| - 1$, and in fact, in the most cases we have equality. However, if $\dim P_{\mathrm{M}}^{(k)}(E) < |E| - 1$, then at least one rank inequality $x(F) \leq r(F)$ with $\varnothing \neq F \subsetneq E$ is an implicit equation. As is easily seen, this implies that an inequality $x(F') \leq r(F')$ (or $x(F') \geq r^k(F')$) does not necessarily induce a facet of $P_{\mathrm{M}}^{(k)}(E)$, although F is inseparable (k-inseparable). To avoid the challenges involved, we only characterize the polytopes $P_{\mathrm{M}}^{(k)}(E)$ of dimension $|E|-1$.

Lemma 2.3. Let $M=(E,\mathcal{I})$ be a matroid and for any $k \in \mathbb{N}$, $0<k<r(E)$, $M_k=(E,\mathcal{I}_k)$ the k-truncation of M with rank function r_k. Then, E is inseparable with respect to r_k.

Proof. Let $E=F_1 \dot\cup F_2$ with $F_1 \neq \varnothing \neq F_2$ be any partition of E. We have to show that $r_k(F_1)+r_k(F_2) > r_k(E)$. By definition, $r_k(E)=k$. First, let $r(F_i) \leq k$ for $i=1,2$. Then, $r_k(F_i)=r(F_i)$ and consequently, $r_k(F_1)+r_k(F_2)=r(F_1)+r(F_2) \geq r(E) > k$ due to the submodularity of r. Next, let w.l.o.g. $r(F_1)>k$. Then, $r_k(F_1)=k$ and, since $F_2 \neq \varnothing$, $r_k(F_2)>0$. Thus, $r_k(F_1)+r_k(F_2)=k+r_k(F_2)>k$. □

Lemma 2.4. Let $M=(E,\mathcal{I})$ be a matroid, $M_k=(E,\mathcal{I}_k)$ its k-truncation with rank function r_k, $\varnothing \neq F \subseteq E$, and $\bar{F}=E \setminus F$ be closed with $r(\bar{F}) < k < r(E)$. Then, F is k-inseparable with respect to r_k.

Proof. $r(\bar{F})<k$ implies $r_k(\bar{F})=r(\bar{F})$, and since beyond it $\bar{F}$ is closed with respect to r, it is also closed with respect to r_k. Let $F=F_1 \dot\cup F_2$ be a proper partition of F. We have to show that $r_k^k(F_1)+r_k^k(F_2) < r_k^k(F)$. First, suppose that $I \in \mathcal{I}$ with $|I|=k$ and $|I \cap \bar{F}|=r_k(\bar{F})$ implies $I \cap F_1=\varnothing$ or $I \cap F_2=\varnothing$. Since $\bar{F}$ is closed with respect to r_k, it follows that $r_k^k(F_1)=r_k^k(F_2)=0$, while $r_k^k(F)=k-r_k(\bar{F})>0$. So assume that there is some independent set I' of cardinality k such that $|I' \cap \bar{F}|=r_k(\bar{F})$ and $I' \cap F_i \neq \varnothing$ for $i=1,2$. Since $k<r(E)$, there is some element e such that $I:=I' \cup \{e\}$ is independent with respect to r. Set $I_1:=I \setminus \{f_1\}$ and $I_2:=I \setminus \{f_2\}$ for $f_1 \in I \cap F_1, f_2 \in I \cap F_2$. Then, $r_k^k(F_1) \leq |I_1 \cap F_1|$ and $r_k^k(F_2) \leq |I_2 \cap F_2|$. Hence, $r_k^k(F_1)+r_k^k(F_2) \leq |I_1 \cap F_1|+|I_2 \cap F_2| < |I_1 \cap F_1|+|I_1 \cap F_2|=|I_1 \cap F|=r_k^k(F)$. □

Lemma 2.5. Let $M=(E,\mathcal{I})$ be a matroid, $\varnothing \neq F \subseteq E$, and A the matrix whose rows are the incidence vectors of $I \in \mathcal{I}$ with $|I|=k$ that satisfy the inequality $x(F) \geq r^k(F)$ at equality. Moreover, denote by A_F the submatrix of A restricted to F. Then, $\operatorname{rank}(A_F)=|F|$ if and only if $r^k(F) \geq 1$, $\bar{F}:=E \setminus F$ is closed, and (i) F is k-inseparable or (ii) $k<r(E)$.

Proof. Necessity. The inequality $x(F) \geq r^k(F)$ is valid for $P_{\mathrm{M}}^{(k)}(E)$. As is easily seen, if $r^k(F) \leq 0$, then $\operatorname{rank}(A_F) < |F|$. Next, assume that $\bar{F}$ is not closed. Then, there is some $e \in F$ such that $r(\bar{F} \cup \{e\})=r(\bar{F})$ which is equivalent to $r^k(F)=r^k(F \setminus \{e\})$. Thus, $x(F) \geq r^k(F)$ is the sum of the inequalities $x(F \setminus \{e\}) \geq r^k(F \setminus \{e\})$ and $x_e \geq 0$. This implies

$\chi^I_e = 0$ for all incidence vectors of independent sets I with $|I| = k$ satisfying $x(F) \geq r^k(F)$ at equality. Again, it follows $\text{rank}(A_F) < |F|$. Finally, suppose that neither $k < r(E)$ nor F is k-inseparable. Then, $k = r(E)$ and F is $r(E)$-separable. Thus, the inequality $x(F) \geq r^{r(E)}(F)$ is the sum of the valid inequalities $x(F_1) \geq r^{r(E)}(F_1)$ and $x(F_2) \geq r^{r(E)}(F_2)$ for some $F_1 \neq \varnothing \neq F_2$ with $F = F_1 \dot\cup F_2$. Setting $\lambda := r^{r(E)}(F_2)\chi^{F_1}_F - r^{r(E)}(F_1)\chi^{F_2}_F$, we see that for any $|F| \times |F|$ submatrix $\tilde{A}_F$ of A_F we have $\tilde{A}_F\lambda = 0$, that is, the columns of $\tilde{A}_F$ are linearly dependent which implies $\text{rank}(A_F) < |F|$.

Suffiency. First, let $k = r(E)$. Suppose $\text{rank}(A_F) < |F|$. Then, $A_F\lambda = 0$ for some $\lambda \in \mathbb{R}^F$, $\lambda \neq 0$. Since $\bar{F}$ is closed and $r^k(F) \geq 1$ (that is, $r(\bar{F}) < k$), for each $e \in F$ there is an independent set I with $|I| = k$ that contains e and whose incidence vector satisfies $x(F) \geq r^k(F)$ at equality. Thus, A_F does not contain a zero-column. Moreover, $A_F \geq 0$, and hence, $F_1 := \{e \in F : \lambda_e > 0\}$ and $F_2 := \{e \in F : \lambda_e \leq 0\}$ defines a proper partition of F. Let $J \subseteq \bar{F}$ with $|J| = r(\bar{F})$ be an independent set. For $i = 1, 2$, let $B_i \subseteq F$ be an independent set such that $J \cup B_i$ is a basis of E and $J \cup (B_i \cap F_i)$ is a basis of $\bar{F} \cup F_i$. Set $S_i := B_i \cap F_i$ and $T_i := B_i \setminus S_i$ $(i = 1, 2)$. By construction, $T_1 \subseteq F_2$ and $T_2 \subseteq F_1$. By matroid axiom (iii), to $J \cup S_1$ there is some $U_1 \subseteq J \cup B_2$ such that $K := J \cup S_1 \cup U_1$ is a basis of F. Clearly, $U_1 \subseteq (B_2 \cap F_2) = S_2$. Since the incidence vectors of $J \cup B_1$ and K are rows of A, it follows immediately $\lambda(T_1) = \lambda(U_1)$. With an analogous construction one can show that there is some $U_2 \subseteq S_1$ such that $\lambda(U_2) = \lambda(T_2)$. It follows, $\lambda(T_2) = -\lambda(S_2) \geq -\lambda(U_1) = -\lambda(T_1) = \lambda(S_1) \geq \lambda(U_2) = \lambda(T_2)$. Thus, between all terms we have equality implying $\lambda(S_1) = \lambda(U_2)$. Moreover, since $U_2 \subseteq S_1$ and $\lambda_e > 0$ for all $e \in S_1$, it follows $S_1 = U_2$. Hence, $K = J \cup S_1 \cup S_2$. This, in turn, implies that F is k-separable, a contradiction.

It remains to show that the statement is true if $k < r(E)$. Let $M_k = (E, \mathcal{I}_k)$ be the k-truncation of M with rank function r_k. By hypothesis, all conditions of Lemma 2.4 hold. Hence, F is k-inseparable with respect to r_k. Thus, all conditions of the lemma hold for r_k instead of r and hence, $\text{rank}(A_F) = |F|$. □

Theorem 2.6. Let $M = (E, \mathcal{I})$ be a matroid and $k \in \mathbb{N}$, $0 < k \leq r(E)$.

(a) $P^{(k)}_{\text{M}}(E)$ has dimension $|E|-1$ if and only if E is inseparable or $k < r(E)$.

(b) Let $\dim P^{(k)}_{\text{M}}(E) = |E| - 1$ and $\varnothing \neq F \subsetneq E$. The inequality $x(F) \leq r(F)$ defines a facet of $P^{(k)}_{\text{M}}(E)$ if and only if F is closed and inseparable, $r(F) < k$, and (i) $\bar{F} := E \setminus F$ is k-inseparable or (ii) $k < r(E)$.

Proof. (a) First, let $k = r(E)$. For any $\varnothing \neq F \subseteq E$, the rank inequality $x(F) \leq r(F)$ defines a facet of $P_{\text{M}}(E)$ if and only if F is closed and inseparable. Consequently, the polytope $P^{(r(E))}_{\text{M}}(E)$, which is a face of $P_{\text{M}}(E)$, has dimension $|E|-1$ if and only if E is inseparable. Next, let $0 < k < r(E)$. By Lemma 2.3, E is inseparable with respect to the rank function r_k of the k-truncation $M_k = (E, \mathcal{I}_k)$. Consequently, $x(E) \leq r_k(E) = k$ defines a facet of $P_{\mathcal{I}_k}(E)$ and hence, $\dim P_{\mathrm{M}}^{(k)}(E) = |E|-1$.

(b) Clearly, $x(F) \leq r(F)$ does not induce a facet of $P_{\mathrm{M}}^{(k)}(E)$ if F is separable or not closed, since $\dim P_{\mathrm{M}}^{(k)}(E) = |E|-1$, and hence, any inequality that is not facet defining for $P_{\mathrm{M}}(E)$ is also not facet defining for $P_{\mathrm{M}}^{(k)}(E)$. Next, if $r(F) \geq k$, then holds obviously $x(F) \leq x(E) = k \leq r(F)$, that is, either F is not closed, $x(F) \leq r(F)$ is an implicit equation, or the face induced by $x(F) \leq r(F)$ is the emptyset. Finally, assume that F is closed but neither (i) nor (ii) holds. Then, $k = r(E)$ and $\bar{F}$ is k-separable. Thus, there are nonempty subsets $\bar{F}_1, \bar{F}_2$ of $\bar{F}$ with $\bar{F} = \bar{F}_1 \dot\cup \bar{F}_2$ such that $r^k(\bar{F}) = r^k(\bar{F}_1) + r^k(\bar{F}_2)$. Now, the inequality $x(\bar{F}) \geq r^k(\bar{F})$, which is equivalent to $x(F) \leq r(F)$, is the sum of the valid inequalities $x(\bar{F}_i) \geq r^k(\bar{F}_i)$, $i = 1, 2$, both not being implicit equations.

To show the converse, let F satisfy all conditions mentioned in Theorem 2.6 (b). The restriction of $M = (E, \mathcal{I})$ to F is again a matroid. Denote it by $M' = (F, \mathcal{I}')$ and its rank function by r'. F remains inseparable with respect to r'. Thus, the restriction of $x(F) \leq r(F)$ to F, denoted by $x_F(F) \leq r(F) = r'(F)$, induces a facet of $P_{\mathcal{I}'}(F)$. A set of affinely independent vectors whose sum of components is equal to some ℓ, is also linearly independent. Thus, there are $|F|$ linearly independent vectors $\chi^{I'_j}$ of independent sets $I'_j \in \mathcal{I}'$ of cardinality $r'(F)$ ($j = 1, \ldots, |F|$). The sets I'_j are also independent sets with respect to $\mathcal{I}$. Due to the matroid axiom (iii), $P := I'_1$ can be completed to an independent set I_1 of cardinality k. Since $P \subseteq F$ and $|P| = r(F)$, $Q := I_1 \setminus P \subseteq \bar{F}$. Now, $I'_j, I_1 \in \mathcal{I}$, $I'_j \subseteq F$, and $r(F) = |I'_j| < |I_1| = k$. Hence, $I_j := I'_j \cup Q \in \mathcal{I}$ for all j. Consequently, we have $|F|$ linearly independent vectors $\chi^{I_j} \in P_{\mathrm{M}}^{(k)}(E)$ satisfying $x(F) \leq r(F)$ at equality.

Next, let A be the matrix whose rows are the incidence vectors of tight independent sets and $A_{\bar{F}}$ its restriction to $\bar{F}$. By Lemma 2.5, $A_{\bar{F}}$ contains a $|\bar{F}| \times |\bar{F}|$ submatrix B of full rank. By construction, each row B_i of B is an incidence vector of an independent set $J'_i \subseteq \bar{F}$ with $|J'_i| = r^k(\bar{F})$. W.l.o.g. we may assume that $B_1 = \chi^Q$, that is, $Q = J'_1$. By a similar argument as above, the independent sets $J_i := J'_i \cup P$ are tight and its incidence vectors are linearly independent.

Alltogether we have $|F|$ linearly independent vectors χ^{I_j} with $I_j \cap \bar{F} = Q$ and $|\bar{F}|$ linearly independent vectors χ^{J_i} with $J_i \cap F = P$, where $J_1 = I_1$. As is easily seen, this yields a system of $|F| + |\bar{F}| - 1 = |E| - 1$ linearly independent vectors satisfying $x(F) \leq r(F)$ at equality. $\square$

Theorem 2.7. $P_{\mathrm{M}}^{c}(E)$ is fulldimensional unless $c = (0, r(E))$ and E is separable.

Proof. Clearly, $\dim P_{\mathrm{M}}^{c}(E) \geq \dim P_{\mathrm{M}}^{(c_p)}(E) + 1$ for all p, since the equation $x(E) = c_p$ is satisfied by all $y \in P_{\mathrm{M}}^{(c_p)}(E)$ but violated by at least one vector $z \in P_{\mathrm{M}}^{c}(E)$.

If $0 < c_p < r(E)$ for some p, then, by Theorem 2.6, $\dim P_{\mathrm{M}}^{(c_p)}(E) = |E| - 1$, and consequently $\dim P_{\mathrm{M}}^{c}(E) = |E|$. If there is no such p, then $c = (0, r(E))$. Again by Theorem 2.6, $\dim P_{\mathrm{M}}^{(r(E))}(E) = |E| - 1$ if and only if E is inseparable. Since $\dim P_{\mathrm{M}}^{(0,r(E))}(E) = \dim P_{\mathrm{M}}^{(r(E))}(E) + 1$, it follows the claim. □

Theorem 2.8. For any $\varnothing \neq F \subseteq E$, the rank inequality $x(F) \leq r(F)$ defines a facet of $P_{\mathrm{M}}^{c}(E)$ if and only if one of the following conditions holds.

(i) $0 < r(F) < c_{m-1}$ and F is closed and inseparable.

(ii) $0 < c_{m-1} = r(F) < c_m < r(E)$, and F is closed and inseparable.

(iii) $0 < c_{m-1} = r(F) < c_m = r(E)$, F is closed and inseparable, $\bar{F}$ is c_m-inseparable, and E is inseparable.

(iv) $0 < c_{m-1} < c_m = r(F)$, $F = E$, and $c_m < r(E)$ or E inseparable.

(v) $c_{m-1} = c_1 = 0$, $c_m = r(E)$, and $r(F) + r(E \setminus F) = r(E)$.

Proof. We prove the theorem by case by case enumeration.

(a) Let $0 < r(F) < c_{m-1}$. It is not hard to see that if F is separable or not closed, then $x(F) \leq r(F)$ does not define a facet of $P_{\mathrm{M}}^{c}(E)$. So, let F be closed and inseparable. By Theorem 2.6, $x(F) \leq r(F)$ defines a facet of $P_{\mathrm{M}}^{(c_{m-1})}(E)$ and $\dim P_{\mathrm{M}}^{(c_{m-1})}(E) = |E| - 1$. Thus, it defines also a facet of $P_{\mathrm{M}}^{c}(E)$.

(b) Let $0 < c_{m-1} = r(F) < c_m < r(E)$. Clear by interchanging c_{m-1} and c_m in item (a).

(c) Let $0 < c_{m-1} = r(F) < c_m = r(E)$. The conditions mentioned in (iii) are equivalent to the postulation that $x(F) \leq r(F)$ defines a facet of $P_{\mathrm{M}}^{(c_m)}(E)$ and $\dim P_{\mathrm{M}}^{(c_m)}(E) = |E| - 1$. If, indeed, the latter is true, then $x(F) \leq r(F)$ induces a facet also of $P_{\mathrm{M}}^{c}(E)$. To show the converse, suppose, for the sake of contradiction, that $x(F) \leq r(F)$ does not induce a facet of $P_{\mathrm{M}}^{(c_m)}(E)$ or $\dim P_{\mathrm{M}}^{(c_m)}(E) < |E| - 1$. Let $\mathcal{B} := \{\chi^{I_j} : I_j \in \mathcal{I}, |I_j| = c_m, j = 1, \ldots, z, \}$ be an affine basis of the face of $P_{\mathrm{M}}^{(c_m)}(E)$ induced by $x(F) \leq r(F)$. By hypothesis, $z \leq |E| - 2$. Moreover, set $J := I_1 \cap F$ and $K := I_1 \setminus J$. Then, any incidence vector of an independent set $L \subseteq F$ with $|L| = c_{m-1}$ can be obtained as an affine combination of the set $\mathcal{B}' := \mathcal{B} \cup \{\chi^J\}$, which can be seen as follows: $L, I_1 \in \mathcal{I}$, and $|L| = r(F)$ implies $L \cup K \in \mathcal{I}$. Consequently, $\chi^L = \chi^{L\cup K} - \chi^K$. Now, $\chi^K = \chi^{I_1} - \chi^J$ and $\chi^{L\cup K} = \sum_{j=1}^{z} \lambda_j \chi^{I_j}$ with $\sum_{j=1}^{z} \lambda_j = 1$, since $L \cup K$ is tight. Thus, $\chi^L = \sum_{j=1}^{z} \lambda_j \chi^{I_j} - \chi^{I_1} + \chi^J$, that is, χ^L is in the affine hull of $\mathcal{B}'$. Since $|\mathcal{B}'| \leq |E| - 1$, $x(F) \leq r(F)$ is not facet defining for $P^c_{\mathrm{M}}(E)$, a contradiction.

(d) Let $0 < c_{m-1} < r(F) < c_m$. Since none of the independent sets I with $|I| = c_p$ is tight for $p = 1, \ldots, m-1$, $x(F) \leq r(F)$ defines a facet of $P^c_{\mathrm{M}}(E)$ if and only if it is an implicit equation for $P^{(c_m)}_{\mathrm{M}}(E)$ and $\dim P^{(c_m)}_{\mathrm{M}}(E) = |E| - 1$. However, $\dim P^{(c_m)}_{\mathrm{M}}(E) = |E| - 1$ implies $c_m < r(E)$ or E is inseparable. In either case, it follows that $x(F) \leq r(F)$ is an implicit equation for $P^{(c_m)}_{\mathrm{M}}(E)$ if and only if $F = E$. Thus, $r(F) = c_m$, a contradiction.

(e) Let $0 < c_{m-1} < c_m = r(F)$. Clearly, if $F \subset E$, then $x(F) \leq r(F)$ is strictly dominated by the cardinality bound $x(E) \leq c_m$. Consequently, $F = E$ and $x(F) \leq r(F)$ is an implicit equation for $P^{(c_m)}_{\mathrm{M}}(E)$. For the same reasons as in (d), $\dim P^{(c_m)}_{\mathrm{M}}(E) = |E| - 1$. Hence, $c_m < r(E)$ or E is inseparable.

(f) Let $c_{m-1} = c_1 = 0$. Again, $x(F) \leq r(F)$ defines a facet of $P^c_{\mathrm{M}}(E)$ if and only if it is an implicit equation for $P^{(c_m)}_{\mathrm{M}}(E)$. This is the case if and only if $c_m = r(E)$ and $r(F) + r(E \setminus F) = r(E)$.

(g) Let $r(F) > c_m$. Then, $x(F) \leq x(E) \leq c_m < r(F)$, that is, the face induced by $x(F) \leq r(F)$ is the empty set. □

Theorem 2.9. Let $F \subseteq E$ with $c_p < r(F) < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. Then, the rank induced forbidden cardinality inequality $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ defines a facet of $P^c_{\mathrm{M}}(E)$ if and only if

(a) $c_p = c_1 = 0$ and $x(F) \leq r(F)$ defines a facet of $P^{(c_{p+1})}_{\mathrm{M}}(E)$, or

(b) $c_p > 0$, F is closed and (i) $\bar{F} := E \setminus F$ is c_{p+1}-inseparable or (ii) $c_{p+1} < r(E)$.

Proof. For $P^{(c_{p+1})}_{\mathrm{M}}(E)$, the inequality $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ is equivalent to $x(F) \leq r(F)$, while for $P^{(c_p)}_{\mathrm{M}}(E)$, it is equivalent to $x(F) \leq c_p$. Hence, in case $c_p = c_1 = 0$, $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ induces a facet of $P^c_{\mathrm{M}}(E)$ if and only if it induces a facet of $P^{(c_{p+1})}_{\mathrm{M}}(E)$. When $\dim P^{(c_{p+1})}_{\mathrm{M}}(E) = |E| - 1$, this is the case if and only if F is closed and inseparable and (i) $\bar{F}$ is c_{p+1}-inseparable or (ii) $c_{p+1} < r(E)$, see Theorem 2.6 (b).

In the following, let $c_p > 0$. Let A be the matrix whose rows are the incidence vectors of $I \in \mathcal{I}$ with $|I| = c_p$ or $|I| = c_{p+1}$ that satisfy the inequality $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ at equality. Denote by A_F and $A_{\bar{F}}$

the restriction of A to F and $\bar{F}$, respectively. By Theorem 2.7, $P_{\mathrm{M}}^{c}(E)$ is fulldimensional. Hence, $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ is facet defining if and only if the affine rank of A is equal to $|E|$.

If F is not closed, then there is some $e \in \bar{F}$ with $r(F \cup \{e\}) = r(F)$. Thus, $\mathrm{FC}_{F'}(x) \leq c_p(c_{p+1} - r(F'))$ is a valid inequality for $P_{\mathrm{M}}^{c}(E)$, where $F' := F \cup \{e\}$, and $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ is the sum of this inequality and $-(c_{p+1} - c_p)x_e \leq 0$. Next, assume that neither (i) nor (ii) holds. Then, $c_{p+1} = r(E)$ and $\bar{F}$ is $r(E)$-separable. Thus, there is a proper partition $\bar{F} = \bar{F}_1 \dot{\cup} \bar{F}_2$ of $\bar{F}$ with $r^{r(E)}(\bar{F}_1) + r^{r(E)}(\bar{F}_2) = r^{r(E)}(\bar{F})$. Since F is closed, it is not hard to see that $r^{r(E)}(\bar{F}_i) > 0$ which implies $c_p < r(F \cup \bar{F}_i) < r(E)$ for $i = 1, 2$, and hence, the inequalities $\mathrm{FC}_{F \cup \bar{F}_1}(x) \leq c_p(c_{p+1} - r(F \cup \bar{F}_1))$ and $\mathrm{FC}_{F \cup \bar{F}_2}(x) \leq c_p(c_{p+1} - r(F \cup \bar{F}_2))$ are valid. One can check again that then $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ is the sum of these both rank induced forbidden cardinality inequalities.

To show the converse, let $M^F = (F, \mathcal{I}^F)$ with $\mathcal{I}^F := \{I \cap F : I \in \mathcal{I}\}$ be the restriction of M to F and $M^F_{c_p} = (F, \mathcal{I}^F_{c_p})$ the c_p-truncation of M^F. Since $0 < c_p < r(F)$, Lemma 2.3 implies that F is inseparable with respect to the rank function of $M^F_{c_p}$. Consequently, the restriction of $x(F) \leq c_p$ to F defines a facet of $P_{\mathcal{I}^F_{c_p}}(F)$. Hence, A contains an $|F| \times |E|$ submatrix B such that B_F is nonsingular and $B_{\bar{F}} = 0$. Next, since F is closed, $r^{c_{p+1}}(\bar{F}) \geq 1$, and (i) $\bar{F}$ is c_{p+1}-inseparable or (ii) $c_{p+1} < r(E)$, Lemma 2.5 implies that A contains a $|\bar{F}| \times |E|$ submatrix C such that $C_{\bar{F}}$ is nonsingular. Thus,

$$D := \begin{pmatrix} B_F & 0 \\ C_F & C_{\bar{F}} \end{pmatrix}$$

is a nonsingular $|E| \times |E|$ submatrix of A (or a row permutation of A). $\square$

2.1.3 Separation problem

Given any matroid $M = (E, \mathcal{I})$, any cardinality sequence c, and any $x^\star \in \mathbb{R}^E$, the separation problem consists of finding an inequality among (2.2)-(2.6) violated by $x^\star$ if there is any. This problem should be solvable efficiently, due to the polynomial time equivalence of optimization and separation (see Grötschel, Lovász, and Schrijver [46]). By default, we may assume that $x^\star$ satisfies the cardinality bounds (2.3), (2.4) and the nonnegativity constraints (2.6). A violated rank inequality among (2.5) (if there is any) can be found by a polynomial time algorithm proposed by Cunningham [20]. So, we are actually interested only in finding an efficient algorithm that solves the separation problem for the class of rank induced forbidden cardinality inequalities (2.2). If $r(F) = |F|$ for all $F \subseteq E$, then the separation routine proposed by Grötschel [45] can be applied. As is already mentioned, it works as follows. For each forbidden cardinality k one just needs to take the first k greatest weights, say $x^\star_{e_1}, \dots, x^\star_{e_k}$, and check whether the rank induced forbidden cardinality inequality associated with $F := \{e_1, \dots, e_k\}$ is violated by $x^\star$. Otherwise we shall see that the separation problem for the rank induced forbidden cardinality inequalities can be transformed to that for the rank inequalities.

The separation problem for the class of rank induced forbidden cardinality inequalities consists of checking whether or not

$$\begin{array}{l}(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) \ \leq \ c_p(c_{p+1} - r(F)) \\ \text{for all } F \subseteq E \text{ with } c_p < r(F) < c_{p+1} \text{ for some } p \in \{0, \dots, m-1\}.\end{array}$$

For any $F \subseteq E$,

$$\begin{array}{rl} & (c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) \leq c_p(c_{p+1} - r(F)) \\ \Leftrightarrow & (c_{p+1} - c_p)x^\star(F) - (r(F) - c_p)x^\star(E) \leq c_p(c_{p+1} - r(F)) \\ \Leftrightarrow & x^\star(F) \leq \frac{c_p(c_{p+1} - r(F)) + (r(F) - c_p)x^\star(E)}{(c_{p+1} - c_p)} =: \gamma_F. \end{array}$$

Moreover, for any $k \in \{1, \dots, r(E)\}$, the right hand sides of the inequalities $x^\star(F) \leq \gamma_F$ for $F \subseteq E$ with $r(F) = k$ are equal and differ only by a constant to the right hand sides of the corresponding rank inequalities $x(F) \leq r(F) = k$. Thus, both the separation problem for the rank inequalities and rank induced forbidden cardinality inequalities could be solved by finding, for each $k \in \{1, \dots, |E|\}$, a set $F^\star \subseteq E$ of rank k that maximizes $x^\star(F)$. If $x^\star(F^\star) > k$, then the inequality $x(F^\star) \leq r(F^\star)$ is violated by $x^\star$. If, in addition, $c_p < k < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$ and $x^\star(F^\star) > \gamma_{F^\star}$, then $x^\star$ violates the rank induced forbidden cardinality inequality associated with $F^\star$.

This natural generalization of Grötschel's separation algorithm, however, seems usually not to result in an efficient separation routine. In order to mark the difficulties, we investigate the above approach for the class of rank inequalities, when $M = (E, \mathcal{I})$ is the graphic matroid defined on some graph $G = (V, E)$. It is well known that the closed and inseparable rank inequalities for the graphic matroid are of the form $x(E(W)) \leq |W| - 1$ for $\varnothing \neq W \subseteq V$. If we would tackle the separation problem for this class of inequalities by finding, for each $k \in \{1, \dots, |V|\}$ separately, a set $W^\star_k$ that maximizes $x^\star(E(W))$ such that $|W| = k$, then we would run into trouble, since for each k, such a problem is the weighted version of the densest k-subgraph problem which is known to be NP-hard (see Feige and Seltser [31]).

The last line of argument indicates that it is probably not a good idea to split the separation problem for the rank induced forbidden cardinality inequalities (2.2) into many separation problems by defining for each forbidden cardinality k between c_1 and c_m a subclass of inequalities

$$\mathrm{FC}_F(x) \leq c_p(c_{p+1} - k) \text{ for all } F \subseteq E \text{ with } r(F) = k.$$

It would be rather better to approach it as "non-cardinality constrained" problem. And this is exactly what Cunningham did for the rank inequalities.

In what follows, we firstly remind of some important facts regarding Cunningham's algorithm for the separation of the rank inequalities. Afterwards, we show how the separation problem for the rank induced forbidden cardinality inequalities can be reduced to that for the rank inequalities.

The theoretical background of Cunningham's separation routine is the following min-max relation.

Theorem 2.10 (Edmonds [28])**.** For any $x^\star \in \mathbb{R}^E_+$,

$$\max\{y(E) : y \in P_{\mathrm{M}}(E),\, y \leq x^\star\} = \min\{r(F) + x^\star(E \setminus F) : F \subseteq E\}.$$

□

Indeed, for any $y \in P_{\mathrm{M}}(E)$ with $y \leq x^\star$, $y(E) = y(F) + y(E \setminus F) \leq r(F) + x^\star(E \setminus F)$, and equality will be attained if only if $y(F) = r(F)$ and $y(E \setminus F) = x^\star(E \setminus F)$. Theorem 2.10 guarantees that any F minimizing $r(F) + x^\star(E \setminus F)$ maximizes $x^\star(F) - r(F)$. For any matroid $M = (E, \mathcal{I})$ given by an independence testing oracle and any $x^\star \in \mathbb{R}^E_+$, Cunningham's algorithm finds a $y \in P_{\mathrm{M}}(E)$ with $y \leq x^\star$ maximizing $y(E)$, a decomposition of y as convex combination of incidence vectors of independent sets, and a set $F^\star \subseteq E$ with $r(F^\star) + x^\star(E \setminus F^\star) = y(E)$ in strongly polynomial time. The vector y will be constructed by path augmentations along shortest paths in an auxiliary digraph.

Next, we return to the separation problem for the rank induced forbidden cardinality inequalities (2.2). In what follows, we suppose that $x^\star$ satisfies the rank inequalities (2.5).

Lemma 2.11. Let $x^\star \in \mathbb{R}^E_+$ satisfying all rank inequalities (2.5). If a rank induced forbidden cardinality inequality $\mathrm{FC}_F(x) \leq c_p(c_{p+1} - r(F))$ with $c_p < r(F) < c_{p+1}$ is violated by $x^\star$, then $c_p < x^\star(E) < c_{p+1}$.

Proof. First, assume that $x^\star(E) \leq c_p$. Then $x^\star(F) \leq c_p$, and hence,

$$\begin{aligned}
&(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) \\
\leq\ &(c_{p+1} - r(F))c_p - (r(F) - c_p)x^\star(E \setminus F) \\
\leq\ &c_p(c_{p+1} - r(F)).
\end{aligned}$$

Next, assume that $x^\star(E) \geq c_{p+1}$. By hypothesis, $x^\star$ satisfies all rank inequalities (2.5), in particular, $x(F) \leq r(F)$. Thus,

$$\begin{aligned}
&(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F)\\
&= (c_{p+1} - c_p)x^\star(F) - (r(F) - c_p)x^\star(E)\\
&\leq (c_{p+1} - c_p)r(F) - (r(F) - c_p)x^\star(E)\\
&\leq (c_{p+1} - c_p)r(F) - (r(F) - c_p)c_{p+1}\\
&= c_p(c_{p+1} - r(F)).
\end{aligned}$$

□

Lemma 2.12. Let $x^\star \in \mathbb{R}^E_+$ satisfying all rank inequalities (2.5), and let $c_p < x^\star(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Then for any $F \subseteq E$ we have: If $(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) > c_p(c_{p+1} - r(F))$, then $c_p < r(F) < c_{p+1}$.

Proof. Let $F \subseteq E$, and assume that $r(F) \leq c_p$. Then,

$$\begin{aligned}
&(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) - c_p(c_{p+1} - r(F))\\
&= (c_{p+1} - c_p)x^\star(F) - (r(F) - c_p)x^\star(E) - c_p(c_{p+1} - r(F))\\
&\leq (c_{p+1} - c_p)r(F) - (r(F) - c_p)x^\star(E) - c_p(c_{p+1} - r(F))\\
&= \underbrace{(c_{p+1} - x^\star(E))}_{>0}\underbrace{(r(F) - c_p)}_{\leq 0} \;\leq\; 0.
\end{aligned}$$

Next, if $r(F) \geq c_{p+1}$, then

$$\begin{aligned}
&(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) - c_p(c_{p+1} - r(F))\\
&= (c_{p+1} - c_p)x^\star(F) - (r(F) - c_p)x^\star(E) - c_p(c_{p+1} - r(F))\\
&\leq (c_{p+1} - c_p)x^\star(E) - (r(F) - c_p)x^\star(E) - c_p(c_{p+1} - r(F))\\
&= \underbrace{(c_{p+1} - r(F))}_{\leq 0}\underbrace{(x^\star(E) - c_p)}_{>0} \;\leq\; 0.
\end{aligned}$$

Thus, $(c_{p+1} - r(F))x^\star(F) - (r(F) - c_p)x^\star(E \setminus F) > c_p(c_{p+1} - r(F))$ only if $c_p < r(F) < c_{p+1}$. □

Theorem 2.13. For any $P^c_{\mathrm{M}}(E)$ and any $x^\star \in \mathbb{R}^E_+$ satisfying all rank inequalities (2.5), the separation problem for $x^\star$ and the rank induced forbidden cardinality inequalities (2.2) can be solved in strongly polynomial time.

Proof. By Lemmas 2.11 and 2.12 we know that $x^\star$ violates a rank induced forbidden cardinality inequality only if $c_p < x^\star(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Thus, if $x^\star(E) = c_q$ for some $q \in \{1, \dots, m\}$, then $x^\star \in P^c_{\mathrm{M}}(E)$.

Suppose that $c_p < x^\star(E) < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. We would like to find some $F' \subseteq E$ such that

$$(c_{p+1} - r(F'))x^\star(F') - (r(F') - c_p)x^\star(E \setminus F') - c_p(c_{p+1} - r(F')) > 0$$

if there is any. Lemma 2.12 says that $c_p < r(F') < c_{p+1}$, and thus, the inequality $\mathrm{FC}_{F'}(x) \leq c_p(c_{p+1} - r(F'))$ is indeed a rank induced forbidden cardinality inequality among (2.2) violated by $x^\star$. If there is no such F', then for all $F \subseteq E$ with $c_p < r(F) < c_{p+1}$ the associated rank induced forbidden cardinality inequality with F is satisfied by $x^\star$, and by Lemma 2.11, all other rank induced forbidden cardinality inequalities among (2.2) are also satisfied by $x^\star$.

To find such a subset F' of E, set $\delta := \frac{x^\star(E) - c_p}{c_{p+1} - c_p}$. Since $c_p < x^\star(E) < c_{p+1}$, $0 < \delta < 1$. Moreover, $\frac{c_{p+1} - x^\star(E)}{c_{p+1} - c_p} = 1 - \delta$. For any $F \subseteq E$ it now follows:

$$\begin{array}{lr} & (c_{p+1} - c_p)x^\star(F) - (r(F) - c_p)x^\star(E) - c_p(c_{p+1} - r(F)) > 0 \\ \Leftrightarrow & x^\star(F) - \frac{r(F)x^\star(E) + c_p x^\star(E) - c_p c_{p+1} + c_p r(F)}{c_{p+1} - c_p} > 0 \\ \Leftrightarrow & x^\star(F) - r(F)\frac{x^\star(E) - c_p}{c_{p+1} - c_p} - c_p \frac{c_{p+1} - x^\star(E)}{c_{p+1} - c_p} > 0 \\ \Leftrightarrow & x^\star(F) - r(F)\delta > c_p(1 - \delta) \\ \Leftrightarrow & \frac{x^\star(F)}{\delta} - r(F) > c_p \frac{(1-\delta)}{\delta}. \end{array}$$

Setting $x' := \frac{1}{\delta}x^\star$, we see that the last inequality is equivalent to $x'(F) - r(F) > c_p \frac{(1-\delta)}{\delta}$. Thus, we can apply Cunningham's algorithm to find some $F \subseteq E$ that maximizes $x'(F) - r(F)$. If $x'(F) - r(F) > c_p \frac{(1-\delta)}{\delta}$, then $c_p < r(F) < c_{p+1}$ and the rank induced forbidden cardinality inequality associated with F is violated by $x^\star$. □

Consequently, we suggest a separation routine that works as follows. Assume that the fractional point $x^\star$ satisfies the nonnegativity constraints and the cardinality bounds. First, compute with Cunningham's algorithm a subset F of E maximizing $x^\star(F) - r(F)$. If $x^\star(F) - r(F) > 0$, then the associated rank inequality $x(F) \leq r(F)$ is violated by $x^\star$. If $x^\star(F) - r(F) \leq 0$, then $x^\star$ satisfies all rank inequalities (2.5), and if, in addition, $x^\star(E) = c_p$ for some p, then we know that $x^\star \in P^c_{\mathrm{M}}(E)$. Otherwise, i.e., if $c_p < x^\star(E) < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$, we check whether or not there is a violated rank induced forbidden cardinality inequality among (2.2) by applying Cunningham's algorithm on $M = (E, \mathcal{I})$ and $x' = \frac{1}{\delta}x^\star$.

Corollary 2.14. Given a matroid $M = (E, \mathcal{I})$ by an independence testing oracle, a cardinality sequence c, and a vector $x^\star \in \mathbb{R}^E_+$, the separation problem for $x^\star$ and $P^c_{\mathrm{M}}(E)$ can be solved in strongly polynomial time. □

2.1.4 Extensions

The cardinality constrained matroid polytope turns out to be a useful object to enhance the theory of polyhedra associated with cardinality constrained combinatorial optimization problems. Imposing cardinality constraints on a combinatorial optimization problem does not necessarily turn it into a harder problem: The cardinality constrained version of the maximum weight independent set problem in a matroid is manageable on the algorithmic as well as on the polyhedral side without any difficulties. Facets related to cardinality restrictions (rank induced forbidden cardinality inequalities) are linked to well known notions of matroid theory (closed subsets of E). The analysis of the separation problem for the rank induced forbidden cardinality inequalities discloses that it is sometimes better not to split a cardinality constrained problem into "simpler" cardinality constrained problems but to transform it into one or more non-cardinality restricted problems.

It stands to reason to investigate the intersection of two matroids with regard to cardinality restrictions. As it is well known, if an independence system $\mathcal{I}$ defined on some ground set E can be described as the intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, then the optimization problem $\max w(I)$, $I \in \mathcal{I}$ can be solved in polynomial time, for instance with Lawler's weighted matroid intersection algorithm [56]. This algorithm solves also the cardinality constrained version $\max w(I)$, $I \in \mathcal{I} \cap \mathrm{CHS}^c(E)$, since for each cardinality $p \leq r(E)$ it generates an independent set I of cardinality p which is optimal among all independent sets J of cardinality p. Thus, from an algorithmic point of view the problem is well studied. However, there is an open question regarding the associated polytope $P^c_{\text{IND}}(E)$. As it is well known, $P_{\text{IND}}(E) = P_{\text{M}_1}(E) \cap P_{\text{M}_2}(E)$, that is, the non-cardinality constrained independent set polytope $P_{\text{IND}}(E)$ is determined by the nonnegativity constraints $x_e \geq 0$, $e \in E$, and the rank inequalities $x(F) \leq r_j(F)$, $\varnothing \neq F \subseteq E$, $j = 1, 2$, where r_j is the rank function with respect to $\mathcal{I}_j$. We do not know, however, whether or not $P^c_{\text{IND}}(E) = P^c_{\text{M}_1}(E) \cap P^c_{\text{M}_2}(E)$ holds. So far, we have not found any counterexample contradicting the hypothesis that equality holds.

2.2 THE CARDINALITY CONSTRAINED POLYMATROID

Let S be a finite set and f a *set function* on S, that is, $f : 2^S \rightarrow \mathbb{R}$. The function f is called *submodular* if

$$f(T) + f(U) \geq f(T \cap U) + f(T \cup U)$$

for all $T, U \subseteq S$. For instance, the rank function of a matroid is submodular. The function f is called *integer* if $f(T) \in \mathbb{Z}$ for all $T \subseteq S$. It is said to be *nondecreasing* if $f(T) \leq f(U)$ whenever $T \subseteq U \subseteq S$.

Let f be a submodular set function on S. Then, the polyhedron

$$P_f(S) := \{x \in \mathbb{R}^S : x \geq 0,\ x(T) \leq f(T) \text{ for all } T \subseteq S\}$$

is called the *polymatroid associated with* f. $P_f(S)$ is bounded, and thus a polytope.

In the following let f be an integer nondecreasing submodular set function on S with $f(\varnothing) = 0$. Since f is integer, $P_f(S)$ is integer. Thus it makes sense to introduce cardinality constraints. Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $c_m \leq f(S)$. We call

$$P_f^c(S) := \operatorname{conv}\{x \in P_f(S) \cap \mathbb{Z}^S : x(S) = c_p \text{ for some } p\}$$

the *cardinality constrained polymatroid*.

We notice that, strictly speaking, we leave the context of combinatorial optimization, since the feasible integer points are not necessarily binary. However, integer polymatroids can be reduced to matroids. We present the reduction, which, for instance, can be found in Schrijver [76, vol. 2, p. 776]. Associate with each $s \in S$ a set E_s of size $f(\{s\})$, such that the set $E := \bigcup_{s \in S} E_s$ is a disjoint union of the sets E_s. Define a set function r on E by

$$r(F) := \min_{T \subseteq S}(|F \setminus \bigcup_{s \in T} E_s| + f(T))$$

for $F \subseteq E$. Then, r is the rank function of a matroid $M = (E, \mathcal{I})$,

$$f(T) = r(\bigcup_{s \in T} E_s) \tag{2.9}$$

for all $T \subseteq S$, and

$$P_f(S) = \operatorname{conv}\{x \in \mathbb{Z}_+^S : \exists\, I \in \mathcal{I} \text{ with } x_s = |I \cap E_s|\ \forall\, s \in S\}, \tag{2.10}$$

see [76]. We say that M *induces* the polymatroid $P_f(S)$.

2.2.1 Complete linear description

Theorem 2.15. Let S be a finite set and f an integer nondecreasing submodular set function on S with $f(\varnothing) = 0$. Moreover, let $M = (E, \mathcal{I})$ be a matroid that induces $P_f(S)$. Then

$$P_f^c(S) = \{x \in \mathbb{R}^S : \exists\, y \in P_{\mathrm{M}}^c(E) \text{ such that } x_s = y(E_s)\ \forall\, s \in S\}. \tag{2.11}$$

Proof. Clearly, equation (2.11) is equivalent to

$$P_f^c(S) = \mathrm{conv}\{x \in \mathbb{Z}_+^S : \exists\, I \in \mathcal{I} \cap \mathrm{CHS}^c(E) \text{ with } x_s = |I \cap E_s|\ \forall\, s \in S\},$$

Denote the polyhedron on the right hand side of the equation by Q. First, we show that $P_f^c(S) \subseteq Q$. Since $P_f^c(S)$ is an integer polyhedron, it suffices to prove that $x \in P_f^c(S) \cap \mathbb{Z}^S$ implies $x \in Q$. Let $x \in P_f^c(S) \cap \mathbb{Z}^S$. Since $P_f^c(S) \subseteq P_f(S)$, it follows that $x \in P_f(S) \cap \mathbb{Z}^S$, and hence exists $I \in \mathcal{I}$ such that $x_s = |I \cap E_s|$ for all $s \in S$ by equation (2.10). Moreover,

$$|I| = \sum_{s \in S} |I \cap E_s| = \sum_{s \in S} x_s = x(S),$$

and thus $|I| = c_p$ for some p. This implies $x \in Q$.

Next, let $x \in Q$ be an integer point. By definition of Q there is $I \in \mathcal{I} \cap \mathrm{CHS}^c(E)$ such that $x_s = |I \cap E_s|$ for all $s \in S$. $I \in \mathcal{I}$ implies $x \in P_f(S)$ by equation (2.10), while $I \in \mathrm{CHS}^c(E)$ implies $x(S) = c_p$ for some p. Consequently, $x \in P_f^c(S)$. □

Lemma 2.16. Let S be a finite set and f an integer nondecreasing submodular set function on S with $f(\varnothing) = 0$.

(i) Let $u \in P_f(S) \cap \mathbb{Z}^S$, $T \subseteq S$, and $k \in \mathbb{Z}_+$ such that $u(T) \leq k \leq f(T)$. Then exists $v \in P_f(S) \cap \mathbb{Z}^S$ such that $v(S) = k$, $v_s \geq u_s$ for $s \in T$, and $v_s = 0$ for $s \in S \setminus T$.

(ii) Let $u \in P_f(S) \cap \mathbb{Z}^S$ and $T \subseteq S$ such that $f(T) \leq u(S)$. Then exists $v \in P_f(S) \cap \mathbb{Z}^S$ such that $v(S) = u(S)$, $v(T) = f(T)$, $v_s \geq u_s$ for $s \in T$, and $v_s \leq u_s$ for $s \in S \setminus T$.

Proof. Let $M = (E, \mathcal{I})$ be a matroid that induces $P_f(S)$.

(i) Since $u \in P_f(S) \cap \mathbb{Z}^S$, it exists $I \in \mathcal{I}$ such that $u_s = |I \cap E_s|$ for all $s \in S$. Let $E_T := \bigcup_{s \in T} E_s$ and $J := I \cap E_T$. It follows immediately from the matroid axioms that $J \in \mathcal{I}$ and that there is some $K \in \mathcal{I}$ such that $J \subseteq K \subseteq E_T$ and $|K| = k$. Consequently, v defined by $v_s := |K \cap E_s|$ for $s \in S$ is a point as required.

(ii) It follows immediately from (i) that there is some $w \in P_f(S) \cap \mathbb{Z}^S$ such that $w(T) = f(T)$, $v_s \geq u_s$ for $s \in T$, and $v_s = 0$ for $s \in S \setminus T$. Since $u, w \in P_f(S) \cap \mathbb{Z}^S$, it exist $I, J \in \mathcal{I}$ with $u_s := |I \cap E_s|$ and $w_s := |J \cap E_s|$ for $s \in S$. By construction, $|J| = r(E_T) \leq |I|$. By the third matroid axiom, there is some $K \subseteq I \setminus J$ such that $J \cup K \in \mathcal{I}$ and $|J \cup K| = |I|$. Since $|J| = r(E_T)$, we have $K \subseteq I \setminus T$. Consequently, w defined by $w_s := |(J \cup K) \cap E_s|$ for all $s \in S$ is a point as required. □

Theorem 2.17 (see Schrijver [76], vol. 2, p. 781)**.** Let S be a finite set and f an integer nondecreasing submodular set function on S with $f(\varnothing) = 0$. Moreover, let $c = (0, 1, \ldots, k)$ for some $k \in \mathbb{Z}_+$. Then $P_f^c(S)$ is determined by the system

$$\begin{aligned} x(S) &\leq k, && && (2.12)\\ x(T) &\leq f(T) && \text{for all } \varnothing \neq T \subseteq S, && (2.13)\\ x_s &\geq 0 && \text{for all } s \in S. && (2.14) \end{aligned}$$

□

Theorem 2.18. Let S be a finite set and f an integer nondecreasing submodular set function on S with $f(\varnothing) = 0$. Then $P_f^c(S)$ is determined by the inequalities (2.13), (2.14),

$$x(S) \geq c_1, \tag{2.15}$$

$$x(S) \leq c_m, \tag{2.16}$$

$$(c_{p+1} - f(T))x(T) - (f(T) - c_p)x(S \setminus T) \leq c_p(c_{p+1} - f(T)) \tag{2.17}$$

$$\text{for all } T \subseteq S \text{ with } c_p < f(T) < c_{p+1} \text{ for some } p.$$

Inequalities (2.17) are called *f-induced forbidden cardinality inequalities.*

Proof. To prove Theorem 2.18 we adapt large parts of the proof to Theorem 2.2. In the following let $M = (E, \mathcal{I})$ be a matroid that induces $P_f(S)$.

Clearly, inequalities (2.13)-(2.16) are valid. To prove the validity of inequalities (2.17), let T be any subset of S such that $c_p < f(T) < c_{p+1}$ for some p. By equation (2.9), $f(T) = r(E_T)$, where $E_T := \bigcup_{s \in T} E_s$. Thus,

$$(c_{p+1} - r(E_T))y(E_T) - (r(E_T) - c_p)y(E \setminus E_T) \leq c_p(c_{p+1} - r(E_T))$$

is a valid inequality for $P_{\mathrm{M}}^c(E)$. By Theorem 2.15, the projection is given by the equations $x_s = E_s$ for $s \in S$. Therefore, the inequality

$$(c_{p+1} - f(T))x(T) - (f(T) - c_p)x(S \setminus T) \leq c_p(c_{p+1} - f(T))$$

is valid for $P_f^c(S)$.

Since all inequalities of system (2.13)-(2.17) are valid, $P_{\mathrm{M}}^c(E)$ is contained in the polyhedron defined by (2.13)-(2.17). To show the converse, we consider any valid inequality $b^T x \leq b_0$ for $P_f^c(S)$ and associate with the inequality the following subsets of S:

$$\begin{aligned} P &:= \{s \in S : b_s > 0\},\\ Z &:= \{s \in S : b_s = 0\},\\ N &:= \{s \in S : b_s < 0\}. \end{aligned}$$

We will show by case by case enumeration that the inequality $b^T x \leq b_0$ is dominated by some inequality of the system (2.12)-(2.16). By definition, $S = P \dot\cup Z \dot\cup N$, and hence, if $P = Z = N = \varnothing$, then $S = \varnothing$, and there is nothing to show. By a scaling argument we may assume that either $b_0 = 1$, $b_0 = 0$, or $b_0 = -1$.

(1) $b_0 = -1$.

(1.1) $c_1 = 0$. Then $0 \in P_f^c(E)$, and hence $0 = b^T 0 \leq -1$, a contradiction.

(1.2) $c_1 > 0$.

(1.2.1) $P = Z = \varnothing$, $N \neq \varnothing$. Assume that there is some tight $x \in P_f^c(S) \cap \mathbb{Z}^S$ with $x(S) = c_p$, $p \geq 2$. Then, for any $x' \in \mathbb{Z}_+^S$ with $x' \leq x$ and $x'(S) = c_1$ holds: $x' \in P_f^c(S)$ and $b^T x' > b^T x = -1$, a contradiction. Therefore, if any $x \in P_f^c(S) \cap \mathbb{Z}^S$ is tight, then $x(S) = c_1$. Thus, the inequality $b^T x \leq -1$ is dominated by the cardinality bound $x(S) \geq c_1$.

(1.2.2) $P \cup Z \neq \varnothing$, $N = \varnothing$. Then, $b^T x \geq 0$ for all $x \in P_f^c(S)$, a contradiction.

(1.2.3) $P \cup Z \neq \varnothing$, $N \neq \varnothing$. First, assume that $c_1 \leq f(P \cup Z)$. With $u = 0$, $T = P \cup Z$, and $k = c_1$ we see by Lemma 2.16 that there is $x' \in P_f^{(c_1)}(S)$ such that $\operatorname{supp}(x') \subseteq P \cup Z$. Thus $b^T x' \geq 0$, which is a contradiction, and hence, $c_1 > f(P \cup Z)$. Assume, for the sake of contradiction, that there is some tight $x' \in P_f^c(S) \in \mathbb{Z}^S$ such that $x'(S) = c_p$ for $p \geq 2$. If $\operatorname{supp}(x') \subseteq N$, then any $\hat{x} \in \mathbb{Z}_+^S$ with $\hat{x} \leq x'$ and $\hat{x}(S) = c_1$ is a point of $P_f^c(S)$ and violates the inequality $b^T x \leq -1$. Hence, $\operatorname{supp}(x') \cap (P \cup Z) \neq \varnothing$. On the other hand, $\operatorname{supp}(x') \cap N \neq \varnothing$ due to $c_p > c_1 > f(P \cup Z)$. The latter inequality chain implies that $\sum_{s \in N} x'_s \geq c_p - c_1$. Consequently, there exists $x'' \in P_f^c(S) \cap \mathbb{Z}^S$ such that $x''_s = x'_s$ for $s \in P \cup Z$ and $x''(S) = c_1$. Clearly, x'' violates the inequality $b^T x \leq -1$, a contradiction. Therefore, if any $x' \in P_f^c(S)$ is tight, then $x'(S) = c_1$. Thus, $b^T x \leq -1$ is dominated by the cardinality bound $x(S) \geq c_1$.

(2) $b_0 = 0$.

(2.1) $P \cup Z \neq \varnothing$, $N = \varnothing$. Then, either $b^T x \leq 0$ is not valid or $b = 0$.

(2.2) $P = \varnothing$, $Z \cup N \neq \varnothing$. Then, $b^T x \leq 0$ is dominated by the nonnegativity constraints $x_s \geq 0$ for $s \in N$ or $b = 0$.

(2.3) $P \neq \varnothing$, $N \neq \varnothing$.

(2.3.1) $c_1 > 0$. If $c_1 \leq f(P \cup Z)$, then by Lemma 2.16 there is some $x' \in P_f^c(S) \cap \mathbb{Z}^S$ such that $\text{supp}(x') \subseteq P \cup Z$, $\text{supp}(x') \cap P \neq \varnothing$, and $x'(S) = c_1$. Hence, $b^T x' > 0$, a contradiction. Thus, $c_1 > f(P \cup Z)$. Assume, for the sake of contradiction, that there is some tight $x' \in P_f^c(S)$ such that $x'(S) = c_p$ for $p \geq 2$. Since $c_p > c_1 > f(P \cup Z)$ and x' is tight, $\text{supp}(x') \cap (P \cup Z) \neq \varnothing \neq \text{supp}(x') \cap N$. From here, the proof for this case can be finished as the proof for the case (1.2.3) with $b_0 = 0$ instead of $b_0 = -1$ in order to show that $b^T x \leq 0$ is dominated by the cardinality bound $x(E) \geq c_1$.

(2.3.2) $c_1 = 0$. As in case (2.3.1), it follows immediately that $c_2 > f(P \cup Z)$, and if $x' \in P_f^c(S)$ is tight, then $x'(S) = c_1 = 0$, that is, $x' = 0$, or $x'(S) = c_2$. Moreover, if $x' \in P_f^c(S)$ with $x'(S) = c_2$ is tight, then follows $\sum_{s \in P \cup Z} x'_s = f(P \cup Z)$. Hence, $b^T x \leq b_0$ is dominated by the f-induced forbidden cardinality inequality $\text{FC}_F(x) \leq 0$ with $F = P \cup Z$.

$$(c_2 - f(T))x(T) - f(T)x(S \setminus T) \leq 0,$$

where $T := P \cup Z$.

(3) $b_0 = 1$.

(3.1) $P = \varnothing$, $Z \cup N \neq \varnothing$. Then, $b \leq 0$, and hence $b^T x \leq 1$ is dominated by any nonnegativity constraint $x_s \geq 0$, $s \in S$.

(3.2) $P \cup Z \neq \varnothing$, $N = \varnothing$. Assume that there is some $x' \in P_f(S) \cap \mathbb{Z}^S$ such that $x'(S) \neq c_p$ for $p = 1, \ldots, m$, $x'(S) < c_m$, and $b^T x' > 1$. Then, of course, all $x'' \in P_f(S) \in \mathbb{Z}^S$ with $x'' \geq x'$ violate $b^T x \leq 1$, in particular, those x'' with $x''(S) = c_m$, a contradiction. Hence, the inequality $b^T x \leq 1$ is not only a valid inequality for $P_f^c(S)$ but also for $P_f^{(0,1,\ldots,c_m)}(S)$, that is, $b^T x \leq 1$ is dominated by some inequality of the system (2.12)-(2.14), where $k = c_m$.

(3.3) $P \neq \varnothing$, $N \neq \varnothing$. Let $p \in \{1, \ldots, m\}$ be minimal such that there is $x^\star \in P_f^c(S) \in \mathbb{Z}^S$ with $x^\star(S) = c_p$. Of course, $c_p > 0$, because otherwise $x^\star$ could not be tight. If $p = m$, then $b^T x \leq 1$ is dominated by the cardinality bound $x(S) \leq c_m$, because then all tight $x' \in P_f^c(S)$ satisfy $x'(S) = c_m$. So, let $0 < c_p < c_m$. We distinguish 2 subcases.

(3.3.1) $c_p \geq f(P \cup Z)$. Suppose, for the sake of contradiction, that there is some tight $x' \in P_f^c(S) \in \mathbb{Z}^S$ such that $x'(S) = c_p$ and $\sum_{s \in P \cup Z} x'_s < f(P \cup Z)$. By Lemma 2.16, it exists $\hat{x} \in P_f \cap \mathbb{Z}^S$

such that $\hat{x}(S) = f(P \cup Z)$, $\hat{x}_s \geq x'_s$ for $s \in P \cup Z$, and $\hat{x}_s = 0$ for $s \in N$. Since $\hat{x}(S) \leq c_p$, there is some $\bar{x} \in P_f^c(S) \cap \mathbb{Z}^S$ with $\bar{x}(S) = c_p$, $\bar{x}_s = \hat{x}_s$ for $s \in P \cup Z$, and $\bar{x}_s \leq x'_s$ for $s \in N$ by Lemma 2.16. By construction, $\bar{x}$ violates the inequality $b^T x \leq 1$. Thus, $\sum_{s \in P \cup Z} x'_s = f(P \cup Z)$. For the same reason, any tight $\tilde{x} \in P_f^c(S) \cap \mathbb{Z}^S$ satisfies $\sum_{s \in P \cup Z} \tilde{x}_s = f(P \cup Z)$, and since p is minimal, $\tilde{x}(S) \geq c_p$. Now, with similar arguments as in case (1.2.3) one can show that if $\tilde{x} \in P_f^c(S) \cap \mathbb{Z}^S$ is tight, then $\tilde{x}(S) = c_p$. Thus, $c_p = c_1 > 0$ and $b^T x \leq 1$ is dominated by the cardinality bound $x(E) \geq c_1$.

(3.3.2) $c_p < f(P \cup Z)$. Following the argumentation line in (3.3.1), we see that $\mathrm{supp}(x') \subseteq P \cup Z$ and $\sum_{s \in P} x'_s$ has to be maximal for any tight $x' \in P_f^c(S) \in \mathbb{Z}^S$ with $x'(S) = c_p$. Assume that $c_{p+1} \leq f(P \cup Z)$. One easily observes that from any tight $x' \in P_f^c(S) \in \mathbb{Z}^S$ with $x'(S) = c_p$ we can construct a tight vector $x'' \in P_f^c(S) \in \mathbb{Z}^S$ with $x'(S)' = c_{p+1}$ such that $x''_s = x'_s$ for $s \in P \cup N$ and $x''_s \geq x'_s$ for $s \in Z$. However, it is not hard to see that there is no tight $\tilde{x} \in P_f^c(S) \in \mathbb{Z}^S$ such that $\tilde{x}_s > 0$ for some $s \in N$. Thus, if $c_{p+1} \leq f(P \cup Z)$, then $b^T x \leq 1$ is dominated by the nonnegativity constraints $x_s \geq 0$, $s \in N$. Therefore, $c_{p+1} > f(P \cup Z)$. The following is now immediate: If $x' \in P_f^c(S) \in \mathbb{Z}^S$ is tight, then $x'(S) = c_p$ or $x'(S) = c_{p+1}$; if $x'(S) = c_p$, then $\mathrm{supp}(x') \subset P \cup Z$, and if $x'(S) = c_{p+1}$, then $\sum_{s \in P \cup Z} x'_s = f(P \cup Z)$ and $c_{p+1} > f(P \cup Z)$. Thus, $b^T x \leq 1$ is dominated by the f-induced forbidden cardinality inequality

$$(c_{p+1} - f(T))x(T) - (f(T) - c_p)x(S \setminus T) \leq c_p(c_{p+1} - f(T)),$$

where $T := P \cup Z$.

□

2.2.2 Extensions

The separation problem for the f-induced forbidden cardinality inequalities (2.17) can be reduced to that for the inequalities (2.13). The construction is straightforward along the lines of Section 2.1.3. The optimization version of the separation problem for inequalities (2.13) is known as submodular function minimization, which can be performed with combinatorial algorithms in strongly polynomial time, see Iwata, Fleischer, and Fujishige [51] and Schrijver [75]). Thus, we have the following

Theorem 2.19. Let $f : S \to \mathbb{R}$ be an integer nondecreasing submodular function given by a value giving oracle. Moreover, let c be a cardinality sequence and $x^\star \in \mathbb{R}^S_+$. Then, the separation problem for $x^\star$ and $P^c_f(S)$ can be solved in strongly polynomial time. $\square$

We pass on a characterization of facets.

Chapter 3

Cardinality constrained paths and cycles

This chapter analyzes polyhedra associated with cardinality constrained versions of path and cycle problems defined on directed and undirected graphs. Let $D = (N, A)$ be a directed graph on n nodes that has neither loops nor parallel arcs. An (s,t)-*walk* is a sequence of arcs $W = (a_1, a_2, \dots, a_r)$ such that $a_i = (i_{p-1}, i_p)$ for $p = 1, \dots, r$ with $i_0 = s$ and $i_r = t$. If all nodes i_p are distinct, then W is called a *path*. If $s = t$, then W is a *cycle*, and if, in addition, all other nodes are distinct, then W is called a *simple cycle*. Usually we refer to simple cycles as cycles. In what follows, we perceive paths and cycles as subsets of the arc set A. However, directed simple paths and cycles will be sometimes denoted by a tuple of nodes. For instance, the tuple (i_1, i_2, i_3, i_4) denotes the path $\{(i_1, i_2), (i_2, i_3), (i_3, i_4)\}$; the tuple $(i_1, i_2, i_3, i_4, i_1)$ denotes the cycle $\{(i_1, i_2), (i_2, i_3), (i_3, i_4), (i_4, i_1)\}$. Moreover, denote by $\mathcal{P}_{s,t}(D)$ and $\mathcal{C}(D)$ the collection of all simple directed (s,t)-paths and the collection of all simple directed cycles in D, respectively.

Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $c_1 \geq 1$ and $c_m \leq n$. The *directed cardinality constrained* (s,t)*-path polytope*, which will be denoted by $P^c_{s,t\text{-path}}(D)$, is the convex hull of the incidence vectors of simple directed (s,t)-paths P such that $|P| = c_p$ holds for some $p \in \{1, \dots, m\}$, that is,

$$P^c_{s,t\text{-path}}(D) := \{\chi^P \in \mathbb{R}^A : P \in \mathcal{P}_{s,t}(D) \cap \mathrm{CHS}^c(A)\}.$$

The *directed cardinality constrained cycle polytope* $P^c_C(D)$, similar defined, is the convex hull of the incidence vectors of simple directed cycles $C \in \mathcal{C}(D)$ with $|C| = c_p$ for some p. Note, since D does not have loops, we may assume $c_1 \geq 2$ when we investigate cycle polytopes. If it is clear from the context, $P^c_{s,t\text{-path}}(D)$ and $P^c_C(D)$ are usually just called *cardinality constrained path polytope* and *cardinality constrained cycle polytope*, respectively. The undirected counterparts of these polytopes are defined similarly. We denote them by $P^c_{s,t\text{-path}}(G)$ and $P^c_C(G)$, where G is an undirected graph. The associated polytopes without cardinality restrictions are denoted by $P_{s,t\text{-path}}(D)$, $P_{s,t\text{-path}}(G)$, $P_C(D)$, and $P_C(G)$.

Since solving the associated linear problems is in general NP-hard, we cannot expect to obtain complete and tractable linear characterization of

Table 3.1: Literature survey on path and cycle polyhedra. D denotes a directed graph, G an undirected graph.

Schrijver [76], chapter 13:	*dominant of the directed path polytope* $\mathrm{dmt}(P_{s,t\text{-path}}(D)) := P_{s,t\text{-path}}(D) + \mathbb{R}^A_+$
Dahl, Gouveia [23]:	*hop constrained path polytope* $P^{\leq k}_{s,t\text{-path}}(D) := P^{(1,\ldots,k)}_{s,t\text{-path}}(D)$
Dahl, Realfsen [25]:	$P^{\leq k}_{s,t\text{-path}}(D)$, D acyclic
Dahl, Foldnes, Gouveia [22]:	*hop constrained walk polytope* $P^{\leq k}_{s,t\text{-walk}}(D)$
Nguyen [68]:	*dominant of the undirected hop constrained path polytope* $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(G)) := P^{\leq k}_{s,t\text{-path}}(G) + \mathbb{R}^A_+$
Balas, Oosten [4]:	directed cycle polytope $P_C(D)$
Balas, Stephan [5]:	$P_C(D)$ and relaxations
Coullard, Pulleyblank [19], Bauer [9]:	undirected cycle polytope $P_C(G)$
Hartmann, Özlük [48]:	*directed k-cycle polytope* $P^{(k)}_C(D)$
Nguyen, Maurras [63, 64], Girlich et al. [42]:	*undirected k-cycle polytope* $P^{(k)}_C(G)$
Bauer, Savelsbergh, Linderoth [10]:	*undirected hop constrained cycle polytope* $P^{\leq k}_C(G)$
Huygens, Mahjoub, Pesneau [50], Dahl, Huygens, Mahjoub, Pesneau [24], Huygens, Labbé, Mahjoub, Pesneau [49]:	k edge-disjoint hop constrained path polyhedra

these polytopes neither for the ordinary polytopes nor for the cardinality constrained versions.

Cycle and path polytopes, with and without cardinality restrictions, are already well studied. For a literature survey on these polytopes see Table 3. However, those publications that treat cardinality restrictions, discuss only the cases $\leq k$ or $= k$.

The main contribution of this chapter will be the presentation of IP-models (or IP-formulations) for cardinality constrained path and cycle polytopes whose inequalities generally define facets with respect to complete graphs and digraphs. Moreover, the associated separation problems can be solved in polynomial time.

The basic idea to derive strong integer characterizations for cardinality constrained path and cycle polytopes can be presented best for cycle polytopes. According to Balas and Oosten [4], the integer points of the ordinary cycle polytope $P_C(D)$ can be characterized by the system

$$\begin{array}{rll} x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) = 0 & \text{for all } i \in N, & \\ x(\delta^{\text{out}}(i)) \le 1 & \text{for all } i \in N, & \\ -x((S : N \setminus S)) + x(\delta^{\text{out}}(i)) + x(\delta^{\text{out}}(j)) \le 1 & \text{for all } S \subset N, & \\ & 2 \le |S| \le n-2, & (3.1) \\ & i \in S, j \in N \setminus S, & \\ x(A) \ge 2, & & \\ x_{ij} \in \{0,1\} & \text{for all } (i,j) \in A. & \end{array}$$

Here, $\delta^{\text{out}}(i)$ and $\delta^{\text{in}}(i)$ denote the set of arcs leaving and entering node i, respectively; for any subsets S and T of N, $(S : T)$ denotes the set of arcs $\{(i,j) \in A : i \in S,\ j \in T\}$. In case of $T = N \setminus S$, the arc set $(S : T)$ is called a *directed cut*. If, in addition, $s \in S$ and $t \in T$, $(S : T)$ is also said to be a *(directed)* (s,t)*-cut*.

Next, for any $S \subseteq N$, we denote by $A(S)$ the subset of arcs whose both endnodes are in S. Moreover, for any $B \subseteq A$, $N(B)$ denotes the set of nodes covered by B.

In order to obtain a characterization of the integer points of $P_C^c(D)$ we just have to add the cardinality bounds (1.1)

$$c_1 \le x(A) \le c_m$$

and the forbidden cardinality inequalities (1.2)

$$\begin{array}{l} (c_{p+1} - |F|)x(F) \ - \ (|F| - c_p)x(A \setminus F) \le c_p(c_{p+1} - |F|) \\ \quad \text{for all } F \subseteq A \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \in \{1, \dots, m-1\}, \end{array}$$

by Theorem 1.2. However, as is already mentioned, the forbidden cardinality inequalities in this form are quite weak, that is, they define very low dimensional faces of $P_C^c(D)$. The key for obtaining stronger forbidden cardinality inequalities for $P_C^c(D)$ is to count the nodes of a cycle rather than its arcs. The trivial, but crucial observation here is that, for the incidence vector $x \in \{0,1\}^A$ of a cycle in D and for every node $i \in N$, we have $x(\delta^{\text{out}}(i)) = 1$ if the cycle contains node i, and $x(\delta^{\text{out}}(i)) = 0$ if it does not. Thus, for every $W \subseteq N$ with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$, the *node induced forbidden cardinality inequality*

$$(c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) - (|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \le c_p(c_{p+1} - |W|), \quad (3.2)$$

is valid for $P_C^c(D)$, cuts off all cycles C, with $c_p < |C| < c_{p+1}$, that visit $\min\{|C|, |W|\}$ nodes of W, and is satisfied with equation by all cycles of cardinality c_p or c_{p+1} that visit $\min\{|C|, |W|\}$ nodes of W. Using these inequalities one obtains the following integer characterization for $P_C^c(D)$:

$$\begin{aligned}
x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= 0 && \text{for all } i \in N,\\
x(\delta^{\text{out}}(i)) &\leq 1 && \text{for all } i \in N,\\
-x((S : N \setminus S)) + x(\delta^{\text{out}}(i)) + x(\delta^{\text{out}}(j)) &\leq 1 && \text{for all } S \subset N,\\
& && 2 \leq |S| \leq n-2,\\
& && i \in S, j \in N \setminus S,\\
c_1 \leq x(A) &\leq c_m, &&\\
(c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) & &&\\
-(|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) & &&\\
-c_p(c_{p+1} - |W|) &\leq 0 && \forall\, W \subseteq N : \exists\, p :\\
& && c_p < |W| < c_{p+1},\\
x_{ij} &\in \{0,1\} && \text{for all } (i,j) \in A.
\end{aligned} \tag{3.3}$$

However, in the polyhedral analysis of cardinality constrained path and cycle polytopes we will focus on the directed cardinality constrained path polytope for a simple reason: valid inequalities for $P^c_{s,t\text{-path}}(D)$ can easily be transformed into valid inequalities for the other polytopes. In particular, from the IP-model for $P^c_{s,t\text{-path}}(D)$ that we present in Section 3.4 we derive IP-models for the remaining polytopes $\mathcal{P}$, as illustrated in Figure 3.1, such that a transformed inequality is facet defining for $\mathcal{P}$ when the original inequality is facet defining for $P^c_{s,t\text{-path}}(D)$.

We describe briefly the tools that are used to derive facet defining inequalities for the polytopes $\mathcal{P}$ from facet defining inequalities for $P^c_{s,t\text{-path}}(D)$.

The first tool is Theorem 3.6, which can be used to lift facet defining inequalities for the directed cardinality constrained path polytope to facet defining inequalities for the directed cardinality constrained cycle polytope provided both polytopes are defined on appropriate digraphs. It uses the fact that the former polytope is isomorphic to a facet of the latter polytope.

The second tool uses the concept of symmetric inequalities, which will be explained using the example of the cycle polytopes. Suppose that the directed cardinality constrained cycle polytope is defined on the complete digraph $D_n = (N, A)$. An inequality $b^T x \leq \beta$, with $b \in \mathbb{R}^A$, is called *symmetric* if

$b_{ij} = b_{ji}$ for all $i < j$. One can show that the undirected counterpart

$$\sum_{1 \le i < j \le n} b_{ij} y_{ij} \le \beta$$

of a symmetric valid inequality $b^T x \le \beta$ is valid for $P_C^c(K_n)$, where K_n denotes the complete undirected graph on n nodes. Moreover, it induces a facet of $P_C^c(K_n)$ if $b^T x \le \beta$ induces a facet of $P_C^c(D_n)$. This follows from an argument of Fischetti [32], originally stated for the ATSP and STSP, which is also mentioned in Hartmann and Özlük [48] in the context of directed and undirected k-cycle polytopes $P_C^{(k)}(D_n)$ and $P_C^{(k)}(K_n)$. This concept can be adapted to the directed and undirected path polytopes in a modified version. For this, we refer to Subsection 3.3.2. Since the nontrivial inequalities presented in the IP-models for the directed cardinality constrained cycle and path polytopes are facet defining and equivalent to symmetric inequalities, their undirected counterparts induce facets of the undirected cardinality constrained cycle and path polytopes, respectively.

We do not start off with the polyhedral analysis of the directed cardinality constrained path polytope $P_{s,t\text{-path}}^c(D)$ itself, but with its subpolytopes $P_{s,t\text{-path}}^{(c_p)}(D)$. Theorem 3.4 and Table 3.2 imply that they are of codimension 1 whenever $4 \le c_p \le n-1$, provided that we have an appropriate digraph D. Thus, any facet defining inequality $b^T x \le \beta$ for $P_{s,t\text{-path}}^{(c_p)}(D)$ which is also valid for $P_{s,t\text{-path}}^c(D)$ can easily be shown to be facet defining also for $P_{s,t\text{-path}}^c(D)$ if $b^T x' = \beta$ holds for some $x' \in P_{s,t\text{-path}}^c(D) \setminus P_{s,t\text{-path}}^{(c_p)}(D)$.

According to the previous observations, this chapter is subdivided into the following eight parts. Section 3.1 shows that the directed cardinality constrained path polytope is usually isomorphic to a facet of the directed cardinality constrained cycle polytope, which makes the application of standard lifting techniques very easy. Next, Sections 3.2 and 3.3 study the single cardinality constrained path polytope $P_{s,t\text{-path}}^{(k)}(D)$ and related polytopes. In Section 3.4 we present an IP-model for the directed cardinality constrained (s,t)-path polytope $P_{s,t\text{-path}}^c(D)$ and give necessary and sufficient conditions for the inequalities of this model to be facet defining. Moreover, we derive a couple of further valid inequalities for this polytope. Most of these inequalities can be transformed into valid inequalities for the undirected counterpart of this polytope as well as the directed and undirected cardinality constrained cycle polytope by application of polyhedral standard techniques. This is the topic of Sections 3.5-3.7. Moreover, we derive some further strong valid inequalities for the directed cardinality constrained cycle polytope that are not derived from the associated path polytope. Finally, Section 3.8 briefly sketches the separation problems for some interesting inequalities.

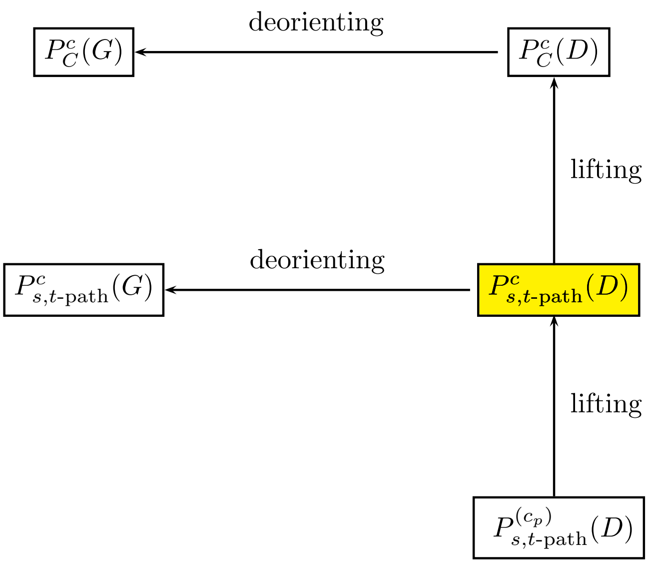


Figure 3.1: $P^c_{s,t\text{-path}}(D)$ and related polytopes.

This is joint work with Volker Kaibel with the exception of Section 3.2, 3.3 and some parts of Section 3.5 and Section 3.8. The contents appear in [52, 77, 78].

3.1 THE RELATIONSHIP BETWEEN DIRECTED PATH AND CYCLE POLYTOPES

In the following we investigate the cardinality constrained path polytope $P^c_{0,n\text{-path}}(D)$ defined on a digraph $D = (N, A)$ with node set $N = \{0, \dots, n\}$. In particular, $s = 0$ and $t = n$. Since $(0, n)$-paths do not use arcs entering 0 or leaving n, we may assume that $\delta^{\text{in}}(0) = \delta^{\text{out}}(n) = \varnothing$. Next, suppose that A contains the arc $(0, n)$ and the cardinality sequence c starts with $c_1 = 1$. Then,

$$\dim P^{(c_1,c_2,\dots,c_m)}_{0,n\text{-path}}(D) = \dim P^{(c_2,\dots,c_m)}_{0,n\text{-path}}(D) + 1.$$

Moreover, an inequality $\alpha^T x \le \alpha_0$ defines a facet of $P^{(c_2,\dots,c_m)}_{0,n\text{-path}}(D)$ if and only if the inequality $\alpha^T x + \alpha_0 x_{0n} \le \alpha_0$ defines a facet of $P^{(1,c_2,\dots,c_m)}_{0,n\text{-path}}(D)$. Thus, the consideration of cardinality sequences starting with 1 does not give any new insights into the facial structure of cardinality constrained path polytopes. So we may assume that A does not contain the arc $(0, n)$. So, for our purposes it suffices to suppose that the arc set A of D is given by

$$A = \{(0, i), (i, n) : i = 1, \dots, n-1\} \bigcup \{(i, j) : 1 \le i, j \le n-1, i \ne j\}. \quad (3.4)$$

Therefore, by default, we will deal with the directed graph $\tilde{D}_n = (\tilde{N}_n, \tilde{A}_n)$, where $\tilde{N}_n = \{0, 1, \ldots, n\}$ and $\tilde{A}_n = A$ in (3.4).

Let D' be the digraph that arises by removing node 0 from $\tilde{D}_n$ and identifying $\delta^{\text{out}}(0)$ with $\delta^{\text{out}}(n)$. Then, D' is a complete digraph on node set $\{1, \ldots, n\}$ and the set $\mathcal{P}(\tilde{D}_n)$ of simple $(0, n)$-paths becomes the set $\mathcal{C}^n(D')$ of simple cycles that visit node n. The convex hull of the incidence vectors of cycles $C \in \mathcal{C}^n(D')$ in turn is the restriction of the cycle polytope defined on D' to the hyperplane $x(\delta^{\text{out}}(n)) = 1$. Balas and Oosten [4] showed that the *degree constraint*

$$x(\delta^{\text{out}}(i)) \leq 1$$

induces a facet of the cycle polytope defined on a complete digraph. Hence, the path polytope $P_{0,n\text{-path}}(\tilde{D}_n)$ is isomorphic to a facet of the cycle polytope $P_C(D')$. From the next theorems we conclude that this relation holds also for cardinality constrained path and cycle polytopes. We start with some preliminary statements from linear algebra.

Lemma 3.1. Let $k \neq \ell$ be natural numbers, let $x^1, x^2, \ldots, x^r \in \mathbb{R}^p$ be vectors satisfying the equation $1^T x^i = k$, where 1^T is the vector of all ones, and let $y \in \mathbb{R}^p$ be a vector satisfying the equation $1^T y = \ell$. Then the following holds:

(i) y is not in the affine hull of the set $\{x^1, \ldots, x^r\}$.

(ii) The points $x^1, \ldots, x^r$ are affinely independent if and only if they are linearly independent.

Proof. (i) Assume that $y = \sum_{i=1}^r \lambda_i x^i$ for some $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ with $\sum_{i=1}^r \lambda_i = 1$. Then it follows that $\ell = 1^T y = \sum_{i=1}^r \lambda_i (1^T x^i) = k \sum_{i=1}^r \lambda_i = k$, a contradiction.

(ii) The suffiency is clear. To show the necessity, assume that the points are not linearly independent. Then for one point, say x^r, it exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{r-1}$ such that

$$x^r = \sum_{i=1}^{r-1} \lambda_i x^i.$$

It follows

$$\begin{aligned} k = 1^T x^r = \sum_{i=1}^{r-1} \lambda_i \underbrace{1^T x^i}_{=k} &= k \sum_{i=1}^{r-1} \lambda_i \\ \Rightarrow \qquad \sum_{i=1}^{r-1} \lambda_i &= 1, \end{aligned}$$

a contradiction. $\square$

According to the terminology of Balas and Oosten [4], for any digraph $D = (N, A)$ on n nodes we call the polytope

$$P^c_{CL}(D) := \{(x, y) \in P^c_C(D) \times \mathbb{R}^n : y_i = 1 - x(\delta^{\text{out}}(i)), i = 1, \ldots, n\}$$

the *cardinality constrained cycle-and-loops polytope.* Its integer points are the incidence vectors of spanning unions of a simple cycle and loops.

Lemma 3.2. The points $x^1, \ldots, x^p \in P^c_C(D)$ are affinely independent if and only if the corresponding points $(x^1, y^1), \ldots, (x^p, y^p) \in P^c_{CL}(D)$ are affinely independent.

Proof. The map $f : P^c_{CL}(D) \to P_C(D),\ (x, y) \mapsto x$ is an affine isomorphism. $\square$

Theorem 3.3 (Hartmann and Özlük [48])**.** Let $D_n = (N, A)$ be the complete digraph on n nodes and $k \in \mathbb{N}$.

(i)
$$\dim P^{(k)}_C(D_n) = \begin{cases} |A|/2 - 1, & \text{if } k = 2 \text{ and } n \geq 2, \\ n^2 - 2n, & \text{if } 2 < k < n \text{ and } n \geq 5, \\ n^2 - 3n + 1, & \text{if } k = n \text{ and } n \geq 3, \end{cases} \tag{3.5}$$

and $\dim P^{(3)}_C(D_4) = 6$.

(ii) For any node $i \in N$, the degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P^{(k)}_C(D_n)$ whenever $4 \leq k < n$. $\square$

Theorem 3.4. Let $D_n = (N, A)$ be the complete digraph on $n \geq 3$ nodes and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $c_1 \geq 2$, $c_m \leq n$, and $m \geq 2$. Then the following holds:

(i) The dimension of $P^c_C(D_n)$ is $(n-1)^2$.

(ii) For any node $i \in N$, the degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P^c_C(D_n)$.

Proof. (i) Balas and Oosten [4] proved that $\dim P_C(D_n) = (n-1)^2$. Since $P^c_C(D_n) \subseteq P_C(D_n)$, it follows immediately that $\dim P^c_C(D_n) \leq (n-1)^2$. When $n = 3$, $m \geq 2$ implies $P^c_C(D_n) = P_C(D_n)$, and thus $\dim P^{(2,3)}_C(D_3) = 4$. When $n = 4$, the statement can be verified using a computer program, for instance, with `polymake` [41]. For $n \geq 5$ the claim follows from Lemma 3.1 (i) and Theorem 3.3 (i) unless $c = (2, n)$: it exists some cardinality c_p, with

$2 < c_p < n$, and thus there are $n^2 - 2n + 1$ affinely independent vectors $x^r \in P_C^{(c_p)}(D_n) \subset P_C^c(D_n)$. Moreover, since $m \geq 2$, there is a vector $y \in P_C^c(D_n)$ of another cardinality which is affinely independent from the points x^r. Hence, $P_C^c(D_n)$ contains $n^2 - 2n + 2$ affinely independent points proving $\dim P_C^c(D_n) = (n-1)^2$.

The case $c = (2, n)$ requires a higher amount of technical detail, because the dimensions of both polytopes $P_C^{(2)}(D_n)$ and $P_C^{(n)}(D_n)$ are less than $n^2 - 2n$. Setting $d_n := \dim P_C^{(n)}(D_n)$, we see that there are $d_n + 1 = n^2 - 3n + 2$ linearly independent points $x^r \in P_C^{(2,n)}(D_n) \cap P_C^{(n)}(D_n)$ satisfying $x^r(A) = n$. Clearly, the points $(x^r, y^r) \in P_{CL}^{(2,n)}$ are also linearly independent. Next, consider the point (x^{23}, y^{23}), where x^{23} is the incidence vector of the 2-cycle $\{(2,3), (3,2)\}$, and $n-1$ further points (x^{1i}, y^{1i}), where x^{1i} is the incidence vector of the 2-cycle $\{(1,i), (i,1)\}$. The incidence matrix Z whose rows are the vectors (x^r, y^r), $r = 1, 2, \ldots, d_n + 1$, (x^{23}, y^{23}), and (x^{1i}, y^{1i}), $i = 2, 3, \ldots, n$, is of the form

$$Z = \begin{pmatrix} X & 0 \\ Y & Q \end{pmatrix},$$

where

$$Q = \left(\begin{array}{c|c} 1 & 0 \ 0 \ 1 \cdots 1 \\ \hline 0 & E - I \end{array} \right).$$

E is the $(n-1) \times (n-1)$ matrix of all ones and I the $(n-1) \times (n-1)$ identity matrix. $E - I$ is nonsingular, and thus Q is of rank n. X is of rank $d_n + 1$, and hence rank $(Z) = d_n + 1 + n = n^2 - 2n + 2$. Together with Lemma 3.2, this yields the desired result.

(ii) When $n \leq 4$, the statement can be verified using a computer program. When $n \geq 5$ and $4 \leq c_p < n$ for some index $p \in \{1, \ldots, m\}$, the claim can be showed along the lines of the proof to part (i) using Theorem 11 of Hartmann and Özlük [48] saying that the degree constraint defines a facet of $P_C^{(c_p)}(D_n)$.

It remains to show that the claim is true for

$$c \in \{(2,3), (2,n), (3,n), (2,3,n)\},$$

where $n \geq 5$. W.l.o.g. consider the inequality $x(\delta^{\text{out}}(1)) \leq 1$. When $c = (2,3)$, consider all 2- and 3-cycles whose incidence vectors satisfy $x(\delta^{\text{out}}(1)) = 1$. This are exactly $n^2 - 2n + 1$ cycles, namely the 2-cycles $\{(1,j), (j,1)\}$, $j = 2, \ldots, n$, and the 3-cycles $\{(1,j), (j,h), (h,1)\}$ for all arcs (j,h) that are not incident with node 1. Their incidence vectors are affinely independent, and hence, the degree constraint is facet defining for $P_C^{(2,3)}(D_n)$. This implies also that it induces a facet of $P_C^{(2,3,n)}(D_n)$. Turning to the case $c = (2,n)$, note that the degree constraint is satisfied with equality by all Hamiltonian cycles. Hence, we have $d_n + 1$ linearly independent Hamiltonian cycles and again, the 2-cycles $\{(1,i),(i,1)\}$, which are linearly independent of them. Finally, let $c = (3,n)$. Beside $d_n + 1$ Hamiltonian cycles, consider the 3-cycles $(1,3),(3,4),(4,1)$ and $\{(1,2),(2,j),(j,1)\}$, $j = 3,\dots,n$. Then the $n^2 - 2n + 1$ corresponding points in $P^c_{CL}(D_n)$ build a nonsingular matrix. Hence, by Lemma 3.2, it follows the desired result. □

Corollary 3.5. (i) Let $4 \leq k < n$. Then,

$$\dim P^{(k)}_{0,n\text{-path}}(\tilde{D}_n) = n^2 - 2n - 1.$$

(ii) Let $c = (c_1,\dots,c_m)$ be a cardinality sequence with $m \geq 2$ and $c_1 \geq 2$. Then,

$$\dim P^c_{0,n\text{-path}}(\tilde{D}_n) = n^2 - 2n.$$

Another important fact can be derived from both theorems. Facet defining inequalities for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ can easily be lifted to facet defining inequalities for $P^c_C(D_n)$. For sequential lifting, see Nemhauser and Wolsey [67].

Theorem 3.6. Let $c = (c_1,\dots,c_m)$ be a cardinality sequence with $4 \leq c_1 < n$ if $m = 1$, and $2 \leq c_1 < \dots < c_m \leq n$ otherwise. Moreover, let $\alpha^T x \leq \alpha_0$ be a facet defining inequality for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and γ the maximum of $\alpha(C)$ over all cycles C in $\tilde{D}_n$ with $|C| = c_p$ for some p. Setting $\alpha_{ni} := \alpha_{0i}$ for $i = 1,\dots,n-1$, the inequality

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} x_{ij} + (\gamma - \alpha_0) x(\delta^{\text{out}}(n)) \leq \gamma \tag{3.6}$$

defines a facet of $P^c_C(D_n)$. □

This fact implies that it would be profitably to study first the facial structure of the cardinality constrained directed path polytope and afterwards that of the corresponding cycle polytope. However, for the $(0,n)$-k-path polytope we will proceed in the opposite direction, since the k-cycle polytope is already well studied. This means, starting from the results for the k-cycle polytope $P^{(k)}_C(D_n)$ given by Hartmann and Özlük [48] we will prove in many cases analogous results for the $(0,n)$-k-path polytope $(\tilde{D}_n)$ and it is not surprising that this can often be done along the lines of the proofs of the authors mentioned above.

No similar relationship seems to hold between undirected cycle and path polytopes.

3.2 The directed $(0,n)$-k-path polytope

Since the directed k-cycle polytope is already well studied by Hartmann and Özlük [48] and this and the directed $(0,n)$-k-path polytope are closely related to each other, it seems likely that many theorems and proof methods used in this section are very similar to that in [48]. Indeed, one part of this section is to translate those results in [48] into related results for $P^{(k)}_{0,n\text{-path}}(D)$.

Let $D = (N, A)$ be a digraph on node set $N = \{0, \dots, n\}$ whose arc set A contains neither loops nor parallel arcs. The integer points of $P^{(k)}_{0,n\text{-path}}(D)$ are characterized by the system

$$x(\delta^{\text{in}}(0)) = 0, \tag{3.7}$$

$$x(\delta^{\text{out}}(n)) = 0, \tag{3.8}$$

$$x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) = \begin{cases} 1 \text{ if } i = 0, \\ 0 \text{ if } i \in N \setminus \{0, n\}, \\ -1 \text{ if } i = n, \end{cases} \tag{3.9}$$

$$x(A) = k, \tag{3.10}$$

$$x(\delta^{\text{out}}(i)) \le 1 \qquad \forall\, i \in N \setminus \{0, n\}, \tag{3.11}$$

$$x((S : N \setminus S)) \ge x(\delta^{\text{out}}(j)) \qquad \forall\, S \subset N, 3 \le |S| \le n-2, \; 0, n \in S, j \in N \setminus S, \tag{3.12}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\, (i,j) \in A. \tag{3.13}$$

The incidence vectors of node-disjoint unions of a $(0,n)$-path and cycles on node set $N \setminus \{(0,n)\}$ are described by the equations (3.7)-(3.8), the *flow conservation constraints* (3.9), degree constraints (3.11), and the integrality constraints (3.13). The *one-sided min-cut inequalities* (3.12) are satisfied by all $(0,n)$-paths but violated by the union of a $(0,n)$-path and cycles on $N \setminus \{0,n\}$. Finally, the cardinality constraint (3.10) ensures that all $(0,n)$-paths are of cardinality k.

Complete linear descriptions of $P^{(k)}_{0,n\text{-path}}(D)$ for $k = 1, 2, 3$ are given in Table 3.2, where $D = \tilde{D}_n$. The results for $k = 2$ and $k = 3$ follow from the fact that a $(0,n)$-2-path visits exactly one internal node and a $(0,n)$-3-path contains exactly one internal arc. Here, a node $i \in \tilde{N}_n \setminus \{0,n\}$ will be called an *internal node*. Arcs connecting two internal nodes are called *internal arcs*. Since the number of internal nodes is $n-1$, the dimension of $P^{(2)}_{0,n\text{-path}}(\tilde{D}_n)$ is $n-2$, and since the number of internal arcs is $(n-1)(n-2)$, the dimension of $P^{(3)}_{0,n\text{-path}}(\tilde{D}_n)$ is equal to $(n-1)(n-2)-1 = n-3n+1$. The $(0,n)$-1-path polytope $P^{(1)}_{0,n\text{-path}}(\tilde{D}_n)$ has clearly dimension 0 and is determined by the

Table 3.2: Polyhedral Analysis of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$.

k	Dimension	Complete linear description
1	0	$x_{0n} = 1$ $x_{ij} = 0 \quad \forall\,(i,j) \in \tilde{A}_n \setminus \{(0,1)\}$
2	$n-2$	$x(\delta^{\text{in}}(0)) = 0$ $x(\delta^{\text{out}}(n)) = 0$ $x_{ij} = 0 \quad \forall\,(i,j) \in \tilde{A}_n(\tilde{N}_n \setminus \{0,n\})$ $x(\delta^{\text{out}}(0)) = 1$ $x_{0j} - x_{jn} = 0 \quad \forall\, j \in \tilde{N}_n \setminus \{0,n\}$ $x_{0j} \geq 0 \quad \forall\, j \in \tilde{N}_n \setminus \{0,n\}$
3	n^2-3n+1	$x(\delta^{\text{in}}(0)) = 0$ $x(\delta^{\text{out}}(n)) = 0$ $x(\tilde{A}'_n) = 1, \quad \tilde{A}'_n := \tilde{A}_n(\tilde{N}_n \setminus \{0,n\})$ $x(\delta^{\text{out}}(i)) = x_{0i} + x_{in} \quad \forall\, i \in \tilde{N}_n \setminus \{0,n\}$ $x(\delta^{\text{in}}(i)) = x_{0i} + x_{in} \quad \forall\, i \in \tilde{N}_n \setminus \{0,n\}$ $x_{ij} \geq 0 \quad \forall\,(i,j) \in \tilde{A}_n(\tilde{N}_n \setminus \{0,n\})$
		Partial linear description
4 $\vdots$ $n-1$	n^2-2n-1	equations (3.7)-(3.10) see Section 3.2.2
		Remark
n	n^2-3n+1	equivalent to the ATSP

equations $x_{0n} = 1$ and $x_{ij} = 0$ for all $(i,j) \in \tilde{A}_n \setminus \{(0,n)\}$. As is already mentioned in Section 3.1, the dimension of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ is equal to n^2-2n-1 whenever $4 \leq k < n$. Finally, when $k = n$, $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ is isomorphic to the asymmetric traveling salesman polytope (ATSP) which has dimension $n^2 - 3n + 1$ (see [44]).

3.2.1 Basic results

This subsection adapts essentially Section 2 of Hartmann and Özlük [48]. Lemma 3.7 amalgamates Lemmas 2 and 6 of [48] for our purposes. The other statements of this section can be proved in the same manner as the original statements in [48]; so we omit their proofs.

Lemma 3.7 (cf. Lemmas 2 and 6 of Hartmann and Özlük [48])**.** Let $3 \leq k < n$, $c \in \mathbb{R}^{\tilde{A}_n}$, $s, t \in \tilde{N}_n$, $s \neq t$, and $R \subseteq \tilde{N}_n \setminus \{s, t, 0, n\}$. There are λ, π_s, π_t, and π_j, $j \in R$, with

$$\begin{aligned} c_{si} &= \lambda + \pi_s - \pi_i \quad \forall\, i \in R, \\ c_{it} &= \lambda + \pi_i - \pi_t \quad \forall\, i \in R, \\ c_{ij} &= \lambda + \pi_i - \pi_j \quad \forall\, (i,j) \in \tilde{A}_n(R), \end{aligned}$$

if one of the following conditions holds:

(i) $|R| \geq 5$ and $c_{ih} + c_{hj} = c_{i\ell} + c_{\ell j}$ for all distinct nodes $i \in R \cup \{s\}$, $j \in R \cup \{t\}$, $h, \ell \in R$.

(ii) $|R| \geq k \geq 4$ and $c(P) = \gamma$ for all (s,t)-k-paths P, whose internal nodes are all in R.

(iii) $|R| = k - 1$, $c(P) = \gamma$ for all (s,t)-k-paths P, whose internal nodes are all the nodes of R, and $c(P) = \delta$ for all (s,t)-r-paths P, all $r - 1$ of whose internal are in R, for some $2 \leq r < k$.

(iv) $k = 3$, $|R| \geq 3$, $c(P) = \gamma$ for all (s,t)-3-paths P, whose internal nodes are all in R, and $c(P) = \delta$ for each (s,t)-2-path P whose inner node is in R.

Proof. (i) In particular, $c_{ih} + c_{hj} = c_{i\ell} + c_{\ell j}$ for all distinct nodes $i, j, h, \ell \in R$. Using Lemma 2 of Hartmann and Özlük [48], it follows that there are λ and π_j, $j \in R$, with

$$c_{ij} = \lambda + \pi_i - \pi_j \quad \forall\, (i,j) \in \tilde{A}_n(R).$$

Next, setting $\pi_s := c_{sh} + \pi_h - \lambda$ and $\pi_t := \lambda + \pi_h - c_{ht}$ for some $h \in R$, we derive

$$\begin{aligned} c_{si} &= c_{sh} + c_{h\ell} - c_{i\ell} = \lambda + \pi_s - \pi_i, \\ c_{it} &= c_{ht} + c_{\ell h} - c_{\ell i} = \lambda + \pi_i - \pi_t \end{aligned}$$

for all $i \in R$.

(ii) First, let $|R| \geq 5$. Since $|R| \geq k$, for all distinct nodes $i, j, h, \ell \in R$ there is an (s,t)-k-path that contains the arcs (i,h) and (h,j) but does not visit node ℓ. Replacing node h by node ℓ in P yields another (s,t)-k-path and thus $c_{ih} + c_{hj} = c_{i\ell} + c_{\ell j}$ for all distinct nodes $i, j, h, \ell \in R$. Lemma 2 of Hartmann and Özlük implies that there are λ and π_j, $j \in R$, such that $c_{ij} = \lambda + \pi_i - \pi_j$ for all $(i,j) \in \tilde{A}_n(R)$. Set $\pi_s := c_{sh} + \pi_h - \lambda$ and $\pi_t := \lambda + \pi_\ell - c_{\ell t}$ for some $h \neq \ell \in R$. Any (s,t)-k-path whose internal nodes are in R and that uses the arcs $(s,h), (\ell, t)$ yields $\gamma = k\lambda + \pi_s - \pi_t$. Further, considering for $i \in R$ an (s,t)-k-path P whose internal nodes are in R and that uses the arcs $(s,i), (\ell,t)$ yields $c_{si} = \lambda + \pi_s - \pi_i$ for all $i \in R$. Analogously, it follows that $c_{jt} = \lambda + \pi_j - \pi_t$ for all $j \in R$.

Next, let $|R| = k = 4$. Without loss of generality, we may assume that $R = \{1,2,3,4\}$. Setting $Q := \{1,2,3\}$ and identifying the nodes s and t, Theorem 23 of Grötschel and Padberg [47] implies that there are α_s, β_t, $\{\alpha_j : j \in Q\}$, and $\{\beta_j : j \in Q\}$ such that

$$\begin{aligned} c_{si} &= \alpha_s + \beta_i & &\forall\, i \in Q, \\ c_{ij} &= \alpha_i + \beta_j & &\forall\, (i,j) \in \tilde{A}_n(Q), \\ c_{it} &= \alpha_i + \beta_t & &\forall\, i \in Q. \end{aligned}$$

Considering for any two nodes $i \neq j \in Q$ the (s,t)-4-paths $(s,4,h,i,t)$ and $(s,4,h,j,t)$, where h is the remaining node in Q, we see that $c_{hi} + c_{it} = c_{hj} + c_{jt}$ which implies that $\alpha_i + \beta_i = \alpha_j + \beta_j$ for all $i, j \in Q$. Denoting by λ this common value and setting $\pi_s := \alpha_s$, $\pi_j := \alpha_j$ for $j = 1,2,3$, and $\pi_t := \lambda - \beta_t$, yields $c_{si} = \lambda + \pi_s - \pi_i$, $c_{it} = \lambda + \pi_i - \pi_t$ for $i = 1,2,3$, and $c_{ij} = \lambda + \pi_i - \pi_j$ for all $(i,j) \in \tilde{A}_n(Q)$. Now setting $\pi_4 := \lambda + \pi_s - c_{s4}$, we see that $c_{4t} = \lambda + \pi_4 - \pi_t$, $c_{i4} = \lambda + \pi_i - \pi_4$, and $c_{4i} = \lambda + \pi_4 - \pi_i$ for $i = 1,2,3$.

(iii) This is Lemma 6 of Hartmann and Özlük [48].

(iv) Without loss of generality, let $1, 2 \in R$. Condition (iii) implies that there are λ, π_s, π_1, π_2, and π_t with the required property restricted on $Q := \{1,2\}$. Further, it follows that $\gamma = 3\lambda + \pi_s - \pi_t$ and $\delta = 2\lambda + \pi_s - \pi_t$. Setting $\pi_i := \lambda + \pi_s - c_{si}$ for all $i \in R \setminus Q$, we see immediately that $c_{it} = \lambda + \pi_i - \pi_t$ for all $i \in R \setminus Q$. Thus we also obtain $c_{ij} = \lambda + \pi_i - \pi_j$ for all $(i,j) \in \tilde{A}_n(R)$. □

Equivalence of inequalities is an important matter when studying polyhedra. The next results can be used to identify equivalent inequalities with respect to $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Before stating them, we introduce some notions and recall some facts.

First, note that two valid inequalities for the polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ are equivalent if one can be obtained from the other by multiplication with a positive scalar and adding appropriate multiples of the flow conservation constraints (3.9) and the cardinality constraint (3.10). Clearly, two valid inequalities define the same facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if they are equivalent, because the mentioned equations are all implicit equations for this system.

Next, let $Cy = d$ be a linear equation system with $C \in \mathbb{R}^{k\times r}$ and $d \in \mathbb{R}^r$, let $\mathcal{J}$ be the column index set of C, and denote the jth column of C by C_j. A subset $\mathcal{K}$ of $\mathcal{J}$ or the set of variables associated with $\mathcal{K}$ is said to be a *basis* of the system $Cy = d$ if the columns C_j, $j \in \mathcal{K}$ are linearly independent and if the set $\{C_j \mid j \in \mathcal{K}\}$ spans the linear hull of A, that is, of $\{A_j \mid j \in \mathcal{J}\}$.

Finally, we introduce the following two definitions: a *balanced cycle* is a (not necessarily directed) simple cycle that contains the same number of forward and backward arcs and an *unbalanced 1-tree* is a subgraph of D consisting of a spanning tree T plus an arc (h,ℓ) whose fundamental cycle $C(h,\ell)$ is not balanced.

Theorem 3.8 (cf. Theorem 3 of Hartmann and Özlük [48])**.** Let $n \geq 2$ and let H be a subgraph of D. The variables corresponding to the arcs of H form a basis for the linear equality system (3.9), (3.10) if and only if H is an unbalanced 1-tree. □

Corollary 3.9 (cf. Corollary 4 of Hartmann and Özlük [48])**.** Let $c^Tx \leq c_0$ be a valid inequality for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$, and let values b_{ij} be specified for the arcs (i,j) in an unbalanced 1-tree H. Then there is an equivalent inequality $\tilde{c}^T x \leq \tilde{c}_0$ for which $\tilde{c}_{ij} = b_{ij}$ for all arcs $(i,j) \in H$. □

Corollary 3.10 (cf. Corollary 5 of Hartmann and Özlük [48])**.** Let $3 \leq k < n$, $c \in \mathbb{R}^{\tilde{A}_n}$, $s \in \tilde{N}_n \setminus \{n\}$, $t \in \tilde{N}_n \setminus \{0\}$, $s \neq t$, $R \subseteq \tilde{N}_n \setminus \{s,t,0,n\}$ with $|R| \geq 2$, let either of the conditions of Lemma 3.7 be satisfied, and suppose that $c_{ij} = \beta$ holds for all (i,j) in an unbalanced 1-tree H on R. Then $c_{ij} = \beta$ for all $i,j \in R$. Moreover, there are σ and τ with $c_{si} = \sigma$ and $c_{it} = \tau$ for all $i \in R$.

Proof. In either case, Lemma 3.7 implies that there are λ, π_j, $j \in R$, π_s, and π_t with

$$\begin{aligned} c_{si} &= \lambda + \pi_s - \pi_i \quad \forall\, i \in R,\\ c_{it} &= \lambda + \pi_i - \pi_t \quad \forall\, i \in R,\\ c_{ij} &= \lambda + \pi_i - \pi_j \quad \forall\, (i,j) \in \tilde{A}_n(R), \end{aligned}$$

Without loss of generality, let $\pi_h = 0$ for some $h \in R$. Theorem 3.8 then implies that $\lambda = \beta$ and $\pi_j = 0$ for all $j \in R$. Thus, $c_{si} = \beta + \pi_s$ and $c_{it} = \beta - \pi_t$ for all $i \in R$. □

The next result can be used to lift facet defining inequalities for the $(0,n)$-k-path polytope defined on $\tilde{D}_n$ into facet defining inequalities for the $(0,n)$-k-path polytope defined on $D_{n+r+1} - (\delta^-(0) \cup \delta^+(n))$. Here, D_{n+r+1}

denotes the complete digraph on node set $\{0, 1, \ldots, n + r\}$. Moreover, for any subset B of the arc set A of a directed graph $D = (N, A)$, $D - B$ denotes edge deletion: $D - B := (N, A \setminus B)$.

Before stating the next theorem we need some definitions. A subset $B \subseteq \tilde{A}_n$ of cardinality k is called a *k-bowtie* if it is the union of a $(0, n)$-path P and a simple cycle C connected at exactly one node. The k-bowtie B is said to be *tied* at node h if $\tilde{N}_n(P) \cap \tilde{N}_n(C) = \{h\}$. A facet F of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ is called *regular* if it is defined by an inequality $c^T x \leq c_0$ that is not equivalent to a nonnegativity constraint $x_{ij} \geq 0$ or a *broom inequality*

$$x((\delta^{\text{out}}(i)) \geq x_{ji} + x_{ih} \tag{3.14}$$

for some internal node i, where $j = h$ is an internal node or $j = 0$ and $h = n$. Note that F is already regular if for each internal node h, there is a $(0, n)$-k-path P with $c(P) < c_0$ that does not visit node h (see [48]).

Theorem 3.11 (cf. Theorem 8 of Hartmann and Özlük [48])**.** Suppose that $c^T x \leq c_0$ induces a regular facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$, where $3 < k < n$. Let h be an internal node such that $c(B) \leq c_0$ for all k-bowties B tied at node h and let δ_h be the maximum of $c(\Gamma)$ over all $0, n$-paths Γ of cardinality $k - 1$ that visit node h. Then

$$c^T x + \sum_{\substack{i=0 \\ i \neq h}}^{n-1} c_{ih} x_{i,n+1} + \sum_{\substack{j=1 \\ j \neq h}}^{n} c_{hj} x_{n+1,j} + (c_0 - \delta_h)[x_{h,n+1} + x_{n+1,h}] \leq c_0 \tag{3.15}$$

defines a regular facet of $P^{(k)}_{0,n\text{-path}}(D')$, where D' is the digraph obtained by subtracting from the complete digraph on node set $\{0, \ldots, n+1\}$ the arc sets ($\delta^{\text{in}}(0)$ and $\delta^{\text{out}}(n)$). $\square$

Since inequality (3.15) is obtained by copying the coefficient structure of node h, one refers to this process as "lifting by cloning node h". Clearly, repeating this process r times, we end up with a facet defining inequality for the $(0, n)$-k-path polytope defined on the digraph $D_{n+r+1} - (\delta^-(0) \cup \delta^+(n))$ provided we started with a facet defining inequality for the $(0, n)$-k-path polytope defined on $\tilde{D}_n$. In order to show that a class $\mathcal{K}$ of regular inequalities define facets of the $(0, n)$-k-path polytope it suffices to show it for a subclass $\mathcal{K}' \subset \mathcal{K}$ from which the remaining inequalities in $\mathcal{K} \setminus \mathcal{K}'$ can be obtained by cloning internal nodes. The members of a minimal subclass $\mathcal{K}'$ (minimal with respect to set inclusion) are said to be *primitive*.

Before stating the last theorem of this section we need again some definitions. Let F be a subset of $\tilde{A}_n$, the *auxiliary graph* G_F is an undirected bipartite graph on $2n$ nodes $v_0, \dots, v_{n-1}$, $w_1, \dots, w_n$, with the property that $(i,j) \in F$ if and only if G_F contains the arc (v_i, w_j). Moreover, we define the following equivalence relation on the arc set $\tilde{A}_n$: two arcs (i,j) and (h,ℓ) are *related with respect to* $c^T x \le c_0$, if there is an arc $(f,g) \in \tilde{A}_n$ with $a_{ij} = a_{fg} = a_{h\ell}$ and two tight $(0,n)$-k-paths P_{ij}, $P_{h\ell}$ such that $(i,j),(f,g) \in P_{ij}$ and $(h,\ell),(f,g) \in P_{h\ell}$.

Theorem 3.12 (cf. Theorem 9 of Hartmann and Özlük [48])**.** Let $b \in \mathbb{R}_+^{\tilde{A}_n}$ and $b^T x \le \beta$ be a facet defining inequality for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$, where $3 < k < n$. Suppose that the auxiliary graph G_Z for the arc set $Z := \{(i,j) \in \tilde{A}_n : b_{ij} = 0\}$ is connected, every tight $(0,n)$-k-path with respect to $b^T x \le \beta$ contains at least one arc $(i,j) \in Z$, and every arc (i,j) belongs to the same equivalence class with respect to $b^T x \le \beta$. Let R be a set of nodes, set $q := k + |R|$, and let t be the smallest number such that

$$b^T x + t \sum_{j \in R} x(\delta^{\text{out}}(j)) \le \beta + |R| t \tag{3.16}$$

is valid for all $(0,n)$-q-paths on $\tilde{N}_n \cup R$, and if $|R| \ge 2$ suppose that at least one tight $(0,n)$-q-path with respect to (3.16) visits r nodes in R with $0 < r < |R|$. Then (3.16) is facet defining for the $(0,n)$-q-path polytope on $\tilde{N}_n \cup R$. □

3.2.2 Facets and valid inequalities

In what follows, we will show that the inequalities given in the IP-formulation, the nonnegativity constraints $x_{ij} \ge 0$, as well as some more inequalities are in general facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Throughout, we assume that $4 \le k \le n-1$. The inequalities considered in Theorems 3.13 - 3.17 were shown to be valid for the k-cycle polytope in Hartmann and Özlük [48]. So they are also valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$, since the $(0,n)$-k-path polytope on D can be interpreted as the restriction of the k-cycle polytope on D_n to the hyperplane defined by $x(\delta^{\text{out}}(n)) = 1$.

Trivial inequalities

Theorem 3.13 (cf. Theorem 10 of Hartmann and Özlük [48])**.** The nonnegativity constraint

$$x_{ij} \ge 0 \tag{3.17}$$

is valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ and induces a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ whenever $4 \le k \le n-1$.

Proof. When $n \leq 6$ and $k = 4$ or $k = 5$, (3.17) can be proved to induce a facet by application of a convex hull code (e.g. Polymake [41]), so we assume that $n \geq 7$. Suppose that $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ with $x_{ij} = 0$. At least one of the two nodes i and j is an internal node, because $(0, n) \notin \tilde{A}_n$. Without loss of generality, we may assume that $j \in \{1, \ldots, n-1\}$ and set $R := \tilde{N}_n \setminus \{0, n, j\}$. By Corollary 3.9, we may assume that $c_{jw} = c_{0w} = c_{wn} = 0$ for some $w \in R$ and $c_{h\ell} = 0$ for all arcs (h, ℓ) in some unbalanced 1-tree on R.

Let $q \in R \cup \{0\}$, $r, s \in R$, $t \in R \cup \{n\}$ be distinct nodes, and let P be a $(0, n)$-k-path that contains the arcs (q, r) and (r, t) but does not visit node s or use the arc (i, j). Substituting node r by node s in P we obtain another $(0, n)$-k-path that does not use (i, j). Hence condition (3.7) of Lemma 3.7 holds, and Corollary 3.10 implies that $c_{h\ell} = 0$ for all $(h, \ell) \in \tilde{A}_n(\tilde{N}_n \setminus \{j\})$ which also implies that $c_0 = 0$.

Each $(0, n)$-k-path that uses the arc (j, w) but does not use the arc (i, j) also satisfies (3.17) with equality, so $c_{hj} = 0$ for all $h \in \tilde{N}_n \setminus \{i, n, w\}$. Similar considerations yield $c_{jh} = 0$ for all $h \in \tilde{N}_n \setminus \{0\}$ and $c_{wj} = 0$ if $w \neq i$. Thus, $c_{h\ell} = 0$ for all arcs $(h, \ell) \neq (i, j)$ and therefore $c^T x = c_0$ is simply $c_{ij} x_{ij} = 0$. □

Theorem 3.14 (cf. Theorem 11 of Hartmann and Özlük [48])**.** Let j be an internal node. The degree constraint

$$x(\delta^{\text{out}}(j)) \leq 1 \tag{3.18}$$

is valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ and induces a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ whenever $4 \leq k \leq n - 1$.

Proof. Without loss of generality, we will show that $x(\delta^{\text{out}}(1)) \leq 1$ defines a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. First we will show that Theorem 3.14 holds when $k = 4$. If $n = 5$, $x(\delta^{\text{out}}(1)) \leq 1$ can be proved to define a facet using a convex hull code. Theorem 3.11 applied to node 2 yields then the result when $n \geq 6$.

Secondly, we will investigate the case $k \geq 5$. Suppose that $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ with $x(\delta^{\text{out}}(1)) = 1$. By Corollary 3.9, we may assume that $c_{21} = c_{02} = c_{2n} = 0$ and $c_{ij} = 0$ in some unbalanced 1-tree on $R := \{2, 3, \ldots, n-1\}$. Since $|R| \geq k - 1 \geq 4$ and $c(P) = c_0 - c_{01}$ for all $(1, n)$-paths P of cardinality $k - 1$ whose internal nodes are all in R, condition (3.7) of Lemma 3.7 holds. Thus, $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(R \cup \{n\})$ and $c_{1j} = 1$ for all $j \in R$ using Corollary 3.10. Now it is easy to see that $c^T x = c_0$ is simply $x(\delta^{\text{out}}(1)) = 1$. □

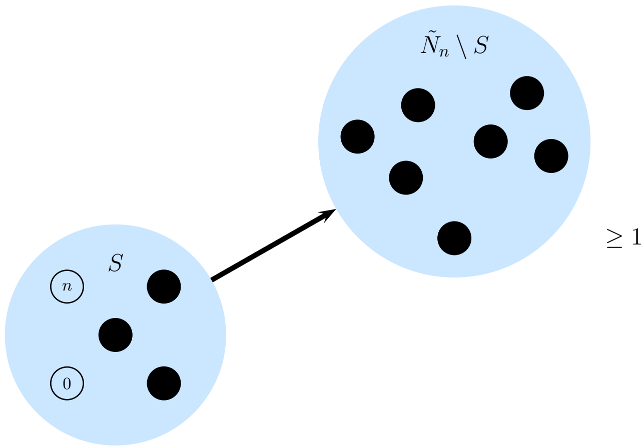


Figure 3.2: Illustration of a min-cut inequality for $k = 5$ and $|S| = 5$.

Cut inequalities

Theorem 3.15 (cf. Theorem 12 of Hartmann and Özlük [48]). Let $S \subset \tilde{N}_n$ and $0, n \in S$. The *min-cut inequality*

$$x((S : \tilde{N}_n \setminus S)) \geq 1 \tag{3.19}$$

is valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $|S| \leq k$. Furthermore, it induces a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $3 \leq |S| \leq k$ and $|\tilde{N}_n \setminus S| \geq 2$.

Figure 3.2 sketches the support graph of a min-cut inequality for $k = 5$ and $|S| = 5$. As is easily seen, each $(0, n)$-5-path has to use at least one cut arc $(i, j) \in (S : \tilde{N}_n \setminus S)$ depicted by the arrow from S to $\tilde{N}_n \setminus S$.

Proof of Theorem 3.15. The min-cut inequality (3.19) is valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $|S| \leq k$, since a $(0, n)$-k-path can be obtained in S if and only if $|S| \geq k+1$. When $|S| = 2$, (3.19) is an implicit equation. When $|\tilde{N}_n \setminus S| = 1$, $n \leq k$. So we suppose that $3 \leq |S| \leq k$ and $|\tilde{N}_n \setminus S| \geq 2$.

First let $|S| = 3$. When $|\tilde{N}_n \setminus S| \leq 4$, (3.19) can be shown to be facet defining by means of a convex hull code, so let $|\tilde{N}_n \setminus S| \geq 5$. Let w.l.o.g. $S = \{0, 1, n\}$ and suppose that $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.19) with equality. Using Corollary 3.9, we may assume that $c_{01} = 0$, $c_{0w} = c_0$ and $c_{wn} = 0$ for some $w \in \tilde{N}_n \setminus S$, as well as $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree H on $\tilde{N}_n \setminus S$.

Let $i \in (\tilde{N}_n \setminus S) \cup \{0\}$, $j \in (\tilde{N}_n \setminus S) \cup \{n\}$, $h, l \in \tilde{N}_n \setminus S$ be distinct nodes, and let P be a tight $(0, n)$-k-path that contains the arcs $(i, h), (h, j)$ but does not visit node ℓ. Such a path P exists even when $k = 4$. Replacing node h by node ℓ yields another tight $(0, n)$-k-path, and hence condition (3.7) of Lemma 3.7 holds. Corollary 3.10 implies that $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(\tilde{N}_n \setminus S)$, $c_{0i} = c_0$, and $c_{in} = 0$ for all $i \in \tilde{N}_n \setminus S$. Now it is easy to see that $c_{1i} = c_0$ and $c_{i1} + c_{1n} = 0$ for all $i \in \tilde{N}_n \setminus S$. Subtracting c_{1n} times the equation $x(\delta^{\text{in}}(n)) = 1$ and adding c_{1n} times the equation $x((\tilde{N}_n \setminus S : S)) - x((S : \tilde{N}_n \setminus S)) = 0$, we see that $c^T x = c_0$ is equivalent to $(c_0 - c_{1n})x(S : \tilde{N}_n \setminus S) = c_0 - c_{1n}$.

Secondly, let $|S| \geq 4$. Let w.l.o.g. $S = \{0, 1, 2, \ldots, q, n\}$ for some $q < k$ and suppose that $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.19) with equality. Using Corollary 3.9, we may assume that $c_{01} = c_{1n} = 0$, $c_{1i} = c_0$ for all $i \in (\tilde{N}_n \setminus S)$, and $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree on $R := S \setminus \{0, n\}$.

Let P be the path $(q + 1, \ldots, k - 1, n)$ and Q the path $(q + 1, \ldots, k, n)$. Then $c(\Gamma) = c_0 - c(P)$ for all $(0, q + 1)$-paths Γ, whose internal nodes are all the nodes of R. Further, $c(\Delta) = c_0 - c(Q)$ for all $(0, q + 1)$-paths Δ, all q of whose internal nodes are in R. Therefore, condition (3.7) of Lemma 3.7 holds and Corollary 3.10 implies that $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(R \cup \{0\})$ and $c_{i,q+1} = c_0$ for all $i \in R$. Replacing node $q + 1$ by any other node in $\tilde{N}_n \setminus S$ (in the above argumentation), we obtain $c_{ij} = c_0$ for all $(i, j) \in (R : \tilde{N}_n \setminus S)$.

Next, consider for any arc $(i, j) \in \tilde{A}_n(\tilde{N}_n \setminus S)$ a tight $(0, n)$-k-path P that uses the arcs $(0, 1), (1, 2), (2, j)$ and skips node i. Then, the $(0, n)$-k-path $P' := (P \setminus \{(0, 1), (1, 2), (2, j)\}) \cup \{(0, 2), (2, i), (i, j)\}$ is also tight. Thus, we derive that $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(\tilde{N}_n \setminus S)$. Furthermore, from the tight $(0, n)$-k-paths that start with the arc $(0, 1)$ and use some arc (i, n) with $i \in \tilde{N}_n \setminus S$ we deduce $c_{in} = 0$ for all those arcs (i, j). Moreover, from the tight $(0, n)$-k-paths starting with the arc $(0, 2)$ and ending with the arcs $(i, 1), (1, n)$ for some $i \in \tilde{N}_n \setminus S$ we obtain $c_{i1} = 0$ for $i \in \tilde{N}_n \setminus S$. It is now easy to see that $c_{0i} = c_0$ for all $i \in \tilde{N}_n \setminus S$, $c_{jn} = 0$ for all $j \in R$, and $c_{ij} = 0$ for all $(i, j) \in (\tilde{N}_n \setminus S : R)$ (distinguish the cases $k = 4$ and $k \geq 5$). Therefore $c^T x = c_0$ is simply $c_0 x((S : \tilde{N}_n \setminus S)) = c_0$. □

Theorem 3.16 (cf. Theorem 13 of Hartmann and Özlük [48]). Let $S \subset \tilde{N}_n$ and $0, n \in S$. The one-sided min-cut inequality

$$x((S : \tilde{N}_n \setminus S)) \geq x(\delta^{\text{out}}(\ell)) \tag{3.20}$$

is valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ for all $\ell \in \tilde{N}_n \setminus S$, and facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $|S| \geq k + 1$ and $|\tilde{N}_n \setminus S| \geq 2$.

Proof. The one-sided min-cut inequality (3.20) is valid, because all $(0, n)$-k-paths that visit some node $\ell \in \tilde{N}_n \setminus S$ use at least one arc in $(S : \tilde{N}_n \setminus S)$. If $|\tilde{N}_n \setminus S| = 1$, then (3.20) is the flow constraint $x(\delta^{\text{in}}(\ell)) - x(\delta^{\text{out}}(\ell)) = 0$. If, indeed, $|\tilde{N}_n \setminus S| \geq 2$ but $|S| \leq k$, then (3.20) can be obtained by summing up the min-cut inequality (3.19) and the degree constraint $-x(\delta^{\text{out}}(\ell)) \geq -1$.

So suppose that $|S| \geq k+1$ and $|\tilde{N}_n \setminus S| \geq 2$. Let w.l.o.g. $\ell = 1$ and set $R := S \cup \{1\}$. By adding to (3.20) the flow constraint $x(\delta^{\text{out}}(1)) - x(\delta^{\text{in}}(1)) = 0$, it can be easily seen that (3.20) is equivalent to

$$x((S : \tilde{N}_n \setminus R)) - \sum_{i \in \tilde{N}_n \setminus R} x_{i1} \geq 0. \tag{3.21}$$

Suppose that $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.21) with equality. By Corollary 3.9, we may assume that $c_{in} = 0$ for all $i \in \tilde{N}_n \setminus R$ and $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree on R. Condition (3.7) of Lemma 3.7 is satisfied; hence, from Corollary 3.10 follows that $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(R)$ which also implies that $c_0 = 0$.

Any $(0, n)$-k-path that contains the arcs $(1, i), (i, n)$ for some $i \in \tilde{N}_n \setminus R$ and whose remaining arcs are in $\tilde{A}_n(R)$ satisfies (3.21) with equality. Since $c_{in} = 0$ and $c_a = 0$ for all $a \in \tilde{A}_n(R)$, it follows that $c_{1i} = 0$ for all $i \in \tilde{N}_n \setminus R$. Now considering tight $(0, n)$-k-paths that contain the arcs $(1, i), (i, j), (j, n)$ for some $(i, j) \in \tilde{A}_n(\tilde{N}_n \setminus R)$ and whose remaining arcs are in $\tilde{A}_n(R)$, we see that $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(\tilde{N}_n \setminus R)$. Further, the $(0, n)$-k-paths that use the arcs $(1, i), (i, j)$ for $i \in \tilde{N}_n \setminus R, j \in S \setminus \{n\}$ and whose remaining arcs are in $\tilde{A}_n(R)$ yield $c_{ij} = 0$ for all $(i, j) \in (\tilde{N}_n \setminus R : S \setminus \{n\})$. Finally, considering for each $(i, j) \in (S : \tilde{N}_n \setminus R)$ and $h \in \tilde{N}_n \setminus R$ a tight $(0, n)$-k-path that contains the arcs $(i, j), (j, 1)$ and a tight $(0, n)$-k-path that contains the arcs $(i, j), (j, h), (h, 1)$, we see that $c_{j1} = c_{h1}$ for all $j, h \in \tilde{N}_n \setminus R$, $c_{ij} = c_{hg}$ for all $(i, j), (h, g) \in (S : \tilde{N}_n \setminus R)$, and $c_{ij} + c_{h1} = 0$ for all $(i, j) \in (S : \tilde{N}_n \setminus R)$, $h \in \tilde{N}_n \setminus R$. Thus $c^T x = c_0$ is simply $c_{jh} x((S : \tilde{N}_n \setminus R)) - c_{jh} \sum_{i \in \tilde{N}_n \setminus R} x_{i1} = 0$ for some $(j, h) \in (S : \tilde{N}_n \setminus R)$. □

Theorem 3.17 (cf. Theorem 15 of Hartmann and Özlük [48])**.** *Let $\tilde{N}_n = R \dot{\cup} S \dot{\cup} T$ be a partition of $\tilde{N}_n$ and let $0, n \in S$. The generalized max-cut inequality*

$$x((S : T)) + \sum_{i \in R} x(\delta^{\text{out}}(i)) \leq \lfloor (k + |R|)/2 \rfloor \tag{3.22}$$

is valid for the $(0, n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ for $k \geq 4$ and facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $k + |R|$ is odd, $|S \setminus \{n\}| > (k - |R|)/2$, $|T| > (k - |R|)/2$, and

(i) *$k = |R| + 3$, $|R| \geq 2$, and $|S| = 3$, or*

(ii) *$k \geq |R| + 5$.*

Proof. Necessity. From $x((S:T)) \leq x((T:S)) + x((T:R))$ and $x(\tilde{A}_n) = k$ we derive the inequality $2x((S:T)) + \sum_{i\in R} x(\delta^{\text{out}}(i)) \leq k$. Adding the inequality $\sum_{i\in R} x(\delta^{\text{out}}(i)) \leq |R|$, dividing by two, and rounding down, we obtain (3.22). When $k+|R|$ is even, then (3.22) is obtained with no rounding, and hence it is not facet defining. When $|S \setminus \{n\}| \leq (k-|R|)/2$ or $|T| \leq (k-|R|)/2$, then (3.22) is implied by degree constraints $x(\delta^{\text{out}}(i)) \leq 1$.

Let P be any $(0,n)$-k-path and denote by r the number of nodes in R visited by P. Then $|v(P) \cap (S \setminus \{n\} \cup T)| = k - r$ and hence $\chi^P((S:T)) \leq (k-r)/2$. This in turn implies that there is no tight $(0,n)$-k-path if $r \leq |R| - 2$, where $|R| \geq 2$. Now, when $k = |R| + 3$ and $|S| \geq 4$, (3.22) is dominated by nonnegativity constraints $x_{ij} \geq 0$ for $(i,j) \in \tilde{A}_n(S \setminus \{0,n\})$. Further, when $k = |R| + 3$, $|S| = 3$, and $|R| = 1$, (3.22) is dominated by the inequality (3.28). Finally, when $k \leq |R| + 1$, (3.22) is dominated by some nonnegativity constraints, for example, $c_{in} = 0$ for some $i \in T$.

Suffiency. First we will show that (3.22) is facet defining if $R = \varnothing$. In this case, the resulting inequality

$$x((S:T)) \leq \lfloor k/2 \rfloor = q \tag{3.23}$$

where $k = 2q+1$, is called *max-cut inequality.* First, we show that (3.23) is facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. If $k = 5$ and $|S \setminus \{n\}| = 3$ or $|T| = 3$, we will show that (3.23) defines a facet using Theorem 3.11. The only primitive inequalities are those with $n = 6$ and by application of a convex hull code, we see that in this case (3.23) is facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Moreover, (3.23) is regular, since for each inner node h there is a non-tight $(0,n)$-k-path that does not visit h. Without loss of generality, let $T = \{1,2,\ldots,t\}$ and $S = \{t+1,\ldots,n,0\}$ for some $4 \leq t \leq n-4$.

Suppose that $c^T x = c_0$ holds for all $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ satisfying (3.23) with equality. By Corollary 3.9, we may assume that $c_{02} = 1$, $c_{t+1,n} = 0$, $c_{j1} = 1$ for all $j \in S \setminus \{n\}$, and $c_{1i} = 0$ for all $i \in T$.

First, consider any $(0,n)$-$2q$-path P that alternates between nodes in S and nodes in T, but does not visit node 1. Replacing any arc $(i,j) \in P$ with $i \in S$, $j \in T$ by the arcs $(i,1),(1,j)$ we obtain a tight $(0,n)$-k-path, and therefore $c(P) - c_{ij} = c_0 - 1$ holds for all $(i,j) \in P \cap (S:T)$. This in turn implies that $c_{ij} = 1$ for all $(i,j) \in (S:T)$, since we have $3 \leq t \leq n-3$ and $c_{02} = 1$. Next, consider any tight $(0,n)$-k-path that uses arcs $(i,h),(h,j)$ for $i,j \in S \setminus \{0,n\}$, $h \in T$ but does not visit some node $\ell \in T$. Replacing node h by node ℓ yields another tight path which implies immediately $c_{ih} + c_{hj} = c_{i\ell} + c_{\ell j}$. Similarly we obtain $c_{hi} + c_{i\ell} = c_{hj} + c_{j\ell}$ and thus $c_{ih} + c_{hi} = c_{j\ell} + c_{\ell j}$ for all $i,j \in S \setminus \{0,n\}$ and $h,\ell \in T$. Since $t \geq 3$ and $c_{ih} = c_{j\ell} = 1$, we see that there is some σ with $c_{hi} = \sigma$ for all $h \in T$, $i \in S \setminus \{0,n\}$. Now consider any tight path that contains the arcs $(1, t+1), (t+1, n)$ and does not visit some node $\ell \in T$. Replacing node $t+1$ by node ℓ yields another tight $(0, n)$-k-path and hence $c_{1,t+1} + c_{t+1,n} = c_{1\ell} + c_{\ell n}$. Since $c_{1,t+1} = \sigma$ and $c_{t+1,n} = c_{1\ell} = 0$, this implies $c_{\ell n} = \sigma$ for all $\ell \in T$, $\ell \neq 1$. Of course, it follows also that $c_{1n} = \sigma$.

Finally, any tight $(0, n)$-k-path contains exactly one arc $(i, j) \in \tilde{A}_n(S) \cup \tilde{A}_n(T)$, so $c_{ij} = c_0 - q(1+\sigma)$ for all $(i, j) \in \tilde{A}_n(S) \cup \tilde{A}_n(T)$. Due to $c_{t+1,n} = 0$, this implies that $c_{ij} = 0$ for all $(i, j) \in \tilde{A}_n(S) \cup \tilde{A}_n(T)$. Adding σ times the equation $x((S : T)) - x((T : S)) = 0$, we see that $c^T x = c_0$ is equivalent $x((S : T)) = q$. This proves that (3.23) is also facet defining when $0, n \in T$.

When $R \neq \varnothing$, we prove the claim by showing that the conditions of Theorem 3.12 hold for (3.23). Since $w = k - |R|$ is odd and $w \geq 5$, the inequality $x((S : T)) \leq \lfloor w/2 \rfloor$ induces a facet of the $(0, n)$-w-path polytope defined on the digraph $D^+ = (\tilde{N}_n \setminus R, \tilde{A}_n(\tilde{N}_n \setminus R))$. Let us denote this inequality by $d^T x \leq d_0$. It is easy to see that the auxiliary graph G_Z for the arc set $Z = \{(i, j) : d_{ij} = 0\}$ is connected (cf. [48]). Furthermore, each tight $(0, n)$-w-path contains two arcs (i, j) and (h, ℓ) which are not adjacent, and hence all arcs in Z are in the same equivalency class with respect to $d^T x \leq d_0$. Since there are tight $(0, n)$-k-paths with respect to (3.22) that visit $|R| - 1$ of the nodes in R, Theorem 3.12 implies that (3.22) induces a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ unless $k = |R| + 3$, $|R| \geq 2$, and $|S| = 3$.

Finally, suppose that $k = |R| + 3$, $|R| \geq 2$, and $|S| = 3$. Without loss of generality, we may assume that $S = \{0, 1, n\}$, $2, 3 \in R$, and $4, 5 \in T$. Suppose that $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.22) with equality. By Corollary 3.9, we may assume that $c_{2j} = 1$ for all $j \in R$, $c_{i2} = 0$ for all $i \in T$, $c_{32} = 1$, $c_{21} = 1$, $c_{1n} = 0$, and $c_{04} = 1$. There are tight $(0, n)$-k-paths that visit a node $\ell \in T$ followed by all $|R|$ (or any $|r| - 1$) nodes in R and a node 1. Applying Lemma 3.7, we see that

$$\begin{aligned} c_{\ell j} &= \lambda + \pi_\ell - \pi_j \quad && (j \in R) \\ c_{ij} &= \lambda + \pi_i - \pi_j \quad && (i, j \in R) \\ c_{im} &= \lambda + \pi_i - \pi_m \quad && (i \in R) \end{aligned}$$

for some λ, π_j, $j \in R$, π_ℓ, and π_1. Let w.l.o.g. $\pi_2 = 0$. Theorem 3.8 then implies that $\lambda = 1$ and $\pi_j = 0$ for all $j \in R$, $c_{i2} = 0$ implies that $\pi_\ell = -1$, and $c_{21} = 1$ implies that $\pi_1 = 0$. Thus, $c_{ij} = 1$ for all $(i, j) \in \tilde{A}_n(R)$, $c_{ij} = 0$ for all $i \in T, j \in R$, and $c_{i1} = 1$ for all $i \in R$. Next, considering any tight $(0, n)$-k-path P that uses the arcs $(0, 4), (2, 1), (1, n)$ and visits all $|R|$ nodes in R yields $c_0 = |R| + 1$. Replacing node 4 by another node $j \in T$ yields $c_{0j} = 1$ for all $j \in T$. Next, consider any tight $(0, n)$-k-path P that uses the arcs $(0, i), (i, j), (j, 1)$ for some $i, j \in R$. Then the $(0, n)$-k-path $P' := (P \setminus \{(0, i), (i, j), (j, 1)\}) \cup \{(0, j), (j, i), (i, 1)\}$ is also tight, and hence,

$c_{0i} = c_{0j}$ for all $i, j \in R$. Denote this common value by σ. From the tight $(0,n)$-k-paths that visit the nodes 1 and t for some $t \in T$ and all nodes in R, we derive $c_{ij} = 1 - \sigma$ for all $i \in R$, $j \in T$. Now it is easy to see that $c_{in} = 1 + \sigma$ for all $i \in T$. Considering any tight $(0,n)$-k-path that uses the arcs $(0,2), (2,1), (1,4), (4,3)$, and (m,n) for an appropriate $m \in R$ yields $\sigma = 0$. Thus, $c_{0i} = 0$ and $c_{in} = 1$ for all $i \in R$, $c_{1j} = 1$ for all $j \in T$, and $c_{ij} = 1$ for all $i \in R, j \in T$. Determining the coefficients of the remaining arcs is an easy task. So we see that $c^T x = c_0$ is simply (3.22). □

Theorem 3.18. Let $\tilde{N}_n = R \dot\cup S \dot\cup T$ be a partition of $\tilde{N}_n$ and let $0, n \in T$. The generalized max-cut inequality

$$x((S:T)) + \sum_{i \in R} x(\delta^{\text{out}}(i)) \leq \lfloor (k + |R|)/2 \rfloor \tag{3.24}$$

is valid for the $(0,n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ for $k \geq 4$ and facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $k + |R|$ is odd, $|S| > (k - |R|)/2$, $|T \setminus 0| > (k - |R|)/2$, and

(i) $k = |R| + 3$, $|R| \geq 2$, and $|T| = 3$, or

(ii) $k \geq |R| + 5$.

□

Theorem 3.19. Let $\tilde{N}_n = R \dot\cup S \dot\cup T$ be a partition of $\tilde{N}_n$, let $0 \in S$, and let $n \in T$. The generalized max-cut inequality

$$x((S:T)) + \sum_{i \in R} x(\delta^{\text{out}}(i)) \leq \lfloor (k + |R| + 1)/2 \rfloor \tag{3.25}$$

is valid for the $(0,n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ for $k \geq 4$ and facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $k+|R|$ is even, $k \geq |R|+4$, $|S| > (k-|R|)/2$, and $|T| > (k - |R|)/2$.

Proof. From the inequality $x((S:T)) \leq x((T:S)) + x((T:R)) + 1$ and the equation $x(\tilde{A}_n) = k$ we derive the inequality $2x((S:T)) + \sum_{i \in R} x(\delta^{\text{out}}(i)) \leq k + 1$. Adding the inequality $\sum_{i \in R} x(\delta^{\text{out}}(i)) \leq |R|$, dividing by two, and rounding down yields (3.25). If $k + |R|$ is odd we obtain (3.25) without rounding and hence it is not facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. When $|S| \leq (k - |R|)/2$ or $|T| \leq (k - |R|)/2$, (3.25) is dominated by degree constraints $x(\delta^{\text{out}}(j)) \leq 1$. Furthermore, we have to show that (3.25) is not facet defining if $k \le |R|+2$. When $R = \varnothing$, this is clear. Otherwise consider any $(0,n)$-k-path P and denote the number of nodes in R visited by P by r. It is easy to see that P is tight only if $r \ge |R|-1$. For the sake of contradiction, assume that $k \le |R|$ and P is tight. Then we have $r = |R|-1$ and thus $k = |R|$ which implies $\lfloor (k+|R|+1)/2 \rfloor = |R|$. But $\chi^P((S:T)) + \sum_{i \in R} \chi^P(\delta^{\text{out}}(i)) = |R|-1$, so P is not tight, a contradiction. Hence, the only possibility is that $k = |R|+2$. Now, $k = |R|+2$ implies that $|S|, |T| \ge 2$ and $\lfloor (k+|R|+1)/2 \rfloor = |R|+1$. But then (3.25) is dominated by the nonnegativity constraints $x_{ij} \ge 0$ for all $(i,j) \in \tilde{A}_n(S) \cup \tilde{A}_n(T)$.

First, we show that (3.25) is facet defining when $R = \varnothing$. In this case, k is even and (3.25) is the max-cut inequality

$$x((S : \tilde{N}_n \setminus S)) \le \lfloor (k+1)/2 \rfloor = k/2. \tag{3.26}$$

If $k = 4$ and $|S| = 3$ or $|\tilde{N}_n \setminus S| = 3$, we will show that (3.26) defines a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ using Theorem 3.11. The only primitive members of family (3.26) with $k = 4$ are those with $|S| = |\tilde{N}_n \setminus S| = 3$. Inequality (3.26) is obviously regular, and using a convex hull code, we see that (3.26) defines a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Moreover, all k-bowties tied at an inner node satisfy (3.26).

If $k \ge 6$ suppose that the equation $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.26) with equality. Let w.l.o.g. $1, 2 \in \tilde{N}_n \setminus S$. By Corollary 3.9, we may assume that $c_{02} = 1$, $c_{i1} = 1$ for all $i \in S$, and $c_{1j} = 0$ for all $j \in \tilde{N}_n \setminus S$, $j \ne 1$. Since $|S|, |\tilde{N}_n \setminus S| \ge 4$, we can apply the same argumentation as in the proof to Theorem 3.17. Thus $c_{ij} = 1$ for all $(i,j) \in (S : \tilde{N}_n \setminus S)$, $c_{ij} = \sigma$ for all $(i,j) \in (\tilde{N}_n \setminus (S \cup \{n\}) : S \setminus \{0\})$, for some σ, and $c_{ij} = 0$ for all $(i,j) \in \tilde{A}_n(S) \cup \tilde{A}_n(T)$. Evaluating the costs of tight $(0,n)$-k-paths yields $c_0 = \frac{k}{2} + (\frac{k}{2} - 1)\sigma$ which implies that $c^T x = c_0$ is the equation $x((S : \tilde{N}_n \setminus S)) + \sigma x((\tilde{N}_n \setminus S : S)) = \frac{k}{2} + \sigma(\frac{k}{2} - 1)$. Adding σ times the equation $x((S : \tilde{N}_n \setminus S)) - x((\tilde{N}_n \setminus S : S)) = 1$, we see that (3.26) is equivalent to $x((S : \tilde{N}_n \setminus S)) = k/2$.

Applying Theorem 3.12 to the $(0,n)$-w-path polytope defined on the digraph $D^* = (\tilde{N}_n \setminus R, \tilde{A}_n(\tilde{N}_n \setminus R))$, where $w = k - |R|$, proves that (3.25) is facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ even for $R \ne \varnothing$. □

Theorem 3.20. Let $\tilde{N}_n = R \,\dot\cup\, S \,\dot\cup\, T$ be a partition of $\tilde{N}_n$, let $0 \in T$, and let $n \in S$. The generalized max-cut inequality

$$x((S : T)) + \sum_{i \in R} x(\delta^{\text{out}}(i)) \le \lfloor (k + |R| - 1)/2 \rfloor \tag{3.27}$$

is valid for the $(0, n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ for $k \geq 4$ and facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $k+|R|$ is even, $k \geq |R|+4$, $|S| > (k-|R|)/2$, and $|T| > (k - |R|)/2$. □

Remark 3.21. If $R = \varnothing$, inequality (3.27) is equivalent to the inequality

$$x((T : S)) \leq \lfloor (k+1)/2 \rfloor,$$

since in this case holds the equation $x((S : T)) = x((T : S)) - 1$.

Theorem 3.22. Let $\varnothing \neq T = \tilde{N}_n \setminus \{0, 1, 2, 3, n\}$. The inequality

$$\begin{gathered} x_{03} - x_{3n} + 3x_{12} - x_{21} + 2x_{13} - 2x_{31} - 2x_{2n} + 2x((T : \{3\})) \\ +x(\tilde{A}_n(T)) + x((\{1\} : T)) - x((T : \{1\})) + x((T : \{2\})) - x((\{2\} : T) \geq 0 \end{gathered} \tag{3.28}$$

is facet defining for $P^{(4)}_{(s,t)\text{-path}}(D)$.

Proof. When $|T| = 1$, the claim can be verified with a convex hull code. For $|T| \geq 2$ we apply Theorem 3.11. □

Jump inequalities

Dahl and Gouveia [23] introduced a class of valid inequalities for the hop constrained path polytope $P^{\leq k}_{0,n\text{-path}}(\tilde{D}_n)$ they called jump and lifted jump inequalities. Given a partition

$$\tilde{N}_n = \bigcup_{p=0}^{k+1} S_p$$

of $\tilde{N}_n$ into $k+2$ node sets, where $S_0 = \{0\}$ and $S_{k+1} = \{n\}$, the associated *jump inequality* encode the fact that a $(0, n)$-path P of cardinality at most k must make at least one "jump" from a node set S_i to a node set S_j, with $j - i \geq 2$:

$$\sum_{i=0}^{k-1} \sum_{j=i+2}^{k+1} x((S_i : S_j)) \geq 1. \tag{3.29}$$

By Dahl, Foldnes, and Gouveia [22], a jump inequality (3.29) induces a facet of the dominant of $P^{\leq k}_{0,n\text{-path}}(\tilde{D}_n)$, see Theorem 5.28. However, it is not facet defining for $P^{\leq k}_{0,n\text{-path}}(\tilde{D}_n)$ and $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Following an idea of Dahl and Gouveia [23], decreasing the coefficients on arcs in $(S_{k-1} \cup S_k : S_1 \cup S_2)$

by 1, we obtain a facet defining inequality for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. The resulting inequality

$$\sum_{i=0}^{k-1}\sum_{j=i+2}^{k+1} x((S_i : S_j)) - x((S_{k-1}\cup S_k : S_1 \cup S_2)) \geq 1 \tag{3.30}$$

is called *lifted jump inequality*. For a deeper investigation of these inequalities we refer to Chapter 5.6.

Theorem 3.23. Let

$$\tilde{N}_n = \bigcup_{p=0}^{k+1} S_p$$

be a partition of $\tilde{N}_n$, where $S_0 = \{0\}$ and $S_{k+1} = \{n\}$. Then, the associated lifted jump inequality (3.30) is facet defining for the $(0,n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if $|S_i| \geq 2$ for $i = 1,\dots,k$.

Proof. We refer to an arc (i,j) as *forward arc* if $(i,j) \in (S_h : S_\ell)$ for some $h < \ell$ and as *backward arc* if $(i,j) \in (S_q : S_r)$ for some $q > r$. We say, the $(0,n)$-k-path P makes a "jump" with respect to (3.30) if P uses an arc $(i,j) \in (S_h : S_\ell)$ for some $0 \leq h < \ell \leq k+1$ with $\ell \geq h+2$.

The lifted jump inequality (3.30) is valid for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$, since it is valid for the path polytope $P^{\leq k}_{0,n\text{-path}}(\tilde{D}_n)$ (see [23]).

To show that (3.30) is facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$, we apply Theorem 3.11. So we have to verify that the conditions of Theorem 3.11 hold for the primitive members of (3.30), that is, when $|S_i| = 2$ for $i = 1,\dots,k$, which implies $n = 2k+1$. In what follows, let $d^T x \geq 1$ be such a lifted jump inequality.

Let $B = P \cup C$ be any k-bowtie , where C is a simple cycle and P is a simple $(0,n)$-path. Since $|P| \leq k$, $d(P) \geq 1$. When $d(C) \geq 0$, it follows $d(B) \geq 1$, too. Otherwise $d(C) = -1$ and C is a cycle in

$$\left(\bigcup_{j=2}^{k-2}(S_j : S_{j+1})\right) \cup (S_{k-1} : S_2),$$

since $|C| \leq k-2$. Thus, the cardinality of C is equal to $k-2$ and P is a $(0,n)$-2-path that makes two "jumps". Therefore, the lifted jump inequality $d^T x \geq 1$ is satisfied by all k-bowties.

Furthermore, $d^T x \geq 1$ is regular, since to each internal node h there exists a non-tight $(0,n)$-k-path that does not visit node h.

It remains to be shown that $d^Tx \geq 1$ is facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Without loss of generality, let $S_i = \{i, k+i\}$ for $i = 1, \ldots, k$. When $k = 4$ or $k = 5$, the inequality $d^Tx \geq 1$ can be seen to be facet defining using a convex hull code. So let $k \geq 6$. Suppose that $c^Tx = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.30) with equality. Denoting by P the $(0, 2k+1)$-path $(0, \ldots, k, 2k+1)$, we may assume by Corollary 3.10 that $c(P) = 0$, $c_{0,k+1} = 0$, and $c_{i,k+i} = 0$ for $i = 1, \ldots, k$. Substituting two connected arcs $(i, j), (j, h) \in P$ by the arc (i, h), we see that $c_{m-1,m+1} = c_0$ for $m = 1, \ldots, k-1$, and $c_{k-1,2k+1} = c_0$. Next, replacing three connected arcs $(i, j), (j, h), (h, \ell) \in P$ with $i > 0$ by the arcs $(i, k+i), (k+i, \ell)$, we see that $c_{2k-2,2k+1} = c_0$ and $c_{k+i,i+3} = c_0$ for $i = 1, \ldots, k-3$. Furthermore, replacing in these $(0, n)$-k-paths node i by node $k+i-1$ (for $i \geq 2$) yields $c_{m,m+1} = 0$ for $m = k+1, \ldots, 2k-3$, and considering successively the $(0, n)$-k-paths

$$(0, k+1, 4, \ldots, q, k+q, \ldots, 2k+1)$$

for $q = k, \ldots, 4$, we see that even $c_{m,m+1} = 0$ for $m = k+1, \ldots, 2k$, since $k \geq 6$. We can now easily deduce that $c_{i,k+i+1} = c_{k+i,i+1} = c_{k+i,i} = 0$ for $i = 1, \ldots, k$, $c_a = c_0$ for all $a \in (S_i : S_{i+2})$ $(i = 0, \ldots, k-1)$, and $c_a = c_0$ for all $a \in (S_i : S_{i+3})$ $(i = 0, \ldots, k-2)$. Furthermore, for each arc $a \in (S_i : S_{i+4})$, $i = 0, \ldots, k-3$, there is a tight $(0, n)$-k-path containing a that does not use any backward arc, which implies that $c_a = c_0$ for all those arcs a. Moreover, for each arc $a \in (S_m : S_{m-1})$ there is a tight $(0, n)$-k-path that uses a, makes a jump from S_i to S_{i+4} for some i, and does not use any further backward arcs. Hence, $c_a = 0$ for all $a \in (S_m : S_{m-1})$, $m = 2, \ldots, k+1$. It is now easy to see that the remaining coefficients can be determined as required, and therefore, $c^Tx = c_0$ is simply $c_0 d^Tx = c_0$. □

Cardinality-path inequalities

The cardinality-path inequalities were originally formulated for the cardinality constrained cycle polytope $P_C^{\leq k}(G)$ defined on an undirected graph $G = (N, E)$. They say that an undirected simple cycle of cardinality at most k never uses more edges of an undirected simple path P of cardinality k than internal nodes of P. Denoting by $\dot{P}$ the internal nodes of such a path, this can be expressed as

$$y(P) \leq \frac{1}{2} \sum_{v \in \dot{P}} y(\delta(v)). \tag{3.31}$$

Inequality (3.31) is called *cardinality-path inequality.* This idea can be transferred to the directed $(0, n)$-k-path polytope.

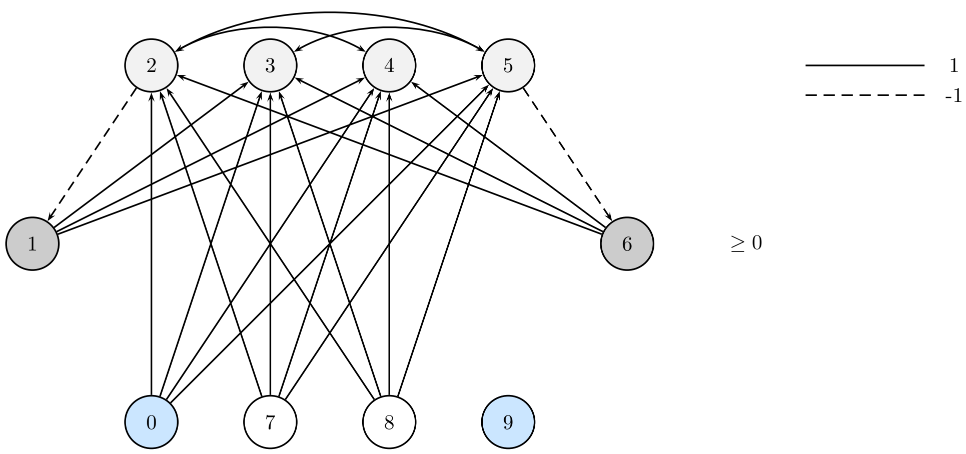


Figure 3.3: Support graph of a cardinality-path inequality for $n = 9$, $k = 6$, and the path $P = (1, 2, 3, 4, 5, 6)$.

Theorem 3.24. Let s, t be internal nodes, P an (s, t)-path of cardinality $k-1$, and $\text{bid}(P) := P \cup \{(i, j) : (j, i) \in P\}$. The cardinality-path inequality

$$\sum_{i \in \dot{P}} x(\delta^{\text{in}}(i)) - x(\text{bid}(P)) \geq 0 \tag{3.32}$$

is valid for the $(0, n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ and induces a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $k \in \{4, 5\}$ and $n \geq k + 2$ or $k \geq 6$ and $n \geq 2k - 3$.

Proof. Without loss of generality, let $P = (1, 2, \ldots, k)$. An illustration of the inequality for $n = 9$ and $k = 6$ is given in Figure 3.3. Missing arcs have coefficients 0.

Necessity. When $k \in \{4, 5\}$ and $n = k+1$, (3.32) can be seen not to induce a facet using a convex hull code. When $k \geq 6$ and $k + 1 \leq n \leq 2k - 4$, (3.32) is dominated by the nonnegativity constraints $x_{2,k-1} \geq 0$ and $x_{k-1,2} \geq 0$.

Suffiency. When the conditions in Theorem 3.24 are satisfied and the cardinality of the node set $S := \{1, k, k + 1, \ldots, n - 1\}$ is at most 4, (3.32) can be seen to induce a facet using a convex hull code. So suppose that $|S| \geq 5$ and that an equation $c^T x = c_0$ is satisfied by every $x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.32) with equality. By Corollary 3.10 we may assume that $c_{j,j+1} = 0$ for $j = 1, \ldots, k - 2$, $c_{0,n-1} = c_{n-1,n} = 0$, and $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree on S.

For any four distinct nodes $i \in S \cup \{0\}$, $j, h \in S$, and $\ell \in S \cup \{n\}$ there is a tight $(0, n)$-k-path that uses the arcs $(i, h), (h, j)$ and skips node ℓ. Replacing node h by node ℓ yields another tight $(0, n)$-k-path and thus

$c_{ih} + c_{hj} = c_{i\ell} + c_{\ell j}$. Using Corollary 3.10 we obtain $c_{ij} = 0$ for all $(i,j) \in \tilde{A}_n(S \cup \{0,n\})$ and therefore also $c_0 = 0$.

In the following we distinguish the three cases $k = 4$, $k = 5$, and $k \geq 6$.

CASE 1: $k = 4$.

From the $(0,n)$-4-paths $(0,5,1,2,n)$ and $(0,1,2,3,n)$ we derive $c_{2n} = c_{3n} = 0$, and from the $(0,n)$-4-paths $(0,1,2,i,n)$ for $i = 4,\dots,n-1$ we derive $c_{2i} = 0$.

Next, considering the $(0,n)$-4-paths $(0,5,4,3,n)$ and $(0,4,3,2,n)$ yields $c_{43} = c_{32} = 0$. Hence, we can also deduce that $c_{3j} = 0$ for all $j \in S \setminus \{4\}$.

Further, from all tight $(0,n)$-4-paths that use the arc $(3,4)$ we deduce that $c_{ij} + c_{34} = 0$ for $(i,j) \in \{(0,2),(0,3),(1,3)\} \cup (S : \{3\})$. Analogously, it follows that $c_{h\ell} + c_{21} = 0$ for all $(h,\ell) \in \{(0,2),(0,3),(4,2)\} \cup (S : \{2\})$. In particular, $c_{02} + c_{21} = c_{02} + c_{34} = 0$ which implies that $c_{21} = c_{34}$ and hence, $c_{ij} + c_{h\ell} = 0$ for all $(i,j) \in \{(1,3),(4,2)\} \cup (S \cup \{0\} : \{2,3\})$ and $(h,\ell) \in \{(2,1),(3,4)\}$. So $c^T x = c_0$ is obviously equivalent to (3.32).

CASE 2: $k = 5$.

This case can be carried out similar as the case $k = 4$; so we omit this part of the proof.

CASE 3: $k \geq 6$.

From the $(0,n)$-k-path $(0,\dots,k-1,n)$ we derive that $c_{k-1,n} = 0$. Furthermore, setting $T := \{3,\dots,k-2\}$, it can be easily seen that $c_{ij} = 0$ for all $i \in T, j \in (S \setminus \{1\}) \cup \{n\}$. Next, for any arc $(i,j) \in (\dot{P} \setminus \{k-1\} : S \cup \{n\} \cup \{(k-1,n)\})$ there is a tight $(0,n)$-k-path that uses the arcs (i,j) and $(h,h+1)$ for $h = 1,\dots,i-1$ and whose remaining arcs are in $\tilde{A}_n(S \cup \{0,n\})$. Hence, $c_{ij} = 0$ for all those arcs (i,j). Further, from the $(0,n)$-k-path $(0,\dots,k-3,k,k-1,n)$ we derive that $c_{k,k-1} = 0$. Moreover, for any node $i \in S \setminus \{1\}$ there is a tight $(0,n)$-k-path that uses the arcs $(0,1),(1,2),(2,i),(k,k-1),(k-1,n)$ and whose remaining arcs are in $\tilde{A}_n(S)$. Thus, $c_{2i} = 0$ for all $i \in S \setminus \{1\}$. Considering further tight $(0,n)$-k-paths on node set $S \cup \{0,2,k-1,n\}$, we see that also $c_{k-1,i} = 0$ for all $i \in S \setminus \{k\}$ and $c_{2n} = 0$. Finally, considering successively the $(0,n)$-k-paths $(0,\dots,i-2,k,k-1,\dots,i,n)$ for $i = k-2,\dots,2$, we find that $c_{i+1,i} = 0$ for $i = 2,\dots,k-2$.

It remains to be shown that $c_{21} = c_{k-1,k} = \sigma$ and $c_{ij} = -\sigma$ for all arcs (i,j) in $\bigcup_{h=2}^{k-1} \delta^{\text{in}}(h) \setminus \text{bid}(P)$ for some σ. From the two tight $(0,n)$-k-paths $(0,4,5,\dots,k+2,n)$ and $(0,4,3,2,1,k+1,k+2,\dots,n)$ we derive that $c_{21} = c_{k-1,k}$. Denote this common value by σ. Since to each arc $(i,j) \in$

$\bigcup_{h=2}^{k-1} \delta^{\text{in}}(h) \setminus \text{bid}(P)$ there is a tight $(0,n)$-k-path that uses either the arc $(2,1)$ or $(k-1,k)$, and therefore, $c_{ij} = -\sigma$ for all those arcs (i,j). Thus, $c^T x = c_0$ is simply

$$\sigma x(\text{bid}(P)) - \sigma \sum_{i \in \tilde{N}_n(\dot{P})} x(\delta^{\text{in}}(i)) = 0.$$

□

3.3 Facets of polytopes related to the $(0,n)$-k-path polytope

In this section, we derive some new facet defining inequalities for the directed k-cycle polytope using Theorem 3.6. Furthermore, we derive facet defining inequalities for the undirected $[0,n]$-k-path polytope from facet defining inequalities for the $(0,n)$-k-path polytope using the concept of symmetric inequalities. At the beginning of this chapter we have explained this concept in connection with cycle polytopes. Fortunately, it can be easily adapted to path polytopes. For this, we refer to Section 3.3.2.

To the best of the knowledge of the author, the undirected $[0,n]$-k-path polytope has not been studied before. Hence, we also give a short polyhedral analysis of this polytope.

3.3.1 New facets of the directed k-cycle polytope

Hartmann and Özlük have presented many nontrivial and interesting inequalities that are facet defining for the directed k-cycle polytope $P_C^{(k)}(D_n)$. Applying Theorem 3.6 to Theorems 3.19 and 3.20, we obtain some new facet defining inequalities for $P_C^{(k)}(D_n)$.

Corollary 3.25. Let $N = \{j\} \,\dot\cup\, R \,\dot\cup\, S \,\dot\cup\, T$ be a partition of N. The inequality

$$\begin{aligned} x((S:\{j\})) + x((\{j\}:T)) + x((S:T)) \\ + \textstyle\sum_{i\in R} x(\delta^{\text{out}}(i)) \le \lfloor (k+|R|+1)/2 \rfloor \end{aligned} \tag{3.33}$$

defines a facet of the k-cycle polytope $P_C^{(k)}(D_n)$ if $k+|R|$ is even, $k \ge |R|+4$, $|S| > (k-|R|)/2-1$, and $|T| > (k-|R|)/2-1$. □

Corollary 3.26. Let $N = \{j\} \,\dot\cup\, R \,\dot\cup\, S \,\dot\cup\, T$ be a partition of N. The inequality

$$x(\delta^{\text{out}}(r)) + x((S:T)) + \sum_{i\in R} x(\delta^{\text{out}}(i)) \le \lfloor (k+|R|+1)/2 \rfloor \tag{3.34}$$

defines a facet of the k-cycle polytope $P_C^{(k)}(D_n)$ if $k+|R|$ is even, $k \geq |R|+4$, $|S| > (k-|R|)/2-1$, and $|T| > (k-|R|)/2-1$. □

3.3.2 Facets of the undirected $[0,n]$*-*k*-path polytope*

The undirected $[0,n]$-k-path polytope $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ is the symmetric counterpart of the directed $(0,n)$-k-path polytope $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. Here, $K_{n+1} = (N,E)$ denotes the complete graph on node set $N = \{0,\dots,n\}$. Table 3.3 gives linear descriptions of $P^{(1)}_{0,n\text{-path}}(K_{n+1})$ and $P^{(2)}_{0,n\text{-path}}(K_{n+1})$. The complete polyhedral analysis of the $[0,n]$-k-path polytope $P^{(3)}_{0,n\text{-path}}(K_{n+1})$ begins with the next theorem. Afterwards we will turn to the $[0,n]$-k-path polytopes $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ with $4 \leq k \leq n-1$.

Theorem 3.27. Let $K_{n+1} = (N,E)$ be the complete graph on node set $N = \{0,\dots,n\}$. Then

$$\dim P^{(3)}_{0,n\text{-path}}(K_{n+1}) = |E| - n - 2.$$

Proof. First note that each internal edge $e = [i,j]$ corresponds to two incidence vectors $P^{(i,j)}$ and $P^{(j,i)}$ of $[0,n]$-3-paths as follows: $P^{(i,j)} = \chi^{\{[0,i],[i,j],[j,n]\}}$ and $P^{(j,i)} = \chi^{\{[0,j],[j,i],[i,n]\}}$. Consider the points $P^{(h,n-1)}$, $P^{(n-1,h)}$ for $h = 1,\dots,n-2$ and $P^{(i,j)}$ for $1 \leq i < j \leq n-2$. It is easy to see that these $|E|-n-1$ points are linearly independent and thus, $\dim P^{(3)}_{0,n\text{-path}}(K_{n+1}) \geq |E|-n-2$.

Next, all incidence vectors of $[0,n]$-3-paths satisfy the following system of linearly independent equations:

$$y_{0n} = 0, \tag{3.35}$$
$$y(\delta(0)) = 1, \tag{3.36}$$
$$y(\delta(n)) = 1, \tag{3.37}$$
$$y(\delta(i)) - 2(y_{0i} + y_{in}) = 0, \qquad i = 1,\dots,n-1, \tag{3.38}$$

where $\delta(j)$ denotes the set of edges which are incident with node j and $y(F) = \sum_{e\in F} y_e$ for any $F \subseteq E$. This implies that $\dim P^{(3)}_{0,n\text{-path}}(K_{n+1}) \leq |E|-n-2$, which completes the proof. □

Remark 3.28. Adding the equations (3.36)-(3.38), subtracting two times (3.35), and dividing by two, yields the equation

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} y_{ij} = 1. \tag{3.39}$$

Table 3.3: Polyhedral analysis of $P^{(1)}_{0,n\text{-path}}(K_{n+1})$ and $P^{(2)}_{0,n\text{-path}}(K_{n+1})$.

p	Dimension	Complete linear description	
1	0	$y_{0n} = 1$	
		$y_{ij} = 0$	$\forall\ [i,j] \in E \setminus \{[0,n]\}$
2	$n-2$	$y_{0n} = 0$	
		$y(\delta(0)) = 1$	
		$y_{0i} - y_{in} = 0$	$i = 1, \dots, n-1$
		$y_{0i} \geq 0$	$i = 1, \dots, n-1$
		$y_{ij} = 0$	$1 \leq i < j \leq n-1$

Theorem 3.29. A complete and nonredundant linear description of the $[0,n]$-3-path polytope $P^{(3)}_{0,n\text{-path}}(K_{n+1})$ is given by the equations (3.35)-(3.38), the nonnegativity constraints $y_{ij} \geq 0$ for $1 \leq i < j \leq n$, and the inequalities

$$\sum_{i \in W} y_{in} + \sum_{[i,j] \in E^W} y_{ij} \leq 1 \tag{3.40}$$

for all $W \subseteq \{1, \dots, n-1\}$ with $1 \leq |W| \leq n-2$, where $E^W := \{[i,j] \in E : 1 \leq i,j \leq n-1, i,j \notin W\}$.

Proof. Validity. Let $W \subseteq \{1, \dots, n-1\}$, let $1 \leq |W| \leq n-2$, and let $c^T y \leq 1$ be the inequality of family (3.40) associated with W. The edge set of the support graph $G = (N, F)$, defined by $F := \{e \in E : c_e = 1\}$, decomposes into two disconnected subsets $F^n := \{[i,n] \in F : i \in W\}$ and $F^{\neg n} := F \setminus F^n$. As is easily seen, each $[0,n]$-3-path P uses at most one edge of F in the subgraph $G \subset K_{n+1}$. Hence, $c^T y \leq 1$ is valid for $P^{(k)}_{0,n\text{-path}}(K_{n+1})$.

Nonredundancy. Since the equations (3.35)-(3.38) are linearly independent, they induce a nonredundant description of the lineality space of the polytope $P^{(3)}_{0,n\text{-path}}(K_{n+1})$.

Next, we prove that the inequalities given in Theorem 3.29 are nonredundant by showing that the set of induced faces is an anti-chain. Let F_1 and F_2 be two faces of $P^{(3)}_{0,n\text{-path}}(K_{n+1})$ that are induced by different inequalities. When F_1 and F_2 are induced by nonnegativity constraints, they are clearly not contained into each other. If only one of them is induced by a nonnegativity constraint $y_{ij} \geq 0$ $(1 \leq i < j \leq n)$, say F_1, it follows immediately that $F_2 \not\subset F_1$. Since $|N(F^{\neg n})| \geq 2$, there is also a point $P^{(h,\ell)}$ in F_1 that is not in F_2 and thus, $F_1 \not\subset F_2$.

Finally, let both faces not induced by nonnegativity constraints. Denote the edge sets of the support graphs corresponding to F_1 and F_2 by E_1 and

E_2, respectively. Since $E_1 \not\subset E_2$ and $E_2 \not\subset E_1$, it follows also that $F_1 \not\subset F_2$ and $F_2 \not\subset F_1$.

Completeness. We will show that each facet defining inequality $d^T x \leq d_0$ for $P^{(3)}_{0,n\text{-path}}(K_{n+1})$ is equivalent to a nonnegativity constraint $y_{ij} \geq 0$ or an inequality of family (3.40).

(i) $d_{0i} = 0$ for $i = 1, \ldots, n$,

(ii) $d_{zn} = 0$ for some internal node z,

(iii) $d_{uw} = 0$ for some internal edge $[u, w]$, and

(iv) $d_{ij} \geq 0$ for $1 \leq i < j \leq n$.

This immediately implies that $d_0 > 0$ and $0 \leq d_e \leq d_0$ for all $e \in E$.

Next, we will show that $d_e \in \{0, d_0\}$ for all $e \in E$. Suppose, for the sake of contradiction, that $M := \{[i,j] \in E : 0 < d_{ij} < d_0\} \neq \varnothing$. Assuming that there is some internal edge $[h, \ell] \in M$ with $[h, n], [\ell, n] \notin M$, we see that $d_{hn} = d_{\ell n} = 0$, since $d_{h\ell} + d_{\ell n} \leq d_0$ and $d_{h\ell} + d_{hn} \leq d_0$. Thus, $d^T y \leq d_0$ is dominated by the inequality $\tilde{d}^T y \leq d_0$, where $\tilde{d}_{h\ell} = d_0$ and $\tilde{d}_e = d_e$ for all $e \in E \setminus \{[h, \ell]\}$. Assuming that there is some edge $[m, n]$ such that $[i, m] \notin M$ for all internal nodes $i \neq m$, yields $d_{im} = 0$ for all internal nodes $i \neq m$. Therefore $d^T y \leq d_0$ is dominated by the inequality $\hat{d}^T y \leq d_0$, where $\hat{d}_{mn} = d_0$ and $\hat{d}_e = d_e$ for all $e \in E \setminus \{[m, n]\}$. So, in what follows, we may assume:

(a) $[i, n] \in M$ or $[j, n] \in M$ for each internal edge $[i, j] \in M$;

(b) for each edge $[h, n] \in M$ there is an internal edge $[i, h] \in M$.

In particular, we deduce that $M \cap \{[i,j] : 1 \leq i < j \leq n-1\} \neq \varnothing$ and $M \cap \{[i,n] : 1 \leq i \leq n-1\} \neq \varnothing$.

Let d_{rs} be the minimum over all edges in $M \cap \{[i,j] : 1 \leq i < j \leq n-1\}$ and d_{vn} be the minimum over all edges in $M \cap \{[i,n] : 1 \leq i \leq n-1\}$. We now construct two different inequalities $a^T y \leq a_0$ and $b^T y \leq b_0$ that together imply $d^T y \leq d_0$. The coefficients of the both inequalities we set as follows:

$$
\begin{array}{rclll}
a_0 & = & b_0 & = d_0, & \\
a_{ij} & = & b_{ij} & = d_{ij} & \forall\, [i,j] \in E \setminus M, \\
a_{ij} & = & & d_{ij} - d_{rs} & \text{for } 1 \leq i < j \leq n-1, \\
a_{hn} & = & & d_{hn} + d_{rs} & \text{for } 1 \leq h \leq n-1, \\
 & & b_{ij} & = d_{ij} + d_{vn} & \text{for } 1 \leq i < j \leq n-1, \\
 & & b_{hn} & = d_{hn} - d_{vn} & \text{for } 1 \leq h \leq n-1.
\end{array}
$$

It can be easily seen that $d^Ty \leq d_0$ is a convex combination of $a^Ty \leq a_0$ and $b^Ty \leq b_0$:

$$(d, d_0) = \frac{d_{vn}}{d_{rs} + d_{vn}}(a, a_0) + \frac{d_{rs}}{d_{rs} + d_{vn}}(b, b_0).$$

Furthermore, all three inequalities are pairwise nonequivalent; so it remains to be shown that the inequalities $a^Ty \leq a_0$ and $b^Ty \leq b_0$ are valid for $P^{(3)}_{0,n\text{-path}}(K_{n+1})$. This can be done by checking $a_{ij}+a_{jn} \leq a_0$ and $b_{ij}+b_{jn} \leq b_0$ for all $1 \leq i, j \leq n-1$ with $i \neq j$.

Let i and j be distinct nodes in $\{1, \dots, n-1\}$.

CASE 1: $[i, j], [j, n] \notin M$.

We have $a_{ij} = b_{ij} = d_{ij}$ and $a_{jn} = b_{jn} = d_{jn}$. Thus, $a_{ij} + a_{jn} \leq a_0$ and $b_{ij} + b_{jn} \leq b_0$, since $d_{ij} + d_{jn} \leq d_0$.

CASE 2: $[i, j] \in M$, $[j, n] \notin M$.

Since $0 < d_{ij} < d_0$, $d_{jn} \in \{0, d_0\}$, and $d_{ij} + d_{jn} \leq d_0$, we deduce that $d_{jn} = 0$. Hence, also $a_{jn} = b_{jn} = 0$. Since $a_{ij} = d_{ij} - d_{rs} < d_{ij}$, it follows that $a_{ij} + a_{jn} \leq a_0$. Due to (a), $[i, n] \in M$, and since $d_{in} \geq d_{vn}$, we deduce that $d_{ij} \leq d_0 - d_{vn}$. Thus, $b_{ij} + b_{jn} = d_{ij} + d_{vn} \leq d_0 = b_0$.

CASE 3: $[i, j] \notin M$, $[j, n] \in M$.

This implies that $a_{ij} = b_{ij} = d_{ij} = 0$ and thus, $b_{ij} + b_{jn} \leq b_0$. Due to (b), there is some internal node ℓ such that $[\ell, j] \in M$. Since $d_{\ell j} \geq d_{rs}$, we deduce that $d_{jn} \leq d_0 - d_{rs}$ and hence, $a_{ij} + a_{jn} = d_{jn} + d_{rs} \leq d_0 = a_0$.

CASE 4: $[i, j], [j, n] \in M$.

Clear.

Thus, in all four cases, the inequalities $a^Ty \leq a_0$ and $b^Ty \leq b_0$ are valid for $P^{(3)}_{0,n\text{-path}}(K_{n+1})$. So we have shown that $d_e \in \{0, d_0\}$ for all $e \in E$ and without loss of generality, we may assume that $d_0 = 1$.

We resume: the facet defining inequality $d^Ty \leq d_0$ satisfies (i)-(iii), $d_0 = 1$, and $d_e \in \{0, 1\}$ for all $e \in E$. Note that $d_{\ell n} = 1$ for some internal node ℓ implies that $d_{i\ell} = 0$ for all internal nodes $i \neq \ell$.

When $d_{in} = 0$ for $1 \leq i \leq n-1$, we deduce that $d_e = 1$ for all internal edges $e \neq [u, w]$, i.e., $d^Ty \leq d_0$ is equivalent to the nonnegativity constraint $y_{uw} \geq 0$.

When $d_{in} = 1$ for all internal nodes $i \neq z$, we see that $d_e = 0$ for all internal edges e. Then, $d^Ty \leq d_0$ is equivalent to the nonnegativity constraint $y_{zn} \geq 0$.

In all other cases, i.e., for $1 \leq \sum_{i=1}^{n-1} d_{in} \leq n-2$, the inequality $d^T y \leq d_0$ is not equivalent to a nonnegativity constraint which implies that for each edge e there is a tight $[0,n]$-3-path containing e. Thus, $d_{ij} = 1$ for all internal edges $[i,j]$ for which $d_{in} = d_{jn} = 0$. Therefore, $d^T y \leq d_0$ is the member of family (3.40) associated with $W := \{i \in \{1,\dots,n-1\} : d_{in} = 1\}$. □

Next, we turn to the polytopes $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ when $4 \leq k \leq n-1$. The integer points in $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ are characterized by the following model:

$$
\begin{aligned}
y_{0n} &= 0, && && (3.41)\\
y(\delta(0)) &= 1, && && (3.42)\\
y(\delta(n)) &= 1, && && (3.43)\\
y(\delta(j)) &\leq 2 && \text{for all } j \in N \setminus \{0,n\}, && (3.44)\\
y(\delta(j) \setminus \{e\}) - y_e &\geq 0 && \text{for all } j \in N \setminus \{0,n\}, e \in \delta(j), && (3.45)\\
y((S : N \setminus S)) &\geq y(\delta(j)) && \text{for all } S \subset N, 3 \leq |S| \leq n-2, && (3.46)\\
& && 0, n \in S, j \in N \setminus S, &&\\
y(E) &= k, && && (3.47)\\
x_e &\in \{0,1\} && \text{for all } e \in E. && (3.48)
\end{aligned}
$$

Here, for any node sets S, T of N, $y((S:T))$ is short for $\sum_{i\in S}\sum_{j\in T} y_{ij}$.

The *parity constraints* (3.45) together with the degree (3.44) and the integrality constraints (3.48) ensure that every internal node has degree 0 or 2. Hence, constraints (3.41)-(3.45) and the integrality constraint (3.48) are satisfied by the incidence vector of the node disjoint union of a simple $[0,n]$-path and simple cycles on the set of internal nodes. The one-sided min-cut inequality (3.46) is satisfied by the incidence vectors of simple $[0,n]$-paths but violated by the incidence vectors of the union of a simple $[0,n]$-path and simple cycles. Finally, the cardinality constraint (3.47) excludes all incidence vectors of $[0,n]$-paths which have a cardinality that is not equal to k.

Lemma 3.30. Let $4 \leq k \leq n-1$ and $n \geq 6$. If the equation

$$c^T y = c_0$$

is satisfied by all $[0,n]$-k-paths, then there are α, β, γ, such that $c_{0i} = \alpha$, $c_{in} = \beta$ for all $i \in \{1,\dots,n-1\}$ and $c_{ij} = \gamma$ for all $i,j \in \{1,\dots,n-1\}$.

Proof. Set $S := \{1,\dots,n-1\}$, let i,j,h,ℓ be any distinct nodes in S, and consider any $[0,n]$-k-path P that uses the edges $[i,j],[j,h]$ but does not visit node ℓ. Replacing node j by node ℓ yields $c_{ij}+c_{jh} = c_{i\ell}+c_{h\ell}$. Next, consider any $[0,n]$-k-path P' that uses the edges $[j,i],[i,\ell]$ but does not visit the node

h. Replacing node i by node h yields $c_{ij} + c_{i\ell} = c_{jh} + c_{h\ell}$. We deduce that $c_{ij} =_{h\ell}$ and since $|S| \geq 5$, we see that $c_{ij} = c_{h\ell}$ for all distinct nodes $i, j, h, \ell \in S$. Denoting this common value by γ, it follows immediately that there are α, β with $c_{0i} = \alpha$ and $c_{in} = \beta$ for all $i \in S$. □

We are now well prepared to determine the dimension of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ in dependence of n and k. For the sake of completeness we determine also the dimension of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ when $k = n$.

Theorem 3.31. Let $n \geq k \geq 4$. Then

$$\dim P^{(k)}_{0,n\text{-path}}(K_{n+1}) = \begin{cases} |E| - 4 & \text{if } k \leq n-1, \\ |E| - n - 2 & \text{if } k = n \geq 4. \end{cases}$$

Proof. Using a convex hull code we see that $\dim P^{(4)}_{0,6\text{-path}}(K_6) = 11$. Next, suppose that $n \geq 6$ and $4 \leq k \leq n-1$. We will show that (3.41)-(3.43) and (3.47) is a minimal equality subsystem for $P^{(k)}_{0,n\text{-path}}(K_{n+1})$. Since the equations (3.41)-(3.43) and (3.47) are linearly independent, $\dim P^{(k)}_{0,n\text{-path}}(K_{n+1}) \leq \frac{(n+1)n}{2} - 4$. It remains to be shown that any equation that is satisfied by all $y \in P^{(k)}_{0,n\text{-path}}(K_{n+1})$ is a linear combination of (3.41)-(3.43) and (3.47). Let $c^T y = c_0$ be such an equation. By Lemma 3.30, there are α, β, γ with $c_{0i} = \alpha$, $c_{in} = \beta$ for all internal nodes i and $c_{ij} = \gamma$ for all internal nodes $i \neq j$. Thus,

$$\begin{aligned}(c^T y, c_0) = {} & \gamma(y(E), k) \\ & +(\alpha - \gamma)(y(\delta(0)), 1) \\ & +(\beta - \gamma)(y(\delta(n)), 1) \\ & +(c_{0n} + \gamma - \alpha - \beta)(y_{0n}, 0).\end{aligned}$$

Finally, let $k = n \geq 4$. Theorem 7 of Grötschel and Padberg [47] implies that the dimension of the traveling salesman polytope $P^{(n+1)}_{C}(K_{n+1})$ defined on the complete graph on node set N is equal to $|E| - n - 1$ for $n \geq 2$ and Theorem 8 of the same authors says that the inequalities $x_e \leq 1$ induce facets F_e of $P^{(n+1)}_{C}(K_{n+1})$ for $n \geq 3$. Since F_{0n} is isomorphic to $P^{(n)}_{0,n\text{-path}}(K_{n+1})$, we obtain the required result. □

Next, we use the concept of symmetric inequalities to derive facet defining inequalities for the undirected $[0, n]$-k-path polytope $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ from facet defining inequalities for the directed $(0, n)$-k-path polytope$P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$. An inequality $c^T x \leq c_0$ with $c \in \mathbb{R}^{\tilde{A}_n}$ is said to be *symmetric* if $c_{ij} = c_{ji}$ for all $1 \leq i < j \leq n-1$. It is easy to see that the undirected counterpart $\bar{c}^T y \leq c_0$ of a symmetric inequality $c^T x \leq c_0$ (obtained by setting $\bar{c}_{0i} := c_{0i}, \bar{c}_{in} := c_{in}$

for all internal nodes i, and $\bar{c}_{ij} := c_{ij} = c_{ij}$ for all $1 \leq i < j \leq n-1$) is facet defining for $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ if $c^T x \leq c_0$ is facet defining for $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ (cf. [48]). The argument that can be used to prove the statement is the following: assuming that $\bar{c}^T y \leq c_0$ does not induce a facet of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$, then there is a facet inducing inequality $\bar{d}^T y \leq d_0$ for $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ such that $\{y \in P^{(k)}_{0,n\text{-path}}(K_{n+1}) : \bar{c}^T y = c_0\} \subsetneq \{y \in P^{(k)}_{0,n\text{-path}}(K_{n+1}) : \bar{d}^T y = d_0\}$. But then $\{x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n) : c^T y = c_0\} \subsetneq \{x \in P^{(k)}_{0,n\text{-path}}(\tilde{D}_n) : d^T y = d_0\}$, where $d^T x \leq d_0$ is the directed counterpart of $\bar{d}^T y \leq d_0$ (obtained by setting $d_{0i} := \bar{d}_{0i}$, $d_{in} := \bar{d}_{in}$ for all $i \in \{1, \ldots, n-1\}$, and $d_{ij} := d_{ji} := \bar{d}_{ij}$ for all $1 \leq i < j \leq n-1$). The same argumentation holds for the general cardinality constrained path polytopes $P^{c}_{0,n\text{-path}}(\tilde{D}_n)$ and $P^{c}_{0,n\text{-path}}(K_{n+1})$.

Following an argument of Boros et al. [15] and Hartmann and Özlük [48], an inequality $c^T x \leq c_0$ with $c \in \mathbb{R}^{\tilde{A}_n}$ is equivalent to a symmetric inequality if and only if the system $t_i - t_j = c_{ij} - c_{ji}$ for $1 \leq i < j \leq n-1$ is consistent, since the flow conservation constraints are not symmetric themselves. The degree constraint (3.18) and the cut inequalities (3.19), (3.20), (3.23), and (3.26) are equivalent to symmetric inequalities. Hence, their undirected counterparts are facet defining for $P^{(k)}_{0,n\text{-path}}(K_{n+1})$.

Corollary 3.32. Let $4 \leq k < n$.

(i) The degree constraint

$$y(\delta(j)) \leq 2 \tag{3.49}$$

induces a facet of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ for every internal node j of G.

(ii) Let $S \subset N$ and $0, n \in S$. The min-cut inequality

$$y((S : N \setminus S)) \geq 2 \tag{3.50}$$

induces a facet of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ if $3 \leq |S| \leq k$.

(iii) Let $S \subset N$ and $0, n \in S$. The one-sided min-cut inequality

$$y((S : N \setminus S)) \geq y(\delta(j)) \tag{3.51}$$

defines a facet of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ for every node $j \in N \setminus S$.

(iv) Let $S \subset N$ and $0, n \in S$. The max-cut inequality

$$y((S : T)) \leq k - 1 \tag{3.52}$$

defines a facet of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ if k is odd, $S \setminus \{n\} > k/2$, and $T > k/2$.

(v) Let $S \subset N$ and $0 \in S$ and $n \in T$. The max-cut inequality

$$y((S:T)) \leq k-1 \tag{3.53}$$

induces a facet of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ if k is even, $|S| > k/2$, and $|T| > k/2$.

□

Finally, we show that the nonnegativity constraints $x_e \geq 0$ define facets of the $[0,n]$-k-path polytope $P^{(k)}_{0,n\text{-path}}(K_{n+1})$.

Theorem 3.33. Let $4 \leq k < n$. The nonnegativity constraint

$$y_e \geq 0 \tag{3.54}$$

defines a facet of the $[0,n]$-k-path polytope $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ for all edges $e \neq [0,n]$ of K_{n+1}.

Proof. When $n \leq 5$, (3.54) can be seen to be facet defining using a convex hull code; so assume that $n \geq 6$. Let $c^T y = c_0$ be an equation that is satisfied by every $y \in P^{(k)}_{0,n\text{-path}}(K_{n+1})$ with $y_e = 0$. Since the lineality space of $P^{(k)}_{0,n\text{-path}}(K_{n+1})$ is determined by the equations (3.41)-(3.43) and (3.47), we may assume that $c_{0n} = 0$, $c_{0m} = c_{mn} = 0$ for some internal node m with $[0,m] \neq e \neq [m,n]$, and $c_d = 0$ for some internal edge $d \neq e$.

Let $f = [i,j]$, $g = [h,\ell] \in E \setminus \{e\}$ be non-adjacent edges. Without loss of generality, we may assume that the nodes j and ℓ are not incident with edge e. Let P be any tight $[0,n]$-k-path that uses the edges $[i,j]$, $[j,h]$ but does not visit node ℓ. Replacing node j by node ℓ yields another tight path and hence, $c_{ij} + c_{jh} = c_{i\ell} + c_{\ell h}$. Next, consider any tight $[0,n]$-k-path P' that uses the edges $[j,i]$, $[i,\ell]$ and does not visit node h. Replacing node i by node h yields another tight path and thus, $c_{ij} + c_{jh} = c_{i\ell} + c_{\ell h}$. Adding both equations, we obtain $c_f = c_g$, and since $|N \setminus \{0,n\}| \geq 5$, this implies $c_f = c_g$ for all internal edges f, g that are not equal to e. Now it is easy to see that also $c_{0i} = c_{0j}$ and $c_{hn} = c_{\ell n}$ for all edges $[0,i]$, $[0,j]$, $[h,n]$, $[\ell,n]$ not equal to e. Since $c_{0m} = c_{mn} = 0$ and $c_d = 0$, it follows that $c_f = 0$ for all edges $f \neq e$ which implies also $c_0 = 0$. Hence, $c^T x = c_0$ is simply the equation $c_e y_e = 0$. □

3.4 Facets of $P^c_{0,n\text{-PATH}}(\tilde{D}_n)$

In this section we consider the inequalities of the below-mentioned IP-model for the directed cardinality constrained path polytope $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and give necessary and sufficient conditions for them to be facet defining. Most of the inequalities studied in Section (3.2) are shown to be facet defining for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ as well. Moreover, we present some further classes of inequalities that cut off paths of forbidden cardinality.

Let $D = (N, A)$ be a digraph on node set $N = \{0, \ldots, n\}$. The integer points of $P^c_{0,n\text{-path}}(D)$ are characterized by the following system:

$$
\begin{aligned}
x(\delta^{\text{in}}(0)) &= 0, \\
x(\delta^{\text{out}}(n)) &= 0, \\
x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= \begin{cases} 1 \text{ if } i = 0, \\ 0 \text{ if } i \in N \setminus \{0, n\}, \\ -1 \text{ if } i = n, \end{cases} \\
x(\delta^{\text{out}}(i)) &\le 1 \qquad \forall\, i \in N \setminus \{0, n\}, \\
x((S : N \setminus S)) - x(\delta^{\text{in}}(j)) &\ge 0 \qquad \forall\, S \subset N : 0, n \in S, j \in N \setminus S,
\end{aligned}
$$

$$
\begin{aligned}
x(A) &\ge c_1, \\
x(A) &\le c_m,
\end{aligned} \tag{3.55}
$$

$$
\begin{aligned}
&(c_{p+1} - |W| + 1) \sum_{i \in W} x(\delta^{\text{out}}(i)) \\
&-(|W| - 1 - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \\
&\qquad -c_p(c_{p+1} - |W| + 1) \le 0 \qquad \forall\, W \subseteq N:\ 0, n \in W,\ \exists\, p \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{with } c_p < |W| - 1 < c_{p+1},
\end{aligned}
$$

$$
x_{ij} \in \{0, 1\} \qquad \text{for all } (i, j) \in A.
$$

Here, the forbidden cardinality inequalities arise in a form different from inequalities (3.2), since the number of nodes that are visited by a simple path is one more than the number of arcs in difference to a simple cycle. The first three and the integrality constraints ensure that x is the incidence vector of a simple $(0, n)$-path P (cf. (3.7)-(3.13)). The cardinality bounds and the forbidden cardinality inequalities guarantee that $|P| = c_p$ for some p.

Dahl and Gouveia [23] gave a complete linear description of $P^{(1,2,3)}_{0,n\text{-path}}(D')$, where $D' = D \cup \{(0, n)\}$. So, we have also one for $P^{(2,3)}_{0,n\text{-path}}(D)$. Consequently, from now on we exclude the case $c = (2, 3)$ with respect to directed path polytopes. More precisely, in what follows, we consider only the set of cardinality sequences

$$\text{CS} := \{c = (c_1, \ldots, c_m) : m \geq 2, 2 \leq c_1 < \cdots < c_m \leq n, c \neq (2,3)\}.$$

Due to the flow conservation constraints, two different inequalities that are valid for $P^c_{0,n\text{-path}}(D)$ may define the same face. The next theorem is the counterpart of Theorem 3.9 for $P^c_{0,n\text{-path}}(D)$.

Theorem 3.34. Let $\alpha^T x \geq \alpha_0$ be a valid inequality for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and let T be a spanning tree of D. Then for any specified set of coefficients β_{ij} for the arcs $(i,j) \in T$, there is an equivalent inequality $\bar{\alpha}^T x \geq \alpha_0$ for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ such that $\bar{\alpha}_{ij} = \beta_{ij}$ for $(i,j) \in T$. □

3.4.1 Facets related to general cardinality restrictions

The cardinality bounds $x(\tilde{A}_n) \geq c_1$ and $x(\tilde{A}_n) \leq c_m$ define facets of the cardinality constrained path polytope $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $4 \leq c_i \leq n-1$ for $i = 1, m$ (see Table 3.2).

Next, we turn to the forbidden cardinality inequalities. Due to the easier notation, we analyze them for the polytope $P^* := \{x \in P^c_C(D_n) | x(\delta^{\text{out}}(1)) = 1\}$ which is isomorphic to $P^c_{0,n\text{-path}}(\tilde{D}_n)$.

Theorem 3.35. Let $D_n = (N, A)$ be the complete digraph on $n \geq 4$ nodes and W a subset of N with $1 \in W$ and $c_p < |W| < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. The node induced forbidden cardinality inequality

$$(c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) - (|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \leq c_p(c_{p+1} - |W|) \tag{3.56}$$

defines a facet of P^* if and only if $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n-1$.

Proof. Assuming that $|W| + 1 = c_{p+1} < n$, we see that (3.56) is dominated by nonnegativity constraints $x_{ij} \geq 0$ for $(i,j) \in N \setminus W$. When $c_{p+1} = n$ and $n - |W| \geq 2$, (3.56) is dominated by another inequality of the same form for some $W' \supset W$ with $|W'| = n-1$. Therefore, if inequalities (3.56) are not facet defining, then they are dominated by other inequalities of the IP-model that are facet defining for P^*.

Suppose that $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$. By choice, $|W| \geq 3$ and $|N \setminus W| \geq 3$. Moreover, assume that the equation $b^T x = b_0$ is satisfied by all points that satisfy (3.56) at equality. Setting $\iota := c_{p+1} - |W|$, we will show that

$$\begin{aligned} b_{1i} &= \iota \quad \forall\, i \in N \setminus \{1\} \\ b_{i1} &= \iota \quad \forall\, i \in W \setminus \{1\}, \\ b_{ij} &= \kappa \quad \forall\, i \in W \setminus \{1\}, j \in N \setminus \{1\}, \\ b_{ij} &= \lambda \quad \forall\, i \in N \setminus W, j \in N \setminus \{1\}, \\ b_{i1} &= \mu \quad \forall\, i \in N \setminus W \end{aligned} \tag{3.57}$$

for some $\kappa \neq 0, \lambda, \mu$. Then, considering a tight cycle of cardinality c_p and two tight cycles of cardinality c_{p+1}, one using an arc in $(N \setminus W : \{1\})$, the other not, yields the equation system

$$\begin{aligned} b_0 &= 2\iota + (c_p - 2)\kappa \\ b_0 &= \iota + (|W| - 1)\kappa + (c_{p+1} - |W| - 1)\lambda + \mu \\ b_0 &= 2\iota + (|W| - 2)\kappa + (c_{p+1} - |W|)\lambda \end{aligned}$$

which solves to

$$\begin{aligned} b_0 &= 2\iota + (c_p - 2)\kappa \\ \mu &= \iota + (\tfrac{|W|-c_p}{|W|-c_{p+1}} - 1)\kappa \\ \lambda &= \tfrac{|W|-c_p}{|W|-c_{p+1}}\kappa. \end{aligned}$$

Thus, $b^T x = b_0$ is the equation

$$\begin{aligned} &\iota x(\delta^{\text{out}}(1)) + \iota x(\delta^{\text{in}}(1)) + (\tfrac{|W|-c_p}{|W|-c_{p+1}} - 1)\kappa \sum_{i \in N \setminus W} x_{i1} \\ &\quad + \kappa \sum_{i \in W \setminus \{1\}} x(\delta_1^{\text{out}}(i)) + \tfrac{|W|-c_p}{|W|-c_{p+1}}\kappa \sum_{i \in N \setminus W} x(\delta_1^{\text{out}}(i)) = 2\iota + (c_p - 2)\kappa, \end{aligned}$$

where $\delta_1^{\text{out}}(i) := \delta^{\text{out}}(i) \setminus \{(i,1)\}$. Adding $\kappa - \iota$ times the equations $x(\delta^{\text{out}}(1)) = 1$ and $x(\delta^{\text{in}}(1)) = 1$ and multiplying the resulting equation with $-\frac{|W|-c_{p+1}}{\kappa}$, we see that $b^T x = b_0$ is equivalent to (3.56).

To show (3.57), we may assume without loss of generality that $2 \in W$ and $b_{1i} = c_{p+1} - |W|$, $i \in N \setminus \{1\}$, and $b_{21} = c_{p+1} - |W|$, by Theorem 3.34. Next, let $\mathcal{R}$ be the set of subsets of N of cardinality c_{p+1} that contain W, i.e.,

$$\mathcal{R} := \{R \subset N : |R| = c_{p+1},\ R \supset W\}.$$

For any $R \in \mathcal{R}$, the c_{p+1}-cycles on R are tight tours on R. Theorem 23 of Grötschel and Padberg [47] implies that there are $\tilde{\alpha}_i^R, \tilde{\beta}_i^R$ for $i \in R$ such that $b_{ij} = \tilde{\alpha}_i^R + \tilde{\beta}_j^R$ for all $(i,j) \in A(R)$. Setting

$$\begin{aligned} \alpha_i^R &:= \tilde{\alpha}_i^R - \tilde{\alpha}_1^R \quad (i \in R), \\ \beta_i^R &:= \tilde{\beta}_i^R - \tilde{\alpha}_1^R \quad (i \in R), \end{aligned} \tag{3.58}$$

yields $\alpha_i^R + \beta_j^R = b_{ij}$ for all $(i,j) \in A(R)$. Since $\alpha_1^R = 0$ and $b_{1i} = \iota$, it follows that $\beta_i^R = \iota$ for all $i \in R \setminus \{1\}$. In a similar manner one can show for any $S \in \mathcal{R}$ the existence of α_i^S, β_i^S for $i \in S$ with $\alpha_1^S = 0$, $\beta_j^S = \iota$ for $j \in S \setminus \{1\}$, and $\alpha_i^S + \beta_i^S = b_{ij}$ for all $(i,j) \in A(S)$. This implies immediately that $\alpha_i^R = \alpha_i^S$ and $\beta_i^R = \beta_i^S$ for all $i \in R \cap S$. Thus, there are α_i, β_i for all $i \in N$ such that $\alpha_1 = 0$, $\beta_i = \iota$ for $i \in N \setminus \{1\}$, and $b_{ij} = \alpha_i + \beta_j$ for all $(i,j) \in A$.

Next, consider a tight c_p-cycle that contains the arcs $(1,k),(k,j)$ but does not visit node ℓ for some $j,k,\ell \in W$. Replacing node k by node ℓ yields another tight c_p-cycle, and therefore $b_{1k} + b_{kj} = b_{1\ell} + b_{\ell j}$, which implies that $\alpha_k = \alpha_\ell$ for all $k,\ell \in W \setminus \{1\}$. Thus, there is κ such that $b_{ij} = \kappa$ for all $i \in W \setminus \{1\}$, $j \in N \setminus \{1\}$. Moreover, it follows immediately that $b_{i1} = \iota$ for all $i \in W \setminus \{1\}$. One can show analogously that $\alpha_i = \alpha_j$ for all $i,j \in N \setminus W$. This implies the existence of λ, μ with $b_{ij} = \lambda$ for all $i \in N \setminus W$, $j \in N \setminus \{1\}$ and $b_{i1} = \mu$ for all $i \in N \setminus W$.

Finally, when $|W|+1 = c_{p+1} = n$, we show that there are $n^2 - 2n$ affinely independent points $x \in P^*$ satisfying (3.56) at equality. Without loss of generality, let $W = \{1,\ldots,n-1\}$. Because each tour is tight with respect to (3.56), it exist n^2-3n+2 linearly independent points $(x^r,y^r) \in Q := \{(x,y) \in P^c_{CL}(D_n) | x(\delta^{\text{out}}(1) = 1)\}$ with $y^r = 0$. Furthermore, consider the incidence vectors of the $n-2$ cycles $(1,2,\ldots,c_p)$, $(1,3,4,\ldots,c_p+1),\ldots,(1,n-2,n-1,2,3,\ldots,c_p-2)$, $(1,n-1,2,3,\ldots,c_p-1)$. The corresponding points in Q are linearly independent and they are also linearly independent of the points (x^r,y^r). Hence, (3.56) is also facet defining if $|W|+1 = c_{p+1} = n$. □

Theorem 3.36. Let $D_n = (N,A)$ be the complete digraph on n nodes, and let $1 \in W \subset N$ with $c_p < |W| < c_{p+1}$ for some $p \in \{1,\ldots,m-1\}$. The *cardinality-subgraph inequality*

$$2x(A(W)) - (|W| - c_p - 1)[x((W : N \setminus W)) + x((N \setminus W : W))] \leq 2c_p \quad (3.59)$$

is valid for P^* and induces a facet of P^* if and only if $p+1 < m$ or $c_{p+1} = n = |W|+1$.

Figure 3.4 sketches the support graph of a cardinality-subgraph inequality with respect to P^*. The coefficients associated with the red arcs (the arcs $(i,j) \in A(W)$) are 2. Since $|W| = 4$ and $c = (2,6,8)$, that is, $c_p = c_1 = 2$, it follows $-[|W| - c_p - 1] = -1$ for the coefficients associated with the arcs $(i,j) \in (W : N \setminus W) \cup (N \setminus W : W)$. This is illustrated by the green dashed arrows.

Proof of Theorem 3.36. A cycle of cardinality less or equal to c_p uses at most c_p arcs of $A(W)$ and thus its incidence vector satisfies (3.59). A cycle

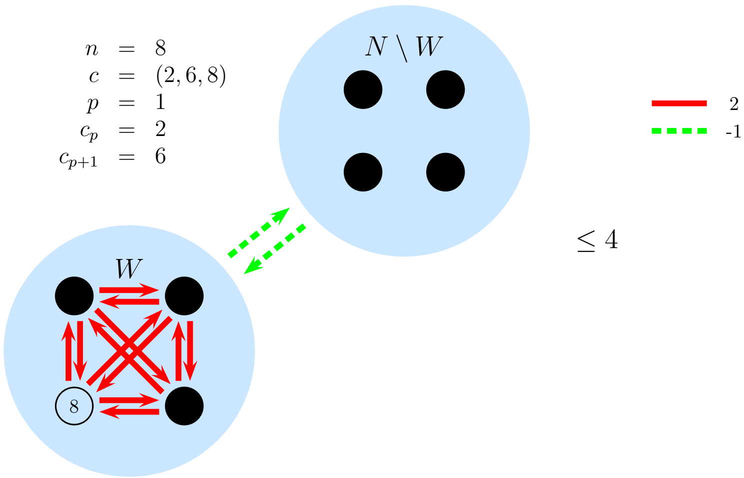


Figure 3.4: Support graph of a cardinality-subgraph inequality with respect to P^*.

C of cardinality greater or equal to c_{p+1} uses at most $|W| - 1$ arcs in $A(W)$ and if C indeed visits any node in W, then it uses at least 2 arcs in $(W : N \setminus W) \cup (N \setminus W : W)$ and hence,

$$2\chi^C(A(W)) - (|W| - c_p - 1)[\chi^C((W : N \setminus W)) + \chi^C((N \setminus W : W))] \\ \leq 2(|W| - 1) - 2(|W| - c_p - 1) = 2c_p.$$

In particular, all cycles of feasible cardinality that visit node 1 satisfy (3.59).

To prove that (3.59) is facet defining, assume that $p+1 = m$ and $c_m < n$. When $c_{p+1} - c_p = 2$ holds, then (3.59) does not induce a facet of P^* for the same reason as the corresponding node induced forbidden cardinality inequality does not induce a facet of P^*. Indeed, both inequalities define the same face. When $c_{p+1} - c_p > 2$, then it is easy to see that the face induced by (3.59) is a proper subset of the face defined by the node induced forbidden cardinality inequality (3.56), and thus, it is not facet defining. The same argumentation holds when $p + 1 = m$, $c_m = n$, and $n - |W| > 1$.

To show that (3.59) defines a facet, when the conditions are satisfied, we suppose that the equation $b^T x = b_0$ is satisfied by every $x \in P^*$ that satisfies (3.59) at equality. Using Theorem 3.34 we may assume that $b_{w1} = 2$ for some $w \in W$, $b_{1i} = 2$ for all $i \in W$, and $b_{iw} = -(|W| - c_p - 1)$ for all $i \in N \setminus W$.

Let $q, r \in N \setminus W$ be two nodes that are equal if $c_{p+1} = |W| + 1$ and otherwise different. Then, all (q, r)-paths of cardinality $|W|+1$ whose internal nodes are all the nodes of W satisfies the equation $b^T x = b_0$. (Note, in case $c_{p+1} = |W| + 1$, the paths are Hamiltonian cycles.) Thus, it exist α_q, β_r, and

α_j, β_j for $j \in W$ with

$$\begin{aligned} b_{qj} &= \alpha_q + \beta_j \quad (j \in W) \\ b_{ir} &= \alpha_i + \beta_r \quad (i \in W) \\ b_{ij} &= \alpha_i + \beta_j \quad ((i,j) \in A(W)). \end{aligned}$$

Without loss of generality we may assume that $\beta_w = 0$. Since $b_{1j} = 2$, it follows that $\alpha_1 = 2$, $\beta_j = 0$ for all $j \in W \setminus \{1\}$, and $\alpha_q = |W| - c_p - 1$. When $c_p = 2$, then the cycles $\{(1,j),(j,1)\}$ for $j \in W \setminus \{1\}$. When $c_p \geq 3$, then consider a tight c_p-cycle that starts with $(1,i),(i,j)$ and skips node k for some $i,j,k \in W \setminus \{1\}$. Replacing the arcs $(1,i),(i,j)$ by $(1,k),(k,j)$ yields another tight c_p-cycle, and thus the equation $b_{1i} + b_{ij} = b_{1k} + b_{kj}$. In either case, it follows that $b_{j1} = 2$ for $j \in W \setminus \{1\}$ and there is λ such that $b_{ij} = \lambda$ for all $(i,j) \in A(W \setminus \{1\})$. Summarizing our intermediate results and adding further, easy obtainable observations, we see that

$$\begin{aligned} b_{1i} &= 2 && (i \in W \setminus \{1\}) \\ b_{i1} &= 2 && (i \in W \setminus \{1\}) \\ b_{ij} &= \lambda && ((i,j) \in A(W \setminus \{1\})) \\ b_{qi} &= -(|W| - c_p - 1) && (i \in W \setminus \{1\}) \\ b_{q1} &= -(|W| - c_p - 1) + 2 - \lambda && \\ b_{ir} &= -(|W| - c_p - 1)(\lambda - 1) && (i \in W \setminus \{1\}) \\ b_{1r} &= -(|W| - c_p - 1)(\lambda - 1) + 2 - \lambda && \\ b_0 &= 4 + (c_p - 2)\lambda && \end{aligned} \tag{3.60}$$

holds.

So, when $c_{p+1} = n$, we have $q = r$ and $N \setminus W = \{q\}$, and thus, $b^T x = b_0$ is the equation

$$\begin{aligned} 2x(\delta^{\text{out}}(1)) - \lambda x_{1q} + 2x(\delta^{\text{in}}(1)) - \lambda x_{q1} + \lambda x(A(W \setminus \{1\})) \\ -(|W| - c_p - 1)x(\delta^{\text{out}}(q)) \\ -(|W| - c_p - 1)(\lambda - 1)x(\delta^{\text{in}}(q)) = 4 + (c_p - 2)\lambda. \end{aligned}$$

Adding $(1 - \frac{\lambda}{2})(|W| - c_p - 1)$ times the equation $x(\delta^{\text{out}}(q)) - x(\delta^{\text{in}}(q)) = 0$ and $(\lambda - 2)$ times the equations $x(\delta^{\text{out}}(1)) = 1$ and $x(\delta^{\text{in}}(1)) = 1$, we see that $b^T x = b_0$ is equivalent to (3.59), and hence (3.59) is facet defining.

Otherwise, that is, if $p + 1 < m$, (3.60) holds for each pair of nodes $q, r \in N \setminus W$. Moreover, letting $k \neq l \in W \setminus \{1\}$, it can be seen that every (k,l)-path P of cardinality $c_{p+1} - |W| + 1$ or $c_m - |W| + 1$ whose internal nodes are in $N \setminus W$ satisfies the equation $b^T x = -\lambda(|W| - c_p - 1)$. Thus, there are π_k, π_l, and π_j, $j \in N \setminus W$, such that

$$\begin{aligned} b_{kj} &= \pi_k - \pi_j \quad (j \in N \setminus W) \\ b_{jl} &= \pi_j - \pi_l \quad (j \in N \setminus W) \\ b_{ij} &= \pi_i - \pi_j \quad ((i,j) \in A(N \setminus W)). \end{aligned}$$

Since $b_{kj} = -(|W| - c_p - 1)(\lambda - 1)$ for $j \in N \setminus W$, it follows that $\pi_i = \pi_j$ for all $i, j \in N \setminus W$ which implies that $b_{ij} = 0$. Hence, $b^T x = b_0$ is the equation

$$\begin{aligned} &2x(\delta^{\text{out}}(1)) + 2x(\delta^{\text{in}}(1)) - \lambda \textstyle\sum_{i \in N\setminus W}(x_{1i} + x_{i1}) \\ &+\lambda x(A(W \setminus \{1\})) - (|W| - c_p - 1)x((N \setminus W : W)) \\ &\qquad -(|W| - c_p - 1)(\lambda - 1)x((W : N \setminus W)) = 4 + (c_p - 2)\lambda. \end{aligned}$$

Adding $(1 - \frac{\lambda}{2})(|W| - c_p - 1)$ times the equation

$$x((N \setminus W : W)) - x((W : N \setminus W)) = 0$$

and $(\lambda - 2)$ times the equations $x(\delta^{\text{out}}(1)) = 1$ and $x(\delta^{\text{in}}(1)) = 1$, we see that $b^T x = b_0$ is equivalent to (3.59), and hence (3.59) is facet defining. □

3.4.2 Facets unrelated to cardinality restrictions

Theorem 3.37. Let $c \in \text{CS}$ and $n \geq 4$. The nonnegativity constraint

$$x_{ij} \geq 0 \tag{3.61}$$

defines a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $c \neq (2, n)$ or $c = (2, n)$, $n \geq 5$, and (i, j) is an inner arc.

Proof. By Theorem 3.13, (3.61) defines a facet of $PP^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ whenever $4 \leq k \leq n - 1$. Hence, Lemma 3.1 implies that (3.61) is facet defining for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if $n \geq 5$ and there is an index p with $4 \leq c_p \leq n - 1$.

It remains to examine the cases $c = (2, n), (3, n)$, and $(2, 3, n)$. When $n = 4$, the assertion can be verified using a computer program. For $n \geq 5$, inequalities (3.61) are facet defining for the asymmetric traveling salesman polytope$P^{(n)}_C(D_n)$ (see Grötschel and Padberg [47]). Thus, there are $n^2 - 3n + 1$ linearly independent points x^r in $P^c_{0,n\text{-path}}(\tilde{D}_n)$ satisfying $\mathbb{1}^T x^r = n$ and $x^r_{ij} = 0$. When $c = (2, n)$ and the arc (i, j) is incident with a terminal node, there are only $n^2 - 2n - 1$ affinely independent points in $P^c_{0,n\text{-path}}(\tilde{D}_n)$ satisfying (3.61) with equality, since we have only $n - 2$ paths from 0 to n of cardinality 2 that do not use (i, j). When $c = (2, n)$ and (i, j) is an inner arc or $c = (3, n)$, (3.61) can be shown to induce a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ applying the same techniques as in the proof to Theorem 3.4. Clearly then, (3.61) induces also a facet of $P^{(2,3,n)}_{s,t\text{-path}}(D)$. □

Theorem 3.38. Let $c \in \text{CS}$, $n \geq 4$, and i be an internal node of $\tilde{D}_n$. The degree constraint

$$x(\delta^{\text{out}}(i)) \leq 1 \tag{3.62}$$

induces a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ unless $c = (2, n)$.

Proof. When $n \geq 5$ and $4 \leq c_p \leq n-1$ for some index p, (3.62) can be shown to induce a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ using Lemma 3.1 and Theorem 3.14, saying that (3.62) induces a facet of $P^{(c_p)}_{0,n\text{-path}}(\tilde{D}_n)$. So, let $c \in \{(2,n),(3,n),(2,3,n)\}$. All Hamiltonian $(0,n)$-paths satisfy $x(\delta^{\text{out}}(i)) = 1$, and n^2-3n+2 of them are linearly independent. So, when $c = (2,n)$, we have only one further affinely independent $(0,n)$-path, namely the path $\{(0,i),(i,n)\}$. Thus, in this case (3.62) is not facet defining. Against it, when $c = (3,n)$, we consider the paths $\{(0,i),(i,j),(j,n)\}$ for all internal nodes $j \neq i$. These $n-2$ paths are linearly independent, and they are also linearly independent of the n^2-3n+2 Hamiltonian paths which can be proved with the same approach as in the proof to Theorem 3.4. Consequently, (3.62) induces a facet of $P^{(3,n)}_{0,n\text{-path}}(\tilde{D}_n)$ and $P^{(2,3,n)}_{0,n\text{-path}}(\tilde{D}_n)$. □

Lemma 3.39. Let $4 \leq p \leq n-1$ and $\tilde{N}_n = \{0\} \,\dot\cup\, Q \,\dot\cup\, \{v\} \,\dot\cup\, R \,\dot\cup\, \{n\}$ be a partition of $\tilde{N}_n$ with Q and R nonempty and $|Q| \leq p-2$. Moreover, let $b^Tx = b_0$ be an equation, with $b_{0i} = 0$ for $i \in Q \cup \{v\}$ and $b_{0j} = 1$ for $j \in R$, that is satisfied by each $(0,n)$-path of cardinality p that satisfies the equation $x((Q \cup \{0\} : R)) - x((R : \{v\})) = 0$. If $|Q| \geq 5$ or $|R| \geq 5$, then there are $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, and θ such that

$$\begin{array}{ll}
b_{ij} = \alpha & \forall\,(i,j) \in \tilde{A}_n(Q),\\
b_{ij} = \alpha & \forall\,(i,j) \in \tilde{A}_n(R),\\
b_{ij} = 1+\alpha & \forall\,(i,j) \in (Q:R),\\
b_{ij} = \beta & \forall\,(i,j) \in (R:Q),\\
b_{in} = \gamma & \forall\, i \in Q,\\
b_{iv} = \delta & \forall\, i \in Q,\\
b_{vi} = \epsilon & \forall\, i \in Q,\\
b_{in} = \zeta & \forall\, i \in R,\\
b_{iv} = \eta & \forall\, i \in R,\\
b_{vi} = \theta & \forall\, i \in R.
\end{array} \tag{3.63}$$

Proof. First, we assume that $|Q| \geq 5$ which implies $p \geq 7$. Let P_1 be any $(0,v)$-path of cardinality $|Q|$ that visits all nodes in Q but one, say l. Complete P_1 to a $(0,n)$-path of cardinality p that satisfies the equation $x((Q\cup\{0\} : R))-x((R : \{v\})) = 0$, and hence also $b^Tx = b_0$. Substituting any two arcs $(i,k),(k,j) \in \tilde{A}_n(Q) \cap P$ by the arcs $(i,l),(l,j)$ yields a new $(0,n)$-path that satisfies also the equation $b^Tx = b_0$, and hence, $b_{ik} + b_{kj} = b_{il} + b_{li}$. Obviously, we can derive such an equation for all distinct nodes $i,j,k,l \in Q$. By Lemma 2 of Hartmann and Özlük [48], it exist α, π_0, π_v, and π_j, $j \in Q$, with

$$
\begin{aligned}
b_{0i} &= \alpha + \pi_0 - \pi_i \quad \forall\, i \in R,\\
b_{iv} &= \alpha + \pi_i - \pi_v \quad \forall\, i \in R,\\
b_{ij} &= \alpha + \pi_i - \pi_j \quad \forall\, (i,j) \in \tilde{A}_n(Q).
\end{aligned}
$$

Without loss of generality, we may assume that $\pi_z = 0$ for some $z \in Q$, and since $b_{0i} = 0$ for $i \in Q$, we have even $\pi_i = 0$ for all $i \in Q$. Consequently, $b_{ij} = \alpha$ for all $(i,j) \in \tilde{A}_n(Q)$ and $b_{iv} = \alpha - \pi_v =: \delta$ for all $i \in Q$.

Next, consider a $(0,n)$- path P of cardinality p with $b(P) = b_0$ starting with the arc $(0,j)$ for some $j \in R$ and ending with the arcs $(k,l),(l,m),(m,n)$ for any $k,l,m \in Q$ that skips some node $i \in Q$. Substituting the arcs $(0,j),(k,l),(l,m)$ by the arcs $(0,i),(i,j),(k,m)$, yields a path $\tilde{P}$ for which $b(\tilde{P}) = b_0$ holds. Thus, $b_{0j} + b_{kl} + b_{lm} = b_{0i} + b_{ij} + bl,m$, which implies that $b_{ij} = 1 + \alpha$ for all $(i,j) \in (Q:R)$.

Next, consider a $(0,n)$-path P of cardinality p with $b(P) = b_0$ that ends with the arcs $(i,j),(j,n)$ and skips node k for some $i,j,k \in Q$. Then the path $\tilde{P} := (P \setminus \{(i,j),(j,n)\}) \cup \{(i,k),(k,n)\}$ satisfies $b(\tilde{P}) = b_0$, and thus, $b_{ij}+b_{jn} = b_{ik}+b_{kn}$ which implies $b_{jn} = b_{kn}$, since $b_{ij} = b_{ik} = \alpha$. Consequently, $b_{in} = \gamma$ for some γ and all $i \in Q$. With a similar construction one can show that there is ϵ such that $b_{vi} = \epsilon$ for $i \in Q$. When $|R| = 1$, then we have no more to show. Otherwise, let $(i,j) \in R$ and P be a $(0,n)$-path starting with $(0,i),(i,j)$, skipping a node $k \in Q$, and satisfying $b(P) = b_0$. Substituting node i by k, yields the equation $b_{0i} + b_{ij} = b_{0k} + b_{kj}$, Since $b_{0i} = 1$, $b_{0k} = 0$, and $b_{kj} = 1 + \alpha$, it follows $b_{ij} = \alpha$. It can be shown with similar arguments that the remaining equations of system (3.63) also holds.

Second, let $|R| \geq 5$. When $p \geq 5$, the proof can be performed similarly by showing firstly that the equation $b_{ik}+b_{kj} = b_{il}+b_{lj}$ holds for any distinct nodes $i,j,k,l \in R$, and applying Lemma 2 of Hartmann and Özlük [48]. When $p = 4$, the method fails, because it does not exist a $(0,n)$-path of cardinality 4 that satisfies the equation $b^T x = b_0$ and visits 3 nodes in R. So, let $p = 4$ which implies $|Q| \leq 2$. Consider the paths $\{(0,i),(i,v),(v,k),(k,n)\}$ and $\{(0,j),(j,v),(v,k),(k,n)\}$ for some $i,j \in R$ and $k \in Q$. Since both paths satisfy the equation $b^T x = b_0$, it follows that $b_{iv} = b_{jv}$. Hence, $b_{iv} = \eta$ for some η, for $i \in R$. This in turn implies the existence of α such that $b_{ij} = \alpha$ holds for all $(i,j) \in \tilde{A}_n(R)$, by considering the paths $\{(0,i),(i,j),(j,v),(v,n)\}$. Analogous constructions show successively that also the remaining equations of system (3.63) hold. □

Theorem 3.40. Let $c = (c_1,\ldots,c_m) \in \mathrm{CS}$, $n \geq 4$, $S \subset \tilde{N}_n$, $0,n \in S$, and $v \in \tilde{N}_n \setminus S$. The one-sided min-cut inequality

$$x((S:\tilde{N}_n \setminus S)) - x(\delta^{\mathrm{in}}(v)) \geq 0 \tag{3.64}$$

induces a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $|\tilde{N}_n \setminus S| \geq 2$, $|S| \geq c_1 + 1$, and $c \neq (2, n)$.

Proof. Necessity. When $\tilde{N}_n \setminus S = \{v\}$, (3.64) becomes the trivial inequality $0x \geq 0$, and thus it is not facet defining. When $|S| \leq c_1$, all feasible $(0, n)$-paths P satisfy $|P \cap (S : \tilde{N}_n \setminus S)| \geq 1$, and hence, (3.64) can be obtained by summing up the inequality $x((S : \tilde{N}_n \setminus S)) \geq 1$ and the degree constraint $-x(\delta^{\text{in}}(v)) \geq -1$. Suppose that $c = (2, n)$, $|\tilde{N}_n \setminus S| \geq 2$, and $|S| \geq 3$. With respect to the polytope $P^{(n)}_{0,n\text{-path}}(\tilde{N}_n)$, (3.64) is equivalent to the inequality $x(\tilde{A}_n(S)) \leq |S| - 1$. The latter inequality is an equivalent of a subtour elimination constraint defined on D_n which is known to be facet defining for the ATSP. Thus, (3.64) induces a facet of $P^{(n)}_{0,n\text{-path}}(\tilde{D}_n)$ and consequently, there are $n^2 - 3n + 1$ linear independent incidence vectors of Hamiltonian $(0, n)$-paths satisfying (3.64) at equality. From the $(0, n)$-paths of cardinality 2 are only the paths $\{(0, i), (i, n)\}$, $i \in (S \setminus \{0, n\}) \cup \{v\}$ tight which are at most $n - 2$. Consequently, there are not enough affinely independent points satisfying (3.64) at equality, and hence, (3.64) does not induce a facet of $P^{(2,n)}_{0,n\text{-path}}(\tilde{D}_n)$.

Sufficiency. By Theorem 3.16, (3.64) induces a facet of $P^{(k)}_{0,n\text{-path}}(\tilde{D}_n)$ for $4 \leq k \leq n - 2$ if and only if $|S| \geq k + 1$ and $|\tilde{N}_n \setminus S| \geq 2$. Hence, when $|S| \geq c_i + 1$ for some index $i \in \{1, \ldots, m\}$ with $c_i \geq 4$ and $|\tilde{D}_n \setminus S| \geq 2$, inequality (3.64) is facet defining for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ by applying Lemma 3.1. In particular, this finishes the proof when $i = 1$. Note that in case $i = m$, $c_i \geq 4$ and $|S| \geq c_i + 1$ imply $4 \leq c_m \leq n - 2$, since $|S| \leq n - 1$.

It remains to consider the cases $c_1 = 2$ and $c_1 = 3$. Set $Q := S \setminus \{0, n\}$ and $R := \tilde{N}_n \setminus (S \cup \{v\})$. Suppose that $R \neq \varnothing$, $|Q| \geq c_1 - 1$, and $c \neq (2, n)$.

First, let $c = (3, n)$ and $2 \leq |Q| \leq n - 3$. We have already mentioned that inequality (3.64) induces a facet of $P^{(n)}_{0,n\text{-path}}(\tilde{D}_n)$, which implies that there are $n^2 - 3n + 1$ linearly independent incidence vectors x^r of Hamiltonian paths satisfying (3.64) at equality. In addition, consider the paths $(0, i), (i, j), (j, n)$ for some $(i, j) \in \tilde{A}_n(Q)$, and $(0, k), (k, v), (v, n)$ for each $k \in Q \cup R$. All these $n - 1$ paths are tight and with linearly independent incidence vectors, and considering the corresponding points in $P^c_{PL}(\tilde{D}_n)$ one can see that they are also linearly independent of the points x^r. Consequently, we have found $n^2 - 2n$ affinely independent points satisfying (3.64) with equality.

Next, let $c_1 = 3$, $c \neq (3, n)$, and $2 \leq |Q| \leq c_2 - 2$. Since $c \neq (3, n)$, it follows that $4 \leq c_2 \leq n - 1$. When $n \leq 10$, the assertion can be verified using a computer program. So, let $n \geq 11$, which implies that Q or R contains at least five nodes. Furthermore, let $b^T x = b_0$ be an equation that is satisfied by all $(0, n)$-paths of feasible cardinality satisfying (3.64) at equality. By

Theorem 3.34, we may assume that $b_{0i} = 0$ for $i \in Q \cup \{v\}$, $b_{vn} = 0$, and $b_{0j} = 1$ for $j \in R$. By Lemma 3.39, there are $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, and θ such that equation system (3.63) holds. Since, $b_{vn} = 0$, a path $(0,i),(i,v),(v,n)$ with $i \in Q \cup R$ yields $\delta = b_0$ and $\eta = b_0 - 1$. Next, consider a tight $(0,n)$-path of cardinality c_2 that visits firstly all nodes in Q, then $q \geq 0$ nodes in R, and finally node v. In either case, such a path yields the equation $(c_2 - 3)\alpha = 0$, and hence, $\alpha = 0$. Moreover, from the paths $(0,i),(i,j),(j,n)$ and $(0,v),(v,k),(k,n)$ for some $i,j,k \in Q$ we deduce that $\gamma = b_0$ and $\epsilon = 0$. Again, consider a tight $(0,n)$-path of cardinality c_2 that visits firstly some nodes in R, then v, and finally some nodes in Q. Such a path yields the equation $2b_0 = b_0$, and thus, $b_0 = 0$. Now it is easy to see that $\beta = \zeta$ and $\zeta + \theta = 0$. Summarizing the results, we conclude that $b^T x = b_0$ is the equation

$$x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) + \theta[x(\delta^{\text{out}}(v)) - x((\tilde{N}_n \setminus S : S))] = 0.$$

Since $x(\delta^{\text{out}}(v)) = x(\delta^{\text{in}}(v))$ and $x((S : \tilde{N}_n \setminus S)) = x((\tilde{N}_n \setminus S : S)$ for all $x \in P^c_{0,n\text{-path}}(\tilde{D}_n)$, $b^T x = b_0$ and

$$x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0.$$

are equivalent which proves that (3.64) defines a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ for all c with $c_1 = 3$ and Q with $2 \leq |Q| \leq c_2 - 2$.

Finally, let $c_1 = 2$ and $c \neq (2,n)$. When $c = (2,3,n)$ and $2 \leq |Q| \leq n-3$, the assertion follows immediately, because (3.64) defines already a facet of $P^{(3,n)}_{0,n\text{-path}}(\tilde{D}_n)$. When $c = (2,3,n)$ and $|Q| = 1$, say $Q = \{q\}$, then we have beside $n^2 - 3n + 1$ linearly independent tight Hamiltonian paths, $n-2$ paths $(0,i),(i,v),(v,n)$ for $i \in Q \cup R$, and the path $(0,q),(q,n)$. These $n^2 - 2n$ paths can be shown to be linearly independent proving the assertion. Finally, let $4 \leq c_2 \leq n-1$ and $1 \leq |Q| \leq c_2 - 2$. When $n \leq 10$, the claim can be shown using a computer program. When $n \geq 11$, again we assume that an equation $b^T x = b_0$ is satisfied by all points $y \in P^c_{0,n\text{-path}}(\tilde{D}_n)$ that satisfies (3.64) at equality. Moreover, we suppose that $b_{0i} = 0$ for $i \in Q \cup \{v\}$, $b_{vn} = 0$, $b_{0i} = 1$ for $i \in R$, and there are $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, and θ such that (3.63) holds. This time, it follows immediately $b_0 = 0$ and $\gamma = 0$ by considering the paths $(0,i),(i,n)$ for $i \in Q \cup \{v\}$. We distinguish three cases:

$|R| = 1$: Since $|R| = 1$ implies $|Q| = n-3$ and $c_2 = n-1$, the $(0,n)$-paths of cardinality c_2 whose internal node are in $Q \cup \{v\}$ yield the equations $(c_2 - 3)\alpha + \delta$, $(c_2 - 3)\alpha + \epsilon = 0$, and $(c_2 - 4)\alpha + \delta + \epsilon = 0$. This in turn gives $\alpha = \delta = \epsilon = 0$. Now it can be easily derived that $\eta = -1$, $\beta = \zeta$, and $\zeta + \theta = 0$. Hence, the equation $b^T x = b_0$ is equivalent to the equation $x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0$.

$|R| \geq 2$ and $p = 4$: The paths

$$\begin{aligned}
&\{(0,q),(q,i),(i,v),(v,n)\},\\
&\{(0,i),(i,v),(v,q),(q,n)\},\\
&\{(0,v),(v,i),(i,j),(j,n)\},\\
&\{(0,v),(v,i),(i,q),(q,n)\},\\
&\{(0,i),(i,v),(v,j),(j,n)\},\\
&\{(0,q),(q,v),(v,j),(j,n)\}, \quad q \in Q, i,j \in R
\end{aligned}$$

yield an equation system that solves to $\alpha = \delta = \epsilon = 0$, $\eta = -1$, $\beta = \zeta$, and $\zeta + \theta = 0$. Thus, also in this case we see that $b^T x = b_0$ is equivalent to the equation $x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0$.

$|R| \geq 2$ and $p \geq 5$: For any subsets $N_1, N_2, \ldots, N_p$ of $\tilde{N}_n$ we say that a path P *goes along* $N_1, N_2, \ldots, N_k$ (denoted by $N_1 \to N_2 \to \cdots \to N_k$) when P can be decomposed in subsequences $P_1, P_2, \ldots, P_{k-1}$ such that $P_i \subseteq N_i \cup (N_i : N_{i+1})$ and $P_i \cap (N_i : N_{i+1}) \neq \varnothing$ for $i = 1, \ldots, k-1$. From $(0,n)$-paths of cardinality c_2 going along $0 \to Q \to R \to v \to n$, $0 \to Q \to v \to R \to n$, $0 \to R \to v \to Q \to n$, $0 \to v \to R \to Q \to n$, and $0 \to R \to v \to R \to Q \to n$, we obtain an equation system which solves to $\alpha = \epsilon = 0$, $\eta = -1$, $\theta + \beta = 0$, and $\delta + \zeta + \theta = 0$. When $|Q| = 1$, then any $(0,n)$-path of cardinality c_2 that skips the node in Q and ends with $(v,r),(r,n)$, yields the equation $\zeta + \theta = 0$ which implies also $\delta = 0$ and $\beta = \zeta$. Otherwise, there is a $(0,n)$-path of cardinality c_2 along $0 \to Q \to v \to R \to Q \to n$ from which we obtain the equation $\delta + \theta + \beta = 0$. Hence, we obtain also $\delta = 0$, $\beta = \zeta$, and $\zeta + \theta = 0$. In either case, we see that $b^T x = b_0$ is equivalent to the equation $x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0$.

Thus, inequality (3.64) defines also a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ for all c and Q with $c_1 = 2$ and $1 \leq |Q| \leq c_2 - 2$, which finishes the proof. □

We introduce a further class of inequalities whose undirected pendants we need later for the characterization of the integer points of $P^c_C(K_n)$.

Theorem 3.41. Let $c \in \text{CS}$, $n \geq 4$, $S \subset \tilde{N}_n$, and $0, n \in S$. The min-cut inequality

$$x((S : \tilde{N}_n \setminus S)) \geq 1 \tag{3.65}$$

is valid for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if and only if $|S| \leq c_1$ and facet defining for it if and only if $3 \leq |S| \leq c_1$ and $|\tilde{N}_n \setminus S| \geq 2$.

Proof. When $c \neq (3,n)$, the theorem follows from Theorem 3.15, Lemma 3.1, and the fact that $m \geq 2$. When $c = (3,n)$, then $|S| = 3$ and $|\tilde{N}_n \setminus S| = n-2$.

W.l.o.g. let $S = \{0, 1, n\}$ and $\tilde{N}_n \backslash S = \{2, \ldots, n-1\}$. Identifying nodes 0 and n we see that the min-cut inequality is equivalent to a subtour elimination constraint. Thus, there are $n^2 - 3n + 1$ linearly independent Hamiltonian $(0, n)$-paths in $\tilde{D}_n$ that satisfy (3.65) at equality. To see that (3.65) is facet defining, consider also the 3-paths $\{(0, 1), (1, i), (i, n)\}$ for $i = 2, \ldots, n-1$ and $\{(0, 2), (2, 3), (3, n)\}$. These are $n - 1$ affinely independent points, and they are also affinely independent of the $n^2 - 3n + 1$ Hamiltonian paths which is easily seen by considering the corresponding points in $P^c_{PL}(\tilde{D}_n)$. Thus, we have constructed $n^2 - 2n$ points in $P^c_{0,n\text{-path}}(\tilde{D}_n)$ satisfying (3.65) at equality. □

3.4.3 Inequalities specific to odd or even paths

Theorem 3.42. Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 2$, and c_p even for $1 \leq p \leq m$, and let $\tilde{N}_n = S \,\dot{\cup}\, T$ be a partition of $\tilde{N}_n$ with $0 \in S$, $n \in T$. The *odd path exclusion constraint*

$$x(\tilde{A}_n(S)) + x(\tilde{A}_n(T)) \geq 1 \tag{3.66}$$

is valid for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and defines a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if and only if (i) $c_1 = 2$ and $|S|, |T| \geq \frac{c_2}{2} + 1$, or (ii) $c_1 \geq 4$ and $|S|, |T| \geq \frac{c_2}{2}$.

An illustration of inequalities (3.66) is given in Figure 3.5. It shows the support graph of an odd path exclusion constraint for $n = 6$. The black arcs have coefficients 1; all other coefficients are 0. The red arcs depict an odd path that violates the inequality.

Proof of Theorem 3.42. Clearly, each $(0, n)$-path of even cardinality uses at least one arc in $\tilde{A}_n(S) \cup \tilde{A}_n(T)$. Thus, inequality (3.66) is valid.

When $|S|$ or $|T|$ is less than $\frac{c_2}{2}$, then there is no $(0, n)$-path of cardinality c_p, $p \geq 2$, that satisfies (3.67) at equality which implies that (3.67) cannot be facet defining for $P^c_{0,n\text{-path}}(\tilde{D}_n)$. Thus $|S|, |T| \geq \frac{c_2}{2}$ holds if (3.66) is facet defining. For $c_1 = 2$ we have to require even $|S|, |T| \geq \frac{c_2}{2} + 1$. For the sake of contradiction assume w.l.o.g. that $|S| = \frac{c_2}{2}$. Then follows $|T| \geq \frac{c_2}{2} + 1$. However, for an inner arc $(i, j) \in \tilde{A}_n(S)$ there is no tight $(0, n)$-path of cardinality c_2 that uses (i, j).

Next, let (i) or (ii) be true. The conditions imply that for $p = 1$ or $p = 2$ $c_p \geq 4$ and $|S|, |T| \geq \frac{c_p}{2} + 1$ holds. Restricted to the polytope $P^{(c_p)}_{0,n\text{-path}}(\tilde{D}_n)$ inequality (3.66) is equivalent to the max-cut inequality $x((S : T)) \leq \frac{c_p}{2}$ which were shown to be facet defining for $P^{(c_p)}_{0,n\text{-path}}(\tilde{D}_n)$ (see Theorem 3.17). Thus there are $n^2 - 2n - 1$ linearly independent points in $P^c_{0,n\text{-path}}(\tilde{D}_n) \cap$

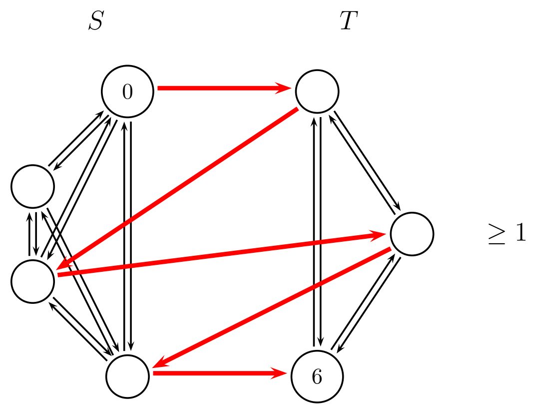


Figure 3.5: Support graph of an odd path exclusion constraint for $n = 6$ and an odd path that violates it.

$P^{(c_p)}_{0,n\text{-path}}(\tilde{D}_n)$ satisfying (3.66) at equality. Moreover, the conditions ensure that there is also a tight $(0, n)$-path of cardinality c_q, where $q = 3 - p$. By Lemma 3.1 (i), the incidence vector of this path is affinely independent of the former points, and hence, (3.66) defines a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$. □

Theorem 3.43. Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $\tilde{N}_n = S \mathbin{\dot\cup} T$ be a partition of $\tilde{N}_n$ with $0, n \in S$. The *even path exclusion constraint*

$$x(\tilde{A}_n(S)) + x(\tilde{A}_n(T)) \geq 1 \tag{3.67}$$

is valid for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and defines a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if and only if (i) $c_1 = 3$, $|S| - 1 \geq \frac{c_2+1}{2}$, and $|T| \geq \frac{c_2-1}{2}$, or (ii) $c_1 \geq 5$ and $\min(|S| - 1, |T|) \geq \frac{c_2-1}{2}$.

Proof. Since the start and end node 0 and n are in the same partition, each $(0, n)$-path of odd cardinality uses at least one arc in $\tilde{A}_n(S) \cup \tilde{A}_n(T)$. Thus, inequality (3.67) is valid.

To show necessity assume that $|S|-1$ or $|T|$ is less than $\frac{c_2-1}{2}$. Then there is no $(0, n)$-path of cardinality c_r, $r \geq 2$, that satisfies (3.67) at equality. Thus, (3.67) cannot be facet defining. Hence, we must have $|S| - 1, |T| \geq \frac{c_2-1}{2}$. In case $c_1 = 3$ we have to require even $|S| - 1 \geq \frac{c_2+1}{2}$, because otherwise there would be no tight $(0, n)$-path using an inner arc $(i, j) \in \tilde{A}_n(S)$.

To show sufficiency we firstly suppose that (i) holds. Let $b^T x = b_0$ be an equation that is satisfied by all $(0, n)$-paths of feasible cardinality satisfying (3.67) at equality. By Theorem 3.34, we may assume that $b_{0i} = 0$

for $i \in T$, $b_{in} = 1$ for $j \in S \setminus \{0\}$, and $b_{uv} = 0$ for some particular arc $(u, v) \in (T : S \setminus \{0, n\})$. From the tight path $(0, u), (u, v), (v, n)$ we deduce that $b_0 = 1$. Furthermore, for $i \in T$, $j \in S$ the paths $(0, i), (i, j), (j, n)$ are tight, $b_{0i} = 0$, and $b_{jn} = 1$. Thus, $b_{ij} = 0$ for all $i \in T$, $j \in S \setminus \{0, n\}$. Next, we show that $b_{ij} = 0$ for all $i \in S \setminus \{0, n\}$, $j \in T$. Let $a_1 = (i_1, j_1), \ldots, a_p = (i_p, j_p) \in (S \setminus \{0, n\} : T)$ be any non-adjacent arcs, where $p := \frac{c_2-1}{2}$. Associate with each $q \in \{1, \ldots, p\}$ a tight $(0, n)$-path P^q of cardinality c_2 that uses all arcs $a_1, \ldots, a_p$ with the exception of a_q and whose associated coefficients b_{ij} of the remaining arcs are known. For instance, if $q = 1$ then

$$(0, j_1), (j_1, i_2), (i_2, j_2), \ldots, (j_{p-1}, i_p), (i_p, j_p), (j_p, i_1), (i_1, n)$$

is such a path. These paths yield an equality system of the form

$$(E - I)b = 0,$$

where E is the $p \times p$ matrix of all ones, I is the $p \times p$ identity matrix, and b is the vector of variables $b_k = b_{a_k}$ for $1 \leq k \leq p$. Since $E - I$ is a nonsingular matrix, we have that $b_{a_k} = 0$ for $1 \leq k \leq p$. Since the arcs a_j were arbitrarily chosen, it follows that $b_{ij} = 0$ for all $i \in S \setminus \{0, n\}$, $j \in T$. Next, considering the paths $(0, i), (i, j), (j, n)$ for $i \in S \setminus \{0, n\}$, $j \in T$ we see that $b_{jn} = \gamma$ and $b_{0i} = 1 - \gamma$ for some γ. Further, from the paths $(0, i), (i, j), (j, n)$ with $(i, j) \in \tilde{A}_n(T)$ we derive that $b_{ij} = 1 - \gamma$ for all $(i, j) \in \tilde{A}_n(T)$. Finally, from any tight path using an arc $(i, j) \in \tilde{A}_n(S \setminus \{0, n\})$, we derive that $b_{ij} = 1 - \gamma$ for those arcs (i, j). Thus, $b^T x = b_0$ is the equation

$$(1 - \gamma)x(\tilde{A}_n(S)) + (1 - \gamma)x(\tilde{A}_n(T)) + \gamma x(\delta^{\text{in}}(n)) = 1.$$

Subtracting γ times the equation $x(\delta^{\text{in}}(n)) = 1$ and dividing by $1 - \gamma$, we see that $b^T x = b_0$ is equivalent to (3.67). Consequently, (3.67) defines a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$.

Finally, suppose that (ii) holds. Note, inequality (3.67) is equivalent to the *max-cut inequality* $x((S : T)) \leq \lfloor \frac{c_1}{2} \rfloor$ with respect to $P^{(c_1)}_{0,n\text{-path}}(\tilde{D}_n)$. From Theorem 3.17 follows that the latter inequality defines a facet of $P^{(c_1)}_{0,n\text{-path}}(\tilde{D}_n)$, hence also the former. Since there is also a tight $(0, n)$-path of cardinality c_2, it follows the claim by Lemma 3.1 (i). □

Theorem 3.44. Let $D_n = (N, A)$ be the complete digraph on $n \geq 6$ nodes and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $m \geq 3$, $c_1 \geq 2$, $c_m \leq n$, and $c_{p+2} = c_{p+1} + 2 = c_p + 4$ for some $2 \leq p \leq m - 2$. Moreover, let $N = P \,\dot{\cup}\, Q \,\dot{\cup}\, \{r\}$ be a partition of N, where P contains node 1 and satisfies

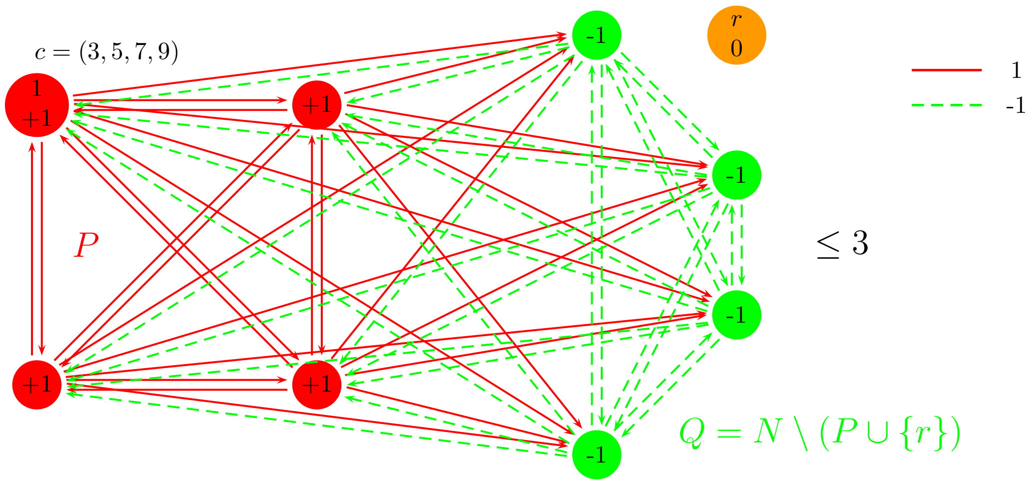


Figure 3.6: Support graph of a modified node induced forbidden cardinality inequality with respect to P^*.

$|P| = c_p + 1 = c_{p+1} - 1$. Then the *modified node induced forbidden cardinality inequality*

$$\sum_{v \in P} x(\delta^{\text{out}}(v)) - \sum_{v \in Q} x(\delta^{\text{out}}(v)) + x((Q : \{r\})) - x((P : \{r\})) \leq c_p \quad (3.68)$$

defines a facet of $P^* = \{x \in P^c_C(D_n) | x(\delta^{\text{out}}(1)) = 1$.

An illustration of these inequalities is given in Figure 3.6. Here, the node set P consists of red nodes, where the bigger node is node 1. Q consists of the green nodes. The coefficients of the missing arcs (the arcs incident with r) are 0. We have the cardinality sequence $c = (3, 5, 7, 9)$, and the cardinality of P is 4, which implies $c_p = 3$. The picture shows that a modified node induced forbidden cardinality inequality is just a node induced forbidden cardinality inequality on $N \setminus \{r\}$.

Proof of Theorem 3.44. The arcs that are incident with node r have coefficients zero. Let C be a cycle that visits node 1 and is of feasible cardinality. If C does not visit node r, C satisfies clearly (3.68), since the restriction of (3.68) to the arc set $A(N \setminus \{r\})$ is an ordinary node induced forbidden cardinality inequality (3.56). When C visits node r and uses at most c_p arcs whose corresponding coefficients are equal to one, then C satisfies also (3.68), since all those coefficients that are not equal to 1 are 0 or -1. So, let C with $|C| \geq c_{p+1}$ visit node r and use as many arcs whose corresponding coefficients are equal to one as possible. That are exactly $|P|$ arcs which are contained in $A(P) \cup (P : Q)$. But then C must use at least one arc in $A(Q) \cup (Q : P)$ whose coefficient is -1. Hence, also in this case C satisfies (3.68), which proves the validity of (3.68).

To show that (3.68) is facet defining, suppose that the equation $b^T x = b_0$ is satisfied by all points that satisfy (3.68) at equality. By Theorem 3.34, we may assume that $b_{1r} = b_{r1} = 0$ and $b_{1i} = 1$ for $i \in N \setminus \{1, r\}$. By considering the c_{p+1}-cycles with respect to $P \cup \{j\}$ for $j \in N \setminus P$, one can show along the lines of the proof of Theorem 3.35 that there are α_k, β_k, $k \in N$, with $b_{ij} = \alpha_i + \beta_j$ for all $(i, j) \in A$, $\alpha_1 = 0$, $\beta_r = 0$, and $\beta_j = 1$. In particular, when $c_p = 2$, the tight 2-cycles $\{(1, i), (i, 1)\}$, $i \in P$ yield $\alpha_k = \alpha_\ell$ for $k, \ell \in P \setminus \{1\}$. Otherwise one can show as in the proof of Theorem 3.35 that $\alpha_k = \alpha_l$ for all $k, l \in P \setminus \{1\}$. Thus, there is κ such that $\alpha_i = \kappa$ for $i \in P$, $i \neq 1$. This in turn implies that there is λ with $\alpha_j = \lambda$ for $j \in Q$ by considering tight c_{p+1}-cycles. Then, the equation $b_{r1} = 0$, a tight cycle of cardinality c_p, and two tight cycles of cardinality c_{p+1}, one visiting node r, the other a node $j \in Q$, yield the equation system

$$\begin{aligned} b_{r1} &= 0 \\ b_0 &= (c_p - 1)(\kappa + 1) + \beta_1 \\ b_0 &= c_p(\kappa + 1) \\ b_0 &= c_p(\kappa + 1) + \lambda + \beta_1 + 1 \end{aligned}$$

which solves to

$$\begin{aligned} b_0 &= c_p(\kappa + 1) \\ \lambda &= -\kappa - 2 \\ \beta_1 &= \kappa + 1 \\ \alpha_r &= -\kappa - 1. \end{aligned}$$

Next, consider for $i \in P \setminus \{1\}$, $j, k \in Q$ a c_{p+2}-cycle C that starts in node 1, then visits all nodes in $P \setminus \{1, i\}$, followed by the nodes j, r, i, k, and finally returns to 1. Since C is tight, we can derive the equation

$$1 + (c_p - 1)(\kappa + 1) + b_j r + (\alpha_r + 1) + (\kappa + 1) + (\lambda + \beta_1) = b_0$$

which solves to $b_{jr} = \kappa$. By considering further tight c_{p+2}-cycles one can deduce that $b_{ri} = -\kappa$ for $i \in Q$ and $b_{jk} = -\kappa - 1$ for $(j, k) \in A(Q)$. Thus, $b^T x = b_0$ is the equation

$$\begin{aligned} x(\delta^{\text{out}}(1) \setminus \{(1, r)\}) - x((Q : \{1\})) \\ +(2\kappa + 1)x((P \setminus \{1\} : \{1\})) \\ +(\kappa + 1) \textstyle\sum_{i \in P \setminus \{1\}} x(\delta^{\text{out}}(i) \setminus \{(i, 1), (i, r)\}) \\ -(\kappa + 1) \textstyle\sum_{i \in Q} x(\delta^{\text{out}}(i) \setminus \{(i, 1), (i, r)\}) \\ -\kappa x(\delta^{\text{out}}(r) \setminus \{(r, 1)\}) + \kappa x(\delta^{\text{in}}(r) \setminus \{(1, r)\}) = c_p(\kappa + 1). \end{aligned}$$

Adding κ times the equations $x(\delta^{\text{out}}(1)) - x(\delta^{\text{in}}(1)) = 0$ and $x(\delta^{\text{out}}(r)) - x(\delta^{\text{in}}(r)) = 0$, we see that $b^T x = b_0$ is equivalent to (3.68), and hence, (3.68) defines a facet. $\square$

3.4.4 Inequalities related to hop constraints

Theorem 3.45. Let $c = (c_1, \ldots, c_m) \in \text{CS}$, and let

$$\tilde{N}_n = \bigcup_{p=0}^{c_m+1} N_i$$

be a partition of $\tilde{N}_n$ such that $N_0 = \{0\}$ and $N_{c_m+1} = \{n\}$. The lifted jump inequality

$$\sum_{i=0}^{c_m-1} \sum_{j=i+2}^{c_m+1} x((N_i : N_j)) - x((N_{c_m-1} \cup N_{c_m} : N_1 \cup N_2)) \geq 1 \qquad (3.69)$$

is valid for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and induces a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if $|N_i| \geq 2$ for $i = 1, \ldots, c_m$.

Proof. This follows immediately from Theorem 3.23, Lemma 3.1, and the fact that $m \geq 2$. $\square$

Theorem 3.46. Let $c = (c_1, \ldots, c_m) \in \text{CS}$, s, t two distinct internal nodes of $\tilde{D}_n = (\tilde{N}_n, \tilde{A}_n)$, P an (s,t)-path of cardinality $c_m - 1$, and $\text{bid}(P) := P \cup \{(i,j) : (j,i) \in P\}$. The cardinality-path inequality

$$\sum_{i \in \dot{P}} x(\delta^{\text{in}}(i)) - x(\text{bid}(P)) \geq 0 \qquad (3.70)$$

is valid for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ and induces a facet of $P^c_{0,n\text{-path}}(\tilde{D}_n)$ if $c_m \in \{4,5\}$ and $n \geq c_m + 2$ or $c_m \geq 6$ and $n \geq 2c_m - 3$.

Proof. This follows immediately from Theorem 3.24, Lemma 3.1, and the fact that $m \geq 2$. $\square$

3.5 Facets of the directed cardinality constrained cycle polytope

In this and the next two sections, we derive facet defining inequalities for related polytopes mentioned in the introduction mostly from facet defining inequalities for the directed cardinality constrained path polytope $P^c_{0,n\text{-path}}(\tilde{D}_n)$.

According to model (3.3), the integer points of the directed cardinality constrained cycle polytope $P_C^c(D)$ defined on a digraph $D = (N, A)$ are characterized by the system

$$\begin{aligned} x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= 0 && \text{for all } i \in N, \\ x(\delta^{\text{out}}(i)) &\leq 1 && \text{for all } i \in N, \\ -x((S : N \setminus S)) + x(\delta^{\text{out}}(i)) + x(\delta^{\text{out}}(j)) &\leq 1 && \text{for all } S \subset N, \\ & && 2 \leq |S| \leq n-2, \\ & && i \in S, j \in N \setminus S, \end{aligned}$$

$$\begin{aligned} x(A) &\geq c_1, \\ x(A) &\leq c_m, \end{aligned}$$

$$\begin{aligned} &(c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) \\ &-(|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \leq c_p(c_{p+1} - |W|) \\ &\text{for all } W \subseteq N \text{ with } c_p < |W| < c_{p+1}, \text{ for some } p \in \{1, \ldots, m-1\}, \end{aligned}$$

$$x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in A.$$

Corollary 3.47. Let $D_n = (N, A)$ be the complete digraph on $n \geq 3$ nodes and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $m \geq 2$ and $c_1 \geq 2$. Then the following statements hold:

(a) The nonnegativity constraint $x_{ij} \geq 0$ defines a facet of $P_C^c(D_n)$.

(b) The degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$ for every $i \in N$.

(c) Let S be a subset of N with $2 \leq |S| \leq n - 2$, let $v \in S$ and $w \in N \setminus S$. The *Two-sided min-cut inequality*

$$x(\delta^{\text{out}}(v)) + x(\delta^{\text{out}}(w)) - x((S : N \setminus S)) \leq 1 \tag{3.71}$$

induces a facet of $P_C^c(D_n)$ if and only if $|S|, |N \setminus S| \geq c_1$ and $c \notin \{(2, 3), (2, n)\}$.

(d) For any $S \subset N$ with $|S|, |N \setminus S| \leq c_1 - 1$, the min-cut inequality

$$x((S : N \setminus S)) \geq 1 \tag{3.72}$$

is valid for $P_C^c(D_n)$ and induces a facet of $P_C^c(D_n)$ if and only if $|S|, |N \setminus S| \geq 2$.

(e) Let S be a subset of N and $j \in N \setminus S$. The one-sided min-cut inequality

$$x((S : N \setminus S)) - x(\delta^{\text{out}}(j)) \geq 0 \tag{3.73}$$

defines a facet of $P_C^c(D_n)$ if and only if $|S| \geq c_1$ and $2 \leq |N \setminus S| \leq c_1 - 1$.

(f) The cardinality bound $x(A) \geq c_1$ defines a facet of $P_C^c(D_n)$ if and only if $c_1 = 3$ and $n \geq 5$ or $4 \leq c_1 \leq n-1$. Analogously, $x(A) \leq c_m$ defines a facet of $P_C^c(D_n)$ if and only if $c_m = 3$ and $n \geq 5$ or $4 \leq c_m \leq n-1$.

(g) Let W be a subset of N with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. The node induced forbidden cardinality inequality (3.56) defines a facet of $P_C^c(D_n)$ if and only if $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n-1$.

(h) Let W be a subset of N such that $c_p < |W| < c_{p+1}$ holds for some $p \in \{1, \ldots, m-1\}$. The cardinality-subgraph inequality (3.59) is valid for $P_C^c(D_n)$ and induces a facet of $P_C^c(D_n)$ if and only if $p+1 < m$ or $c_{p+1} = n = |W|+1$.

(i) Let c_p be even for $1 \leq p \leq m$ and $N = S \,\dot\cup\, T \,\dot\cup\, \{n\}$ a partition of N. The *odd cycle exclusion constraint*

$$x(A(S)) + x(A(T)) + x((T : \{n\})) - x((\{n\} : T)) \geq 0 \tag{3.74}$$

is valid for $P_C^c(D_n)$ and defines a facet of $P_C^c(D_n)$ if and only if (α) $c_1 = 2$ and $|S|, |T| \geq \frac{c_2}{2}$, or ($\beta$) $c_1 \geq 4$ and $|S|, |T| \geq \frac{c_2}{2} - 1$.

(j) Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $N = S \,\dot\cup\, T$ be a partition of N. The *even cycle exclusion constraint*

$$x(A(S)) + x(A(T)) \geq 1 \tag{3.75}$$

is valid for $P_C^c(D_n)$ and defines a facet of $P_C^c(D_n)$ if and only if $|S|, |T| \geq \frac{c_2-1}{2}$.

(k) Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 3$, $c_1 \geq 2$, $c_m \leq n$, $n \geq 6$, and $c_{p+2} = c_{p+1} + 2 = c_p + 4$ for some $2 \leq p \leq m-2$. Moreover, let $N = P \,\dot\cup\, Q \,\dot\cup\, \{r\}$ be a partition of N, with $|P| = c_p + 1 = c_{p+1} - 1$. Then the modified node induced forbidden cardinality inequality (3.68) defines a facet of $P_C^c(D_n)$.

(l) Let $j \in N$, and let

$$N = \bigcup_{p=0}^{c_m} N_p$$

be a partition of N such that $N_0 = \{j\}$. The lifted jump inequality

$$\begin{aligned} \sum_{i=0}^{c_m-2} \sum_{j=i+2}^{c_m} x((N_i : N_j)) - x((N_{c_m-1} \cup N_{c_m} : N_1 \cup N_2)) \\ + \sum_{i=1}^{c_m-1} x((N_i : N_0)) - 2x(\delta^{\text{in}}(j)) \geq -1 \end{aligned} \tag{3.76}$$

is valid for $P_C^c(D_n)$ and induces a facet of $P_C^c(D_n)$ if $|N_i| \geq 2$ for $i = 1, \ldots, c_m$.

(m) Let r, s, t be three distinct nodes of D_n, P an (s,t)-path of cardinality $c_m - 1$ such that $r \notin N(P)$, and $\text{bid}(P) := P \cup \{(i,j) : (j,i) \in P\}$. The cardinality-path inequality

$$\sum_{i \in \dot{P} \cup \{r\}} x(\delta^{\text{in}}(i)) - x(\text{bid}(P)) \geq -1 \tag{3.77}$$

is valid for $P_C^c(D_n)$ and induces a facet of $P_C^c(D_n)$ if $c_m \in \{4,5\}$ and $n \geq c_m+2$ or $c_m \geq 6$ and $n \geq 2c_m - 3$.

Proof. (a) When $n \leq 4$, the statement can be verified using a computer program. When $c = (2,3)$ and $n \geq 5$, we apply Theorem 10 of Hartmann and Özlük [48] which says that $x_{ij} \geq 0$ defines a facet of $P_C^{(p)}(D_n)$ whenever $p \geq 3$ and $n \geq p+1$. Thus, there are $n^2 - 2n$ 3-cycles satisfying $x_{ij} \geq 0$ at equality. Together with Lemma 3.1 applied on these tight 3-cycles and any 2-cycle not using arc (i,j), we get the desired result. The remainder statements of (a) follow by application of Theorem 3.37 and Theorem 3.6.

(b) First, when $c = (2,3)$ one can show along the lines of the proof to Proposition 5 of Balas and Oosten [4] that $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$. Next, when $(2,3) \neq c \neq (2,n)$, the degree constraint can be shown to induce a facet using theorems 3.38 and 3.6. Finally, when $c = (2,n)$, consider any $n^2 - 3n + 2$ affinely independent tours and the $n-1$ additional 2-cycles $\{(i,j),(j,i)\}$ for $j \in N \setminus \{i\}$. Note, all tours satisfy the degree constraint at equation, and hence, these $n^2 - 2n + 1$ constructed points are tight. Moreover, using Lemma 3.2 they can be easily shown to be facet defining for $P_C^c(D_n)$.

(c) Supposing that $c = (2,3)$, the inequality (3.71) is dominated by the nonnegativity constraint $x_{ij} \geq 0$ for any arc $(i,j) \in (S : N \setminus S) \cup (N \setminus S : S)$ that is neither incident with v nor with w. Next, suppose that $c = (2,n)$. Inequality (3.71) is equivalent to the subtour elimination constraint $x(A(S)) \leq |S| - 1$ with respect to the ATSP $P_C^{(n)}(D_n)$. Thus, we have $n^2 - 3n + 1$ tours satisfying (3.71) at equality. But we have only $n-1$ tight 2-cycles, and consequently, (3.71) does not induce a facet. Next, if $|S| \leq c_1 - 1$, then (3.71) is the sum of the valid inequalities $x(\delta^{\text{out}}(v)) - x((S : N \setminus S)) \leq 0$ and $x(\delta^{\text{out}}(w)) \leq 1$. Finally, if $|N \setminus S| \leq c_1 - 1$, then (3.71) is the sum of the inequalities $x(\delta^{\text{out}}(w)) - x((S : N \setminus S)) \leq 0$ and $x(\delta^{\text{out}}(v)) \leq 1$ (cf. Hartmann and Özlük [48, p. 162]).

Suppose that the conditions in (c) are satisfied. First, consider the inequality (3.71) on the polytope $Q := \{x \in P_C^c(D_n) : x(\delta^{\text{out}}(1)) = 1\}$ which is isomorphic to the path polytope $P_{0,n\text{-path}}^c(\tilde{D}_n)$. Then, (3.71) is equivalent to the one-sided min-cut inequality (3.64) which defines a facet of Q by Theorem 3.40. Thus, also (3.71) defines a facet of Q. Now, by application of Theorem 3.6 on Q and (3.71) we obtain the desired result. (When $c_1 \geq 4$, then the statement can be proved also with Theorem 14 of Hartmann and Özlük [48].

(d) Assuming $|S| = 1$ or $|N \setminus S| = 1$ implies that (3.72) is an implicit equation. So, let $|S|, |N \setminus S| \geq 2$ which implies that $c_1 \geq 3$. From Theorem 3.41 follows that (3.72) defines a facet of $Q := \{x \in P_C^c(D_n) : x(\delta^{\text{out}}(i)) = 1\}$, and hence, by Theorem 3.6, it defines also a facet of $P_C^c(D_n)$.

(e) When $|N \setminus S| \geq c_1$, (3.73) is obviously not valid. When $|N \setminus S| = 1$, (3.73) is the flow constraint $x(\delta^{\text{in}}(j)) - x(\delta^{\text{out}}(j)) = 0$. When $|S| \leq c_1 - 1$ and $|N \setminus S| \leq c_1 - 1$, (3.73) is the sum of the valid inequalities $x((S : N \setminus S)) \geq 1$ and $-x(\delta^{\text{out}}(j)) \geq -1$.

Suppose that $|S| \geq c_1$ and $2 \leq |N \setminus S| \leq c_1 - 1$. Then in particular $c_1 \geq 3$ holds. For any node $i \in S$, (3.73) defines a facet of $Q := \{x \in P_C^c(D_n) : x(\delta^{\text{out}}(i)) = 1\}$, by Theorem 3.40. Applying Theorem 3.6 we see that therefore (3.73) defines also a facet of $P_C^c(D_n)$.

(f) Since $\dim\{x \in P_C^c(D_n) : x(A) = c_i\} = \dim P_C^{(c_i)}(D_n)$, the claim follows directly from Theorem 1 of Hartmann and Özlük [48].

(g)-(i) Necessity can be proved as in the corresponding part of the proof to Theorem 3.35 (3.36, 3.42) while suffiency can be shown by applying Theorem 3.6 to Theorem 3.35 (3.36, 3.42).

(j) By Theorem 15 of Hartmann and Özlük [48], (3.75) defines a facet of $P_C^{(c_1)}(D_n)$. Moreover, the cardinality conditions for S and T ensure that there is a tight cycle of cardinality c_2, and hence, by Lemma 3.1, (3.75) defines a facet of $P_C^c(D_n)$.

(k)-(m) Apply Theorem 3.6 to the Theorems 3.44, 3.45, and 3.46, respectively. □

Next, we investigate some inequalities that induce facets of $P_C^c(D_n)$ but have no analogue for $P^c_{0,n\text{-path}}(\tilde{D}_n)$. We start with two preliminary results. The first one is an adaption of Theorem 3.9 for $P_C^c(D_n)$.

Theorem 3.48. Let $\alpha^T x \geq \alpha_0$ be a valid inequality for $P_C^c(D_n)$ and let T be a spanning tree of D. Then for any specified set of coefficients β_{ij} for the arcs $(i,j) \in T$, there is an equivalent inequality $\bar{\alpha}^T x \geq \alpha_0$ for $P_C^c(D_n)$ such that $\bar{\alpha}_{ij} = \beta_{ij}$ for $(i,j) \in T$. □

Lemma 3.49. Let $\gamma^T x \leq \gamma_0$ define a facet F of $P_C^c(D_n)$ which is not equivalent to a nonnegativity constraint. Then, for any two arcs $a, b \in A$, $a \neq b$, there are at least two points $x^a, x^b \in F$ such that $x^a_a > 0$, $x^a_b = 0$ and $x^b_b > 0$, $x^b_a = 0$.

Proof. Since $\gamma^T x \leq \gamma_0$ is not equivalent to a nonnegativity constraint, there are points $x^a, x^b \in F$ with $x^a_a > 0$ and $x^b_b > 0$. Assume that $x_a = x_b$ for all $x \in F$. Then exist $\lambda > 0$ and $x^* \in P^c_C(D_n) \setminus F$ such that

(i) $\delta^T x := \gamma^T x + \lambda x_a - \lambda x_b \leq \gamma_0$ is valid for $P^c_C(D_n)$ and

(ii) $\gamma^T x^* + \lambda x^*_a - \lambda x^*_b = \gamma_0$.

Since $\delta^T x = \gamma_0$ for all $x \in F$, it follows $F \subsetneq \{x \in P^c_C(D_n) : \delta^T x = \gamma_0\}$. Hence, F is not a facet of $P^c_C(D_n)$, a contradiction. □

Linear ordering constraints

At the beginning of this chapter we introduced the inequality

$$x(A) \geq 2$$

as part of an integer programming formulation for the ordinary cycle polytope $P_C(D)$ defined on a directed graph $D = (N, A)$. Of course, this inequality is valid. However, it can be substituted by a whole class of valid inequalities. For simplicity assume that D is the complete digraph on n nodes. Then, for any permutation π of N, the inequality

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{\pi(i),\pi(j)} \geq 1 \tag{3.78}$$

is valid for $P_C(D)$, see Balas and Ooosten [4]. Moreover, the inequality $x(A) \geq 2$ is the sum of the inequalities (3.78) associated with any permutation π and its reversal.

Inequalities (3.78) can be generalized for the cardinality constrained cycle polytope $P^c_C(D)$ as follows. If C is a directed cycle of length at least c_1 and $N = \bigcup_{i=1}^r N_i$ a partition of N such that the subsets N_i are of cardinality less than c_1, then the inequality

$$\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} x((N_i : N_j)) \geq 1 \tag{3.79}$$

says that C uses at least one arc in $\bigcup_{1 \leq i < j \leq r} (N_i : N_j)$.

Inequalities (3.78) and (3.79) are called *linear ordering constraints*. The next two theorems show that inequalities (3.78) and (3.79) define usually facets of $P^c_C(D_n)$.

Theorem 3.50. Let $D_n = (N, A)$ be the complete digraph on n nodes and $c = (c_1, \ldots, c_m)$ a cardinality sequence. For any permutation π of N, the inequality (3.78) defines a facet of $P_C^c(D_n)$ if and only if one of the following conditions holds:

(i) $P_C^c(D_n) = P_C(D_3)$;

(ii) $k \geq 4$, $c = \{2, k\}$, and $n \geq 2k - 2$;

(iii) $m \geq 3$, $c_1 = 2$, and $n \geq 2c_2 - 3$.

Proof. Assume w.l.o.g. that $(\pi(1), \ldots, \pi(n)) = (1, \ldots, n)$, that is, we consider the inequality

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} \geq 1 \tag{3.80}$$

Furthermore, set $A^* := \{(3, 2), \ldots, (n, n-1)\}$.

Sufficiency.

(i): Constraint (3.80) defines obviously a facet of $P_C(D_3)$.

(ii) and (iii): Let $b^T x = b_0$ be an equation that is satisfied by all cycles $C \in \mathcal{C}(D_n) \cap \mathrm{CHS}^c(A)$ satisfying (3.80) at equality.

Due to Theorem (3.48) we may assume that $b_{i1} = 0$ for $i = 2, \ldots, n$. Moreover, since every 2-cycle satisfies (3.80) at equality, we derive $b_{1i} = b_0$ for $i = 2, \ldots, n$.

Next, consider the coefficients b_a, $a \in A^*$. If (ii) is true, we consider the k-cycles $(n-1, n-2, \ldots, n-k+3, v+1, v, 1, n-1)$ and $(n, n-1, \ldots, n-k+3, v+1, 1, n)$ for $v = 2, \ldots, k-1$. We obtain $b_{v+1,v} = b_{n,n-1}$, $v = 2, \ldots, k-1$, and thus $b_{32} = \cdots = b_{k,k-1}$. The cycle $(k, k-1, \ldots, 1, k)$ yields then $b_{32} = \cdots = b_{k,k-1} = 0$, since $b_{1k} = b_0$ and $b_{21} = 0$, and with the cycles $(v, v-1, \ldots, v-k+2, 1, v)$, $v = k+1, \ldots, n$, we conclude successively $b_{v,v-1} = 0$, $v = k+1, \ldots, n$.

If (iii) is true, we consider the k-cycles

$$(n-1, n-2, \ldots, n-k+3, v+1, v, 1, n-1)$$

and

$$(n, n-1, \ldots, n-k+3, v+1, 1, n)$$

for $v = 2, \ldots, k-2$. We obtain

$$b_{32} = \cdots = b_{k-1,k-2} = b_{n,n-1}. \tag{3.81}$$

Furthermore, we derive from the cycles $(w, w-1, \ldots, w-k+4, 3, 2, 1, w)$ and $(w+1, w, \ldots, w-k+4, 3, 1, w+1)$ the equations

$$b_{32} = b_{k+1,k} = \ldots = b_{n,n-1}, \qquad w = k, \ldots, n-1. \tag{3.82}$$

It follows from (3.81) and (3.82) that

$$b_{32} = \cdots = b_{k-1,k-2} = b_{k+1,k} = \cdots = b_{n,n-1}. \tag{3.83}$$

Since $c_p > k$ for some p, we derive from the cycles $(k, k-1, \ldots, 1, k)$ and $(m, m-1, \ldots, 1, m)$

$$\begin{aligned} &\sum_{i=k}^{m-1} b_{i+1,i} &&= 0 && \\ \overset{(3.82)}{\Longrightarrow}\quad & b_{i+1,i} &&= 0, && i = k, \ldots, m-1 \\ \overset{(3.83)}{\Longrightarrow}\quad & b_{i+1,i} &&= 0, && i = 2, \ldots, k-2, k, \ldots, m-1, \end{aligned}$$

and again the cycle $(k, k-1, \ldots, 1, k)$ yields also $b_{k,k-1} = 0$.

Next, consider the coefficients b_{uv} and b_{vu} for $1 < u < v \leq n$ such that $2 \leq v-u \leq n-k+2$. Since the node set $\{1, \ldots, u, v, \ldots, n\}$ is of cardinality at least k, there is a k-cycle containing (v, u) whose remaining arcs are in

$$(A^* \setminus \{(v, v-1), \ldots, (u+1, u)\}) \cup \{(2, 1), \ldots, (u, 1)\} \cup \{(1, v), \ldots, (1, n)\}.$$

Clearly, it follows immediately that $b_{vu} = 0$, and hence we conclude $b_{uv} = b_0$.

Finally, consider the coefficients b_{uv} and b_{vu} for $1 < u < v \leq n$ such that $n-k+3 \leq v-u \leq n$. The node set $\{u, u+1, \ldots, v\}$ is of cardinality at least $k+1$. From the k-cycle $(v, v-1, \ldots, v-k+2, u, v)$ we obtain $b_{uv} = b_0$, since $b_{v,v-1} = \cdots = b_{v-k+3,v-k+2} = b_{v-k+2,u} = 0$. Moreover, this implies $b_{vu} = 0$.

To summarize, we have shown that the equation $b^T x = b_0$ is equivalent to (3.80). This proves that (3.80), and hence, (3.78), is facet defining.

Necessity. For $2 \leq n \leq 3$ the statement is obviously true. Hence let $n \geq 4$, and let us suppose, for the sake of contradiction, that (ii) or (iii) is not true.

a) Assume that $c_1 > 2$.

Then there is no point $x \in P_C^c(D_n)$ satisfying (3.80) with $x_{12} > 0$, a contradiction.

b) Assume that $m = 1$.

Then follows $c = (2)$ and $P_C^c(D_n) = P_C^{(2)}(D_n)$. Thus, every $x \in P_C^c(D_n)$ is tight with respect to (3.80), that is, (3.80) is an implicit equation, a contradiction. Consequently, $m \geq 2$.

c) Assume that $k \leq n \leq 2k - 4$.

Consider the arcs (u, v) and (v, u) given by $u := 2$ and $v := n - k + 4$. Since $2 \leq v - u = n - k + 2 \leq k - 2$, there is neither a c_p-cycle containing (u, v) nor a c_q-cycle containing (v, u) whose incidence vectors satisfy (3.80) at equality for all p, q, with $c_p, c_q \geq k$, that is, only the 2-cycle (u, v, u) is tight. But this is a contradiction to Lemma 3.49.

d) Suppose that $n = 2k - 3$ and $c = (2, k)$.

We consider the polytope P^* defined by

$$P^* := \operatorname{conv}(P_C^c(D_n) \cup C^*)$$

where C^* is the triangle $(1, 3, 2, 1)$. Note that $\dim P^* = \dim P_C^c(D_n)$. First we will show that inequality (3.80) defines a facet F^* of P^* and then that it is not facet defining for $P_C^{(2,k)}(D_n)$.

Let us assume that the equation $b^T x = b_0$ is satisfied by all $x \in P^*$ that satisfy (3.80) at equality. As is easily seen, it follows $b_{1v} = b_0$, $v = 2, \dots, n$. In order to show $b_{v+1,v} = 0$, $v = 2, \dots, n-1$, consider the k-cycles

$$\begin{array}{l} (n-1, n-2, \dots, n-k+3, v+1, v, 1, n-1), \\ (n, n-1, \dots, n-k+3, v+1, 1, n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v = 2, \dots, k-2. \end{array}$$

We obtain $b_{n,n-1} = b_{32} = b_{43} = \cdots = b_{k-1,k-2}$. Now the triangle $(1, 3, 2, 1)$ yields $b_{32} = 0$, and hence $b_{n,n-1} = b_{32} = b_{43} = \cdots = b_{k-1,k-2} = 0$. Further, the cycle $(k, k-1, \dots, 1, k)$ yields $b_{k,k-1} = 0$.

The remaining coefficients can be determined as in (ii) and (iii) of the part *Sufficiency*, since all arguments hold also for $n = 2k - 3$ and $c = (2, k)$. Hence, F^* is a facet of P^*.

Now we will prove that (3.80) is not facet defining for $P_C^c(D_n)$, $n = 2k - 3$, by showing that $C^* \notin \operatorname{aff}(F)$. The crucial point is that the k-cycles satisfying (3.80) at equality are linearly independent of the 2-cycles for $n = 2k - 3$, while for $n \geq 2k - 2$ this is no longer true.

Let us denote by $F(k)$ the k-cycles whose incidence vectors satisfy (3.80) at equality. Suppose, for the sake of contradiction, that

$$\begin{aligned}\chi^{(1,3,2,1)} &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \lambda_{ij}\chi^{(i,j,i)} + \sum_{C\in F(k)} \mu_C\chi^C \\ &\quad \text{and } \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \lambda_{ij} + \sum_{C\in F(k)} \mu_C = 1.\end{aligned}$$

Since the arc $(1,3)$ is contained only in the cycles $(1,3,2,1)$ and $(1,3,1)$, it follows $\lambda_{13} = 1$. However, $\chi_{ij}^{(1,3,2,1)} = 0$ and $\{C \in F(k) : (i,j) \in C\} = \varnothing$ for all $1 \le i < j \le n$, $(i,j) \ne (1,3)$, with $j - i \le k-1$, and hence $\lambda_{ij} = 0$ for those components ij. Similarly, it follows $\lambda_{ij} = 0$ for all $1 \le i < j \le n$ with $j - i \ge k$, since $\chi_{ij}^{(1,3,2,1)} = 0$ and $\{C \in F(k) \mid (i,j) \in C\} = \varnothing$ for those components ij. Thus,

$$\begin{aligned} \chi^{(1,3,2,1)} &= \chi^{(1,3,1)} + \sum_{C\in F(k)} \mu_C\chi^C \\ \Leftrightarrow \quad e_{32} + e_{21} - e_{31} &= \sum_{C\in F(k)} \mu_C\chi^C \\ \Rightarrow \quad 1 &= 1^T \sum_{C\in F(k)} \mu_C\chi^C \\ \Rightarrow \quad 1 &= \sum_{C\in F(k)} \mu_C \underbrace{1^T\chi^C}_{=k} \\ \Rightarrow \quad 1 &= k \underbrace{\sum_{C\in F(k)} \mu_C}_{=0} \\ \Rightarrow \quad 1 &= 0. \qquad \text{Contradiction!}\end{aligned}$$

□

Theorem 3.51 (cf. Theorem 16 of Hartmann and Özlük [48])**.** Let $c = (c_1, \dots, c_m)$ be a cardinality sequence such that $c_1 \ge 3$ and $m \ge 2$. Furthermore, let $r \in \mathbb{N}$, $r \ge 3$, and $N = \bigcup_{i=1}^{r} N_i$ be a partition of N with $|N_i| < c_1$ for $1 \le i \le r$. Then, the following statements are equivalent.

(i) The inequality (3.79) defines a facet of $P_C^c(D_n)$.

(ii) $|N_1| + |N_r| \ge c_1$ and $|N_i| + |N_{i+1}| \ge c_1$ for $i = 1, \dots, r-1$.

Proof.
"$(i) \Rightarrow (ii)$:" This can be shown along the lines of the proof to Theorem 16 of Hartmann and Özlük [48].

"$(ii) \Rightarrow (i)$:" This follows directly from Theorem 16 of Hartmann and Özlük [48] and Lemma 3.1. $\square$

Corollary 3.52. Let $D_n = (N, A)$ be the complete digraph on $n \geq 5$ nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $m \geq 2$. Next,

(a) let $3 \leq c_m < n$, and let $N = \bigcup_{i=1}^{r} N_i$ be a partition of N with $|N_i| < c_m$ for $i = 1, \dots, r$ for some $r \geq 3$ such that conditions (ii) in Theorem 3.51 are satisfied;

(b) let $c_1 \leq \gamma$ and $q = \max\{p \in \{1, \dots . m-1\} : c_p \leq \gamma\}$, where $\gamma := \max\{|N_i| : i = 1, \dots, r\}$.

Then, the inequality

$$x(A) - (c_m - c_q) \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} x((N_i : N_j)) \leq c_q \tag{3.84}$$

defines a facet of $P_C^c(D_n)$.

Proof. Consider inequality (3.79).

$$\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} x((N_i : N_j)) \geq 1.$$

Due to (a), (3.79) induces a facet of $P_C^{(c_m)}(D_n)$. Multiplying (3.79) with $-(c_m - c_q)$ and adding the equation

$$x(A) = c_m,$$

we see that (3.79) is equivalent to (3.84) with respect to $P_C^{(c_m)}(D_n)$.

Thus, it remains to show that (3.84) is valid and that exists a tight cycle $\mathcal{C}(D_n) \cap \mathrm{CHS}^{(c_1, \dots, c_{m-1})}(A)$, by Lemma 3.1. Both is guaranteed by condition (b). The incidence vectors of all cycles of cardinality at most c_q satisfy the inequality $x(A) \leq c_q$ and hence, all the more (3.84). Next, the incidence vectors of all cycles of cardinality greater than c_q satisfy the cardinality bound $x(A) \leq c_m$ and (3.79). Thus, they also satisfy (3.84). Finally, $c_1 \leq \gamma$ guarantees that the set $Q := \{p \in \{1, \dots . m-1\} : c_p \leq \gamma\}$ is nonempty. Since $q \in Q$, this implies that there is a c_q-cycle C such that $C \subseteq A(N_\ell)$ for some ℓ, and since $\chi^C(A) = c_q$, C is tight with respect to (3.84). $\square$

Note that inequality (3.84) can be found by applying standard sequential lifting to (3.79) and $P_C^{(c_m)}(D_n)$.

3.6 Facets of the undirected cardinality constrained cycle polytope

Let $G = (N, E)$ be a graph and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $3 \leq c_1 < \cdots < c_m \leq n$. As is easily seen, the integer points of the undirected cardinality constrained cycle polytope $P_C^c(G)$ are characterized by the system

$$
\begin{aligned}
y(\delta(j)) &\leq 2 && \text{for all } j \in N,\\
y(\delta(j) \setminus \{e\}) - y_e &\geq 0 && \text{for all } j \in N,\ e \in \delta(j),\\
y(\delta(i)) + y(\delta(j)) - y((S : N \setminus S)) &\leq 2 && \text{for all } S \subset N,\\
& && i \in S, j \in N \setminus S,\\
y(E) &\geq c_1, &&\\
y(E) &\leq c_m, &&\\
(c_{p+1} - |W|) \sum_{i \in W} y(\delta(i)) & &&\\
-(|W| - c_p) \sum_{i \in N \setminus W} y(\delta(i)) &\leq 2c_p(c_{p+1} - |W|) &&\\
\text{for all } W \subset N \text{ with } c_p < |W| < c_{p+1} & \text{ for some } p \in \{1, \ldots, m-1\}, &&\\
x_e &\in \{0, 1\} && \text{for all } e \in E.
\end{aligned}
\tag{3.85}
$$

In what follows, we assume that $m \geq 2$ and that the undirected cardinality constrained cycle polytope is defined on the complete graph K_n on n nodes, that is, $G = K_n$. It was shown in [58] and [64] that $\dim P_C^{(p)}(K_n) = |E| - 1$ for $3 \leq p \leq n-1$ and $n \geq 5$. Thus, it is easy to verify that $\dim P_C^c(K_n) = |E| = n(n-1)/2$ for all $n \geq 4$, since $m \geq 2$. Note, in case of $n = 4$, $P_C^c(K_n) = P_C(K_n)$, and by Theorem 2.3 of Bauer [9], $\dim P_C(K_4) = 6 = |E|$.

Facet defining inequalities for $P_C^c(K_n)$ can be derived directly from the inequalities mentioned in Corollary 3.47 (b)-(h), since these inequalities are equivalent to symmetric inequalities. Recall that an inequality $c^T x \leq \gamma$ with $c \in \mathbb{R}^A$ is symmetric if $c_{ij} = c_{ji}$ for all $i < j$. Furthermore, the inequality $c^T x \leq \gamma$ is equivalent to a symmetric inequality if the system $t_i - t_j = c_{ij} - c_{ji}$ is consistent, see Hartmann and Özlük [48] and Boros et al. [15].

Corollary 3.53. Let $K_n = (N, E)$ be the complete graph on $n \geq 3$ nodes and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $m \geq 2$ and $c_1 \geq 3$. Then holds:

(a) For any $e \in E$, the nonnegativity constraint $y_e \geq 0$ defines a facet of $P_C^c(K_n)$ if and only if $n \geq 5$.

(b) The degree constraint $y(\delta(i)) \leq 2$ defines a facet of $P_C^c(K_n)$ for every

$i \in N$.

(c) Let S be a subset of N with $c_1 \leq |S| \leq n - c_1$, let $v \in S$ and $w \in N \setminus S$. Then, the two-sided min-cut inequality

$$y(\delta(v)) + y(\delta(w)) - y((S : N \setminus S)) \leq 2 \tag{3.86}$$

induces a facet of $P_C^c(K_n)$.

(d) For any $S \subset N$ with $|S|, |N \setminus S| \leq c_1 - 1$, the min-cut inequality

$$y((S : N \setminus S)) \geq 2 \tag{3.87}$$

is valid for $P_C^c(K_n)$ and induces a facet of $P_C^c(K_n)$ if and only if $|S|, |N \setminus S| \geq 2$.

(e) Let S be a subset of N such that $|N \setminus S| < c_1$ and $j \in N \setminus S$. The one-sided min-cut inequality

$$y((S : N \setminus S)) - y(\delta(j)) \geq 0 \tag{3.88}$$

is valid for $P_C^c(K_n)$ and defines a facet of $P_C^c(K_n)$ if and only if $|S| \geq c_1$ and $|N \setminus S| \geq 2$.

(f) The cardinality bound $y(E) \geq c_1$ defines a facet of $P_C^c(K_n)$. The cardinality bound $y(E) \leq c_m$ defines a facet of $P_C^c(K_n)$ if and only if $c_m < n$.

(g) Let W be a subset of N with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. The node induced forbidden cardinality inequality

$$(c_{p+1} - |W|) \sum_{i \in W} y(\delta(i)) - (|W| - c_p) \sum_{i \in N \setminus W} y(\delta(i)) \leq 2c_p(c_{p+1} - |W|) \tag{3.89}$$

defines a facet of $P_C^c(K_n)$ if and only if $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n - 1$.

(h) Let W be a subset of N such that $c_p < |W| < c_{p+1}$ holds for some $p \in \{1, \ldots, m - 1\}$. The cardinality-subgraph inequality

$$2y(E(W)) - (|W| - c_p - 1)y((W : N \setminus W)) \leq 2c_p \tag{3.90}$$

is valid for $P_C^c(K_n)$ and induces a facet of $P_C^c(K_n)$ if and only if $p + 1 < m$ or $c_{p+1} = n = |W| + 1$.

(i) Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $N = S \,\dot\cup\, T$ be a partition of N. The even cycle exclusion constraint

$$y(E(S)) + y(E(T)) \geq 1 \tag{3.91}$$

is valid for $P_C^c(K_n)$ and defines a facet of $P_C^c(K_n)$ if and only if $|S|, |T| \geq \frac{c_2 - 1}{2}$.

(j) Let r, s, t be three distinct nodes of K_n and P an undirected (s, t)-path of cardinality $c_m - 1$ such that $r \notin N(P)$. The cardinality-path inequality

$$\sum_{i \in \dot{P} \cup \{r\}} \frac{1}{2} y(\delta(i)) - y(P) \geq -1 \tag{3.92}$$

is valid for $P_C^c(K_n)$ and induces a facet of $P_C^c(K_n)$ if $c_m \in \{4, 5\}$ and $n \geq c_m + 2$ or $c_m \geq 6$ and $n \geq 2c_m - 3$.

Proof. (a) When $n \leq 5$ the statement can be verified using a computer program. When $n \geq 6$, the claim follows from Proposition 2 of Kovalev, Maurras, and Vaxés [58], Proposition 2 of Maurras and Nguyen [64], and the fact that $m \geq 2$.

(b)-(j) All directed inequalities occurring in Corollary 3.47 (b)-(h) and (j) are equivalent to symmetric inequalities. For example, the degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ is equivalent to $x(\delta^{\text{out}}(i)) + x(\delta^{\text{in}}(i)) \leq 2$. Via the identification $y(\delta(i)) \cong x(\delta^{\text{out}}(i)) + x(\delta^{\text{in}}(i))$ we see that $y(\delta(i)) \leq 2$ defines a facet of $P_C^c(K_n)$ if $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$.

Necessity can be shown with similar arguments as for the directed counterparts of these inequalities. □

Note that if $|N \setminus S| = 2$, the inequalities in (e) are equivalent to the parity constraints

$$y(\delta(j) \setminus \{e\}) - y_e \geq 0 \qquad (j \in N, e \in \delta(j))$$

mentioned in the IP-model (3.85).

The odd cycle exclusion constraints (3.74), the modified node induced forbidden cardinality inequalities (3.68), and the lifted jump inequalities (3.76) from Corollary 3.47 are not symmetric nor equivalent to symmetric inequalities. Hence, we did not derive counterparts of these inequalities for $P_C^c(K_n)$. Of course, given a valid inequality $c^T x \leq c_0$ for $P_C^c(D_n)$, one obtains a valid inequality $\tilde{c}^T y \leq 2c_0$ for $P_C^c(K_n)$ by setting $\tilde{c}_{ij} := c_{ij} + c_{ji}$ for $i < j$. However, it turns out that the counterparts of these two classes of inequalities are irrelevant for a linear description of $P_C^c(K_n)$.

3.7 FACETS OF THE UNDIRECTED CARDINALITY CONSTRAINED PATH POLYTOPE

Let $G = (N, E)$ be a graph with node set $N = \{0, 1, \ldots, n\}$ and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $1 \leq c_1 < \ldots < c_m \leq n$. The integer points of the undirected cardinality constrained path polytope $P^c_{0,n\text{-path}}(G)$ are characterized by the system

$$
\begin{array}{rlr}
y(\delta(0)) = 1, & & \\
y(\delta(n)) = 1, & & \\
y(\delta(j)) \leq 2 & \text{for all } j \in N \setminus \{0,n\}, & \\
y(\delta(j) \setminus \{e\}) - y_e \geq 0 & \text{for all } j \in N \setminus \{0,n\}, e \in \delta(j), & \\
y((S : N \setminus S)) \geq y(\delta(j)) & \text{for all } S \subset N,\ 0,n \in S,\ j \in N \setminus S, & \\
y(E) \geq c_1, & & \\
y(E) \leq c_m, & & \\
(c_{p+1} - |W|) \sum_{i \in W} y(\delta(i)) & & (3.93) \\
\quad -(|W| - c_p) \sum_{i \in N \setminus W} y(\delta(i)) \leq 2c_p(c_{p+1} - |W|) & & \\
\text{for all } W \subset N \text{ with } 0,n \in W,\ c_p < |W| - 1 < c_{p+1} & & \\
\text{for some } p \in \{1, \ldots, m-1\}, & & \\
x_e \in \{0,1\} & \text{for all } e \in E. &
\end{array}
$$

Recall that the term $y((S : T))$ is short for $\sum_{i \in S} \sum_{j \in T} y_{ij}$.

Theorem 3.54. Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \ldots, n\}$, $n \geq 3$, and let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 2$ and $c_1 \geq 2$. Then

$$\dim P^c_{0,n\text{-path}}(K_{n+1}) = |E| - 3.$$

Proof. All points $y \in P^c_{0,n\text{-path}}(K_{n+1})$ satisfy the equations

$$y_{0n} = 0, \tag{3.94}$$
$$y(\delta(0)) = 1, \tag{3.95}$$
$$y(\delta(n)) = 1. \tag{3.96}$$

Thus, the dimension of $P^c_{0,n\text{-path}}(K_{n+1})$ is at most $|E| - 3$. When $4 \leq c_i < n$ for some $i \in \{1, \ldots, m\}$, then the statement is implied by Theorem 3.31, saying that $\dim P^{(c_i)}_{0,n\text{-path}}(K_{n+1}) = |E| - 4$, and the fact that $m \geq 2$. So, assume that $c = (2,3)$, $c = (2,n)$, $c = (3,n)$, or $c = (2,3,n)$.

When $c = (2,3)$, assume that the equation $by = b_0$ is satisfied by $[0,n]$-paths of cardinality 2 and 3. Then the equations

$$
\begin{array}{llllll}
y_{0i} & + & y_{ij} & + & y_{jn} & = 0, \\
y_{0j} & + & y_{ij} & + & y_{in} & = 0, \\
y_{0i} & & & + & y_{in} & = 0, \\
y_{0j} & & & + & y_{in} & = 0
\end{array}
$$

imply $b_{ij} = 0$, $b_{0i} = b_{0j}$, and $b_{in} = b_{jn}$ for $1 \le i, j \le n-1$. Thus, $b^T x = b_0$ is a linear combination of the equations (3.94)-(3.96).

When $c = (2, n)$, $c = (3, n)$, or $c = (2, 3, n)$, the statement can be verified using Theorem 3.31 saying that $\dim P^{(n)}_{0,n\text{-path}}(K_{n+1}) = |E| - n - 2$. Thus, there are $|E| - n - 1$ linearly independent Hamiltonian paths. Via the approach on the path-and-loops polytope defined on an undirected graph we see that indeed $\dim P^c_{0,n\text{-path}}(K_{n+1}) = |E| - 3$. $\square$

In what follows, we confine ourselves to the set CS of cardinality sequences $c = (c_1, \ldots, c_m)$ with $m \ge 2$, $c_1 \ge 2$, and $c \ne (2, 3)$. Next, we show that the nonnegativity constraints $y_e \ge 0$ define facets of $P^c_{0,n\text{-path}}(K_{n+1})$. Since this requires for those path polytopes with $c \in \{(3, n), (2, 3, n)\}$ a higher amount of technical detail we start with a preparing Lemma and a following definition.

Lemma 3.55. Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \ldots, n\}$, $n \ge 4$. For any $e \in E, e \ne [0, n]$, the nonnegativity constraint $y_e \ge 0$ defines a facet of $P^{(n)}_{0,n\text{-path}}(K_{n+1})$ if and only if e is incident to 0 or n and $n \ge 4$ or e is not incident to both 0 and n and $n \ge 5$.

Proof. Let $n \ge 5$ and $e = [i, j]$ be an edge whose endnodes are not incident with 0 and n, say w.l.o.g. $e = [n-2, n-1]$. Consider the Hamiltonian $[n-2, n-1]$-path polytope $P^{(n-2)}_{n-2,n-1\text{-path}}(G)$ defined on the complete subgraph $G = (N', E')$ induced by the node set $N' := \{1, \ldots, n-1\}$. Its dimension is $|E'| - n$ by Theorem 3.31. Thus, there are $|E'| - n + 1$ linearly independent Hamiltonian $[n-1, n-2]$-paths in G that can be extended to Hamiltonian $[0, n]$-paths in K_{n+1} by adding to each path the edges $[0, n-2]$ and $[n-1, n]$. Their incidence vectors, say X^r, $r = 1, \ldots, |E'| - n + 1$, are linearly independent and satisfy $x^r_{n-1,n-2} = 0$ by construction. But they satisfy also $x^r_{0,n-1} = x^r_{n-2,n} = 0$ and $x^r_{0,k} = x^r_{k,n} = 0$ for $k = 1, \ldots, n-3$. Next, denote by $Y^{[0,k],[l,n]}$ the incidence vector of a Hamiltonian $[0, n]$-path that does not use edge $[n-2, n-1]$, starts with $[0, k]$, and ends with $[l, n]$. Then, it is easy to see that the points $Y^{[0,n-1],[1,n]}$, $P^{[0,1],[n-2,n]}$, and $P^{[0,i],[n-1,n]}$, $P^{[0,n-2],[i,n]}$ for $i = 1, \ldots, n-3$ are linearly independent, and they are also linearly independent of the points X^r. Since it is a total of $(|E'| - n + 1) + (2n - 4) = |E| - n - 2$ points, the inequality $y_{n-2,n-1} \ge 0$ defines a facet of $P^{(n)}_{0,n\text{-path}}(K_{n+1})$. When $n \le 4$, then it follows immediately that $y_{n-2,n-1} \ge 0$ is not facet defining.

When the edge e is incident to 0 or n, then a simpler construction starting with Hamiltonian $[i, j]$-paths of the complete subgraph $G = (N', E')$ induced by the node set $N' := (N \setminus \{0, n\}) \cup \{i, j\}$ yields the desired result. $\square$

The *undirected cardinality constrained* $[0,n]$*-path-and-loops polytope* is the polytope

$$P^c_{PL}(K_{n+1}) := \left\{ (y,z) \in P^c_{0,n\text{-path}}(K_{n+1}) \times \mathbb{R}^{n-1} \;\middle|\; \begin{array}{l} z_i = 2 - y(\delta(i)), \\ i = 1, \ldots, n-1 \end{array} \right\}.$$

Some points $y^1, \ldots, y^p \in P^c_{0,n\text{-path}}(K_{n+1})$ are affinely independent if and only if the corresponding points $(y^1, z^1), \ldots, (y^p, z^p) \in P^c_{PL}(K_{n+1})$ are affinely independent.

Theorem 3.56. Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \ldots, n\}$ with $n \geq 4$, and let $c = (c_1, \ldots, c_m) \in \text{CS}$ be a cardinality sequence. For any $e \in E, e \neq [0,n]$, the nonnegativity constraint $y_e \geq 0$ defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $c \neq (2,n)$ or $c = (2,n)$ and e is an internal edge.

Proof. When $4 \leq c_i < n$ for some $i \in \{1, \ldots, m\}$, then the claim follows from Theorem 3.33 and the fact that $m \geq 2$. Otherwise, $c = (2,n)$, $c = (3,n)$, or $c = (2,3,n)$ for $n \geq 4$. By Lemma 3.55, $y_e \geq 0$ defines a facet of $P^{(n)}_{0,n-\text{path}}(K_{n+1})$. Thus, there are $|E| - n - 2$ linearly independent points $y^r \in P^{(3,n)}_{0,n-\text{path}}(K_{n+1})$ satisfying $y^r_e = 0$ and $1^T y^r = n$. Considering the corresponding points in the path-and-loops polytope $P^{(3,n)}_{PL}(K_{n+1})$, we see that they can easily completed to a set of $|E| - 3$ affinely independent points in $P^{(3,n)}_{0,n-\text{path}}(K_{n+1})$ satisfying $y_e = 0$. Thus, $y_e \geq 0$ defines a facet. Thus, it induces also a facet of $P^{(2,3,n)}_{0,n-\text{path}}(K_{n+1})$. Finally, when $c = (2,n)$, we have $n-1$ 2-paths satisfying $y_e = 0$ if e is an internal edge, and otherwise only $n-2$. In the former case, these $n-1$ 2-paths are affinely independent of the points y^r (consider the corresponding points in the path-and-loops polytope $P^{(2,n)}_{PL}(K_{n+1})$), and hence, $y_e \geq 0$ is facet defining. In the latter case, we have only a total of $|E| - 4$ affinely independent points satisfying $y_e = 0$, and thus, $y_e \geq 0$ does not induce a facet of $P^{(2,n)}_{0,n-\text{path}}(K_{n+1})$. ☐

Finally, we use the concept of symmetric inequalities to transform facet defining inequalities for $P^c_{0,n\text{-path}}(\tilde{D}_n)$ into those for $P^c_{0,n\text{-path}}(K_{n+1})$, see Section 3.3.2.

Corollary 3.57. Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \ldots, n\}$ with $n \geq 4$, and let $c = (c_1, \ldots, c_m) \in \text{CS}$ be a cardinality sequence. Then we have:

(a) The degree constraint $y(\delta(i)) \leq 2$ defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ for every node $i \in N \setminus \{0, n\}$ unless $c = (2,n)$.

(b) Let S be a subset of N with $0, n \in S$ and $|S| \leq c_1$. Then, the min-cut inequality

$$y((S : N \setminus S)) \geq 2 \tag{3.97}$$

induces a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $|S| \geq 3$ and $|V \setminus S| \geq 2$.

(c) Let $S \subset N$ with $0, n \in S$ and $j \in N \setminus S$. Then, the one-sided min-cut inequality

$$y((S : N \setminus S)) - y(\delta(j)) \geq 0 \tag{3.98}$$

is valid for $P^c_{0,n\text{-path}}(K_{n+1})$ and induces a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $|S| \geq c_1 + 1$ and $|N \setminus S| \geq 2$.

(d) The cardinality bound $y(E) \geq c_1$ defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $c_1 \geq 4$. The cardinality bound $y(E) \leq c_m$ defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $c_m < n$.

(e) Let W be a subset of N with $0, n \in W$ and $c_p < |W| - 1 < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. The node induced forbidden cardinality inequality

$$\begin{aligned} &(c_{p+1} - |W| + 1) \textstyle\sum_{i \in W} y(\delta(i)) \\ &\quad -(|W| - c_p - 1) \textstyle\sum_{i \in N \setminus W} y(\delta(i)) \leq 2c_p(c_{p+1} - |W| + 1) \end{aligned} \tag{3.99}$$

defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $c_{p+1} - |W| + 1 \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n$.

(f) Let W be a subset of N such that $0, n \in W$ and $c_p < |W| - 1 < c_{p+1}$ for some $p \in \{1, \ldots, m-1\}$. The cardinality-subgraph inequality

$$2y(E(W)) - (|W| - c_p - 2)y((W : N \setminus W)) \leq 2c_p \tag{3.100}$$

is valid for $P^c_{0,n\text{-path}}(K_{n+1})$ and induces a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if $p + 1 < m$ or $c_{p+1} = n = |W|$.

(g) Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 2$, and c_p even for $1 \leq p \leq m$, and let $N = S \,\dot{\cup}\, T$ be a partition of N with $0 \in S$, $n \in T$. The odd path exclusion constraint

$$y(E(S)) + y(E(T)) \geq 1 \tag{3.101}$$

is valid for $P^c_{0,n\text{-path}}(K_{n+1})$ and defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if (i) $c_1 = 2$ and $|S|, |T| \geq \frac{c_2}{2} + 1$, or (ii) $c_1 \geq 4$ and $|S|, |T| \geq \frac{c_2}{2}$.

(h) Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $N = S \,\dot{\cup}\, T$ be a partition of N with $0, n \in S$. The even path exclusion constraint

$$y(E(S)) + y(E(T)) \geq 1 \tag{3.102}$$

is valid for $P^c_{0,n\text{-path}}(K_{n+1})$ and defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if and only if (α) $c_1 = 3$, $|S|-1 \geq \frac{c_2+1}{2}$, and $|T| \geq \frac{c_2-1}{2}$, or (β) $c_1 \geq 5$ and $\min(|S|-1, |T|) \geq \frac{c_2-1}{2}$.

(i) Let $s, t \in N \setminus \{0, n\}$ be two distinct nodes and P an (s,t)-path of cardinality $c_m - 1$. The cardinality-path inequality

$$\sum_{i \in \dot{P}} \frac{1}{2} y(\delta(i)) - y(P) \geq 0 \tag{3.103}$$

is valid for $P^c_{0,n\text{-path}}(K_{n+1})$ and defines a facet of $P^c_{0,n\text{-path}}(K_{n+1})$ if $c_m \in \{4, 5\}$ and $n \geq c_m + 2$ or $c_m \geq 6$ and $n \geq 2c_m - 3$. □

As is already mentioned, the modified node induced forbidden cardinality inequalities (3.68) and the lifted jump inequalities (3.69) are not equivalent to symmetric inequalities.

3.8 Separation

In this section we investigate briefly separation problems for the most inequalities studied in this chapter. As this thesis is restricted to theoretical aspects of polyhedral combinatorics associated with CCCOP, we do not present heuristics for NP-hard separation problems.

The concept of symmetric inequalities, which we used to derive valid inequalities for the undirected counterparts of the cardinality constrained directed cycle and path polytopes, can also be applied to separation. More precisely, the separation problem for a class of symmetric inequalities can be traced back to that for its undirected counterpart. We demonstrate the mechanism for the cardinality constrained cycle polytopes. Let $D = (N, A)$ be a directed graph, $x^\star \in \mathbb{R}^A$, and $b^T x \leq b_0$ a member of a family $\mathcal{F}$ of symmetric inequalities. Since D is not necessarily complete, symmetric means that $b_{ij} = b_{ji}$ only if $(i,j), (j,i) \in A$. The edge set of the underlying graph $G = (N, E)$ is given by $E := \{[i,j] : A(\{i,j\}) \neq \varnothing\}$. Defining $y^\star \in \mathbb{R}^E$ by $y^\star_{ij} := x^\star(A(\{i,j\}))$ and $\bar{b} \in \mathbb{R}^E$ by $\bar{b}_{ij} := b(A(\{i,j\}))$ for $[i,j] \in E$, we see that

$$\begin{aligned}
& b^T x^\star > b_0 \\
\Leftrightarrow\quad & \sum_{(i,j)\in A} b_{ij} x^\star_{ij} > b_0 \\
\Leftrightarrow\quad & \sum_{i<j} b(A(\{i,j\})) x^\star(A(\{i,j\})) > b_0 \\
\Leftrightarrow\quad & \sum_{[i,j]\in E} b_{ij} y^\star_{ij} > b_0,
\end{aligned}$$

i.e., $x^\star$ violates the inequality $b^T x \leq b_0$ if and only if $y^\star$ violates $\bar{b}^T y \leq b_0$. In other words, the separation problem for $\mathcal{F}$ can be solved with a separation routine for its undirected counterpart.

Since the most inequalities that are valid for cardinality constrained cycle or path polytopes are equivalent to symmetric inequalities, we proceed as follows. Whenever possible we study the separation problem of a collection of inequality classes with respect to the undirected cardinality constrained cycle polytope. Thus, an instance of the separation problem for a class of inequalities is usually given by an undirected graph $G = (N, E)$ defined on n nodes, a cardinality sequence $c = (c_1, \ldots, c_m)$ with $c_1 \geq 3$, and a vector $y^\star \in \mathbb{R}^E_+$.

Cut inequalities

Class:	Min-cut inequalities (3.87) $\{y((S : N \setminus S)) \geq 2 : \lvert S\rvert, \lvert N \setminus S\rvert < c_1\}$
Related versions:	(3.19), (3.50), (3.65), (3.72), (3.97)
Complexity:	NP-hard (see Hartmann and Özlük [48])

Class:	One-sided min-cut inequalities (3.88) $\{y((S : N \setminus S)) \geq y(\delta(j)) :$ $S \subset N,\, j \in N \setminus S, \lvert N \setminus S\rvert < c_1\}$
Related versions:	(3.20), (3.51), (3.64), (3.73), (3.98)
Complexity:	depends on the side constraints

For instance, the separation problem for inequalities (3.88) is NP-hard due to the restriction that $|N \setminus S| < c_1$ (see Hartmann and Özlük [48]), while for the inequalities (3.98)

$$\{y((S : N \setminus S)) - y(\delta(j)) \geq 0 : S \subset N,\, 0, n \in S,\, j \in N \setminus S\},$$

which are valid for the undirected cardinality constrained path polytope, it can be solved in polynomial time by computing a minimum $\{0, n\}$-i-cut in G for each node $i \in N \setminus \{0, n\}$.

Class:	Two-sided min-cut inequality (3.86) $\{y(\delta(v)) + y(\delta(w)) - y((S : N \setminus S)) \leq 2 :$ $v \in S,\, w \in N \setminus S\}$
Related versions:	(3.71)
Complexity:	polynomial (see Hartmann and Özlük [48] or Bauer [9])

Forbidden cardinality inequalities

Class: Node induced forbidden cardinality inequalities (3.89)
$$\{(c_{p+1} - |W|) \sum_{i \in W} y(\delta(i)) - (|W| - c_p) \sum_{i \in N \setminus W} y(\delta(i)) \leq 2c_p(c_{p+1} - |W|) : W \subseteq N,\ c_p < |W| < c_{p+1},\ p \in \{1, \dots, m-1\}\}$$
Related versions: (3.56), (3.68), (3.99)
Complexity: polynomial

The node induced forbidden cardinality inequalities can be separated with a greedy algorithm. To this end, set $z_i^\star := y^\star(\delta(i))$ for all $i \in N$ and apply the greedy separation algorithm 8.27 of Grötschel [45] on input data $z^\star$, N, and c.

To separate the modified node induced forbidden cardinality inequalities (3.68) the algorithm in the previous paragraph can be applied $n-1$ times as subroutine, namely: for each internal node r of N, apply it to the subgraph induced by $N \setminus \{r\}$ of the underlying graph $G = (N, E)$. Note that these inequalities are valid for the directed cardinality constrained path and cycle polytope, but they are not symmetric.

Cardinality-subgraph inequalities

Class: Cardinality-subgraph inequalities (3.90)
$$\{2y(E(W)) - (|W| - c_p - 1) y((W : N \setminus W)) \leq 2c_p : W \subseteq N,\ c_p < |W| < c_{p+1},\ p \in \{1, \dots, m-1\}\}$$
Related versions: (3.59), (3.100)
Complexity: probably NP-hard

It seems to be very unlikely that there is a polynomial time algorithm that solves the separation problem for this class of inequalities. In the special case of $m = 2$ and $c_m = c_2 - c_1 = 2$ the separation problem for the inequalities (3.90) reduces to find a subset W^* of N of cardinality $k := c_1 + 1$ such that $y^\star(E(W^*)) > c_1$. The associated optimization problem $\max\{y(E(W)) : W \subseteq N,\ |W| = k\}$, is the weighted version of the densest k-subgraph problem which is known to be NP-hard (see Feige and Seltser [31]).

Odd/even path/cycle exclusion constraints

Class:	Even cycle exclusion constraint (3.91) $\{y(E(S)) + y(E(T)) \geq 1 : N = S \dot\cup T\}$ Here, c_p is odd for $p = 1, \dots, m$
Related versions:	(3.23), (3.26), (3.52), (3.53), (3.66), (3.67), (3.74), (3.75), (3.101), (3.102)
Complexity:	NP-hard

The separation problems for these classes of inequalities are equivalent to the maximum cut problem which is known to be NP-hard.

Cardinality-path inequalities

Class:	Cardinality-path inequalities (3.31) $\{y(\mathrm{dir}(P)) - \frac{1}{2} \sum_{v \in \dot{P}} y(\delta(v)) \leq 0 :$ P path in G, $\lvert P \rvert = k\}$
Related versions:	(3.32), (3.70), (3.77), (3.92), (3.103)
Complexity:	NP-hard (see Bauer, Linderoth, and Savelsbergh [10])

Linear ordering constraints

Class:	Linear ordering constraints (3.79) $\left\{\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} x((N_i : N_j)) \geq 1 : N = \bigcup_{i=1}^{r} N_i, \lvert N_i \rvert < c_1, i = 1, \dots, r\right\}$
Related versions:	(3.78)
Complexity:	NP-hard (see Hartmann and Özlük [48])

Here, an instance of the separation problem is given by a directed graph $D = (N, A)$, a cardinality sequence $c = (c_1, \dots, c_m)$, and a vector $x^\star \in \mathbb{R}^A_+$.

Lifted jump inequalities

Class: Lifted jump inequalities (3.45)

$$\left\{\sum_{i=0}^{c_m-1}\sum_{j=i+2}^{c_m+1} x((N_i : N_j)) - x((N_{c_m-1} \cup N_{c_m} : N_1 \cup N_2)) \geq 1 : N = \bigcup_{p=0}^{c_m+1} N_i,\ N_0 = \{0\},\ N_{c_m+1} = \{n\}\right\}$$

Related versions: (3.30), (3.76)

Complexity: probably NP-hard (see Dahl, Flatberg, Foldnes, and Gouveia [21])

Here, an instance of the separation problem is given by a directed graph $D = (N, A)$ on node set $N = \{0, \ldots, n\}$, a cardinality sequence c, and a vector $x^\star \in \mathbb{R}^A_+$.

We conclude that all inequalities of the IP-models (3.3), (3.55), (3.85), and (3.93) for the directed and undirected cardinality constrained cycle and path polytopes can be separated in polynomial time.

Chapter 4

Recommendations for deriving strong valid inequalities related to cardinality constraints

In the previous chapters we have seen that an appropriate modification of the ordinary forbidden cardinality inequalities results in facet defining inequalities for polyhedra associated with cardinality constrained combinatorial optimization problems. In this chapter we will give three recommendations how to derive stronger inequalities than inequalities (1.2) to cut off solutions of forbidden cardinality.

In what follows, $\Pi = (E, \mathcal{I}, w)$ is a combinatorial optimization problem (COP) and $\Pi_c = (E, \mathcal{I}, w, c)$ its cardinality constrained version, where $c = (c_1, \ldots, c_m)$ denotes a cardinality sequence. Moreover, denote by $P_{\mathcal{I}}(E)$ and $P^c_{\mathcal{I}}(E)$ the polytope associated with Π and Π_c, respectively.

4.1 Incorporation of combinatorial structures

In the previous chapters we have observed that the ordinary forbidden cardinality inequalities (1.2)

$$\begin{array}{l}(c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) \leq c_p(c_{p+1} - |F|) \\ \quad \text{for all } F \subseteq E \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \in \{1, \ldots, m-1\}\end{array}$$

are usually quite weak for the polyhedron associated with the cardinality constrained version of a COP $\Pi = (E, \mathcal{I}, w)$. An important reason is that they do no incorporate combinatorial structures of the given problem. To illustrate this statement, let us return to matroids.

By Edmonds [29], a rank inequality $x(F) \leq r(F)$ is facet defining for the matroid polytope if and only if F is closed and inseparable. Recall that any $F \subseteq E$ is said to be closed if $r(F \cup \{e\}) > r(F)$ for all $e \in E \setminus F$. It is called inseparable if $r(F_1) + r(F_2) > r(F)$ for all nonempty partitions $F = F_1 \dot\cup F_2$ of F. When we renounce of these properties and, in addition, substitute the right hand side of the rank inequality by $|F|$, then we obtain a valid inequality, but usually a quite weak inequality. So, the first and most important reason, why the rank inequality associated with a closed and inseparable subset F of E is facet defining, arises from the fact that the bound $r(F)$ is tighter than $|F|$ (unless $r(F) = |F|$). The second reason is connected to the exposed position of F among subsets F' of E with the same rank as F.

The first observation ($r(F)$ instead of $|F|$) can be immediately incorporated into forbidden cardinality inequalities. We obtain the rank induced forbidden cardinality inequalities (2.2)

$$(c_{p+1} - r(F))x(F) - (r(F) - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r(F))$$
$$\text{for all } F \subseteq E \text{ with } c_p < r(F) < c_{p+1} \text{ for some } p \in \{0, \ldots, m-1\}.$$

From the second observation (F closed and inseparable) we can easily adapt the closeness. If F and $F' := F \cup \{e\}$ for some $e \in E \setminus F$ have the same rank k, where $c_p < k < c_{p+1}$, then the rank induced forbidden cardinality inequality associated with F is the sum of the rank induced forbidden cardinality inequality associated with F' and the inequality $-(c_{p+1} - c_p)x_e \leq 0$, which is a multiple of the nonnegativity constraint $-x_e \leq 0$. In contrast, the separability seems not to fit into the framework of cardinality constrained matroids. The essential message of Theorem 2.9 is that the rank induced forbidden cardinality inequality associated with F usually induces a facet of $P^c_{\mathrm{M}}(E)$ if and only if F is closed. It shows that only in special cases separability plays a role for a rank induced forbidden cardinality inequality to be facet defining.

The observations in the previous paragraphs yield the first of three recommendations made in this chapter to find valid inequalities that are specific to cardinality restrictions. In analogy to matroid theory, we define a function ρ, called *rank function*, by $\rho(F) := \max\{|I \cap F| : I \in \mathcal{I}\}$ for all $F \subseteq E$. Moreover, any subset F of E is called *closed* if $\rho(F \cup \{e\}) > \rho(F)$ for all $e \in E \setminus F$.

Recommendation 1. Instead of investigating the original forbidden cardinality inequalities (1.2), analyze the *rank induced forbidden cardinality inequalities*

$$\begin{aligned} \mathrm{FC}_F(x) := (c_{p+1} - \rho(F))x(F) - (\rho(F) - c_p)x(E \setminus F) &\leq c_p(c_{p+1} - \rho(F)), \\ F \subseteq E \text{ closed with } c_p < \rho(F) < c_{p+1} &\text{ for some } p. \end{aligned} \tag{4.1}$$

Evidently, inequalities (4.1) are valid for $P^c_{\mathcal{I}}(E)$. Moreover, they are stronger than inequalities (1.2). To see this, let F and G be subsets of E such that $F \subseteq G$ and $c_p < |F| = \rho(G) < c_{p+1}$ for some p. Then, the forbidden cardinality inequality associated with F is the sum of the rank induced forbidden cardinality inequality associated with G and the inequalities $-(c_{p+1} - c_p)x_e \leq 0$ for $e \in G \setminus F$. Moreover, it is not hard to see that the closeness of F is a necessary condition for inequality (4.1) to be facet defining. Namely, if F is not closed, that is, there exists $e \in E \setminus F$ such that $\rho(F \cup \{e\}) = \rho(F)$, then (4.1) is the sum of the inequalities $\mathrm{FC}_{F\cup\{e\}}(x) \leq c_p(c_{p+1} - \rho(F \cup \{e\}))$ and $-(c_{p+1} - c_p)x_e \leq 0$.

Recommendation 1 can be seen as one possibility to incorporate some combinatorial structure into inequalities that cut off feasible solutions $I \in \mathcal{I}$ of forbidden cardinality.

Our computational results with the convex hull codes `polymake` [41] and PORTA [17] as well as our theoretical results confirm that the forbidden cardinality inequalities in the latter form frequently appear in the linear descriptions of many polyhedra associated with CCCOPs. Unfortunately, they are still not necessarily facet defining, in general not separable in polynomial time unless P = NP, and sometimes hard to identify.

We give an example for Recommendation 1.

Example A: Cardinality constrained matchings

A *matching* of a graph $G = (N, E)$ is a set of mutually disjoint edges. A matching of cardinality $|N|/2$ is said to be *perfect*. Given any edge weights $w_e \in \mathbb{R}$, to find a maximum weight (minimum weight perfect) matching in G is one of the hardest combinatorial optimization problems solvable in polynomial time.

As it is well known, the problem of finding a maximum weight matching of cardinality $k \leq \lfloor |N|/2 \rfloor$ can be easily transformed into the perfect matching problem. Add $\ell := |N| - 2k$ new nodes $u_1, \ldots, u_\ell$ and join each of them with every node $v \in N$ by a (zero-weight) edge. Denote the resulting graph by $G' = (N', E')$. Then, the restriction of any perfect matching M in G' to G is a matching of cardinality k, since the node set $\{u_1, \ldots, u_\ell\}$ is a stable set. Consequently, for any cardinality sequence c, the associated cardinality constrained matching problem can be solved in polynomial time.

Let $\mathcal{M}$ be the collection of all matchings of G. The *matching polytope* $P_{\text{MATCH}}(E)$ of $G = (N, E)$ is the convex hull of the incidence vectors of all matchings $M \in \mathcal{M}$. By Edmonds [27], the matching polytope is determined by the following inequalities:

$$
\begin{aligned}
x_e &\geq 0 && \text{for all } e \in E, && (4.2)\\
x(\delta(v)) &\leq 1 && \text{for all } v \in N, && (4.3)\\
x(E(W)) &\leq \left\lfloor \frac{1}{2}|W| \right\rfloor && \text{for all } W \subseteq N,\ |W| \text{ odd.} && (4.4)
\end{aligned}
$$

Adding the equation

$$x(E) = k, \tag{4.5}$$

we obtain a complete linear description of

$$P^{(k)}_{\text{MATCH}}(E) := \text{conv}\{\chi^M \in \mathbb{R}^E : M \in \mathcal{M},\ |M| = k\},$$

as we want to show now.

Theorem 4.1. $P^{(k)}_{\text{MATCH}}(E)$ is determined by the inequalities (4.2)-(4.5).

Proof. By Chvátal [18], two vertices of the matching polytope $P_{\text{MATCH}}(E)$ are adjacent if and only if the symmetric difference of the corresponding matchings is a path or a cycle. In particular, the cardinalities of both matchings differ by at most one. Consequently, the hyperplane H defined by $x(E) = k$ does not intersect an edge of $P_{\text{MATCH}}(E)$ in its relative interior. Hence, all vertices of $P := H \cap P_{\text{MATCH}}(E)$ are also vertices of $P_{\text{MATCH}}(E)$, which implies $P = P^{(k)}_{\text{MATCH}}(E)$. □

The set of all matchings $\mathcal{M}$ of G forms an independence system. Inequalities (4.3) and (4.4) are rank inequalities with respect to the rank function

$$r : 2^E \to \mathbb{Z}_+, \qquad r(F) := \max\{|M| : M \in \mathcal{M}, M \subseteq F\}$$

associated with $\mathcal{M}$. If $F = \delta(v)$ for some $v \in V$, then $r(F) = 1$ provided that $\delta(v) \neq \varnothing$. Next, if $F = E(W)$ for some $W \subseteq V$, then $r(F) \leq \lfloor \frac{1}{2}|W| \rfloor$, and equality holds if, for instance, the subgraph $(W, E(W))$ is connected. Hence, inequalities (4.3) and (4.4) are usually indeed rank inequalities.

The aim in this example is to derive rank induced forbidden cardinality inequalities that are facet defining for the *cardinality constrained matching polytope*

$$P^c_{\text{MATCH}}(E) := \text{conv}\{\chi^M \in \mathbb{R}^E : M \in \mathcal{M} \cap \text{CHS}^c(E)\}.$$

However, it is not so easy to give a meaningful characterization of rank induced forbidden cardinality inequalities, since to determine the rank of a set

$F \subseteq E$ is a nontrivial task. Let F be any subset of E, and let $G' := (N, F)$. The rank of F is the maximum size of a matching contained in F. By the Tutte-Berge formula [13, 80], this number is given by

$$r(F) = \min_{U \subseteq N} \frac{1}{2}(|N| + |U| - o(G' - U)),$$

where for any graph H, $o(H)$ denotes the number of its odd components.

Considering this background, we turn towards an easy case. Let $K_n = (N, E)$ be the complete graph on n nodes, and let $N = \bigcup_{i=1}^k N_i$ be a partition of N. Then, the rank of $F := \bigcup_{i=1}^k E(N_i)$ is given by $r(F) = \sum_{i=1}^k \left\lfloor \frac{1}{2}|N_i| \right\rfloor$. For a subset of such given sets $F \subseteq E$, we present facet defining rank induced forbidden cardinality inequalities.

Theorem 4.2. Let $K_n = (N, E)$ be the complete graph on n nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $m \geq 2$, $c_1 \geq 1$, and $c_m \leq \lfloor \frac{n}{2} \rfloor - 1$. Moreover, let $N = \bigcup_{i=1}^k N_i$ be a partition of N to odd subsets N_i such that $c_p < r := \sum_{i=1}^k \left\lfloor \frac{1}{2}|N_i| \right\rfloor < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Then, the inequality

$$(c_{p+1} - r)x(F) - (r - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r) \tag{4.6}$$

defines a facet of $P^c_{\text{MATCH}}(E)$, where $F := \bigcup_{i=1}^k E(N_i)$.

Proof. Clearly, $r = r(F)$ and $r(F)$ matches also with the value $\rho(F)$. Thus, following the argumentation after Recommendation 1, inequality (4.6) is valid. To show that (4.6) defines a facet of $P^c_{\text{MATCH}}(E)$, we assume that there is an equation $b^T x = b_0$ that is satisfied by all points in $P^c_{\text{MATCH}}(E)$ which satisfy (4.6) at equality. If $c_p = 1$, then it follows that $b_e = b_0 = \frac{b_0}{c_p}$ for all $e \in F$. Next, let $c_p \geq 2$ which implies $r \geq 3$. We will show that for any two edges $e, f \in F$, $b_e = b_f$ holds. If e and f are non-adjacent, then there is a matching M of cardinality $c_p + 1$ with $e, f \in F$ due to $r(F) = r \geq 3$. The matchings $M_e := M \setminus \{e\}$ and $M_f := M \setminus \{f\}$ are tight, that is, the incidence vectors of M_e and M_f satisfy the inequality (4.6) at equality. Hence, $b_0 = b\chi^{M_e} = b\chi^{M_f}$ which implies immediately $b_e = b_f$. If e and f are adjacent, then there is some edge $g \in F$ which is adjacent neither to e nor to f. By the former argumentation, $b_e = b_g$ and $b_f = b_g$, and thus, $b_e = b_f$. Any tight matching $M \subseteq F$ yields now $b_e = \frac{b_0}{c_p}$ for all $e \in F$.

Next, consider the coefficients $b_e, e \in E \setminus F$. If $c_{p+1} = r + 1$, we conclude that that $b_e = -b_0 \frac{r - c_p}{c_p} = -b_0 \frac{r - c_p}{c_p(c_{p+1} - r)}$. So, let $c_{p+1} > r + 1$ and $e^\star \in E \setminus F$ be any edge. Then, one can always find a matching $M^\star \subseteq F$ with $|M^\star| = r$ such that $M^\star \cup \{e^\star\}$ is also a matching. Moreover, $M^\star \cup \{e^\star\}$ can be completed to a matching M' with $|M'| = c_{p+1} + 1$ even if $c_{p+1} = c_m$, since $c_m \leq \lfloor \frac{n}{2} \rfloor - 1$. The matchings $M'_f := M' \setminus \{f\}$, $f \in M' \setminus M^\star$ are tight with respect to (4.6) which implies $b\chi^{M'_f} = b_0$ for all $f \in M' \setminus M\star$. Hence, it is not hard to see that $b_f = -b_0 \frac{r - c_p}{c_p(c_{p+1} - r(F))}$ for all $f \in M' \setminus M^\star$. In particular, $b_{e^\star} = -b_0 \frac{r - c_p}{c_p(c_{p+1} - r(F))}$, and since $e^\star$ were arbitrarily chosen, it follows that $b_e = -b_0 \frac{r - c_p}{c_p(c_{p+1} - r(F))}$ for all $e \in E \setminus F$. Thus, $b^T x = b_0$ is a multiple of (4.6).

Moreover, since inequality (4.6) is not an implicit equation, $P^c_{\text{MATCH}}(E)$ is fulldimensional, and inequality (4.6) is facet defining. □

Inequalities (4.6) turn out to be a cardinality constrained version of inequalities (4.4). Moreover, it is worth mentioning that F as given in Theorem 4.2 is closed.

Also inequalities (4.3) have a natural translation to the context of cardinality restrictions:

$$
\begin{aligned}
&(2c_{p+1} - |W|) \sum_{v \in W} x(\delta(v)) \\
&\qquad -(|W| - 2c_p) \sum_{v \in N \setminus W} x(\delta(v)) \leq 2c_p(2c_{p+1} - |W|) \\
&\qquad\qquad \text{for all } W \subseteq N \text{ with } 2c_p < |W| < 2c_{p+1},\ p = 1, \ldots, m.
\end{aligned}
\tag{4.7}
$$

However, these inequalities cannot be derived as rank induced forbidden cardinality inequalities, since they have up to three different coefficients $4c_{p+1} - 2|W|$, $4c_p - 2|W|$, and $2c_p + 2c_{p+1} - 2|W|$, while (rank induced) forbidden cardinality inequalities have only two. Inequalities (4.7) are easily seen to be valid for the cardinality constrained matching polytope $P^c_{\text{MATCH}}(E)$, since a matching of cardinality c_p covers $2c_p$ nodes for $p \in \{1, \ldots, m\}$. This means that the cardinality sequence $c = (c_1, \ldots, c_m)$ for the number of allowed edges used in a matching can be directly translated to the cardinality sequence $\tilde{c} := (2c_1, \ldots, 2c_m)$ for the number of allowed nodes covered by a matching. We remark that the class of inequalities (4.7) is identically with the class of node induced forbidden cardinality inequalities (3.89) for the undirected cardinality constrained cycle polytope in case that the cardinality sequence only consists of even numbers.

4.2 COMBINATORIAL/MATROIDAL RELAXATIONS

In case that $\Pi = (E, \mathcal{I}, w)$ is the maximum independent set problem over a matroid, the polytope associated with Π_c, that is, the cardinality constrained matroid polytope $P^c_{\text{M}}(E)$, is determined by the system (2.2)-(2.6):

$$
\begin{aligned}
&\mathrm{FC}_F(x) := (c_{p+1} - r(F))x(F) - (r(F) - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r(F)) \\
&\qquad \text{for all } F \subseteq E \text{ with } c_p < r(F) < c_{p+1} \text{ for some } p \in \{0, \ldots, m-1\},
\end{aligned}
$$

$$\begin{aligned}
x(E) &\geq c_1, \\
x(E) &\leq c_m, \\
x(F) &\leq r(F) && \text{for all } \varnothing \neq F \subseteq E, \\
x_e &\geq 0 && \text{for all } e \in E,
\end{aligned}$$

see Theorem 2.2. Moreover, all inequalities are separable in polynomial time. As it is already mentioned in Section 2.1.3 of Chapter 2, the separation problem for the rank inequalities can be solved by a combinatorial algorithm proposed by Cunningham [20], while the separation problem for the rank induced forbidden cardinality inequalities can be traced back to that for the rank inequalities, cf. Theorem 2.13 and Corollary 2.14.

The question, how we can benefit from the nice polyhedral structure of cardinality constrained matroids, leads to the second recommendation.

Recommendation 2. Find a "good" combinatorial relaxation (or matroidal relaxation) $\Pi' = (E, \mathcal{J}, w)$ of the COP of consideration $\Pi = (E, \mathcal{I}, w)$, or even better, directly of its cardinality constrained version $\Pi_c = (E, \mathcal{I}, w, c)$.

Here, a COP $\Pi' = (E, \mathcal{J}, w)$ is called a *combinatorial relaxation* (*matroidal relaxation*) of $\Pi = (E, \mathcal{I}, w)$ if $\mathcal{J} \supseteq \mathcal{I}$ (and $\mathcal{J}$ is a matroid). Of course, $\mathcal{J} \supseteq \mathcal{I}$ or $\mathcal{J} \supseteq (\mathcal{I} \cap \mathrm{CHS}^c(E))$ implies that $(\mathcal{J} \cap \mathrm{CHS}^c(E)) \supseteq (\mathcal{I} \cap \mathrm{CHS}^c(E))$. Hence, valid inequalities for $P^c_{\mathcal{J}}(E)$ are also valid for $P^c_{\mathcal{I}}(E)$.

The hope behind Recommendation 2 is that "good" combinatorial relaxations yield strong inequalities for $P^c_{\mathcal{I}}(E)$. In the best case it means that $P^c_{\mathcal{J}}(E)$ has a tractable facial description, and the facet defining inequalities for $P^c_{\mathcal{J}}(E)$ are also facet defining for $P^c_{\mathcal{I}}(E)$. If, for instance, $\mathcal{J}$ is a matroid, then $P^c_{\mathcal{J}}(E)$ has a tractable facial structure, but this alone does not imply the tightness of its facet defining inequalities for $P^c_{\mathcal{I}}(E)$.

Of course, the quality of a combinatorial (matroidal) relaxation influences the strength of the associated inequalities with respect to $P^c_{\mathcal{I}}(E)$. For instance, an independence system $\mathcal{I}$ defined on some ground set E is the intersection of finitely many matroids defined on the same set E: The circuit system $\mathcal{C}$ associated to $\mathcal{I}$ has only finitely many members. Each circuit $C \in \mathcal{C}$ can be used to define a matroid $\mathcal{I}_C$ by setting $\mathcal{I}_C := \{I \subseteq E : C \nsubseteq I\}$. Then, $\mathcal{I} = \cap_{C \in \mathcal{C}} \mathcal{I}_C$. This is, however, usually not an efficient way to describe $\mathcal{I}$, since in general $\mathcal{I}$ is the intersection of less matroids. From a polyhedral point of view a smaller description of $\mathcal{I}$ by matroids probably leads to stronger inequalities for $P^c_{\mathcal{I}}(E)$.

In Recommendation 2 we suggest to use combinatorial (matroidal) relaxations of Π_c instead of Π. Of course, the combinatorial relaxations of

Π and Π_c are usually the same, but not necessarily. If $\mathcal{J} \supseteq \mathcal{I}$, then $\mathcal{J} \supseteq (\mathcal{I} \cap \mathrm{CHS}^c(E))$. However, $\mathcal{K} \supseteq (\mathcal{I} \cap \mathrm{CHS}^c(E))$ for some $\mathcal{K} \subseteq 2^E$ does not necessarily imply $\mathcal{K} \supseteq \mathcal{I}$. This also affects the facial structures of the polytopes associated with Π and Π_c. Consider an artificial COP as in Section 1.1, for instance, the *embedded directed odd cycle problem* (EDOCP)

$$\min\{w(C) : C \subseteq E, \text{ if } |C| \geq 3 \text{ is odd, then } C \text{ is a simple directed cycle}\}$$

defined on a digraph $D = (N, A)$. The associated polytope, namely the *embedded directed odd cycle polytope* $P_{\text{EDOC}}(A)$ is fulldimensional, since $0 \in P_{\text{EDOC}}(A)$ and $u_a \in P_{\text{EDOC}}(A)$ for all $a \in A$. Here, u_a denotes the ath unit vector. Moreover, a trivial inequality $x_a \leq 1$ defines a facet of $P_{\text{EDOC}}(A)$, since the vectors u_a and $u_a + u_b$ for all $b \in A \setminus \{a\}$ belong to $P_{\text{EDOC}}(A)$, are linearly independent, and satisfy the inequality at equality. The trivial inequalities $x_a \leq 1$ for $a \in A$ can be interpreted as rank inequalities for the trivial matroid $\mathcal{I} = 2^A$. The singletons $\{a\}$, where $a \in A$, are the closed and inseparable sets with respect to $\mathcal{I}$. Now, restricting the feasible solutions of the EDOCP to odd cardinalities ≥ 3, we obtain the so called *directed odd cycle problem* (DOCP). Of course, an inequality $x_{ij} \leq 1$ for $a = (i, j) \in A$ is also valid for the polytope associated with the DOCP, the so called *directed odd cycle polytope*

$$P_{\text{DOC}}(A) := \mathrm{conv}\{\chi^C \in \mathbb{R}^A : C \text{ is a simple directed cycle with } |C| \text{ odd}\},$$

but now the inequality is the consequence of other valid inequalities. Denoting by $\delta^{\text{out}}(i)$ the set of arcs leaving node i, we see that the inequality $x_{ij} \leq 1$ is the sum of the degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ and the nonnegativity constraints $-x_{ik} \leq 0$ for all $k \in \delta^{\text{out}}(i) \setminus \{j\}$, and hence, this inequality is not facet defining if $|\delta^{\text{out}}(i)| \geq 2$. Conversely, $x(\delta^{\text{out}}(i)) \leq 1$ is not valid for $P_{\text{EDOC}}(A)$ unless $|\delta^{\text{out}}(i)| = 1$. Consequently, for this example it is better to use combinatorial/matroidal relaxations for Π_c than Π in order to find strong valid inequalities for the polytope associated with Π_c.

We close the discussion with an example for a favorable application of Recommendation 2 based on cardinality constrained paths, also see Chapter 3.

Example B: Cardinality constrained paths

Let $D = (N, A)$ be a directed graph and $c = (c_1, \ldots, c_m)$ a cardinality sequence. We recall some notations from Chapter 3. For any $v \in N$, we denote by $\delta^{\text{in}}(v)$ and $\delta^{\text{out}}(v)$ the set of arcs entering and leaving node v, respectively.

For any two disjoint nodes $s, t \in N$, let $\mathcal{P}_{s,t}(D)$ be the collection of simple directed (s,t)-paths of D. The polytope

$$P^c_{s,t\text{-path}}(D) := \operatorname{conv}\{\chi^P \in \mathbb{R}^A : P \in \mathcal{P}_{s,t}(D) \cap \mathrm{CHS}^c(A)\}$$

is the directed cardinality constrained path polytope.

According to Section 3.4 of Chapter 3, the integer points of $P^c_{s,t\text{-path}}(D)$ can be described by the system

$$
\begin{aligned}
x_{uv} &\in \{0,1\} && \text{for all } (u,v) \in A, && (4.8)\\
x(\delta^{\text{out}}(s)) = x(\delta^{\text{in}}(t)) &= 1, && && (4.9)\\
x(\delta^{\text{in}}(s)) = x(\delta^{\text{out}}(t)) &= 0, && && (4.10)\\
x(\delta^{\text{out}}(v)) - x(\delta^{\text{in}}(v)) &= 0 && \text{for all } v \in N \setminus \{s,t\}, && (4.11)\\
x(\delta^{\text{out}}(v)) &\le 1 && \text{for all } v \in N \setminus \{s,t\}, && (4.12)\\
x(\delta^{\text{in}}(v)) - x((S : N \setminus S)) &\le 0 && \forall S \subset N : s,t \in S, v \in N \setminus S, && (4.13)\\
x(A) &\ge c_1, && &&\\
x(A) &\le c_m, && &&
\end{aligned}
$$

$$
\begin{aligned}
&(c_{p+1} - |W|) \textstyle\sum_{v \in W} x(\delta^{\text{out}}(v))\\
&\quad -(|W| - c_p) \textstyle\sum_{v \in N \setminus W} x(\delta^{\text{out}}(v)) \le c_p(c_{p+1} - |W|)\\
&\qquad\qquad \text{for all } W \subseteq N : s \in W, t \in N \setminus W\\
&\qquad\qquad \text{with } c_p < |W| < c_{p+1} \text{ for some } p.
\end{aligned}
\tag{4.14}
$$

It is worthwile to have a closer look at the cardinality constrained path polytope from a matroidal point of view, disclosing that the node induced forbidden cardinality inequalities (4.14) are originated from matroids. The collection of all simple (s,t)-paths is contained in the intersection of the same three matroids that are used to formulate the asymmetric traveling salesman problem by matroids. The three matroids are the two partition matroids $M^{\text{out}} = (A, \mathcal{I}^{\text{out}})$ and $M^{\text{in}} = (A, \mathcal{I}^{\text{in}})$ whose independence systems are defined by

$$
\begin{aligned}
\mathcal{I}^{\text{out}} &:= \{B \subseteq A : |B \cap \delta^{\text{out}}(v)| \le 1 \text{ for all } v \in N\},\\
\mathcal{I}^{\text{in}} &:= \{B \subseteq A : |B \cap \delta^{\text{in}}(v)| \le 1 \text{ for all } v \in N\},
\end{aligned}
$$

respectively, and the graphic matroid $M^F = (A, \mathcal{I}^F)$, where $\mathcal{I}^F$ denotes the collection of all forests of D. Consequently, the rank and rank induced forbidden cardinality inequalities associated with these matroids are valid inequalities for $P^c_{s,t\text{-path}}(D)$.

The facet defining rank inequalities for $P^c_{M^{\text{out}}}(A)$ are exactly the inequalities $x(\delta^{\text{out}}(v)) \le 1$ for $v \in N$. Thus, inequalities (4.12) are originated from the partition matroid M^{out}. The facet defining forbidden cardinality inequalities for $P^c_{M^{\text{out}}}(A)$ are of the form

$$(c_{p+1}-|U|)\sum_{v\in U} x(\delta^{\text{out}}(v))-(|U|-c_p)\sum_{v\in N\setminus U} x(\delta^{\text{out}}(v)) \leq c_p(c_{p+1}-|U|), \quad (4.15)$$

where $U \subseteq N$ with $c_p < |U| < c_{p+1}$ for some $p \in \{1,\dots,m-1\}$. If $s \in U$ and $t \in N \setminus U$, then inequality (4.15) is equivalent to

$$(c_{p+1}-|U|)\sum_{v\in U\cup\{t\}} x(\delta^{\text{out}}(v))-(|U|-c_p)\sum_{v\in N\setminus(U\cup\{t\})} x(\delta^{\text{out}}(v)) \leq c_p(c_{p+1}-|U|)$$

due to $x(\delta^{\text{out}}(t)) = 0$. Setting $W := U \cup \{t\}$, we see that this inequality is an inequality among (4.14). Thus, inequalities (4.14) are originated from the cardinality constrained version of M^{out}. By Theorem 3.35, these inequalities define facets of $P^c_{s,t\text{-path}}(D)$.

Due to the flow conservation constraints (4.11), the inequalities that can be derived from facet defining rank and rank induced forbidden cardinality inequalities for $P^c_{M^{\text{in}}}(A)$ are equivalent to any of the inequalities (4.12) and (4.14).

The facet defining rank inequalities for $P^c_{M^F}(A)$ are of the form $x(A(U)) \leq |U| - 1$ for $\varnothing \neq U \subseteq N$. Due to the equations (4.9) and (4.10), the face induced by the rank inequality associated with some U is contained in the face induced by the rank inequality associated with $U' := U \setminus \{s,t\}$ (with respect to $P^c_{s,t\text{-path}}(D)$). However, the inequality $x(A(U')) \leq |U'| - 1$ is still not facet defining for $P^c_{s,t\text{-path}}(D)$. To this end, consider an inequality among (4.13) with $S := N \setminus U'$ and some $u \in U'$:

$$x(\delta^{\text{in}}(u)) - x((S : N \setminus S)) \leq 0.$$

Adding the inequalities $x(\delta^{\text{in}}(v)) \leq 1$ for $v \in U' \setminus \{u\}$, we obtain $x(A(U')) \leq |U'| - 1$. Also, none of the forbidden cardinality inequalities

$$(c_{p+1} - r_F(W))x(W) - (r_F(W) - c_p)x(A \setminus W) \leq c_p(c_{p+1} - r_F(W))$$

for closed sets $W \subseteq A$ with respect to the graphic matroid is facet defining for $P^c_{s,t\text{-path}}(D)$ regardless in which partition $W, A \setminus W$ are s and t.

4.3 ITERATED INEQUALITY-STRENGTHENING

A favorite method to obtain insights about the facial structure of a polytope is to compute the H-representation of a polytope, given by its V-representation, with convex-hull codes such as PORTA [17] or `polymake` [41]. However, since the used routines have exponential running time (indeed, a polynomial algorithm for the convex hull problem does not exist), this approach works only for small problem instances. Here general lifting procedures might come into play. We are perhaps not able to compute the H-representation of a polytope in reasonable time but to determine quickly the affine space associated with the face induced by a valid inequality. For example, PORTA returns a set of linearly independent equations that are satisfied by all points in the face F_a of a polyhedron P induced by a valid inequality $a^T x \leq \alpha$. In other words, the set of equations determines the affine hull of F_a. However, P intersects usually both half spaces induced by such an equation $b^T x = \beta$, which means that neither $b^T x \leq \beta$ nor $b^T x \geq \beta$ are valid for P.

Algorithm 2: Inequality-Strengthening.

Input: A 0-1-polytope $P \in \mathbb{R}^d$ given by its vertex set $\mathcal{V}$, a valid inequality $a^T x \leq \alpha$, and an equation $b^T x = \beta$ that is satisfied by all points $v \in \mathcal{V}$ that satisfy $a^T x \leq \alpha$ at equality.

Output: A valid inequality $c^T x \leq \gamma$ such that the face induced by this inequality contains the face induced by $a^T x \leq \alpha$.

1 Set $\tilde{\mathcal{V}} := \{v \in \mathcal{V} : b^T v > \beta\}$.

2 **if** $\tilde{\mathcal{V}} = \varnothing$ **then**

  return "$a^T x \leq \alpha$".

end

3 Set $\lambda_v := \frac{\alpha - a^T v}{b^T v - \beta}$ for all $v \in \tilde{\mathcal{V}}$.

4 Set $\lambda^\star := \min\{\lambda_v : v \in \tilde{\mathcal{V}}\}$.

5 Define a new inequality $c^T x \leq \gamma$ by $c := a + \lambda^\star \cdot b$ and $\gamma := \alpha + \lambda^\star \cdot \beta$.

return "$c^T x \leq \gamma$".

In this situation, Algorithm 2 can be applied: $\tilde{\mathcal{V}}$ is the set of all vertices of P that violate the inequality $b^T x \leq \beta$. Thus, $a^T v < \alpha$ and $\lambda_v > 0$ for all $v \in \tilde{\mathcal{V}}$, which in turn implies $\lambda^\star > 0$. Now, for every $v \in \mathcal{V} \setminus \tilde{\mathcal{V}}$ we have: $a^T v \leq \alpha$, $b^T v \leq \beta$, and hence $c^T v \leq \gamma$. Moreover, for each $v \in \tilde{\mathcal{V}}$ we have: $(a + \lambda b) \cdot v \leq \alpha + \lambda\beta$ for $0 \leq \lambda \leq \lambda_v$. Hence, $c^T x \leq \gamma$ is satisfied by every $v \in \tilde{\mathcal{V}}$. Consequently, $c^T x \leq \gamma$ is a valid inequality for P. Furthermore, if $\tilde{\mathcal{V}} \neq \varnothing$, then by the choice of $\lambda^\star$ in line 4, the face induced by $a^T x \leq \alpha$ is strictly contained in the face induced by $c^T x \leq \gamma$. So, applying Algorithm 2 iteratively, results in a facet defining inequality for P. Of course, the running time of the procedure is linear in $|\mathcal{V}|$, but in general not polynomial in the dimension d of the space.

Based on these observations, we give the following recommendation.

Recommendation 3. If the forbidden cardinality inequalities are not facet defining for the polytope of consideration, then they can still be used to derive stronger cardinality specific inequalities. This can be tried, for instance, with Algorithm 2.

To give an application for Algorithm 2, we consider cardinality constrained cuts.

Example C: Cardinality constrained cuts

Let $G = (N, E)$ be a graph. For any $S \subseteq N$, we denote by $\delta(S)$ the set of edges connecting S and $N \setminus S$. A subset C of E is called a *cut* if $C = \delta(S)$ for some $S \subseteq N$. The sets S and $N \setminus S$ are the *shores* of C. The collection of all cuts of G is denoted by $\mathcal{C}$. The *max cut problem* $\max\{w(C) : C \in \mathcal{C}\}$, for general weight functions w, is NP-hard. In the following, we consider cardinality restrictions acting on the shores of the cuts.

Let $C \subseteq E$ be a cut with shores S and T. Then, $|T| = |N| - |S|$, that is, the cardinality of S determines that of T and vice versa. Moreover, $\min\{|S|, |T|\} \leq \lfloor \frac{n}{2} \rfloor$. Consequently, it is sufficient to force only the cardinality of the smaller shore which can be done with cardinality sequences $c = (c_1, \ldots, c_m)$ with $1 \leq c_1 < \ldots < c_m \leq \lfloor \frac{n}{2} \rfloor$. These observations give reason to define the *node cardinality constrained cut polytope*

$$P^c_{\text{Cut}}(E) := \text{conv}\{\chi^{\delta(S)} \in \mathbb{R}^E : S \in \text{CHS}^c(N)\}.$$

In what follows, let $P^c_{\text{Cut}}(E)$ be defined on the complete graph $K_n = (N, E)$ on n nodes. Since the cardinality sequence restricts N and not E, it seems to be hard to incorporate the forbidden cardinality inequalities. However, requiring not only $S \in \text{CHS}^c(N)$ for a shore S of a cut, but also $s \notin S$ for a fixed node s, gives rise to these inequalities. Denote by $\mathcal{S}$ the collection of all subsets of N not containing s. Then, $\bar{P}^c_{\text{Cut}}(E) := \text{conv}\{\chi^{\delta(U)} \in \mathbb{R}^E : U \in \mathcal{S} \cap \text{CHS}^c(N)\}$ is a slight variation of $P^c_{\text{Cut}}(E)$. Both polytopes are related as follows: Let $c = (c_1, \ldots, c_m)$ be a cardinality sequence with $1 \leq c_m < \ldots < c_m \leq \lfloor \frac{n}{2} \rfloor$ and $\bar{c} = (c_1, \ldots, c_m, n - c_m, \ldots, n - c_1)$. Then, $P^c_{\text{Cut}}(E) = P^{\bar{c}}_{\text{Cut}}(E)$ and $P^{\bar{c}}_{\text{Cut}}(E) = \bar{P}^{\bar{c}}_{\text{Cut}}(E)$, but for cardinality sequences $c = (c_1, \ldots, c_m)$ with $1 \leq c_1 < c_2 < \ldots < c_m \leq n - 1$, $P^c_{\text{Cut}}(E)$ strictly contains $\bar{P}^c_{\text{Cut}}(E)$ in general. Thus, $\bar{P}^c_{\text{Cut}}(E)$ generalizes $P^c_{\text{Cut}}(E)$, that is, the collection of polytopes $\{P^{(c_1,\ldots,c_m)}_{\text{Cut}}(E) : 1 \leq c_1 < c_2 < \ldots < c_m \leq n - 1\}$ is contained in the collection of polytopes $\{\bar{P}^{(c_1,\ldots,c_m)}_{\text{Cut}}(E) : 1 \leq c_1 < c_2 < \ldots < c_m \leq n - 1\}$. The difference in the facial structure, however, is small. For instance, if we restrict ourselves to $c = (k)$, then $\bar{P}^{(k)}_{\text{Cut}}(E)$ is a face of

$P_{\text{Cut}}^{(k)}(E)$. If $k \leq \lfloor \frac{n}{2} \rfloor$, then it is induced by $x(\delta(s)) \geq k$, if $k > \lfloor \frac{n}{2} \rfloor$, then by $x(\delta(s)) \leq k$.

Since $[s,v] \in E$ for all $v \in N \setminus \{s\}$, we have $|\delta(U) \cap \delta(s)| = |U|$ for all $U \in \mathcal{S} \cap \text{CHS}^c(N)$. Thus, the forbidden cardinality inequality

$$(c_{p+1} - |W|)\, x(\delta(s) \cap \delta(W)) - (|W| - c_p)\, x(\delta(s) \cap \delta(N \setminus W)) \leq c_p(c_{p+1} - |W|)$$

is valid for all $W \in \mathcal{S}$ with $c_p < |W| < c_{p+1}$ for some p. The inequalities are not facet defining, but by the application of Algorithm 2 and the right choice of equations generated by PORTA they can be strengthened. With this approach we identified a very simple class of $n \cdot (m-1)$ inequalities

$$x(E) - (n - c_p - c_{p+1})x(\delta(s)) \leq c_p c_{p+1} \quad \forall\, s \in N,\, p \in \{1, \ldots, m-1\}, \tag{4.16}$$

which will be shown to be facet defining in Theorem 4.6.

Example. We consider the complete graph $K_{13} = (N, E)$, the cardinality sequence $c = (2, 6, 7, 11)$, and the forbidden cardinality inequality

$$2 \sum_{v \in Y} x_{sv} - 2 \sum_{v \in Z} x_{sv} \leq 4,$$

where $s = 1$, $Y = \{2, 3, 4, 5\}$, and $Z = \{6, \ldots, 13\}$. All points in $Q := P_{\text{Cut}}^c(E)$ satisfying the inequality at equality satisfy the equation

$$x(E(Z)) - 6x((Z : \{s\})) = 0.$$

The minimum value $\lambda^\star$ in Algorithm 2 will be attained by 4. This results in the inequality

$$2 \sum_{v \in Y} x_{sv} - 26 \sum_{v \in Z} x_{sv} + 4x(E(Z)) \leq 4.$$

Iterating Algorithm 2 with input data

$$\begin{gathered} 2 \sum_{v \in Y} x_{sv} - 26 \sum_{v \in Z} x_{sv} + 4x(E(Z)) \leq 4, \\ 8\, x(E(Y)) + x((Y : Z)) + 2\, x(E(Z)) = 48, \end{gathered}$$

we obtain $\lambda^\star = \frac{2}{3}$ and after scaling the inequality

$$6 \sum_{v \in Y} x_{sv} - 78 \sum_{v \in Z} x_{sv} + 16\, x(E(Y)) + 2\, x((Y : Z)) + 16\, x(E(Z)) \leq 108.$$

Finally, with input data

$$6\sum_{v\in Y} x_{sv} - 78\sum_{v\in Z} x_{sv} + 16\,x(E(Y)) + 2\,x((Y:Z)) + 16\,x(E(Z)) \le 108,$$
$$-6\sum_{v\in Y} x_{sv} + 6\sum_{v\in Z} x_{sv} - x(E(Y)) + x((Y:Z)) - x(E(Z)) = 0,$$

we obtain $\lambda^\star = 7$ and

$$-36x(\delta(s)) + 9x(E(N\setminus\{s\})) \le 108.$$

The last inequality is a multiple of an inequality of the form (4.16).

The goal of the remainder of this section is to show that inequalities (4.16) define facets of $P^c_{\mathrm{Cut}}(E)$. With respect to $P^{(c_p)}_{\mathrm{Cut}}(E)$, an inequality among (4.16) is equivalent to $x(\delta(s)) \ge c_p$, since $x(E) = c_p(n-c_p)$ for all $x \in P^{(c_p)}_{\mathrm{Cut}}(E)$. An analogous observation holds for c_{p+1}. Thus, in order to show that the inequalities (4.16) are indeed facet defining, we first study the inequalities $x(\delta(s)) \ge k$ with respect to $P^{(k)}_{\mathrm{Cut}}(E)$.

To simplify the following proofs we recall some facts from Linear Algebra. Denote the kernel and the image of a matrix $A \in \mathbb{R}^{m\times n}$ by $\ker(A)$ and $\mathrm{im}(A)$, respectively. Denote by A_i the ith column of A. Let $v_1,\dots,v_k \in \ker(A)$ and $v_{k+1},\dots,v_r \in \mathbb{R}^n$. In order to show that these vectors are linearly independent, it is sufficient to do so for the vectors $v_1,\dots,v_k$ and $Av_{k+1},\dots,Av_r$ separately. Moreover, we need the following lemma.

Lemma 4.3. Let $n\in\mathbb{N}$, $n\ge 2$, and $\alpha,\beta\in\mathbb{R}$. The $n\times n$ matrix A defined by

$$a_{ij} = \begin{cases} \alpha & \text{if } i=j, \\ \beta & \text{otherwise} \end{cases}$$

has full rank if and only if $\alpha\ne\beta$ and $\alpha+(n-1)\beta\ne 0$.

Proof. Clearly, if $\alpha=\beta$, then $\mathrm{rank}(A) < n$. If $\alpha+(n-1)\beta = 0$, then $\sum_{i=1}^n A_i = 0$, which implies that $\mathrm{rank}(A) < n$. To show the converse, define a new column $A_{n+1} := \frac{\beta}{\alpha+(n-1)\beta}\sum_{i=1}^n A_i$. All entries of A_{n+1} are equal to β. Thus, the matrix B defined by $B_i := A_i - A_{n+1}$ for $i=1,\dots,n$, and $B_{n+1} = A_{n+1}$ has entries $\alpha-\beta$ on the diagonal of the first n entries. Since $\alpha\ne\beta$, it follows that $\mathrm{rank}(B) = n$. We conclude that $\mathrm{rank}(A) = \mathrm{rank}(A, A_{n+1}) = \mathrm{rank}(B) = n$. $\square$

Denoting by $\mathbb{1}_{n,n}$ the $n\times n$ matrix of all ones and by $I_{n,n}$ the $n\times n$ identity matrix, the matrix defined in Lemma 4.3 is equal to $\beta\mathbb{1}_{n,n} + (\alpha-\beta)I_{n,n}$.

Theorem 4.4. Let $K_n = (N, E)$ be the complete graph on n nodes and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ be an integer. Then,

$$\dim P_{\mathrm{Cut}}^{(k)}(E) = \begin{cases} n-1, & \text{if } k = 1, \\ |E| - n, & \text{if } n \text{ is even and } k = n/2, \\ |E| - 1, & \text{otherwise.} \end{cases} \tag{4.17}$$

Proof. In case that $k = 1$, the statement is clearly true. If n is odd and $k = \lfloor \frac{n}{2} \rfloor$, $P_{\mathrm{Cut}}^{(k)}(E)$ is the face of $P_{\mathrm{Cut}}(E)$ induced by the inequality $x(E) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, which by Theorem 2.1 of Barahona and Mahjoub [7] defines a facet of $P_{\mathrm{Cut}}(E)$. Since $P_{\mathrm{Cut}}(E)$ is fulldimensional, it follows $\dim P_{\mathrm{Cut}}^{(k)}(E) = |E| - 1$. Next, let n be even and $k = n/2$. Then, for any $s \in N$, $|N^s|$ is odd and $\ell := k - 1 = \lfloor \frac{|N^s|}{2} \rfloor$. Hence, $\dim P_{\mathrm{Cut}}^{(\ell)}(E^s) = |E^s| - 1 = (|E| - n + 1) - 1 = |E| - n$. Since any cut $\delta^s(W)$ with $|W| = \ell$ in K_n^s can be augmented to a cut in K_n by adding node s to W, it follows immediately that $\dim P_{\mathrm{Cut}}^{(k)}(E) \geq |E| - n$. In order to show equality, we remark that any cut $\delta(W)$ of K_n with $|W| = \frac{n}{2}$ satisfies the n linearly independent equations $x(\delta(v)) = \frac{n}{2}$, $v \in N$.

Finally, let $2 \leq k < \lfloor \frac{n}{2} \rfloor$. Since all points in $P_{\mathrm{Cut}}^{(k)}(E)$ satisfy the equation

$$x(E) = k(n-k), \tag{4.18}$$

it follows that $\dim P_{\mathrm{Cut}}^{(k)}(E) \leq |E| - 1$. To show equality, let $b^T x = \beta$ be an equation that is satisfied by all $x \in P_{\mathrm{Cut}}^{(k)}(E)$. Our goal is to show that $b^T x = \beta$ is a multiple of (4.18).

Let $N = S \,\dot\cup\, T \,\dot\cup\, U \,\dot\cup\, \{v\} \,\dot\cup\, \{w\}$ be a partition of N such that $|S| = |T| = k - 1$ and $|U| = n - 2k$. Similar as in the proof of Lemma 2.5 of Barahona, Grötschel, and Mahjoub [6] one can show that $b((s : U)) = b((t : U))$ by considering the cuts

$$\begin{aligned} C_1 &:= (S \cup \{s\} \cup U : T \cup \{t\}), \qquad & C_2 &:= (S \cup \{s\} : T \cup \{t\} \cup U) \\ C_3 &:= (S \cup \{t\} \cup U : T \cup \{s\}), \qquad & C_4 &:= (S \cup \{t\} : T \cup \{s\} \cup U). \end{aligned}$$

Here, for any $y \in N$ and $Z \subseteq N \setminus \{y\}$, $b((y : Z))$ denotes the sum $\sum_{v \in Z} b_{yv}$. Now let $W \subseteq N \setminus \{s, t\}$ be any node set of cardinality $n - 2k + 1$. Since U was arbitrarily chosen, we have $b((s : W \setminus \{v\})) = b((t : W \setminus \{v\}))$ for each $v \in W$. Defining $z_v := b_{sv} - b_{tv}$ for $v \in W$, we can write this set of equations as $(\mathbb{1}_{|W|,|W|} - I_{|W|,|W|})z = 0$. By Lemma 4.3, $\mathbb{1}_{|W|,|W|} - I_{|W|,|W|}$ is a nonsingular matrix, and thus $z = 0$ is the only solution implying $b_{sv} = b_{tv}$ for all $v \in W$. Since s, t, and W were arbitrarily chosen, we can conclude that $b_e = \sigma$ for all $e \in E$ for some $\sigma \in \mathbb{R}$, and hence $\beta = \sigma k(n-k)$. Consequently, $b^T x = \beta$ is a multiple of (4.18). $\square$

Theorem 4.5. Let $K_n = (N, E)$ be the complete graph on n nodes and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ integer. Then for any $s \in N$, the inequality

$$x(\delta(s)) \geq k \tag{4.19}$$

is valid for $P^{(k)}_{\text{Cut}}(E)$. It defines a facet of $P^{(k)}_{\text{Cut}}(E)$ if and only if $k = 1$, otherwise it is a face of dimension $|E| - n$.

Proof. Let F be the face induced by (4.19). In case $k = 1$, the cuts $\delta(w)$ for $w \in N \setminus \{s\}$ are tight with respect to (4.19). Since their incidence vectors are linearly independent, we conclude that F has dimension $n - 2$, that is, (4.19) defines a facet. When n is even and $k = \frac{n}{2}$, then (4.19) is satisfied with equality by all points $x \in P^{(k)}_{\text{Cut}}(E)$. Consequently, $F = P^{(k)}_{\text{Cut}}(E)$ which by Theorem 4.4 implies $\dim F = |E| - n$.

Next, let $2 \leq k \leq \frac{n}{2} - 1$. The incidence vector of a feasible tight cut satisfies equation (4.18) and the $n - 1$ equations

$$x_{sv} - \frac{x(\delta(v)) - k}{n - 2k} = 0, \qquad v \in N^s. \tag{4.20}$$

Since these equations are linearly independent, it follows that $\dim F \leq |E| - n$. The inequality $\dim F \geq |E| - n$ follows from the fact that $\dim P^{(k)}_{\text{Cut}}(E^s) = |E^s| - 1 = |E| - n$ and any cut $\delta^s(W)$ with $|W| = k$ in K^s_n corresponds to a tight cut $\delta(W)$ in K_n.

Finally, let n be odd and $k = \lfloor \frac{n}{2} \rfloor$. Clearly, $\dim F \leq |E| - n$, since the incidence vector of a tight cut satisfies the equations (4.18) and (4.20). To show equality, consider the cut polytope $P^{(k)}_{\text{Cut}}(E^s)$ which can be obtained by projecting F to $\mathbb{R}^{E^s}$. Its dimension is $|E^s| - |N^s|$. Consequently, there are $r := |E^s| - |N^s| + 1$ linearly independent incidence vectors of cuts, say $\delta^s(W_1), \ldots, \delta^s(W_r)$ of K^s_n with $|W_i| = k$. Since the shores of a cut $\delta^s(W_i)$ have the same cardinality, we may assume w.l.o.g. that for some $t \in N^s$, $t \in W_i$ for $i = 1, \ldots, r$. Of course, the cuts $C_i := \delta(W_i \cup \{s\})$, $i = 1, \ldots, r$ are tight with respect to (4.19), and their incidence vectors are linearly independent, too. In addition, besides (4.18) and (4.20), these vectors satisfy the $n - 2$ equations

$$x_{sv} - x_{tv} + x_{st} = 0 \qquad \text{for all } v \in \tilde{N}, \tag{4.21}$$

where $\tilde{N} := N^s \setminus \{t\}$. Since $r + (n - 2) = |E| - n + 1$, it suffices to construct $(n - 2)$ further tight cuts whose incidence vectors are linearly independent and linearly independent of the former points. To this end, let w.l.o.g. $N' = \{1, \ldots, n-2\}$ and $U = \{s, 1, \ldots, n-k\}$. For each v with $1 \leq v \leq n-k$, $\tilde{C}_v :=$

$\delta(U \setminus \{v\})$ is a feasible tight cut. Moreover, for each $v \in \{n-k+1, \ldots, n-2\}$ and any $u, \tilde{u} \in \{1, \ldots, n-k\}$, the cut $\tilde{C}_v := \delta((U \cup \{v\}) \setminus \{u, \tilde{u}\})$ is tight. Let A be the matrix associated with the left hand side of the equations (4.21). Since $\chi^{C_i} \in \ker(A)$, it remains to show that the matrix $B := [A\chi^{\tilde{C}_1}, \ldots, A\chi^{\tilde{C}_{n-2}}]$ has full rank. Indeed, B is of the form

$$B = \begin{bmatrix} 2I_{n-k,n-k} & 0 \\ * & 2\left(\mathbb{1}_{k-2,k-2} - I_{k-2,k-2}\right) \end{bmatrix},$$

which implies immediately $\operatorname{rank}(B) = n-2$. $\square$

Now we prove that inequalities (4.16) induces facets of $P^c_{\text{Cut}}(E)$.

Theorem 4.6. Let $K_n = (N, E)$ be the complete graph on n nodes and $c = (c_1, \ldots, c_m)$ a cardinality sequence with $1 \le c_1 < \ldots c_m \le \lfloor \frac{n}{2} \rfloor$. Then, $P^c_{\text{Cut}}(E)$ is fulldimensional. Moreover, the inequality (4.16)

$$x(E) - (n - c_p - c_{p+1})x(\delta(s)) \le c_p c_{p+1}$$

defines a facet of $P^c_{\text{Cut}}(E)$ for all $s \in N$.

Proof. W.l.o.g, let $N^s = \{1, \ldots, n-1\}$. Let F be the face of $P^c_{\text{Cut}}(E)$ induced by (4.16). We will show that $\dim F = |E| - 1$, which implies that $P^c_{\text{Cut}}(E)$ is fulldimensional due to the fact that not all feasible cuts are tight. It follows that $c_{p+1} \ge 2$, since $m \ge 2$. Inequality (4.16) is equivalent to $x(\delta(s)) \ge c_{p+1}$ with respect to $P^{(c_{p+1})}_{\text{Cut}}(E)$. Hence, by Theorem 4.5, there are $q := |E| - n + 1$ linearly independent incidence vectors of tight cuts $C_i := \delta(W_i)$ with $s \in W_i$ and $|W| = n - c_{p+1}$. In what follows, we distinguish three cases.

(1) Let $c_{p+1} < \lfloor \frac{n}{2} \rfloor$. Then, the vectors χ^{C_i} satisfy the $n-1$ equations

$$x_{sv} + \frac{x(\delta(s)) - x(\delta^s(v))}{n - 2c_{p+1} - 1} = 0 \qquad \forall\, v \in N^s. \tag{4.22}$$

Denote by A the matrix associated with the left hand side of (4.22). To construct $n-1$ further points, consider the set $U = \{s, 1, \ldots, r\}$ with $r = n - c_p - 2$. For each node $v \in \{r+1, \ldots, n-1\}$, the cut $\tilde{C}_v := \delta(U \cup \{v\})$ is tight, and for each node $v \in \{1, \ldots, r\}$ and any two disjoint nodes $t, u \in \{r+1, \ldots, n-1\}$, the cut $\tilde{C}_v := \delta((U \cup \{t, u\}) \setminus \{v\})$ is tight. Since $\chi^{C_i} \in \ker(A)$ for $i = 1, \ldots, q$, it is sufficient to prove that the matrix $B := \left[A\chi^{\tilde{C}_1}, \ldots, A\chi^{\tilde{C}_{n-1}}\right]$ has full rank. It is not hard to see that B is of the form

$$B = \begin{bmatrix} 0 & \sigma \cdot I_{r,r} \\ \sigma(\mathbb{1}_{\bar{r},\bar{r}} - I_{\bar{r},\bar{r}}) & * \end{bmatrix},$$

where $\bar{r} = n - 1 - r$ and $\sigma = 1 + \frac{1+2c_p-n}{n-2c_{p+1}-1}$. Clearly, $I_{r,r}$ and $\mathbb{1}_{\bar{r},\bar{r}} - I_{\bar{r},\bar{r}}$ have full rank, and hence, also B.

(2) Let $c_{p+1} = \frac{n}{2}$. This time the vectors χ^{C_i} satisfy the equations

$$x(\delta(v)) - x(\delta(s)) = 0 \qquad \forall\, v \in N^s. \tag{4.23}$$

Let A be the matrix associated with the left hand side of (4.23). Of course, $\chi^{C_i} \in \ker(A)$ for $i = 1, \dots, q$. Next, consider again the cuts $\tilde{C}_v$, $v = 1, \dots, n-1$. The matrix $B := \left[A\chi^{\tilde{C}_1}, \dots, A\chi^{\tilde{C}_{n-1}}\right]$ is of the form

$$B = \begin{bmatrix} 0 & (n-2c_p)I_{r,r} \\ (n-2c_p)(\mathbb{1}_{\bar{r},\bar{r}} - I_{\bar{r},\bar{r}}) & * \end{bmatrix}.$$

Since B has obviously full rank, (4.16) defines a facet if $c_{p+1} = \frac{n}{2}$.

(3) Let $c_{p+1} = \frac{n-1}{2}$. When $c_p > 1$, reverse the roles of c_p and c_{p+1} and apply (1). In what follows, we may assume that $c_p = 1$. The vectors χ^{C_i} satisfy the equations

$$x(\delta^s(v)) - x(\delta(s)) = 0 \qquad \forall\, v \in N^s. \tag{4.24}$$

Let A denote the left hand side of the system (4.24). Since $\chi^{C_i} \in \ker(A)$, it remains to show that the images of the incidence vectors of the cuts $\delta(i)$, $i = 1, \dots, n-1$ are linearly independent. Now, for each $i \in \{1, \dots, n-1\}$,

$$x(\delta^s(j)) - x(\delta(s)) = \begin{cases} n-3 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $A \cdot \left[\chi^{\delta(1)}, \dots, \chi^{\delta(n-1)}\right] = (n-3)I_{n-1,n-1}$. □

Since the argumentation for showing that inequalities (4.16) are facet defining uses the facial structure of the polytopes $P^{(k)}_{\text{Cut}}(E)$, very similar results can be obtained for $\bar{P}^c_{\text{Cut}}(E)$.

4.4 EXTENSIONS

The chapter shows that the incorporation of the combinatorial structure of a COP $\Pi = (E, \mathcal{I}, w)$ into forbidden cardinality inequalities may result in strong inequalities that cut off feasible solutions $I \in \mathcal{I}$ of forbidden cardinality. In particular, well-known attributes of matroid theory (closedness) and matroidal relaxations might play an important role in this context.

It suggests itself to search for complete linear descriptions of polyhedra $P^c_{\mathcal{I}}(E)$ associated with CCCOPs for those problems for which a complete linear description of the polyhedron $P_{\mathcal{I}}(E)$ associated with the ordinary COP is known. For instance, the matching polytope $P_{\text{MATCH}}(E)$ is determined by the inequalities (4.2)-(4.4). However, we do not know whether it is sufficient to add inequalities (4.7), (4.6), and the cardinality bound $c_1 \leq x(E) \leq c_m$ in order to obtain a complete linear description of $P^c_{\text{MATCH}}(E)$.

If a complete linear description of $P^{(c_i)}_{\mathcal{I}}(E)$ is known for $i = 1, \ldots, m$, then an extended formulation for $P^c_{\mathcal{I}}(E)$ can be obtained via *disjunctive programming*, which is optimization over the union of polyhedra. The notion *extended formulation* will be defined in Chapter 5.1. Below we restate a well-known result of Balas [2].

Theorem 4.7. Given r polyhedra $P^i = \{x \in \mathbb{R}^n : A^i x \geq b^i\} = \text{conv}(V^i) + \text{cone}(R^i)$, the following system:

$$\begin{aligned} y &= \sum_{i=1}^r x^i \\ A^i x^i &\geq \lambda^i b^i, & i = 1, \ldots, r \\ \sum_{i=1}^r \lambda^i &= 1 \\ \lambda_i &\geq 0, & i = 1, \ldots, r \end{aligned} \tag{4.25}$$

provides an extended formulation for the polyhedron

$$P := \text{conv}\left(\bigcup_{i=1}^r V^i\right) + \text{cone}\left(\bigcup_{i=1}^r R^i\right). \qquad \square$$

In our context, $r = m$ and $P^i = P^{(c_i)}_{\mathcal{I}}(E)$ for $i = 1, \ldots, m$. In addition, in many cases the linear descriptions of the polyhedra $P^{(c_i)}_{\mathcal{I}}(E)$ will only differ in the cardinality constraints $x(E) = c_i$, that is, there is some common constraint system $Ax \geq b$ such that $P^{(c_i)}_{\mathcal{I}}(E) = \{x \in \mathbb{R}^n : Ax \geq b,\, x(E) = c_i\}$ for $i = 1, \ldots, m$. This results in the system

$$\begin{aligned} y &= \sum_{i=1}^m x^i \\ Ax^i &\geq \lambda^i b, & i = 1, \ldots, m \\ x^i(E) &= \lambda^i c_i, & i = 1, \ldots, m \\ \sum_{i=1}^m \lambda^i &= 1 \\ \lambda_i &\geq 0, & i = 1, \ldots, m. \end{aligned} \tag{4.26}$$

For example, cardinality constrained matchings or the intersection of two cardinality constrained matroids can be modeled this way. However, our attempts to derive complete linear descriptions of the associated polytopes via projection were without success.

Chapter 5

Dynamic programming, projection, and the hop constrained path polytope

A frequently occurring phenomenon in combinatorial optimization is that hard problems can be solved in polynomial time for some cases using special algorithms. For instance, the shortest path problem defined on a directed graph $D = (N, A)$ with arc weights $w_a \in \mathbb{R}$ is known to be NP-hard, but if the weights are nonnegative, it can be solved in polynomial time with Dijkstra's algorithm [26]. Or, in order to give another example, the maximum independent set problem, where the independence system is given by an independence oracle, is NP-hard, but if the independence system forms a matroid, then the problem can be solved in oracle-polynomial time with the greedy algorithm. For further aspects of this discussion, we refer to Chapter 1.

Although such algorithms for limited models are interesting in their own right, they are even of greater value if there is a way for translating results to assist with more involved instances. One such way is to derive a partial or even complete linear description of the polyhedron associated with the limited model. Members of such a linear description are usually valid inequalities for the polyhedron associated with the harder problem. Hence, they may provide cutting planes to strengthen the linear programming relaxation of the model for the harder problem. Moreover, the separation problem for those inequalities is quite often efficiently solvable. Aspects of such an approach in connection with matroidal relaxations of (cardinality constrained) combinatorial optimization problems have been discussed in Chapter 4.2.

In this chapter, we consider such an approach for the *hop constrained shortest path problem* (HCSPP). This problem is defined as follows. Given a directed graph $D = (N, A)$, a length function $d : A \to \mathbb{R}$, two distinct nodes $s, t \in N$, and a number $k \in \mathbb{N}$, find a shortest (s, t)-path (with respect to d) using at most k arcs. Clearly, this problem is NP-hard for an arbitrary length function d. However, if D has no negative cycles with respect to d, it can be solved with the Moore-Bellman-Ford algorithm [11, 36, 66] – a dynamic programming algorithm –, in polynomial time.

In the following, we will describe how this algorithm could possibly help in finding facet defining inequalities for the polytope associated with the hop constrained shortest path problem, namely the hop constrained path polytope. In short, we use the inherent structure of the Moore-Bellman-Ford algorithm, which can be essentially expressed by the well known Bellman equations, to provide (compact) extended formulations for two relaxations of the hop constrained path polytope: the dominant of the hop constrained path polytope and the hop constrained walk polytope. By means of projection we derive facet defining inequalities for these relaxations and show how these inequalities are related to the hop constrained path polytope itself.

The chapter is structured as follows. Section 5.1 briefly sketches the main ideas behind projection. Section 5.2 gives an overview over polyhedral aspects of dynamic programming. In particular, we review the fundamental work of Martin, Rardin, and Campbell [60] who have provided a framework for deriving linear characterizations of dynamic programs. In Section 5.3, we review the most important results from the literature related to the hop constrained path polytope. Moreover, we introduce two relaxations of this polytope mentioned above corresponding to the cases that the given length function d is nonnegative or has no negative cycles. Section 5.4 breaks down the dynamic programming paradigm to the hop constrained shortest path problem. Next, Section 5.5 relates the separation problems of the two relaxations of the hop constrained path polytope to the multicommodity flow problem and length-bounded cut and flow problems. The results there imply that it will be quite hard to design combinatorial algorithms that solve these separation problems in polynomial time. Finally, Section 5.6 characterizes all 0/1-facet defining inequalities for the dominant of the hop constrained path polytope and all facet defining inequalities for the hop constrained walk polytope with coefficients in $\{-1, 0, 1\}$ using the dynamic program paradigm. The attained results will be related to already known results.

5.1 Projection

In this section, we briefly describe the general idea from the literature for obtaining a linear characterization of a polyhedron via projection and address the separation problem for such a polyhedron. For projection methods in various settings, we refer to the review article of Balas [3].

Given a polyhedron of the form

$$Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \geq a\},$$

the *projection of Q onto the x-space* is defined as

$$\mathrm{Proj}_x(Q) := \{x \in \mathbb{R}^p : \exists\, y \in \mathbb{R}^q \text{ with } (x, y) \in Q\}.$$

Conversely, a system of the form $Ax + Bx \geq a$ is said to be an *extended formulation* for a polyhedron $P \subseteq \mathbb{R}^p$ if $P = \text{Proj}_x(Q)$, where $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + Bx \geq a\}$.

Given a polyhedron of the form

$$Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \geq a\}$$

and the projection $\text{Proj}_x(Q)$ of Q onto the x-space, the polyhedral cone $\mathcal{C} := \{v : v^T B = 0^T, v \geq 0\}$ is called the *projection cone*.

Given an extended formulation for a polyhedron $P \subseteq \mathbb{R}^p$, the following theorem due to Balas [3] addresses the task to derive a complete linear description (in the space $\mathbb{R}^p$) for this polyhedron. It is based on Benders' Decomposition Theorem [12].

Theorem 5.1 (Balas [3])**.** Let $Q = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \geq a\}$ be a polyhedron. Then,

$$\text{Proj}_x(Q) = \{x \in \mathbb{R}^p : (v^T A)x \geq v^T a \text{ for all } v \in \text{extr}(\mathcal{C})\},$$

where $\text{extr}(\mathcal{C})$ denotes the set of extreme rays of the projection cone $\mathcal{C}$. $\square$

Usually, it is difficult to determine all extreme rays of $\mathcal{C}$ or all those extreme rays $v \in \text{extr}(\mathcal{C})$, whose corresponding inequalities $(v^T A)x \geq v^T a$ define facets of $\text{Proj}_x(Q)$. However, sometimes the extreme rays or a subset of them have a convenient structure.

Another result due to Liu [57] concerns the separation problem for a polyhedron characterized by an extended formulation. Before stating a reformulation of this result, we introduce a definition. We say that the separation problem for a polyhedron $P \subseteq \mathbb{R}^p$ and a point $x^\star \in \mathbb{R}^p$ is *equivalent* to a linear program of the form

$$\begin{array}{ll} \min & (Cx^\star - c)^T v \\ \text{s.t.} & D^T v = 0,\ v \geq 0 \end{array}$$

if the following holds:

(i) $x^\star \in P$ if and only if an optimal solution of the LP is $v^\star = 0$, and

(ii) in case of $x^\star \notin P$, every feasible point $\tilde{v}$ with $(Cx^\star - c)^T \tilde{v} < 0$ defines a hyperplane $(\tilde{v}^T C)x = \tilde{v}^T c$ that separates $x^\star$ from P.

Theorem 5.2 (Liu [57])**.** Let $P \subseteq \mathbb{R}^p$ and $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + By \geq a\}$ be two polyhedra. Then,

$$P = \text{Proj}_x(Q)$$

if and only if the separation problem for P and any point $x^\star \in \mathbb{R}^p$ is equivalent to the linear program

$$\begin{array}{ll} \min & (Ax^\star - a)^T v \\ \text{s.t.} & B^T v = 0,\ v \geq 0. \end{array} \tag{5.1}$$

Proof. The linear program (5.1) has either an optimal solution, – and then $v^\star = 0$ is an optimal solution –, or it is unbounded.

To show the necessity, let $P = \mathrm{Proj}_x(Q)$ and $x^\star \in \mathbb{R}^p$ be any point. By Theorem 5.1, $P = \mathrm{Proj}_x(Q)$ implies immediately $P = \{x \in \mathbb{R}^p : (v^T A)x \geq v^T a$ for all $v \in \mathcal{C}\}$. Thus, $x^\star \in P$ if and only if the LP (5.1) has as optimal solution $v^\star = 0$. Moreover, in case of $x^\star \notin P$, the LP (5.1) is unbounded. Thus, $(Ax^\star - a)^T \tilde{v} < 0$ for some feasible point $\tilde{v}$ of the linear program. Indeed, each such point $\tilde{v}$ defines a hyperplane $(\tilde{v}^T A)x = \tilde{v}^T a$ that separates $x^\star$ from P.

To show the sufficiency, we have to verify that $P = \mathrm{Proj}_x(Q)$. Let $z \in P$. Since the separation problem for P and $x^\star = z$ is equivalent to the LP (5.1) and $z \in P$, we conclude that $v^\star = 0$ is an optimal solution of (5.1) with $x^\star = z$. Consequently, $(v^T A)z \geq v^T a$ for all v satisfying $B^T v = 0,\ v \geq 0$. Thus, $z \in P = \mathrm{Proj}_x(Q)$. Hence, if $z \in P = \mathrm{Proj}_x(Q)$, then $(v^T A)z \geq v^T a$ for all v satisfying $B^T v = 0,\ v \geq 0$ Consequently, the LP (5.1) is bounded for $x^\star = z$ and $v^\star = 0$ is an optimal solution. This implies $z \in P$. □

5.2 Polyhedral aspects of dynamic programming

In this section, we focus on polyhedral aspects of dynamic programming, also termed *divide-and-conquer*. A dynamic programming algorithm decomposes a problem into a recursive sequence of smaller subproblems, solves these subproblems, and assembles a solution for the original problem from the solutions of the subproblems. In the simplest case this scheme can be modeled by solving a shortest path problem in an acyclic digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$. The different states or interim phases that can be taken during the algorithm are represented by nodes of $\mathcal{D}$, decisions that can be made to switch from state i to j by arcs $(i, j) \in \mathcal{A}$, and the costs of these changes by arc lengths. In this context, $\mathcal{D}$ is called *dynamic programming graph*, or abbreviated, *DP-graph.* A solution for the original problem corresponds to a path from the first node, say s, to the terminal node, say t. Consequently, the original problem can be solved by finding a shortest path from s to t in $\mathcal{D}$.

If the dynamic program can be modeled in such a way, a polyhedral description can be easily derived by writing down the shortest path problem as a linear program. Martin, Rardin, and Campbell [60] provide a framework to derive linear characterizations of more general dynamic programs, namely if a state relies not only on a single other state but on several ones. This can be fittingly modeled by an acyclic hypergraph $\mathcal{H} = (\mathcal{N}, \mathcal{E})$, called *dynamic programming hypergraph* or *DP-hypergraph*. As in the simple case states of the dynamic program correspond to nodes in the hypergraph, while the composition of a set J of states to a state ℓ will be represented by a hyperarc (J, ℓ). The cost of decision (J, ℓ) will be denoted by $c[J, \ell]$. The acyclicity of $\mathcal{H}$ will be expressed by a numbering function

$$\sigma : \mathcal{N} \to \{1, 2, \dots, m\},$$

with $m := |\mathcal{N}|$, such that

$$\sigma(j) < \sigma(\ell) \qquad \text{for all } (J, \ell) \in \mathcal{E},\ j \in J.$$

As in the simple case of a shortest path conceptualization, an optimal solution will include exactly one hyperarc pointing into the final state $t := \sigma^{-1}(m)$. The associated integer characterization of such a dynamic program is as follows. For each hyperarc (J, ℓ) introduce a binary variable $y[J, \ell]$ which takes the value 1 if decision (J, ℓ) is part of the solution, otherwise 0. In terms of these variables, the model reads:

$$\begin{aligned}
&\min \sum_{(J,\ell)\in\mathcal{E}} c[J,\ell]\,y[J,\ell] \\
&\text{subject to} \sum_{(J,t)\in\mathcal{E}} y[J,t] = 1, \\
&\sum_{(J,\ell)\in\mathcal{E}} y[J,\ell] - \sum_{\substack{(J,\bar{\ell})\in\mathcal{E}:\\ \ell\in J}} y[J,\bar{\ell}] = 0 && \text{for } \sigma(\ell) = 1, 2, \dots, m-1, \\
&y[J,\ell] \in \{0,1\} && \text{for all } (J,\ell) \in \mathcal{E}.
\end{aligned}$$

The first equation ensures that exactly one decision results in the final state t. The remaining equations guarantee that whenever some state $\ell \in \mathcal{N} \setminus \{t\}$ is part of the solution, then it has been accessed via a decision $(J, \ell) \in \mathcal{E}$ and has to contribute to a decision $(\bar{J}, \bar{\ell}) \in \mathcal{E}$. Martin, Rardin, and Campbell [60] showed that the integrality conditions $y[J, \ell] \in \{0, 1\}$ can be replaced by the nonnegativity constraints $y[J, \ell] \geq 0$ for all $(J, \ell) \in \mathcal{E}$ if each state ℓ can be connected to a subset $I[\ell]$ of a so-called *reference set* I such that

(i) the subsets are consistent with the acyclic order:

$$I[j] \subseteq I[p] \qquad \text{for all } (J, p) \in \mathcal{E},\ j \in J,$$

(ii) subsets for different tails of the same hyperarc are disjoint:

$$I[j] \cap I[j'] = \varnothing \qquad \text{for all } (J, p) \in \mathcal{E},\ j, j' \in J,\ j \neq j'.$$

Consequently, in this case, we obtain an LP-formulation for the original problem in terms of the decision variables $y(J, \ell)$ associated with the dynamic program:

$$\begin{aligned}
\min \sum_{(J,\ell)\in\mathcal{E}} c[J,\ell]y[J,\ell] & & \\
\text{subject to} \sum_{(J,t)\in\mathcal{E}} y[J,t] = 1, & & \\
\sum_{(J,\ell)\in\mathcal{E}} y[J,\ell] - \sum_{\substack{(J,\bar{\ell})\in\mathcal{E}:\\ \ell\in J}} y[J,\bar{\ell}] = 0 & \qquad \text{for } \sigma(\ell) = 1, 2, \dots, m-1, \\
y[J,\ell] \geq 0 & \qquad \text{for all } (J,\ell)\in\mathcal{E}.
\end{aligned} \tag{5.2}$$

We collect some easy but important facts about the above LP observed by Martin, Rardin, and Campbell [60].

Observation 1. If $\mathcal{H}$ is just an acyclic digraph, we obtain the usual LP-formulation for the shortest path problem.

Observation 2. If the dynamic program is polynomial in the input size of the original problem, then the LP (5.2) is *compact*, which means that the linear program is posed over polynomially many variables and constraints.

Observation 3. Let ν be the dual multiplier for the first equation of the LP (5.2) and $u[\ell]$ the dual variable for the flow conservation constraint associated with ℓ. The construction of an optimal solution of the dual

$$\begin{aligned}
\max \quad & \nu & & \\
\text{subject to} \quad & \nu - \sum_{j\in J} u[j] \leq c[J,t] & & \text{for all } [J,t]\in\mathcal{E}, \\
& u[\ell] - \sum_{j\in J} u[j] \leq c[J,\ell] & & \text{for all } [J,\ell]\in\mathcal{E}, \ell\neq t
\end{aligned} \tag{5.3}$$

in Algorithm 3 is essentially the dynamic programming computation itself.

Algorithm 3: Construction of an optimal solution $\nu^\star$, $u^\star$ of (5.3).

for $\sigma(\ell) := 1$ **to** t **do**
  Set $u^\star[\ell] := \min\{c[J,\ell] + \sum_{j\in J} u^\star[j] : (J,\ell)\in\mathcal{E}\}$.
end
Set $\nu^\star := u^\star[t]$.

In view of Observation 2 we remark that a compact linear program can be solved in polynomial time (e.g., Khachiyan [54], Karmarkar [53]), or more precisely, in strongly polynomial time (e.g., Orlin [69] and Tardos [79]).

The topic of deriving linear characterizations for combinatorial optimization problems via dynamic programs, has been paid some attention, mainly in the 1980's and 1990's. Prodon, Liebling, and Groflin [71] give a dynamic programming based polyhedral characterization for Steiner trees on directed series-parallel graphs (see also Goemans [43]). Barany, Van Roy, and Wolsey [8], Eppen and Martin [30], and Martin, Rardin, and Campbell [60] provide such formulations for various kinds of lot sizing problems. Further examples are Martin et al. [60] for k-terminal graphs, Liu [57] for 2-terminal Steiner trees, and Raffensperger [72] for the cutting stock, the tank scheduling, and the traveling salesman problem. Recently, Kaibel and Loos (personal communication) provided a dynamic programming based extended formulation for *full orbitopes*.

To derive such linear descriptions for hard combinatorial optimization problems as the TSP is, however, usually of small use, since their input sizes become very large already for small problem instances. More interesting is the question how one can benefit from such linear characterizations for polynomial solvable cases of hard combinatorial optimization problems. The entire problem is usually formulated in another decision space than the dynamic program. Hence, one can include the entire characterization of the dynamic program (5.2) in the relaxation of the harder case in form of an extended formulation. In addition, one can try to derive a linear characterization of the relaxation in the original space via projection of the extended formulation and include it in the formulation for the whole problem if possible.

The decision vectors of the dynamic program and of the relaxation can be usually connected by a transformation matrix T. Denoting the decision vectors of the relaxation by x and writing down the equations of the LP (5.2) in matrix form, the extended formulation reads as follows:

$$\begin{array}{rl} \min & d^T x \\ \text{s.t.} & Ay = e \\ & Ty = x \\ & x \geq 0,\ y \geq 0. \end{array} \tag{5.4}$$

For the remainder of this section, denote by Q the set of all feasible points (x, y) of (5.4). Projecting out the y-variables, we obtain a polyhedral characterization of the relaxation only in terms of x-variables. For the special case of (5.4), Martin, Rardin, and Campbell [60] showed the following.

Theorem 5.3 (Martin et al. [60]).

$$\text{Proj}_x(Q) = \{x \geq 0 : b^T x \leq -u_m \,\forall\, (u, b) \text{ satisfying } u^T A + b^T T \leq 0\},$$

where u_m denotes the last entry of u. □

Martin, Rardin, and Campbell [60] discuss several advantages and disadvantages to using model (5.4) instead of the projected model given by Theorem 5.3. Most of their arguments favour the use of the unprojected model: If the DP-graph is of polynomial size, then also the extended formulation (5.4), while the projected model often consists of exponential many inequalities, which requires separation routines. Moreover, the coefficients of the constraints of (5.4) are very simple, while the projection often leads to most unwieldy constraints.

In our opinion, there is no final answer to this question. For example, it is also worth to be considered to incorporate the extended formulation of the relaxation of a hard problem implicitly. Namely, it is also possible to work in the natural decision space of the hard combinatorial optimization problem and use the equivalence of the separation problem to a linear program, see Theorem 5.2. In this context, we obtain the following LP:

$$\begin{array}{ll} \min & -u^T e - v^T x^\star + z^T x^\star \\ \text{s.t.} & u^T A + v^T T + w^T I = 0 \\ & \qquad\qquad w \geq 0,\ z \geq 0. \end{array} \tag{5.5}$$

Note that, if the extended formulation (5.4) is compact, then also the LP (5.5), i.e. polynomial in the input size of the original problem and $x^\star$. Thus, in this case, the separation problem for $\text{Proj}_x(Q)$ and $x^\star$ can be solved in polynomial time.

Nevertheless, the relationship between the LP-formulation over the DP-graph and the corresponding model in the original space is at least of theoretical interest. Hence, we continue the investigation of projection mechanisms of dynamic programming based LP-formulations using the example of the hop constrained shortest path problem defined on a directed graph.

5.3 Facts on the hop constrained path polytope

Let $D = (N, A)$ be a simple digraph. Recall some basic definitions from Chapter 3. An (s,t)-walk is a sequence of arcs $W = (a_1, a_2, \ldots, a_q)$ such that $a_i = (i_{p-1}, i_p) \in A$ for $p = 1, \ldots, q$ with $i_0 = s$ and $i_q = t$. If all nodes i_p are distinct, then W is a path. If $s = t$, then W is a cycle. We denote by $\mathcal{W}_{s,t}(D)$ the collection of all (s,t)-walks of D and by $\mathcal{P}_{s,t}(D)$ the set of all

(s,t)-paths. Moreover, for any $k \in \mathbb{N}$, we denote by $\mathcal{W}^{\leq k}_{s,t}(D)$ and $\mathcal{P}^{\leq k}_{s,t}(D)$ the collection of all (s,t)-walks and -paths with at most k arcs, respectively. Paths and walks will be usually denoted only as a sequence of nodes, but their incidence vectors are defined in the arc space $\mathbb{R}^A$. Here, for any walk W, its incidence vector $\chi^W \in \mathbb{R}^A$ is defined by $\chi^W_a :=$ number of times in which the arc a is visited by W, for all $a \in A$. Since each path is a walk, its incidence vector is well-defined. Moreover, this definition is consistent with the usual definition of the incidence vector of a path. Note that different walks may have the same incidence vector, while for paths this statement is not true.

The digraph D possesses a length function $d : A \to \mathbb{R}$. For any path (or walk) $P = (i_0, i_1, i_2, \ldots, i_q)$, the number $d(P) := \sum_{p=1}^{q} d((i_{p-1}, i_p))$ is the *length of* P. In the *hop constrained shortest path (walk) problem* or the k-*hop constrained shortest path (walk) problem* we are looking for a path $P \in \mathcal{P}^{\leq k}_{s,t}(D)$ (a walk $W \in \mathcal{W}^{\leq k}_{s,t}(D)$) of minimum length. The hop constrained shortest walk problem can be solved in polynomial time with the Moore-Bellman-Ford algorithm [11, 36, 66], while the analogous path problem is NP-hard. If, however, D has no negative cycles with respect to d, it can also be solved in polynomial time for every k.

So it is assumed that the hop constrained path polytope $P^{\leq k}_{s,t\text{-path}}(D)$, that is, the convex hull of the incidence vectors of paths $P \in \mathcal{P}^{\leq k}_{s,t}(D)$, does not have a tractable facial structure in general.

According to (3.55), the integer points of the hop constrained path polytope $P^{\leq k}_{s,t\text{-path}}(D)$ are characterized by the system

$$\begin{aligned}
x(\delta^{\text{in}}(s)) &= 0, && && (5.6)\\
x(\delta^{\text{out}}(t)) &= 0, && && (5.7)\\
x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= \begin{cases} 1 \text{ if } i = s, \\ 0 \text{ if } i \in N \setminus \{s,t\}, \\ -1 \text{ if } i = t, \end{cases} && && (5.8)\\
x(A) &\leq k, && && (5.9)\\
x(\delta^{\text{out}}(i)) &\leq 1 && \forall\, i \in N \setminus \{s,t\}, && (5.10)\\
x(\delta^{\text{out}}(S)) &\geq x(\delta^{\text{out}}(j)) && \forall\, S \subset N, 3 \leq |S| \leq |N| - 3, && (5.11)\\
& && s,t \in S, j \in N \setminus S, && \\
x_{ij} &\in \{0,1\} && \forall\, (i,j) \in A. && (5.12)
\end{aligned}$$

Recall that, for any $S \subseteq N$, $\delta^{\text{out}}(S) = \{(i,j) \in A : i \in S,\, j \in N \setminus S\}$ and $\delta^{\text{in}}(S) = \{(i,j) \in A : i \in N \setminus S,\, j \in S\}$. As usual, for nodes $j \in N$, we write $\delta^{\text{out}}(j)$ and $\delta^{\text{in}}(j)$ instead of $\delta^{\text{out}}(\{j\})$ and $\delta^{\text{in}}(\{j\})$, respectively.

The *hop constrained walk polytope* $P^{\leq k}_{s,t\text{-walk}}(D)$, which is the convex hull of the incidence vectors of walks $W \in \mathcal{W}^{\leq k}_{s,t}(D)$, should have a simpler characterization, since the associated linear optimization problem is solvable in polynomial time. Its integer points are characterized by the flow conservation constraints (5.8), the cardinality bound (5.9), the one-sided min-cut inequalities (5.11), and the integrality conditions $x_{ij} \in \mathbb{Z}_+$ for all $(i,j) \in A$. The hop constrained walk polytope can be considered as a relaxation of the hop constrained path polytope that corresponds to the case that the length function $d : A \to \mathbb{R}$ has no negative cycles. In that case, minimizing d over $P^{\leq k}_{s,t\text{-path}}(D)$ is equivalent to minimizing d over $P^{\leq k}_{s,t\text{-walk}}(D)$.

We introduce another relaxation of $P^{\leq k}_{s,t\text{-path}}(D)$ that corresponds to the case that d is nonnegative. In this case, the hop constrained shortest path problem is equivalent to minimizing d over the *dominant of the hop constrained path polytope* $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D)) := P^{\leq k}_{s,t\text{-path}}(D) + \mathbb{R}^A_+$. An integer characterization of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ is given by the integrality constraints

$$x_{ij} \in \mathbb{Z}_+ \qquad \text{for all } (i,j) \in A,$$

the (s,t)*-min-cut inequalities*

$$x(C) \geq 1 \qquad \text{for all } (s,t)\text{-cuts } C, \tag{5.13}$$

and the jump inequalities (3.29)

$$\sum_{i=0}^{k-1} \sum_{j=i+2}^{k+1} x((S_i : S_j)) \geq 1$$

for all partitions

$$N = \bigcup_{p=0}^{k+1} S_p$$

of N with $S_0 = \{s\}$ and $S_{k+1} = \{t\}$ (see Fact 1, 5.6).

A third relaxation of the hop constrained path polytope is the *dominant of the hop constrained walk polytope*, defined as

$$\mathrm{dmt}(P^{\leq k}_{s,t\text{-walk}}(D)) := P^{\leq k}_{s,t\text{-walk}}(D) + \mathbb{R}^A_+.$$

Since each walk $W \in \mathcal{W}^{\leq k}_{s,t}(D)$ contains a path $P \in \mathcal{P}^{\leq k}_{s,t}(D)$, one easily verifies:

Proposition 5.4.

$$\mathrm{dmt}(P^{\leq k}_{s,t\text{-walk}}(D)) = \mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D)).$$

□

The aim of this chapter is to derive strong valid inequalities for $P^{\leq k}_{s,t\text{-path}}(D)$ using the dynamic programming paradigm. The polyhedra that correspond to the case that the hop constrained shortest path problem can be solved with means of dynamic programming in polynomial time have been identified. Hence, the next task is to investigate the facial structure of the introduced relaxations of $P^{\leq k}_{s,t\text{-path}}(D)$.

Although one can optimize over $P^{\leq k}_{s,t\text{-walk}}(D)$ and $\text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ in polynomial time, complete linear descriptions of these polyhedra are known only for a few cases in dependency of k. Following an argument of Dahl and Gouveia [23], we conjecture that these polyhedra do not have simple H-representations even if the underlying digraph is acyclic. Consider the binary knapsack problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} c_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} h_i y_i \leq L, \\ & \qquad y_i \in \{0,1\} \quad i = 1, \ldots, n, \end{aligned} \tag{5.14}$$

where $h_i \in \mathbb{N}$ for $i = 1, \ldots, n$. This problem can be transformed to a hop constrained shortest path problem as follows. The acyclic digraph $G = (N, A)$ to be constructed consists of n blocks, for each item i with height h_i one block. The blocks are defined on a subset of nodes $V := \{v_0, \ldots, v_n\}$. Each block consists of a (v_{i-1}, v_i)-path P^i of length 0 with exactly $h_i + 1$ arcs and the arc (v_{i-1}, v_i) of length c_i. For an illustration, see Figure 5.1.

Feasible solutions of the knapsack problem (5.14) and (v_0, v_n)-paths in G with at most $L + n$ arcs are in 1-1-correspondence. Let P be a (v_0, v_n)-path using at most $L + n$ arcs. For each block, this path consists of either the single arc (v_{i-1}, v_i) or the "long path" P^i with $h_i + 1$ arcs. In the first case, set

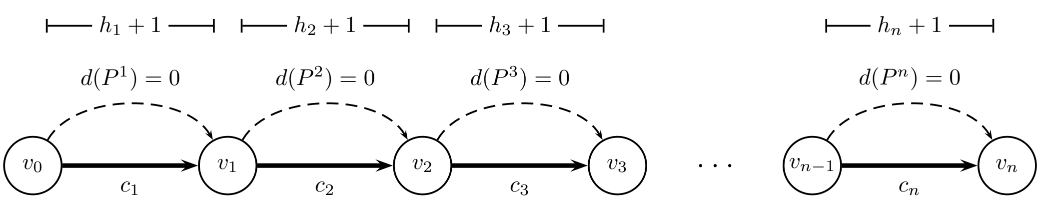


Figure 5.1: Illustration of G: for each item i with height h_i a block consisting of a (v_{i-1}, v_i)-path P^i of length 0 with exactly $h_i + 1$ arcs and the arc (v_{i-1}, v_i) of length c_i.

$y_i := 0$, in the second case $y_i := 1$. By construction, y is a feasible knapsack solution. The length of P is

$$\sum_{i=1}^{n} \underbrace{|P \cap \{(v_{i-1}, v_i)\}|}_{=0 \text{ or } 1} c_i,$$

while the costs of y are

$$\sum_{i=1}^{n} (1 - |P \cap \{(v_{i-1}, v_i)\}|) c_i.$$

Conversely, it is easily seen, each feasible solution of the knapsack problem (5.14) yields a (v_0, v_n)-path with at most $L + n$ arcs. Furthermore, the shortest such path corresponds to a knapsack solution of maximum costs.

From a polyhedral point of view, the knapsack polytope, that is, the convex hull of the feasible solutions of (5.14), is the projection of the polytope

$$\{(x, y) \in \mathbb{R}^A \times \mathbb{R}^n : x \in P^{\leq L+n}_{v_0,v_n\text{-path}}(G),\ y_i = 1 - x_{v_{i-1},v_i}, i = 1, \ldots, n\}$$

onto the y-space. Even more, due to the easy projection mechanism and the special block structure of G, a complete linear description of $P^{\leq L+n}_{v_0,v_n\text{-path}}(G)$ immediately yields one of the knapsack polytope, and vice versa. For instance, let

$$\begin{aligned} x(\delta^{\text{out}}(v_0)) &= 1, \\ x(\delta^{\text{in}}(v_n)) &= 1, \\ x(\delta^{\text{out}}(w)) - x(\delta^{\text{in}}(w)) &= 0 \qquad\qquad \text{for all } w \in N \setminus \{v_0, v_n\}, \\ Bx &\geq b \end{aligned}$$

be a complete linear description of $P^{\leq L+n}_{v_0,v_n\text{-path}}(G)$. Due to the flow conservation constraints, there is an equivalent formulation

$$\begin{aligned} x(\delta^{\text{out}}(v_0)) &= 1, \\ x(\delta^{\text{in}}(v_n)) &= 1, \\ x(\delta^{\text{out}}(w)) - x(\delta^{\text{in}}(w)) &= 0 \qquad\qquad \text{for all } w \in N \setminus \{v_0, v_n\}, \\ \tilde{B}x &\geq \tilde{b} \end{aligned}$$

such that the columns of $\tilde{B}$ associated with the arcs belonging to the "long paths" P^i are zero. The projection can now be performed by just substituting the variables x_{v_{i-1},v_i} by the terms $1 - y_i$. This gives the resulting inequality system

$$B'y \leq b',$$

where $b' := B'\mathbb{1} - b$ and the i-th column of B' is defined to be the column of $\tilde{B}$ associated with (v_{i-1}, v_i), $i = 1, \ldots, n$.

So the facial structure of $P^{\leq L+n}_{v_0,v_n\text{-path}}(G)$ is essentially the same as of the knapsack polytope. Since G is acyclic, the polytopes $P^{\leq L+n}_{v_0,v_n\text{-path}}(G)$ and $P^{\leq L+n}_{v_0,v_n\text{-walk}}(G)$ are identical. Hence, the above observation also holds for $P^{\leq L+n}_{v_0,v_n\text{-walk}}(G)$.

This relation between the knapsack problem and the hop constrained shortest path (walk) problem indicates that the hop constrained walk polytope probably does not have a nice linear characterization. Further aspects supporting this conjecture will be considered in Section 5.5.2. Nevertheless, one has to argue with caution, as in general the transformation from the knapsack problem to the hop constrained shortest path problem from above is not polynomial in the input size of a knapsack problem instance. For instance, due to the polynomial equivalence of optimization and separation (Grötschel, Lovász, and Schrijver [46]), $P^{\leq L+n}_{v_0,v_n\text{-path}}(G)$ can be completely described by valid inequalities whose input sizes are polynomial in the input size of G. However, not all of them have necessarily polynomial input sizes with respect to the input size of the underlying knapsack instance.

Previous work

In the last years, there has been some interest in the hop constrained path polytope itself and some closely related polyhedra, see, for instance, Dahl and Gouveia [23] for $P^{\leq k}_{s,t\text{-path}}(D)$ itself and Schrijver [76] for the dominant of the ordinary path polytope. Moreover, Dahl and Realfsen [25] have investigated $P^{\leq k}_{s,t\text{-path}}(D)$ defined on an acyclic digraph D, in particular, if D is a 2-graph. A digraph $D = (N, A)$ on node set $N = \{v_0, v_1, \ldots, v_n\}$ is called a *2-graph* if $A \subseteq \{(v_i, v_j) : 0 \leq i < j \leq i + 2 \leq n\}$. Dahl, Foldnes, and Gouveia [22] have provided a complete linear description of the hop constrained walk polytope $P^{\leq 4}_{s,t\text{-walk}}(D)$. Nguyen [68] investigated the facial structure of the dominant of the hop constrained path polytope defined on an undirected graph. A couple of papers consider $=k$-cycle and $\leq k$-cycle polytopes defined on directed as well as on undirected graphs. For an overview, see Table 3. Finally, the results of Chapter 3.2 can easily be adapted to $P^{\leq k}_{s,t\text{-path}}(D)$.

In the following, we are listing the most important results of the literature mentioned above in relation to the hop constrained path polytope.

Fact 1 (Schrijver [76])**.** The dominant of the ordinary path polytope, that is, the polyhedron $\mathrm{dmt}(P_{s,t\text{-path}}(D))$, is determined by the system

$$\begin{aligned} x(C) &\geq 1 && \text{for all } (s,t)\text{-cuts } C, \\ x_{ij} &\geq 0 && \text{for all arcs } (i,j) \in A. \end{aligned} \tag{5.15}$$

Fact 2 (Dahl and Realfsen [25])**.** A complete linear description of $P^{\leq k}_{s,t\text{-path}}(D)$ defined on a 2-graph D is given by the equations (5.6)-(5.8), the nonnegativity constraints $x_{ij} \geq 0$ for all $(i,j) \in A$, and the cardinality bound (5.9).

Fact 3 (Dahl and Gouveia [23])**.** The (s,t)-2-path polytope $P^{\leq 2}_{s,t\text{-path}}(D)$ is determined by the equations (5.6)-(5.8) and $x_{ij} = 0$ for all internal arcs (i,j), as well as the nonnegativity constraints $x_{ij} \geq 0$ for all arcs (i,j) with $i = s$ or $j = t$.

Fact 4 (Dahl and Gouveia [23])**.** The nonnegativity constraints $x_{ij} \geq 0$ for all $(i,j) \in A$, the equations (5.6)-(5.8), and the inequalities

$$x_{si} - \sum_{j \in N \setminus \{s,t\}} x_{ij} \geq 0 \qquad \text{for all } i \in N \setminus \{s,t\}$$

provide a complete linear description of $P^{\leq 3}_{s,t\text{-path}}(D)$.

Fact 5 (Dahl, Foldnes, and Gouveia [22])**.** The 4-hop constrained walk polytope $P^{\leq 4}_{s,t\text{-walk}}(D)$ is determined by the flow conservation constraints (5.8), the nonnegativity constraints $x_{ij} \geq 0$ for all $(i,j) \in A$, and the inequalities

$$\sum_{i \in I} x_{si} + \sum_{j \in J} x_{jt} - \sum_{i \in I, j \in J} x_{ij} \geq 0 \qquad \text{for all } I, J \subseteq N \setminus \{s,t\}. \tag{5.16}$$

Fact 6. Provided that the arc set of D is given by

$$A = \{(s,i),(i,t) : i \in N'\} \bigcup \{(i,j) \in N' \times N' : i \neq j\}, \tag{5.17}$$

where $N' := N \setminus \{s,t\}$ is the set of internal nodes, all inequalities of Chapter 3.2, which have been shown to be facet defining for the (s,t)-k-path polytope $P^{(k)}_{s,t\text{-path}}(D)$, are also facet defining for $P^{\leq k}_{s,t\text{-path}}(D)$ or can easily be lifted into facet defining inequalities for $P^{\leq k}_{s,t\text{-path}}(D)$ using the following theorem.

Theorem 5.5 (cf. Theorem 18 of Hartmann and Özlük [48])**.** Let $c^T x \leq c_0$ induce a facet of the (s,t)-k-path polytope $P^{(k)}_{s,t\text{-path}}(D)$, where $4 \leq k < |N|-1$ and the arc set of $D = (N,A)$ is given by (5.17). If μ is the smallest value such that the inequality

$$\mu x(A) + c^T x \leq \mu k + c_0 \tag{5.18}$$

is valid for the hop constrained path polytope $P^{\leq k}_{s,t\text{-path}}(D)$, then this inequality is also facet inducing for $P^{\leq k}_{s,t\text{-path}}(D)$. □

5.4 The dynamic programming paradigm applied to the hop constrained shortest path problem

In this section, we derive extended formulations of the the hop constrained walk polytope and the dominant of the hop constrained path polytope from the structure inherent to the Moore-Bellman-Ford algorithm [11, 36, 66].

Given a simple directed graph $D = (N, A)$, a fixed node $s \in N$, and a length function $d : A \to \mathbb{R}$, the Moore-Bellman-Ford algorithm computes (the value of) a shortest (s, j)-path for each node $j \in N$, provided D has no negative cycles. Using an appropriate variant of this algorithm, it computes in its main loop the length of a shortest (s, j)-path with at most ℓ arcs for $\ell = 1, \ldots, |N|-1$, see Algorithm 4. The correctness of the algorithm is based on the *Bellman Equations*

$$\begin{aligned} u_j^{(\ell)} = \min\{u_j^{(\ell-1)}, \min\{u_i^{(\ell-1)} &+ d((i,j)) : (i,j) \in \delta^{\text{in}}(j)\}\} \\ &\text{for all } j \in N,\ \ell = 1, \ldots, |N|-1, \end{aligned} \tag{5.19}$$

where $u_j^{(\ell)}$ denotes the length of a shortest (s, j)-path with at most ℓ arcs for $\ell = 0, 1, \ldots, |N| - 1$.

Suppose that the Moore-Bellman-Ford algorithm will be used to compute a k-hop constrained shortest (s, t)-path. We construct an acyclic digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ with length function $\tilde{d} : \mathcal{A} \to \mathbb{R}$ as follows. For the state $u_s^{(0)}$, we introduce a node $[s, 0]$, for the states $u_t^{(\ell)}$, $\ell \in \{1, \ldots, k\}$, we introduce a node $[t, k]$, and for each state $u_i^{(\ell)}$, with $i \in N \setminus \{s, t\}$ and $\ell \in \{1, \ldots, k-1\}$, we introduce a node $[i, \ell]$. Next, with each node $i \in N \setminus \{s, t\}$, we associate the arcs $([i, \ell-1], [i, \ell])$ of length 0, $\ell = 2, \ldots, k-1$ (corresponding to the decision $u_i^{(\ell)} = u_i^{(\ell-1)}$), and with each arc $(i, j) \in A\setminus(\{(s,t)\}\cup\delta^{\text{in}}(s)\cup\delta^{\text{out}}(t))$, we associate the arcs $([i, \ell-1], [j, \ell])$ of length $d((i, j))$, where $\ell = 1$ if $i = s$, $\ell = k$ if $j = t$, and ℓ runs from 2 to $k-1$ otherwise. Finally, if $(s, t) \in A$, we introduce the arc $([s, 0], [t, k])$ of length $d((s, t))$. $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ is said to be the *dynamic programming graph associated with* (D, s, t, k) or just *DP-graph associated with* (D, s, t, k). The length function $\tilde{d}$ is called the *extension of d to* $\mathcal{D}$.

The hop constrained shortest path problem is now quite easily viewed as one of finding a shortest $([s, 0], [t, k])$-path in the DP-graph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ provided that D has no negative cycles.

$\mathcal{D} \setminus \{([s, 0], [t, k])\}$ is a layered digraph with layers

$$\begin{aligned} \mathcal{N}_0 &:= \{[s, 0]\}, \\ \mathcal{N}_\ell &:= \{[i, \ell] : i \in N \setminus \{s, t\}\}, \qquad\qquad \ell = 1, \ldots, k-1 \\ \mathcal{N}_n &:= \{[t, k]\}. \end{aligned}$$

Algorithm 4: Moore-Bellman-Ford algorithm

Input: A digraph $D = (N, A)$, a fixed node $s \in N$, and a length function $d : A \to \mathbb{R}$ such that D has no negative cycles with respect to d.

Output: For each node $j \in N$ and each number $\ell \in \{0, \dots, |N| - 1\}$, the length $u_j^{(\ell)}$ of a shortest (s, j)-path using at most ℓ arcs and its predecessor $p(j, \ell)$ on such a path. If j is not reachable from s, then $u_j^{(\ell)} = +\infty$ and $p(j, \ell)$ is undefined for all ℓ.

Set $u_s^{(0)} := 0$ and $u_j^{(0)} := +\infty$ for all $j \in N \setminus \{s\}$.
for $\ell := 1$ **to** $|N| - 1$ **do**
  Set $t_j := u_j^{(\ell-1)}$ for all $j \in N$.
  forall $(i, j) \in A$ **do**
    if $t_j > u_i^{(\ell-1)} + d((i, j))$ **then**
      set $t_j := u_i^{(\ell-1)} + d((i, j))$ and $p(j, \ell) := i$.
    end
  end
  Set $u_j^{(\ell)} := t_j$ for all $j \in N$.
end

The both arc sets A and $\mathcal{A}$ are connected via a set function

$$\varphi : \mathcal{A} \to A \cup \varnothing,\ \varphi(([i, h], [j, \ell])) = \begin{cases} (i, j) & \text{if } i \neq j \\ \varnothing & \text{else.} \end{cases}$$

Suppose that $\delta^{\text{in}}(s) = \delta^{\text{out}}(t) = \varnothing$. Denoting by $\mathcal{P}_{s,t}(\mathcal{D})$ the collection of all $([s, 0], [t, k])$-paths in $\mathcal{D}$, we see that each walk $W \in \mathcal{W}_{s,t}^{\leq k}(D)$ corresponds to at least one path $P' \in \mathcal{P}_{s,t}(\mathcal{D})$ and each path $P \in \mathcal{P}_{s,t}(\mathcal{D})$ corresponds to a walk $W' \in \mathcal{W}_{s,t}^{\leq k}(D)$. Walks in $\mathcal{W}_{s,t}^{\leq k}(D)$ and paths in $\mathcal{P}_{s,t}(\mathcal{D})$ are not in 1-1-correspondence due to the arcs of the form $([i, \ell - 1], [i, \ell])$. These arcs have been incorporated into the model in order to give an easier characterization of facets later. Arcs of this form are called *artificial*. The set of all artificial arcs is denoted by $\hat{\mathcal{A}}$. An illustration of the model is given in Figure 5.2.

The DP-approach provides a linear characterization of the HCSPP as flow formulation in terms of decision variables y_a associated with the arcs $a = ([h, i], [j, \ell]) \in \mathcal{A}$. The x- and y-variables are connected via a transformation matrix $T \in \mathbb{R}^{A \times \mathcal{A}}$ that represents the set function φ. In the following we investigate the polyhedral relationships.

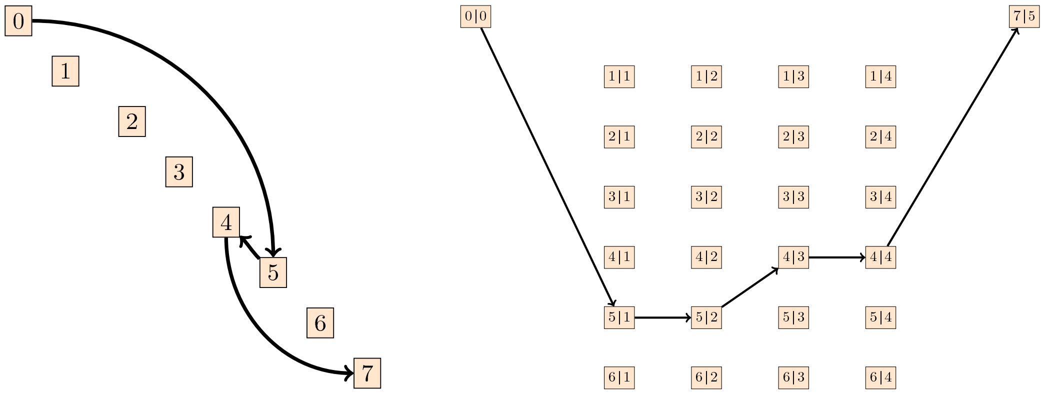


Figure 5.2: A digraph $D = (N, A)$ on node set $N = \{0, 1, \dots, 7\}$ and associated DP-graph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ for $(D, 0, 7, 5)$; arc sets are omitted. Illustration of a $(0, 7)$-walk and one of its counterparts in $\mathcal{D}$.

Theorem 5.6. Let $D = (N, A)$ be a simple directed graph, let $s, t \in N$ be distinct nodes, let $\delta^{\text{in}}(s) = \delta^{\text{out}}(t) = \varnothing$, let k be a natural number, and let $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ be the DP-graph associated with (D, s, t, k). The hop constrained walk polytope $P^{\leq k}_{s,t\text{-walk}}(D)$ is the projection of the polytope defined by

$$
\begin{aligned}
y(\delta^{\text{out}}([s,0])) &= 1, && && (5.20)\\
y(\delta^{\text{in}}([t,k])) &= 1, && && (5.21)\\
y(\delta^{\text{in}}([i,\ell])) - y(\delta^{\text{out}}([i,\ell])) &= 0 && \text{for all } [i,\ell] \in \mathcal{N} \setminus \{[s,0],[t,k]\}, && (5.22)\\
y_a &\geq 0 && \text{for all } a \in \mathcal{A}, && (5.23)\\
x &= Ty && && (5.24)
\end{aligned}
$$

onto the x-space.

Proof. We have to show that

$$P^{\leq k}_{s,t\text{-walk}}(D) = \{x \in \mathbb{R}^A : \exists\, y \in P_{[s,0],[t,k]\text{-path}}(\mathcal{D}) \text{ with } x = Ty\}.$$

By definition, each $x \in P^{\leq k}_{s,t\text{-walk}}(D)$ is a convex combination of the incidence vectors of walks $W \in \mathcal{W}^{\leq k}_{s,t}(D)$. Moreover, by construction of the DP-graph, for each $W \in \mathcal{W}^{\leq k}_{s,t}(D)$ there is some $P' \in \mathcal{P}_{s,t}(\mathcal{D})$ such that $\varphi(P') = W$. Consequently, for appropriate numbers $\lambda_{P'} \geq 0$ with $\sum_{P' \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_{P'} = 1$ we have:

$$x = \sum_{P' \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_{P'} \chi^{\varphi(P')} = \sum_{P' \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_{P'} T \chi^{P'} = T \left(\sum_{P' \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_{P'} \chi^{P'} \right),$$

that is, $x = Ty$ for $y := \sum_{P' \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_{P'} \chi^{P'}$.

Next, let $x \in \mathbb{R}^A$ for which exists $y \in P_{[s,0],[t,k]\text{-path}}(\mathcal{D})$ with $x = Ty$. Since y is a convex combination of the incidence vectors of paths $P \in \mathcal{P}_{s,t}(\mathcal{D})$ and $W' := \varphi(P) \in \mathcal{W}^{\leq k}_{s,t}(D)$ for all $P \in \mathcal{P}_{s,t}(\mathcal{D})$, we obtain for appropriate numbers $\lambda_P \geq 0$ with $\sum_{P \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_P = 1$:

$$x = Ty = T\left(\sum_{P \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_P \chi^P\right) = \sum_{P \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_P T \chi^P = \sum_{P \in \mathcal{P}_{s,t}(\mathcal{D})} \lambda_P \chi^{\varphi(P)},$$

that is, x is a convex combination of incidence vectors of walks of cardinality at most k. □

Since the dynamic programming polytope $P_{[s,0],[t,k]\text{-path}}(\mathcal{D})$ is just an ordinary path polytope, its dominant

$$\operatorname{dmt}(P_{[s,0],[t,k]\text{-path}}(\mathcal{D})) = P_{[s,0],[t,k]\text{-path}}(\mathcal{D}) + \mathbb{R}^{\mathcal{A}}_{+}$$

is determined by the system

$$y(C) \geq 1 \qquad \text{for all } ([s,0],[t,k])\text{-cuts } C \subseteq \mathcal{A}, \tag{5.25}$$
$$y_a \geq 0 \qquad \text{for all arcs } a \in \mathcal{A}. \tag{5.26}$$

Theorem 5.7. Let $D = (N, A)$ be a simple directed graph, let $s, t \in N$ be distinct nodes, let $\delta^{\text{in}}(s) = \delta^{\text{out}}(t) = \varnothing$, let k be a natural number, and let $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ be the DP-graph associated with (D, s, t, k). The polyhedron $\operatorname{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ is the projection of the polyhedron $P^\star$ defined by (5.25), (5.26), and

$$x \geq Ty \tag{5.27}$$

onto the space of x-variables.

Proof. We have to show that

$$\operatorname{dmt}(P^{\leq k}_{s,t\text{-path}}(D)) = \{x \in \mathbb{R}^A : \exists\, y \in \operatorname{dmt}(P_{[s,0],[t,k]\text{-path}}(\mathcal{D})) \text{ with } Ty \leq x\}.$$

Let $u = v + w \in \operatorname{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ with $v \in P^{\leq k}_{s,t\text{-path}}(D)$ and $w \geq 0$. Since $P^{\leq k}_{s,t\text{-path}}(D) \subseteq P^{\leq k}_{s,t\text{-walk}}(D)$, by Theorem 5.6 there is $y \in P_{[s,0],[t,k]\text{-path}}(\mathcal{D})$ such that $Ty = v$. Clearly, for appropriate $z \in \mathbb{R}^{\mathcal{A}}_{+}$, $T(y + z) \geq v + w = u$. To show the other set inclusion, let $x \in \mathbb{R}^A$ such that $x \geq Tu$ for some $u \in \operatorname{dmt}(P_{[s,0],[t,k]\text{-path}}(\mathcal{D}))$. Then, $u = y + z$ for some $y \in P_{[s,0],[t,k]\text{-path}}(\mathcal{D})$ and $z \geq 0$. Setting $v := Ty$ and $w := Tz$, we see that $u \geq v + w \in \operatorname{dmt}(P^{\leq k}_{s,t\text{-walk}}(D))$. Since $\operatorname{dmt}(P^{\leq k}_{s,t\text{-walk}}(D)) = \operatorname{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, it follows the statement of the theorem. □

In case of $k \geq |N| - 1$, $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ equals the dominant of the ordinary path polytope, and hence it is determined by the inequality system (5.15).

5.5 Separation, arc set capacitated and length-bounded cuts and flows

This section is subdivided into four parts. In the first subsection, we present compact linear programming formulations for the separation problems for $P^{\leq k}_{s,t\text{-walk}}(D)$ and $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$. In the second and third subsection, we embed the separation problem for $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ into the greater context of length-bounded and arc set capacitated cut and flow problems. The attained insights will essentially explain why it seems to be difficult to design a polynomial combinatorial algorithm that solves the separation problem at least for $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$. Moreover, they enhance the theory on length-bounded flows and cuts by some aspects. Finally, in the fourth subsection, we point to well-known approximation algorithms for fractional packing and covering problems that can be applied to the above-mentioned flow and cut problems. Some parts of this section are joint work with Maren Martens.

5.5.1 Separation

The separation problem for $P^{\leq k}_{s,t\text{-walk}}(D)$ can be formulated as a compact linear program, which, of course, is not surprising in view of the compact extended formulation for this polytope.

Theorem 5.8. Let $D = (N, A)$ be a simple directed graph, $s \neq t \in N$, $\delta^{\text{in}}(s) = \delta^{\text{out}}(t) = \varnothing$, and $k \in \mathbb{N}$. Moreover, let $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ be the DP-graph associated with (D, s, t, k).

(i) For any $x^\star \in \mathbb{R}^A_+$ satisfying the flow conservation constraints (5.8), the linear program

$$
\begin{array}{lrl}
\min & \displaystyle\sum_{(i,j)\in A} x^\star_{ij}\tau_{ij} & \\
\text{s.t.} & \pi_{i\ell} - \pi_{jm} + \tau_{ij} \geq 0 & \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A} \setminus \hat{\mathcal{A}}, \\
 & \pi_{i,\ell-1} - \pi_{i\ell} \geq 0 & \text{for all } i \in N \setminus \{s,t\}, \\
 & & \ell \in \{2,3,\dots,k-1\}, \\
 & \pi_{s0} = 0, & \\
 & \pi_{tk} = 1, & \\
 & 0 \leq \pi_{i\ell} \leq 1 & \text{for all } [i,\ell] \in \mathcal{N} \setminus \{[s,0],[t,k]\},
\end{array}
\tag{5.28}
$$

where $\hat{\mathcal{A}}$ denotes the set of artificial arcs, always has an optimal solution.

(ii) Let $(\pi^\star, \tau^\star)$ be an optimal solution of (5.28) and $\nu^\star$ its objective value. Then, $x^\star \in P^{\le k}_{s,t\text{-walk}}(D)$ if and only if $\nu^\star \ge 1$. In case of $x^\star \notin P^{\le k}_{s,t\text{-walk}}(D)$, the inequality $\sum_{(i,j)\in A} \tau^\star_{ij} x_{ij} \ge 1$ is a valid inequality for $P^{\le k}_{s,t\text{-walk}}(D)$ violated by $x^\star$.

Proof. (i) As is easily seen, the feasible region of the linear program (5.28) is nonempty. Moreover, due to the constraints $\pi_{s0} = 0$, $\pi_{tk} = 1$, and $0 \le \pi_{vw} \le 1$ for $[v, w] \in \mathcal{N} \setminus \{[s, 0], [t, k]\}$, it follows that $\tau_{ij} \ge -1$ for all $(i, j) \in A$. This implies that the objective value of each feasible solution (π, τ) is at least $-\sum_{(i,j)\in A} x^\star_{ij}$. Thus, the linear program (5.28) always has an optimal solution.

(ii) We introduce dual variables π_{s0} and π_{tk} for equation (5.20) multiplied by -1 and equation (5.21), respectively, a dual variable $\pi_{i\ell}$ for each flow conservation constraint (5.22) associated with a node $[i, \ell] \in \mathcal{N} \setminus \{[s, 0], [t, k]\}$, a dual variable σ_a for each nonnegativity constraint (5.23) associated with an arc $a \in \mathcal{A}$, and a dual variable τ_{ij} for each equation of the system (5.24) associated with an arc $(i, j) \in A$. Then, by Theorems 5.2 and 5.6 we see that the separation problem for $P^{\le k}_{s,t\text{-walk}}(D)$ and any $x^\star \in \mathbb{R}^A$ is equivalent to the linear program

$$\begin{array}{lrl} \min & \pi_{s0} - \pi_{tk} + \sum\limits_{(i,j)\in A} x^\star_{ij} \tau_{ij} & \\ \text{s.t.} & \pi_{jm} - \pi_{i\ell} + \sigma_a - \tau_{ij} = 0 & \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A} \setminus \hat{\mathcal{A}}, \\ & \pi_{jm} - \pi_{i\ell} + \sigma_a = 0 & \text{for all } a = ([i,\ell],[j,m]) \in \hat{\mathcal{A}}, \\ & \sigma_a \ge 0 & \text{for all } a \in \mathcal{A}. \end{array} \tag{5.29}$$

Adding a constant M to each entry $\pi_{i\ell}$ of a feasible solution (π, σ, τ) for (5.29), we obtain a feasible solution with the same objective value. Thus, the variables $\pi_{i\ell}$ can be assumed to be nonnegative. Next, recall that either $(\pi, \sigma, \tau) = (0, 0, 0)$ is an optimal solution of (5.29) or the LP is unbounded. In the latter case, any feasible solution (π, σ, τ) with objective value less than zero provides a valid inequality $\pi_{s0} - \pi_{tk} + \sum_{(i,j)\in A} x_{ij}\tau_{ij} \ge 0$ for $P^{\le k}_{s,t\text{-walk}}(D)$ violated by $x^\star$. By the former observation, we may assume that $\pi \ge 0$. Moreover, we may assume that $\mu := \max\{\pi_{i\ell} : [i, \ell] \in \mathcal{N}\} > 0$, because otherwise the objective value of (π, σ, τ) would be nonnegative. Hence, $(\pi', \sigma', \tau') := \frac{1}{\mu}(\pi, \sigma, \tau)$ is also a feasible solution with objective value less than zero, it provides a valid inequality that separates $x^\star$ from $P^{\le k}_{s,t\text{-walk}}(D)$, and it satisfies the trivial inequalities $0 \le \pi_{i\ell} \le 1$ for all $[i, \ell] \in \mathcal{N}$. Hence, adding these inequalities to (5.29), removing the slack variables σ_a, $a \in \mathcal{A}$, and observing that the artificial arcs are of the form $([i, \ell - 1], [i, \ell])$, we see that the separation problem for $P^{\leq k}_{s,t\text{-walk}}(D)$ and $x^\star$ can be expressed as the linear program

$$\begin{array}{lll} \min & \pi_{s0} - \pi_{tk} + \sum\limits_{(i,j)\in A} x^\star_{ij}\tau_{ij} & \\ \text{s.t.} & \pi_{i\ell} - \pi_{jm} + \tau_{ij} \geq 0 & \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A}\setminus\hat{\mathcal{A}}, \\ & \pi_{i,\ell-1} - \pi_{i\ell} \geq 0 & \text{for all } i \in N\setminus\{s,t\}, \\ & & \ell \in \{2,3,\dots,k-1\}, \\ & 0 \leq \pi_{i\ell} \leq 1 & \text{for all } [i,\ell] \in \mathcal{N}. \end{array} \tag{5.30}$$

Finally, observe that $x^\star$ satisfies the flow constraint $x(\delta^+(s)) = 1$. Thus, $\pi_{s0} + \sum_{a\in\delta^+(s)} x^\star_a\tau_a = (\pi_{s0} - \lambda) + \sum_{a\in\delta^+(s)} x^\star_a(\tau_a + \lambda)$ for all $\lambda \in \mathbb{R}$, in particular, for $\lambda = \pi_{s0}$. Hence, we may assume that π_{s0} is fixed to zero. For a similar reason, we may assume that $\pi_{tk} = 1$. This leads to the statement (ii) of the theorem. □

Analogously, the separation problem for $\operatorname{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ and any point $x^\star \in \mathbb{R}^A$ can be formulated as an LP. Writing down the $([s,0],[t,k])$-min-cut inequalities (5.25) in matrix form $Zy \geq \mathbb{1}$ and applying Theorems 5.2 and 5.7, we derive the LP

$$\begin{array}{ll} \min & \tau^T x^\star - \rho^T\mathbb{1} \\ \text{s.t.} & \rho^T Z + \sigma^T - \tau^T T = 0 \\ & \rho \geq 0,\ \sigma \geq 0,\ \tau \geq 0, \end{array}$$

which, however, has exponential size. The following theorem implies that the separation problem for $\operatorname{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ can be written as a slight variation of the compact linear program (5.28).

Theorem 5.9. Let $D = (N,A)$ be a simple directed graph, $s \neq t \in N$, $\delta^{\text{in}}(s) = \delta^{\text{out}}(t) = \varnothing$, and $k \in \mathbb{N}$. Moreover, let $\mathcal{D} = (\mathcal{N},\mathcal{A})$ be the DP-graph associated with (D,s,t,k), $\hat{\mathcal{A}}$ the set of its artificial arcs, and $\mathcal{N}'$ its set of internal nodes.

(i) For any $x^\star \in \mathbb{R}^A_+$, the dual programs

$$\begin{array}{lrl} \max & z_{tk} & \\ \text{s.t.} & z_{s0} + y(\delta^{\text{out}}([s,0])) = 0, & \\ & z_{tk} - y(\delta^{\text{in}}([t,k])) = 0, & \\ & y(\delta^{\text{out}}([i,\ell])) - y(\delta^{\text{in}}([i,\ell])) = 0 & \text{for all } [i,\ell] \in \mathcal{N}', \\ & \sum\limits_{\substack{a\in\mathcal{A}:\\ \varphi(a)=(i,j)}} y_a \leq x^\star_{ij} & \text{for all } (i,j) \in A, \\ & y_a \geq 0 & \text{for all } a \in \mathcal{A} \end{array} \tag{5.31}$$

and

$$
\begin{array}{lrr}
\min & \sum_{(i,j)\in A} x^\star_{ij}\tau_{ij} & \\
\text{s.t.} & \pi_{s0} = 0, & \\
& \pi_{tk} = 1, & \\
& \pi_{i\ell} - \pi_{jm} + \tau_{ij} \geq 0 & \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A}\setminus\hat{\mathcal{A}}, \\
& \pi_{i,\ell-1} - \pi_{i\ell} \geq 0 & \text{for all } i \in N\setminus\{s,t\}, \\
& & \ell \in \{2,3,\dots,k-1\}, \\
& \tau_{ij} \geq 0 & \text{for all } (i,j) \in A
\end{array}
\tag{5.32}
$$

always have optimal solutions.

(ii) Let $(\pi^\star, \tau^\star)$ be an optimal solution of (5.32) and $\nu^\star$ its objective value. Then, $x^\star \in \text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ if and only if $\nu^\star \geq 1$. In case that $x^\star$ does not belong to $\text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, the inequality $\sum_{(i,j)\in A}\tau^\star_{ij}x_{ij} \geq 1$ is a valid inequality for $\text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ violated by $x^\star$.

Proof. (i) As is easily seen, $(y,z) = (0,0)$ and (π,τ) defined by $\pi_{i\ell} := 0$ for all $[i,\ell] \in \mathcal{N}\setminus\{[t,k]\}$, $\pi_{tk} := 1$, and $\tau_{ij} := 1$ for all $(i,j) \in A$ are feasible solutions of (5.31) and (5.32), respectively. The Duality Theorem (e.g. [46]) now implies the proposed claim.

(ii) The Flow Decomposition Theorem (Ford and Fulkerson [35], Gallai [37]) implies that each $([s,0],[t,k])$-flow y is a conic combination of the incidence vectors of paths $P \in \mathcal{P}_{s,t}(\mathcal{D})$, since $\mathcal{D}$ is acyclic. Thus, for each $P \in \mathcal{P}_{s,t}(\mathcal{D})$, there exists $\lambda_P \geq 0$ such that $y = \sum_{P\in\mathcal{P}_{s,t}(\mathcal{D})}\lambda_P\chi^P$. The value ν of y is determined by $\nu = \sum_{P\in\mathcal{P}_{s,t}(\mathcal{D})}\lambda_P$. Its projection, $x := \sum_{P\in\mathcal{P}_{s,t}(\mathcal{D})}\lambda_P\chi^{\varphi(P)}$, is a conic combination of walks $W \in \mathcal{W}^{\leq k}_{s,t}(D)$.

Suppose that $(\pi^\star, \tau^\star)$ is an optimal solution of (5.32) and $\nu^\star$ its objective value. By the Duality Theorem, the linear program (5.31) has an optimal solution $(y^\star, z^\star)$ with the same objective value $\nu^\star = z^\star_{tk}$. Since $y^\star$ is a flow of value $\nu^\star$, it follows immediately that $x^\star \in \text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ if and only if $\nu^\star \geq 1$, since $\text{dmt}(P^{\leq k}_{s,t\text{-walk}}(D)) = \text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$. Moreover, in case of $x^\star \notin \text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, it follows that $\nu^\star < 1$, and hence the inequality $\sum_{(i,j)\in A}\tau^\star_{ij}x_{ij} \geq 1$ is violated by $x^\star$. The validity of this inequality also follows by duality. For any $x' \in \text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, consider the dual linear programs (5.31) and (5.32) with x' instead of $x^\star$. The objective value of an optimal solution of the new linear program (5.31)

$$\begin{aligned}
\max\quad & z_{tk} \\
\text{s.t.}\quad & z_{s0} + y(\delta^{\text{out}}([s,0])) = 0, \\
& z_{tk} - y(\delta^{\text{in}}([t,k])) = 0, \\
& y(\delta^{\text{out}}([i,\ell])) - y(\delta^{\text{in}}([i,\ell])) = 0 \quad \text{for all } [i,\ell] \in \mathcal{N} \setminus \{[s,0],[t,k]\}, \\
& \sum_{a \in \mathcal{A}:\varphi(a)=(i,j)} y_a \le x'_{ij} \quad \text{for all } (i,j) \in A, \\
& y_a \ge 0 \quad \text{for all } a \in \mathcal{A}
\end{aligned}$$

is at least 1, while $(\pi^\star, \tau^\star)$ is a feasible solution of its dual, the new linear program (5.32):

$$\begin{aligned}
\min\quad & \sum_{(i,j)\in A} x'_{ij}\tau_{ij} \\
\text{s.t.}\quad & \pi_{s0} = 0, \\
& \pi_{tk} = 1, \\
& \pi_{i\ell} - \pi_{jm} + \tau_{ij} \ge 0 \quad \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A} \setminus \hat{\mathcal{A}}, \\
& \pi_{i,\ell-1} - \pi_{i\ell} \ge 0 \quad \text{for all } i \in N \setminus \{s,t\}, \ell \in \{2,3,\dots,k-1\}, \\
& \tau_{ij} \ge 0 \quad \text{for all } (i,j) \in A.
\end{aligned}$$

Thus, $\sum_{(i,j)\in A} \tau^\star_{ij} x'_{ij} \ge 1$. □

We remark that the membership problems for the polyhedra $P^{\le k}_{s,t\text{-walk}}(D)$ and $\operatorname{dmt}(P^{\le k}_{s,t\text{-path}}(D))$ are exhaustively described by the linear program (5.31) and the dual of the linear program (5.28), respectively.

5.5.2 Length-bounded cuts and flows

Here and in the following subsection, we collect some arguments indicating that it is quite hard to design a polynomial combinatorial algorithm for $\operatorname{dmt}(P^{\le k}_{s,t\text{-path}}(D))$ to solve the separation or membership problem. However, our observations do not necessarily imply that it is maybe easier to design a polynomial combinatorial separation (membership) algorithm for $P^{\le k}_{s,t\text{-walk}}(D)$. The trivial inequalities $0 \le \pi_{i\ell} \le 1$ can be added to the linear program (5.32), since there always exists an optimal solution satisfying these constraints, see Section 5.6.1. Thus, the linear programs (5.28) and (5.32) differ only in the nonnegativity constraints For the τ_{ij}. However, this might be an important aspect. For instance, Dahl, Foldnes, and Gouveia [22] observed that the facial structure of the 4-hop constrained walk polytope is easier to describe than that of its dominant. It is an open question whether this difference will be reflected in the separation problems for both such polyhedra not only for

$k = 4$. (Also, see the discussion about polymatroids and extended polymatroids in [46].) However, we leave these aspects for future research.

In this subsection, we consider the membership and separation problem for $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ in the greater context of length-bounded cut and flow problems. For a broader presentation of length-bounded cut and flow problems, we refer to Baier et al. [1].

Let $D = (N, A)$ be a directed graph that possesses two independent weight functions, namely a length function $d : A \to \mathbb{R}_+$ and a capacity function $u : A \to \mathbb{R}_+$. Given two distinct nodes $s, t \in N$ and a number $L \in \mathbb{R}_+$, an (s,t)-path (-walk) P is *L-length-bounded* if $\sum_{a\in P} d(a) \leq L$. An (s,t)-cut C is said to be *L-length-bounded* if $\mathcal{P}^L_{s,t}(D \setminus C, d|_{D\setminus C}) = \varnothing$, that is, deleting C destroys all (s,t)-paths (and all (s,t)-walks) with length at most L in D. Here, $d|_{D\setminus C}$ is the restriction of d to $D \setminus C$. The set of all L-length-bounded (s,t)-paths (-walks, -cuts) will be denoted by $\mathcal{P}^L_{s,t}(D, d)$ ($\mathcal{W}^L_{s,t}(D, d)$, $\mathcal{C}^L_{s,t}(D, d)$). The *capacity* of a (length-bounded) cut C is $u(C)$. An *L-length-bounded (s,t)-flow* is a function $f : \mathcal{P}^L_{s,t}(D, d) \to \mathbb{R}_+$ assigning a flow value f_P to each path $P \in \mathcal{P}^L_{s,t}(D, d)$. The flow f is *feasible*, if the total flow on each arc a, $\sum_{P:a\in P} f_P$, does not exceed its capacity $u(a)$.

The problem of finding a feasible L-length-bounded (s,t)-flow of maximum value, called *L-length-bounded maximum (s,t)-flow problem*, can be expressed as a linear program:

$$\begin{array}{lll}
\max & \displaystyle\sum_{P\in\mathcal{P}^L_{s,t}(D,d)} f_P & \\
\text{s.t.} & \displaystyle\sum_{\substack{P\in\mathcal{P}^L_{s,t}(D,d):\\ P\ni a}} f_P \leq u(a) & \text{for all } a \in A, \\
& f_P \geq 0 & \text{for all } P \in \mathcal{P}^L_{s,t}(D, d).
\end{array} \tag{5.33}$$

In the *L-length-bounded minimum (s,t)-cut problem* we search for an L-length-bounded (s,t)-cut of minimum capacity. Its linear programming relaxation is the dual of (5.33):

$$\begin{array}{lll}
\min & \displaystyle\sum_{a\in A} u(a)\tau_a & \\
\text{s.t.} & \displaystyle\sum_{a\in P} \tau_a \geq 1 & \text{for all } P \in \mathcal{P}^L_{s,t}(D, d), \\
& \tau_a \geq 0 & \text{for all } a \in A.
\end{array} \tag{5.34}$$

The optimization problem, expressed by the linear program (5.34), is called the *fractional L-length-bounded minimum (s,t)-cut problem.*

The dual linear programs (5.33) and (5.34) are, of course, not compact. Nevertheless, they always have compact optimal solutions. For the linear program (5.34), this is clear, since it has only m variables, while for the linear program (5.33), it follows from the fact that it has only m constraints.

Proposition 5.10 (Baier et al. [1])**.** Denote by m the number of arcs in D. Given an L-length-bounded (s,t)-flow f in D, then there exists an L-length-bounded (s,t)-flow $\tilde{f}$ with the same length bound and the same flow value per arc such that $\tilde{f}_P > 0$ for at most m paths $P \in \mathcal{P}^L_{s,t}(D,d)$. □

Length-bounded paths and walks obviously generalize hop constrained paths and walks. Moreover, the membership problem for $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ and any point $x^\star \in \mathbb{R}^A$ can be represented by the dual linear programs (5.33) and (5.34) with $L := k$, unit-lengths, and $u(a) := x^\star_a$ for all $a \in A$, and the separation problem by the latter one. Conversely, this means that, given unit-lengths, the dual linear programs (5.33) and (5.34) have also compact pendants. Using the DP-approach, this is an immediate consequence of Theorem 5.9.

In what follows, if unit-lengths are given, we speak of hop constraints instead of length bounds. So length-bounded flows are called *hop constrained flows*, and so on.

Corollary 5.11. Let $D = (N, A)$ be a simple directed graph, $u : A \to \mathbb{R}_+$ a capacity function, $s \neq t \in N$, and $k \in \mathbb{N}$. Moreover, let $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ be the DP-graph associated with (D, s, t, k) and $\hat{A}$ the subset of its artificial arcs. The k-hop constrained maximum (s,t)-flow and the fractional k-hop constrained minimum (s,t)-cut problem on D are equivalent to the linear programs

$$
\begin{array}{llr}
\max & z_{tk} & \\
\text{s.t.} & z_{s0} + y(\delta^{\text{out}}([s,0])) = 0, & \\
& z_{tk} - y(\delta^{\text{in}}([t,k])) = 0, & \\
& y(\delta^{\text{out}}([i,\ell])) - y(\delta^{\text{in}}([i,\ell])) = 0 & \text{for all } [i,\ell] \in \mathcal{N}, \\
& & [s,0] \neq [i,\ell] \neq [t,k], \\
& \displaystyle\sum_{\substack{a \in \mathcal{A}: \\ \varphi(a) = (i,j)}} y_a \leq u((i,j)) & \text{for all } (i,j) \in A, \\
& y_a \geq 0 & \text{for all } a \in \mathcal{A}
\end{array}
\tag{5.35}
$$

and

$$
\begin{array}{llr}
\min & \displaystyle\sum_{(i,j) \in A} u((i,j))\tau_{ij} & \\
\text{s.t.} & \pi_{s0} = 0, & \\
& \pi_{tk} = 1, & \\
& \pi_{i\ell} - \pi_{jm} + \tau_{ij} \geq 0 & \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A} \setminus \hat{\mathcal{A}}, \\
& \pi_{i,\ell-1} - \pi_{i\ell} \geq 0 & \text{for all } i \in N \setminus \{s,t\}, \\
& & \ell \in \{2,3,\ldots,k-1\}, \\
& \tau_{ij} \geq 0 & \text{for all } (i,j) \in A,
\end{array}
\tag{5.36}
$$

respectively. □

Note that, in case of hop constrained flows ($k = \lfloor L \rfloor$, $d \equiv 1$), any feasible (optimal) solution of the linear program (5.35) can be polynomially transformed into a feasible solution of the linear program (5.33) with the same value due to the Flow Decomposition Theorem (Ford and Fulkerson [35], Gallai [37]), and vice versa. An analogous result holds for fractional hop constrained cuts. Clearly, if (π', τ') is a feasible (optimal) solution for (5.36), then τ' is one for (5.34). Conversely, if τ' is a feasible (optimal) solution for (5.34), then one for (5.36) can be computed with Dijkstra's algorithm [26] as follows. W.l.o.g. assume that $\delta^{\text{in}}(s) = \delta^{\text{out}}(t) = \varnothing$. Let $\tilde{\tau}'$ be the extension of τ' to $\mathcal{D}$. Dijkstra's algorithm computes for each node $[i, \ell] \in \mathcal{N}$ the length of a shortest $([s, 0], [i, \ell])$-path with respect to $\tilde{\tau}'$. Denote this value by $\text{dist}([i, \ell])$. Since $\tilde{\tau}'$ is zero along the artificial arcs, it follows that $\text{dist}([i, \ell - 1]) \geq \text{dist}([i, \ell])$ for all $i \in N \setminus \{s, t\}$, $\ell \in \{2, \dots, k - 1\}$. Next, because of $\tau'(P) \geq 1$ for all $P \in \mathcal{P}^{\leq k}_{s,t}(D)$, it follows that $\text{dist}([t, k]) \geq 1$. Finally, for each arc $a = ([i, \ell], [j, m]) \in \mathcal{A} \setminus \hat{\mathcal{A}}$,

$$\text{dist}([j, m]) \leq \text{dist}([i, \ell]) + \tilde{\tau}'(([i, \ell], [j, m]))$$

and $\tilde{\tau}'(([i, \ell], [j, m])) = \tau'_{ij}$ implies $\text{dist}([i, \ell]) - \text{dist}([j, m]) + \tau'_{ij} \geq 0$. Thus, (π', τ'), where $\pi'_{i\ell} := \frac{\text{dist}([i,\ell])}{\text{dist}([t,k])}$ for all nodes $[i, \ell] \in N$, is a feasible (optimal) solution of the linear program (5.36).

We mention two results of Baier et al. [1], which are interesting in our context. The first result concerns the complexity of the transformation between arc- and path-based formulations of length-bounded flows. A given length-bounded path-flow can easily be transformed into an arc-flow, while the reverse transformation is NP-hard in general.

Theorem 5.12 (Baier et al. [1])**.** *Unless* P = NP*, there does not exist a polynomial algorithm to transform an arc-flow which is known to correspond to a length-bounded path-flow into a length-bounded path-flow.* □

In case of unit-lengths, such an algorithm exists. Even more, one can check in polynomial time, if a given arc-flow $x : A \to \mathbb{R}_+$ actually corresponds to a hop constrained path-flow. One way for doing this, is using the DP-model. Set $A' := A \setminus (\delta^{\text{in}}(s) \cup \delta^{\text{out}}(t))$. If $x_a > 0$ for some arc $a \in A \setminus A'$, then x does not correspond to a hop-constrained path-flow, since those flows only ship flow along (s, t)-paths. Otherwise check, whether or not the linear system

$$
\begin{aligned}
y(\delta^{\text{out}}([s,0])) &= x(\delta^{\text{out}}(s)), \\
y(\delta^{\text{out}}([i,\ell])) - y(\delta^{\text{in}}([i,\ell])) &= 0 && \text{for all } [i,\ell] \in \mathcal{N} \setminus \{[s,0],[t,k]\}, \\
\sum_{\substack{a \in \mathcal{A}: \\ \varphi(a)=(i,j)}} y_a &= x_{ij} && \text{for all } (i,j) \in A', \\
y_a &\geq 0 && \text{for all } a \in \mathcal{A}
\end{aligned}
$$

is consistent, a task that can be settled with interior point methods in polynomial time.

The second result addresses the fractionality of length-bounded cuts and flows.

Theorem 5.13 (cf. Theorem 8 of Baier et al. [1])**.** There are unit-capacity (acyclic) digraphs of order n such that every k-hop constrained maximum (s,t)-flow ships more than half of the total flow along paths with flow values $\mathcal{O}(1/n)$. □

The original theorem of Baier et al. [1] has been formulated in the context of outerplanar graphs and was proven by means of the underlying graph of a digraph belonging to the family of acyclic digraphs as illustrated in Figure 5.1. Applied to the directed case, their proof is as follows. Consider the family of acyclic block digraphs depicted in Figure 5.1 with $n = k+2$ and $|P^1| = k+1$, $|P^i| = 2$ for $i = 2, \dots, n$. The unique maximum $(2k+2)$-hop constrained (v_0, v_n)-flow ships flow value $\frac{k}{k+1}$ along the path $(P \setminus \{(v_0,v_1)\}) \cup P^1$ and flow value $\frac{1}{k+1}$ along each path $(P \setminus \{(v_{i-1},v_i)\}) \cup P^i$ for $i = 2, \dots, k+2$, where $P := (v_0, v_1, \dots, v_{k+2})$.

As is mentioned in [1], in view of Theorem 5.13, it seems to be very unlikely that there is a combinatorial algorithm that computes a hop constrained flow of maximum value in polynomial time.

5.5.3 Arc set capacitated cuts and flows

In this subsection, we generalize the hop constrained maximum flow problem in terms of the compact formulation (5.35).

Let $D = (N, A)$ be any directed graph, let s, t be any two distinct nodes of N, let $(A_i,\, i \in \mathcal{I})$ be a finite family of subsets of A, and associate with each subset A_i a capacity $u(A_i) \geq 0$. We call the optimization problem expressed by the linear program

$$\begin{aligned} \max \quad & x(\delta^{\text{out}}(s)) - x(\delta^{\text{in}}(s)) \\ \text{s.t.} \quad & x(\delta^{\text{in}}(v)) - x(\delta^{\text{out}}(v)) = 0 && \text{for all } v \in N \setminus \{s,t\}, \\ & x(A_i) \leq u(A_i) && \text{for all } i \in \mathcal{I}, \\ & x_a \geq 0 && \text{for all } a \in A \end{aligned} \tag{5.37}$$

the *maximum* (s,t)*-flow problem with capacity constraints on sets of arcs*, briefly *arc set capacitated maximum flow problem*. The ordinary maximum flow problem is obtained when $(A_i,\, i \in \mathcal{I})$ consists of the singletons $\{a\}$, $a \in A$.

Considering the equivalent LP

$$\begin{aligned} \max \quad & z_t \\ \text{s.t.} \quad & z_s + x(\delta^{\text{out}}(s)) - x(\delta^{\text{in}}(s)) = 0, \\ & z_t + x(\delta^{\text{out}}(t)) - x(\delta^{\text{in}}(t)) = 0, \\ & x(\delta^{\text{out}}(v)) - x(\delta^{\text{in}}(v)) = 0 && \text{for all } v \in N \setminus \{s,t\}, \\ & x(A_i) \leq u(A_i) && \text{for all } i \in \mathcal{I}, \\ & x_a \geq 0 && \text{for all } a \in A, \end{aligned} \tag{5.38}$$

we see that the dual of (5.37) can be expressed as

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{I}} u(A_i)\tau_i \\ \text{s.t.} \quad & \pi_s = 0, \\ & \pi_t = 1, \\ & \pi_u - \pi_v + \sum_{\substack{i \in \mathcal{I}: \\ (u,v) \in A_i}} \tau_i \geq 0 && \text{for all } (u,v) \in A, \\ & \tau_i \geq 0 && \text{for all } i \in \mathcal{I}. \end{aligned} \tag{5.39}$$

The optimization problem corresponding to the linear program (5.39) is called the *fractional minimum* (s,t)*-cut problem with capacity constraints on arc sets*, or briefly, *fractional arc set capacitated minimum cut problem.*

As is easily seen, the arc set capacitated maximum flow problem subsumes the hop constrained maximum flow problem in terms of its compact formulation (5.35). However, it also subsumes a reformulation of the multicommodity flow problem:

Given a directed graph $D = (N, A)$ and a collection of commodities in form of source-sink pairs $(s_i, t_i) \in N \times N$, $i = 1, \dots, k$, a *multicommodity flow* is a k-tupel $f = (x^1, \dots, x^k)$ of (s_i, t_i)-flows x^i. If arc capacities $u(a) \geq 0$, $a \in A$, are given, then the multicommodity flow f is said to be *feasible* if the total flow on each arc a, $\sum_{i=1}^k x_a^i$, does not exceed its capacity $u(a)$.

In the *maximum multicommodity flow problem* one is interested in finding a feasible multicommodity flow $f = (x^1, \ldots, x^k)$ of maximum total value $\sum_{i=1}^k [x^i(\delta^{\text{out}}(s_i)) - x^i(\delta^{\text{in}}(s_i))]$. If, in addition, demands $d_1, \ldots, d_k$ are given, then one looks for a feasible multicommodity flow $f = (x^1, \ldots, x^k)$ such that flow x^i has value d_i, for $i = 1, \ldots, k$. The latter problem is called the *multicommodity flow problem*.

It is straightforward to write down the maximum multicommodity flow problem as a linear program in polynomially many variables and constraints:

$$\begin{array}{lll}
\max & \sum_{i=1}^{k} [x^i(\delta^{\text{out}}(s_i)) - x^i(\delta^{\text{in}}(s_i))] & \\
\text{s.t.} & x^i(\delta^{\text{out}}(v)) - x^i(\delta^{\text{in}}(v)) = 0 & \text{for all } v \in V \setminus \{s_i, t_i\}, \\
& & i = 1, \ldots, k, \\
& \sum_{i=1}^{k} x_a^i \leq u(a) & \text{for all } a \in A, \\
& x_a^i \geq 0 & \text{for all } a \in A, i \in \{1, \ldots, k\}.
\end{array} \tag{5.40}$$

The multicommodity flow problem can be expressed as

$$\begin{array}{ll}
x^i(\delta^{\text{out}}(s_i)) - x^i(\delta^{\text{in}}(s_i)) = d_i & i = 1, \ldots, k, \\
x^i(\delta^{\text{out}}(v)) - x^i(\delta^{\text{in}}(v)) = 0 & \text{for all } v \in V \setminus \{s_i, t_i\}, \\
& i = 1, \ldots, k, \\
\sum_{i=1}^{k} x_a^i \leq u(a) & \text{for all } a \in A, \\
x_a^i \geq 0 & \text{for all } a \in A, i \in \{1, \ldots, k\}.
\end{array} \tag{5.41}$$

The (maximum) multicommodity flow problem can be transformed into the arc set capacitated maximum flow problem as follows. Introduce k disjoint copies of D, for each commodity (s_i, t_i) one copy $D^i = (N^i, A^i)$ of D. Denote by $v^i \in N^i$ and $a^i \in A^i$ the i-th copy of node $v \in N$ and arc $a \in A$, respectively. Next, introduce two additional nodes s and t and join each node s_i^i with s by the arc (s, s_i^i) and each node t_i^i with t by the arc (t_i^i, t). Denote the resulting graph by $D' = (N', A')$. Finally, we introduce a family $(A'_j,\ j \in \mathcal{I})$ of subsets of A' by

$$\begin{aligned}
\mathcal{I} &:= A, \\
A'_j &:= \{j^i : i = 1, \ldots, k\} \text{ for all } j \in A.
\end{aligned}$$

Setting capacities $u'(A'_j) := u(j)$ for $j \in A$, we see that all copies of arc j share a common capacity. Clearly, if $\bar{f} = (\bar{x}^1, \ldots, \bar{x}^k)$ is an optimal solution of (5.40), then $\bar{x} \in \mathbb{R}^{A'}$ defined by

$$\begin{aligned} \bar{x}_{a^i} &:= \bar{x}^i_a, && \text{for all } a \in A, i = 1, \dots, k, \\ \bar{x}_{s,s^i_i} &:= \bar{x}^i(\delta^{\text{out}}(s_i)) - \bar{x}^i(\delta^{\text{in}}(s_i)), && i = 1, \dots, k, \\ \bar{x}_{t^i_i,t} &:= \bar{x}^i(\delta^{\text{in}}(t_i)) - \bar{x}^i(\delta^{\text{out}}(t_i)), && i = 1, \dots, k, \end{aligned}$$

is an optimal solution of the arc set capacitated maximum flow problem defined on D'. Note that the values $\bar{x}_{s,s^i_i}$ and $\bar{x}_{t^i_i,t}$ are nonnegative, because otherwise the multicommodity flow $\bar{f}$ would not be maximal. Conversely, if $\bar{x}$ is an arc set capacitated maximum flow in D', then $\bar{f} = (\bar{x}^1, \dots, \bar{x}^k)$ defined by

$$\bar{x}^i_a := \bar{x}_{a^i}, \qquad \text{for all } a \in A, i = 1, \dots, k,$$

is a maximum multicommodity flow in D.

Also if demands $d_1, \dots, d_k$ are given for the commodities, the multicommodity flow problem defined on D can be transformed into the arc set capacitated maximum flow problem defined on D' by adding the capacity constraints $u'((s, s^i_i)) := d_i$ for $i = 1, \dots, k$. Clearly, the value of any feasible flow with respect to D' is bounded by $d := \sum_{i=1}^k d_i$, and system (5.41) is consistent if and only if the value of the maximum flow is equal to d.

Since a combinatorial algorithm for the multicommodity flow problem is unknown and due to its close relationship to the hop constrained maximum flow problem, it seems to be quite hard to find a polynomial combinatorial algorithm that solves the latter problem given by the compact formulation (5.35). Clearly, this formulation has a special structure that could simplify the problem. For instance, the DP-graph $\mathcal{D}$ is acyclic and layered. Moreover, the capacities are defined on a disjoint union of arc sets. These arc sets themselves consist of non-adjacent arcs. On the other hand, the two latter properties also hold for the reformulation of the multicommodity flow problem as the arc set capacitated flow problem. Also, the argument that $\mathcal{D}$ is an acyclic layered network, can be very likely undermined. Ramachandran [73] has shown that the ordinary maximum flow (as well as the ordinary minimum cut problem) has the same complexity in acyclic digraphs as in a general digraphs. Moreover, to the best of our knowledge, the ordinary maximum flow problem in layered networks is not known to be easier than in general digraphs. Hence, we suspect that the special structure of $\mathcal{D}$ does not help to simplify the hop constrained maximum flow problem in terms of the compact linear program (5.35).

5.5.4 Approximation algorithms

The previous subsections make it appear quite difficult to find polynomial combinatorial algorithms for arc set capacitated or hop constrained maximum flow problems, and hence also for the membership problem for the dominant of the hop constrained path polytope. Although we have focused on flow problems, we suspect that their dual problems, which subsume the separation problem for $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, are not easier to handle. With this as a backdrop, we briefly refer to a class of combinatorial approximation algorithms available for hop constrained and arc set capacitated flow and fractional cut problems. The approximation algorithms that will be derived in this subsection are based on formulations of the above mentioned problems as packing and covering LPs, respectively. Since they are only specializations of well-known approximation algorithms by Fleischer [33] and Garg and Könemann [40] for packing and covering LPs, we do not go into the details. See e.g. [14, 34, 59, 70], for further literature.

A *packing LP* is a linear program of the form

$$\max\{c^T x : Ax \leq b,\, x \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$ is a nonnegative matrix and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are nonnegative vectors. Garg and Könemann [40] describe a primal-dual approximation algorithm that computes a feasible primal solution with objective at least $(1-\varepsilon)$ times the optimum by $\mathcal{O}(\varepsilon^{-2} m \log m)$ calls to an oracle. Given a dual variable vector y', this oracle either returns a column A_i of A such that $y^T A_i \geq c_i$ is a most violated inequality by y', or asserts that there is no such column. Using path-formulations, the above mentioned flow problems can be described by packing LPs, and hence the algorithm of Garg and Könemann [40] can be applied to compute approximative solutions. However, we prefer to revert to the multicommodity flow approximation algorithm of the same authors to derive slightly better running times.

Given a directed graph $D = (N, A)$ with capacities $u : A \to \mathbb{R}_+$ and k source-sink pairs $(s_1, t_1), \ldots, (s_r, t_r)$, the maximum multicommodity flow problem can be expressed by the packing LP

$$\begin{array}{lll} \max & \sum\limits_{P\in\mathcal{P}} f_P & \\ \text{s.t.} & \sum\limits_{\substack{P\in\mathcal{P}:\\ P\ni a}} \leq u(a) & \text{for all } a \in A, \\ & f_P \geq 0 & \text{for all } P \in \mathcal{P}. \end{array}$$

Here, $\mathcal{P}$ denotes the union of the sets $\mathcal{P}_{s_i,t_i}(D)$, $i = 1, \ldots, r$, and f_P is a variable for the flow sent along the path $P \in \mathcal{P}$. Algorithm 5 by Garg and

Könemann [40] computes a feasible multicommodity flow whose total value is at least $\frac{1}{1+\varepsilon}$ times the optimum. The algorithm consists of $\mathcal{O}(\varepsilon^{-2} m \log L)$ iterations, where $m := |A|$ and L is the maximum number of arcs on any path $P \in \mathcal{P}$. Starting with primal solution $f = 0$ and an infeasible dual solution z, it computes in each iteration a shortest path P with respect to z and increases f_P by the minimum capacity of an arc $a \in P$. Also the numbers z_a, $a \in P$, will be increased by certain values. Since the new primal solution f is likely to be infeasible, at the very end, it is scaled to feasibility. Note that for this step, the capacities are required to be strictly positive. However, this can be assumed w.l.o.g. as we can always delete zero capacity arcs. The most time consuming step in each iteration consists of r shortest paths computations. For the initial dual values, Korte and Vygen [55] suggest $z_a := (n(1+\varepsilon))^{-\lceil \frac{5}{\varepsilon} \rceil}(1+\varepsilon)$ for all $a \in A$, where $0 < \varepsilon \frac{1}{2}$. By an argument of Garg and Könemann [40], n can be replaced by the maximum number of arcs of any path $P \in \mathcal{P}$.

Theorem 5.14 (Garg and Könemann [40], cf. Korte and Vygen [55])**.** Algorithm 5 computes a feasible multicommodity flow whose total value is at least $\frac{1}{1+\varepsilon}$ times the optimum. Its running time is $\mathcal{O}(\varepsilon^{-2} m r T_{\mathrm{sp}} \log L)$, where m is the number of arcs, L is the maximum number of arcs of any path $P \in \mathcal{P}$, r is the number of commodities, and T_{sp} is the time required to compute a shortest (s,t)-path in a digraph with nonnegative arc lengths. □

Using, for instance, Dijkstra's algorithm as a subroutine, we see that the multicommodity flow approximation algorithm is a fully polynomial time approximation scheme.

Algorithm 5 can be used as prototype to derive fully polynomial time approximation schemes for hop constrained and arc set capacitated maximum flow problems. For the k-hop constrained maximum (s,t)-flow problem, which is just a single commodity flow problem with additional constraints, one only has to replace $\mathcal{P}$ by $\mathcal{P}^{\leq k}_{s,t}(D)$. Thus, r shortest path computations in each iteration will be replaced by one call of an algorithm that computes a hop constrained shortest (s,t)-path. Moreover, in case of the single commodity maximum flow problem, L denotes the maximum number of arcs on any simple (s,t)-path. Clearly, if $L \leq k$, the hop constrained maximum flow problem turns out to be an ordinary maximum flow problem, which can be solved with any max-flow algorithm to optimality. Hence, w.l.o.g. we may assume that $k < L$. This number L can be replaced by k in Algorithm 5. Along the lines of the original proof of Theorem 5.14, one easily verifies the following result.

Corollary 5.15. There is an algorithm that computes a feasible k-hop constrained (s,t)-flow whose total value is at least $\frac{1}{1+\varepsilon}$ times the optimum in

Algorithm 5: Multicommodity flow approximation scheme

Input: A digraph $D = (N, A)$ with arc capacities $u : A \to \mathbb{R}_+ \setminus \{0\}$, r source-sink pairs $(s_1, t_1), \ldots, (s_r, t_r) \in N \times N$, and a number ε with $0 < \varepsilon \leq \frac{1}{2}$.
Output: A feasible multicommodity flow $f : \mathcal{P} \to \mathbb{R}_+$.
Let $f_P := 0$ for all $P \in \mathcal{P}$.
Let L be the maximum number of arcs on any path $P \in \mathcal{P}$.
Set $\delta := (L(1+\varepsilon))^{-\lceil \frac{5}{\varepsilon} \rceil}(1+\varepsilon)$ and $z_a := \delta$ for all $a \in A$.
while *true* **do**
  Choose $P \in \mathcal{P}$ such that $z(P)$ is minimum.
  if $z(P) < 1$ **then**
    Set $\gamma := \min_{a \in P} u(a)$.
    Set $f_P := f_P + \gamma$.
    Set $z_a := z_a \left(1 + \frac{\varepsilon\gamma}{u(a)}\right)$ for all $a \in P$.
  else
    Set $\xi := \max_{a \in A} \frac{1}{u(a)} \sum_{P \in \mathcal{P} : a \in P} f_P$.
    Set $f_P := f_P / \xi$ for all $P \in \mathcal{P}$.
    return f.
  end
end

time $\mathcal{O}(\varepsilon^{-2} m T_{\text{hcsp}} \log k)$, where m is the number of arcs and T_{hcsp} is the time required to compute a k-hop constrained shortest (s,t)-path in a digraph with nonnegative arc lengths. □

Using Algorithm 4 as subroutine, this results in a fully polynomial time approximation scheme for the hop constrained maximum flow problem.

Another slight modification of the multicommodity flow approximation scheme results in an approximation algorithm for the arc set capacitated maximum flow problem, see Algorithm 6. Given a directed graph $D = (N, A)$ with two distinct nodes $s, t \in N$ and capacities $u(A_i) \geq 0$ on a family of arc sets $(A_i,\ i \in \mathcal{I})$, the arc set capacitated maximum flow problem can be expressed as the packing LP

$$\begin{aligned} \max \quad & \sum_{P \in \mathcal{P}_{s,t}(D)} f_P \\ \text{s.t.} \quad & \sum_{P \in \mathcal{P}_{s,t}(D)} |P \cap A_i| f_P \leq u(A_i) \qquad \text{for all } i \in \mathcal{I}, \\ & f_P \geq 0 \qquad \text{for all } P \in \mathcal{P}_{s,t}(D). \end{aligned} \tag{5.42}$$

Algorithm 6: Arc set capacitated flow approximation scheme

Input: A digraph $D = (N, A)$ with capacities $u(A_i) > 0$ on a family of arc sets $(A_i,\ i \in \mathcal{I})$, two distinct nodes $s, t \in N$, and a number ε with $0 < \varepsilon \leq \frac{1}{2}$.

Output: A feasible arc set capacitated flow $f : \mathcal{P}_{s,t}(D) \to \mathbb{R}_+$.

1 Let $f_P := 0$ for all $P \in \mathcal{P}_{s,t}(D)$.
2 Let A' be the set of those arcs $a \in A$ not covered by the arc sets $A_i, i \in \mathcal{I}$.
3 **if** A' *contains an* (s,t)*-path* P **then**
4   Set $f_P := \infty$.
5   **return** f.
end
6 Let L be the maximum number of arcs on any path $P \in \mathcal{P}$.
7 Set $\delta := (L(1+\varepsilon))^{-\lceil \frac{5}{\varepsilon} \rceil}(1+\varepsilon)$ and $z_i := \delta$ for all $i \in \mathcal{I}$.
while *true* **do**
8   Set $d_a := \sum_{i \in \mathcal{I}: a \in A_i} z_i$ for all $a \in A$.
9   Let $P \in \mathcal{P}_{s,t}(D)$ such that $d(P)$ is minimum.
10   **if** $d(P) < 1$ **then**
11   Set $\gamma := \min\limits_{i \in \mathcal{I}} \frac{u(A_i)}{|P \cap A_i|}$.
12   Set $f_P := f_P + \gamma$.
13   Set $z_i := z_i(1 + \varepsilon\gamma|P \cap A_i|\frac{1}{u(A_i)})$ for all $i \in \mathcal{I}$.
    else
14   Set $\xi := \max\limits_{i \in \mathcal{I}} \frac{1}{u(A_i)} \sum_{P \in \mathcal{P}_{s,t}(D)} |P \cap A_i| f_P$.
15   Set $f_P := f_P / \xi$ for all $P \in \mathcal{P}_{s,t}(D)$.
16   **return** f.
    end
end

The dual is the linear program

$$
\begin{aligned}
\min \quad & \sum_{i \in \mathcal{I}} u(A_i) z_i \\
\text{s.t.} \quad & \sum_{i \in \mathcal{I}} |P \cap A_i| z_i \geq 1 \quad \text{for all } P \in \mathcal{P}_{s,t}(D), \\
& z_i \geq 0 \qquad\qquad\qquad \text{for all } i \in \mathcal{I}.
\end{aligned}
\tag{5.43}
$$

Checking whether a dual solution z satisfies all path constraints can be settled with a shortest path computation with respect to the lengths $d_a := \sum\limits_{i \in \mathcal{I}: a \in A_i} z_i$ for $a \in A$. This justifies the choice of P in line 9 of Algorithm 6. All modifications can be incorporated in the original proof of Theorem 5.14 given in [40]. Hence we skip the proof of the following result.

Corollary 5.16. Algorithm 6 computes an arc set capacitated flow whose total value is at least $\frac{1}{1+\varepsilon}$ times the optimum in time $\mathcal{O}(\varepsilon^{-2}m(T_{\mathrm{sp}}+qm)\log L)$, where $m := |A|$, $q := |\mathcal{I}|$, and T_{sp} is the time required to compute a shortest (s,t)-path in a digraph with nonnegative arc lengths. □

We now very briefly deal with approximation algorithms for the fractional cut problems considered in this section. A *covering LP with upper bounds on variables* is a linear program of the form

$$\min\{c^T x : Ax \geq b,\, x \leq u,\, x \geq 0\},$$

where $A \in \mathbb{R}^{n\times m}$ is a nonnegative integer matrix and $b \in \mathbb{R}^m$, $c, u \in \mathbb{R}^n$ are nonnegative integer vectors. Fleischer [33] describes an algorithm that computes an ε-optimal solution of such an LP and its dual by at most $\mathcal{O}(\varepsilon^{-2}m\log(c^T u))$ calls to an oracle. Given a value α and a variable vector x, this oracle either returns a row A_i of A such that $A_i x/b_i < \alpha$ or asserts that there is no such row. Thus, her algorithm is based on the same ideas as the approximation algorithm for packing LPs by Garg and Könemann [40], but it differs in the details. In view of the linear programs (5.34) (with $L = k$ and $d \equiv 1$) and (5.43), it is not hard to see that the fractional hop constrained and arc set capacitated minimum cut problems can be formulated as covering LPs with variable upper bounds equal to one provided that the objective functions, that is, the capacity functions, are integer or rational. The oracle calls consist of (hop constrained) shortest path computations.

5.6 Characterization of facets

With a view to the previous section, it is probably quite difficult to design a polynomial combinatorial algorithm that solves the separation problem of the polyhedron $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ unless we find some for the multicommodity flow problem or other fractional flow problems. This indicates that $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ and probably also $P^{\leq k}_{s,t\text{-walk}}(D)$ do not have facial structures which are easy to describe. As a consequence, we only present a partial description of $P^{\leq k}_{s,t\text{-walk}}(D)$ and $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ by facet defining inequalities. More precisely, the aim of Subsections 5.6.1 and 5.6.2 is to characterize all 0/1-facet defining inequalities for $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ and all facet defining inequalities for $P^{\leq k}_{s,t\text{-walk}}(D)$ with coefficients in the set $\{-1, 0, 1\}$. To the best of our knowledge, this completeness result has not been given before. The derived inequalities are not new: (s,t)-min-cut inequalities (5.13) as well as jump (3.29) and lifted jump inequalities (3.30). Nevertheless, the used projection mechanism is worth to be considered, since the coefficient structure of the inequalities defined in the Euclidean space $\mathbb{R}^A$ can be traced back to simpler structures in the Euclidean space $\mathbb{R}^{\mathcal{N}}$. This relation will be used in Subsection 5.6.3 to derive facet defining inequalities for $P^{\leq k}_{s,t\text{-walk}}(D)$ with fractional coefficients. Maybe this approach opens the door to new classes of facet defining inequalities in future.

Throughout this section, we assume that D is the digraph that can be obtained from the complete digraph on $n+1$ nodes by deleting the arcs $a \in \delta^{\text{in}}(s) \cup \delta^{\text{out}}(t)$. Next, we denote by $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ the DP-graph associated with (D, s, t, k) and by $\hat{\mathcal{A}}$ the set of its artificial arcs. For any $\pi \in \mathbb{R}^{\mathcal{N}}$, let $(\tau^\pi, \tau^\pi_0) \in \mathbb{R}^A \times \mathbb{R}$ be defined by

$$\tau^\pi_0 := \pi_{tk} - \pi_{s0}, \tag{5.44}$$

$$\tau^\pi_{ij} := \max\{\pi_{j\ell} - \pi_{ih} : a = ([i,h],[j,\ell]) \in \mathcal{A} \setminus \hat{\mathcal{A}}\}, \qquad (i,j) \in A. \tag{5.45}$$

We say that (τ^π, τ^π_0) or just τ^π is *induced* by π. Moreover, for any $v \in \mathbb{R}^p$, we call v^+ defined by $v^+_i := \max\{0, v_i\}$ for $i = 1, \ldots, p$, the *positive part of* v.

5.6.1 *All $\{0,1\}$-facets of dmt($P^{\leq k}_{s,t\text{-path}}(D)$)*

In this subsection we show that each nontrivial facet defining inequality with coefficients in the set $\{0,1\}$ is either an (s,t)-min-cut inequality among (5.13) or a jump inequality (3.29).

By Theorem 5.7, $\text{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ is the projection of the polyhedron $P^\star$ defined by the inequality system (5.25)-(5.27), that is,

$$\begin{aligned} Zy &\geq \mathbb{1}, \\ y &\geq 0, \\ x - Ty &\geq 0, \end{aligned}$$

onto the space of x-variables. Here, Z is the constraint matrix associated with all $([s,0],[t,k])$-min-cut inequalities, and T is the matrix representing the set function

$$\varphi : \mathcal{A} \to A \cup \varnothing, \ \varphi(([i,h],[j,\ell])) = \begin{cases} (i,j) & \text{if } i \neq j, \\ \varnothing & \text{else.} \end{cases}$$

Thus, the projection cone $\mathcal{C}$ is given by the set of all (ρ, σ, τ) satisfying

$$\begin{aligned} \rho^T Z + \sigma^T - \tau^T T &= 0, \\ \rho \geq 0, \ \sigma \geq 0, \ \tau &\geq 0. \end{aligned} \tag{5.46}$$

In view of Theorem 5.9, saying that the separation problem for the polyhedron $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ and any point $x^\star \in \mathbb{R}^A_+$ can be expressed by the linear program (5.32)

$$
\begin{array}{lrl}
\min & \sum_{(i,j)\in A} x^\star_{ij}\tau_{ij} & \\
\text{s.t.} & \pi_{s0} = 0, & \\
& \pi_{tk} = 1, & \\
& \pi_{i\ell} - \pi_{jm} + \tau_{ij} \geq 0 & \text{for all } a = ([i,\ell],[j,m]) \in \mathcal{A}\setminus\hat{\mathcal{A}}, \\
& \pi_{i,\ell-1} - \pi_{i\ell} \geq 0 & \text{for all } i \in N\setminus\{s,t\}, \ell \in \{2,3,\dots,k-1\}, \\
& \tau_{ij} \geq 0 & \text{for all } (i,j) \in A,
\end{array}
$$

the essential part of the projection can be described in compact form by the polytope Π defined as the set of all $\pi \in \mathbb{R}^{\mathcal{N}}$ satisfying

$$
\begin{aligned}
\pi_{s0} &= 0, \\
\pi_{tk} &= 1, \\
\pi_{i,\ell-1} - \pi_{i\ell} &\geq 0 \quad \text{for all } i \in N\setminus\{s,t\}, \ell \in \{2,3,\dots,k-1\}
\end{aligned}
\tag{5.47}
$$

together with the projection rule

$$
\tau_{ij} = \max\{0, \max\{\pi_{jm} - \pi_{i\ell} : a = ([i,\ell],[j,m]) \in \mathcal{A}\setminus\hat{\mathcal{A}}\}\}. \tag{5.48}
$$

This means, τ is the positive part of τ^π. In what follows, let $\tau^{\pi,+} := (\tau^\pi)^+$. Furthermore, denote by $\mathcal{C}^+(\Pi)$ the set of all vectors that are positive parts of vectors induced by some $\pi \in \Pi$.

Theorem 5.17.

$$
\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D)) = \{x \in \mathbb{R}^A : x \geq 0,\ \tau^T x \geq 1 \text{ for all } \tau \in \mathcal{C}^+(\Pi)\}. \tag{5.49}
$$

Before proving the theorem, let us remark that the projection mechanism can be nicely described via potentials. Recall that given a directed graph $G = (V,E)$ and a weight function $w : E \to \mathbb{R}$, a function $p : V \to \mathbb{R}$ is called a *potential* if $w((i,j)) \geq p(j) - p(i)$ for each arc $(i,j) \in E$. Now each (s,t)-cut $\delta^{\text{out}}(S)$ of G induces via its incidence function

$$
\chi^{\delta^{\text{out}}(S)} : E \to \mathbb{R},\ \chi^{\delta^{\text{out}}(S)}((i,j)) = \begin{cases} 1 & \text{if } (i,j) \in \delta^{\text{out}}(S), \\ 0 & \text{otherwise} \end{cases}
$$

a potential

$$
p^S : V \to \mathbb{R},\ p^S(i) = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{otherwise.} \end{cases}
$$

Clearly, each function $p : V \to \mathbb{R}$ with $p(s) = 0$, $p(t) = 1$, and $0 \le p(i) \le 1$ for all $i \in V \backslash \{s, t\}$ is a convex combination of potentials p^S associated with (s, t)-cuts $\delta^{\text{out}}(S)$: $p = \sum_S \lambda_S p^S$. Moreover, p is a potential for $\sum_S \lambda_S \chi^{\delta^{\text{out}}(S)}$. In our context, this means that each $\pi \in \Pi$ can be interpreted as a potential with respect to certain convex combinations of the rows of Z. Given π, denote by P^π the set of all ρ with $\rho^T \mathbb{1} = 1$ such that π is a potential for $\omega := \rho^T Z$. Each triple $(\rho, \sigma, \tau) \in \mathcal{C}$ such that $\rho \in P^\pi$ provides a valid inequality $\tau^T x \ge 1$. The task to find the strongest under all these inequalities, can be expressed by the projection rule

$$\tau_{ij} := \max\{\omega_a : \varphi(a) = (i, j)\} \qquad \text{for all } (i, j) \in A.$$

Thus, $\tau = \tau^{\pi,+}$.

Proof of Theorem 5.17. Denote the polyhedron on the right hand side of equation (5.49) by P. Let $x' \in \text{dmt}(P^{\le k}_{s,t\text{-path}}(D))$ and $\tau \in \mathcal{C}^+(\Pi)$. Since $\tau = \tau^{\pi,+}$ for some $\pi \in \Pi$, (π, τ) is a feasible solution of the linear program (5.32). This implies $\tau^T x' \ge 1$, by Theorem 5.9. Thus, $x' \in P$.

Now let $x' \in P$. Suppose that $x' \notin \text{dmt}(P^{\le k}_{s,t\text{-path}}(D))$. Then, there exists a feasible solution (π, τ) of the linear program (5.32) such that $\tau^T x' < 1$. Run the Moore-Bellman-Ford algorithm 4 with lengths τ_a, $a \in A$. The algorithm returns in particular for each $[i, \ell] \in \mathcal{N}$ the length $u_i^{(\ell)}$ of a shortest (s, i)-path using at most ℓ arcs. Define $\tilde{\pi} \in \mathbb{R}^{\mathcal{N}}$ by $\tilde{\pi}_{i\ell} = u_i^{(\ell)}$ for $[i, \ell] \in \mathcal{N}$. Then, $\tau = \tau^{\tilde{\pi},+}$. Moreover, $\tilde{\pi}_{s0} = 0$ and $\tilde{\pi}_{i\ell} \ge 0$ for $[i, \ell] \in \mathcal{N} \backslash \{[s, 0]\}$, since $\tau^{\pi,+} \ge 0$. Next, $\tilde{\pi}_{tk} \ge 1$, since $\tau^T x \ge 1$ is a valid inequality for $\text{dmt}(P^{\le k}_{s,t\text{-path}}(D))$, by Theorem 5.9. W.l.o.g., we may assume that $\tilde{\pi}_{tk} = 1$. (Otherwise replace $\tilde{\pi}$ by $\frac{1}{\tilde{\pi}_{tk}} \tilde{\pi}$.)

Finally, suppose that the set $\mathcal{N}' := \{[i, \ell] \in \mathcal{N} : \tilde{\pi}_{i\ell} > 1\}$ is nonempty. Then, we will show that $\sigma \le \tau$, where $\sigma := \tau^{\hat{\pi},+}$ and $\hat{\pi}$ is defined by $\hat{\pi}_{i\ell} := \min\{1, \tilde{\pi}_{i\ell}\}$ for all $[i, \ell] \in \mathcal{N}$. This implies $\sigma^T x' < 1$. Moreover, by definition, $\hat{\pi} \in \Pi$.

Let $[i, \ell] \in \mathcal{N}'$ such that ℓ is maximal. Define the vector $\hat{\pi}^{i\ell}$ by $\hat{\pi}^{i\ell}_{jh} := \tilde{\pi}_{jh}$ for all $[j, h] \in \mathcal{N} \setminus \{[i, \ell]\}$, $\hat{\pi}^{i\ell}_{i\ell} := 1$. Then one can see that $\sigma^{i\ell} \le \tau$, where $\sigma^{i\ell}$ denotes the positive part of the vector induced by $\hat{\pi}^{i\ell}$. $\mathcal{N}'' := \mathcal{N}' \setminus \{[i, \ell]\}$ has one element less than $\mathcal{N}'$. Thus, after $|\mathcal{N}'|$ iterations of this step, we end up with the pair of vectors $(\hat{\pi}, \sigma)$. $\square$

Next, we would like to characterize those 0/1-vectors $\pi \in \Pi$ whose associated vectors $\tau^{\pi,+}$ imply facet defining inequalities. For this, consider the analogy between vectors of Π and length-vectors $u = \left(u_i^{(\ell)}\right)$, where $u_i^{(\ell)}$

denotes the length of a shortest (s,i)-path with at most ℓ arcs. Here, we assume that D has no negative cycles with respect to the given length function $d : A \to \mathbb{R}$. For fixed i, the numbers $u_i^{(1)}, u_i^{(2)}, \ldots$ are nonincreasing. This property has been incorporated into the DP-model via the artificial arcs of $\mathcal{D}$ and has resulted into the inequalities $\pi_{i,\ell-1} - \pi_{i,\ell} \geq 0$. Another important property of the numbers $u_i^{(\ell)}$ is expressed by the Bellman equations (5.19):

$$u_j^{(\ell)} = \min\{u_j^{(\ell-1)}, \min\{u_i^{(\ell-1)} + d((i,j)) : (i,j) \in \delta^{\text{in}}(j)\}\} \\ \text{for all } j \in N,\ \ell = 1, \ldots, |N|-1.$$

Applied to Π this means to require that among the inequalities

$$\pi_{j,\ell-1} - \pi_{j\ell} \geq 0, \qquad \pi_{ih} - \pi_{j\ell} + \tau_{ij}^{\pi} \geq 0$$

of (5.32) at least one is tight for every node $[j,\ell] \in \mathcal{N} \setminus \{[0,0]\}$. This leads to Definition 5.18 (a) and subsequent definitions.

Definition 5.18. Let $\pi \in \mathbb{R}^{\mathcal{N}}, \tau \in \mathbb{R}^{A}$.

(a) π is said to be τ-*in-monotone* if for every node $[j,\ell] \in \mathcal{N} \setminus \{[s,0]\}$, $\pi_{j\ell} = \pi_{ih} + \tau_{ij}$ for some arc $([i,h],[j,\ell]) \in \delta^{\text{in}}([j,\ell])$, where $\tau_{jj} := 0$.

(b) π is said to be τ-*out-monotone* if for every node $[i,h] \in \mathcal{N} \setminus \{[t,k]\}$, $\pi_{j\ell} = \pi_{ih} + \tau_{ij}$ for some arc $([i,h],[j,\ell]) \in \delta^{\text{in}}([i,h])$, where $\tau_{ii} := 0$.

(c) π is said to be τ-*in-and-out-monotone* if π is τ-in- and τ-out-monotone.

If it is clear from context, we omit the prefix τ in the above definitions. We now present the main result of this subsection.

Theorem 5.19. $\tau^{\pi,+}$-in-and-out-monotone 0/1-vectors $\pi \in \Pi$ and nontrivial facet defining 0/1-inequalities for $\text{dmt}(P_{s,t\text{-path}}^{\leq k}(D))$ are in 1-1-correspondence. This means,

(a) each in-and-out-monotone 0/1-vector $\pi \in \Pi$ induces a facet defining 0/1-inequality for $\text{dmt}(P_{s,t\text{-path}}^{\leq k}(D))$;

(b) each nontrivial facet defining 0/1-inequality for $\text{dmt}(P_{s,t\text{-path}}^{\leq k}(D))$ is induced by an in-and-out-monotone 0/1-vector $\pi \in \Pi$;

(c) if two in-and-out-monotone 0/1-vectors $\pi, \tilde{\pi} \in \Pi$ induce the same facet defining inequality for $\text{dmt}(P_{s,t\text{-path}}^{\leq k}(D))$, then $\pi = \tilde{\pi}$.

The remainder of this subsection is dedicated to prove Theorem 5.19.

Lemma 5.20. Let $\pi \in \Pi$, $i \in N \setminus \{s,t\}$, and $0 \le \alpha \le 1$ such that

$$\mathcal{N}^{\pi}_{i,\alpha} := \{\ell \in \{1,\dots,k-1\} : \pi_{i\ell} = \alpha\} \neq \varnothing.$$

(a) Let $h := \min \mathcal{N}^{\pi}_{i,\alpha}$. If π is $\tau^{\pi,+}$-in-monotone, then there exists an $([s,0],[i,h])$-path $P \subseteq \mathcal{A}$ such that $\tau^{\pi,+}(\varphi(P)) = \alpha$.

(b) Let $h := \max \mathcal{N}^{\pi}_{i,\alpha}$. If π is $\tau^{\pi,+}$-out-monotone, then there exists an $([i,h],[t,k])$-path $P \subseteq \mathcal{A}$ such that $\tau^{\pi,+}(\varphi(P)) = 1-\alpha$.

Proof. (a) We prove the result by induction on $h = 1,\dots,k-1$. For any internal node $i \in N$ and $h = 1$ set $P := \{([s,0],[i,1])\}$. Then, $\tau^{\pi,+}(\varphi(P)) = \pi_{i1} - \pi_{s0} = \alpha - 0 = \alpha$.

Let $h \in \{1,\dots,k-2\}$, and assume the statement to be true for all internal nodes $[i,\ell] \in \mathcal{N}$ with $\ell \le h$. Consider any internal node $[i,h+1] \in \mathcal{N}$ such that $h+1 = \min \mathcal{N}^{\pi}_{i,\alpha}$. Since $h+1$ is the minimum number, this implies $\pi_{ih} > \pi_{i,h+1}$. Next, since π is in-monotone, it follows that $\pi_{i,h+1} - \pi_{jh} = \tau^{\pi,+}_{ji}$ for some internal node $j \in N$, $j \neq i$. Consider the set $\mathcal{N}^{\pi}_{j,\beta}$ for $\beta := \pi_{jh}$, and assume that $h > \min \mathcal{N}^{\pi}_{j,\beta}$. Then, we have at the same time $\pi_{j,h-1} = \pi_{jh}$ and $\pi_{ih} > \pi_{i,h+1}$, which implies $\pi_{ih} - \pi_{j,h-1} > \tau^{\pi,+}_{ji}$, a contradiction. Consequently, $h = \min \mathcal{N}^{\pi}_{j,\beta}$, and hence, there exists an $([s,0],[j,h])$-path $P' \subseteq \mathcal{A}$ such that $\tau^{\pi,+}(\varphi(P')) = \pi_{jh}$, by hypothesis. Thus, the path $P := P' \cup \{([j,h],[i,h+1])\}$ satisfies $\tau^{\pi,+}(\varphi(P)) = \pi_{i,h+1} = \alpha$.

(b) can be proved analogously. $\square$

Let

$$\bar{\mathcal{N}} := \mathcal{N} \bigcup \{[i,\ell] : i \in N \setminus \{s,t\}, \ell \in \{0,k\}\}.$$

For any $\pi \in \Pi$ define its *extension* $\bar{\pi} \in \mathbb{R}^{\bar{\mathcal{N}}}$ by $\bar{\pi}_{i\ell} := \pi_{i\ell}$ for all $[i,\ell] \in \mathcal{N}$ and $\bar{\pi}_{i0} := 1$ and $\bar{\pi}_{ik} := 0$ for $i \in N \setminus \{s,t\}$. Next, for any $\pi \in \Pi$ with extension $\bar{\pi}$, any $i \in N \setminus \{s,t\}$, and any $\alpha \in [0,1]$, we define

$$\bar{\mathcal{N}}^{\bar{\pi}}_{i,\alpha} := \{\ell \in \{0,\dots,k\} : \bar{\pi}_{i\ell} = \alpha\}.$$

In addition, for each $\ell \in \{0,\dots,k\}$ associate subsets of internal nodes of N defined by

$$\begin{aligned} N^{\bar{\pi},\min}_{\ell,\alpha} &:= \{i \in N \setminus \{s,t\} : \min \bar{\mathcal{N}}^{\bar{\pi}}_{i,\alpha} = \ell\} \\ \text{and } N^{\bar{\pi},\max}_{\ell,\alpha} &:= \{i \in N \setminus \{s,t\} : \max \bar{\mathcal{N}}^{\bar{\pi}}_{i,\alpha} = \ell\}. \end{aligned}$$

In the following, consider $\alpha = 0$. Clearly, $N^{\bar{\pi},\min}_{0,0} = \varnothing$, and the remaining sets $N^{\bar{\pi},\min}_{\ell,0}$, $\ell \in \{1,\dots,k\}$, define a partition of the internal nodes:

$$N \setminus \{s,t\} = \bigcup_{\ell=1}^{k} N^{\bar{\pi},\min}_{\ell,0}.$$

The following lemma is immediate. For an illustration of the lemma see Figure 5.3.

Lemma 5.21. Let $\pi \in \Pi$ be a 0/1-vector and $\bar{\pi} \in \mathbb{R}^{\bar{\mathcal{N}}}$ its extension. Then,

(a) π is $\tau^{\pi,+}$-in-monotone if and only if $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing$ for $\ell \in \{2,\dots,k-1\}$ implies $N^{\bar{\pi},\min}_{h,0} \neq \varnothing$ for all $h \in \{1,\dots,\ell-1\}$.

(b) π is $\tau^{\pi,+}$-out-monotone if and only if $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing$ for $\ell \in \{2,\dots,k-1\}$ implies $N^{\bar{\pi},\min}_{h,0} \neq \varnothing$ for all $h \in \{\ell+1,\dots,k\}$.

(c) π is $\tau^{\pi,+}$-in-and-out-monotone if and only if

$$N \setminus \{s,t\} = N^{\bar{\pi},\min}_{1,0} \,\dot{\cup}\, N^{\bar{\pi},\min}_{k,0}$$

(that is, $N^{\bar{\pi},\min}_{\ell,0} = \varnothing$ for $\ell = 2,\dots,k-1$) or $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing$ for $\ell = 1,\dots,k$.

□

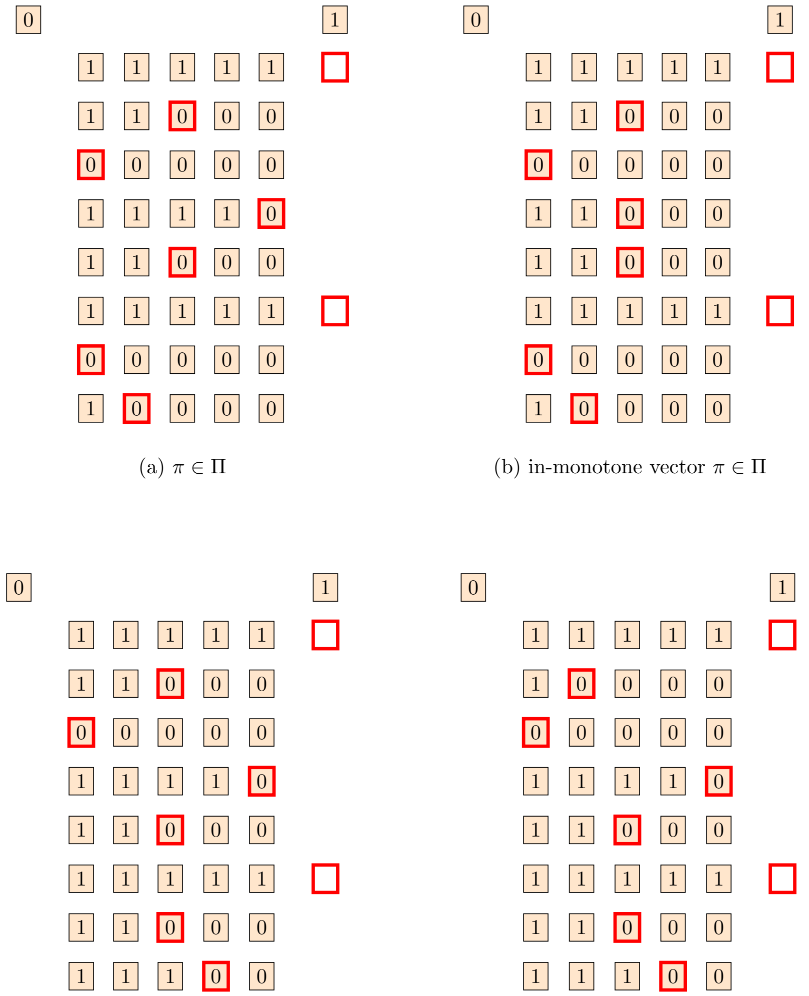
For any $\pi \in \mathbb{R}^{\mathcal{N}}$ and any internal node $i \in N$, we define the component vector $\pi_i := [\pi_{i1}, \pi_{i2,\dots,\pi_{i,k-1}}]$.

Proof of Theorem 5.19. For any $\mu \in \{0,1\}^A$, let $A^\mu_0 := \{(i,j) \in A : \mu_{ij} = 0\}$ and $A^\mu_1 := \{(i,j) \in A : \mu_{ij} = 1\}$.

(a) Let $\pi \in \Pi \cap \{0,1\}^{\mathcal{N}}$ be an in-and-out-monotone vector and set $\tau := \tau^{\pi,+} \in \mathbb{R}^A$. Clearly, the inequality $\tau^T x \geq 1$ is valid, and $\pi \in \{0,1\}^{\mathcal{N}}$ implies $\tau \in \{0,1\}^A$. The polyhedron $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ is fulldimensional. In order to show that $\tau^T x \geq 1$ induces a facet of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, we construct $|A|$ affinely independent points of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$ satisfying $\tau^T x \geq 1$ at equality.

Partition the arc set A into the two subsets $A_0 := A^\tau_0$ and $A_1 := A^\tau_1$. For each arc $(i,j) \in A_1$ we will construct a path $P \in \mathcal{P}^{\leq k}_{s,t}(D)$ such that $P \cap A_1 = \{(i,j)\}$. Denoting the incidence vector of P by x^{ij}, we see that $\tau^T x^{ij} = 1$. Moreover, the points x^{ij}, $(i,j) \in A_1$, are linearly independent. Next, for any arc $(p,q) \in A_1$ and each $(i,j) \in A_0$ define the vector $x^{ij} := x^{pq} + e_{ij}$, where e_{ij} is the ij-th unit vector. These points are obviously affinely independent, and they are also affinely independent of the former points x^{ij}, $(i,j) \in A_1$. This completes the construction of $|A|$ affinely independent points.

It remains to show that such a path really exists for $(i,j) \in A_1$. We construct an (s,i)-path $P' \subseteq A_0$ and an (j,t)-path $P'' \subseteq A_0$ such that $P := P' \cup \{(i,j)\} \cup P''$ is an (s,t)-path with at most k arcs. Then, $\tau(P) = 1$ by construction. If $i = s$, we set $P' := \varnothing$. So, let i be an internal node.



(a) $\pi \in \Pi$

(b) in-monotone vector $\pi \in \Pi$

(c) out-monotone vector $\pi \in \Pi$

(d) in-and-out-monotone vector $\pi \in \Pi$

Figure 5.3: Illustration of {in}-{out}-monotone 0/1-vectors $\pi \in \Pi$.

Apply Lemma 5.20 (a) with $\alpha = 0$. Since $\tau_{ij} = 1$, the set $\mathcal{N}^{\pi}_{i,0}$ is nonempty. Let $g = \min \mathcal{N}^{\pi}_{i,0}$. Then there exists an $([s,0],[i,g])$-path $\tilde{P}' \subseteq \mathcal{A}$ such that $\tau(\varphi(\tilde{P}')) = \alpha = 0$. Thus, $P' := \varphi(\tilde{P}') \subseteq A_0$. Note that P' consists of exactly g arcs. Next, consider node j. If $j = t$, set $P'' := \varnothing$. Otherwise apply Lemma 5.20 (b) with $\alpha = 1$. Since $\tau_{ij} = 1$, the set $\mathcal{N}^{\pi}_{j,1}$ is nonempty. Moreover, it exists an $([i,h],[t,k])$-path $\tilde{P}'' \subseteq \mathcal{A}$ such that $\tau(\varphi(\tilde{P}'')) = 1-\alpha = 0$ for $h = \max \mathcal{N}^{\pi}_{j,1}$. Thus, $P'' := \varphi(\tilde{P}'') \subseteq A_0$. Moreover, by construction of

$\tilde{P}''$, $|P''| \leq k-h$. Thus, $|P| = |P'| + 1 + |P''| = g + 1 + k - h \leq k$, where it is assumed that $g = 0$ if $P' = \varnothing$ and $h = k$ if $P'' = \varnothing$.

(b) Let $\tau^T x \geq \tau_0$ be an inequality with coefficients $\tau_{ij} \in \{0,1\}$ for all $(i,j) \in A$ that induces a nontrivial facet of $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$. Since $\tau \geq 0$ and $\tau^T x \geq \tau_0$ is not equivalent to a nonnegativity constraint, it follows that $\tau_0 > 0$. Moreover, since $\tau_{st} = 1$ and $\tau^T x \geq \tau_0$ is facet defining, we have $\tau_0 = 1$.

Run the Moore-Bellman-Ford algorithm 4 with length function $d : A \to \mathbb{R}$, $(i,j) \mapsto \tau_{ij}$. The algorithm especially returns for each $[i,\ell] \in \mathcal{N}$ the length of a shortest (s,i)-path using at most ℓ arcs, $u_i^{(\ell)}$. Define the vector $\pi \in \mathbb{R}^{\mathcal{N}}$ by $\pi_{i\ell} := u_i^{(\ell)}$, $[i,\ell] \in \mathcal{N}$. Since $\tau \in \{0,1\}^A$, it follows that $\pi_{i\ell} \in \mathbb{Z}_+$ for all $[i,\ell] \in \mathcal{N}$ and, in particular, $\pi_{s0} = 0$. Next, since $\tau_{st} = 1$, we have $\pi_{tk} \leq 1$. Assume that $\pi_{tk} = 0$. Since π_{tk} denotes the length of a shortest path $P \in \mathcal{P}^{\leq k}_{s,t}(D)$, it follows immediately $\tau^T \chi^P = 0 < 1 = \tau_0$, a contradiction. Consequently, $\pi_{tk} = 1$. Moreover, since $u_i^{(1)} \in \{0,1\}$ and the sequence $u_i^{(1)}, u_i^{(2)}, \ldots, u_i^{(k-1)}$ is nonincreasing for every internal node $i \in N$, it follows immediately that $\pi \in \Pi \cap \{0,1\}^{\mathcal{N}}$.

Next, $\tau^{\pi,+} \leq \tau$, by definition of $\tau^{\pi,+}$. Thus, $A_1^{\tau^{\pi,+}} \subseteq A_1^{\tau}$. Since the inequality $\tau^T x \geq 1$ induces a facet, it follows equality. Thus, $\tau^{\pi,+} = \tau$.

It remains to be shown that π is in-and-out-monotone. The Bellman equations (5.19) imply that π is in-monotone. We show, by contraposition, that π is also out-monotone. Suppose not. Then, it exists an internal node $[i,\ell] \in \mathcal{N}$ with $\ell < k-1$ such that $\pi_{j,\ell+1} - \pi_{i\ell} < \tau_{ij}$ for all internal nodes $j \in N$, where $\tau_{ii} := 0$. In particular, since $\pi_{i,\ell+1} - \pi_{i\ell} < 0$, it follows immediately that $\pi_{i\ell} = 1$ and $\pi_{i,\ell+1} = 0$.

First, suppose that $\ell = 1$. Consider the vector $\rho \in \mathbb{R}^{\mathcal{N}}$ defined by $\rho_{i1} := 0$ and $\rho_{jh} := \pi_{jh}$ for all $[j,h] \in \mathcal{N} \setminus \{[i,1]\}$, and let $\psi \in \mathbb{R}^A$ be the positive part of the vector induced by ρ. Clearly, $\psi \in \{0,1\}^A$ and $A_1^{\psi} \subsetneq A_1^{\tau}$. Hence the inequality $\psi^T x \geq 1$ strictly dominates the inequality $\tau^T x \geq 1$, and thus, the latter one is not facet defining.

Next, assume that $1 < \ell < k-1$. Clearly, one of the four following configurations holds for every row $j \neq i$.

(a)	$\pi_{i,\ell-1} = 1$	$\pi_{i\ell} = 1$	$\pi_{i,\ell+1} = 0$
	$\pi_{j,\ell-1} = 1$	$\pi_{j\ell} = 1$	$\pi_{j,\ell+1} = 1$

(b)

$\pi_{i,\ell-1} = 1$	$\pi_{i\ell} = 1$	$\pi_{i,\ell+1} = 0$
$\pi_{j,\ell-1} = 1$	$\pi_{j\ell} = 1$	$\pi_{j,\ell+1} = 0$

(c)

$\pi_{i,\ell-1} = 1$	$\pi_{i\ell} = 1$	$\pi_{i,\ell+1} = 0$
$\pi_{j,\ell-1} = 1$	$\pi_{j\ell} = 0$	$\pi_{j,\ell+1} = 0$

(d)

$\pi_{i,\ell-1} = 1$	$\pi_{i\ell} = 1$	$\pi_{i,\ell+1} = 0$
$\pi_{j,\ell-1} = 0$	$\pi_{j\ell} = 0$	$\pi_{j,\ell+1} = 0$

Configuration (a) implies $\pi_{j,\ell+2} = 1$, since otherwise it follows that $\tau_{ij} = 0$. This in turn implies $\pi_{j,\ell+1} - \pi_{i\ell} = \tau_{ij}$, a contradiction. Thus, $\tau_{ij} = 1$ and $\tau_{ji} = 0$. Next, configurations (b) and (c) imply $\tau_{ij} = \tau_{ji} = 0$, while (d) implies $\tau_{ij} = 0$ and $\tau_{ji} = 1$.

Define a partition of the internal nodes of N by

$$N \setminus \{s,t\} = N^0 \,\dot{\cup}\, N^1,$$

where

$$\begin{aligned} N^0 &:= \{j \in N \setminus \{s,t\} : \pi_{j,\ell+1} = 0\}, \\ N^1 &:= \{j \in N \setminus \{s,t\} : \pi_{j,\ell+1} = 1\}. \end{aligned}$$

Moreover, define $\sigma \in \mathbb{R}^{\mathcal{N}}$ by $\sigma_{j\ell} := 0$ for all $j \in N^0$, $\ell \in \{1,\ldots,k-1\}$ and $\sigma_{j\ell} := \pi_{j\ell}$ otherwise. Let $\omega \in \mathbb{R}^A$ be the positive part of the vector induced by σ. Then, it follows immediately that $\omega_{vw} = 0$ for all $(v,w) \in A \cap (N \times (N^0 \cup \{s\}))$ and $\omega_{vw} = \tau_{vw}$ for all $(v,w) \in A \cap (N^1 \cup \{s,t\})$. Since $\tau_{vw} = 1$ for all $(v,w) \in N^0 \times N^1$, we see that also $\omega_{vw} = \tau_{vw}$ for all $(v,w) \in N^0 \times N^1$. Now, one easily verifies that $A_1^\omega \subsetneq A_1^\tau$, and hence the inequality $\tau^T x \geq 1$ were not facet defining, a contradiction.

(c) Let π, $\rho \in \Pi$ be two in-and-out-monotone 0/1-vectors such that $\tau^{\pi,+} = \tau^{\rho,+}$ and $\sum_{(i,j)\in A} \tau_{ij}^{\pi,+} x_{ij} \geq 1$ is facet defining. We have to show that $\pi = \rho$.

Consider the sets $N_{\ell,0}^{\bar{\pi},\min}$, $N_{\ell,0}^{\bar{\rho},\min}$ for $\ell = 1,\ldots,k$, where $\bar{\pi}$ and $\bar{\rho}$ are the extensions of π and ρ, respectively. Since π and ρ are in-and-out-monotone, $N_{\ell,0}^{\bar{\pi},\min} \neq \varnothing \neq N_{\ell,0}^{\bar{\rho},\min}$ for $\ell = 1,k$, see Lemma 5.21 (c). Furthermore, since $\tau_{si}^{\pi,+} = \tau_{si}^{\rho,+}$ and $\tau_{it}^{\pi,+} = \tau_{it}^{\rho,+}$ for all $i \in N \setminus \{s,t\}$, we derive $N_{\ell,0}^{\bar{\pi},\min} = N_{\ell,0}^{\bar{\rho},\min}$ for $\ell = 1,k$. Thus, $\pi_i = \rho_i = 0^T$ for $i \in N_{1,0}^{\bar{\pi},\min}$ and $\pi_i = \rho_i = \mathbb{1}^T$ for $i \in N_{k,0}^{\bar{\pi},\min}$. Moreover, Lemma 5.21 (c) implies that either $N_{\ell,0}^{\bar{\pi},\min} = N_{\ell,0}^{\bar{\rho},\min} = \varnothing$ for

$\ell = 2, \ldots, k-1$ or $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing \neq N^{\bar{\rho},\min}_{\ell,0}$ for $\ell = 2, \ldots, k-1$. In the first case, we are done. So, consider the second case. We show by induction on ℓ that $N^{\bar{\pi},\min}_{\ell,0} = N^{\bar{\rho},\min}_{\ell,0}$ for $\ell = 1, \ldots, k-1$. The initial step is already made.

Letting $\ell \in \{1, \ldots, k-2\}$ and $N^{\bar{\pi},\min}_{i,0} = N^{\bar{\rho},\min}_{i,0}$ for $i = 1, \ldots, \ell$, we have to show that $N^{\bar{\pi},\min}_{\ell+1,0} = N^{\bar{\rho},\min}_{\ell+1,0}$. Let $j \in N^{\bar{\pi},\min}_{\ell+1,0}$. Since $N^{\bar{\pi},\min}_{i,0} = N^{\bar{\rho},\min}_{i,0}$ for $i = 1, \ldots, \ell$ and $N^{\bar{\pi},\min}_{p,0} \cap N^{\bar{\pi},\min}_{q,0} = \varnothing$ for $p \neq q$, it follows immediately that $j \in N^{\bar{\rho},\min}_{r,0}$ for some $r \in \{\ell+1, \ldots, k\}$. Assuming $r \geq \ell + 2$ yields $\tau^{\rho,+}_{hj} = 1$ for all $h \in N^{\bar{\rho},\min}_{\ell,0}$, while $\tau^{\pi,+}_{hj} = 0$, a contradiction. Thus $r = \ell + 1$ and hence, $j \in N^{\bar{\rho},\min}_{\ell+1,0}$. Interchanging π and ρ in the above argumentation, we see that $j \in N^{r\bar{h}o,\min}_{\ell+1,0}$ also implies $j \in N^{\bar{\pi},\min}_{\ell+1,0}$. Hence, $j \in N^{\bar{\pi},\min}_{\ell+1,0}$.

It follows that $\pi_i = \rho_i$ for all internal nodes $i \in N$, and hence, $\pi = \rho$. □

We close this subsection with the identification of in-and-out-monotone 0/1-vectors $\pi \in \Pi$ with already well-known valid inequalities.

Observation 4. Let $\pi \in \Pi$ be a τ-in-and-out-monotone 0/1-vector, where $\tau := \tau^{\pi,+}$.

(a) $\tau^T x \geq 1$ is an (s,t)-min-cut inequality among (5.13) if and only if $N \setminus \{s,t\} = N^{\bar{\pi},\min}_{1,0} \,\dot{\cup}\, N^{\bar{\pi},\min}_{k,0}$, that is, $N^{\bar{\pi},\min}_{\ell,0} = \varnothing$ for $\ell = 2, \ldots, k-1$. The shores of the associated (s,t)-cut are given by $S := \{s\} \cup N^{\bar{\pi},\min}_{1,0}$ and $T := \{t\} \cup N^{\bar{\pi},\min}_{k,0}$.

(b) $\tau^T x \geq 1$ is a jump inequality (3.29) if and only if $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing$ for $\ell = 1, \ldots, k$. The partition associated with this jump inequality is given by $S_0 := \{s\}$, $S_\ell := N^{\bar{\pi},\min}_{\ell,0}$, $\ell = 1, \ldots, k$, and $S_{k+1} := \{t\}$.

5.6.2 All $\{-1,0,1\}$-facets of $P^{\leq k}_{s,t\text{-walk}}(D)$

In this subsection, we will identify all nontrivial facet defining $\{-1,0,1\}$-inequalities for $P^{\leq k}_{s,t\text{-walk}}(D)$ with inequalities that are essentially equivalent to lifted jump inequalities (3.30). They are of the following form:

$$\sum_{i=0}^{k-3} \sum_{j=i+2}^{k-1} x((S_i : S_j)) - x((S_{k-1} : S_1 \cup S_2)) \geq 0,$$

where

$$N = \bigcup_{p=0}^{k} S_p$$

is a partition of N into $k+1$ node sets with $S_0 = \{s\}$ and $S_k = \{t\}$. Such an inequality is equivalent to a lifted jump inequality (3.30) if $|S_1| \geq 2$, which can be seen as follows. Let $S_1 = \tilde{S}_1 \cup \tilde{S}_k$ be any partition of S_1. Set $\tilde{S}_{k+1} := S_k$ and $\tilde{S}_i := S_i$ for $i = 0, 2, 3, \ldots, k-1$. Then, we derive a lifted jump inequality from the above inequality by adding the equations $x(\delta^{\text{in}}(t)) = 1$ and $x(\delta^{\text{in}}(v)) - x(\delta^{\text{out}}(v)) = 0$ for all $v \in \tilde{S}_k$.

We begin with some preliminary observations about $P^{\leq k}_{s,t\text{-walk}}(D)$. Recall that D is obtained from the complete digraph on $n+1$ nodes by deleting the arc set $\delta^{\text{in}}(s) \cup \delta^{\text{out}}(t)$. Clearly, since $P^{\leq k}_{s,t\text{-walk}}(D)$ contains $P^{\leq k}_{s,t\text{-path}}(D)$ and all $x \in P^{\leq k}_{s,t\text{-walk}}(D)$ satisfy the flow conservation constraints (5.8), Corollary (3.5) implies the following statement.

Corollary 5.22. Let $4 \leq k < n$. Then,

$$\dim P^{\leq k}_{s,t\text{-walk}}(D) = \dim P^{\leq k}_{s,t\text{-path}}(D) = n^2 - 2n + 1 = (n-1)^2. \qquad \square$$

We remark that in difference to the digraph in Corollary (3.5), D contains the arc (s,t).

Due to the flow conservation constraints (5.8), equivalent valid inequalities for $P^{\leq k}_{s,t\text{-walk}}(D)$ can be identified as in case of $P^{\leq k}_{s,t\text{-path}}(D)$.

Corollary 5.23 (cf. Theorem 3.34)**.** Let $4 \leq k < n$, let $\alpha^T x \geq \alpha_0$ be a valid inequality for $P^{\leq k}_{s,t\text{-walk}}(D)$, and let T be a spanning tree of D. Then for any specified set of coefficients β_{ij} for the arcs $(i,j) \in T$, there is an equivalent inequality $\bar{\alpha}^T x \geq \alpha_0$ for $P^{\leq k}_{s,t\text{-walk}}(D)$ such that $\bar{\alpha}_{ij} = \beta_{ij}$ for $(i,j) \in T$. $\square$

Next, we show that inequalities (5.50) are indeed facet defining. For this, an inequality $\tau^T x \geq \tau_0$ (or equation $\tau^T x = \tau_0$) with $\tau \in \mathbb{R}^A$ is said to be *t-rooted* if $\tau_{it} = 0$ for all $i \in N \setminus \{t\}$.

Theorem 5.24. Let $4 \leq k < n$ and let

$$N = \bigcup_{p=0}^{k} S_p$$

be a partition of N with $S_0 = \{s\}$ and $S_k = \{t\}$.

(i) The inequality

$$\sum_{i=0}^{k-3} \sum_{j=i+2}^{k-1} x((S_i : S_j)) - x((S_{k-1} : S_1 \cup S_2)) \geq 0, \tag{5.50}$$

induces a facet of $P^{\leq k}_{s,t\text{-path}}(D)$ if and only if either $k = 4$ or $k \geq 5$ and $|S_{k-1}| \geq 2$.

(ii) Let $S_{k-1} = \{z\}$ for some node $z \in N \setminus \{s,t\}$. Then, the inequality

$$\sum_{i=0}^{k-3} \sum_{j=i+2}^{k-1} x((S_i : S_j)) - x(\delta^{\text{out}}(z) \setminus \{(z,t)\}) \geq 0, \tag{5.51}$$

induces a facet of $P^{\leq k}_{s,t\text{-path}}(D)$.

(iii) The inequality (5.50) induces a facet of $P^{\leq k}_{s,t\text{-walk}}(D)$.

Proof. We prove statement (i), indicating the necessary modifications for statements (ii) and (iii). Note that in case of $k = 4$, inequalities (5.50) and (5.51) are identical.

The validity of inequality (5.50) follows from Corollary 5.25 and Theorem 5.27.

If $k \geq 5$ and $|S_{k-1}| = 1$, that is, $S_{k-1} = \{z\}$ for some $z \in N \setminus \{s,t\}$, then inequality (5.50) is the sum of inequality (5.51) and the nonnegativity constraints $x_{uz} \geq 0$ for all $u \in \bigcup_{p=3}^{k-2} S_p$.

To show the converse, let either $k = 4$ or $k \geq 5$ and $|S_{k-1}| \geq 2$. Denote inequality (5.50) by $b^T x \geq 0$. Suppose that the equation $c^T x = c_0$ is satisfied by every $x \in P^{\leq k}_{s,t\text{-path}}(D)$ ($P^{\leq k}_{s,t\text{-walk}}(D)$) that satisfies the inequality $b^T x \geq 0$ with equality. By Theorem 3.34 (Corollary 5.23) we may assume that $c^T x = c_0$ is t-rooted. In particular, $c_{st} = 0$, which implies $c_0 = 0$.

First, we show by induction on p that $c_{ij} = 0$ for $(i,j) \in (S_p : S_{p+1})$, $p = 0, 1, \ldots, k-1$. For $p = 0$, this is true, since the 2-paths (s,v,t) with $v \in S_1$ are tight with respect to $b^T x \geq 0$. Thus, $c_{sv} = 0$ for all $v \in S_1$. Assume the statement to be true for some $p \in \{0, 1, \ldots, k-2\}$, and consider any coefficient c_{uv} with $(u,v) \in (S_p : S_{p+1})$. Then, any path $P = (s, v_1, v_2, \ldots, v_{p-1}, u, v, t)$ with $v_j \in S_j$ for $j = 1, \ldots, p-1$ is tight with respect to $b^T x \geq 0$, and hence, $c_{uv} = 0$.

Next, we prove that $c_{ij} = \sigma$ for all $(i,j) \in (S_{k-1} : S_1 \cup S_2)$, for some $\sigma \in \mathbb{R}$. In case of $k \geq 5$, consider any two arcs $(v,w), (\tilde{v}, \tilde{w}) \in (S_{k-1} : S_1 \cup S_2)$. The paths (s,u,v,w,t) and $(s,u,\tilde{v},\tilde{w},t)$ for some node $u \in S_{k-2}$ are tight, which immediately implies $c_{vw} = c_{\tilde{v}\tilde{w}}$. Thus, it exists $\sigma \in \mathbb{R}$ such that $c_{ij} = \sigma$ for all $(i,j) \in (S_{k-1} : S_1 \cup S_2)$. (If $S_{k-1} = \{z\}$, then one can choose a similar construction to show that $c_{zv} = \sigma$ for all $v \in N \setminus \{s,t,z\}$.) In case of $k = 4$ and $|S_3| = 1$, consider the paths (s,v,w,t) and $(s,\tilde{v},\tilde{w},t)$. Since, $v = \tilde{v}$, it follows again that $c_{ij} = \sigma$ for all $(i,j) \in (S_{k-1} : S_1 \cup S_2)$, for some $\sigma \in \mathbb{R}$. In case of $k = 4$ and $|S_3| \geq 2$, let first $(v,w), (\tilde{v},\tilde{w}) \in (S_{k-1} : S_1)$. Considering the paths (s,u,v,w,t) and $(s,u,\tilde{v},\tilde{w},t)$ for some $u \in S_2$, yields $c_{vw} = c_{\tilde{v}\tilde{w}}$. Next, the paths (s,v,w,t) and $(s,v,\hat{w},t)$, where $\hat{w} \in S_2$ imply $c_{vw} = c_{v\hat{w}}$.

Thus, $c_{\tilde{v}\tilde{w}} = c_{v\hat{w}}$, and hence, $c_{ij} = \sigma$ for all $(i,j) \in (S_{k-1} : S_1 \cup S_2)$, for some $\sigma \in \mathbb{R}$.

It is now easy to see that $c_{ij} = -\sigma$ for all $(i,j) \in (S_p : S_q)$ with $0 \leq p < p+2 \leq q \leq k-1$, $c_{ij} = 0$ for all $(i,j) \in A(S_p)$, $p = 1,\ldots,k-1$, and $c_{ij} = 0$ for all $(i,j) \in (S_q : S_p)$, $1 \leq p < q \leq k-2$.

Finally, we prove that $c_{ij} = 0$ for all arcs $(i,j) \in (S_{k-1} : S_p)$, $3 \leq p \leq k-2$. In case of $k = 4$, there is nothing to show. So let $k \geq 5$. Consider any arc $(v_{k-1}, v_p) \in (S_{k-1} : S_p)$ with $3 \leq p \leq k-2$. Since $|S_{k-1}| \geq 2$, there is $\tilde{v}_{k-1} \in S_{k-1}$, $\tilde{v}_{k-1} \neq v_{k-1}$. (For walks, the restriction $|S_{k-1}| \geq 2$ is not necessary, as walks may visit a node more than one time.) The path $(s, v_{k-1}, v_p, v_{p+1}, \ldots, v_{k-2}, \tilde{v}_{k-1}, v_1, t)$ uses at most k arcs and is tight with respect to the inequality $b^T x \geq 0$, where $v_i \in S_i$ for $i = 1, p, p+1, \ldots, k-2$. We conclude that $c_{v_{k-1},v_p} = 0$. Therefore, $c^T x = c_0$ is simply $-\sigma b^T x = 0$. □

We now turn to the characterization of $\{-1,0,1\}$-facets by using the DP-approach. Let Π^0 be the set of all $0 \neq \pi \in [0,1]^{\mathcal{N}}$ such that $\pi_{s0} = \pi_{tk} = 0$, $\pi_{i,k-1} = 0$ for all internal nodes $i \in N$, and $\pi_{i,\ell} \geq \pi_{i,\ell+1}$ for all $\ell \in \{1,\ldots,k-2\}$, $i \in N \setminus \{s,t\}$. Denote by $\mathcal{C}(\Pi)$ ($\mathcal{C}(\Pi^0)$) the set of all $\tau \in \mathbb{R}^A$ that are induced by some $\pi \in \Pi$ ($\pi \in \Pi^0$).

Corollary 5.25.

$$P^{\leq k}_{s,t\text{-walk}}(D) = \left\{ x \in \mathbb{R}^A : \begin{array}{l} x \geq 0,\ x \text{ satisfies equations (5.8)}, \\ \tau^T x \geq 0 \text{ for all } \tau \in \mathcal{C}(\Pi^0) \end{array} \right\}. \tag{5.52}$$

Proof. By Theorem 5.8,

$$P^{\leq k}_{s,t\text{-walk}}(D) = \left\{ x \in \mathbb{R}^A : \begin{array}{l} x \geq 0,\ x \text{ satisfies equations (5.8)}, \\ \tau^T x \geq 1 \text{ for all } \tau \in \mathcal{C}(\Pi) \end{array} \right\}.$$

Next, due to the flow conservation constraints (5.8), two vectors $\pi, \tilde{\pi} \in \mathbb{R}^N$ project into equivalent inequalities if for every node $i \in N \setminus \{s,t\}$, $\pi_i - \tilde{\pi}_i \equiv \lambda_i$ for some $\lambda_i \in \mathbb{R}$. This implies that the values π_{s0}, π_{tk}, and $\pi_{i,k-1}$ for all internal nodes $i \in N$ can be fixed to 0. The right hand side of a projected inequality is then 0. This implies Corollary 5.25. □

Let $\pi \in \Pi^0 \cap \{0,1\}^{\mathcal{N}}$ and $\bar{\pi} \in \mathbb{R}^{\bar{\mathcal{N}}}$ its extension. Then, $N^{\bar{\pi},\min}_{0,0} = N^{\bar{\pi},\min}_{k,0} = \varnothing$. The following lemma is immediate.

Lemma 5.26. Let $\pi \in \Pi^0 \cap \{0,1\}^{\mathcal{N}}$ and $\bar{\pi} \in \mathbb{R}^{\bar{\mathcal{N}}}$ its extension. Then, π is τ^π-in-and-out-monotone if and only if $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing$ for $\ell = 1,\ldots,k-1$. □

Theorem 5.27. Let $k \geq 4$. τ^π-in-and-out-monotone 0/1-vectors $\pi \in \Pi^0$ and nontrivial facet defining t-rooted $0/\pm 1$-inequalities for $P^{\leq k}_{s,t\text{-walk}}(D)$ are in 1-1-correspondence. This means,

(a) each in-and-out-monotone 0/1-vector $\pi \in \Pi^0$ induces a facet defining t-rooted $0/\pm 1$-inequality for $P^{\leq k}_{s,t\text{-walk}}(D)$;

(b) each nontrivial facet defining t-rooted $0/\pm 1$-inequality for $P^{\leq k}_{s,t\text{-walk}}(D)$ is induced by an in-and-out-monotone 0/1-vector $\pi \in \Pi^0$;

(c) if two row-in-and-out-monotone 0/1-vectors π, $\tilde{\pi} \in \Pi^0$ induce the same facet defining inequality for $P^{\leq k}_{s,t\text{-walk}}(D)$, then $\pi = \tilde{\pi}$.

Proof. (a) Let $\pi \in \Pi^0$ be an in-and-out-monotone 0/1-vector and $\bar{\pi} \in \mathbb{R}^{\bar{\mathcal{N}}}$ its extension. By Lemma 5.27, $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing$ for $\ell = 1, \ldots, k-1$. Thus, the resulting inequality $\tau^T x \geq 0$, with $\tau := \tau^\pi$, is an inequality of the form (5.50), which is facet defining by Theorem 5.24 (iii).

(b) For any $\mu \in \mathbb{Z}^A$ and any $z \in \mathbb{Z}$, let $A^\mu_z := \{(i,j) \in A : \mu_{ij} = z\}$.

Let $\tau^T x \geq \tau_0$ be a nontrivial facet defining t-rooted $0/\pm 1$-inequality for $P^{\leq k}_{s,t\text{-walk}}(D)$. In particular, $\tau_{st} = 0$, $\tau^T x \geq 0$ is valid and not the sum of nonnegativity constraints. Hence, $\tau_0 = 0$ and $A^\tau_{-1} \neq \varnothing \neq A^\tau_1$.

Next, define $\pi \in \mathbb{R}^{\mathcal{N}}$ by $\pi_{i\ell} := u^{(\ell)}_i$, $[i,\ell] \in \mathcal{N}$, where the numbers $u^{(\ell)}_i$ are the values returned by the Moore-Bellman-Ford algorithm 4 for the length function $d : A \to \mathbb{R}$, $(i,j) \mapsto \tau_{ij}$. Clearly, $\pi \neq 0$ and $\pi_{s0} = 0$. Moreover, the validity of $\tau^T x \geq \tau_0$ means that $\tau(W) \geq 0$ for all $W \in \mathcal{W}^{\leq k}_{s,t}(D)$. This implies $\pi_{tk} \geq 0$; $\tau_{st} = 0$ implies even $\pi_{tk} = 0$. This in turn implies $\pi_{i\ell} \geq 0$ for all internal nodes $[i,\ell] \in \mathcal{N}$, since the inequality is t-rooted. With the same or similar arguments as in the proof to Theorem 5.19 (b) one now can close the gaps to show that π is in-and-out-monotone, $\pi \in \Pi^0 \cap \{0,1\}^{\mathcal{N}}$, and $\tau = \tau^\pi$.

(c) This statement can be shown along the lines of the proof to Theorem 5.19 (c). So we only give a sketch of the proof. Let π, $\rho \in \Pi$ be two in-and-out-monotone 0/1-vectors such that $\tau^\pi = \tau^\rho$ and $\sum_{(i,j)\in A} \tau^\pi_{ij} x_{ij} \geq 0$ is facet defining. We have to show that $\pi = \rho$. It immediately follows that $N^{\bar{\pi},\min}_{1,0} = N^{\bar{\rho},\min}_{1,0}$, $N^{\bar{\pi},\min}_{\ell,0} \neq \varnothing \neq N^{\bar{\rho},\min}_{\ell,0}$ for $\ell = 2, \ldots, k-1$, and $N^{\bar{\pi},\min}_{\ell,0} = N^{\bar{\rho},\min}_{\ell,0} = \varnothing$ for $\ell = 0, k$ (see Lemma 5.26). Now one shows again by induction on ℓ that $N^{\bar{\pi},\min}_{\ell,0} = N^{\bar{\rho},\min}_{\ell,0}$ for $\ell = 1, \ldots, k-1$. It follows that $\pi_i = \rho_i$ for all internal nodes $i \in N$, and hence, $\pi = \rho$. $\square$

5.6.3 Extensions

Dahl and Gouveia [23] describe a generalization of the jump inequalities called r-jump inequalities. Let

$$N = \bigcup_{p=0}^{k+r} S_p$$

be a partition of node set N, where $r \in \mathbb{N}$, $1 \le r \le n-k$, $S_0 = \{s\}$, and $S_{k+r} = \{t\}$. The *r-jump inequality* associated with this partition is defined as

$$\sum_{p=0}^{k+r-1} \sum_{q=p+1}^{k+r} \alpha_{pq} x((S_p : S_q)) \ge r, \tag{5.53}$$

where for $p < q$, $\alpha_{pq} := \min\{q-p-1, r\}$. Dahl, Foldnes, and Gouveia [23] have shown that the r-jump inequalities induce facets of $\mathrm{dmt}(P^{\le k}_{s,t\text{-walk}}(D))$ and consequently of $\mathrm{dmt}(P^{\le k}_{s,t\text{-path}}(D))$. [1]

Theorem 5.28 (Dahl, Foldnes, and Gouveia [22])**.** For $k \ge 4$, the r-jump inequality (5.53) is facet defining for $\mathrm{dmt}(P^{\le k}_{s,t\text{-path}}(D))$. □

Dahl and Gouveia [23] deal with the problem to strengthen r-jump inequalities for $P^{\le k}_{s,t\text{-path}}(D)$, since these inequalities are not facet defining for $P^{\le k}_{s,t\text{-path}}(D)$ in general. The resulting inequalities are called *lifted r-jump inequalities.* For $r = 1$, that is, in case of the ordinary jump inequalities, they obtain stronger inequalities by decreasing some coefficients on the backward arcs, which results into the class of lifted jump inequalities (3.30)

$$\sum_{p=0}^{k-1} \sum_{q=p+2}^{k+1} x((S_p : S_q)) - x((S_{k-1} \cup S_k : S_1 \cup S_2)) \ge 1.$$

They also discuss the case $r = 2$ in connection with $k = 4$. Following roughly the same idea of decreasing coefficients associated with backward arcs, they derive a class of lifted 2-jump inequalities

$$\begin{aligned} &\sum_{p=0}^{5} \sum_{q=p+1}^{6} \alpha_{pq} x((S_p : S_q)) - 2x((S_4 \cup S_5 : S_1 \cup S_2)) \\ &-x((S_4 \cup S_5 : S_3)) - x((S_3 : S_2 \cup S_1)) - x(A(S_3)) \ge 2 \end{aligned} \tag{5.54}$$

for all partitions

$$N = \bigcup_{p=0}^{6} S_p$$

[1] In difference to our definition, Dahl, Foldnes, and Gouveia [23] introduce walks as node-arc sequences, where nodes but not arcs may be repeated. However, the two k-hop constrained walk polytopes resulting from the different definitons of walks have the same dominant.

with $S_0 = \{s\}$ and $S_6 = \{t\}$. Experiments of Dahl and Gouveia with the convex-hull code PORTA [17] indicate that these inequalities define facets of $P^{\leq 4}_{s,t\text{-path}}(D)$. Moreover, they show that these inequalities are equivalent to the inequalities (5.16):

$$\sum_{i \in I} x_{si} + \sum_{j \in J} x_{jt} - \sum_{i \in I, j \in J} x_{ij} \geq 0 \qquad \text{for all } I, J \subseteq N \setminus \{s,t\}.$$

With a view to the completeness result for the 4-hop constrained walk polytope $P^{\leq 4}_{s,t\text{-walk}}(D)$ (see Fact 5), Dahl, Foldnes, and Gouveia [22] point to the interesting fact that to describe $\mathrm{dmt}(P^{\leq k}_{s,t\text{-path}}(D))$, one needs the whole class of r-jump inequalities, while to describe $P^{\leq 4}_{s,t\text{-walk}}(D)$, one only needs a suitable class of lifted 2-jump inequalities.

Using the DP-approach, its seems to be natural to consider the coefficient vector of an r-jump inequality as $\tau^{\pi,+}$-vector. The corresponding τ^{π}-vector gives a valid inequality for $P^{\leq k}_{s,t\text{-walk}}(D)$ that dominates the inequality associated with $\tau^{\pi,+}$ (with respect $P^{\leq k}_{s,t\text{-walk}}(D)$). So, this is a systematic way to strengthen r-jump inequalities. The π-vector associated with an r-jump inequality (5.53) has the following configuration:

$$\begin{aligned}
\pi_{s0} &:= 0, \\
\pi_{tk} &:= r, \\
\pi_{i\ell} &:= \begin{cases} 0 & \text{if } p - \ell \leq 0, \\ r & \text{if } p - \ell \geq r, \\ p - \ell & \text{otherwise,} \end{cases} \\
&= \min\{r, \max\{p-\ell, 0\}\} \qquad\qquad i \in S_p,\ p = 1, \dots, k+r-1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell = 1, \dots, k-1.
\end{aligned}$$

Note that $\pi' := \frac{1}{r}\pi \in \Pi$.

For example, for $k = 4$ and $r = 2$, the π-vector is of the following form:

$$\begin{array}{ccccc}
\boxed{0} & & & & \boxed{2} \\
& \boxed{0} & \boxed{0} & \boxed{0} & \\
& \boxed{1} & \boxed{0} & \boxed{0} & \\
& \boxed{2} & \boxed{1} & \boxed{0} & \\
& \boxed{2} & \boxed{2} & \boxed{1} & \\
& \boxed{2} & \boxed{2} & \boxed{2} &
\end{array}$$

Here, the $(p+1)$-th row represents all row vectors π_i, $i \in S_p$ for $p = 1, \dots 5$. The inequality $\sum_{a\in A} \tau_a^{\pi,+} x_a \geq 2$ is a 2-jump inequality; the inequality $\sum_{a\in A} \tau_a^{\pi} x_a \geq 2$ is a lifted 2-jump inequality among (5.54).

We could check now whether the inequality $\tau^T x \geq r$, where $\tau := \tau^\pi$, induces a facet of $P^{\leq k}_{s,t\text{-walk}}(D)$. However, we prefer to study the inequality that results from the unique equivalent vector to π' in Π^0 given by $\tilde{\pi}' := \frac{1}{\mu}\tilde{\pi}$, where $\tilde{\pi}$ is defined by

$$\begin{aligned}
&\tilde{\pi}_{s0} := 0,\\
&\tilde{\pi}_{tk} := 0,\\
&\tilde{\pi}_{i\ell} := \pi_{i\ell} - \pi_{i,k-1}, \qquad i \in S_p,\ p = 1, \dots, k+r-1, \ell = 1, \dots, k-1,
\end{aligned}$$

and $\mu := \max\limits_{[i.\ell]\in\mathcal{N}} \tilde{\pi}_{i\ell}$. This means that for $\ell \in \{1, \dots, k-1\}$,

$$\begin{aligned}
\tilde{\pi}_{i\ell} &= 0, && i \in S_1 \cup S_{k+r-1},\\
\tilde{\pi}_{i\ell} &= \min\{r, \max\{p-\ell, 0\}\}, && i \in S_p,\ p = 2, \dots, k-1,\\
\tilde{\pi}_{i\ell} &= k-1-\ell, && i \in \bigcup_{p=k}^{r+1} S_p,\\
\tilde{\pi}_{i\ell} &= \min\{p-1, k-1-\ell\}, && i \in S_{k+r-p},\ p = 2, \dots, z,
\end{aligned}$$

where $z := \min\{k-2, r\}$. Note that $\bigcup\limits_{p=k}^{r+1} S_p = \varnothing$ in case of $r < k-1$. An illustration of π and $\tilde{\pi}$ for $k = 6$ and $r = 7$ is given in Figure 5.4. Consider the rows of size $5 = k-1$ in (a) and (b) in Figure 5.4. While π consists of $12 = k+r-1$ different rows, $\tilde{\pi}$ only has $8 = k+z-2$ different rows.

Theorem 5.29. Let $k, r \in \mathbb{N}$, $4 \leq k < n$, $1 \leq r \leq n-k$, and $z := \min\{k-2, r\}$. Moreover, let

$$N = \bigcup_{p=0}^{k+z-1} T_p$$

be a partition of N such that $T_0 := \{s\}$ and $T_{k+z-1} := \{t\}$. Define $\tilde{\pi} \in \mathbb{R}^{\mathcal{N}}$ by

$$\begin{aligned}
&\tilde{\pi}_{s0} := 0,\\
&\tilde{\pi}_{tk} := 0,\\
&\tilde{\pi}_{i\ell} = \min\{r, \max\{p-\ell, 0\}\}, && i \in T_p,\ p = 1, \dots, k-1,\\
&\tilde{\pi}_{i\ell} = \min\{p, k-1-\ell\}, && i \in T_{k+z-1-p},\ p = 1, \dots, z-1,
\end{aligned}$$

where $\ell \in \{1, \dots, k-1\}$, and set $\tau := \tau^{\tilde{\pi}}$.

0						7
	0	0	0	0	0	
	1	0	0	0	0	
	2	1	0	0	0	
	3	2	1	0	0	
	4	3	2	1	0	
	5	4	3	2	1	
	6	5	4	3	2	
	7	6	5	4	3	
	7	7	6	5	4	
	7	7	7	6	5	
	7	7	7	7	6	
	7	7	7	7	7	

(a) π-vector associated with 7-jump inequality

0						0
	0	0	0	0	0	
	1	0	0	0	0	
	2	1	0	0	0	
	3	2	1	0	0	
	4	3	2	1	0	
	4	3	2	1	0	
	4	3	2	1	0	
	4	3	2	1	0	
	3	3	2	1	0	
	2	2	2	1	0	
	1	1	1	1	0	
	0	0	0	0	0	

(b) associated vector $\tilde{\pi}$

Figure 5.4: Vectors π and $\tilde{\pi}$ associated with an r-jump inequality for $r = 7$, $k = 6$.

(i) The inequality $\tau^T x \geq 0$ induces a facet of $P^{\leq k}_{s,t\text{-walk}}(D)$.

(ii) The inequality $\tau^T x \geq 0$ induces a facet of $P^{\leq k}_{s,t\text{-path}}(D)$ if $|T_p| \geq 2$ for $p = 1, 2, \ldots, k + z - 2$.

Proof. (i) In what follows, whenever a node i is indexed with p, that is, $i = i_p$, it means that $i \in T_p$. For any $\gamma \in \mathbb{R}$, $b(p,q) \equiv \gamma$ is defined to be $b_{ij} = \gamma$ for all $(i,j) \in (T_p : T_q)$. Moreover, all walks that will be considered in this proof are tight with respect to the inequality $\tau^T x \geq 0$.

When $r = 1$, $\tau^T x \geq 0$ is an ordinary jump inequality which has been shown to be facet defining in Theorem 5.24. So, let $r \geq 2$, which implies $z \geq 2$, and let $bx = b_0$ be an equation that is satisfied by all $x \in P^{\leq k}_{s,t\text{-walk}}(D)$ that satisfy $\tau^T x \geq 0$ at equality. W.l.o.g. we may assume that the equation $bx = b_0$ is in t-rooted form, that is, $b_{it} = 0$ for all $i \in N \setminus \{t\}$. Clearly, $b_{st} = 0$ implies $b_0 = 0$.

First, from walks of the form

$$(s, i_1, i_2, \ldots, i_p, t),$$
$$(s, i_1, i_2, \ldots, i_p, i_q, t)$$

for $p = 1, \ldots, k-2$, $q = p, p-1, \ldots, 1$, we derive that $b(p-1, p) \equiv 0$ and $b(p, q) \equiv 0$ for those p and q. Then, walks of the form

$$(s, i_1, i_2, \ldots, i_{k-2}, i_p, t)$$

imply $b(k-2, p) \equiv 0$ for $p = k-1, \ldots, k+z-2$.

Secondly, for any node $i \in T_2$, set $\gamma := b_{si}$. Consider all walks of the form

$$(s = i_0, i_1, \ldots, i_h, i_{h+2}, \ldots, i_k, t),$$
$$(s, i_2, \ldots, i_{k-2}, i_p, i_q, t),$$
$$(s, i_2, i_3, \ldots, i_{k-2}, i_{k+z-2}, i_1, t),$$

with $h \in \{0, 1, \ldots, k-3\}$, $p \in \{k-1, \ldots, k+z-3\}$, and $q \in \{p+1, \ldots, k+z-2\}$. It follows that $b(h, h+2) \equiv \gamma$ and $b(p, q) \equiv -\gamma$ for all those h, p, and q, as well as $b(k+z-2, 1) \equiv -\gamma$.

Thirdly, for $p = 1, \ldots, z$, $q = p, \ldots, z$, consider walks of the form

$$(s, i_{k+z-1-p}, \ldots, i_{k+z-2}, i_1, t)$$

as well as

$$\begin{array}{ll} (s, i_{k+z-1-p}, i_{k+z-1-q}, i_{k+z-q}, \ldots, i_{k+z-2}, i_1, t) & \text{if } z < k-2 \text{ or } q \leq z-1, \\ (s, i_{k+z-1-p}, i_{k+z-1-q}, i_{k+z-q}, \ldots, i_{k+z-2}, t) & \text{if } q = z = k-2. \end{array}$$

We conclude that $b(0, k+z-1-p) \equiv p\gamma$ and

$$b(k+z-1-p, k+z-1-q) \equiv \begin{cases} (q-p)\gamma & \text{if } z < k-2 \text{ or } q \leq z-1, \\ (q-p-1)\gamma & \text{if } q = z = k-2. \end{cases}$$

Fourthly, for any $p \in \{3, \ldots, k-2\}$, consider walks of the form

$$\{(s, i_p)\} \cup W,$$

with $W = (i_p, i_{p+1}, \ldots, i_{p+k-2}, t)$ if $p \leq z$, and $W = (i_p, i_{p+1}, \ldots, i_{k+z-2}, i_1, t)$ otherwise. It follows that $b(0, p) \equiv (p-1)\gamma$ for $p = 3, \ldots, z$, and $b(0, p) \equiv z\gamma$ for $p = z+1, \ldots, k-2$. Furthermore, let $h \in \{1, \ldots, k-3\}$, $p \in \{h+$

$2, \dots, k-2\}$, and $q \in \{1, \dots, z\}$. Set $W := (i_p, \dots, i_{p+k-3}, t)$ if $p \leq z+1$ and $W := (i_p, \dots, i_{k+z-2, i_1, t})$ otherwise. Then, from walks of the form

$$\begin{aligned}&\{(s, i_h), (i_h, i_p)\} \cup W,\\&(s, i_1, \dots, i_h, i_{k+z-1-q}, \dots, i_{k+z-2}, i_1, t)\end{aligned}$$

we derive that $b(h,p) \equiv \min\{z\gamma, (p-h-1)\gamma\}$ as well as $b(h, k+z-1-q) \equiv \min\{q\gamma, (k-2-h)\gamma\}$ for all those h, p, and q.

Fifthly, by considering further appropriate walks, we see that

$$b(k+z-1-p, q) \equiv \begin{cases} -1\gamma, & q = 1, \dots, p+1, \\ (q-2-p)\gamma, & q = p+2, \dots, z+1, \\ (z-p)\gamma, & q = z+2, \dots, k-2, \end{cases}$$

for $p = 1, \dots, z$.

This implies that $b = \gamma\tau$, and thus, the inequality $\tau^T x \geq 0$ induces a facet of $P^{\leq k}_{s,t\text{-walk}}(D)$.

(ii) If $|T_p| \geq 2$ for $p = 1, 2, \dots, k+z-2$, then the construction in the proof to statement (i) can be realized with paths. □

Chapter 6

Conclusion

In this thesis we studied several cardinality constrained combinatorial optimization problems from a polyhedral point of view. Given a combinatorial optimization problem $\Pi = (E, \mathcal{I}, w)$ and a cardinality sequence $c = (c_1, \ldots, c_m)$, that is, a finite sequence of increasing natural numbers, the general idea of this thesis was to investigate the facial structure of the associated polytope $P^c_{\mathcal{I}}(E) = \text{conv}\{\chi^I : I \in \mathcal{I}, |I| = c_p \text{ for some } p\}$. In particular, we were interested in those inequalities that are valid for $P^c_{\mathcal{I}}(E)$ but not for $P_{\mathcal{I}}(E) = \text{conv}\{\chi^I : I \in \mathcal{I}\}$.

The polyhedral investigation of cardinality constrained combinatorial optimization problems in the presented sense has its seeds in Maurras' PhD Thesis [61] from 1976 in which a complete linear description of the cardinality constrained matroid polytope has been presented. Since that time only Grötschel's paper on *Cardinality homogeneous set systems, cycles in matroids, and associated polytopes* [45] has contributed to this area of research, essentially by presenting a separation routine for the forbidden cardinality inequalities (1.2). This thesis calls the attention to cardinality constrained versions of other combinatorial structures than only matroids: paths, cycles, cuts, matchings, polymatroids.

Although it is daring to derive general statements from the polyhedral analysis of special polytopes, we would like to summarize the results in three key observations.

First, Chapter 2 and Chapter 3 indicate that facet defining inequalities for the ordinary polytope $P_{\mathcal{I}}(E)$ usually define facets of its cardinality constrained version $P^c_{\mathcal{I}}(E)$.

Second, the forbidden cardinality inequalities (1.2) together with the cardinality bounds (1.1) always cut off feasible solutions $I \in \mathcal{I}$ of forbidden cardinality, but inequalities (1.2) usually do not define facets of $P^c_{\mathcal{I}}(E)$. The incorporation of combinatorial structures of Π into inequalities (1.2), however, often yields strong valid inequalities with respect to the cardinality constrained version of Π. For cardinality constrained matroids, this attempt results in the class of rank induced forbidden cardinality inequalities (2.2), which first has been introduced by Maurras [61]. For directed cardinality constrained cycles (or paths), this results, for instance, in node induced forbidden cardinality inequalities (3.2) ((3.56)).

Third, the polyhedral investigation of the hop constrained path polytope in Chapter 5 confirms: even if we consider polynomial time solvable versions of a combinatorial optimization problem (the shortest path problem) and restrict ourselves to a simple cardinality constraint such as $c = (1, 2, \dots, k)$, we already may be confronted with widely not understood valid inequalities for $P^c_{\mathcal{I}}(E)$ that cut off feasible solutions of forbidden cardinality. Nevertheless, Chapter 5 shows that sometimes algorithmic and polyhedral structures can be brought to a fruitful interplay to derive new polyhedral insights: We considered the class of r-jump inequalities (5.53). By a theorem of Dahl, Foldnes, and Gouveia [22], they are facet defining for the dominant of the hop constrained path polytope. They are, however, not facet defining for the hop constrained path polytope itself. Using their characterization implied by the dynamic programming approach, we presented a canonical way to lift them into facet defining inequalities for the hop constrained path polytope.

The polyhedral analysis of cardinality constrained combinatorial optimization problems offers many possibilities for future research. It suggest itself to search complete linear descriptions of the cardinality constrained versions of those polytopes for which such ones are known in the ordinary case. We refer to the discussion on the cardinality constrained matching polytope or the polytope associated with the intersection of two matroids in Subsection 2.1.4 and Section 4.4.

Another question arises when analyzing the coefficients of inequalities related to cardinality constraints. Forbidden cardinality inequalities (1.2), cardinality-subgraph inequalities (3.59), and other inequalities have in common that their coefficients can be obtained by arithmetic operations such that at most two (adjacent) members of the cardinality sequence c are involved. It would be interesting to find inequalities, where more than two members are required in order to explain the coefficients, or to characterize under which conditions those inequalities are redundant.

Regarding to the hop constrained path polytope: The dynamic programming approach offers a chance to find new facet defining inequalities for this polytope. First computational experiments indicate that there are facet defining inequalities for the dominant different from r-jump inequalities (5.53), but with π-vectors closely related to those of the latter class of inequalities. Using the dynamic programming approach, we hope to be able to generalize the class of r-jump inequalities (5.53). Anyway, we believe that there should be other interesting combinatorial optimization problems where a combinatorial algorithm to its solution helps in some way to find strong valid inequalities for the associated polytope.

The way we derived important results of this thesis indicates that, at least in the best case, polyhedral structures of the cardinality constrained version of a combinatorial optimization problem can be traced back to polyhedral and combinatorial structures of the ordinary problem. For example, by Theorem 2.13, the separation problem of the rank induced forbidden cardinality inequalities (2.2) can be transformed to that of the rank inequalities (2.5) in linear time. Or, to give another example, the interpretation of the node induced forbidden cardinality inequalities (3.2) (or (3.56)) as rank induced forbidden cardinality inequalities is based on the observation that the collection of all directed cycles (paths) is contained in the independence system of a partition matroid, see Chapter 4.

However, trying to detach oneself from special combinatorial structures, one quickly reaches the limits of polyhedral (and complexity) theory associated with cardinality constrained combinatorial optimization problems. As the Chapters 1 and 4 approve, one cannot automatically draw conclusions about the cardinality constrained problem based on the ordinary combinatorial optimization problem. In our opinion, the most important reason for this obstruction lies in the very general definition of the notion of 'combinatorial optimization problem' itself. As long as such different structures like matchings, cycles, cuts, and so on, can be subsumed under one and the same combinatorial optimization problem ("if $|I| = 3$, I is a matching; if $|I| = 4$, I is a cycle; if $|I| = 5$, I is a cut, ..."), it will be hard to derive any interesting results. In other words, meaningful universal results require a reasonable restriction of the notion of 'combinatorial optimization problem'. Of course, it would be desirable if future research focuses on this difficulty.

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Notation

0	number, vector, or matrix with all entries equal to 0
$\varnothing$	empty set
2^E	power set of E
$\mathbb{1}$	vector with all entries equal to 1
$\mathbb{N}$	set of natural numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Q}_+$	set of nonnegative rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{R}^n$	set of real n-vectors
$\mathbb{R}^E$	set of real vectors indexed by E: $x \in \mathbb{R}^E \Leftrightarrow (x_e)_{e \in E}$ with $x_e \in \mathbb{R}$
$\mathbb{Z}$	set of integral numbers
$\mathbb{Z}_+$	set of nonnegative integral numbers
$\mathbb{Z}^E$	set of integral vectors indexed by E: $x \in \mathbb{R}^E \Leftrightarrow (x_e)_{e \in E}$ with $x_e \in \mathbb{R}$
$\lfloor \alpha \rfloor$	lower integer part of α
$\lceil \alpha \rceil$	upper integer part of α
$\delta(i)$	set of edges incident with node i
$\delta(S)$	cut induced by S
$\delta^{\text{in}}(i)$	set of arcs leaving node i
$\delta^{\text{out}}(i)$	set of arcs leaving node i
$\delta^{\text{out}}(S)$	directed cut induced by S
$\Pi = (E, \mathcal{I}, w)$	comb. opt. problem: E ground set, $\mathcal{I}$ set of feasible solutions, w weight function
$\Pi_c = (E, \mathcal{I}, w, c)$	card. constr. comb. opt. problem: E ground set, $\mathcal{I}$ set of feasible solutions, w weight function, c cardinality sequence
Π_k	k-COP

$\Pi_N = (E, \mathcal{I}, w, N)$	card. constr. comb. opt. problem: E ground set, $\mathcal{I}$ set of feasible solutions, w weight function, N finite subset of nonnegative integral numbers
τ^{π}	vector induced by π
$\tau^{\pi,+}$	positive part of τ^{π}
χ^F	incidence vector of F
$\text{aff}(S)$	affine hull of S
$A(W)$	set of arcs with both endnodes in W
$\text{bid}(B)$	union of arc set B and its reversal arcs
$c = (c_1, c_2, \ldots, c_m)$	cardinality sequence
$\text{cone}(S)$	conical hull of S
$\text{conv}(S)$, $\text{conv}\, S$	convex hull of S
CS	set of cardinality sequences $c = (c_1, \ldots, c_m)$ with $m \geq 2$, $2 \leq c_1 < \cdots < c_m \leq n$, and $c \neq (2, 3)$
$D - B$	subdigraph of D obtained by deleting arc set B
$\tilde{D}_n$	special directed graph with node set $\tilde{N}_n := \{0, 1, \ldots, n\}$: $\tilde{D}_n := D' - (\delta^{\text{in}}(0) \cup \delta^{\text{out}}(n) \cup \{0, n\})$, where D' is the complete directed graph on $\tilde{N}_n$
$\dim P$	dimension of polyhedron P
$\text{dmt}(P)$	dominant of polyhedron $P \subseteq \mathbb{R}^n$: $\text{dmt}(P) := P + \mathbb{R}^n_+$
$E(W)$	set of edges with both endnodes in W
$N(B)$	set of nodes covered by arcs (edges) in B
$\binom{n}{k}$	binomial coefficient: n over k
$\dot{P}$	internal nodes of path P
$P_C(D)$	directed cycle polytope defined on digraph D
$P_C^c(D)$	cardinality constrained directed cycle polytop defined on digraph D
$P_C^{(k)}(D)$	directed k-cycle polytope defined on digraph D
$P_C(G)$	undirected cycle polytope defined on graph G
$P_C^c(G)$	cardinality constrained undirected cycle polytop defined on graph G
$P_C^{(k)}(G)$	undirected k-cycle polytope defined on graph G
$P_C^{\leq k}(G)$	undirected hop constrained cycle polytope defined on graph G

$P^c_{\text{Cut}}(E)$	node cardinality constrained cut polytope
$P_{\text{DOC}}(A)$	directed odd cycle polytope
$P_{\text{EDOC}}(A)$	embedded directed odd cycle polytope
$P_{\text{IND}}(E)$	independent set polytope
$P^c_{\text{IND}}(E)$	cardinality constrained independent set polytope
$P_{\text{MATCH}}(E)$	matching polytope
$P^c_{\text{MATCH}}(E)$	cardinality constrained matching polytope
$P_{\text{M}}(E)$	matroid polytope
$P^c_{\text{M}}(E)$	cardinality constrained matroid polytope
$P_{s,t\text{-path}}(D)$	directed (s,t)-path polytop defined on digraph D
$P^c_{s,t\text{-path}}(D)$	cardinality constrained directed (s,t)-path polytop defined on digraph D
$P^{(k)}_{s,t\text{-path}}(D)$	directed (s,t)-k-path polytope defined on digraph D
$P^{\leq k}_{s,t\text{-path}}(D)$	directed k-hop constrained (s,t)-path polytope defined on digraph D
$P_{s,t\text{-path}}(G)$	undirected $[s,t]$-path polytop defined on graph G
$P^c_{s,t\text{-path}}(G)$	cardinality constrained undirected $[s,t]$-path polytop defined on graph G
$P^{(k)}_{s,t\text{-path}}(G)$	undirected $[s,t]$-k-path polytope defined on graph G
$P^{\leq k}_{s,t\text{-path}}(G)$	undirected k-hop constrained $[s,t]$-path polytope defined on graph G
$P^{\leq k}_{s,t\text{-walk}}(D)$	directed k-hop constrained (s,t)-walk polytope defined on digraph D
$\text{Proj}_x(P)$	Projection of P onto the x-space
$\text{rank}(A)$	rank of matrix A
$r(F)$, $r_{\mathcal{I}}(F)$	rank of F
$r^k(F)$	k-rank of $F \subseteq E$: $r^k(F) := k - r(E \setminus F)$
$(S : T)$	set of arcs (i,j) with $i \in S$, $j \in T$
$\text{supp}(v)$	support of vector $v \in R^E$: $\{e \in E : v_e \neq 0\}$
v^+	positive part of vector $v \in \mathbb{R}^n$: $v_i^+ := \max\{v_i, 0\}$ for $i = 1, \dots, n$
$v(F)$	sum of all entries v_j, $j \in F$

v^T	transpose vector
$\mathcal{C}(D)$	set of all directed simple cycles of digraph D
$\mathcal{C}^L_{s,t}(D,d)$	set of all L-length-bounded (s,t)-cuts of digraph D
$\mathcal{P}_{s,t}(D)$	set of all directed (s,t)-paths of digraph D
$\mathcal{P}^{\leq k}_{s,t}(D)$	set of all k-hop constrained directed (s,t)-paths of digraph D
$\mathcal{P}^L_{s,t}(D,d)$	set of all L-length-bounded (s,t)-paths of digraph D
$\mathcal{W}_{s,t}(D)$	set of all directed (s,t)-walks of digraph D
$\mathcal{W}^{\leq k}_{s,t}(D)$	set of all k-hop constrained directed (s,t)-walks of digraph D
$\mathcal{W}^L_{s,t}(D,d)$	set of all L-length-bounded (s,t)-walks of digraph D

Abbreviations

k-COP	k-combinatorial optimization problem
ATSP	asymmetric traveling salesman polytope
CCCOP	cardinality constrained combinatorial optimization problem
COP	combinatorial optimization problem
CYCLE	shortest cycle problem
DOCP	directed odd cycle problem
DP	dynamic program, dynamic programming
EDOCP	embedded directed odd cycle problem
FC	forbidden cardinality
HCSPP	hop constrained shortest path problem
IND	maximum weight independent set problem
IP	integer program, integer programming
LP	linear program, linear programming
STSP	symmetric traveling salesman polytope
TSP	traveling salesman problem

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