Combinatorics of Tropical Linear Programming

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Contents

1 Introduction

A *tropical linear inequality* is an inequality of the form

$$\min \{ a_i + x_i \mid i \in I \} \leq \min \{ a_\ell + x_\ell \mid \ell \in [d] \setminus I \} ,$$

where $[d] = \{1, 2, \ldots, d\}$ for a natural number d, $I \subseteq [d]$ and $a_1, \ldots, a_d$ are elements of the *tropical numbers* $\mathbb{T}_{\min} = \mathbb{R} \cup \{\infty\}$. We study systems of tropical linear inequalities and in particular, the algorithmic question of finding a *feasible* point $(x_1, \ldots, x_d)^\top \in \mathbb{T}_{\min}^d$ which fulfills all the inequalities in a given system. This is the *tropical feasibility problem*. Methods to solve this are referred as *tropical linear programming*. It is a tropical analogue of classical linear programming.

All advances for tropical linear programming also yield new insights in classical linear programming, mean payoff games and scheduling problems through the results in [MSS04, AGG12, ABGJ15, Sch09]. It provides a promising approach for the deep open questions to decide if mean payoff games are polynomially solvable and if linear programming can be solved in strongly-polynomial time.

The *covector graph* of a point $x \in \mathbb{T}_{\min}^d$ is a bipartite graph encoding the entry where the minimum in an inequality is attained. By additionally marking on which side of the inequality the minimum is attained, we can determine if a point is feasible just from its covector graph. The set of covector graphs forms the *covector decomposition*. We introduce *signed tropical matroids* as a generalization of the collection of covector graphs with an additional sign information. This allows us to formulate a generalization of the tropical feasibility problem in terms of signed tropical matroids.

By polyhedral means, we deduce properties of covector graphs and extend the covector decomposition to points with infinite coordinates. Furthermore, they enable us to characterize the matching structure of the covector graphs. Along the way, we resolve a conjecture of Develin and Yu (2007).

We design an algorithm to solve a generalization of the feasibility problem for signed tropical matroids. The algorithm resembles the classical simplex method, it only operates on sets of indices called *bases*. Moreover, executed on tropical linear inequality systems, which is the *realizable case*, it is even at least pseudo-polynomial. By its combinatorial nature, it provides new insights into the structure of the tropical feasibility problem.

Chapter 2 is concerned with covector graphs in the realizable case which leads to several new results concerning *covector decompositions*. This is complemented by the introduction of the field of *Puiseux fractions* in Chapter 3 which is helpful for the connection between classical linear programming and tropical linear programming. With this as helpful intuition, we develop the theory for abstract tropical linear programming in Chapter 4. The three chapters are essentially self-contained.

The tropical feasibility problem originates from considerations of two-sided equations in max-plus linear algebra. This field arising from optimization and economics goes back several decades; for an overview see, e.g., Litvinov, Maslov and Shpiz [LMS01], Cohen, Gaubert and Quadrat [CGQ04] or Butkovič [But10] and their references.

The interest in the feasibility problem was boosted as it was shown to be equivalent to finding the winning states of mean payoff games [AGG12]. These are two player games with perfect information on a directed graph with weights on the arcs. The problem of deciding if a given position is winning is known to be in NP $\cap$ co-NP, and even more in UP $\cap$ co-UP, see [Jur98, ZP96, EM79, MSS04]. This decision problem is conjectured to be polynomial-time solvable [GKK88] but no such algorithm has been found. *Parity games* [Jur98] form a subclass of mean payoff games. They are polynomial time equivalent to the model-checking problem for the modal μ-calculus [EJ91]. Furthermore, they proved useful to construct hard instances for several algorithms see, e.g., [Fri11]. Recently, it was shown in [CJK$^+$] that deciding if a position is winning in a parity game can be solved in quasipolynomial time. However, it still remains open if this can be solved in polynomial time.

Geometric combinatorics entered the scene under the name of “tropical convexity” through the work by Develin and Sturmfels in their landmark paper [DS04]. In their work on configurations of points $v^{(1)}, \dots, v^{(n)} \in \mathbb{R}^d$ and their tropical span

$$\left\{ \min(\lambda_1 + v^{(1)}, \dots, \lambda_n + v^{(n)}) \;\middle|\; \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\} ,$$

where the minimum is taken componentwise, they revealed the polyhedral structure and the translation to subdivisions of the product of two simplices $\Delta_{n-1} \times \Delta_{d-1}$. This line of research has been continued in [Jos05], [AD09], [FR15], among other references.

Furthermore, tropical geometry occurs as a discrete limit or piecewise-linear image of algebraic geometry. The dequantization process [Mas86, Vir01] and the logarithmic limit [GKZ94, Mik00, IM12] preserve several properties and allow a combinatorial study of algebraic varieties. Considering polyhedra as *semi-algebraic sets* we obtain *tropical polyhedra* also through this dequantization process. In this spirit, tropical polyhedra were examined as images of classical polyhedra under a *valuation map* in [DY07]. Moreover, this leads to *tropical spectrahedra* whose study was originated in [AGS16].

Our approach is based on the connection with classical polyhedra but uses mainly the perspective of geometric combinatorics enriched with tools from combinatorial optimization.

Since infinite coordinates appear naturally in tropical linear programming we extend the work [DS04] to point configurations in the *tropical projective space* which is a compactification of the *tropical torus*. In particular, we give a new proof of the *Structure Theorem for Tropical Convexity* [DS04, FR15] which relates the *covector decomposition* of tropical point configurations and subdivisions of subpolytopes of products of simplices $\Delta_{n-1} \times \Delta_{d-1}$. This connection is established through *covector graphs* which encode the local structure of a tropical point configuration. Our characterization by *minimal matchings* and *non-negative cycles* is an important tool in several proofs later on. A similar result appears independently in [JK16] in connection with face monoids. Our unifying approach for finite and infinite coordinates is based on *weighted digraph polyhedra* which are associated to weighted directed graphs. These appear as *shortest path polyhedra* in combinatorial optimization and are related to *flow polytopes*. We show that their recession cones are in bijection with the *order polytope* given by the reachability order in a directed graph. This encodes the polyhedral combinatorics of the covector decomposition in the boundary of the tropical projective space. By these means, we derive covector decompositions for the tropical projective space. Moreover, we apply the results for covector graphs to *tropical halfspace arrangements*. Thereby, we resolve a conjecture by Develin and Yu [DY07, Conjecture 2.11]. This also gives rise to new tools to study tropical linear programs because the *feasible region* is described by an intersection of tropical halfspaces.

Over the years, several variants of tropical linear programs were considered, see, e.g., [CG79, Zim74, BA09] and Butkovič [But10] for an introduction. Only recently in [ABGJ15] and [ABGJ14a], Allamigeon, Benchimol, Gaubert and Joswig translated the classical simplex algorithm by Dantzig [Dan63] to the tropical setting and showed that combinatorial pivoting rules have the same complexity in the classical setting as in the tropical analogue. The tropicalization of the classical simplex method is obtained by applying a *valuation map* to all the intermediate values of a linear program over fields of Puiseux or, more generally, Hahn series. It is well known and not difficult to see that the standard concepts from linear programming (LP), e.g., the Farkas Lemma and LP duality, carry over to an arbitrary ordered field; see, e.g., [CK70, Section II] or [Jer73, §2.1]. Traces of this can already be found in Dantzig's monograph [Dan63, Chapter 22]. In particular, the correctness of the simplex method and usual convex hull algorithms is valid over any ordered field. A classical construction, due to Hilbert, turns a field of rational functions, e.g., with real coefficients, into an ordered field; see [vdW93, §147]. In [Jer73] Jeroslow discussed these fields in the context of linear programming in order to provide a rigorous foundation of the so-called "big M method".

In contrast to classical linear programming, there is no polynomial time algorithm known for tropical linear programming. Even if no pivoting rule has yet been found, for which the classical simplex method solves classical linear programming in polynomial time, there are several algorithms [Chu15, Kar84, Kha79] which succeed in weakly polynomial time. All these algorithms rely on complicated numerical operations on the input. Therefore, the tempting approach to "tropicalize" one of these weakly polynomial algorithms turned out to be a dead end.

Real Puiseux series $\mathbb{R}\{\!\{t\}\!\}$ are infinite series with arbitrary real coefficients and real exponents. Therefore, exact computations are not possible. Hence, we restrict to a subfield $\mathbb{Q}\{t\}$, which is the field of rational functions with rational coefficients and rational exponents, called *Puiseux fractions* as it consists of the fractions of polynomials with rational exponents. This field allows for exact computations and is implemented in `polymake` [GJ00]. We compute algorithmically challenging examples from [GS79] and [ABGJ14b], thereby demonstrating that the parameters in these examples are indeed only required to be "sufficiently small or big". Our experiments show that the actual runtime is quite bad due to the technical overhead. However, this is not a flaw of our implementation but just comes with the construction. As theoretical applications, we prove that tropical convex hulls can be calculated through this approach, although, [AGG10] describes a more appropriate algorithm for this. Furthermore, we give a proof that polyhedra over Puiseux fractions have the same combinatorics as real polyhedra for which the parameter t is evaluated by a sufficiently big real number. The latter connection gives an intuition for the tropicalization of the simplex method through linear programming over Puiseux series.

Taking this connection as a starting point we develop *abstract tropical linear programming* as a generalization of tropical linear programming inspired by oriented matroid programming, cf. [Bla77, Fuk82, Tod85, Ter85], which represents the abstract analogue for classical linear programming. A generalization of the covector decomposition for finite tropical point configurations was developed under the name *tropical oriented matroid* in [AD09]. It was further studied in [OY11] and finally shown to be equivalent to not necessarily regular subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$ as well as tropical pseudohyperplane arrangements in [Hor16]. However, an additional sign information is needed to encode a halfspace structure. Motivated by the equivalence of the covector decomposition with regular subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$, we define signed tropical matroids in terms of not necessarily regular subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$. An abstract covector graph is the bipartite graph G associated to the cell $\operatorname{conv}\{(e_j, e_i) \mid (i,j) \text{ edge of } G\}$ in a subdivision of $\Delta_{n-1} \times \Delta_{d-1}$. By assigning a sign to each edge of a covector graph, we are able to give an abstract description of the combinatorics of a tropical linear inequality system.

To introduce further algorithmic ideas, we recall the important problems related to tropical linear programming and embed AND-OR-networks [MSS04] as well as mean payoff games [EM79, ZP96, AGG12] into our setting. Additionally, the presented variant of the simplex variant is adapted to the feasibility problem, and it is similar to our feasibility algorithm for signed tropical matroids. [Ben14, Proposition 3.22 & Proposition 5.1] yield that a run of the tropicalization of the simplex method [ABGJ15], for appropriate pivoting rules, only depends on tropical signs of minors of the coefficient matrix and not on the actual size of the coefficients. This suggests to apply this method also to determine the feasibility of a signed tropical matroid. However, for a non-realizable signed tropical matroid, it is not clear if the tropicalized simplex method terminates and produces a reasonable result. This motivates us to walk along a more combinatorial path by employing the matching structure of the covector graphs.

Additionally, the connection of covector graphs with products of simplices enables us to use polyhedral techniques [DLRS10] to examine tropical linear inequality systems. Thus, we can resolve the technical obstacles with degenericity and infinite coordinates by *extension* and *refinement* of subdivisions. We examine the structure of a signed tropical matroid with combinatorial means based on matchings and alternating paths. In particular, the generalization of Cramer determinants [AGG14, ACG+90, RGST05] plays an important role. This leads to a simple algorithm which uses only modifications of index sets, analogously to the simplex method. Unlike the simplex method, where either the entering or the leaving index has to be determined by reduced costs, both can be read off directly from the current covector graph in the iteration. In the abstract setting, we assume a signed tropical matroid to be given by an oracle which yields a covector graph with a prescribed degree sequence. When we apply the algorithm to the feasibility problem for tropical linear inequality systems the oracle is replaced by minimal matching computations. It turns out that, even if the algorithm works in the more general setting, it has an appealing complexity in the realizable case. It is pseudopolynomial but depends only on the structure of a corresponding triangulation. Therefore, a further examination of the minimal integer vectors in the cones of the secondary fan of $\Delta_{n-1} \times \Delta_{d-1}$ will provide new results for the complexity of this problem. As an additional upshot, the algorithm results in certificates for feasibility and infeasibility of a signed tropical matroid through geometric arguments, independently of the equivalence with mean payoff games.

2 Weighted digraphs and tropical cones

This chapter is taken from the paper "Weighted digraphs and tropical cones" [JL16] by Michael Joswig and Georg Loho which is published in "Linear Algebra and its Applications", volume 501, pages 304 – 343. The published version is available at
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2.1 Introduction

The tradition of max-plus linear algebra in optimization and related areas goes back several decades; for an overview, e.g., see Litvinov, Maslov and Shpiz [LMS01], Cohen, Gaubert and Quadrat [CGQ04] or Butkovič [But10] and their references. Develin and Sturmfels connected max-plus linear algebra under the name of *tropical convexity* to geometric combinatorics in their landmark paper [DS04]; see also [MS15, Chapter 5]. This line of research has been continued in [Jos05], [DY07], [AD09], [FR15] and other references. The interest in a more geometric perspective comes from several directions. One source is tropical geometry, which, e.g., relates tropical convexity to the combinatorics of the Grassmannians [SS04], [HJS14], [FR15]. A second independent source is the study of tropical analogues of linear programming [ABGJ15] which, e.g., is motivated by its connections to deep open problems in computational complexity [AGG12].

Since the paper [DS04] by Develin and Sturmfels more than ten years ago some of the strands of research still seem to diverge. The main purpose of this chapter is to help bridging this gap. Our point of departure is [DS04, Theorem 1], which establishes a fundamental correspondence between the configurations of n points in the *tropical projective torus* $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ and the regular subdivisions of the product of simplices $\Delta_{d-1} \times \Delta_{n-1}$. We suggest to call this result the *Structure Theorem of Tropical Convexity*. It was recently extended by Fink and Rincón [FR15, Corollary 4.2] to include regular subdivisions of subpolytopes of products of simplices. For the tropical point configurations this amounts to taking ∞ as a coordinate into account. Our first contribution is a new proof of that result (Corollary 2.34). Moreover, in [DS04] and [FR15] only tropical convex hulls of points (or dually, arrangements of tropical hyperplanes) are considered, whereas here we also bring exterior descriptions in terms of tropical half-spaces [Jos05], [GK11] into the picture. Arrangements of max-tropical halfspaces correspond to the 'two-sided max-linear systems' in the max-plus literature [But10, §7]. As an additional benefit our methods allow us to resolve a previously open question raised by Develin and Yu, who conjectured that a finitely generated tropical convex hull is pure and full-dimensional if and only if it has a half-space description in which the apices of these tropical half-spaces are in general position [DY07, Conjecture 2.11]. We show that, indeed, general position implies pureness and full-dimensionality (Theorem 2.46), and we give a counter-example to the converse (Example 2.47). The approach through tropical convex hulls on the one hand and the approach through systems of tropical inequalities on the other hand gives rise to two interesting cell decompositions of the *tropical projective spaces* (Theorem 2.51 and Corollary 2.54). This ties in with compactifications of tropical varieties; see Mikhalkin [Mik06, §3.4].

As in [DS04] it turns out to be convenient to examine the regular subdivisions of products of simplices and their subpolytopes in terms of a dual ordinary convex polyhedron, which we call the *envelope* of the tropical point configuration. In fact, it is even fruitful to see this envelope as a special case of a more general class of ordinary polyhedra which are associated with directed graphs with weighted arcs. These *weighted digraph polyhedra* are defined by linear inequalities of the form

$$x_i - x_j \ \leq \ w_{ij} \ ,$$

where w_{ij} is the weight on the arc from the node i to the node j. Their feasible points are well known as *potentials* in the optimization literature, and the weighted digraph polyhedra are sometimes called 'shortest path polyhedra'; e.g., see [Sch03, §8.2] for an overview. Recently potentials and weighted digraph polyhedra starred prominently in the work of Khachiyan and al. [KBB$^+$08] on hardness results in the context of vertex enumeration. Specializing all arc weights to zero yields the *braid cones* of Postnikov, Reiner and Williams [PRW08], which are closely related to *order polytopes* of partially ordered sets. By applying a celebrated result of Stanley [Sta86, Theorem 1.2] we obtain a combinatorial characterization of the entire face lattice of any digraph cone (Theorem 2.11).

This chapter is organized as follows. Section 2.2 starts out with investigating a general weighted digraph polyhedron $Q(W)$ associated with a $k{\times}k$-matrix W, which we read as a directed graph $\Gamma = \Gamma(W)$ equipped with a weight function. The braid cones, with all finite entries equal to zero, naturally come in as their recession cones. We show that the face lattice of a braid cone is isomorphic to a face figure of the order polytope associated with the acyclic reduction of Γ and, via Stanley's result [Sta86, Theorem 1.2], to a partially ordered set of partitions of the node set of Γ ordered by refinement. It is a key observation that the faces of a weighted digraph polyhedron are again weighted digraph polyhedra. The envelope of an arbitrary $d{\times}n$-matrix V is the weighted digraph polyhedron for a specific $(d{+}n){\times}(d{+}n)$-matrix constructed from V.

In Section 2.3 we direct our attention to tropical convexity, which is essentially the same as linear algebra over the tropical semi-ring $\mathbb{T}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$. Clearly, it is just a matter of taste if one prefers min or max as the tropical addition. More importantly though, it turns out to be occasionally convenient to use both these operations together to be able to phrase some of our results in a natural way. So we usually consider tropical linear spans of vectors in the min-tropical setting and intersections of tropical half-spaces in the max-setting. With any matrix $V \in \mathbb{R}^{d\times n}$ Develin and Sturmfels associate a polyhedral decomposition of the tropical projective torus $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ [DS04, §3]; here $\mathbf{1}$ denotes the all ones vector. We follow Fink and Rincón [FR15] in calling this polyhedral complex the *covector decomposition*. The cells of the covector decomposition are naturally indexed by subgraphs of the digraph $\Gamma(W)$, where W is the $(d{+}n){\times}(d{+}n)$-matrix mentioned above. Moreover, these cells arise as orthogonal projections of the faces of the envelope of V. If V is finite then (in the tropical projective torus) the union of the bounded cells of the type decomposition is exactly the tropical convex hull of the columns of V. Further, the covector decomposition is dual to a regular subdivision of the product of simplices $\Delta_{d-1} \times \Delta_{n-1}$. If V has infinite coordinates, it still makes sense to talk about the *tropical cone* generated by the columns, but $\Delta_{d-1} \times \Delta_{n-1}$ gets replaced by the subpolytope corresponding to the finite entries of V; see [FR15]. This leads to studying point configurations in the *tropical projective space*; see Mikhalkin [Mik06, §3.4] and Section 2.3.5 below. Another way of interpreting the matrix V, with coefficients in $\mathbb{T}_{\min}$, is as an arrangement of max-tropical hyperplanes. The covector decomposition arises as the common refinement of the affine fans corresponding to these tropical hyperplanes. Equipping such a tropical hyperplane arrangement with a certain graph encoding the feasibility of a cell gives rise to a max-tropical cone described as the intersection of finitely many *tropical half-spaces*; see [Jos05] and [GK11]. This is how tropical cones naturally arise in the context of tropical linear programming. In [ABGJ15] a tropical version of the simplex method is described. The pivoting operation proposed there can be explained in terms of operations on the graph $\Gamma(W)$, the crucial object being the *tangent digraph* from [ABGJ15, §3.1], which carries the same information as the 'tangent hypergraphs' of Allamigeon, Gaubert and Goubault [AGG13]. We show how the tangent digraph encodes the local combinatorics of the covector decomposition induced by V in the neighborhood of a given point. Finally, we recall the *signed cell decompositions* from [ABGJ15, §3.2] which form the tropical analogues of the polyhedral complexes generated from a system of ordinary affine hyperplanes.

The upshot is that all the remarkable combinatorial properties of tropical convexity can be inferred from the weighted digraph polyhedra. It is worth noting that the facet normals of their defining inequalities are precisely the roots of a type A root system. Lam and Postnikov [LP07] introduced 'alcoved polytopes' which are exactly the weighted digraph polyhedra which are bounded (modulo projecting out the subspace $\mathbb{R}\mathbf{1}$). These are also the *polytropes* in [JK10]. Section 2.3.4 gives more details.

2.2 Weighted digraph polyhedra

2.2.1 The construction

Let $W = (w_{ij})$ be an arbitrary $k \times k$-matrix with coefficients in $\mathbb{T}_{\min} = \mathbb{R} \cup \{\infty\}$. This yields a digraph $\Gamma(W)$ with node set $[k]$ and an arc from i to j whenever the coefficient w_{ij} is finite. Notice that $\Gamma(W)$ may have loops, corresponding to finite entries on the diagonal. Also (i,j) and (j,i) both may be arcs, but there are no other multiple edges. The matrix W induces a map, γ, which assigns to each arc (i,j) of $\Gamma(W)$ its *weight* w_{ij}. We call the pair $(\Gamma(W), \gamma(W))$ the *weighted digraph* associated with W. Conversely, each finite directed graph Γ endowed with a weight function γ on its arcs has a *weighted adjacency matrix* $W(\Gamma, \gamma)$. Often we will not distinguish between the matrix W and the digraph Γ equipped with the weight function γ.

Our key player is the *weighted digraph polyhedron* $Q(W)$ in $\mathbb{R}^k$ which is defined by the linear inequalities

$$x_i - x_j \ \leq \ w_{ij} \qquad \text{for each arc } (i,j) \text{ in } \Gamma(W) \ . \tag{2.1}$$

For a directed graph Γ with a weight function γ we also write $Q(\Gamma,\gamma)$ instead of $Q(W(\Gamma,\gamma))$. Observe that $-Q(W) = Q(W^\top)$. A feasible point in $Q(W)$ is sometimes called a *potential* on the digraph Γ; e.g., see [Sch03, §8.2]. The following result of Gallai [Gal58] clarifies the feasibility of the constraints; see also [Sch03, Theorem 8.2] and [But10, §2.1].

Lemma 2.1. *The weighted digraph polyhedron $Q(W)$ is empty if and only if the weighted digraph (Γ,γ) has a negative cycle.*

If the weighted digraph (Γ,γ) does not have any negative cycle there is a directed shortest path between any two nodes. Let $W^* = (w^*_{ij})$ be the $k \times k$-matrix which records the weights of these shortest paths. Following Butkovič [But10, §1.6.2] we call the shortest path matrix W^* the *Kleene star* of W. The tropical addition $\oplus = \min$ extends to vectors and matrices coefficientwise. Moreover, the tropical addition and the tropical multiplication give rise to a tropical matrix multiplication, which we also write as $\odot$. Matrix powers of W with respect to $\odot$ are written as $W^{\odot \ell}$ where $W^{\odot 0} = I$ is the min-tropical unit matrix, which has zero coefficients on the diagonal and ∞ otherwise, and $W^{\odot(\ell+1)} = W^{\odot \ell} \odot W$. With this notation we have the formula

$$W^* \ = \ I \oplus W \oplus W^{\odot 2} \oplus \cdots \oplus W^{\odot k} \ ,$$

whose direct evaluation amounts to applying the Bellman-Ford method for computing all shortest paths [Sch03, §8.3]. The next lemma points out a special property of the inequality description given by W^*; see [Sch03, Theorem 8.3].

Lemma 2.2. *Each of the defining inequalities from (2.1) for the weighted digraph polyhedron of the matrix W^* is tight.*

Proof. Let $x_i - x_j \leq w^*_{ij}$ be an inequality defining $Q(W^*)$. The vector of weights w^*_{pj} for $p \in [k]$, i.e., the jth column of W^*, satisfies each inequality by the shortest path property $w^*_{pj} \leq w^*_{pq} + w^*_{qj}$. Equivalently we have $w^*_{pj} - w^*_{qj} \leq w^*_{pq}$. Due to $w^*_{jj} = 0$, this vector satisfies the equality $x_i - x_j = w^*_{ij}$. □

Throughout the following we assume that (Γ,γ) does not have a negative cycle. In view of Lemma 2.1 this is equivalent to the feasibility of $Q(W)$, and the Kleene star W^* is defined. Further, let $E(W)$ be the *equality graph* of W, which is the undirected graph on the node set $[k]$ and which has an edge between i and j if $Q(W)$ satisfies $x_i - x_j = w^*_{ij} < \infty$ or $x_j - x_i = w^*_{ji} < \infty$.

Lemma 2.3.

(a) We have $Q(W^*) = Q(W)$ and $E(W^*) = E(W)$.

(b) Two distinct nodes i and j are contained in a directed cycle of weight zero in $\Gamma(W)$ if and only if $\{i,j\}$ is contained in the equality graph $E(W)$ if and only if $w^*_{ij} = -w^*_{ji} < \infty$.

Proof. The proof for both statements is essentially the same. Let $\pi = (i_0, i_1, \ldots, i_m)$ be a directed path in Γ. This corresponds to the inequalities $x_{i_{\ell-1}} \leq x_{i_\ell} + w_{i_{\ell-1}i_\ell}$ for $\ell \in \{1, \ldots, m\}$. By transitivity we obtain

$$x_{i_0} \;\leq\; x_{i_m} + \sum_{\ell=1}^{m} w_{i_{\ell-1}i_\ell}$$

as a valid inequality for $Q(W)$. Restricting to shortest paths shows $Q(W^*) \supseteq Q(W)$. The other inclusion is obvious. Notice that this readily implies that the equality graphs $E(W)$ and $E(W^*)$ are the same.

Now suppose that π is a directed cycle of weight zero. In particular, $i_0 = i_m$ is the same node and because of the presumed feasibility, the cycle contains the shortest path for any pair of its nodes. The above yields for each $\mu \in \{0, \ldots, m\}$ the inequalities

$$x_{i_0} \;\leq\; x_{i_\mu} + \sum_{\ell=1}^{\mu} w_{i_{\ell-1}i_\ell} \;=\; x_{i_\mu} + w^*_{i_0,i_\mu} \text{ and } x_{i_\mu} \;\leq\; x_{i_m} + \sum_{\ell=\mu+1}^{m} w_{i_{\ell-1}i_\ell} \;=\; x_{i_0} + w^*_{i_\mu,i_0} \;.$$

With $w^*_{i_0,i_\mu} + w^*_{i_\mu,i_0} = 0$ we obtain

$$x_{i_0} - x_{i_\mu} \;\leq\; w^*_{i_0,i_\mu} \;=\; -w^*_{i_\mu,i_0} \;\leq\; x_{i_0} - x_{i_\mu}$$

and hence the equality $x_{i_0} - x_{i_\mu} = w^*_{i_0,i_\mu}$. This shows that the edge $\{i_0, i_\mu\}$ is contained in the equality graph $E(W^*) = E(W)$.

Finally, let $\{i, j\}$ be an edge in $E(W) = E(W^*)$. Then $x_i - x_j = w^*_{ij} < \infty$, and it follows that also $x_j - x_i = -w^*_{ij}$ is finite. Since the inequality $x_j - x_i \leq w^*_{ji}$ is tight by Lemma 2.2 we obtain $w^*_{ji} = -w^*_{ij}$. Therefore, there is a directed path from j to i in $\Gamma(W)$, and hence (i, j, i) is a directed cycle of weight zero in $\Gamma(W^*)$. From this we infer our claim. $\square$

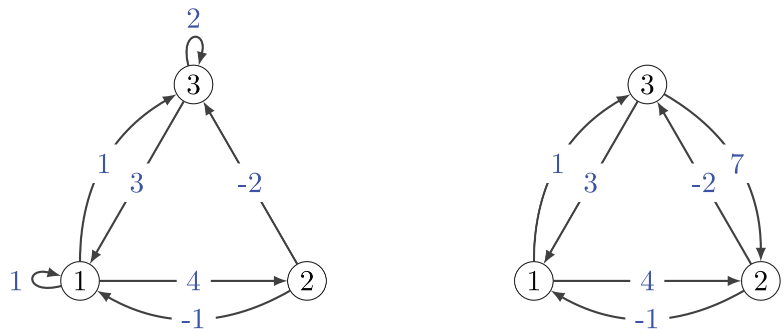


Figure 2.1: The directed graphs defined by the matrices W and W^* from Example 2.4

Example 2.4. The 3×3 matrix

$$W \;=\; \begin{pmatrix} 1 & 4 & 1 \\ -1 & 0 & -2 \\ 3 & \infty & 2 \end{pmatrix} \tag{2.2}$$

defines a directed graph without any cycles of weight zero. Its Kleene star is the matrix

$$W^* = \begin{pmatrix} 0 & 4 & 1 \\ -1 & 0 & -2 \\ 3 & 7 & 0 \end{pmatrix} .$$

The graphs of W and W^* are displayed in Figure 2.1, while Figure 2.2 shows the corresponding weighted digraph polyhedron. Our convention for drawing digraphs is to omit loops of weight zero and arbitrary arcs of infinite weight. Since each weighted digraph polyhedron contains the one-dimensional linear subspace $\mathbb{R}\mathbf{1}$ in its lineality space, throughout we draw pictures in the quotient $\mathbb{R}^d/\mathbb{R}\mathbf{1}$, which is called the *tropical projective* $(d-1)$*-torus* in [MS15, §5.2]. More precisely, for a feasible point $x + \mathbb{R}\mathbf{1}$ in the quotient we draw the unique representative with $x_1 = 0$. This is the same as drawing the intersection of $Q(W)$ with the hyperplane $x_1 = 0$. As the polyhedron $Q(W)$ corresponding to the matrix (2.2) is not contained in any hyperplane its equality graph $E(W)$ is the undirected graph with three isolated nodes.

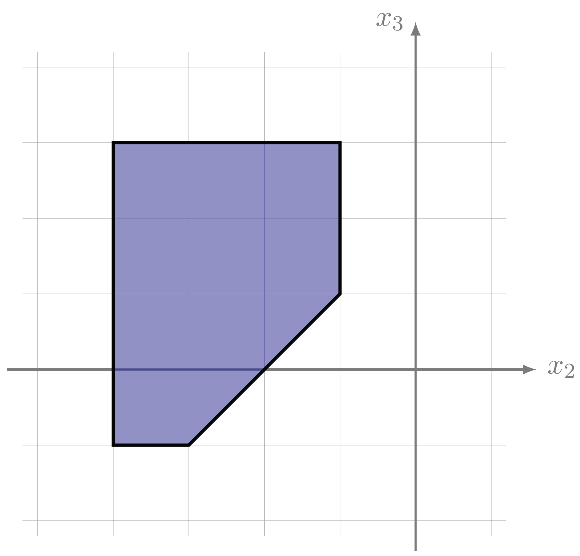


Figure 2.2: The weighted digraph polyhedron $Q(W) = Q(W^*)$ for the matrices W and W^* from Example 2.4, shown in the tropical projective 2-torus

We return to studying general matrices W.

Lemma 2.5. *The connected components of the equality graph of $E(W)$ are complete graphs, and their number is the dimension of the polyhedron $Q(W)$.*

Proof. The equalities $x_i - x_j = w^*_{ij}$ and $x_j - x_\ell = w^*_{j\ell}$ imply $x_i - x_\ell = w^*_{ij} + w^*_{j\ell} \geq w^*_{i\ell}$ and therefore $x_i - x_\ell = w^*_{i\ell}$ for any three nodes i, j, ℓ in the equality graph. So there is an edge between any two nodes in a connected component of $E(W)$. The statement about the dimension follows as the equality graph summarizes exactly those inequalities which are attained with equality and the connected components form a partition of the node set. □

The lemma above says that the equality graph encodes an equivalence relation on the node set $[k]$. The partition into the connected components is the *equality partition*. Abusing our notation, again we denote this partition as $E(W)$.

2.2.2 Intersections and faces

Throughout the following we will frequently consider several graphs which share the same set of nodes. In this case it makes sense to identify such a graph with its set of edges (or arcs, in the directed case). This allows to talk about intersections and unions of such graphs.

Lemma 2.6. *Let U and W be $k \times k$-matrices. The intersection of the weighted digraph polyhedra $Q(U)$ and $Q(W)$ is the weighted digraph polyhedron $Q(U \oplus W)$. The arc set of the graph $\Gamma(U \oplus W)$ is the union of $\Gamma(U)$ and $\Gamma(W)$.*

Proof. The intersection of two polyhedra is given by the union of their defining inequalities. The two inequalities of the form $x_i - x_j \leq u_{ij}$ and $x_i - x_j \leq w_{ij}$ are both satisfied if and only if the inequality $x_i - x_j \leq \min(u_{ij}, w_{ij})$ holds. □

Again we assume that the graph $\Gamma(W)$ does not contain any negative cycle, and thus $Q(W)$ is feasible. Each face of the polyhedron $Q(W)$ is obtained by turning some of the defining inequalities into equalities. More precisely, for any subgraph G of Γ let

$$F_G \;=\; F_G(W) \;=\; F_G(\Gamma, \gamma) \;=\; \{x \in Q(W) \mid x_i - x_j = w_{ij} \text{ for all } (i,j) \in G\} \;.$$

By construction F_G is a face of $Q(W)$, and conversely each face of $Q(W)$ arises in this way. We define a new $k \times k$-matrix, denoted $W \# G$; it is constructed from W by replacing the entries w_{ji} with $-w_{ij}$ for each $(i,j) \in G$. If G contains both (i,j) and (j,i) as arcs, this operation is only defined provided that $w_{ij} + w_{ji} = 0$. The reason is that this equality is implied by $x_i - x_j = w_{ij}$ combined with $x_j - x_i = w_{ji}$. The following is immediate.

Lemma 2.7. *Faces of weighted digraph polyhedra are weighted digraph polyhedra. More precisely,*

$$\begin{aligned} F_G(W) \;&=\; Q(W) \cap \left\{ x \in \mathbb{R}^k \;\middle|\; x_i - x_j = w_{ij} \text{ for } (i,j) \in G \right\} \\ &=\; Q(W) \cap \left\{ x \in \mathbb{R}^k \;\middle|\; x_j - x_i \leq -w_{ij} \text{ for } (i,j) \in G \right\} \;=\; Q(W \# G) \;. \end{aligned}$$

Furthermore, the equality partition $E(W \# G)$ of a face $F_G(W)$ is obtained from the equality partition $E(W)$ by uniting the two parts which contain i and j if (i,j) is an arc in G.

By Lemma 2.5 the dimension of the face $F_G(W)$ equals the size of the partition $E(W \# G)$.

Example 2.8. If W is the matrix from Example 2.4 and G consists of the single arc $(2,3)$ then we have

$$W\#G \;=\; \begin{pmatrix} 1 & 4 & 1 \\ -1 & 0 & -2 \\ 3 & 2 & 2 \end{pmatrix} \;.$$

The equality graph $E(W\#G)$ consists of the isolated node 1, and the nodes 2 and 3 are joined by an edge. This reflects that $Q(W\#G)$ is contained in the supporting hyperplane induced by the equality from G. Finally, the equality partition is $\{\{1\},\{2,3\}\}$.

2.2.3 Braid Cones

We will now apply our previous results to the situation where the weight function is constantly zero on the arcs. Then for an arbitrary digraph Γ the weighted digraph polyhedron

$$Q(\Gamma,\mathbf{0}) \;=\; \left\{ x \in \mathbb{R}^k \;\middle|\; x_i \;\le\; x_j \text{ for all } (i,j) \in \Gamma \right\}$$

is a polyhedral cone, the *braid cone* of Γ studied by Postnikov, Reiner and Williams [PRW08]. See, in particular, [PRW08, §3.4] for detailed information about their combinatorial structure. Here we wish to relate braid cones to order polytopes.

All points in the subspace $\mathbb{R}\mathbf{1}$ are feasible. Since every cycle has weight zero, applying Lemma 2.3(b) to the cone $Q(\Gamma,\mathbf{0})$ yields the following.

Proposition 2.9. *The parts of the equality partition $E(W(\Gamma,\mathbf{0}))$ are exactly the strong components of Γ. In particular, the dimension of the braid cone $Q(\Gamma,\mathbf{0})$ equals the number of strong components of Γ.*

Any hyperplane of the form $x_i = x_j$ defines a *split* of the unit cube $[0,1]^k$, i.e., it defines a (regular) subdivision of the unit cube into two subpolytopes; see [HJ08]. Notice that such a split hyperplane does not separate any edge of the unit cube. Let us look at the map κ which sends each face F of the braid cone $Q(\Gamma,\mathbf{0})$ to the intersection $F \cap [0,1]^k$. Clearly, this intersection is never empty (unless F is).

Now suppose that Γ is acyclic. Then those inequalities which define facets of $Q(\Gamma,\mathbf{0})$ correspond to the covering relations of the partially ordered set $P(\Gamma)$ on the node set $[k]$ of Γ induced by the arcs. It follows that $\kappa(Q(\Gamma,\mathbf{0})) = Q(\Gamma,\mathbf{0}) \cap [0,1]^k$ is the *order polytope* $\operatorname{Ord}(\Gamma)$ of the poset $P(\Gamma)$. The poset $P(\Gamma)$ describes the transitive closure of the relation defined on the set $[k]$ by the arcs of Γ. Conversely, each finite poset gives rise to a directed graph whose nodes are the elements and the arcs are given by the covering relations directed, say, upwards.

The order polytope $\operatorname{Ord}(\Gamma)$ contains the points $\mathbf{0}$ and $\mathbf{1}$ as vertices. Therefore there exists a unique minimal face which contains both of them; denote this face by F_{01}. Note that the dimension of F_{01} can be any number between 1 (if F_{01} is the edge $[\mathbf{0},\mathbf{1}]$) and k (if the graph Γ does not contain any edges). The *face figure* of F_{01}, written as $\mathcal{F}_{01}$, is the principal filter of the element F_{01} in the face poset of the order polytope $\operatorname{Ord}(\Gamma)$. The subposet $\mathcal{F}_{01}$ is the face poset of a polytope of dimension $k - \dim F_{01} - 1$. The face figure $\mathcal{F}_{01}$ consists of exactly those faces of $\operatorname{Ord}(\Gamma)$ which are not contained in any facet of the cube $[0,1]^k$. It is immediate that κ maps faces of the braid cone $Q(\Gamma,\mathbf{0})$ to the faces of the order polytope $\mathrm{Ord}(\Gamma)$ which lie in the face figure $\mathcal{F}_{01}$.

Lemma 2.10. *If Γ is acyclic then the map κ is a poset isomorphism from $\mathcal{F}(Q(\Gamma,\mathbf{0}))$ to the face figure $\mathcal{F}_{01}$ of the face F_{01} of the order polytope* $\mathrm{Ord}(\Gamma)$.

Proof. For any face $G \in \mathcal{F}_{01}$ let $\lambda(G)$ be the cone $\mathrm{pos}(G)+\mathbb{R}\mathbf{1}$. Since G is a face which is not contained in any facet of $[0,1]^k$ it is the intersection of facets of type $x_i \leq x_j$. These inequalities are homogeneous, and so they also hold for $\lambda(G)$. Those inequalities are tight for $Q(\Gamma,\mathbf{0})$, and so λ defines a map from $\mathcal{F}_{01}$ to $\mathcal{F}(Q(\Gamma,\mathbf{0}))$. This also shows that, for any face F of $Q(\Gamma,\mathbf{0})$ we have $\lambda(\kappa(F)) = F$ which means that κ is one-to-one. Conversely, let G be a face of $\mathrm{Ord}(\Gamma)$ which is contained in $\mathcal{F}_{01}$. Then G is defined in terms of split equations of the form $x_i = x_j$. These equations are valid for $\lambda(G) = \mathrm{pos}(G)+\mathbb{R}\mathbf{1}$, which yields $\kappa(\lambda(G)) = G$. Hence κ is surjective, and λ is the inverse map. $\square$

Stanley gave a concise description of the face lattices of order polytopes in terms of partitions [Sta86, Theorem 1.2], and this can be used to derive the following result. This should be compared with [PRW08, Proposition 3.5] which also characterizes the faces of the braid cones, but in a different language.

Theorem 2.11. *Let Γ be an arbitrary directed graph on the node set $[k]$. Then a partition E of $[k]$ is the equality partition of a face of the braid cone $Q(\Gamma,\mathbf{0})$ if and only if*

(i) for each part K of E the induced subgraph of Γ on K is weakly connected, and

(ii) the minor of Γ which results from simultaneously contracting each part of E does not contain any directed cycle.

Proof. Let us first assume that Γ is acyclic. By Lemma 2.7, together with the fact that every cycle has weight zero, the faces of $Q(\Gamma,\mathbf{0})$ are given in terms of the equality partitions of $[k]$. In the acyclic case Lemma 2.10 translates faces of $Q(\Gamma,\mathbf{0})$ into faces of the order polytope $\mathrm{Ord}(\Gamma)$ which contain the special face F_{01}. The property (i) is the connectedness, and property (ii) is the 'compatibility' condition in Stanley's result [Sta86, Theorem 1.2].

We now turn to the general case. If Γ has directed cycles we consider its *acyclic reduction*. The latter graph, occasionally also called 'condensation' in the literature, is obtained by identifying the nodes in each strong component. Since strong components are weakly connected and gather all the directed cycles the same reasoning applies as before. It is easy to see that this digraph is indeed acyclic [Sha81, Corollary 5]. Each partition of $[k]$ which describes a face of $Q(\Gamma,\mathbf{0})$ refines the partition by strong components. $\square$

Notice that there are always two partitions which trivially satisfy the conditions above: The partition of $[k]$ by weak components corresponds to the unique minimal face (which is the lineality space); the partition by strong components corresponds to the entire cone.

Example 2.12. The smallest non-trivial case is $k = 2$, and Γ is the directed graph with two nodes, labeled 1 and 2, with one arc from 1 to 2. The order polytope is the triangle

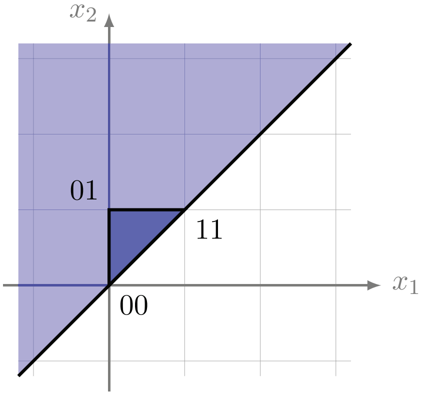


Figure 2.3: Braid cone of a single arc and the corresponding order polytope; see Example 2.12

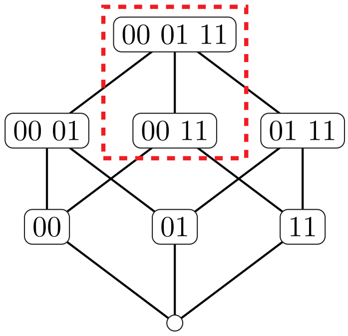


Figure 2.4: Hasse diagram of the triangle conv$\{00, 01, 11\}$ with face figure of conv$\{00, 11\}$ marked

conv$\{00, 01, 11\}$, and the face F_{01} is the edge from 00 to 11. The braid cone $Q(\Gamma, \mathbf{0})$ is the linear half-space $x_1 \leq x_2$, and its lineality space is $\mathbb{R}\mathbf{1}$. The braid cone and the order polytope are shown in Figure 2.3. The node set of Γ only admits the two trivial partitions. The Hasse diagram of the face lattice of $\mathrm{Ord}(\Gamma)$ and the face figure $\mathcal{F}_{01}$ are displayed in Figure 2.4.

Example 2.13. Figure 2.5 shows a digraph on eight nodes and its acyclic reduction, which has six nodes. Figure 2.6 shows the Hasse diagram of the braid cone. That cone is 6-dimensional with a 1-dimensional lineality space. Modulo its lineality space every cone is projectively equivalent to a pyramid over its face at infinity. In this case the braid cone inherits the combinatorics of a 4-simplex.

Remark 2.14. Two distinct digraphs on the node set $[k]$ may induce the same braid cone. This is the case if and only if they induce the same poset. For instance, in Figure 2.5 the arc $(1, 3)$ in the graph on the left and the arc $(1, 378)$ in the graph on the right are redundant. In the acyclic reduction (on the right) we obtain a tree with directed edges. Every tree on ℓ nodes has $\ell - 1$ edges, the braid cone is a simplex cone of dimension $\ell - 1$.

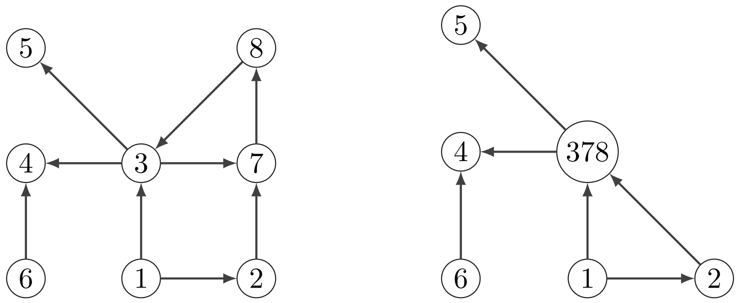


Figure 2.5: Digraph (left) and its acyclic reduction (right)

2.2.4 Weyl–Minkowski decomposition

Now we want to use the Theorem 2.11 on braid cones to describe digraph polyhedra for arbitrary weights. Again we pick a $k \times k$-matrix W, and we assume that $Q(W)$ is feasible. The classical theorem of Weyl and Minkowski (cf. [Zie95, §1]) states that any ordinary polyhedron Q decomposes as the Minkowski sum

$$Q \;=\; P + L + C \ , \tag{2.3}$$

where P is a polytope, L is a linear subspace and C is a pointed polyhedral cone. An ordinary polyhedral cone is *pointed* if it does not contain any affine line (and thus no affine subspace of positive dimension). In the decomposition (2.3) the maximal linear subspace L is unique, while, in general, there may be many choices for C and P. The *recession cone* (which is again unique) is the Minkowski sum of the two unbounded parts, L and C. The *pointed part* is the Minkowski sum $P + C$ (which is unique up to an affine transformation). Next we will decompose a weighted digraph polyhedron in this fashion. We decompose W into the graph Γ and the weight function γ such that $W = W(\Gamma, \gamma)$.

Lemma 2.15. *The recession cone of the weighted digraph polyhedron $Q(\Gamma, \gamma)$ is the braid cone $Q(\Gamma, \mathbf{0})$, and $Q(W(\Gamma, \mathbf{0})\#\Gamma)$ forms the maximal linear subspace.*

Proof. Let x be some point in the recession cone of Q. Then there exists a vector t such that $x + \lambda t \in Q$ for all $\lambda \geq 0$. This means that

$$x_i - x_j + \lambda(t_i - t_j) \;\leq\; w_{ij} \qquad \text{for all } (i,j) \in \Gamma \text{ and } \lambda \geq 0 \ .$$

This forces $t_i - t_j \leq 0$ for all $(i,j) \in \Gamma$, and so t lies in $Q(\Gamma, \mathbf{0})$. The reverse inclusion is similar, and we conclude that the braid cone $Q(\Gamma, \mathbf{0})$ is the recession cone of Q.

Again let $t \in Q(\Gamma, \mathbf{0})$. Then its negative $-t$ is also contained in $Q(\Gamma, \mathbf{0})$ if and only if

$$t_i - t_j \;=\; 0 \qquad \text{for all } (i,j) \in \Gamma$$

if and only if $t \in Q(W(\Gamma, \mathbf{0})\#\Gamma)$. We infer that the braid cone $Q(W(\Gamma, \mathbf{0})\#\Gamma)$ forms the maximal linear subspace of Q. □

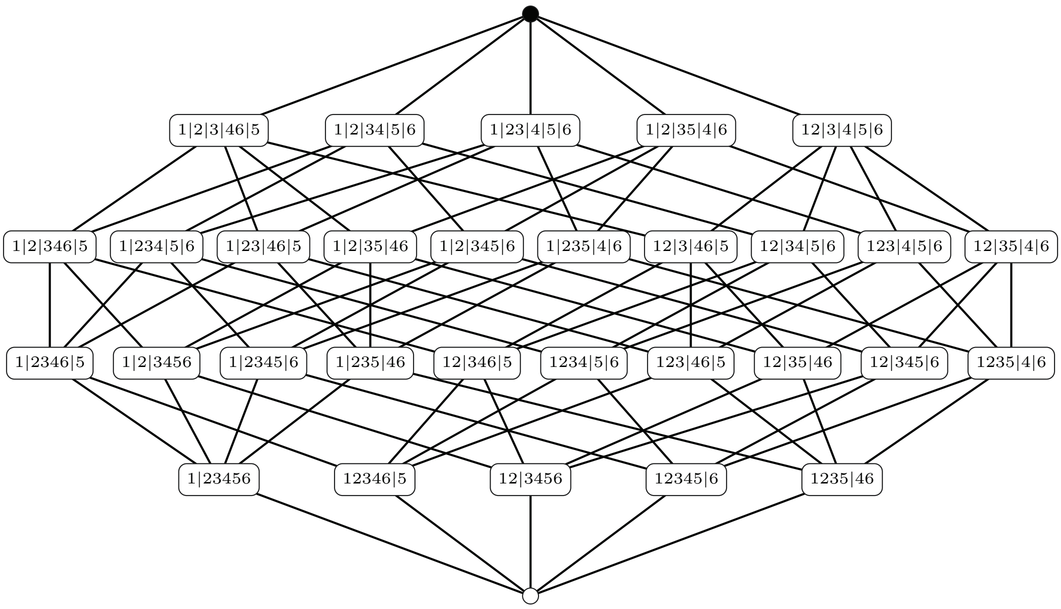


Figure 2.6: Hasse diagram of the braid cone corresponding to the graph in Figure 2.5. For improved readability the node 378 of the acyclic reduction is represented as 3

As a corollary we obtain a slight generalization of [DS04, Corollary 12].

Corollary 2.16. *The weighted digraph polyhedron $Q(\Gamma, \gamma)$ is bounded in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ if and only if Γ consists of one strong component.*

Proof. If Γ has only one strong component, then the recession cone $(\Gamma, \mathbf{0})$ is exactly the one-dimensional lineality space $\mathbb{R}\mathbf{1}$ by Proposition 2.9. Hence, $Q(\Gamma, \gamma)$ is bounded in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. Otherwise, the recession cone is higher-dimensional and the weighted digraph polyhedron is unbounded. □

Our next goal is to describe a minimal system of generators for a braid cone. Recall that a pointed cone is projectively equivalent to a pyramid over its far face. The minimal generators of a pointed cone correspond to the vertices of the far face. For any subset $K \subseteq [k]$, let $\chi(K) \in \mathbb{R}^k$ be the characteristic vector. That is, the ith coordinate of $\chi(K)$ is one if $i \in K$, and it is zero otherwise. With this notation, e.g., we have $\chi([k]) = \mathbf{1}$ and $\chi(\emptyset) = \mathbf{0}$.

Proposition 2.17. *A minimal system of generators of the pointed part of the braid cone $Q(\Gamma, \mathbf{0})$ is given by the vectors $\chi(K)$ with $K \subseteq [k]$ so that the induced subgraph on K is connected, its complement in its weak component in Γ is also connected and every arc in the cut-set of this partition is directed from $[k] \setminus K$ to K.*

Proof. Let $K_1, \ldots, K_\ell$ be the weak components of Γ. In particular, by applying Proposition 2.9 to $Q(W(\Gamma, \mathbf{0})\#\Gamma)$, the dimension of the lineality space of $Q(\Gamma, \mathbf{0})$ equals ℓ. Let

F be a minimal non-trivial face of the cone $Q(\Gamma, \mathbf{0})$. This is a Minkowski sum of the lineality space with a single ray. By Theorem 2.11 the latter corresponds to a partition with $\ell + 1$ parts. Among these exactly $\ell - 1$ parts are weak components of Γ, while the remaining weak component is split into two. Let us assume that the remaining component decomposes as $K_u = K \cup (K_u \setminus K)$, where every arc in the cut-set is directed from $K_u \setminus K$ to K. The characteristic vectors $\chi(K_i)$ for $i \in [\ell]$ linearly span the lineality space of $Q(\Gamma, \mathbf{0})$, while $\chi(K)$ generates the pointed part of F. □

2.2.5 Envelopes and duality

We now turn to the construction of a special class of digraph polyhedra which were introduced by Develin and Sturmfels for studying tropical convexity from the viewpoint of geometric combinatorics [DS04]. For a $d \times n$-matrix V with coefficients in $\mathbb{T}_{\min} = \mathbb{R} \cup \{\infty\}$ we look at the ordinary polyhedron

$$\begin{aligned} \mathcal{E}(V) &= \left\{ (y,z) \in \mathbb{R}^d \times \mathbb{R}^n \;\middle|\; y_i - z_j \leq v_{ij} \text{ for all } i \in [d] \text{ and } j \in [n] \right\} \\ &= \left\{ (y,z) \in \mathbb{R}^d \times \mathbb{R}^n \;\middle|\; y_i - z_j \leq v_{ij} \text{ for all } (i,j) \in \mathrm{B} \right\} \ , \end{aligned}$$

where

$$\mathrm{B}(V) \;=\; \{ (i,j) \in [d] \times [n] \mid v_{ij} \neq \infty \} \tag{2.4}$$

is a (bipartite) directed graph recording the finite entries of V. We call $\mathcal{E}(V)$ the *envelope* of the matrix V. We may see the envelope as a weighted digraph polyhedron via the matrix $(d+n) \times (d+n)$-matrix W which is defined as

$$W \;=\; \begin{pmatrix} \infty_{d \times d} & V \\ \infty_{n \times d} & \infty_{n \times n} \end{pmatrix} \ . \tag{2.5}$$

Up to an obvious relabeling of the nodes $\mathrm{B}(V)$ is the same as $\Gamma(W)$ for the matrix W defined above, and thus we can identify $\mathcal{E}(V)$ with $Q(W)$. Applying Lemma 2.15 and Proposition 2.17 to the envelope we obtain the following.

Corollary 2.18. *The minimal generators of the pointed part of the recession cone of the envelope are given by the partitions $D' \sqcup D'' = [d]$ and $N' \sqcup N'' = [n]$ so that*

(i) the induced subgraph on $D' \sqcup N'$ has the same number of weak components as B,

(ii) the induced subgraph on $D'' \sqcup N''$ is connected, and

(iii) there are no arcs from D'' to N'.

The characteristic vector of $D'' \sqcup N''$ now yields one such generator.

Similarly we obtain from Proposition 2.17 the following corollary which will be helpful in section 2.3.5. A ray can be scaled modulo $\mathbf{1}$ so that it has only non-negative entries and at least one zero entry. Then the *support* of the ray is the set of indices of the non-zero entries. We keep the notation of the former corollary and consider a face of the envelope $\mathcal{E}(V)$ defined by the graph G that contains a minimal generator with support $D''\sqcup N''$. Notice that the arcs of $\Gamma(W\#G)$ which are not arcs of B are arcs from N' to D' or from N'' to D' or from N'' to D''. That is, there are no arcs from N' to D''.

Corollary 2.19. *Let M be the set of column indices j of the matrix V such that $v_{ij} = \infty$ for all $i \in D''$. Then M equals N', and none of the shortest paths in $\Gamma(W\#G)$ between any two nodes in D' contains a node in $D'' \sqcup ([n] \setminus M)$.*

Proof. Observe that M is exactly the subset of the nodes in $[n]$ without an arc between D'' and M in $\Gamma(W\#G)$. Hence, we obtain $N' \subseteq M$ and with Corollary 2.18(ii) even $N' = M$. This yields $[n] \setminus M = N''$. Hence, by Proposition 2.17, there is no arc from $[n] \setminus M$ to D' in $\Gamma(W\#G)$. This implies that every shortest path between two nodes in D' avoids the set $D'' \sqcup ([n] \setminus M)$. □

The graph $\mathrm{B}(V)$ has two kinds of nodes, those which correspond to the rows and those which represent columns of V. In our drawings, like Figure 2.7, we show row nodes as rectangles and column nodes as circles. Moreover, we always draw the row nodes above the column nodes. Therefore, if we want to distinguish them we sometimes talk about the *top* and the *bottom shore* of the bipartite graph.

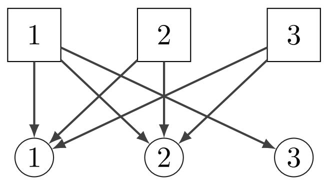


Figure 2.7: Bipartite digraph $\mathrm{B}(V)$ for the matrix in Example 2.20

Example 2.20. For $d = n = 3$ consider the 3×3-matrix

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \infty \\ 0 & 2 & \infty \end{pmatrix} .$$

The lineality space of the envelope $\mathcal{E}(V)$ is spanned by $\mathbf{1}$. The quotient $\mathcal{E}(V)/\mathbf{1}$ is 5-dimensional, and it has exactly two vertices: $(0,1,0;0,0,0)$ and $(0,1,2;2,0,0)$. Its recession cone has six minimal generators, which arise from partitioning the bipartite graph $\mathrm{B}(V)$, which is a subgraph of $K_{3,3}$, into two induced subgraphs which meet the criteria of Corollary 2.18, see Figure 2.7. The sets of the form $D'' \sqcup N''$ read

$$\emptyset\sqcup 1\,, \quad \emptyset\sqcup 2\,, \quad \emptyset\sqcup 3\,, \quad 12\sqcup 123\,, \quad 13\sqcup 123\,, \quad 23\sqcup 12\ .$$

The complementary parts are given by $D' = \{1,2,3\} \setminus D''$ and $N' = \{1,2,3\} \setminus N''$. Notice that, e.g., $23\sqcup 123$ does not occur in the list above since $v_{23} = \infty = v_{33}$; this implies that the induced subgraph is not connected. For instance, $23\sqcup 12$ yields the generator $(0,1,1;1,1,0)$.

A *subpolytope* of a polytope P is the convex hull of some subset of the vertices of P. Each face is a subpolytope, but the converse does not hold. We write e_i for the ith standard basis vector of $\mathbb{R}^k$, for any k, and we write vectors in the product space $\mathbb{R}^d \times \mathbb{R}^n$ as (x, y) where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$. With this notation

$$\Delta_{d-1} \times \Delta_{n-1} \;=\; \operatorname{conv}\{(e_i, e_j) \mid (i,j) \in [d] \times [n]\}$$

is a product of simplices. Develin and Sturmfels established that a tropical configuration of n points induces a polyhedral subdivision of $\mathbb{R}^d$ which is dual to a regular subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ [DS04, Theorem 1]. A polytopal subdivision is *regular* if it is induced by a height function; for details see [DLRS10]. The following statement will be instrumental in Section 2.3.2 below for obtaining a natural generalization to subpolytopes of products of simplices. Notice that those subpolytopes naturally correspond to subgraphs of the complete bipartite graph $[d] \times [n]$.

Theorem 2.21. *The boundary complex of the envelope $\mathcal{E}(V)$ is dual to the regular subdivision of the polytope*

$$\operatorname{conv}\left\{(e_i, e_j) \in \mathbb{R}^d \times \mathbb{R}^n \;\middle|\; (i,j) \in \mathrm{B}(V)\right\}$$

with height function V.

Proof. We abbreviate $\mathrm{B} = \mathrm{B}(V)$. Homogenizing the envelope $\mathcal{E}(V)$ (with leading homogenizing coordinate) yields the cone

$$\left\{(\alpha, y, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^n \;\middle|\; \langle (v_{ij}, -e_i, e_j), (\alpha, y, z)\rangle \geq 0 \text{ for all } (i,j) \in \mathrm{B}\right\} \quad .$$

Hence the polar cone with the dual face lattice can be written as

$$\operatorname{pos}\{(1, \mathbf{0}, \mathbf{0})\} + \operatorname{pos}\{(v_{ij}, -e_i, e_j) \mid (i,j) \in \mathrm{B}\} \quad .$$

Intersecting with the affine hyperplane $H = \{(\alpha, y, z) \mid \langle (0, -\mathbf{1}, \mathbf{1}), (\alpha, y, z)\rangle = 2\}$ gives the polytope

$$P \;=\; \operatorname{conv}\{(v_{ij}, -e_i, e_j) \mid (i,j) \in \mathrm{B}\} \quad ,$$

because all these vectors lie in H and the origin does not.

The orthogonal projection of the lower convex hull of P with respect to $(1, \mathbf{0}, \mathbf{0})$ defines a regular subdivision of the subpolytope of $\Delta_{d-1} \times \Delta_{n-1}$ corresponding to B. If B is the complete bipartite graph or equivalently no entry of V is ∞, that subpolytope is the entire product of simplices. ∎

Any regular subdivision of a subpolytope extends to a regular subdivision of the superpolytope, e.g., by successive placing of the remaining vertices [DLRS10, §4.3.1]. In our situation a regular subdivision of the superpolytope $\Delta_{d-1} \times \Delta_{n-1}$ is obtained by replacing the infinite coefficients in the matrix V with sufficiently large real numbers. Note that this extension is not unique.

2.2.6 Projections

In this section we investigate orthogonal projections of weighted digraph polyhedra and envelopes into the coordinate directions. To this end we let π_I be the projection onto the coordinates in $[k] \setminus I$ for $I \subseteq [k]$. For a $k{\times}k$-matrix W we define W/I by removing the rows and columns whose indices lie in I. We write π_i and W/i if $I = \{i\}$ is a singleton.

Lemma 2.22. *The image of $Q(W) = Q(W^*)$ under the linear projection π_I is the weighted digraph polyhedron $Q(W^*/I)$.*

Proof. By induction it suffices to consider the case where $I = \{k\}$. That $\pi_k(Q(W^*))$ is contained in $Q(W^*/k)$ is clear. We want to show the reverse inclusion. For $(x_1, \dots, x_{k-1}) \in Q(W^*/k)$ we need to find a real number y so that $(x_1, \dots, x_{k-1}, y) \in Q(W) = Q(W^*)$. The latter condition is equivalent to

$$x_i - w^*_{ik} \ \leq \ y \quad \text{and} \quad y \ \leq \ x_i + w^*_{ki} \quad \text{for all} \quad i \in [k-1] \ .$$

So, the claim follows if we can show that

$$\max_{i \in [k-1]} (x_i - w^*_{ik}) \ \leq \ \min_{i \in [k-1]} (x_i + w^*_{ki}) \ . \tag{2.6}$$

Let p and q be indices for which the maximum and the minimum in (2.6), respectively, are attained. Now w^*_{pq} is the length of the shortest path from p to q in the weighted digraph $\Gamma(W)$. This yields

$$x_p - x_q \ \leq \ w^*_{pq} \ \leq \ w^*_{pk} + w^*_{kq} \quad \text{and hence} \quad x_p - w^*_{pk} \ \leq \ x_q + w^*_{kq} \ .$$

□

Now we turn to studying projections of faces of the envelope $\mathcal{E}(V)$ of a not necessarily square $d{\times}n$-matrix. With W defined as in (2.5) we have $\mathcal{E}(V) = Q(W)$. By Lemma 2.7 for any face F of the envelope there is a subgraph G of $\Gamma = \Gamma(W)$ such that $F = Q(W\#G)$. Since, up to a relabeling of the nodes, we can identify the directed graph Γ with the bipartite graph $\mathrm{B} = \mathrm{B}(V)$ and we may read G as a subgraph of B. We define the $n{\times}d$-matrix $V[G]$ with coefficients

$$v'_{ji} \ = \ \begin{cases} -v_{ij} & \text{if } (i,j) \in G \\ \infty & \text{otherwise} \end{cases} \ .$$

The following lemma is similar to [DS04, Lemma 10]. Notice that the tropical matrix product $V \odot V[G]$ yields a $d{\times}d$-matrix.

Lemma 2.23. *The image of the face F of $\mathcal{E}(V) \subset \mathbb{R}^d \times \mathbb{R}^n$ under the orthogonal projection $\pi_{[n]}$ onto the first component is the weighted digraph polyhedron $Q(V \odot V[G])$.*

Proof. For $i, \ell \in [d]$ let $u_{i\ell}$ be a coefficient of $V \odot V[G]$. We have

$$u_{i\ell} = \min_{j\in[n]}(v_{ij} + v'_{j\ell}) = \min_{j\in[n],\, v_{ij}\neq\infty,\, v_{\ell j}\neq\infty}(v_{ij} - v_{\ell j}) ,$$

which is exactly the length of a shortest path from i to ℓ with two arcs in the digraph $\Gamma(W\#G)$. Since the directed graph $\Gamma(W\#G)$ is bipartite the shortest path from i to ℓ (over arbitrarily many arcs) is a concatenation of the two-arc-paths above. Now the claim follows from the previous lemma. $\square$

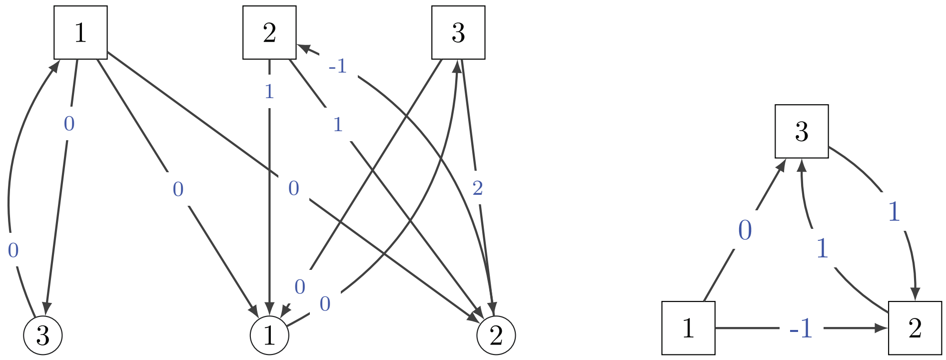


Figure 2.8: Weighted digraphs corresponding to a face of $\mathcal{E}(V)$ from Example 2.24. The first graph corresponds to a face in $\mathbb{R}^d \times \mathbb{R}^n$ whereas the second corresponds to its projection onto $\mathbb{R}^d$. The nodes on the bottom shore are not in their natural ordering to reduce the number of arcs crossing

Example 2.24. We consider the same matrix V as in Example 2.20. For the bipartite graph G on the six nodes $\{1,2,3\} \sqcup \{1,2,3\}$ with arcs $(1,3), (2,2), (3,1)$ we obtain

$$V[G] = \begin{pmatrix} \infty & \infty & 0 \\ \infty & -1 & \infty \\ 0 & \infty & \infty \end{pmatrix} .$$

This yields the product

$$V \odot V[G] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \infty \\ 0 & 2 & \infty \end{pmatrix} \odot \begin{pmatrix} \infty & \infty & 0 \\ \infty & -1 & \infty \\ 0 & \infty & \infty \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \infty & 0 & 1 \\ \infty & 1 & 0 \end{pmatrix} .$$

The corresponding graph is depicted in Figure 2.8 on the right whereas the left one shows the graph $\Gamma(W\#G)$.

2.3 Tropical cones and polyhedral cells

2.3.1 Polyhedral sectors

As before let V be a $d\times n$-matrix with coefficients in $\mathbb{T}_{\min}$. We write $v^{(j)}$ for the jth column of V, and therefore we can identify V with $(v^{(1)}, v^{(2)}, \ldots, v^{(n)})$, the sequence of

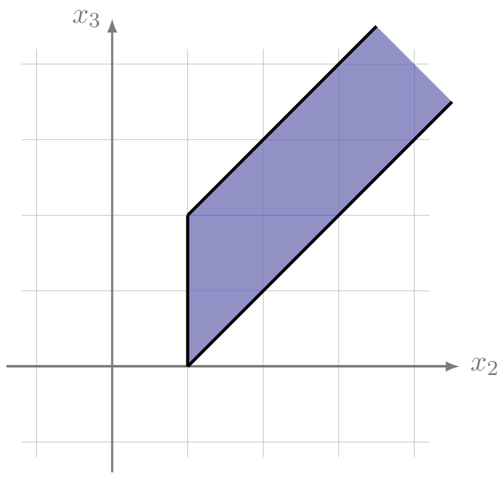


Figure 2.9: Weighted digraph polyhedron given by the matrix $V \odot V[G]$ in Example 2.24 which is unbounded in the tropical projective 2-torus

column vectors. The $(\min, +)$-linear span of the columns of V is the min-*tropical cone*

$$\operatorname{tcone}(V) \;=\; \left\{ (\lambda_1 \odot v^{(1)}) \oplus \cdots \oplus (\lambda_n \odot v^{(n)}) \;\middle|\; \lambda_j \in \mathbb{T}_{\min} \right\} \;.$$

Put in a more algebraic language, a tropical cone is the same as a finitely generated subsemimodule of the semimodule $(\mathbb{T}^d_{\min}, \oplus, \odot)$. A subset M of $\mathbb{R}^d$ is min-*tropically convex* if for any two points $u, v \in M$ we have $\operatorname{tcone}(u, v) \subseteq M$. Any tropically convex set contains $\mathbb{R}\mathbf{1}$, and so we can study its image under the canonical projection to the tropical projective torus. Up to this projection tropical cones generated by vectors with finite entries are precisely the 'tropical polytopes' of Develin and Sturmfels [DS04]. In this section we will generalize key results from that paper to the case where ∞ may occur as a coordinate. By homogenization our results also apply to the formally more general 'tropical polyhedra' studied, e.g., in [AGG12] and [ABGJ15].

Remark 2.25. For an arbitrary $k \times k$-matrix with coefficients in $\mathbb{T}_{\min}$ the weighted digraph polyhedron $Q(W) = Q(W^*)$ coincides with the min-tropical span $\operatorname{tcone}(W^*)$. See also [But10, Theorem 2.1.1] and the Section 2.3.4 on polytropes below.

For $u \in \mathbb{T}^d_{\min}$ and $i \in [d]$ with $u_i \neq \infty$ we define the ith *sector* $S_i(u)$ with respect to *max* as

$$\left\{ z \in \mathbb{R}^d \;\middle|\; \max_{\ell \in [d]} (z_\ell - u_\ell) = z_i - u_i \right\} \;=\; \left\{ z \in \mathbb{R}^d \;\middle|\; \min_{\ell \in [d]} (u_\ell - z_\ell) = u_i - z_i \right\} \;.$$

Notice that the above equality of sets is a consequence of the elementary fact

$$-\max(u, v) \;=\; \min(-u, -v) \;.$$

Moreover, the equation $\min_{\ell \in [d]}(u_\ell - z_\ell) = u_i - z_i$ is equivalent to $z_\ell - z_i \leq u_\ell - u_i$ for each $\ell \in [d]$. As $u_i < \infty$ that minimum cannot be attained for any $\ell \in [d]$ with $u_\ell = \infty$.

We have

$$S_i(u) \;=\; \bigcap_{\ell \in [d],\ u_\ell \neq \infty} \left\{ z \in \mathbb{R}^d \;\middle|\; z_\ell - z_i \leq u_\ell - u_i \right\} \ , \tag{2.7}$$

which means that this sector is the weighted digraph polyhedron for the graph with node set $[d]$ and arc set $\{(\ell, i) \,|\, \ell \in [d], u_\ell \neq \infty\}$, where the arc (ℓ, i) has weight $u_\ell - u_i$.

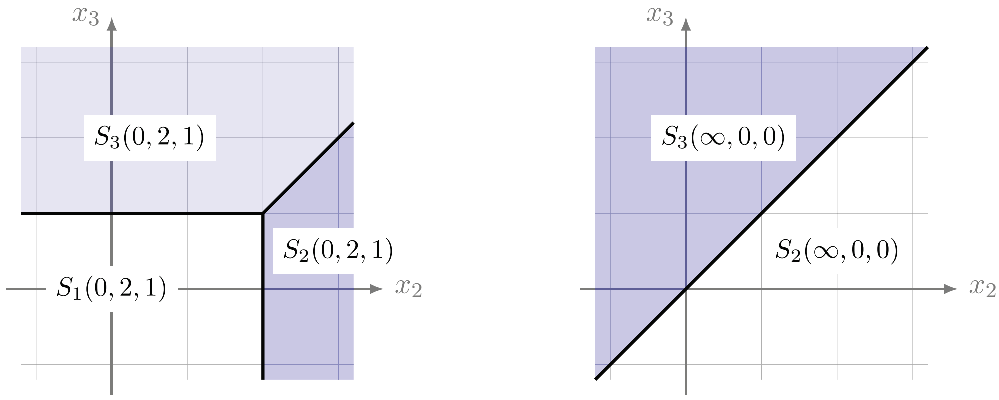


Figure 2.10: Polyhedral decomposition of $\mathbb{R}^3$ as in Lemma 2.26 induced by $(0, 2, 1)$ and $(\infty, 0, 0)$, respectively. Compare the image on the right with Figure 2.3

Lemma 2.26. *The sectors $\{S_i(u) \,|\, u_i \neq \infty\}$ are the maximal cells of a polyhedral decomposition of $\mathbb{R}^d$.*

Proof. Considering the column vector u as a $d \times 1$-matrix, we obtain the envelope $\mathcal{E}(u)$ as a subset of $\mathbb{R}^{d+1}$. The sector $S_i(u)$ is the orthogonal projection of the face defined by the single arc $(i, 1)$ in the bipartite graph $\mathrm{B}(u)$. □

We denote the polyhedral complex arising from the previous lemma by $\Delta(u)$; see also [DS04, Proposition 16]. The negative $-u$ of the vector $u \in \mathbb{T}^d_{\min}$ defines a max-tropical linear form and thus a max-tropical hyperplane. The sectors $S_i(u)$ for $u_i \neq \infty$ are precisely the topological closures of the connected components of the complement of that tropical hyperplane.

Example 2.27. The white sector $S_1(0, 2, 1)$ in Figure 2.10 is the orthogonal projection on $\{1, 2, 3\}$ of the weighted digraph polyhedron given by the bipartite graph with node set $\{1, 2, 3\} \sqcup \{1'\}$ where the arc $(1, 1')$ has weight zero, $(2, 1')$ has weight 2, $(3, 1')$ has weight 1 and $(1', 1)$ has weight zero.

The following result characterizes the solvability of a system of tropical linear equations in $\mathbb{R}^d$. For matrices with finite coordinates this is the Tropical Farkas Lemma [DS04, Proposition 9], a version of which already occurs in [Vor67]. We indicate a short proof for the sake of completeness.

Lemma 2.28. *A point $z \in \mathbb{R}^d$ is contained in* $\mathrm{tcone}(V)$ *if and only if for every $i \in [d]$ there is an index $s \in [n]$ with $z \in S_i(v^{(s)})$.*

Proof. Let $z \in \mathbb{R}^d$ be a point in $\operatorname{tcone}(V)$. Then there is a vector $\lambda \in \mathbb{T}_{\min}^n$ so that $\bigoplus_{j=1}^n \lambda_j \odot v^{(j)} = z$ or, equivalently,

$$\min\{\lambda_j + v_{ij} \mid j \in [n]\} = z_i \qquad \text{for each } i \in [d] \ . \tag{2.8}$$

Now fix $i \in [d]$ and let s be an index j for which the minimum in (2.8) is attained; that is, $z_i = \lambda_s + v_{is}$. If $\ell \in [d]$ with $v_{\ell s} \neq \infty$ this gives

$$z_\ell - z_i = z_\ell - \lambda_s - v_{is} \leq \lambda_j + v_{\ell j} - \lambda_s - v_{is} \qquad \text{for each } j \in [n] \ .$$

Specializing to $j = s$ entails $z_\ell - z_i \leq v_{\ell s} - v_{is}$ and thus $z \in S_i(v^{(s)})$. The entire argument can be reversed to prove the converse. □

2.3.2 The covector decomposition

Again let $V \in \mathbb{T}_{\min}^{d\times n}$, and let $W \in \mathbb{T}_{\min}^{(d+n)\times(d+n)}$ be the matrix which is associated via (2.5). We assume in the following that V has no column equal to the all ∞ vector $(\infty, \ldots, \infty)^\top$; hence, none of the complexes $\Delta(v^{(j)})$ is empty. We do admit rows which solely contain ∞ entries. They add to the lineality of the occurring polyhedra. However, there may also be other contributions to the lineality space; see Lemma 2.15. The weighted bipartite graph $\mathrm{B} = \mathrm{B}(V)$ and the weighted digraph $\Gamma = \Gamma(W)$ are defined as before. For an arbitrary subgraph G of B we define the polyhedron

$$X_G(V) = \bigcap_{(i,j)\in G} S_i(v^{(j)}) \tag{2.9}$$

in $\mathbb{R}^d$.

Remark 2.29. Right from the definition, we obtain $X_{G\cup H}(V) = X_G(V) \cap X_H(V)$ for any two graphs $G, H \subseteq \mathrm{B}(V)$. If, furthermore, $G \subseteq H$ then $X_H(V) \subseteq X_G(V)$. This occurs also in [DS04, Corollary 11 and 13]. It should be stressed that the cells $X_G(V)$ and $X_H(V)$ may coincide even if the graphs G and H are distinct.

Proposition 2.30. *Let G be an arbitrary subgraph of* B *(which we may also read as a subgraph of Γ). Then the orthogonal projection of the face $F_G(W)$ onto $\mathbb{R}^d$ equals $X_G(V)$. If no node in $[n]$ is isolated in G that projection is an affine isomorphism.*

Proof. Our goal is to exploit what we know about weighted digraph polyhedra. To this end we define several digraphs with the same node set $[d] \sqcup G$. Recall that we identify the subgraph G of Γ with its set of edges. However, in the class of digraphs to be defined now, those edges (along with the nodes in $[d]$) play the role of nodes.

Pick $(i, j) \in G$. We let Φ_{ij} be the weighted digraph which results from $\mathrm{B}(v^{(j)})$, which has $[d] \sqcup \{1\}$ as its node set, by renaming the node 1 on the bottom shore by (i, j) and adding an isolated node for each other arc in G. The graph Φ_{ij} has one extra arc in the reverse direction, namely from (i, j) to i. The weights on the arcs from top to bottom are the same as in $\mathrm{B}(v^{(j)})$, while the weight on the single reverse arc is $-v_{ij}$. Compare this with Lemma 2.26 and Example 2.27. By construction the weighted digraph Φ_{ij} is bipartite and thus can be identified with a square matrix of size $d + |G|$. By Lemma 2.23 the weighted digraph polyhedron $Q(\Phi_{ij}) \subset \mathbb{R}^d \times \mathbb{R}^G$ projects orthogonally onto the sector $S_i(v^{(j)}) \subset \mathbb{R}^d$.

Let Φ be the digraph with node set $[d] \sqcup G$ which is obtained as the union of the digraphs Φ_{ij} for $(i,j) \in G$. Notice that by our construction the choice of the weights for the individual graphs Φ_{ij} is consistent. This way we obtain a natural weight function on Φ. Due to Lemma 2.6 we have

$$\pi_G\big(Q(\Phi)\big) \;=\; \pi_G\Big(\bigcap_{(i,j)\in G} Q(\Phi_{ij})\Big) \;=\; \bigcap_{(i,j)\in G} S_i(v^{(j)}) \ .$$

If $\Gamma(W\#G)$ has a negative cycle, so has Φ and by Lemma 2.1 then $F_G(W)$ as well as $X_G(V)$ are empty. If there are no negative cycles, there exists a shortest path between two nodes i and ℓ in $[d]$, and it does not matter if we consider $\Gamma(W\#G)$ or Φ. So, the claim follows with Lemma 2.22.

For the rest, assume that $\Gamma(W\#G)$ has no negative cycle. Since $\Gamma(W\#G)$ is bipartite, any two nodes $i, \ell \in [d]$ are contained in a directed cycle of weight zero of G if this also holds for the graph $\Gamma(\pi_{[n]}(Q(W\#G))$ of the projection of $F_G(W)$ by Lemma 2.22. If no node in $[n]$ is isolated in G, every node in $[n]$ is contained in a directed cycle of weight zero, as every arc from $[n]$ to $[d]$ in $\Gamma(W\#G)$ induces a cycle of length zero. Hence, the equality partition of $F_G(W)$ and of its projection $\Gamma(\pi_{[n]}(Q(W\#G))$ have the same number of parts by Lemma 2.3(b). Therefore, if no node in $[n]$ is isolated in G, we get that $F_G(W)$ has the same dimension as $X_G(V)$. $\square$

The *covector decomposition* $\mathcal{T}(V)$ of $\mathbb{R}^d$ is the common refinement of the polyhedral complexes $\Delta(v^{(j)})$ for $j \in [n]$. For every cell C in the covector decomposition there is a unique maximal subgraph $\mathrm{T}(C)$ of the complete bipartite graph $[d] \times [n]$, called the *covector graph* of C, such that $C = X_{\mathrm{T}(C)}(V)$. This graph is equivalent to the *covector* $(t_1, t_2, \ldots, t_d) \in [n]^d$ where $t_i \subseteq [n]$ consists of the nodes adjacent to i. While the covector notation is concise in most proofs it is convenient to keep the interpretation as a directed bipartite graph. Notice that our cells are closed by definition. By Proposition 2.30, each covector (graph) also uniquely determines a face of $\mathcal{E}(V)$ and every face, for which no node in $[n]$ is isolated, occurs in this way. By Lemma 2.28 the *covector decomposition* $\mathcal{T}(V)$ of $\mathbb{R}^d$ induces a covector decomposition of the tropical cone $\mathrm{tcone}(V)$. The covector graphs correspond to the 'types' of [DS04].

Example 2.31. Figure 2.11 shows an example for the matrix

$$V \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \infty \\ 2 & -1 & \infty \end{pmatrix} \ .$$

The points corresponding to the columns of V are marked 1, 2 and 3. Notice that the third column has ∞ as a coordinate, which is why this point lies outside the tropical projective torus. In fact, it is a boundary point of the *tropical projective plane*; see Section 2.3.5 and Figure 2.16 below.

Only the covectors of the full-dimensional cells are indicated since the covectors of the other cells can directly be deduced from them by Remark 2.29.

The covector decomposition of $\operatorname{tcone}(V)$ has precisely two cells which are maximal with respect to inclusion: the 2-dimensional cell with covector $(3, 2, 1)$ and the 1-dimensional cell with covector $(13, 2, 2) = (13, -, 2) \cup (13, 2, -)$.

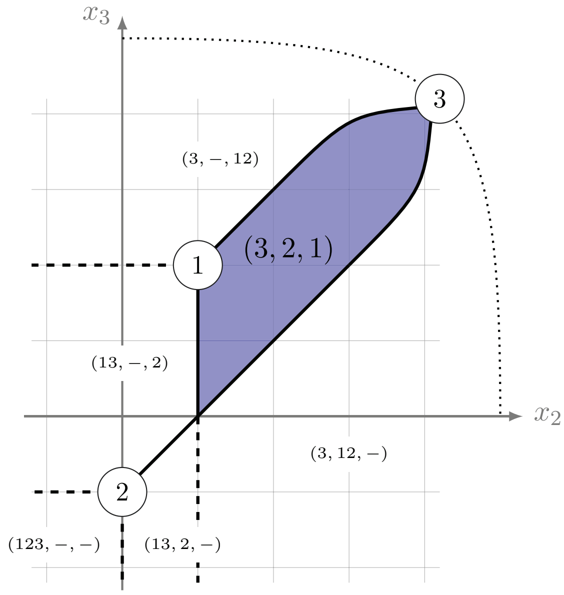


Figure 2.11: Tropical cone in the tropical projective 2-torus, from Example 2.31. The dotted line represents the boundary, which is not part of the tropical projective torus

Remark 2.32. From the viewpoint of tropical geometry the decomposition $\mathcal{T}(V)$ can be deduced from the max-tropical linear forms corresponding to the columns of V. For this, we pick variables $x_{1j}, x_{2j}, \ldots, x_{dj}$ for each column $v^{(j)}$ of V. The product of the tropical linear forms $\max(x_{1j} - v_{1j}, x_{2j} - v_{2j}, \ldots, x_{dj} - v_{dj})$ yields a homogeneous tropical polynomial p in $d \cdot n$ variables x_{ij}. This defines a tropical hypersurface in $\mathbb{R}^{d \cdot n}/\mathbb{R}\mathbf{1}$ where the covectors come into play as the exponent vectors of (tropical) monomials in p. Substituting x_{ij} by y_i gives rise to the tropical hypersurface in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ which induces the cell decomposition of this space.

Theorem 2.33. *The orthogonal projection from the boundary complex of $\mathcal{E}(V)$ onto $\mathbb{R}^d$ induces a bijection between the envelope faces whose covector graph have no isolated node in $[n]$ and the cells in the covector decomposition $\mathcal{T}(V)$ of $\mathbb{R}^d$. This map is a piecewise linear isomorphism of polyhedral complexes.*

Each face whose covector graph neither has an isolated node in $[d]$ (nor an isolated node in $[n]$) maps to a cell in the covector decomposition of $\operatorname{tcone}(V)$.

Proof. Ranging over all the faces whose covector graph has no isolated node in $[n]$ we obtain the bijection with Proposition 2.30. The definition of the covector of a cell combined with Lemma 2.28 characterizes when a cell in $\mathcal{T}(V)$ is contained in the tropical cone generated by the columns of V. $\square$

With Theorem 2.21 the former implies the following.

Corollary 2.34 (Structure Theorem of Tropical Convexity)**.** *The covector decomposition $\mathcal{T}(V)$ of $\mathbb{R}^d$ is dual to the regular subdivision of the polytope*

$$\operatorname{conv}\left\{(e_i, e_j) \in \mathbb{R}^d \times \mathbb{R}^n \;\middle|\; (i,j) \in \mathrm{B}(V)\right\}$$

with weights given by V. Moreover, the covector decomposition of $\operatorname{tcone}(V)$ *is dual to the poset of interior cells.*

The result above is the same as [FR15, Corollary 4.2]; their proof is based on mixed subdivisions and the Cayley Trick [DLRS10, §9.2].

Note that the envelope of a matrix whose coefficients are 0 or ∞ is a braid cone, and so Theorem 2.11 applies to describe the combinatorics. The min-tropical cones corresponding to these matrices are tropical analogues of ordinary 0/1-polytopes.

Corollary 2.35. *Let V be a $d \times n$-matrix whose coefficients are ∞ or 0. A partition E of $[d] \sqcup [n]$ defines a face of the polyhedral fan $\mathcal{T}(V) \subseteq \mathbb{R}^d$ with apex $\mathbf{0}$ if and only if*

(i) for each part K of E the induced subgraph of $\mathrm{B}(V)$ *on K is weakly connected,*

(ii) the minor of $\mathrm{B}(V)$ *which results from simultaneously contracting each part of E does not contain any directed cycle, and*

(iii) no part of E is a single element of $[n]$.

As projections of the faces of the envelope $\mathcal{E}(V)$ the cones in such a fan can encode an arbitrary digraph on d nodes.

Example 2.36. The maximal cell in Figure 2.9 is the intersection of the sectors $S_3((0,1,0)^\top)$, $S_2((0,1,2)^\top)$ and $S_1((0,\infty,\infty)^\top)$. On the other hand, it is the projection of the face of the envelope $\mathcal{E}(V)$ corresponding to the graph on three nodes with the arcs $(1,3),(2,2),(3,1)$ for the matrix V from Example 2.20.

The recession cone of this face is given by the graph in Figure 2.12. It has the strong components 1×3 and 23×12. Hence, a minimal generator of the pointed part of the cone is $(0,1,1;1,1,0)^\top$ by Proposition 2.17. This projects to the ray generated as the positive span of $(0,1,1)^\top$ which is indeed contained in the tropical cone $\operatorname{tcone}(V)$.

Remark 2.37. Clearly, we can also project the envelope $\mathcal{E}(V)$ onto the $[n]$ coordinates of the lower shore. This yields a covector decomposition of $\mathbb{R}^n$ induced by the d rows of the matrix V. Applying Theorem 2.33 to the transpose $V^\top$ gives an isomorphism between the envelope faces without any isolated node in $[d]$ and the cells in the covector decomposition of $\mathbb{R}^n$ induced by the rows of V.

Therefore, the cells whose covector graphs do not have any isolated node in their covector graphs project affinely isomorphic to $\mathbb{R}^d$ as well as to $\mathbb{R}^n$. This entails an isomorphism between the covector decompositions of $\operatorname{tcone}(V)$ and $\operatorname{tcone}(V^\top)$.

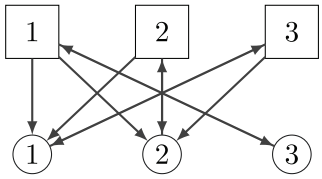


Figure 2.12: Bipartite graph for the face projecting to the maximal cell in Figure 2.9

Proposition 2.38. *Let G be a subgraph of $[d] \times [n]$. Then the following statements are equivalent.*

(i) There is a point $(y, z) \in \mathcal{E}(V) = Q(W)$ for which the inequality corresponding to $(i, j) \in \Gamma(W)$ is attained with equality if and only if $(i, j) \in G$.

(ii) (a) For every pair of subsets $D \subseteq [d]$ and $N \subseteq [n]$ with $|D| = |N|$, every perfect matching of G restricted to $D \sqcup N$ is a minimal matching of the complete bipartite graph $D \times N$ with the weights given by the corresponding submatrix of V;

(b) if there are more minimal perfect matchings in $D \times N$ then each of them is contained in G.

(iii) (a) The graph $\Gamma(W\#G)$ does not have any negative cycle, and

(b) every arc of $\Gamma(W)$ in $\Gamma(W\#G)$ that is contained in a cycle of weight zero is contained in G.

Proof. To conclude (ii) from (i) let $D \subseteq [d]$ and $N \subseteq [n]$ with $|D| = |N|$ so that there is a perfect matching $\mathcal{M}_0$ in $D \times N \cap G$. Let $\mathcal{M}_1$ be any other perfect matching in $D \times N$. Then considering the corresponding inequalities and equations implies after summing up and reordering

$$\sum_{(i,j)\in\mathcal{M}_0} v_{ij} \;=\; \sum_{i\in D} y_i - \sum_{j\in N} z_j \;\leq\; \sum_{(i,j)\in\mathcal{M}_1} v_{ij} \;.$$

Therefore, $\mathcal{M}_0$ is a minimal perfect matching. Furthermore, if $\mathcal{M}_1$ is also a minimal perfect matching, then equality follows in the former inequality. That implies the equations $y_i - z_j = v_{ij}$ for every $(i, j) \in \mathcal{M}_1$. Hence, every arc in $\mathcal{M}_1$ has to be contained in G.

We now want to show that this implies (iii). For this, we consider a non-positive cycle in $\Gamma(W\#G)$ with vertex set $D \sqcup N$. Let A_W be the set of arcs directed from $[d]$ to $[n]$ and A_G the set of arcs directed from $[n]$ to $[d]$. Since $\Gamma(W\#G)$ is bipartite, this implies $|D| = |N| = |A_W| = |A_G|$ and the arc sets A_W and A_G define perfect matchings in $D \times N$.

By definition of $\Gamma(W\#G)$ we obtain for the weight of the cycle

$$\sum_{(i,j)\in A_W} v_{ij} + \sum_{(j,i)\in A_G} (-v_{ij}) \;\leq\; 0 \qquad \text{or, equivalently,} \qquad \sum_{(i,j)\in A_W} v_{ij} \;\leq\; \sum_{(j,i)\in A_G} v_{ij} \;.$$

If the inequality is strict, this contradicts the minimality of the matching via (ii). If the cycle has weight zero and the inequality becomes an equality, this implies that A_W also represents a minimal perfect matching. With (ii) every arc in A_W is also in G then.

The final goal is to lead (iii) back to (i). If $\Gamma(W\#G)$ does not contain a negative cycle, the weighted digraph polyhedron $Q(W\#G)$ is not empty. Therefore, there is (y, z) in the interior of the face $Q(W\#G) \subseteq \mathbb{R}^d \times \mathbb{R}^n$. Let (i, j) be some arc of $\Gamma(W)$. If the equality $y_i - z_j = v_{ij}$ holds, Lemma 2.3(b) yields that there is a cycle of weight zero containing the arc (i, j). With (iii) we obtain $(i, j) \in G$. On the other hand, for $(i, j) \in G$, the graph $\Gamma(W\#G)$ contains the cycle (i, j, i) of weight zero, and the claim follows. □

Together with Proposition 2.30 this also gives a characterization for the covector graphs which are contained in the tropical cone $\operatorname{tcone}(V)$. Furthermore, we obtain a corollary concerning the dimension of a cell.

Corollary 2.39. *If $G \subseteq \mathrm{B}(V)$ is a covector graph for V, the dimension of $F_G(W)$ and thus of $X_G(V)$ equals the number of weak components of G.*

Proof. By property (iii) of Proposition 2.38 two nodes in $[d] \sqcup [n]$ are connected by a path in G if and only if they are in a cycle of weight zero in $\Gamma(W\#G)$. By Lemma 2.3(b) these cycles exactly define the equality partition of $F_G(W)$. Finally, Lemma 2.5 connects this to the dimension. Furthermore, Proposition 2.30 shows the equality for $F_G(W)$ and $X_G(V)$. □

Remark 2.40. The envelope of V is the set of points (y, z) satisfying

$$y_i - z_j \;\leq\; v_{ij} \qquad \text{for } (i, j) \in \mathrm{B}\,.$$

Substituting z_j by $-z_j$ yields

$$y_i + z_j \;\leq\; v_{ij} \qquad \text{for } (i, j) \in \mathrm{B}\;\;, \tag{2.10}$$

which is the form of the envelope in [DS04]. Maximizing the coordinate sum over the polyhedron defined in (2.10) is dual to finding a minimum weight matching by Egerváry's Theorem [Sch03, Theorem 17.1]. This gives rise to a primal-dual algorithm for computing matchings and vertex covers; the method is explained in detail in [PS82, Theorem 11.1]. A partial matching of minimal weight in a subgraph can be expanded by growing so-called 'Hungarian trees', which are shortest path trees in a modified graph. The partial matchings, which encode tight inequalities in the dual description, are collected in the equality subgraphs. By Proposition 2.38 one can deduce that these equality subgraphs are exactly the covector graphs of the dual points (y, z).

2.3.3 Tropical half-spaces

The sectors $S_i(u)$ with $u_i \neq \infty$ from Lemma 2.26, which are responsible for the combinatorial properties of min-tropical point configurations, are precisely the (closures of the) complements of the max-tropical hyperplane with apex u. The same combinatorial objects also control systems of tropical linear inequalities. To see this it is convenient to switch to max as the tropical addition now.

Let $c \in \mathbb{T}_{\min}^d$ and let I be a non-empty proper subset of $[d]$, i.e., $I \neq \emptyset$ and $I \neq [d]$. Then the set $\bigcup_{\ell \in I} S_\ell(c)$ is a max-*tropical half-space* with *apex* c. This is exactly the set of points in $\mathbb{R}^d$ which satisfies the homogeneous max-tropical linear inequality

$$\max_{\ell \in [d] \setminus I} (-c_\ell + x_\ell) \ \leq \ \max_{\ell \in I} (-c_\ell + x_\ell) \ .$$

Since here we allow for ∞ as a coordinate in c this definition is more general than the one in [Jos05]. Notice that $-c$ is an element of $\mathbb{T}_{\max}^d$ and that the halfspaces are defined over the max-tropical semiring. Each tropical cone is the intersection of finitely many tropical half-spaces and conversely. This is proved in [GK11, Theorem 1], based on [Gau92]; note that the proof of [Jos05, Theorem 3.6] (which claims the same) is not valid as it rests on [Jos05, Proposition 3.3], which is false. In [But10, §7.6], referring to [BH84], it is shown that the solution set of any system of max-tropical linear equalities is finitely generated. Since $u \leq v$ holds if and only if $\max(u, v) = v$, i.e., since in the tropical setting studying systems of linear equalities amounts to the same as studying systems of linear inequalities, that result shows one direction of [GK11, Theorem 1].

Remark 2.41. Let W be a $k \times k$-matrix. Each defining inequality (2.1) of the weighted digraph polyhedron $Q(W)$ can be rewritten as

$$x_i - w_{ij} \ \leq \ x_j \qquad \text{for each arc } (i,j) \text{ in } \Gamma(W) \ .$$

Fixing j and varying i then yields

$$\max_{i \in [k]} (x_i - w_{ij}) \ \leq \ x_j \qquad \text{for each } j \in [k] \ .$$

Looking at all j simultaneously we obtain the inequality

$$(-W^\top) \odot_{\max} x \ \leq \ x$$

of column vectors. This means that each weighted digraph polyhedron is a max-tropical cone. In [But10, §1.6.2 and §2] a vector x satisfying the inequality above is called a 'subeigenvector' of the matrix $-W^\top$.

We now want to introduce notation for inequality descriptions of tropical cones which is suitable for our combinatorial approach. Let $V = (v_{ij}) \in \mathbb{T}_{\min}^{d \times n}$ and let Ψ be a subgraph of the complete bipartite graph $[d] \times [n]$ with arcs directed from $[d]$ to $[n]$. We define

$$\operatorname{thalf}(V, \Psi) \ = \ \bigcap_{j \in [n]} \ \bigcup_{(i,j) \in \Psi} S_i(v^{(j)}) \ . \tag{2.11}$$

That is, $\operatorname{thalf}(V, \Psi)$ comprises those points $x \in \mathbb{R}^d$ which satisfy the homogeneous max-tropical linear inequalities

$$\max_{i \in [d],\, (i,j) \notin \Psi} (-v_{ij} + x_i) \ \leq \ \max_{i \in [d],\, (i,j) \in \Psi} (-v_{ij} + x_i)$$

for each $j \in [n]$. In our notation the columns of the matrix V collect the apices of the tropical half-spaces, and the graph Ψ lists the sectors per half-space. In [But10, §7] exterior descriptions of tropical cones like (2.11) are discussed under the name 'two-sided max-linear systems'. To phrase our results below it is convenient to introduce two sets of subgraphs of $[d] \times [n]$, both of which depend on Ψ. We let

$$\begin{aligned} \mathcal{G}_\Psi &= \{G \subseteq \Psi \mid \text{every node in } [n] \text{ has degree 1 in } G\} \qquad \text{and} \\ \mathcal{H}_\Psi &= \{H \subseteq [d] \times [n] \mid \text{every node in } [n] \text{ has degree } \geq 1 \text{ in } \Psi \cap H\} \quad , \end{aligned}$$

which gives the following.

Proposition 2.42. *For each graph $H \in \mathcal{H}_\Psi$ the cell X_H, which may be empty, is contained in* $\operatorname{thalf}(V, \Psi)$. *Moreover, $\mathcal{G}_\Psi \subseteq \mathcal{H}_\Psi$, and we have*

$$\operatorname{thalf}(V, \Psi) \;=\; \bigcup_{G \in \mathcal{G}_\Psi} \bigcap_{(i,j) \in G} S_i(v^{(j)}) \;=\; \bigcup_{H \in \mathcal{H}_\Psi} \bigcap_{(i,j) \in H} S_i(v^{(j)}) \enspace .$$

Proof. Here the first equality is obtained by reordering the intersections and unions in the Definition (2.11). For the second equality notice that $\mathcal{G}_\Psi \subseteq \mathcal{H}_\Psi$. Since for every graph $H \in \mathcal{H}_\Psi$ there is a graph $G \in \mathcal{G}_\Psi$ so that $X_H(V) \subseteq X_G(V)$ the claim follows. $\square$

The preceding proposition says that a cell $X_G(V) = \bigcap_{(i,j) \in G} S_i(v^{(j)})$ in the covector decomposition $\mathcal{T}(V)$ of $\mathbb{R}^d$ with covector graph $G \subseteq [d] \times [n]$ is contained in the max-tropical cone $\operatorname{thalf}(V, \Psi)$ if and only if no node in $[n]$ is isolated in the intersection of G and Ψ. Moreover, $\operatorname{thalf}(V, \Psi)$ is a union of cells. In this way the Proposition 2.42 can be seen as some kind of a dual version of [DS04, Theorem 15], which is a key structural result in tropical convexity.

Corollary 2.43. *The covector decomposition of* $\operatorname{thalf}(V, \Psi)$ *induced by the columns of V is dual to a subcomplex of the regular subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ with weights given by V.*

Example 2.44. The apices $(0,1,1)^\top$ and $(0,2,1)^\top$ induce the cell decomposition depicted in Figure 2.13. Every node in the bottom shore in the graph G to the right has degree 1. Hence, it is the kind of graph contained in $\mathcal{G}_\Psi$ for some appropriate Ψ (for example G itself). However, the corresponding cell is not full-dimensional since the apices are not in general position. Indeed, the covector graph of this cell is obtained from G by adding the arcs $(3,1)$ and $(1,2)$.

Remark 2.45. The *tangent digraph*, defined in [ABGJ15, §3.1], describes the local combinatorics at a cell C of $\operatorname{thalf}(V, \Psi)$. This is related to the above as follows. Deleting all nodes in $[n]$ (and incident arcs) for which all incident arcs are contained in Ψ in the covector graph $\mathrm{T}(C)$ and forgetting about the orientation yields the *tangent graph* $\mathrm{TG}(C)$ of [ABGJ15, §3.1]. By taking the orientation into account and reversing every arc in $\mathrm{TG}(C)$ which is not in $\mathrm{TG}(C) \cap \Psi$ from the bottom shore $[n]$ (corresponding to the hyperplane apices) to the top shore $[d]$ (corresponding to the coordinate directions) we obtain the tangent digraph.

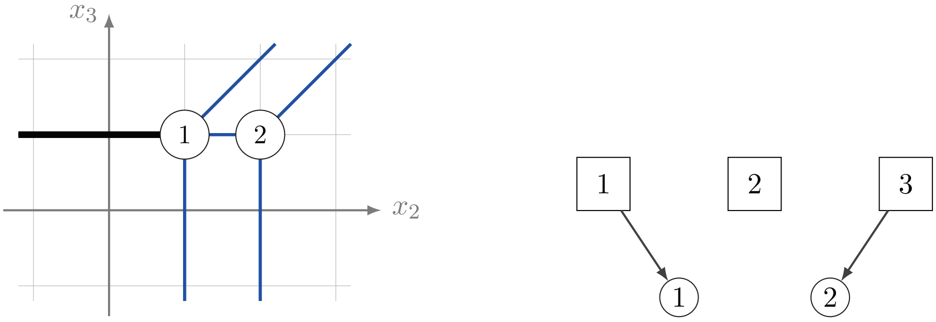


Figure 2.13: The left figure shows the cell decomposition induced by two apices which are not in general position. The right figure depicts a graph G corresponding to the black marked cell X_G on the left

Proposition 2.42 implies that the max-tropical cone $\operatorname{thalf}(V, \Psi)$ is compatible with the covector decomposition of $\mathbb{R}^d$ induced by V. Thus it makes sense to talk about the *covector decomposition* of a max-tropical cone with respect to a fixed system of defining tropical half-spaces. This is the polyhedral decomposition formed by the cells which happen to lie in the tropical cone. A tropical cone is *pure* if each cell in its covector decomposition which is maximal with respect to inclusion shares the same dimension. While the covector decomposition does depend on the choice of the defining inequalities, pureness does not.

The *tropical determinant* of a square matrix $W = (w_{ij}) \in \mathbb{T}_{\min}^{k\times k}$ is

$$\begin{aligned} \operatorname{tdet} W &= \bigoplus_{\sigma\in \operatorname{Sym}(k)} \bigodot_{i\in[k]} w_{i,\sigma(i)} \\ &= \min_{\sigma\in\operatorname{Sym}(k)} \left(w_{1,\sigma(1)} + w_{2,\sigma(2)} + \cdots + w_{k,\sigma(k)}\right) \ , \end{aligned} \tag{2.12}$$

which is the same as the solution to a minimum weight bipartite matching problem in the complete bipartite graph $[k] \times [k]$. The tropical determinant *vanishes* if the minimum in (2.12) equals ∞ or if it is attained at least twice. In [But10, §6.2.1] a square matrix whose tropical determinant does not vanish is called 'strongly regular'. A not necessarily square matrix is *tropically generic* if the tropical determinant of no square submatrix vanishes. A finite set of points is in *tropically general position* if any matrix whose columns (or rows) represent those points is tropically generic. Develin and Yu conjectured that a tropical cone is pure and full-dimensional if and only if it has a half-space description in which the apices of these half-spaces are in general position [DY07, Conjecture 2.11]. The next result confirms one of the two implications.

Theorem 2.46. *Let V and Ψ be as before. If V is tropically generic with respect to the tropical semiring $\mathbb{T}_{\min}$ then the* max*-tropical cone* $\operatorname{thalf}(V, \Psi)$ *is pure and full-dimensional.*

Proof. As in Proposition 2.42 we consider the graph class $\mathcal{G}_\Psi$. If we can show that each ordinary polyhedron $X_G(V) = \bigcap_{(i,j)\in G} S_i(v^{(j)})$ for $G \in \mathcal{G}_\Psi$ is either full-dimensional or empty then the claim follows. Proposition 2.30 implies that $X_G(V)$ is the projection of the weighted digraph polyhedron $Q(W\#G)$, which is a face of $\mathcal{E}(V) = Q(W)$. Assume that $Q(W\#G)$ is feasible. We have to show that $X_G(V)$ is full-dimensional, i.e., it suffices to show that $\dim Q(W\#G) = d$.

In view of Proposition 2.38 together with Corollary 2.39 this will follow if we can show that no two nodes in $[n]$ are contained in a cycle of weight zero in $\Gamma(W\#G)$. Aiming at an indirect argument we suppose that such a cycle exists. Let $D \sqcup N$ be the vertex set of the zero cycle $(d_1, n_1, d_2, n_2, \ldots, d_1)$. We have $|D| = |N|$. Then the arcs $(d_1, n_1), (d_2, n_2), \ldots$ form a perfect matching $\mathcal{M}$ in $D \times N$ whose weight $\sum_i v_{d_i,n_i}$ is minimal by Proposition 2.38. The complementary arcs $(n_1, d_2), (n_2, d_3), \ldots$ of the cycle yield a second matching whose weight is the same as the weight of $\mathcal{M}$ since the total weight of the cycle is zero. This entails that the minimum

$$\min_{\sigma} \sum_{i \in D} v_{i\sigma(i)} \ ,$$

where σ ranges over all bijections from D to N, is attained at least twice for the submatrix of V indexed by $D \times N$. Hence, the apices are not in general position, and this is the desired contradiction. $\square$

Since the matrix V is tropically generic it is immediate that $\operatorname{tcone}(V)$ has at least one full-dimensional cell; e.g., see [But10, Theorem 6.2.18] or [DS04, Proposition 24]. Yet, in general $\operatorname{tcone}(V)$ is not pure; see Example 2.31. The following shows that the reverse direction of Theorem 2.46 does not hold.

Example 2.47. For

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & \infty & \infty \\ 2 & 2 & \infty & 1 & 3 \end{pmatrix}$$

and Ψ as in Figure 2.14 we are interested in the max-tropical cone $C = \operatorname{thalf}(V, \Psi)$. Now C is pure, but the first two columns, $(0,3,2)^\top$ and $(0,2,2)^\top$, of the matrix V are not in general position with respect to min. Notice that each one of the apices of the three remaining tropical half-spaces can be moved without changing the feasible set C. However, the first two tropical half-spaces are *essential* in the sense that they occur in any exterior description of C.

A related conjecture from the same paper [DY07, Conjecture 2.10] was recently resolved by Allamigeon and Katz [AK17].

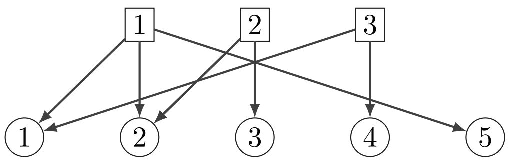


Figure 2.14: The graph Ψ for the max-tropical cone $C = \operatorname{thalf}(V, \Psi)$ from Example 2.47 and Figure 2.15

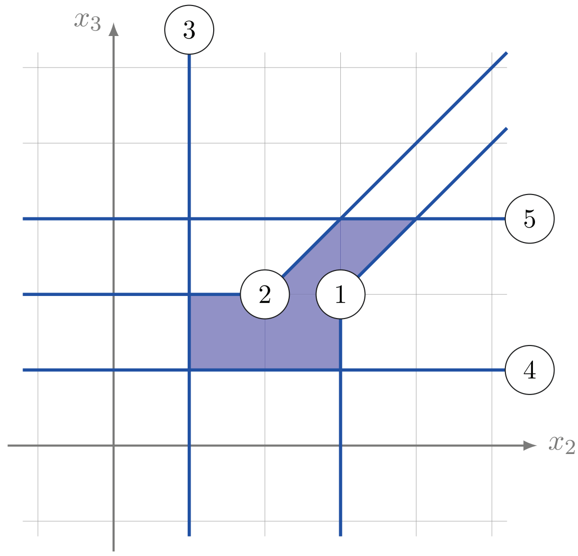


Figure 2.15: The pure max-tropical cone C from Example 2.47. The apices of any max-tropical half-space description are not in general position with respect to min

2.3.4 Polytropes

A *polytrope* is a tropical cone $P = \operatorname{tcone}(V)$ for $V \in \mathbb{R}^{d \times n}$, i.e., with a generating matrix with finite coefficients, which is also convex in the ordinary sense. In that case d generators suffice [DS04, Proposition 18] and [JK10, Theorem 7]. Therefore we may assume that $n = d$. From this we obtain $\operatorname{tcone}(V) = Q(V) = Q(V^*)$ in view of Remark 2.25, and thus any polytrope is a weighted digraph polyhedron; see also [JK10, Proposition 10]. Yet another argument for the same goes through Theorem 2.33 and Lemma 2.7. This is slightly more general as it takes ∞ coefficients into account. Moreover, the covector decomposition of P induced by the square matrix V has a single cell. Its projection to the tropical projective torus $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ is bounded, namely the polytrope P itself. The latter also gives a max-tropical exterior description. The polytropes are exactly the 'alcoved polytopes of type A' of Lam and Postnikov [LP07]. The weighted digraph polyhedra form the natural generalization to polyhedra which are not necessarily bounded. We sum up our discussion in the following statement.

Proposition 2.48. *Let $V \in \mathbb{T}_{\min}^{d \times n}$ such that the* min*-tropical cone* $\operatorname{tcone}(V)$ *is also convex in the ordinary sense. Then there is a $d \times d$-matrix U such that* $\operatorname{tcone}(V) = Q(U)$ *is a weighted digraph polyhedron.*

In the context of proving a hardness result on the vertex-enumeration of polyhedra given in terms of inequalities Khachiyan and al. [KBB+08] study the *circulation polytope* of the digraph Γ, which is the set of all points $u \in \mathbb{R}^\Gamma$ satisfying

$$\begin{aligned} \sum_{j:(i,j)\in\Gamma} u_{ij} - \sum_{\ell:(\ell,i)\in\Gamma} u_{\ell i} &= 0 \quad \text{for all } i \in [k] \\ \sum_{(i,j)\in\Gamma} u_{ij} &= 1 \\ 0 &\leq u_{ij} \quad \text{for all } (i,j) \in \Gamma \ . \end{aligned}$$

The support set $\{(i,j) \in \Gamma \mid u_{ij} \neq 0\}$ of a vertex of the circulation polytope defines a cycle in Γ. Hence, by Lemma 2.1, minimizing the weight function $\gamma(W)$ over the circulation polytope yields a certificate for the feasibility of $Q(W)$. Tran uses this approach to characterize the feasibility of polytropes in terms of ordinary inequalities [Tra17, §2.3].

2.3.5 Covector decompositions of tropical projective spaces

The *tropical projective space* $\mathbb{TP}^{d-1}_{\min}$ is defined as the quotient of $\mathbb{T}^d_{\min} \setminus \{(\infty, \infty, \ldots, \infty)^\top\}$ modulo $\mathbb{R}\mathbf{1}$. That is, its points are equivalence classes of vectors with coefficients in $\mathbb{T}_{\min} = \mathbb{R} \cup \{\infty\}$ with at least one finite entry, up to differences by a real constant; see [Mik06, Example 3.10]. The tropical projective space $\mathbb{TP}^{d-1}_{\min}$ is a natural compactification of the tropical projective torus $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. It is easy to see that the pair $(\mathbb{TP}^{d-1}_{\min}, \mathbb{R}^d/\mathbb{R}\mathbf{1})$ is homeomorphic to the pair of a $(d-1)$-simplex and its interior.

We assume that $V \in \mathbb{T}^{d\times n}_{\min}$ has no column identically ∞. Then V gives rise to a configuration of n labeled points in $\mathbb{TP}^{d-1}_{\min}$. The covector decomposition $\mathcal{T}(V)$ of $\mathbb{R}^d$ does not change if we add a real constant to the entries in any column. So it is an invariant of that point configuration, and, moreover, $\mathcal{T}(V)$ induces a covector decomposition of the tropical projective torus $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. Yet it makes sense to study tropical convexity and tropical cones also within the compactification $\mathbb{TP}^{d-1}_{\min}$. Our goal is to describe a decomposition of the tropical projective space into cells. Let Z be a proper subset of $[d]$. We consider the matrix obtained by removing from V all columns j for which there is an $i \in Z$ with $v_{ij} \neq \infty$. Each row of the resulting matrix with a label in Z has only ∞ as coefficients. Removing these rows yields yet another matrix, which we denote as $V(Z)$. Now this matrix induces a covector decomposition of the *boundary stratum*

$$\mathbb{TP}^{d-1}_{\min}(Z) = \left\{ (p_1, p_2, \ldots, p_d) \in \mathbb{TP}^{d-1}_{\min} \;\middle|\; p_i = \infty \text{ if and only if } i \in Z \right\} ,$$

which is a copy of the tropical projective torus of dimension $d-1-|Z|$. In particular, we have $\mathbb{TP}^{d-1}_{\min}(\emptyset) = \mathbb{R}^d/\mathbb{R}\mathbf{1}$. Notice that for the induced covector decomposition we keep the original labels of the columns and the rows.

For $K \subseteq [d]$ let $b^{(K)}$ be the vector in $\mathbb{T}^d_{\min}$ with

$$b^{(K)}_i = \begin{cases} 0 & \text{for } i \in [d] \setminus K \\ \infty & \text{for } i \in K \end{cases} .$$

Consider $u \in \mathbb{T}^d_{\min}$ and let $\operatorname{supp}(u) = \{i \in [d] \mid u_i \neq \infty\}$ be the support of u. Then the recession cone of the weighted digraph polyhedron $S_i(u)$ is given by the graph on

$[d]$ where the nodes in $[d] \setminus \operatorname{supp}(u)$ are isolated and there are arcs from the nodes in $\operatorname{supp}(u) \setminus \{i\}$ to i, see Equation (2.7). The supports of the rays of $S_i(u)$ are given by the sets in

$$\mathcal{K} \;=\; \{\, A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$$

where

$$\mathcal{A} = \emptyset \cup \{\, M \cup \{i\} \mid M \subseteq \operatorname{supp}(u)\} \quad \text{and} \quad \mathcal{B} = \big\{M \subseteq [d] \setminus \operatorname{supp}(u)\big\} \;.$$

Here, the sets in $\mathcal{A}$ correspond to the faces of the pointed part of the recession cone of $S_i(u)$ described by Theorem 2.11. The set $\mathcal{B}$ encodes rays arising from the lineality space of $S_i(u)$ which was characterized in Lemma 2.15.

So, it is natural to define

$$\overline{S_i(u)} \;=\; S_i(u) \cup \bigcup_{K \in \mathcal{K}} \Big(b^{(K)} + S_i(u)\Big)$$

where the '+'-operator denotes elementwise ordinary addition of $b^{(K)}$ and the set $S_i(u)$.

In the following we will frequently identify subsets of $(\mathbb{R} \cup \{\infty\})^d$ with their images modulo $\mathbb{R}/\mathbf{1}$. In particular, we will typically view $\overline{S_i(u)}$ with $u_i \neq \infty$ as a subset of $\mathbb{TP}^{d-1}_{\min}$.

Lemma 2.49. *The set $\overline{S_i(u)}$ for $u_i \neq \infty$ is the compactification of the sector $S_i(u)$ in $\mathbb{TP}^{d-1}_{\min}$.*

Consider a cell X_G in $\mathcal{T}(V)$ which contains a ray with support Z. Let M be the index set of the columns of V with $v_{ij} = \infty$ for all $i \in Z$ and $j \in M$. Construct the submatrix Y of V indexed by $([d] \setminus Z) \times M$ and the graph H as the restriction of G to the node set $([d] \setminus Z) \sqcup M$.

Lemma 2.50. *The cell decomposition of $\mathbb{TP}_{\min}(Z)^{d-1}$ induced by Y contains the cell $X_G(V) + b^{(Z)}$ which is given by the covector graph H in the decomposition of $\mathbb{R}^{([d]\setminus Z)}$ by Y. Furthermore, we obtain the alternative description*

$$b^{(Z)} + X_G(V) \;=\; b^{(Z)} + \bigcap_{(i,j) \in G} \overline{S_i(v^{(j)})} \;.$$

Proof. The second claim is merely a reformulation with the definition of $\overline{S_i(u)}$.

The first claim follows if we show that

$$\pi_Z(X_G(V)) \;=\; X_H(Y) \tag{2.13}$$

where π_Z is the projection onto the coordinates in $[d] \setminus Z$. Since any ray is generated by the minimal generators of the pointed part of the recession cone and the generators of the lineality space, at first we assume that Z is the support of a minimal generator of the pointed part of the recession cone. Setting $D'' = Z$ in Corollary 2.19 yields that every shortest path is already defined on $([d] \setminus Z) \sqcup M$. Furthermore, the support of a generator of the lineality space is given by a weak component by Lemma 2.15 what implies the same statement about the shortest paths for those generators.

Summarizing, equation (2.13) follows with Lemma 2.22. □

Theorem 2.51. *The union of the covector decompositions induced by the matrices $V(Z)$ where Z ranges over all proper subsets of $[d]$ yields a piecewise linear decomposition of $\mathbb{TP}^{d-1}_{\min}$.*

If the graph $\mathrm{B}(V)$ is weakly connected, then by Lemma 2.15 the intersection poset generated by the sets $\{\overline{S_i(u)} \mid u_i \neq \infty\}$ contains a 0-dimensional cell, whence that piecewise linear decomposition of $\mathbb{TP}^{d-1}_{\min}$ is a cell complex.

Proof. By definition as the common refinement of polyhedral complexes the covector decomposition of $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ induced by V is a polyhedral complex. The bounded cells are polytopes and therefore homeomorphic to closed balls. We need to check that the topology works out right for those cells which are unbounded in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. This is gotten from an induction on d as follows. In the base case $d = 1$ there is nothing to show since the tropical projective torus $\mathbb{R}^1/\mathbb{R}\mathbf{1}$ is a single point. For $d \geq 2$, by induction, we may assume that the covector decomposition induced on the closure

$$\left\{ (p_1, p_2, \ldots, p_d) \in \mathbb{TP}^{d-1}_{\min} \;\middle|\; p_i = \infty \text{ if } i \in Z \right\} ,$$

of $\mathbb{TP}^{d-1}_{\min}(Z)$ yields a cell decomposition if Z is not empty. Now consider $Z = \emptyset$ and let $X_G(V)$ be an unbounded cell with covector $G = (G_1, G_2, \ldots, G_d)$. By Lemma 2.50, the closure of $X_G(V)$ in $\mathbb{TP}^{d-1}_{\min}$ is the union of $X_G(V)$ with all the cells $X_H(V(Z'))$ where Z' ranges over the supports of the rays contained in $X_G(V)$. Here H is the covector which G induces on $\mathbb{TP}^{d-1}_{\min}(Z')$ by omitting those G_i with $i \in Z'$; this union is homeomorphic with a ball. The same argument also shows that intersections of cells are unions of cells. $\square$

By construction one can apply Lemma 2.28 also to the cells in the boundary of the tropical projective space to check for containment in $\operatorname{tcone}(V)$. Consider $z \in \mathbb{T}^d$ and let $\operatorname{supp}(z) = \{i \in [d] \mid z_i \neq \infty\}$ be its support.

Corollary 2.52. *The point z is contained in $\operatorname{tcone}(V)$ if and only if for every $i \in \operatorname{supp}(z)$ there is an index $s \in [n]$ with $z \in \overline{S_i(v^{(s)})}$ and $\operatorname{supp}(v^{(s)}) \subseteq \operatorname{supp}(z)$. A point $z \in \mathbb{T}^d$ is contained in $\operatorname{tcone}(V)$ if and only if for every $i \in [d]$ there is an index $s \in [n]$ with $z \in \overline{S_i(v^{(s)})}$.*

Example 2.53. Let

$$V' = \begin{pmatrix} 0 & 0 & 0 & 0 & \infty & \infty \\ 1 & 0 & \infty & \infty & 0 & \infty \\ 2 & -1 & \infty & \infty & \infty & 0 \end{pmatrix} ,$$

where $d = 3$ and $n = 6$. The third and fourth columns of V' are the same. Notice that the first three columns correspond to the matrix V from Example 2.31. With $Z = \{1\}$ we obtain the matrix

$$V'(Z) = \begin{array}{cc} \begin{pmatrix} 0 & \infty \\ \infty & 0 \end{pmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \\ \begin{array}{cc} 5 & \;\;6 \end{array} & \end{array} ,$$

where we keep the original row and column labels. The one-dimensional tropical projective torus $B = \mathbb{TP}^2_{\min}(\{1\})$ is trivially subdivided; its covector reads $(\bullet, 5, 6)$. To denote cells in the boundary we use the symbol $\bullet$ at the component corresponding to an apex to mark if the cell is in a common boundary stratum with this apex. The union of the 1-dimensional ball B and the unbounded cell in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ with covector $(1234, -, -)$ yields the 2-dimensional cell with covector $(1234, 5, 6)$ in the covector decomposition of $\mathbb{TP}^2_{\min}$ induced by V'; see Figure 2.16 and compare with Figure 2.11.

Notice that, while the tropical projective torus works for min and max alike, the definition of the tropical projective space does depend on the choice of the tropical addition.

2.3.6 Arrangements of tropical halfspaces

So far we associated with a matrix $V \in \mathbb{T}^{d\times n}_{\min}$ the covector decompositions of $\mathbb{R}^d$ and $\mathbb{TP}^{d-1}_{\min}$, respectively, and Theorem 2.33 describes the min-tropical cone $\operatorname{tcone}(V)$ as a union of their cells. Choose a subgraph Ψ of the complete bipartite graph $[d] \times [n]$ (with arcs directed from $[d]$ to $[n]$) as in (2.11). This gives rise to the max-tropical cone $\operatorname{thalf}(V, \Psi) = \bigcap_{j\in[n]} \bigcup_{(i,j)\in\Psi} S_i(v^{(j)})$, which again is a union of cells from the same covector decomposition. Here we want to describe yet another cell decomposition of $\mathbb{R}^d$ (or $\mathbb{TP}^{d-1}_{\min}$), which was introduced in [ABGJ15, §3.2].

For this, we introduce the max-tropical cone with boundary

$$\overline{\operatorname{thalf}}(V, \Psi) \;=\; \bigcap_{j\in[n]} \bigcup_{(i,j)\in\Psi} \overline{S_i(v^{(j)})} \enspace .$$

For a vector $\epsilon \in \{\pm\}^n$ of n signs we consider the directed bipartite graph

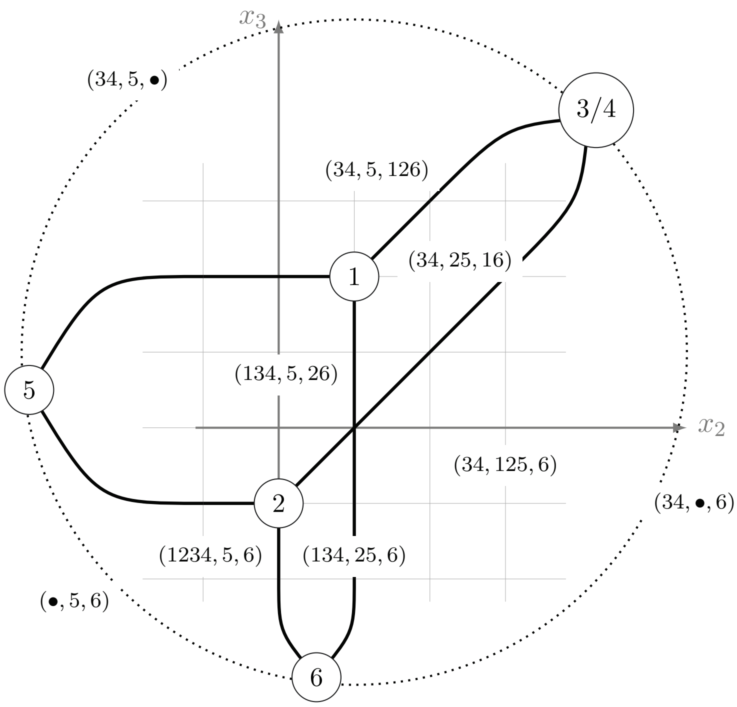
$$\Psi_\epsilon \;=\; \Big\{ (i,j) \in [d] \times [n] \;\Big|\; \big((i,j) \in \Psi \text{ and } \epsilon_j = +\big) \text{ or } \big((i,j) \notin \Psi \text{ and } \epsilon_j = -\big) \Big\} \enspace .$$

The construction of Ψ_ϵ from Ψ amounts to taking the complementary arcs incident to each node $j \in [n]$ with $\epsilon_j = -$. We call the max-tropical cone $\overline{\operatorname{thalf}}(V, \Psi_\epsilon)$ the *inversion* of $\overline{\operatorname{thalf}}(V, \Psi)$ with respect to ϵ. As a subset of $\mathbb{TP}^{d-1}_{\min}$ the inversion may be empty or not. In the latter case $\overline{\operatorname{thalf}}(V, \Psi_\epsilon)$ is the *signed cell* with respect to V, Ψ and ϵ. Each generic point, i.e., a point which does not lie on any of the max-tropical hyperplanes whose apices are columns of V, is contained in a unique signed cell. The trivial inversion with respect to $\epsilon = ++\cdots+$ is the tropical cone $\overline{\operatorname{thalf}}(V, \Psi)$ itself. Each signed cell is a union of cells of the covector decomposition. So Theorem 2.51 together with Proposition 2.42 entails the following.

Corollary 2.54. *The signed cells $\overline{\operatorname{thalf}}(V, \Psi_\epsilon)/\mathbb{R}\mathbf{1}$, where ϵ ranges over all choices of sign vectors, generate a piecewise linear decomposition of $\mathbb{TP}^{d-1}_{\min}$.*

Furthermore, a cell with graph G in the covector decomposition of $\mathbb{TP}^{d-1}_{\min}$ by V is contained in a cell $\overline{\operatorname{thalf}}(V, \Psi_\epsilon)$ if and only if $\Psi_\epsilon \cap G$ has no isolated node.

The decomposition into signed cells is a tropical analogue of the decomposition into polyhedral cells defined by an ordinary affine hyperplane arrangement. As in Theorem 2.51



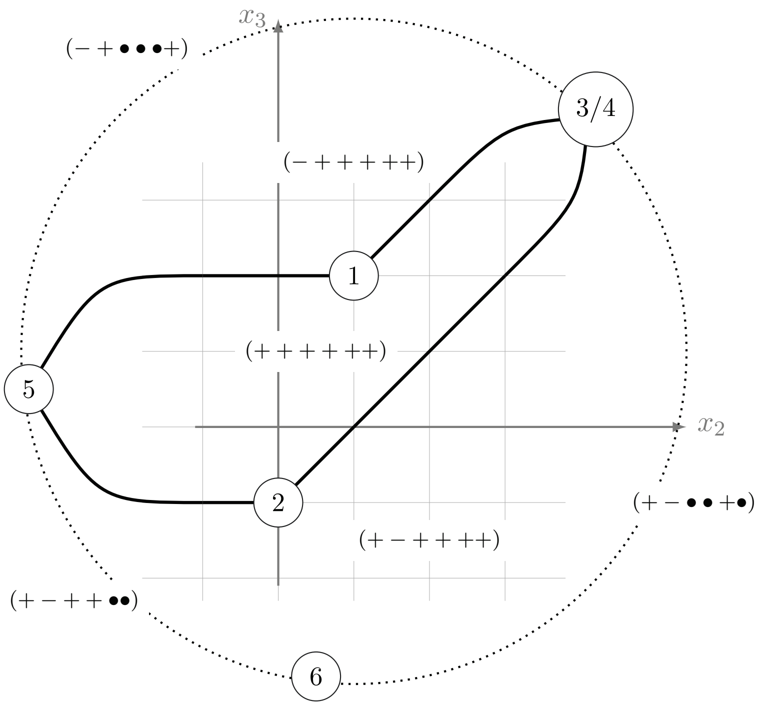


Figure 2.16: Covector decomposition (top) and signed cell decomposition (bottom) in the tropical projective plane

that piecewise linear decomposition is a cell complex, provided that $\mathrm{B}(V)$ is weakly connected.

Example 2.55. Figure 2.16 shows the signed cell decomposition of $\mathbb{TP}^2_{\min}$ induced by the matrix V from Example 2.31 with the extra columns $(\infty, 0, \infty)^\top$ and $(\infty, \infty, 0)^\top$ and the directed bipartite graph $\Psi \subset \{1,2,3\} \times \{1,2,3,4,5\}$ with the six directed edges $(1,1), (2,1), (3,2), (1,3), (2,4), (3,5)$. The six signed cells correspond to the sign vectors $++++++$, $-+++++$ and $+-++++$. The remaining 29 inversions are empty. Finally, the three inversions $-+\bullet\bullet\bullet+$, $+-\bullet\bullet+\bullet$ and $+-++\bullet\bullet$ form a decomposition of the boundary of the tropical projective plane.

3 Linear programming over Puiseux fractions

This chapter is taken from the paper "Linear Programs and Convex Hulls Over Fields of Puiseux Fractions" [JLLS16] by Michael Joswig, Georg Loho, Benjamin Lorenz and Benjamin Schröter. It is published in "Mathematical Aspects of Computer and Information Sciences: 6th International Conference, MACIS 2015", pages 429–445. The final publication is available at Springer via
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3.1 Introduction

It is well known and not difficult to see that the standard concepts from linear programming (LP), e.g., the Farkas Lemma and LP duality, carry over to an arbitrary ordered field; e.g., see [CK70, Section II] or [Jer73, §2.1]. Traces of this can already be found in Dantzig's monograph [Dan63, Chapter 22]. This entails that any algorithm whose correctness rests on these LP corner stones is valid over any ordered field. In particular, this holds for the simplex method and usual convex hull algorithms. A classical construction, due to Hilbert, turns a field of rational functions, e.g., with real coefficients, into an ordered field; see [vdW93, §147]. In [Jer73] Jeroslow discussed these fields in the context of linear programming in order to provide a rigorous foundation of the so-called "big M method". The purpose of this chapter is to describe the implementation of the simplex method and of a convex hull algorithm over fields of this kind in the open source software system `polymake` [GJ00] as well as the related mathematical concepts in relation with tropical linear programming.

Hilbert's ordered field of rational functions is a subfield of the field of formal Puiseux series $\mathbb{R}\{\!\{t\}\!\}$ with real coefficients. The latter field is real-closed by the Artin–Schreier Theorem [SGHL07, Theorem 12.10]; by Tarski's Principle (cf. [Tar48]) this implies that $\mathbb{R}\{\!\{t\}\!\}$ has the same first order properties as the reals. The study of polyhedra over $\mathbb{R}\{\!\{t\}\!\}$ is motivated by tropical geometry [DY07], especially tropical linear programming [ABGJ15]. The connection of the latter with classical linear programming has recently lead to a counter-example [ABGJ14b] to a "continuous analogue of the Hirsch conjecture" by Deza, Terlaky and Zinchenko [DTZ09]. In terms of parameterized linear optimization (and similarly for the convex hull computations) our approach amounts to computing with sufficiently large (or, dually, sufficiently small) positive real numbers. Here we do *not* consider the more general algorithmic problem of stratifying the parameter space to describe all optimal solutions of a linear program for *all* choices of parameters; see, e.g., [JKM08] for work into that direction.

This chapter is organized as follows. We start out with summarizing known facts on ordered fields. Then we describe a specific field, $\mathbb{Q}\{t\}$, which is the field of rational functions with rational coefficients and rational exponents. This is a subfield of $\mathbb{Q}\{\{t\}\}$, which we call the field of *Puiseux fractions*. It is our opinion that this is a subfield of the formal Puiseux series which is particularly well suited for exact computations with (some) Puiseux series; see [MC13] for an entirely different approach. In the context of tropical geometry Markwig [Mar10] constructed a much larger field, which contains the classical Puiseux series as a proper subfield. For our applications it is relevant to study the evaluation of Puiseux fractions at sufficiently large rational numbers. In Section 3.3 we develop what this yields for comparing convex polyhedra over $\mathbb{R}\{\{t\}\}$ with ordinary convex polyhedra over the reals. The tropical geometry point of view enters the picture in Section 2.3. We give an algorithm for solving the dual tropical convex hull problem, i.e., the computation of generators of a tropical cone from an exterior description. Allamigeon, Gaubert and Goubault gave a combinatorial algorithm for this in [AGG13], while we use a classical (dual) convex hull algorithm and apply the valuation map. The benefit of our approach is more geometric than in terms of computational complexity: in this way we will be able to study the fibers of the tropicalization map for classical versus tropical cones for specific examples. Section 3.5 sketches the `polymake` implementation of the Puiseux fraction arithmetic and the LP and convex hull algorithms. The LP solver is a dual simplex algorithm with steepest edge pivoting, and the convex hull algorithm is the classical beneath-and-beyond method [Ede87] [Jos03]. An overview with computational results is given in Section 3.6.

3.2 Ordered Fields and Rational Functions

A field $\mathbb{F}$ is *ordered* if there is a total ordering $\leq$ on the set $\mathbb{F}$ such that for all $a, b, c \in \mathbb{F}$ the following conditions hold:

(i) if $a \leq b$ then $a + c \leq b + c$,

(ii) if $0 \leq a$ and $0 \leq b$ then $0 \leq a \cdot b$.

Any ordered field necessarily has characteristic zero. Examples include the rational numbers $\mathbb{Q}$, the reals $\mathbb{R}$ and any subfield in between.

Given an ordered field $\mathbb{F}$ we can look at the ring of univariate polynomials $\mathbb{F}[t]$ and its quotient field $\mathbb{F}(t)$, the field of rational functions in the indeterminate t with coefficients in $\mathbb{F}$. On the ring $\mathbb{F}[t]$ we obtain a total ordering by declaring $p < q$ whenever the leading coefficient of $q - p$ is a positive element in $\mathbb{F}$. Extending this ordering to the quotient field by letting

$$\frac{u}{v} < \frac{p}{q} \;:\Longleftrightarrow\; uq < vp \ ,$$

where the denominators v and q are assumed positive, turns $\mathbb{F}(t)$ into an ordered field; see, e.g., [vdW93, §147]. This ordered field is called the "Hilbert field" by Jeroslow [Jer73].

By definition, the exponents of the polynomials in $\mathbb{F}[t]$ are natural numbers. However, conceptually, there is no harm in also taking negative integers or even arbitrary rational numbers as exponents into account, as this can be reduced to the former by clearing denominators and subsequent substitution. For example,

$$\frac{2t^{3/2} - t^{-1}}{1 + 3t^{-1/3}} \;=\; \frac{2t^{5/2} - 1}{t + 3t^{2/3}} \;=\; \frac{2s^{15} - 1}{s^6 + 3s^4} \;, \tag{3.1}$$

where $s = t^{1/6}$. In this way that fraction is written as an element in the field $\mathbb{Q}(t^{1/6})$ of rational functions in the indeterminate $s = t^{1/6}$ with rational coefficients. Further, if $p \in \mathbb{F}(t^{1/\alpha})$ and $q \in \mathbb{F}(t^{1/\beta})$, for natural numbers α and β, then the sum $p + q$ and the product $p \cdot q$ are contained in $\mathbb{F}(t^{1/\gcd(\alpha,\beta)})$. This shows that the union

$$\mathbb{F}\{t\} \;=\; \bigcup_{\nu \geq 1} \mathbb{F}(t^{1/\nu}) \tag{3.2}$$

is again an ordered field. We call its elements *Puiseux fractions*. The field $\mathbb{F}\{t\}$ is a subfield of the field $\mathbb{F}\{\!\{t\}\!\}$ of *formal Puiseux series*, i.e., the formal power series with rational exponents of common denominator. For an algorithmic approach to general Puiseux series see [MC13].

The map val which sends the rational function p/q, where $p, q \in \mathbb{F}[t^{1/\nu}]$, to the number $\deg_t p - \deg_t q$ defines a non-Archimedean valuation on $\mathbb{F}(t)$. Here we let $\mathrm{val}(0) = -\infty$. As usual the *degree* is the largest occurring exponent. The valuation map extends to Puiseux series. More precisely, for $f, g \in \mathbb{F}\{t\}$ we have the following:

(i) $\mathrm{val}(f \cdot g) = \mathrm{val}(f) + \mathrm{val}(g)$,

(ii) $\mathrm{val}(f + g) \leq \max(\mathrm{val}(f), \mathrm{val}(g))$.

If $\mathbb{F} = \mathbb{R}$ is the field of real numbers we can evaluate a Puiseux fraction $f \in \mathbb{R}\{t\}$ at a real number τ to obtain the real number $f(\tau)$. This map is defined for all $\tau > 0$ except for the finitely many poles, i.e., zeros of the denominator. Restricting the evaluation to positive numbers is necessary since we are allowing rational exponents. The valuation map satisfies the equation

$$\lim_{\tau \to \infty} \log_\tau |f(\tau)| \;=\; \mathrm{val}(f) \;. \tag{3.3}$$

That is, seen on a logarithmic scale, taking the valuation of f corresponds to interpreting t like an infinitesimally large number. Reading the valuation map in terms of the limit (3.3) is known as *Maslov dequantization*, see [Mas86].

Occasionally, it is also useful to be able to interpret t as a *small* infinitesimal. To this end, one can define the *dual degree* $\deg^*$, which is the smallest occurring exponent. This gives rise to the *dual valuation* map $\mathrm{val}^*(p/q) = \deg_t^* p - \deg_t^* q$ which yields

$$\mathrm{val}^*(f + g) \;\geq\; \min(\mathrm{val}^*(f), \mathrm{val}^*(g)) \quad \text{and} \quad \lim_{\tau \to 0} \log_\tau |f(\tau)| \;=\; \mathrm{val}^*(f) \;.$$

Changing from the primal to the dual valuation is tantamount to substituting t by t^{-1}.

Remark 3.1. The valuation theory literature often employs the dual definition of a valuation. The equation (3.3) is the reason why we usually prefer to work with the primal.

Up to isomorphism of valuated fields the valuation on the field $\mathbb{F}(t)$ of rational functions is unique, e.g., see [vdW93, §147]. As a consequence the valuation on the slightly larger field of Puiseux fractions is unique, too.

To close this section let us look at the algorithmically most relevant case $\mathbb{F} = \mathbb{Q}$. Then, in general, the evaluation map sends positive rationals to not necessarily rational numbers, again due to fractional exponents. By clearing denominators in the exponents one can see that evaluating at $\sigma > 0$ ends up in the totally real number field $\mathbb{Q}(\sqrt[\nu]{\sigma})$ for some positive integer ν. For instance, evaluating the Puiseux fraction from Example (3.1) would give an element of $\mathbb{Q}(\sqrt[6]{\sigma})$.

3.3 Parameterized Polyhedra

Consider a matrix $A \in \mathbb{F}\{t\}^{m\times(d+1)}$. Then the set

$$C \ := \ \left\{ x \in \mathbb{F}\{t\}^{d+1} \;\middle|\; A \cdot x \geq 0 \right\}$$

is a polyhedral cone in the vector space $\mathbb{F}\{t\}^{d+1}$. Equivalently, C is the set of feasible solutions of a linear program with $d+1$ variables over the ordered field $\mathbb{F}\{t\}$ with m homogeneous constraints, the rows of A. The Farkas–Minkowski–Weyl Theorem establishes that each polyhedral cone is finitely generated. A proof for this result on polyhedral cones over the reals can be found in [Zie95, §1.3 and §1.4] under the name "Main theorem for cones". It is immediate to verify that the arguments given hold over any ordered field. Therefore, there is a matrix $B \in \mathbb{F}\{t\}^{(d+1)\times n}$, for some $n \in \mathbb{N}$, such that

$$C \ = \ \{ B \cdot a \mid a \in \mathbb{F}\{t\}^n, \, a \geq 0 \} \quad . \tag{3.4}$$

The columns of B are points and the cone C is the non-negative linear span of those.

Let L be the *lineality space* of C, i.e., L is the unique maximal linear subspace of $\mathbb{F}\{t\}^{d+1}$ which is contained in C. If $\dim L = 0$ the cone C is *pointed*. Otherwise, the set C/L is a pointed polyhedral cone in the quotient space $\mathbb{F}\{t\}^{d+1}/L$. A *face* of C is the intersection of C with a supporting hyperplane. The faces are partially ordered by inclusion. Each face contains the lineality space. Adding the entire cone C as an additional top element we obtain a lattice, the *face lattice* of C. The maximal proper faces are the *facets* which form the co-atoms in the face lattice. The *combinatorial type* of C is the isomorphism class of the face lattice (e.g., as a partially ordered set). Notice that our definition says that each cone is combinatorially equivalent to its quotient modulo its lineality space.

Picking a positive element τ yields matrices $A(\tau) \in \mathbb{F}^{m\times(d+1)}$ and $B(\tau) \in \mathbb{F}^{(d+1)\times n}$ as well as a polyhedral cone $C(\tau) = \{x \in \mathbb{F}^{d+1} \mid A(\tau) \cdot x \geq 0\}$ by evaluating the Puiseux fractions at the parameter τ. Here and below we will assume that τ avoids the at most finitely many poles of the $(m+n)\cdot(d+1)$ coefficients of A and B.

Theorem 3.2. *There is a positive element $\tau_0 \in \mathbb{F}$ so that for every $\tau > \tau_0$ we have*

$$C(\tau) \ = \ \{ B(\tau) \cdot \alpha \mid \alpha \in \mathbb{F}^n, \, \alpha \geq 0 \} \quad ,$$

and evaluating at τ maps the lineality space of C to the lineality space of $C(\tau)$. Moreover, the polyhedral cones C and $C(\tau)$ over $\mathbb{F}\{t\}$ and $\mathbb{F}$, respectively, share the same combinatorial type.

Proof. First we show that an orthogonal basis of the lineality space L evaluates to an orthogonal basis of the lineality space of $C(\tau)$. For this, consider two vectors $x, y \in \mathbb{F}\{t\}^{d+1}$ and pick τ large enough to avoid their poles and zeros. Then, the scalar product of x and y vanishes if and only if the scalar product of $x(\tau)$ and $y(\tau)$ does. Hence, the claim follows.

Now we can assume that the polyhedral cone C is pointed, i.e., it does not contain any linear subspace of positive dimension. If this is not the case the subsequent argument applies to the quotient C/L.

Employing orthogonal bases, as for the lineality spaces above, shows that the evaluation maps the linear hull of C to the linear hull of $C(\tau)$, preserving the dimension. So we may assume that C is full-dimensional, as otherwise the arguments below hold in the linear hull of C.

Let $\ell \leq \binom{m}{d}$ be the number of d-element sets of linearly independent rows of the matrix A. For each such set of rows the set of solutions to the corresponding homogeneous system of linear equations is a one-dimensional subspace of $\mathbb{F}\{t\}^{(d+1)}$. For each such system of homogeneous linear equations pick two non-zero solutions, which are negatives of each other. We arrive at 2ℓ vectors in $\mathbb{F}\{t\}^{(d+1)}$ which we use to form the columns of the matrix $Z \in \mathbb{F}\{t\}^{(d+1)\times 2\ell}$.

By the Farkas–Minkowski–Weyl theorem, we may assume that the columns of B from (3.4) only consist of the rays of C and that the rays of C form a subset of the columns of Z. In particular, the columns of B occur in Z. Since the cone C is pointed, the matrix B contains at most one vector from each opposite pair of the columns of Z. This entails that B has at most ℓ columns.

Further, the real matrix $Z(\tau)$ contains all rays of $C(\tau)$ for each τ that avoids the poles of A and Z. In the following, we want to show that those columns of $Z(\tau)$ which form the rays of $C(\tau)$ are exactly the columns of $B(\tau)$.

We define $s(j,k) \in \mathbb{F}\{t\}$ to be the scalar product of the jth row of A and the kth column of Z. The $m \cdot 2\ell$ signs of the scalar products $s(j,k)$, for $j \in [m]$ and $k \in [2\ell]$, form the *chirotope* of the linear hyperplane arrangement defined by the rows of A (in fact, due to taking two solutions for each homogenous system of linear equations, we duplicate the information of the chirotope). For almost all $\tau \in \mathbb{F}$ evaluating the Puiseux fractions $s(j,k)$ at τ yields an element of $\mathbb{F}$. For sufficiently large τ the sign of $s(j,k)$ agrees with its evaluation. This follows from the definition of the ordering on $\mathbb{F}\{t\}$, cf. [Jer73, Proposition, §1.3].

Let $\tau_0 \in \mathbb{F}$ be larger than all the at most finitely many poles of A and Z. Further, let τ_0 be large enough such that the chirotope of $A(\tau)$ agrees with the chirotope of A for all $\tau > \tau_0$.

By construction the rays of C correspond to the non-negative columns of the chirotope whose support, given by the 0 entries, is maximal with respect to inclusion; these are exactly the columns of B. The corresponding columns of the chirotope of $A(\tau)$, for

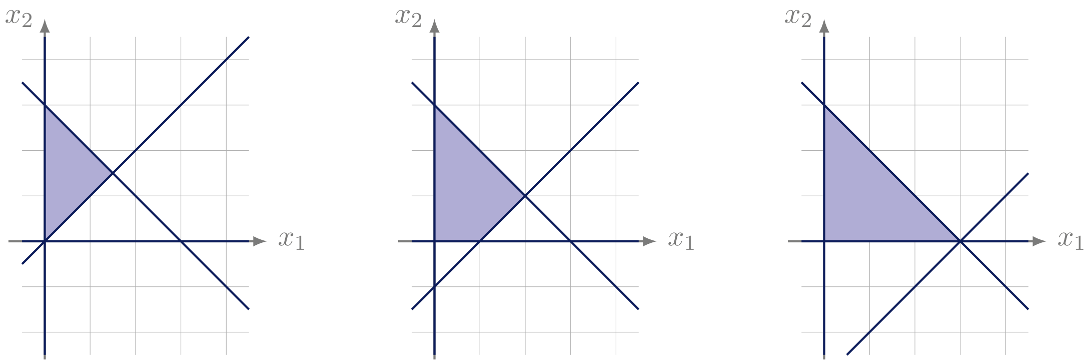

Figure 3.1: Polygon depending on a real parameter as defined in Example 3.3

$\tau > \tau_0$, yield the rays of $C(\tau)$, which, hence, are the columns of $B(\tau)$.

The same holds for the facets of C and $C(\tau)$. The facets of C correspond to the non-negative rows of the chirotope whose support, given by the 0 entries, is maximal with respect to inclusion.

Now the claim follows since the face lattice of a polyhedral cone is determined by the incidences between the facets and the rays. □

A statement related to Theorem 3.2 occurs in Benchimol's PhD thesis [Ben14]. The Proposition 5.12 in [Ben14] discusses the combinatorial structure of tropical polyhedra (arising as the feasible regions of tropical linear programs). Yet here we consider the relationship between the combinatorial structure of Puiseux polyhedra and their evaluations over the reals. As in the proof of [Ben14, Proposition 5.12] we could derive an explicit upper bound on the optimal τ_0. To this end one can estimate the coefficients of the Puiseux fractions in Z, which are given by determinantal expressions arising from submatrices of A. Their poles and zeros are bounded by Cauchy bounds (e.g., see [RS02, Thm. 8.1.3]) depending on those coefficients. We leave the details to the reader.

A *convex polyhedron* is the intersection of finitely many linear inequalities. It is a called a *polytope* if it is bounded. Restricting to cones allows a simple description in terms of homogeneous linear inequalities. Yet this encompasses arbitrary polytopes and polyhedra, as they can equivalently be studied through their homogenizations. In fact, all implementations in `polymake` are based on this principle. For further reading we refer to [Zie95, §1.5]. We visualize Theorem 3.2 with a very simple example.

Example 3.3. Consider the polytope P in $\mathbb{R}\{t\}^2$ for large t defined by the four inequalities

$$x_1, x_2 \geq 0, \qquad x_1 + x_2 \leq 3, \qquad x_1 - x_2 \leq t \ .$$

The evaluations at $\tau \in \{0, 1, 3\}$ are depicted in Figure 3.1. For $\tau = 0$ we obtain a triangle, for $\tau = 1$ a quadrangle and for $\tau \geq 3$ a triangle again. The latter is the combinatorial type of the polytope P over the field of Puiseux fractions with real coefficients.

Corollary 3.4. *The set of combinatorial types of polyhedral cones which can be realized over $\mathbb{F}\{t\}$ is the same as over $\mathbb{F}$.*

Proof. One inclusion is trivial since $\mathbb{F}$ is a subfield of $\mathbb{F}\{t\}$. The other inclusion follows from the preceding result. $\square$

For $A \in \mathbb{F}\{t\}^{m\times d}$, $b \in \mathbb{F}\{t\}^m$ and $c \in \mathbb{F}\{t\}^d$ we consider the linear program $\mathrm{LP}(A, b, c)$ over $\mathbb{F}\{t\}$ which reads as

$$\begin{array}{ll} \text{maximize} & c^\top \cdot x \\ \text{subject to} & A \cdot x = b\,,\ x \geq 0\ . \end{array} \tag{3.5}$$

For each positive $\tau \in \mathbb{F}$ (which avoids the poles of the Puiseux fractions which arise as coefficients) we obtain a linear program $\mathrm{LP}(A(\tau), b(\tau), c(\tau))$ over $\mathbb{F}$. Theorem 3.2 now has the following consequence for parametric linear programming.

Corollary 3.5. *Let $x^* \in \mathbb{F}\{t\}^d$ be an optimal solution to the LP (3.5) with optimal value $v \in \mathbb{F}\{t\}$. Then there is a positive element $\tau_0 \in \mathbb{F}$ so that for every $\tau > \tau_0$ the vector $x^*(\tau)$ is an optimal solution for* $\mathrm{LP}(A(\tau), b(\tau), c(\tau))$ *with optimal value $v(\tau)$.*

The above corollary was proved by Jeroslow [Jer73, §2.3]. His argument, based on controlling signs of determinants, is essentially a local version of our Theorem 3.2. Moreover, determining all the rays of a polyhedral cone can be reduced to solving sufficiently many LPs. This could also be exploited to derive another proof of Theorem 3.2 from Corollary 3.5.

Remark 3.6. It is worth to mention the special case of a linear program over the field $\mathbb{F}\{t\}$, where the coordinates of the linear constraints, in fact, are elements of the field $\mathbb{F}$ of coefficients, but the coordinates of the linear objective function are arbitrary elements in $\mathbb{F}\{t\}$. That is, the feasible domain is a polyhedron, P, over $\mathbb{F}$. Evaluating the objective function at some $\tau \in \mathbb{F}$ makes one of the vertices of P optimal. Solving for all values of τ, in general, amounts to computing the entire normal fan of the polyhedron P. This is equivalent to solving the dual convex hull problem over $\mathbb{F}$ for the given inequality description of P; see also [JKM08]. Here we restrict our attention to solving parametric linear programs via Corollary 3.5.

The next example is a slight variation of a construction of Goldfarb and Sit [GS79]. This is a class of linear optimization problems on which certain versions of the simplex method perform poorly.

Example 3.7. We fix $d > 1$ and pick a positive $\delta \leq \frac{1}{2}$ as well as a positive $\varepsilon < \frac{\delta}{2}$. Consider the linear program

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{d} \delta^{d-i} x_i & \\ \text{subject to} & \quad 0 \leq x_1 \leq \varepsilon^{d-1} & \\ & x_{j-1} \leq \delta x_j \leq \varepsilon^{d-j}\delta - x_{j-1} & \text{for } 2 \leq j \leq d\ . \end{array}$$

The feasible region is combinatorially equivalent to the d-dimensional cube. Applying the simplex method with the "steepest edge" pivoting strategy to this linear program with

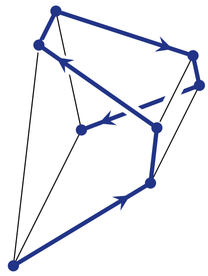

Figure 3.2: The 3-dimensional Goldfarb–Sit cube.

the origin as the start vertex visits all the 2^d vertices. Moreover, the vertex-edge graph with the orientation induced by the objective function is isomorphic to (the oriented vertex-edge graph of) the Klee–Minty cube [KM72]. See Figure 3.2 for a visualization of the 3-dimensional case.

We may interpret this linear program over the reals or over $(\mathbb{R}\{\delta\})\{\varepsilon\}$, the field of Puiseux fractions in the indeterminate ε with coefficients in the field $\mathbb{R}\{\delta\}$. This depends on whether we want to view δ and ε as indeterminates or as real numbers. Here we consider the ordering induced by the dual valuation val*, i.e., δ and ε are *small* infinitesimals, where $\varepsilon \ll \delta$. Two more choices arise from considering ε a constant in $\mathbb{R}\{\delta\}$ or, conversely, δ a constant in $\mathbb{R}\{\varepsilon\}$. Note that our constraints on δ and ε are feasible in all four cases.

Our third and last example is a class of linear programs occurring in [ABGJ14b]. For these the central path of the interior point method with a logarithmic barrier function has a total curvature which is exponential as a function of the dimension.

Example 3.8. Given a positive integer r, we define a linear program over the field $\mathbb{Q}\{t\}$ (with the primal valuation) in the $2r+2$ variables $u_0, v_0, u_1, v_1, \ldots, u_r, v_r$ as follows:

$$\begin{array}{rl} \text{minimize} & v_0 \\ \text{subject to} & u_0 \leq t\,,\ v_0 \leq t^2 \\ & \left.\begin{array}{l} u_i \leq t u_{i-1}\,,\ u_i \leq t v_{i-1} \\ v_i \leq t^{1-\frac{1}{2^i}}(u_{i-1}+v_{i-1}) \end{array}\right\} \qquad \text{for } 1 \leq i \leq r \\ & u_r \geq 0\,,\ v_r \geq 0\,. \end{array}$$

Here it would be interesting to know the exact value for the optimal τ_0 in Theorem 3.2, as a function of r. Experimentally, based on the method described below, we found $\tau_0 = 1$ for $r = 1$ and $\tau_0 = 2^{2^{r-1}}$ for r at most 5. We conjecture the latter to be the true bound in general.

To find the optimal bound for a given constraint matrix A we can use the following method. One can solve the dual convex hull problem for the cone C, which is the feasible region in homogenized form, to obtain a matrix B whose columns are the rays of C. This also yields a submatrix of A corresponding to the rows which define facets of C. Without loss of generality we may assume that that submatrix is A itself. Let τ_0 be the largest zero or pole of any (Puiseux fraction) entry of the matrix $A \cdot B$. Then for every value $\tau > \tau_0$ the sign patterns of $(A \cdot B)(\tau)$ and $A \cdot B$ coincide, and so do the combinatorial types of C and $C(\tau)$. Determining the zeros and poles of a Puiseux fraction amounts to factorizing univariate polynomials.

3.4 Tropical Dual Convex Hulls

Tropical geometry is the study of the piecewise linear images of algebraic varieties, defined over a field with a non-Archimedean valuation, under the valuation map; see [MS15] for an overview. The motivation for research in this area comes from at least two different directions. First, tropical varieties still retain a lot of interesting information about their classical counterparts. Therefore, passing to the tropical limit opens up a path for combinatorial algorithms to be applied to topics in algebraic geometry. Second, the algebraic geometry perspective offers opportunities for optimization and computational geometry. Here we will discuss how classical convex hull algorithms over fields of Puiseux fractions can be applied to compute tropical convex hulls; see [Jos09] for a survey on the subject; a standard algorithm is the tropical double description method of [AGG10].

The *tropical semiring* $\mathbb{T}$ consists of the set $\mathbb{R} \cup \{-\infty\}$ together with $u \oplus v = \max(u, v)$ as the addition and $u \odot v = u + v$ as the multiplication. Extending these operations to vectors turns $\mathbb{T}^{d+1}$ into a semimodule. A *tropical cone* is the sub-semimodule

$$\operatorname{tcone}(G) = \{\, \lambda_1 \odot g_1 \oplus \cdots \oplus \lambda_n \odot g_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{T}\}$$

generated from the columns $g_1, \ldots, g_n$ of the matrix $G \in \mathbb{T}^{(d+1)\times n}$. Similar to classical cones, tropical cones admit an exterior description [GK11]. It is known that every tropical cone is the image of a classical cone under the valuation map $\operatorname{val}\colon \mathbb{R}\{\{t\}\} \to \mathbb{T}$; see [DY07]. Based on this idea, we present an algorithm for computing generators of a tropical cone from a description in terms of tropical linear inequalities; see Algorithm 1 below.

Before we can start to describe that algorithm we first need to discuss matters of general position in the tropical setting. The *tropical determinant* of a square matrix $U \in \mathbb{T}^{\ell\times\ell}$ is given by

$$\operatorname{tdet}(U) \;=\; \bigoplus_{\sigma\in S_\ell} u_{1\pi(1)} \odot \cdots \odot u_{\ell\pi(\ell)} \quad . \tag{3.6}$$

Here S_ℓ is the symmetric group of degree ℓ; computing the tropical determinant is the same as solving a linear assignment optimization problem. Consider a pair of matrices $H^+, H^- \in \mathbb{T}^{m\times(d+1)}$ which serve as an exterior description of the tropical cone

$$Q \;=\; \left\{ z \in \mathbb{T}^{(d+1)} \;\middle|\; H^+ \odot z \geq H^- \odot z \right\} \quad . \tag{3.7}$$

In contrast to the classical situation we have to take two matrices into account. This is due to the lack of an additive inverse operation. We will assume that $\mu(i, j) :=$

$\min(H^+_{ij}, H^-_{ij}) = -\infty$ for any pair $(i,j) \in [m] \times [d+1]$, i.e., for each coordinate position at most one of the corresponding entries in the two matrices is finite. Then we can define

$$\chi(i,j) \ := \begin{cases} 1 & \text{if } \mu(i,j) = H^+_{ij} \neq -\infty \\ -1 & \text{if } \mu(i,j) = H^-_{ij} \neq -\infty \\ 0 & \text{otherwise} \ . \end{cases}$$

For each term $u_{1\pi(1)} \odot \cdots \odot u_{\ell\pi(\ell)}$ in (3.6) we define its *sign* as

$$\operatorname{sgn}(\pi) \cdot \chi(1, \pi(1)) \cdots \chi(\ell, \pi(\ell)) \ ,$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation π. Now the exterior description (3.7) of the tropical cone Q is *tropically sign-generic* if for each square submatrix U of $H^+ \oplus H^-$ we have $\operatorname{tdet}(U) \neq -\infty$ and, moreover, the signs of all terms $u_{1\pi(1)} \odot \cdots \odot u_{\ell\pi(\ell)}$ which attain the maximum in (3.6) agree. By looking at $1{\times}1$-submatrices U we see that in this case all coefficients of the matrix $H^+ \oplus H^-$ are finite and thus $\chi(i,j)$ is never 0.

Algorithm 1 A dual tropical convex hull algorithm

Input: pair of matrices $H^+, H^- \in \mathbb{T}^{m\times(d+1)}$ which provide a tropically sign-generic exterior description of the tropical cone Q from (3.7)

Output: generators for Q

1: pick two matrices $A^+, A^- \in \mathbb{R}\{\{t\}\}^{m\times(d+1)}$ with strictly positive entries such that $\operatorname{val}(A^+) = H^+$ and $\operatorname{val}(A^-) = H^-$

2: apply a classical dual convex hull algorithm to determine a matrix $B \in \mathbb{R}\{\{t\}\}^{(d+1)\times n}$ such that
$$\{\, B \cdot a \mid a \in \mathbb{R}\{\{t\}\}^n,\ a \geq 0\} = \left\{ x \in \mathbb{R}\{\{t\}\}^{(d+1)} \;\middle|\; (A^+ - A^-) \cdot x \geq 0,\ x \geq 0 \right\}$$

3: **return** $\operatorname{val}(B)$

Correctness of Algorithm 1. The main lemma of tropical linear programming [ABGJ15, Theorem 16] says the following. In the tropically sign-generic case, an exterior description of a tropical cone can be obtained from an exterior description of a classical cone over Puiseux series by applying the valuation map to the constraint matrix coefficient-wise. This statement assumes that the classical cone is contained in the non-negative orthant. We infer that

$$\begin{aligned} Q \ &= \ \left\{ z \in \mathbb{T}^{m\times(d+1)} \;\middle|\; H^+ \odot z \geq H^- \odot z \right\} \\ &= \ \operatorname{val}\left(\left\{ x \in \mathbb{R}\{\{t\}\}^{m\times(d+1)} \;\middle|\; A^+ \cdot x \geq A^- \cdot x, x \geq 0 \right\} \right) \\ &= \ \operatorname{val}\left(\left\{ B \cdot a \;\middle|\; a \in \mathbb{R}\{\{t\}\}^n, x \geq 0 \right\} \right) \ . \end{aligned}$$

Now [DY07, Proposition 2.1] yields $Q = \operatorname{val}(\{\, B \cdot a \mid a \in \mathbb{R}\{\{t\}\}^n, x \geq 0\}) = \operatorname{tcone}(\operatorname{val}(B))$. This ends the proof. $\square$

The correctness of our algorithm is not guaranteed if the genericity condition is not satisfied. The crucial properties of the lifted matrices A^+, A^- are not necessarily fulfilled. It is an open question of how an exterior description over $\mathbb{T}$ is related to an exterior description over $\mathbb{R}\{\{t\}\}$ in the general setting. We are even lacking a convincing concept for the "facets" of a general tropical cone.

3.5 Implementation

As a key feature the `polymake` system is designed as a `Perl`/`C++` hybrid, that is, both programming languages are used in the implementation and also both programming languages can be employed by the user to write further code. One main advantage of Perl is the fact that it is interpreted; this makes it suitable as the main front end for the user. Further, Perl has its strengths in the manipulation of strings and file processing. `C++` on the other hand is a compiled language with a powerful template mechanism which allows to write very abstract code which, nonetheless, is executed very fast. Our implementation, in `C++`, makes extensive use of these features. The implementation of the dual steepest edge simplex method, contributed by Thomas Opfer, and the beneath-beyond method for computing convex hulls (see [Ede87] and [Jos03]) are templated. Therefore `polymake` can handle both computations for arbitrary number field types which encode elements in an ordered field.

Based on this mechanism we implemented the type `RationalFunction` which depends on two generic template types for coefficients and exponents. Note that the field of coefficients here does not have to be ordered. Our proof-of-concept implementation employs the classical Euclidean GCD algorithm for normalization. Currently the numerator and the denominator are chosen coprime such that the denominator is normalized with leading coefficient one. For the most interesting case $\mathbb{F} = \mathbb{Q}$ it is known that the coefficients of the intermediate polynomials can grow quite badly, e.g., see [vzGG03, Example 1]. Therefore, as expected, this is the bottleneck of our implementation. In a number field or in a field with a non-Archimedean valuation the most natural choice for a normalization is to pick the elements of the ring of integers as coefficients. The reason for our choice is that this more generic design does not make any assumption on the field of coefficients. This makes it very versatile, and it fits the overall programming style in `polymake`. A fast specialization to the rational coefficient case could be based on [vzGG03, Algorithm 11.4]. This is left for a future version.

The `polymake` implementation of Puiseux fractions $\mathbb{F}\{t\}$ closely follows the construction described in Section 3.2. The new number type is derived from `RationalFunction` with overloaded comparison operators and new features such as evaluating and converting into `TropicalNumber`. An extra template parameter `MinMax` allows to choose whether the indeterminate t is a small or a large infinitesimal.

There are other implementations of Puiseux series arithmetic, e.g., in `Magma` [BCP97] or `MATLAB` [MAT14]. However, they seem to work with finite truncations of Puiseux series and floating-point coefficients. This does not allow for exact computations of the kind we are interested in.

3.6 Computations

We briefly show how our polymake implementation can be used. Further, we report on timings for our LP solver, tested on the Goldfarb–Sit cubes from Example 3.7, and for our (dual) convex hull code, tested on the polytopes with a "long and winding" central path from Example 3.8.

3.6.1 Using polymake

The following code defines a 3-dimensional Goldfarb–Sit cube over the field $\mathbb{Q}\{t\}$, see Example 3.7. We use the parameters $\varepsilon = t$ and $\delta = \frac{1}{2}$. The template parameter Min indicates that the ordering is induced by the dual valuation val*, and hence the indeterminate t plays the role of a small infinitesimal.

```
polytope > $monomial=new UniMonomial<Rational,Rational>(1);
polytope > $t=new PuiseuxFraction<Min>($monomial);
polytope > $p=goldfarb_sit(3,2*$t,1/2);
```

The polytope object, stored in the variable $p, is generated with a facet description from which further properties will be derived below. It is already equipped with a LinearProgram subobject encoding the objective function from Example 3.7. The following lines show the maximal value and corresponding vertex of this linear program as well as the vertices derived from the outer description. Below, we present timings for such calculations.

```
polytope > print $p->LP->MAXIMAL_VALUE;
(1)
polytope > print $p->LP->MAXIMAL_VERTEX;
(1) (0) (0) (1)
polytope > print $p->VERTICES;
(1) (0) (0) (0)
(1) (t^2) (2*t^2) (4*t^2)
(1) (0) (t) (2*t)
(1) (t^2) (t -2*t^2) (2*t -4*t^2)
(1) (0) (0) (1)
(1) (t^2) (2*t^2) (1 -4*t^2)
(1) (0) (t) (1 -2*t)
(1) (t^2) (t -2*t^2) (1 -2*t + 4*t^2)
```

As an additional benefit of our implementation we get numerous other properties for free. For instance, we can compute the parameterized volume, which is a polynomial in t.

```
polytope > print $p->VOLUME;
(t^3 -4*t^4 + 4*t^5)
```

That polynomial, as an element of the field of Puiseux fractions, has a valuation, and we can evaluate it at the rational number $\frac{1}{12}$.

```
polytope > print $p->VOLUME->val;
3
polytope > print $p->VOLUME->evaluate(1/12);
25/62208
```

3.6.2 Linear Programs

We have tested our implementation by computing the linear program of Example 3.7 with polyhedra defined over Puiseux fractions.

The simplex method in `polymake` is an implementation of a (dual) simplex with a (dual) steepest edge pricing. We set up the experiment to make sure our Goldfarb–Sit cube LPs behave as badly as possible. That is, we force our implementation to visit all $n = 2^d$ vertices, when d is the dimension of the input. Table 3.1 illustrates the expected exponential growth of the execution time of the linear program. In three of our four experiments we choose δ as $\frac{1}{2}$. The computation over $\mathbb{Q}\{\varepsilon\}$ costs a factor of about 80 in time, compared with the rational cubes for a modest $\varepsilon = \frac{1}{6}$. However, taking a small ε whose binary encoding takes more than 18,000 bits is substantially more expensive than the computations over the field $\mathbb{Q}\{\varepsilon\}$ of Puiseux fractions. Taking δ as a second small infinitesimal is possible but prohibitively expensive for dimensions larger than twelve.

3.6.3 Convex Hulls

We have also tested our implementation by computing the vertices of the polytope from Example 3.8. For this we used the client `long_and_winding` which creates the $d = (2r + 2)$-dimensional polytope given by $m = 3r + 4$ facet-defining inequalities. Over the rationals we evaluated the inequalities at 2^{2^r} which probably gives the correct combinatorics; see the discussion at the end of Example 3.8. This very choice forces the coordinates of the defining inequalities to be integral, such that the polytope is rational. The number of vertices n is derived from that rational polytope. The running times grow quite dramatically for the parametric input. This overhead could be reduced via a better implementation of the Puiseux fraction arithmetic.

3.6.4 Experimental Setup

Everything was calculated on the same Linux machine with `polymake` perpetual beta version 2.15-beta3 which includes the new number type, the templated simplex algorithm and the templated beneath-and-beyond convex hull algorithm. All timings were measured in CPU seconds and averaged over ten iterations. The simplex algorithm was set to use only one thread.

All tests were done on `openSUSE` 13.1 (x86_64), with Linux kernel 3.11.10-25, `clang` 3.3 and `perl` 5.18.1. The rational numbers use a `C++`-wrapper around the `GMP` library version 5.1.2. As memory allocator `polymake` uses the `pool_allocator` from `libstdc++`, which was version 4.8.1 for the experiments. The hardware for all tests was:

Intel(R) Core(TM) i7-3930K CPU @ 3.20GHz
bogomips: 6400.21
MemTotal: 32928276 kB

Table 3.1: Timings (in seconds) for the Goldfarb–Sit cubes of dimension d with $\delta = \frac{1}{2}$. For ε we tried a small infinitesimal as well as two rational numbers, one with a short binary encoding and another one whose encoding is fairly large. For comparison we also tried both parameters as indeterminates.

d	m	n	$\mathbb{Q}\{\varepsilon\}$ ε	$\mathbb{Q}$ $\varepsilon = \frac{1}{6}$	$\mathbb{Q}$ $\varepsilon = \frac{2}{17^{4500}}$	$(\mathbb{Q}\{\delta\})\{\varepsilon\}$ $\varepsilon \ll \delta$
3	6	8	0.010	0.003	0.005	0.101
4	8	16	0.026	0.001	0.017	0.353
5	10	32	0.064	0.002	0.065	1.034
6	12	64	0.157	0.007	0.253	2.877
7	14	128	0.368	0.006	0.829	7.588
8	16	256	0.843	0.016	2.643	19.226
9	18	512	1.906	0.039	7.703	47.806
10	20	1024	4.258	0.090	21.908	118.106
11	22	2048	9.383	0.191	59.981	287.249
12	24	4096	20.583	0.418	160.894	687.052

Table 3.2: Timings (in seconds) for convex hull computation of the feasibility set from Example 3.8. All timings represent an average over ten iterations. If any test exceeded a one hour time limit this and all larger instances of the experiment were skipped and marked –.

r	d	m	n	$\mathbb{Q}\{t\}$	$\mathbb{Q}$
1	4	7	11	0.018	0.000
2	6	10	28	0.111	0.000
3	8	13	71	0.754	0.010
4	10	16	182	15.445	0.036
5	12	19	471	1603.051	0.150
6	14	22	1226	–	0.737
7	16	25	3201	–	4.001
8	18	28	8370	–	25.093
9	20	31	21901	–	223.240
10	22	34	57324	–	1891.133

4 Abstract tropical linear programming

4.1 Introduction

Tropical linear programming seeks for a point which fulfills all the inequalities in a tropical linear inequality system. These inequalities are of the form

$$\min\{\, a_i + x_i \mid i \in I\} \;\leq\; \min\{\, a_\ell + x_\ell \mid \ell \in [d] \setminus I\} \quad ,$$

where $I \subseteq [d]$ and $a_1, \ldots, a_d$ are elements of the *tropical numbers* $\mathbb{T}_{\min} = \mathbb{R} \cup \{\infty\}$. This is the *tropical feasibility problem* and it is analogous to the feasibility problem in classical linear programming. One can encode scheduling problems with inequality systems of this form [MSS04, But10], and finding feasible points for a tropical linear inequality system is equivalent to determining the winning positions of a mean payoff game [AGG12]. Furthermore, the interplay between classical, tropical linear programming and mean payoff games has proven to be a fruitful approach for obtaining several new complexity results in [ABGJ14a, ABGJ14b, Ben14, Fri11, Han12]. We generalize the tropical feasibility problem to a combinatorial structure, which we call *signed tropical matroids*, and develop an analogue of oriented matroid programming [Bla77, Fuk82, Tod85, Ter85]. In this abstract setting, we derive an algorithm which solves the feasibility problem and also has an interesting complexity for the original problem for tropical linear inequality systems. It is similar to the tropicalization of the simplex method [ABGJ15] but is also guaranteed to terminate and to provide the correct result for signed tropical matroids. This is achieved by a new pivoting concept between bipartite graphs which we call *basic covector* and which correspond to a special subset of bases in the simplex method. A computation of reduced costs to determine the next iteration is not necessary, it can be directly read off the basic covector. Furthermore, we demonstrate that several properties of tropical inequality systems follow from more general results from graph theory and polyhedral geometry. This exhibits the deeper structure of the algorithmic question and brings new combinatorial methods for this problem into play. In particular, it shows the connection between the computational complexity of tropical linear programming as well as deciding the winning position in a mean payoff game and the geometric complexity of polyhedral subdivisions.

Our algorithm relies only on the combinatorial structure of the *signed tropical matroid*. We derive them from "tropical oriented matroids" by adding a sign information which encodes the halfspaces. The latter were originated in the work by Ardila and Develin [AD09] to describe and generalize the combinatorics of tropical point configurations. It was further developed by Oh & Yoo [OY11] and Horn in [Hor16] where a realizability result with "tropical pseudohyperplanes" is shown. Horn also shows that tropical oriented matroids are in bijection with not necessarily regular subdivisions of the product of two simplices $\Delta_{n-1} \times \Delta_{d-1}$. We use this as the starting point for our definition of signed tropical matroids. The required polyhedral notions are elaborated in the core literature [DLRS10] by De Loera, Rambau and Santos. This provides us with the structure to study this abstraction of tropical linear programming. It is an analogue to classical oriented matroid programming which was introduced by Bland [Bla77] and Fukuda [Fuk82] and further extended by Todd [Tod85] and Terlaky [Ter85]. Oriented matroid programming originated from the formulation of the simplex method only in terms of sign vectors.

A *signed tropical matroid* (STM) is a set of bipartite subgraphs, the *covector graphs*, of the complete bipartite graph $K_{d,n}$ with signs on the edges. Many properties follow from purely graph theoretical arguments. An STM is *realizable* if it is derived from a tropical linear inequality system. Non-genericity and the occurrence of ∞ as a coefficient often causes technical obstacles in the study of tropical linear inequality systems. In the generalized setting, it occurs naturally to use polyhedral methods to resolve these issues. We use *extension* and *refinement* of subdivisions [DLRS10] to be able to reduce our arguments to triangulations of $\Delta_{n-1} \times \Delta_{d-1}$. Equivalently, then each bipartite graph is a forest and each edge of $K_{d,n}$ occurs at least once as an edge of a covector graph. *Cramer covectors*, which form a generalization of tropical Cramer solutions [ACG+90, RGST05], play an important role. They describe the *bases* which are analogous to the classical basic points. Like the simplex method, our algorithm iterates over bases which correspond to subsets of the inequalities. In each step, one index is exchanged until a certificate for feasibility or infeasibility is found. In the general setting, we give only a rough upper bound on the number of iterations of the algorithm. For the realizable case, we deduce a pseudopolynomial upper bound. Together, this implies that the running time is determined by the minimal coefficient matrix, which realizes a given signed tropical matroid. Thereby, we connect the study of the complexity with a further investigation of the secondary fan of $\Delta_{n-1} \times \Delta_{d-1}$. We finish by interpreting the results in terms of mean payoff games.

We give a brief overview of the sections. Section 4.2 is dedicated to the introduction of the main concepts for describing the combinatorics of tropical linear inequality systems. In Section 4.3, we show the conversion from AND-OR-networks and mean payoff games to tropical linear inequality systems. Furthermore, we formulate the classical simplex method in such a way that the structural similarity with our algorithm becomes apparent. We move on to signed tropical matroids, the abstraction of tropical linear inequality systems, in Section 4.4. We explain some technical tools for reducing the general case to triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ in Section 4.5. This provides us with the necessary background to derive the algorithms in Section 4.6. Note that the algorithms are rather simple in the required terminology, however, we need the technical tools to prove the correctness. In Section 4.7, we apply the algorithms to the special case of tropical linear inequality systems. For this, we can drop some requirements on the input and deduce upper bounds on the number of iterations. Furthermore, we state some structural implications for tropical linear inequality systems.

The chapter is based on `https://arxiv.org/abs/1612.01890v1`.

4.2 Basic Definitions for Tropical Linear Inequality Systems

Before deriving properties of tropical linear inequality systems, we recall the basic definitions from Chapter 2. Beside the definition of the *tropical semiring*, we introduce *covector graphs* in different flavors as they will be our main tool. They were first defined by Develin and Sturmfels under the name of *types* in [DS04] and further studied as *covectors* in [FR15], as well as in Chapter 2.

4.2.1 Covector graphs for signed systems

The *tropical numbers* consist of the set $\mathbb{T}_{\min} = \mathbb{R} \cup \{\infty\}$. Equipped with the two operations $\oplus$ and $\odot$, where $x \oplus y := \min(x, y)$ and $x \odot y := x + y$ for $x, y \in \mathbb{T}_{\min}$, they form the *tropical semiring*. Just as well, we could consider $\oplus = \max$ as tropical addition. The operations extend to vectors and matrices component-wise and we can define a matrix product analogously to the classical case.

We use the notation $[d] = \{1, \ldots, d\}$ and define the sum over an empty set to be ∞. Furthermore, the symbol $\sqcup$ denotes the disjoint union of the two (color) classes of nodes of a bipartite graph.

We define a *(tropical) signed system* as a pair (A, Σ) with $(a_{ji}) = A \in \mathbb{T}_{\min}^{n \times d}$ and $(\sigma_{ji}) = \Sigma \in \{+, -, \bullet\}^{n \times d}$, where $a_{ji} = \infty \Leftrightarrow \sigma_{ji} = \bullet$. It defines a homogeneous tropical linear inequality system by

$$\bigoplus_{i \in [d],\ \sigma_{ji} = +} a_{ji} \odot x_i \leq \bigoplus_{i \in [d],\ \sigma_{ji} = -} a_{ji} \odot x_i \qquad \text{for } j \in [n] \ . \tag{4.1}$$

A point $x \in \mathbb{T}_{\min}^d$ is *feasible* for (A, Σ) if it fulfills all the inequalities, otherwise we call it *infeasible*. A signed system is *feasible* if there is a feasible point in $\mathbb{TA}^d = \mathbb{T}_{\min}^d \setminus \{(\infty, \ldots, \infty)\}$; otherwise it is *infeasible*. The set of feasible points in $\mathbb{TA}^d$ is the *feasible region*. Such a feasible region is a *tropical cone*, which means that it is closed under tropical addition and scalar multiplication. A *tropical halfspace* is the feasible region of a single tropical linear inequality.

Note that the sign information which we encode in the sign matrix Σ occurs in the patchworking method of Viro [Vir01] and is, alternatively, added to the tropical semiring to form the "symmetrized tropical semiring" [ACG+90].

Definition 4.1. The *(tropical) covector (graph)* $G_A(x)$ of a finite point $x \in \mathbb{R}^d$ is the bipartite graph on the node set $[d] \sqcup [n]$ containing an edge $(i, j) \in [d] \times [n]$ if and only if $a_{ji} + x_i = \min\{a_{jk} + x_k \mid k \in [d], a_{jk} \neq \infty\}$. This means that the covector graph encodes those entries in each row of the product $A \odot x$ where the minimum is attained.

Note that we label the entries of A by pairs $(j, i) \in [n] \times [d]$ and choose the reverse order to denote the edges $(i, j) \in [d] \times [n]$ of a covector graph. We will write pairs for the edges even if we consider it as an undirected graph. Often, we will call tropical covector graphs just covectors.

The nodes in $[d]$ are *coordinate nodes* and in $[n]$ are the *apex nodes*. Coordinate nodes correspond to the variables and are visualized by square nodes. Apex nodes correspond to the rows and the inequalities, respectively. They are depicted by circle nodes. Depending on the sign given by Σ, we call an edge in a covector graph *negative* or *positive*.

Example 4.2. Consider the signed system $(A, \Sigma) = ((0, 0, 0), (+, -, +))$. For each point $x \in \mathbb{R}^3$ with pairwise distinct coordinates, the decomposition in Figure 4.1 shows where the minimum is attained in the product $(0, 0, 0) \odot x = \min(x_1, x_2, x_3)$.

On the left of Figure 4.1, we put the plain covector graphs whereas, on the right, we add the sign information given by Σ.

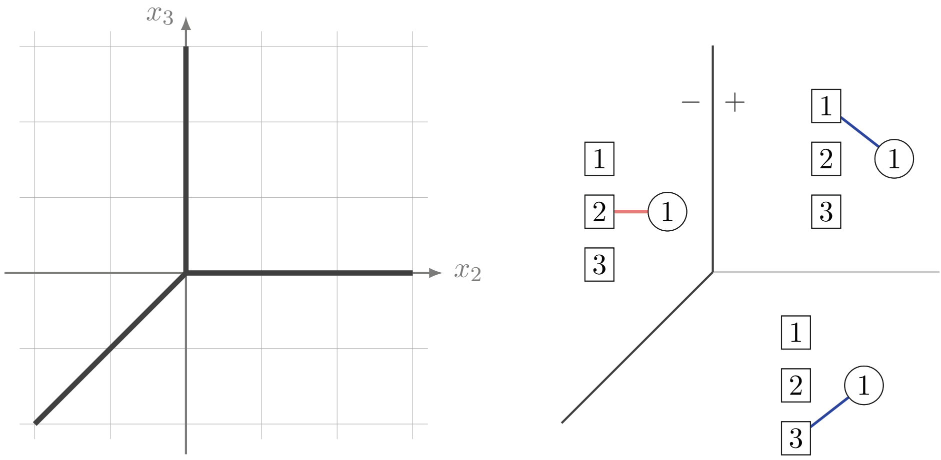

Figure 4.1: We dehomogenize by setting $x_1 = 0$. We depict the covector graphs of the points, where the minimum is attained only once, for $A = (0, 0, 0)$ and $\Sigma = (+, -, +)$, see Example 4.2. Negative edges are red, positive edges are blue.

Directly from the definition, we obtain a characterization of finite feasible points.

Proposition 4.3. *A point $x \in \mathbb{R}^d$ is feasible for the signed system (A, Σ) if and only if no apex node is only incident with negative edges in $G_A(x)$.*

Proof. By definition, a point is infeasible if and only if there is a $j \in [n]$ with

$$\bigoplus_{\sigma_{ji}=+, i \in [d]} a_{ji} \odot x_i > \bigoplus_{\sigma_{ji}=-, i \in [d]} a_{ji} \odot x_i \quad .$$

This means that the minimum is attained only for entries with a minus sign. From this follows the claim with Definition 4.1. $\square$

The cells $\left\{x \in \mathbb{R}^d \;\middle|\; G_A(x) \text{ const}\right\}$ define a *covector decomposition* of $\mathbb{R}^d$. This is the same polyhedral subdivision of $\mathbb{R}^d$ as in Chapter 2 if we replace max by min.

Notice that the covector graphs are homogeneous in the sense that adding an element of $\mathbb{R} \cdot \mathbf{1} = \mathbb{R} \cdot (1, \ldots, 1)$ to a cell yields the same covector graph and the cells in the covector decomposition all contain $\mathbb{R} \cdot \mathbf{1}$ as lineality.

We fix a matrix $A \in \mathbb{T}_{\min}^{n\times d}$, for which every row contains a finite entry, and denote by Γ the complete bipartite graph $K_{d,n}$ on the node set $[d] \sqcup [n]$ with the entries of A as weights on its edges. A *matching* on $D \sqcup N$ with $D \subseteq [d]$ and $N \subseteq [n]$ is a subgraph of $K_{d,n}$ in which each node has degree 1. The *value* of a matching μ with respect to a matrix A is the sum $\sum_{(i,j)\in\mu} a_{ji}$. A matching is *minimal* if all the other matchings in the induced subgraph of $K_{d,n}$ on $D \sqcup N$ have a bigger value.

Combining Proposition 2.30 and Proposition 2.38 yields the following characterization which is similar to [JK16, Theorem 6.1].

Proposition 4.4. *A bipartite graph G on $[d] \sqcup [n]$ is a covector graph of a point $x \in \mathbb{R}^d$ with respect to A if and only if the following three conditions hold:*

1. *No apex node $j \in [n]$ is isolated in G.*
2. *Let μ be a matching in G on a subset $D \sqcup N$ of the nodes with $D \subseteq [d]$, $N \subseteq [n]$ and $|D| = |N|$. Then μ is a minimal matching in Γ.*
3. *Let μ and η be minimal matchings in Γ. If μ is contained in G, so is η.*

4.2.2 Generalized covector graphs

To make use of covector graphs also for points in $\mathbb{T}_{\min}^d$ with ∞ coordinates, we introduce a generalized notion that is slightly different from the approach chosen in Section 2.3.5.

Definition 4.5. The *support* $\operatorname{supp}(x)$ of a point $x \in \mathbb{T}_{\min}^d$ is the set $\{ i \in [d] \mid x_i \neq \infty\}$. Furthermore, the *generalized covector graph* of x is the bipartite graph on the node set $[d] \sqcup [n]$ containing an edge $(i,j) \in [d] \times [n]$ if and only if

$$a_{ji} + x_i = \min \{ a_{jk} + x_k \mid k \in \operatorname{supp}(x), a_{jk} \neq \infty \} \neq \infty \ .$$

We denote it by $G_A(x)$, like the covector graphs from Definition 4.1. In contrast to covector graphs of points in $\mathbb{R}^d$ the generalized covector graphs possibly have isolated apex nodes. A (generalized) covector graph without an isolated apex node is called *proper*.

Note that a generalized covector graph can also be the empty graph and the corresponding point is feasible. The empty graph is the covector graph of $(\infty, \dots, \infty)$ but also for $(0, \infty, \infty)$ with respect to $(\infty, 0, 0)$. This happens, if the support of all the rows is contained in a common proper subset of $[d]$.

Definition 4.6. A (generalized) covector graph G is *infeasible* if there is an apex node which is only incident with negative edges. If G is not infeasible we call it *feasible*.

We obtain the following more general version of Proposition 4.3. It assures that the two notions of feasibility agree for points with finite components and it is the suitable formulation for defining the feasibility in signed tropical matroids, see Section 4.4.

Proposition 4.7. *A point $x \in \mathbb{T}_{\min}^d$ is feasible for the signed system (A, Σ) if and only if no apex node is only incident with negative edges in the generalized covector graph $G_A(x)$.*

Proof. Fix $j \in [n]$ and consider the corresponding inequality Equation 4.1. If j is only incident to negative edges the right hand side is surely smaller and the inequality is not fulfilled. If j has no neighbors in $G_A(x)$ then both sides of the inequality are ∞ and the inequality is fulfilled. Otherwise, it is also a valid inequality. $\square$

This allows as to examine the feasibility of general tropical inequality systems via generalized covector graphs.

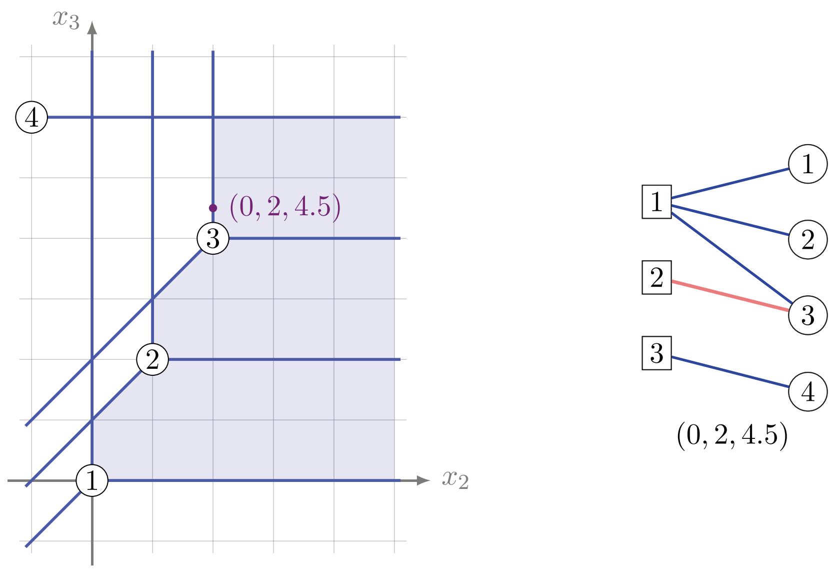

Figure 4.2: As always, we set $x_1 = 0$ to cancel out the lineality $\mathbb{R} \cdot \mathbf{1}$. The shaded area is the *feasible* region of a signed system formed by the four inequalities from Example 4.8. The crooked lines are the boundaries of the *tropical halfspaces*. The bipartite graph is the covector graph of $(0, 2, 4.5)$, where the negative edge is red.

Example 4.8. The left part of Figure 4.2 depicts the feasible region of the signed system (A, Σ) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \\ 0 & \infty & -6 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} + & - & - \\ + & - & + \\ + & - & + \\ - & \bullet & + \end{pmatrix} \quad .$$

This gives rise to the inequality system

$$\begin{aligned} 0 + x_1 &\leq \min(0 + x_2, 0 + x_3) \\ \min(0 + x_1, x_3 - 2) &\leq x_2 - 1 \\ \min(0 + x_1, x_3 - 4) &\leq x_2 - 2 \\ x_3 - 6 &\leq 0 + x_1 \quad . \end{aligned}$$

The covector graph of the point $(0, 2, 4.5)$ is shown in the right part of Figure 4.2. It is feasible since each apex node is incident with a positive edge.

The covector graph of the point $(\infty, 0, \infty)$ has the edges $(2, 1)$, $(2, 2)$ and $(2, 3)$. It is not proper and infeasible.

4.3 Related Algorithmic Problems

The feasibility problem for tropical linear inequality systems is the problem of finding a feasible point of the system. We highlight the relation of this problem to scheduling, mean payoff games and classical linear programming.

The complexity of the decision problems for scheduling AND-OR-networks with arbitrary coefficients and mean payoff games is known to be in NP $\cap$ co-NP and even more in UP $\cap$ co-UP, see [Jur98, ZP96, EM79, MSS04], but there is no polynomial time algorithm known. This was also unclear for classical linear programming while the containment in the complexity class NP $\cap$ co-NP follows easily from linear programming duality. Finally, Khachiyan [Kha79] and, not long after, also Karmarkar [Kar84] provided polynomial-time algorithms. However, it is still unclear if there is a pivoting rule for the simplex method for which it runs in weakly or even strongly polynomial time, see, e. g., [Dan63, Bla77, KM72]. The close relations between tropical linear programming, mean payoff games and classical linear programming, in particular the simplex method, are demonstrated in [Sch09, AGG12, ABGJ15, ABGJ14a].

4.3.1 Scheduling with AND-OR-Networks

Scheduling is concerned with the task of putting several jobs into an order in which they are worked through such that certain constraints are fulfilled. We give a short introduction to a special class of scheduling problems, namely *AND-OR-networks*. They occur in project management with particular temporal dependencies and can be used to model resource constraints. They were extensively studied in, e.g., [MSS04]. In particular, that work contains a formulation of the precedence relations for the starting times with min- and max-inequalities. It also shows the polynomial time equivalence with a decision problem associated to a mean payoff game. We display a tropical geometric relation between the formulation of the set of vectors of starting times and the feasible region of a suitable tropical signed system. For other instances of scheduling problems which can be expressed in terms of tropical inequalities or equations see, e. g., [BA09, §1].

To explain an AND-OR-network we consider the planning of a project. The single jobs depend on each other and are in some precedence relation. We assume that a started job may not be interrupted. If a job can only start if all its predecessor jobs are finished, we call this an *AND-constraint*. If a job can start if at least one of its predecessors is finished, we call this an *OR-constraint*.

In Figure 4.3, one can see the Gantt chart of an AND-constraint and of an OR-constraint visualizing the dependence of the start and finish dates of jobs in these predecessor relations. Here, the dashed line denotes the starting time of the next job which is represented by the bottom bar, its predecessors forming the top three. The lengths of the bars illustrate the processing times.

Figure 4.3: Two types of constraints, OR left, AND right.

Notice that usually one requires the special *starting condition* that every job has to begin after some given point in time. In our model, this is covered by the fact that the expressions are additively homogeneous and hence, one can just mark one node and dehomogenize with respect to this coordinate.

For a broader introduction of scheduling with AND-OR-constraints see [MSS04]. We give a formal definition to work with.

Definition 4.9. An *AND-OR-network* is given by a set of states V and a set of waiting conditions U. The waiting conditions are pairs (X, J) with $J \subseteq V$ and $X \subseteq V \setminus \{J\}$.

The pair (V, U) can be construed to be a directed bipartite graph $\mathcal{B}$ with node set $V \sqcup U$. Each waiting condition (X, J) is expressed by the arcs $(x, (X, J))$ for $x \in X$ and $((X, J), j)$ for $j \in J$. Because of $X \subseteq V \setminus \{J\}$, for each pair $v \in V$ and $u \in U$ there exists at most one of the arcs (u, v) or (v, u). We denote the arc set by $\mathcal{A}$.

Furthermore, we have a weight function $\omega \colon \mathcal{A} \to \mathbb{Q}$ on the arcs to encode processing times, or time lags if the weight is negative.

Then we can describe the precedence constraints for the vector of starting times $t \in \mathbb{T}_{\min}^{V \sqcup U}$ by the inequalities

$$\begin{aligned} t_v &\geq \max_{(u,v)\in\mathcal{A}} (t_u + \omega(u, v)) && \text{for all } v \in V && \text{(AND)} \\ t_u &\geq \min_{(v,u)\in\mathcal{A}} (t_v + \omega(v, u)) && \text{for all } u \in U && \text{(OR)} \ . \end{aligned} \tag{4.2}$$

The max-inequalities correspond to AND-constraints and min-inequalities to OR-constraints.

We can reformulate the first inequality in (4.2) by splitting the maximization into several inequalities to obtain

$$t_v \geq (t_u + \omega(u, v)) \qquad \text{for all } (u, v) \in \mathcal{A} \text{ with } u \in U, v \in V \tag{4.3a}$$

$$t_u \geq \min_{(v,u)\in\mathcal{A}} (t_v + \omega(v, u)) \qquad \text{for all } u \in U \ . \tag{4.3b}$$

Observe that this already yields a signed system.

We can transform the first kind of inequalities (4.3a) further into

$$t_v - \omega(u, v) \geq t_u \qquad \text{for all } (u, v) \in \mathcal{A} \text{ with } u \in U, v \in V \Leftrightarrow \tag{4.3a'}$$

$$\min_{(u,v)\in\mathcal{A}} (t_v - \omega(u, v)) \geq t_u \qquad \text{for all } u \in U \ . \tag{4.3a''}$$

Combining the two kinds of inequalities (4.3b) and (4.3a”) yields

$$\min_{(u,v)\in\mathcal{A}}(t_v - \omega(u,v)) \geq t_u \geq \min_{(v,u)\in\mathcal{A}}(t_v + \omega(v,u)) \qquad \forall u \in U \ .$$

Let $|V| = d$ and $|U| = n$. Then we define matrices $(a_{ji}) = A \in \mathbb{T}_{\min}^{n\times d}$ and $(\sigma_{ji}) = \Sigma \in \{+,-,\bullet\}^{n\times d}$ by identifying each node in V resp. U with indices in $[d]$ resp. $[n]$ and setting

$$(\ a(u,v)\ ,\ \sigma(u,v)\ ) = \begin{cases} (\omega(v,u),+) & (v,u)\in\mathcal{A} \\ (-\omega(u,v),-) & (u,v)\in\mathcal{A} \\ (\infty,\bullet) & \text{else} \end{cases}$$

for $v \in V$ and $u \in U$. This defines a signed system (A,Σ) whose associated inequality system is

$$\min_{\sigma(u,v)=-}(t_v + a(u,v)) \geq \min_{\sigma(u,v)=+}(t_v + a(u,v)) \quad \text{for all} \quad u \in U \ . \tag{4.4}$$

Conversely, if we are given a feasible solution $(t_v)_{v\in V}$ of (4.4) we can define starting times t_u for $u \in U$ by

$$t_u = \min_{\sigma(u,v)=-}(t_v + a(u,v)) \tag{4.5}$$

such that $(t_k)_{k\in U\sqcup V}$ fulfills (4.2). We summarize our considerations in the following theorem.

Theorem 4.10. *The set of feasible points for (4.4) is the projection of the set of feasible starting times for (4.2) on the coordinates labeled by V. Furthermore, for every feasible point of (4.4) we find a feasible point of (4.2).*

Example 4.11. Figure 4.4 depicts the AND-OR-network for the signed system from Example 4.8. For this signed system, we know that $(0,2,4.5)$ is a feasible point. This translates to possible start times for the AND-nodes. With Equation 4.5, we calculate $(2,1,0,0)$ as possible starting times for the OR-nodes.

With the dehomogenization $x_1 = 0$, the coordinatewise minimal point of the feasible region amounts to the point $(0,0,0)$. This yields $(0,-1,-2,0)$ for the resulting start times of the OR-nodes.

Remark 4.12. The pseudopolynomial algorithm in [MSS04, §7.2.2] uses the basic idea to make a violated inequality an equality. If a starting time t_j violates the inequality $t_j \geq \min_{i\in X}(t_i + d_{iw})$ for a waiting condition $w = (X,j)$, one assigns the new value $\min_{i\in X}(t_i + d_{iw})$ to t_j. This yields a pseudopolynomial algorithm as the iteratively computed starting times only increase and can be bounded from above. Similar ideas will come up later on in subsection 4.7.3.

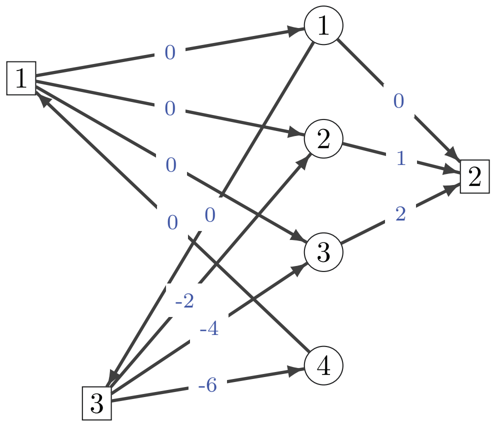

Figure 4.4: The scheduling network derived from the signed system of Example 4.8.

4.3.2 Mean Payoff Games

The connection between mean payoff games and tropical linear inequality systems, which we describe below, was established in [AGG12]. A similar result implicitly occurs in [MSS04, Lemma 7.5] and [Sch09, Lemma 2].

Introduction to Mean Payoff Games

We briefly introduce mean payoff games. Let $\mathcal{G}$ be a finite directed bipartite graph with node set $V_0 \sqcup V_1$, arc set $\mathcal{A}$ and a weight function $\omega\colon \mathcal{A} \to \mathbb{Q}$ on the arcs. Without loss of generality, we can assume that $V_0 = [d]$ and $V_1 = [n]$.

We define a finite two-player game with full information on $\mathcal{G}$, following [ZP96]. At a node in V_p, it is the turn of player p, for $p \in \{0,1\}$. Starting from a fixed node $k \in V_0 \sqcup V_1$, the players alternatingly choose an outgoing arc of the current node and move to the tip of the arc. If a player cannot move because there is no outgoing are, she looses. As soon as the directed path formed in this way produces a cycle, the game finishes. The *outcome* of the game with starting point k is the mean weight of the arcs in that cycle. One player tries to maximize, while the other player tries to minimize the outcome of the game.

A positional *strategy* for player $p \in \{0,1\}$ is a subset τ_p of the arcs $\mathcal{A}$, such that each vertex in V_p is either isolated or incident to exactly one outgoing arc in τ_p. By [EM79], a mean payoff game has an optimal positional strategy.

Following [GP14, §7], we say that a position $i \in V_0$ is *non-losing* for player 1 if there is a strategy for player 1 such that the outcome of the game starting with i is non-negative.

We construct a signed system from the bipartite graph $\mathcal{G}$ with the weights ω similar to Section 4.3.1, but with switched signs.

Let $|V_0| = d$ and $|V_1| = n$. Then we define matrices $(a_{ji}) = A \in \mathbb{T}_{\min}^{n\times d}$ and $(\sigma_{ji}) = \Sigma \in \{+,-,\bullet\}^{n\times d}$ by identifying each node in V_0 resp. V_1 with indices in $[d]$ resp. $[n]$ and

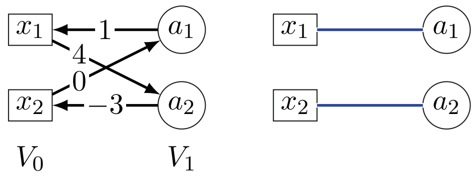

Figure 4.5: A bipartite graph $\mathcal{G}$ depicting the mean payoff game from Example 4.14 and a non-losing strategy τ for player 1 owning the circle nodes.

setting

$$(a(v_1, v_0), \sigma(v_1, v_0)) = \begin{cases} (\omega(v_0, v_1), -) & (v_0, v_1) \in \mathcal{A} \\ (-\omega(v_1, v_0), +) & (v_1, v_0) \in \mathcal{A} \\ (\infty, \bullet) & \text{else} \end{cases} \tag{4.6}$$

for $v_0 \in V_0$ and $v_1 \in V_1$.

Note that the former construction is reversible.

We state the main theorem connecting tropical linear inequality systems and mean payoff games, see [AGG12, Theorem 3.2].

Theorem 4.13. *The set of non-losing states in V_0 for player 1 equals the set of those $i \in [d]$ for which there is a feasible point x for (A, Σ) with $x_i \neq \infty$.*

We sketch one direction of an independent proof to demonstrate how this ties in with the properties of covector graphs. Let $x \in \mathbb{T}^d_{\min}$ be a feasible point for (A, Σ) with support $D \neq \emptyset$. Since its covector graph G is feasible, each node in $[n]$ is either isolated or incident with a positive edge in G. If an apex node $j \in [n]$ is isolated in G, there is no arc between D and j in $\mathcal{G}$ either. For an isolated node, we pick no edge and for a non-isolated apex node, we pick one incident positive edge in G. This yields a strategy τ for player 1.

If a run of the game with starting node in D and fixed strategy τ for player 1 produces a cycle, it can only be a non-negative cycle by Proposition 2.38. This implies the claim.

Example 4.14. The signed system for the graph $\mathcal{G}$ from Figure 4.5 is given by

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} -1 & 0 \\ 4 & 3 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} + & - \\ - & + \end{pmatrix} \end{array} .$$

The corresponding inequality system is $x_1 - 1 \leq x_2, \quad x_2 + 3 \leq x_1 + 4$. The non-losing strategy is obtained from the positive edges of the feasible point $(0, -1)$.

We also relate the example for AND-OR-networks with the corresponding mean payoff game.

Example 4.15. By reversing the arcs and negating the weights in Figure 4.4, we obtain the game graph corresponding to the inequality system from Example 4.8. The blue edges in the covector graph of the feasible point $(0, 2, 4.5)$ yield the non-losing strategy formed by $(1,1), (2,1), (3,1), (4,3)$ (which are directed from circle to square nodes). This, for example, yields the positive cycle 4, $\boxed{3}$, 3, $\boxed{1}$.

Parity Games as Special Mean Payoff Games

Parity games [EJ91, Jur98] also are two player games with perfect information. However, we have no weights on the edges but on the vertices of the game graph. The vertices are assigned to the two players, *even* and *odd*. Even vertices are labeled by an even integer weight, odd vertices by an odd integer weight. Player even wins if the maximal number in the terminating cycle is even, otherwise odd wins.

Let $M = d + n$ be the number of vertices in the two classes. We can consider a parity game as a special mean payoff game where the outgoing edges of a vertex with label $k \in \mathbb{Z}$ get the weight $(-M)^k$. Then the winning states of the so constructed mean payoff game for player 0 resp. 1 are exactly the winning states of player even resp. odd in the parity game. For more details see, e.g., [Jur98].

Recently, it was shown in [CJK$^+$] that parity games can be solved in quasipolynomial time. Parity games have served as suitable instances to demonstrate the worst-case complexity of many algorithms, see, e.g., [Fri11, Han12].

4.3.3 The Simplex Method

In [ABGJ15], it was shown how a run of the classical simplex method translates to a run of a tropical simplex method under some technical assumptions on the input and the requirement that the pivoting rule is combinatorial. This led to a new algorithm for solving mean payoff games presented in [ABGJ14a] which is polynomial time equivalent to the simplex method with the given pivoting rule. A reduction from mean payoff games to linear programming was already given in [Sch09]. However, this approach requires exponentially large coefficients which results in a pseudopolynomial running time due to cost of the arithmetic operations. This is resolved in the approach in [ABGJ15] by considering only the signs determining the pivoting which can be computed directly from the input data.

We give a short introduction to the classical simplex method [Dan63]. We present it as an algorithm to determine the feasibility of a classical linear inequality system. Our exposition is inspired by [Mur76, §4.5].

It is important to observe the similarity between this variant of the simplex method and the algorithms in Section 4.6, in particular Algorithm 3. To obtain that algorithm as a tropicalization of the following variant of the simplex method, one would have to ensure that $x \geq 0$.

The feasibility problem is the task to find an $x \in \mathbb{R}^d$ which fulfills the system

$$A \cdot x \leq b \qquad \text{for } A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n \ .$$

The following is meant to highlight that we can consider it as a method which traverses the vertex-edge graph of the affine hyperplane arrangement given by the equations $a_j \cdot x = b_j$ for $j \in [n]$. Here, a_j is the jth row of A. At each vertex, one is given a rule for choosing the consecutive vertex in a way that guarantees termination.

We assume that the system $(A|b)$ is generic by which we mean that the d-sets $J \subseteq [n]$ are in bijection with the points z which fulfill the subsystem $A_J z = b_J$ with row indices in J. Start with an arbitrary d-set J_0 from $[n]$ and define $x^0 := A_{J_0}^{-1} b_{J_0}$. Then $[n]$ is partitioned into three sets, namely J_0, $K_0^+ := \{ j \in [n] \mid a_j x^0 < b_j \}$ and $K_0^- := \{ j \in [n] \mid a_j x^0 > b_j \}$. The set J_0 denotes the *basic* variables and $[n] \setminus J_0 = K_0^+ \cup K_0^-$ the *non-basic* variables.

Fix an arbitrary vector $y^0 \in \mathbb{R}^n$ with $y^0 \geq 0$ whose support is J_0, e.g. the characteristic vector of J_0 and define

$$c = A^\top y^0 \in \mathbb{R}^d \ .$$

In this way, we obtain a *primal linear program* (P) and its *dual linear program* (D)

$$(P) \quad \begin{array}{l} \max c^\top x \\ Ax \leq b \end{array} \qquad\qquad (D) \quad \begin{array}{l} \min b^\top y \\ A^\top y = c, \quad y \geq 0 \end{array} \ .$$

By construction, y^0 is a feasible point of the dual linear program. Therefore, we can apply “Phase II” of the simplex method as we are already equipped with a feasible point. We want to consider it as a feasibility algorithm for (P). In particular, we want to reach a point x_ℓ where $K_\ell^- = \emptyset$.

First, pick an index $r_0 \in K_0^-$. We want to change x^0 such that the index r_0 of the violated inequality *enters* the *basis*. This means that r_0 becomes a basic variable.

Define

$$i_0 = \arg\min \left\{ \frac{(({A_{J_0}}^\top)^{-1} c)_i}{(({A_{J_0}}^\top)^{-1} {a_{r_0}}^\top)_i} \geq 0 \;\middle|\; i \in [d] \right\} \ ,$$

and λ_0 as the value of this minimum. In the generic case, this minimum is attained at most once. If this minimum does not exist, the inequality system of (P) is infeasible. Note that the existence of this minimum is independent of the choice of c since the occurring numerators are the positive components of y^0. Let j_0 be the i_0-th element of J_0 considered as an ordered index tuple for the rows of A_{J_0}. Then j_0 is the *leaving* variable and $J_1 = J_0 \setminus \{j_0\} \cup \{r_0\}$ becomes the new basis. Now, we can restart the iteration. However, we keep c fixed and for $\ell \geq 1$ choose y^ℓ iteratively in the following way:

$$y_j^\ell = \begin{cases} (({A_{J_\ell}}^\top)^{-1} c)_j & \text{for } j \in J_\ell \\ 0 & \text{for } j \in [n] \setminus J_\ell \ . \end{cases} \tag{4.7}$$

Theorem 4.16. *The vector $y^1 \in \mathbb{R}^n$ fulfills $y^1 \geq 0$, $c = A^\top y^1$ and $b^\top y^1 < b^\top y^0$.*

Proof. Consider the linear equality system

$$c = {A_{J_0 \cup r_0}}^\top z \ .$$

For $z_{d+1} = 0$ we get the solution $y_{J_0}^0 = ({A_{J_0}}^\top)^{-1} c$ and for $z_{i_0} = 0$ we obtain the solution $y_{J_1}^1 = ({A_{J_1}}^\top)^{-1} c$ (up to relabeling of the coordinates).

Furthermore, by multiplying both sides with $A_{J_0}^{-1}$ from the left, we obtain

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & \ddots & 0 & 0 & \\ \vdots & 0 & \ddots & 0 & ({A_{J_0}}^\top)^{-1}{a_{r_0}}^\top \\ 0 & \cdots & 0 & 1 & \end{array}\right) \cdot z = ({A_{J_0}}^\top)^{-1} c \ .$$

This is equivalent to

$$z_{[d]} = \left(({A_{J_0}}^\top)^{-1} c\right) - z_{d+1}\left(({A_{J_0}}^\top)^{-1}{a_{r_0}}^\top\right) \ . \tag{4.8}$$

Choosing z_{d+1} as λ_0, we obtain $z_{i_0} = 0$ and hence, $y^1_{J_1} = z_{[d]\setminus i_0}$. Moreover, Equation 4.8 implies $y^1 \geq 0$ and $c = A^\top y^1$. Finally, we obtain the difference

$$b^\top \cdot y^1 - b^\top \cdot y^0 = {b_{J_0 \cup r_0}}^\top \left(\begin{pmatrix} y^0_{J_0} - \lambda_0\left(({A_{J_0}}^\top)^{-1} \cdot {a_{r_0}}^\top\right) \\ \lambda_0 \end{pmatrix} - \begin{pmatrix} y^0_{J_0} \\ 0 \end{pmatrix}\right) \ .$$

This simplifies to

$$\lambda_0 \left(b_{r_0} - {b_{J_0}}^\top ({A_{J_0}}^\top)^{-1} \cdot {a_{r_0}}^\top\right) \ .$$

With $x^0 = A_{J_0}^{-1} b_{J_0}$, $a_{r_0} \cdot x^0 > b_{r_0}$ and $\lambda_0 \geq 0$, the claim follows. □

If we continue the iteration with y^1 we obtain a sequence of d-subsets $J_0, J_1, \ldots, J_m$ of $[n]$. The sets in this sequence are pairwise disjoint since the sequence of the values $b^\top \cdot y^\ell$, which is defined by J_ℓ via Equation 4.7, is strictly decreasing. This implies the termination of the iteration as there are only finitely many subsets of $[n]$.

Remark 4.17. We could change y^ℓ after each iteration in a way that preserves the objective function value $b^\top \cdot y^\ell$ and the support. This would require a new computation of c. All the statements, in particular the ones concerning the termination of the algorithm, would remain valid.

4.4 Signed Tropical Matroids

As discussed in the previous section, the feasibility problem for tropical linear inequality systems is related to several other important algorithmic problems.

The generalization of the simplex method to oriented matroids in [Bla77, Fuk82, Tod85, Ter85], was a powerful step in the understanding of linear programming. In Section 4.6, we will present an algorithm which finds a feasible cell in a tropical analogue of an oriented matroid and does not cycle. For this, we will introduce an abstract version of covector graphs.

A purely axiomatic approach to grasp the crucial properties of the collection of covector graphs was started by Ardila and Develin in [AD09]. They introduced the name *tropical oriented matroid*. This approach was further developed in [OY11] and [Hor16]. Finally, Horn proved in [Hor16] that tropical oriented matroids encode exactly all subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$, not only regular ones, and also the so called *tropical pseudo-hyperplane arrangements*.

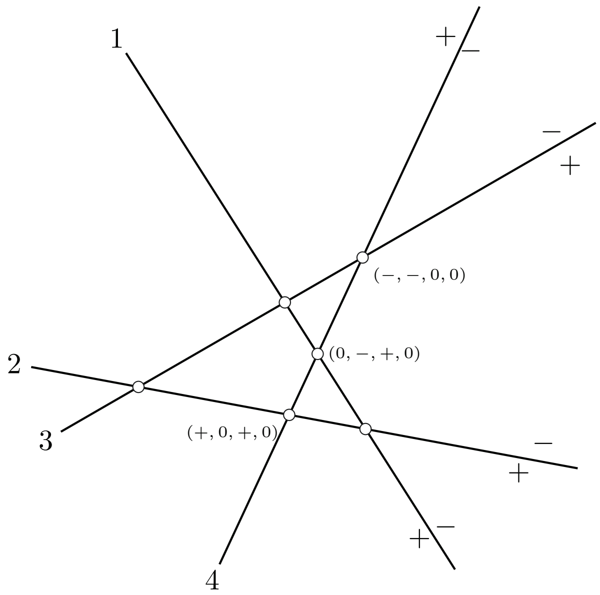

Figure 4.6: An affine halfspace arrangement in $\mathbb{R}^2$. The sign vectors denote in which halfspace of $1, 2, 3, 4$ the vertex of the arrangement lies. These signs form the sets J, K^+ and K^- in the explanation before Theorem 4.16.

4.4.1 A Description via Polytopes and Graphs

We briefly recall the basic polyhedral notions and point to [Zie95, DLRS10] for further reading. A *polytope* is the convex hull of finitely many points and a *polyhedron* is the intersection of finitely many halfspaces. By the Minkowski-Weyl theorem, polytopes are exactly the bounded polyhedra. The *face* of a polyhedron P is the intersection of P with a halfspace that does not contain an interior point of P. A *subpolytope* of a polytope P is the convex hull of a subset of the vertices of P. The convex hull of k affinely independent points, for $k \in \mathbb{N}$, is a $(k-1)$-*simplex* and is denoted by Δ_{k-1}. In the following, Δ_{k-1} stands for the convex hull of the k *standard basis vectors* $e_1, e_2, \ldots, e_k$ in $\mathbb{R}^k$, which is an instance of a $(k-1)$-simplex. The *product* of two polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^n$ is the convex hull of the pairs $(p, q) \in \mathbb{R}^{d+n}$ where p resp. q ranges over all the vertices of P resp. Q. Finally, a *polyhedral complex* is a finite set of polyhedra for which each face of a polyhedron is also contained in the set and the intersection of two polytopes is empty or a face of both. A polyhedral complex is a *(polyhedral) subdivision* of a polyhedron P if the union of all the occurring polyhedra is P. A polyhedral subdivision is a *triangulation* if every polytope is a simplex. A subdivision of a polytope $P \subset \mathbb{R}^d$ is *regular* if it is the orthogonal projection, omitting the last coordinate, of the bounded cells of the polyhedron $\operatorname{conv}\{(x, h(x)) \mid x \text{ vertex of } P\} + \mathbb{R}_{\geq 0} \cdot e_{d+1}$ for some height function $h \colon \mathbb{R}^d \to \mathbb{R}$.

We already saw in Proposition 4.3 and Proposition 4.7 that the feasibility of a point can be characterized by its covector graph with the signs on its edges. We aim to study a generalization of the collection of these covector graphs.

For a matrix $A \in \mathbb{R}^{n\times d}$, it was shown in [DS04, Theorem 1] that the collection of covectors is in bijection with the cells in the regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ with height function A. This was generalized in [FR15] and Chapter 2 to matrices with ∞ entries. For those, the collection of covectors defines a regular subdivision of a subpolytope of $\Delta_{n-1} \times \Delta_{d-1}$, see Corollary 2.34.

On the other hand, we start with a not necessarily regular subdivision of a subpolytope of $\Delta_{n-1} \times \Delta_{d-1}$ and will derive a *signed tropical matroid* from this. Note that non-regular triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ exist if and only if $(n-2)(d-2) \geq 4$, see [DLRS10, Theorem 6.2.19].

4.4.2 Axiom systems

Let $\mathcal{R}$ be a subdivision of a subpolytope $\mathcal{F}$ of $\Delta_{n-1} \times \Delta_{d-1}$. We identify subpolytopes of $\Delta_{n-1} \times \Delta_{d-1}$ and therefore the cells in $\mathcal{R}$ with subgraphs of the complete bipartite graph $K_{d,n}$ via the identification of the vertex (e_j, e_i) with the edge $(i,j) \in [d] \times [n]$. In this spirit, we define $\operatorname{conv}(G) = \operatorname{conv}\{(e_j, e_i) \mid (i,j) \in G\}$ for each subgraph G of $K_{d,n}$. Since all these graphs share the same node set $[d] \sqcup [n]$, we will often even identify them with their set of edges.

Let Σ be a sign matrix $(\sigma_{ji}) \in \{+,-,\bullet\}^{n\times d}$ for which $\sigma_{ji} = \bullet$ if and only if $(i,j) \notin \mathcal{F}$. Moreover, let $\mathcal{S}$ be the set of bipartite graphs without isolated nodes in $[n]$, which correspond to cells in $\mathcal{R}$.

We summarize the required properties which mostly are just adaptions of the definition of a polyhedral subdivision, see [DLRS10, Definition 2.3.1].

Definition 4.18. A signed tropical matroid (STM) is a pair $(\mathcal{S}, \Sigma)$ where $\mathcal{S}$ is a set of subgraphs of $K_{d,n}$ and $(\sigma_{ji}) = \Sigma$ is a matrix in $\{+,-,\bullet\}^{n\times d}$. It has an associated *finity graph* $\mathcal{F} = \bigcup_{G\in\mathcal{S}} G$, which represents the union over all the edges occurring in the graphs in $\mathcal{S}$. Additionally, Σ fulfills $\sigma_{ji} = \bullet \Leftrightarrow (i,j) \notin \mathcal{F}$. We require:

1. No graph in $\mathcal{S}$ has an isolated node in $[n]$.
2. If H is contained in $\mathcal{S}$ then so are all the subgraphs G of H that do not have an isolated node in $[n]$ and for which $\operatorname{conv}(G)$ is a face of $\operatorname{conv}(H)$.
3. For each point $x \in \operatorname{conv}(\mathcal{F})$ there is an $H \in \mathcal{S}$ such that $x \in \operatorname{conv}(H)$.
4. For all H and G in $\mathcal{S}$ with $H \neq G$, the intersection $\operatorname{conv}(H) \cap \operatorname{conv}(G)$ is a face of $\operatorname{conv}(H)$ and $\operatorname{conv}(G)$ or empty.

To emphasize the dependence on n and d we also say that $(\mathcal{S}, \Sigma)$ is a signed tropical (n,d)-matroid. We will often identify $\mathcal{S}$ with the subdivision corresponding to the set of bipartite graphs. The bipartite graphs are the *covector graphs* or just *covectors* in analogy with classical oriented matroids. An STM is *realizable* if it is induced by a matrix A, which means that the covector graphs are generalized covector graphs in the sense of Definition 4.5 or, equivalently, that the polyhedral subdivision corresponding to $\mathcal{S}$ is regular. In this case, we will also use the notation $\mathcal{S}(A)$. Note that the collection of generalized covectors graphs in the realizable case fulfills all the properties which are listed in the last definition.

As in the realizable case, we consider the entries of Σ as signs on the edges; we call an edge with $+$ a *positive edge* and with $-$ a *negative edge*. *Apex nodes* are the nodes in $[n]$ and *coordinate nodes* are those in $[d]$.

Remark 4.19. For each apex node $j \in [n]$, the set of covector graphs, in which j is only incident with negative edges, and the set of covector graphs, in which j is only incident with positive edges, form complementary *pseudohalfspaces* in the sense of [Hor12, Definition 5.5.8].

Example 4.20. The three full-dimensional simplices in the regular subdivision of $\Delta_1 \times \Delta_2$ in Figure 4.7 correspond to the three trees on $[2] \sqcup [3]$ with edge sets

$$\{(1,1),(1,2),(1,3),(2,3)\},\{(1,1),(1,2),(2,2),(2,3)\},\{(1,1),(2,1),(2,2),(2,3)\} \ .$$

The vertex of $\Delta_1 \times \Delta_2$ with label $(2,1)$ is hidden in the figure.

On the other hand, Figure 4.8 depicts a non-regular mixed subdivision of $4 \cdot \Delta_3$. By the Cayley trick ([DLRS10, §9]), triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ are in bijection with fine mixed subdivisions of $n\Delta_{d-1}$. In particular, the full-dimensional cells in the subdivision in Figure 4.8 are in bijection with the full-dimensional cells in a subdivision of $\Delta_3 \times \Delta_3$ and furthermore, the trees in an STM on $[4] \sqcup [4]$.

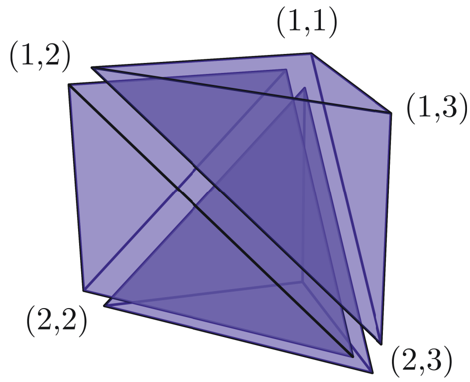

Figure 4.7: A regular subdivision of $\Delta_1 \times \Delta_2$. The vertices are labeled by the corresponding edges in $K_{3,2}$. This picture was created with `polymake` [GJ00].

Definition 4.21. An STM $(\mathcal{S}, \Sigma)$ is *full* if the finity graph is $K_{d,n}$. In this case, Σ contains only $-$ and $+$. For the realizable case, the definition means that all the entries of the coefficient matrix are finite. The STM is *generic* if the subdivision is a triangulation or equivalently by [DLRS10, Lemma 6.2.8], all the graphs are forests.

In Section 4.5, we describe a way to modify a given signed tropical matroid $(\mathcal{S}, \Sigma)$ to obtain a generic full signed tropical matroid $(\mathcal{T}, \Xi)$ with sparsely distributed signs. In the generic full case, we have a particularly nice characterization of the bipartite graphs which are trees and correspond to the maximal cells in the subdivision.

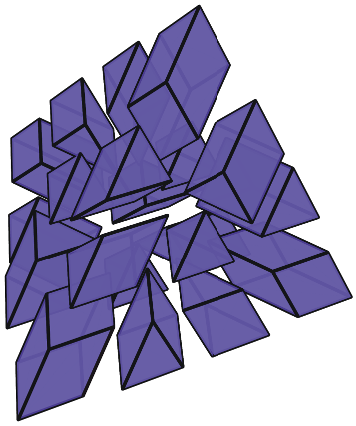

Figure 4.8: A non-regular subdivision of $\Delta_3 \times \Delta_3$. It is visualized as a non-regular mixed subdivision of $4\Delta_3$. This picture was created with the `polymake` extension `tropmat` by Silke Horn [Hor].

Proposition 4.22 (Proposition 7.2, [AB07])**.** *The trees in a full generic signed tropical matroid satisfy:*

1. *Each tree G is a spanning tree.*
2. *For each tree G and each edge e of G either $G - e$ has an isolated node or there is another tree G containing $G - e$.*
3. *There do not exist two distinct trees G and H, and a cycle of $K_{d,n}$ which alternates between edges of G and H.*

Condition (3) is essentially the same as the *comparability* in the axiom system for tropical oriented matroids in [AD09] and we will use this terminology in the following. Equivalently to (3) one could require, that for all $D \subseteq [d]$ and $N \subseteq [n]$ with $|D| = |N|$ there is at most one matching on $D \sqcup N$ which is contained in a tree in $\mathcal{T}$.

Proposition 4.7 justifies the following definition.

Definition 4.23. A covector graph G is *infeasible* if and only if there is an apex node in G which is only incident with negative edges. If G is not infeasible we call it *feasible*.

G is *totally infeasible*, if it is infeasible and every coordinate node is incident with a negative edge.

4.4.3 Matroid Operations and Feasibility

The following operations are useful for inductive arguments and yield the polyhedral methods to examine the boundary strata of the tropical projective space.

Analogously to classical oriented matroids one can define a tropical variant of the operations *deletion* and *contraction*, like in [AD09]. In the following, let $(\mathcal{S}, \Sigma)$ be a signed tropical (n, d)-matroid

For an apex node $j \in [n]$, the *deletion* $\mathcal{S}_{\setminus j}$ is the set of graphs which arise from the graphs of $\mathcal{S}$ by deleting the node j and the incident edges. These graphs describe the cells on the face $\{ e_\ell \mid \ell \in [n] \setminus j\} \times \Delta_{d-1}$ of $\Delta_{n-1} \times \Delta_{d-1}$. We delete the jth row in the sign matrix. If $(\mathcal{S}, \Sigma)$ is induced by a signed system (A, Σ) then the operation corresponds to deleting the jth row of A.

For a coordinate node $i \in [d]$, the *contraction* $\mathcal{S}_{/i}$ is the set of graphs which arise from those graphs of $\mathcal{S}$ for which i is isolated by deleting the node i. These graphs describe the cells on the face $\Delta_{n-1} \times \{ e_\ell \mid \ell \in [d] \setminus i\}$ of $\Delta_{n-1} \times \Delta_{d-1}$. We delete the ith column in the sign matrix. If $(\mathcal{S}, \Sigma)$ is induced by a signed system (A, Σ) then the operation corresponds to deleting the i column of A.

By construction, a deletion and a contraction of an STM is again an STM.

Remark 4.24. Note that the formerly described operations are also related to classical matroid operations since products of simplices are matroid polytopes in the classical sense; see [GGMS87]. However, there is no direct translation and one should be careful not to confuse the tropical with the classical operation.

For the contraction $\mathcal{S}_{/([d]\setminus D)}$, where $\mathcal{S}$ is defined on $[d]$ and $D \neq \emptyset$, we will also write $\mathcal{S}|_D$. In the realizable case, these are the covectors of the points with support D. We only consider points in $\mathbb{TA}^d = \mathbb{T}^d_{\min} \setminus \{(\infty, \ldots, \infty\}$ which corresponds to $D \neq \emptyset$.

Lemma 4.25. *For the finite matrix $A \in \mathbb{R}^{n \times d}$, the covector graphs in the contraction $\mathcal{S}(A)|_D$ for any non-empty $D \subseteq [d]$ are exactly the generalized covectors of the points with support D.*

Proof. Fix a point $x \in \mathbb{TA}^d$ with support $D \subseteq [d]$ and let

$$\omega > 2 \cdot \max\left(\max\{ x_\ell \mid \ell \in \operatorname{supp}(x)\}, \max\{ |a_{ji}| \mid (i,j) \in [d] \times [n]\}\right) .$$

Then the generalized covector graph of x is the same as the proper covector graph of the point $z \in \mathbb{R}^d$ with

$$z_i = \begin{cases} x_i & \text{for } i \in \operatorname{supp}(x) \\ \omega & \text{else} \end{cases} .$$

The other inclusion follows by setting the coordinates of isolated coordinate nodes to ∞. $\square$

With the definition we can now formulate an important consequence of the existence of a totally infeasible covector in a generic full STM. This is visualized in Figure 4.9.

Lemma 4.26. *If a covector graph G in a generic full STM $(\mathcal{T}, \Xi)$ is totally infeasible, then in every covector graph H of any contraction of $(\mathcal{T}, \Xi)$ there is a node in $[n]$ which is only incident with a negative edge.*

Proof. By definition, G is infeasible and there is a matching of negative edges μ on $[d] \sqcup N$ for some subset $N \subseteq [n]$.

Notice that each covector graph in a contraction is constructed from a covector graph of $(\mathcal{T}, \Xi)$. Since one only removes isolated coordinate nodes, feasibility or infeasibility carries over to the contracted covector.

Now, let H be any covector graph in $(\mathcal{T}, \Xi)$. Assume H is feasible. This implies that each apex node $j \in N$ is incident with an edge which does not lie in μ and, hence, is positive. Pick for each node in N one incident positive edge from H. This forms a cover η of N. Moreover, let D be the subset of the coordinate nodes $[d]$ which is covered by η. Then the graph on $D \sqcup N$ with edge set $\mu|_D \cup \eta$, where $\mu|_D$ are those edges in μ incident with D, contains a cycle C. This follows as it has $|D| + |N|$ nodes and $|\mu|_D| + |\eta| \geq |D| + |N|$ edges. Since every node in D is only incident with one edge from $\mu|_D$ and every node in N is only incident with one edge from η and at most one edge from $\mu|_D$, the cycle C has to be alternating between μ and η. However, this contradicts the comparability in Proposition 4.22. $\square$

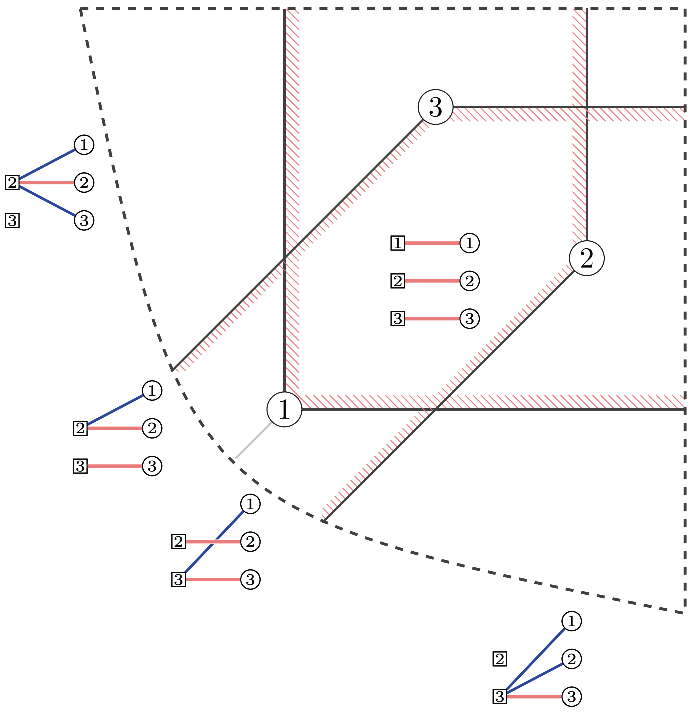

Figure 4.9: A configuration which contains a totally infeasible covector. The shaded bars indicate the *infeasible regions*. The dashed lines denote the boundary strata of the tropical projective space. The covectors on the boundary stratum corresponding to the contraction $\mathcal{T}|_{\{2,3\}}$ are also depicted and infeasible.

4.4.4 Existence of Particular Covector Graphs

We start with a Menger-type lemma; see [Bol98, §3] for similar results. It is purely graph theoretic but contains an important property for covector graphs.

Lemma 4.27. *Let G be a bipartite tree on the node set $D \sqcup N$ for arbitrary sets D and N with $|D| = k + 1$ and $|N| = k$ with a positive integer k. If the nodes in N all have* *degree 2 then, for each $i \in D$, the graph G with i deleted contains a perfect matching. Furthermore, G is the union of these matchings.*

Proof. Fix an arbitrary $i_0 \in D$. Since G is a tree, it has at least two leafs. In particular, there is an $i \in D \setminus \{i_0\}$ which is a leaf in G. Let $j \in N$ be the node adjacent to i. Deleting i and j yields a graph H on $(D \setminus \{i\}) \sqcup (N \setminus \{j\})$ for which each node in $N \setminus \{j\}$ has degree 2.

Proceeding by induction implies the claim about the containment of the matchings.

Furthermore, each edge is contained in such a perfect matching. For this, pick an arbitrary edge $(i, j) \in G$. Let $\ell \in D$ be the node distinct from i which is adjacent to j. Then (i, j) is contained in the perfect matching on $(D \setminus \{\ell\}) \sqcup N$. □

The following result guarantees the existence of covector graphs with specific degree conditions. It is crucial in the transition from realizable to non-realizable considerations.

For the rest of this subsection let $\mathcal{T}$ be a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$

Recall that, by the Cayley trick ([DLRS10, §9]), triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ are in bijection with fine mixed subdivisions of $n\Delta_{d-1}$. This implies the following for the collection of bipartite graphs which correspond to the full-dimensional simplices in $\mathcal{T}$.

Proposition 4.28 ([OY11, Proposition 2.5])**.** *Let $(d_1, \ldots, d_n) \in [d]^n$ with $\sum_{j=1}^{n} d_j = n + d - 1$. There is exactly one tree in $\mathcal{T}$ for which each node $j \in [n]$ has degree d_j.*

Note that a similar statement was proven in [DJS12, Proposition 4.2]. Because of the importance to us, we give a proof independently of [OY11].

Proof. Let the *right degree sequence (RDS)* be the sequence of degrees of the apex nodes.

By [DLRS10, Theorem 6.2.13], which uses the unimodularity, respectively the equidecomposability, of $\Delta_{n-1} \times \Delta_{d-1}$, the number of full-dimensional simplices in a triangulation is $\binom{n+d-2}{n-1}$.

Furthermore, the number of compositions of $n + d - 1$ in n parts is $\binom{n+d-2}{n-1}$.

Hence, it suffices to prove that each sequence $(d_1, \ldots, d_n) \in [d]^n$ with $\sum_{j=1}^{n} d_j = n+d-1$ occurs at most once as an RDS. We describe a construction to find a canonical form for a covector graph with a given RDS which will imply the claim. This approach is depicted in Figure 4.10.

Next, note that we can omit apex nodes of degree 1 as the graph remaining after this removal is still a tree. So, consider two distinct trees t_0 and t_1 with the same RDS $(d_1, \ldots, d_n)$ for which each degree is bigger than 1. From these trees, we construct trees s_0 and s_1 for which each apex node has degree 2. For this, we replace each apex node $j \in [n]$ of degree $d_j > 2$ with $d_j - 1$ nodes $k_1^j, \ldots, k_{d_j-1}^j$. Furthermore, if $i_{j_1} \leq \ldots \leq i_{j_{d_k}}$ are the neighbors of j, we add the edges

$$(i_{j_1}, k_1^j), (i_{j_2}, k_1^j), (i_{j_2}, k_2^j), \ldots, (i_{j_{d_k-1}}, k_{d_j-1}^j), (i_{j_{d_k}}, k_{d_j-1}^j) \ .$$

Hence, s_0 and s_1 are trees on the vertices $[d] \sqcup R$, where R is the d-set formed by the old apex nodes of degree 2 and the new apex nodes which arose from replacing apex nodes of degree > 2. By Lemma 4.27, these trees are the union of $(d-1) \times (d-1)$-matchings on $[d] \setminus \{i\} \sqcup R$ for all $i \in [d]$. From the uniqueness of the construction of s_0 resp. s_1 from t_0 resp. t_1 we deduce that s_0 and s_1 are also distinct. Therefore, there is an $i \in [d]$ for which the perfect matching μ_0 in s_0 on $[d] \setminus \{i\} \sqcup R$ and the perfect matching μ_1 in s_1 on $[d] \setminus \{i\} \sqcup R$ disagree. We conclude that their symmetric difference contains a non-trivial simple cycle C. If we contract the nodes $k_1^j, \ldots, k_{d_j-1}^j$ back to the single node j for each apex node $j \in [n]$ of degree $d_j > 2$, then C becomes a cycle (where a node can appear multiple times). Since t_0 and t_1 are distinct, the cycle has to contain more than 1 apex node. Such a cycle is an alternating cycle in the sense of the comparability in Proposition 4.22. This implies that t_0 and t_1 cannot both occur in the same triangulation. $\square$

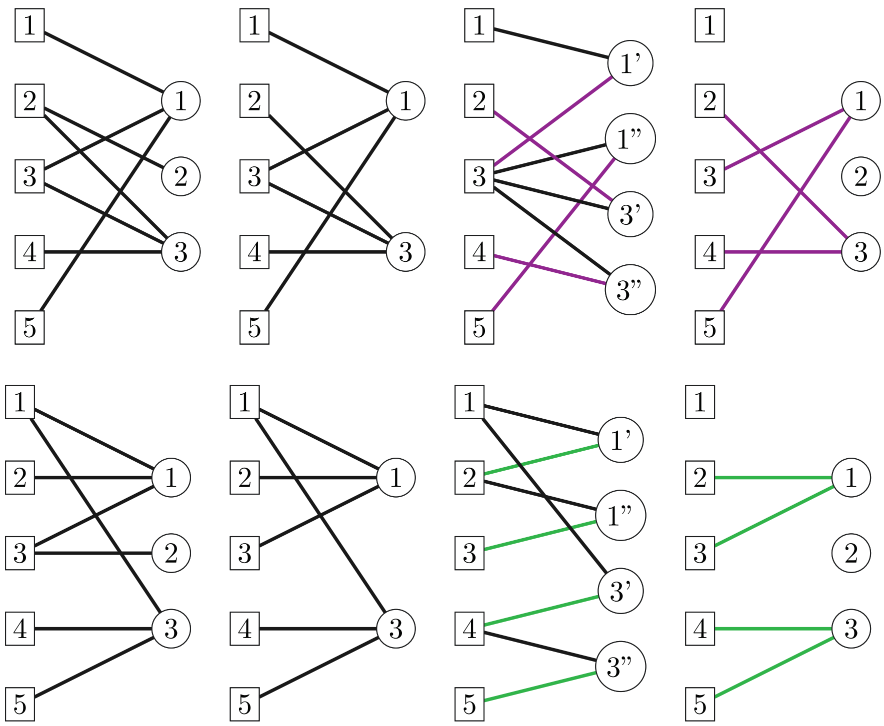

Figure 4.10: The construction to find an alternating cycle from the proof of Proposition 4.28.

We define *Cramer covectors* $\mathcal{C}(N, D \cup \{\delta\})$, where $\delta \in [d]$, $D \subseteq [d] \setminus \{\delta\}$ and $N \subseteq [n]$ with $|D| = |N|$, as the covector graphs in the contraction $\mathcal{T}|_{\{D \cup \delta\}}$ for which each node in N has degree 2. The former lemma guarantees the existence of Cramer covectors in a full generic STM which does not have to be realizable. Note that it is also valid for $D = N = \emptyset$. The Cramer covectors are similar to linkage trees in the sense of [SZ93]; the difference is that the colors of the edges of the linkage tree are replaced by a second class of nodes which yields the bipartite Cramer covectors.

We saw already in Lemma 4.28 and Lemma 4.27 that Cramer covectors have a particularly useful structure. We exploit this to construct Cramer covectors in a fixed STM inductively.

Proposition 4.29. *Let $D \subseteq [d]$, $\delta \in [d] \setminus D$ and $N \subseteq [n]$ with $|N| = |D|$. Furthermore, let y be a covector graph in the contraction $\mathcal{T}|_D$ containing a perfect matching μ on $D \sqcup N$. Then $\mathcal{C}(N, D \cup \{\delta\})$ contains μ.*

Proof. Applying Proposition 4.28 to $\mathcal{T}|_{(D\cup\{\delta\})}$ yields the existence of the covector graph $\mathcal{C}(N, D \cup \{\delta\})$ which has degree 2 for every node in N and degree 1 for the nodes in $[n] \setminus N$. By Lemma 4.27, the induced subgraph of $\mathcal{C}(N, D \cup \{\delta\})$ on $(D \cup \{\delta\}) \sqcup N$ contains a matching on $D' \sqcup N$ for every $|D|$-element subset D' of $(D \cup \{\delta\})$. Especially, it contains a perfect matching on $D \sqcup N$.

By the definition of the contraction $\mathcal{T}|_D$, there is a covector graph $\overline{y}$ in $\mathcal{T}|_{(D\cup\{\delta\})}$ extending y. The comparability condition from Proposition 4.22 yields that the two graphs $\overline{y}$ and $\mathcal{C}(N, D \cup \{\delta\})$ must contain the same matching μ on $D \sqcup N$. ☐

4.4.5 Computations for Realizable Covector Graphs

Starting from a proper covector graph, the next lemma allows us to compute a point with given covector graph.

Let G be a connected covector graph with respect to $A \in \mathbb{T}_{\min}^{n\times d}$ and $\delta \in [d]$ a coordinate node. For any other coordinate $i \in [d]$, let $\delta = i_1, j_1, i_2, \ldots, i_s, j_s, i_{s+1} = i$ be any path from δ to i in G. By the definition of a covector graph, we obtain the sequence of equations $a_{j_t i_t} + x_{i_t} = a_{j_t i_{t+1}} + x_{i_{t+1}}$ for all the tuples (i_t, j_t, i_{t+1}) with $t \in [s]$. Summing up these equations yields $\sum_{t=1}^{s}(a_{j_t i_t} + x_{i_t}) = \sum_{t=1}^{s}(a_{j_t i_{t+1}} + x_{i_{t+1}})$. Equivalently, we obtain

$$\sum_{t=1}^{s} x_{i_{t+1}} - \sum_{t=1}^{s} x_{i_t} = \sum_{t=1}^{s} a_{j_t i_t} - \sum_{t=1}^{s} a_{j_t i_{t+1}}$$

and hence, $x_i - x_\delta = x_{i_{s+1}} - x_{i_1} = \sum_{t=1}^{s} a_{j_t i_t} - \sum_{t=1}^{s} a_{j_t i_{t+1}}$. These equations define x uniquely up to adding multiples of the all ones vector. Since we assumed G to be a covector graph, these necessary conditions are also sufficient. This construction is visualized in Figure 4.11. It proves the following.

Lemma 4.30. *The covector graph of x with respect to A is G.*

For subsets $I \subseteq [d]$ and $J \subseteq [n]$ with $|J| = |I| - 1$ we define the *tropical Cramer solution* $A[J|I] \in \mathbb{T}^d$ by

$$A[J|I]_i = \begin{cases} \operatorname{tdet}(A_{J,I\setminus\{i\}}) & \text{for each } i \in I \\ \infty & \text{else} \quad . \end{cases}$$

To cover the case $J = \emptyset$, we set $\operatorname{tdet}(A_{\emptyset,\emptyset}) = 0$.

These vectors occur as solutions to homogeneous tropical equality systems, see, e.g., [GP97, Theorem 18], [RGST05, Corollary 5.4]. For an extensive study of this computational problem see [AGG14].

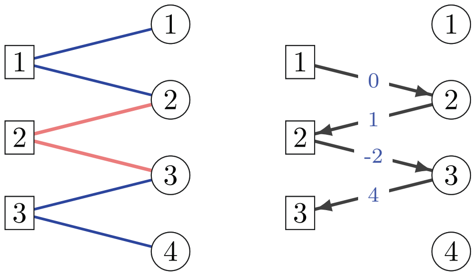

Figure 4.11: The computation of the point $(0, 1, 3)$ for a prescribed covector graph from Example 4.8.

Remark 4.31. [AGG14, Theorem 4.18] implies that the covector graph of $A[J|I]$ for a generic, finite A is just the Cramer covector $\mathcal{C}(J, I)$ since there is a unique covector graph with the prescribed degree sequence. We will determine the covector graph for the non-generic case in Lemma 4.36.

Now, let $A \in \mathbb{T}^{n\times d}$ be an arbitrary matrix. We denote the generalized covector graph of $A[J|I]$ by $\mathcal{C}_A(J, I)$.

Example 4.32. Consider again the matrix A from Example 4.8. The point $(0, 1, 3)$ has the covector graph depicted on the left of Figure 4.11. On the right is the auxiliary weighted directed graph for computing the point from the covector graph.

It is the Cramer solution $\mathcal{C}_A(\{2, 3\}, \{1, 2, 3\})$.

Lemma 4.33. *Let $A \in \mathbb{T}^{(d-1)\times d}$ with $d \in \mathbb{N}$ and x the Cramer solution for this matrix. Then $|x_i - x_h| \leq 2 \cdot d \cdot \max\{|a_{ij}| \mid a_{ij} \neq \infty, (i, j) \in [d] \times [n]\}$ for any $i, k \in [d]$ with $x_i \neq \infty \neq x_k$.*

Proof. This follows from the definition of Cramer solution with the triangle inequality. $\square$

4.5 Polyhedral Constructions

4.5.1 Refinement

The graphs in an STM $(\mathcal{S}, \Sigma)$ have a particularly simple form if $\mathcal{S}$ is a triangulation. Recall from Definition 4.21 that, in this case, we call the STM *generic* and [DLRS10, Lemma 6.2.8] tells us that all the graphs are forests and, especially, that the maximal polytopes in the subdivision are represented by trees. A method to construct a generic STM is by *refining* our subdivision $\mathcal{S}$. This means that we construct a triangulation $\mathcal{T}$ such that each polytope in $\mathcal{S}$ is the union of simplices in $\mathcal{T}$. Hence, every covector graph of $\mathcal{T}$ is a forest and contained in a covector graph of $\mathcal{S}$. This idea is implicitly used in [ABGJ14a] in the perturbation of tropical linear inequality systems.

Since we want to preserve the feasibility of our system, we choose to refine our subdivision with heights defining a *lexicographic triangulation*. By [DLRS10, Definition 4.3.8], the lexicographic triangulation for a point configuration with $k \in \mathbb{N}$ points is the regular triangulation with heights $\psi_i \cdot c^i$ for $i \in [k]$ where $(\psi_1, \dots, \psi_k) \in \{-, +\}^k$ is a sign vector and c is a sufficiently big positive number.

Now, let the matrix $(m_{ji}) = M \in \mathbb{R}^{n \times d}$ contain the heights for a lexicographic triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ for which we only require that the sign pattern of M is the negative of the sign pattern of Σ and that $m_{ji} = \infty \Leftrightarrow \sigma_{ji} = \bullet$.

By [DLRS10, Lemma 2.3.16 & Corollary 2.3.18], we obtain a refinement of $\mathcal{S}$ with respect to M by taking the union of the subdivisions arising by restricting M to the cells of $\mathcal{S}$. Formally this means: Restricting M to the vertices of a cell C in $\mathcal{S}$ induces a regular subdivision of C which we denote by $C|_M$. The union $\bigcup_{C \in \mathcal{S}} C|_M$ of the simplices in each triangulation $C|_M$ is a triangulation of $\mathcal{F}$ which refines $\mathcal{S}$.

In the realizable case, [DLRS10, Lemma 2.3.16] implies that the height matrix corresponding to the refined subdivision is obtained by adding a small multiple of the perturbation matrix M.

The refinement $\mathcal{T}$ of the subdivision $\mathcal{S}$ with the matrix M fulfills the following:

Lemma 4.34. *Let G be a maximal covector graph of $\mathcal{S}$ and $G_1, \dots, G_k$ the maximal covector graphs of $\mathcal{T}$ contained in G. Then G is infeasible if and only if G_ℓ is infeasible for every $\ell \in [k]$.*

Proof. If G is infeasible, there is an apex node which is only incident with negative edges. Since each G_ℓ is a connected subgraph of G without isolated nodes it also contains an apex node which is only incident to negative edges. Hence, it is infeasible.

Now, let G be feasible. For the covector graph G we define the matrix $M|_G$ by replacing every entry m_{ji} of M by ∞ for which (i, j) is not an edge of G. By construction, the polytope in the subdivision $\mathcal{S}$ corresponding to the covector graph G is split up in those polytopes whose corresponding graphs occur as maximal covector graphs in the covector decomposition with respect to $M|_G$. Since no apex node in G is only incident with negative arcs, the signed system $(M|_G, \Sigma)$ has the feasible point $\mathbf{0}$ by the choice of M. Then the maximal covector graphs which contain the covector of $\mathbf{0}$ are feasible. This implies the existence of a feasible covector. $\square$

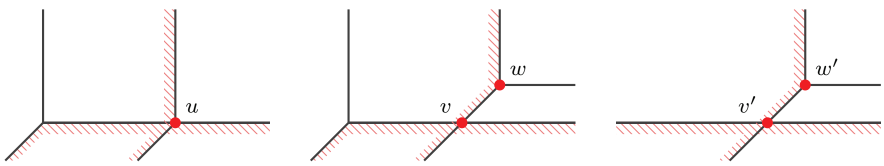

Figure 4.12: The perturbation of the signed system for the left picture yields the middle one which locally looks like the right one. See Example 4.35.

Example 4.35. Consider the signed system (A, Σ) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} + & + & - \\ + & - & + \end{pmatrix} .$$

For a sufficiently big $c \gg 1$ we construct the matrix

$$M = \begin{pmatrix} -c^1 & -c^2 & c^3 \\ -c^4 & c^5 & -c^6 \end{pmatrix}$$

with the negative of the sign pattern of Σ. For the covector graph G of the point $(0, 2, 0)$ on the left of Figure 4.13 this yields (with $M|_G$ as in the proof of Lemma 4.34)

$$A + \varepsilon \cdot M = \begin{pmatrix} -\varepsilon c^1 & -\varepsilon c^2 & \varepsilon c^3 \\ -\varepsilon c^4 & -2 + \varepsilon c^5 & -\varepsilon c^6 \end{pmatrix} \quad \text{and} \quad M|_G = \begin{pmatrix} -c^1 & \infty & c^3 \\ -c^4 & c^5 & -c^6 \end{pmatrix} ,$$

where $\varepsilon > 0$ is sufficiently small. Figure 4.12 shows the original configuration for A, the perturbed configuration for $A + \varepsilon \cdot M$ and the local configuration for $M|_G$. The points are $u = (0, 2, 0)$, $v = (\varepsilon(c^3 + c^5), 2 - \varepsilon(c^1 + c^6), \varepsilon(-c^1 + c^5))$, $w = (\varepsilon c^4, 2 - \varepsilon c^5, \varepsilon c^6)$, $v' = (c^3 + c^5, -c^1 - c^6, -c^1 + c^5)$ and $w' = (c^4, -c^5, c^6)$. Figure 4.13 depicts their covector graphs. The left one is the covector graph of u, the middle one of v and v', the right one of w and w'.

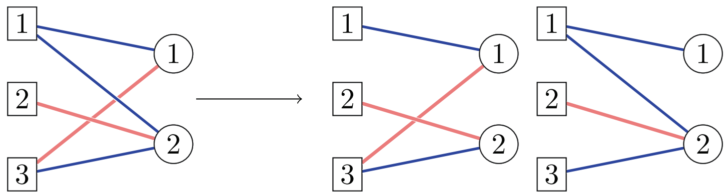

Figure 4.13: The covector graph is replaced by two trees in the refinement.

We also apply the perturbation technique to get a description of a Cramer solution in the non-generic case.

Lemma 4.36. *For a finite matrix $A \in \mathbb{R}^{n \times d}$, which is not necessarily generic, the Cramer covector $\mathcal{C}_A(J, I)$ is the union of all minimal matchings on $(I \setminus \{i\}) \sqcup J$ for all $i \in I$ and on $I \sqcup (J \cup \{j\})$ for all $j \in [n] \setminus J$.*

Proof. Let $\widehat{A}$ be any matrix which induces a triangulation that refines the subdivision induced by A in the sense of [DLRS10, Lemma 2.3.16]. Then there is a covector graph H with respect to A, which contains $G = \mathcal{C}_{\widehat{A}}(J, I)$.

By Proposition 4.4, each matching in H is a minimal matching. Since H contains matchings on $(I \setminus \{i\}) \sqcup J$ for all $i \in I$ and $I \sqcup (J \cup \{j\})$ for all $j \in [n] \setminus J$, it contains all minimal matchings on these vertex sets by the same Proposition. Therefore, we have to show that $H = \mathcal{C}_A(J, I)$.

Since G is connected, so is H, and we can apply Lemma 4.30 to construct a point $x \in \mathbb{R}^d$ which has H as covector graph with respect to A.

Fix a coordinate node $\delta \in I$. For any $i \in I \setminus \{\delta\}$, the path from δ to i is the symmetric sum of the perfect matchings in G on $(I \setminus \{\delta\}) \sqcup J$ and $(I \setminus \{i\}) \sqcup J$. With Lemma 4.30, we obtain that $x_i - x_\delta$ is the difference of the values of the two matchings. As these are minimal matchings, the values equal the determinants. This implies $x_i - x_\delta = \operatorname{tdet}(A_{J,(I \setminus \{i\})}) - \operatorname{tdet}(A_{J,(I \setminus \{\delta\})})$. As x is defined by its covector graph only up to addition of multiples of $\mathbf{1}$, the claim follows. $\square$

4.5.2 Extension from a Subpolytope to $\Delta_{n-1} \times \Delta_{d-1}$

We introduce a construction which allows us to reduce the general case, where the finity graph is a subgraph of $K_{d,n}$, to the complete bipartite graph. This is particularly important as we define the algorithms in Section 4.6 only for a full STM. We give the justification for why we do not lose generality, and provide technical details for later reductions. We achieve this again by polyhedral means. The following technique was also applied to tropical oriented matroids in [Hor16].

Let $\mathcal{F}$ be a subpolytope of $\Delta_{n-1} \times \Delta_{d-1}$ and $\mathcal{S}$ a subdivision of $\mathcal{F}$. An *extension* of $\mathcal{S}$ is a subdivision $\mathcal{T}$ of $\Delta_{n-1} \times \Delta_{d-1}$ which coincides with $\mathcal{S}$ on $\mathcal{F}$.

Placing triangulations provide a tool to construct an extension of a subdivision, see [DLRS10, Lemma 4.3.2]. In particular, for each subdivision of a subpolytope of $\Delta_{n-1} \times \Delta_{d-1}$ there is always an extension. To resolve the $\bullet$ entries of the sign matrix, we just replace them by $+$. We denote the modified sign matrix by Ξ. Note that the (in)feasibility of the covector graphs in $\mathcal{S}$ is preserved in $\mathcal{T}$.

We summarize these considerations.

Proposition 4.37. *The set of covectors in the STM defined by $(\mathcal{S}, \Sigma)$ is contained in the set of covectors defined by $(\mathcal{T}, \Xi)$.*

We study in more detail how an extension can be produced in the realizable case.

[DLRS10, Lemma 4.3.4] shows that a placing triangulation can be obtained by taking a rapidly increasing height function. Namely, if there are $k < n \cdot d$ entries with ∞ in $A \in \mathbb{T}_{\min}^{n \times d}$, let $\Omega = (\Omega_1, \ldots, \Omega_k)$ be a vector of "big" numbers. We require that

$$\Omega_1 > \sum_{a_{ji} \neq \infty} |a_{ji}| \quad \text{and} \quad \Omega_{\ell+1} > \sum_{a_{ji} \neq \infty} |a_{ji}| + \sum_{h=1}^{\ell} \Omega_h \quad \text{for all } \ell \in [k-1] \ . \tag{4.9}$$

We will calculate with the entries of Ω just formally and denote the resulting matrix by $A(\Omega)$.

Remark 4.38. One can think of these Ω_ℓ as artificial infinities. One approach to formalize this is by successively adjoining elements to $\mathbb{R}$. Here, the order extends the natural order on $\mathbb{R}$ such that Ω_ℓ is the biggest element in each extension step. In [ABGJ14a, §3.2], a similar technique with "infinitely small" values is used to reduce the case with $-\infty$ to the finite case.

To show that the matrix $A(\Omega)$ induces an extension of the subdivision of $\mathcal{F}$ by A, we iteratively replace the ∞ entries by the entries of Ω. Let A^1 be obtained from A by replacing one ∞ entry, which belongs to the edge e, with a positive number Ω_1 which is bigger than the sum of the absolute values of the finite entries of A. Consider an arbitrary maximal covector graph G with respect to A and let μ be a perfect matching on $D \sqcup N \subseteq [d] \sqcup [n]$ in G. By Proposition 4.4, the matching μ is minimal with respect to the coefficients of A^1. Hence, by definition of Ω_1, the edge e cannot be contained in μ. Since this is true for any matching in G, again by Proposition 4.4, the graph G is also a covector with respect to A^1. By iteratively inserting $\Omega_1, \Omega_2, \ldots, \Omega_k$ for the ∞ entries, this implies that the subdivision induced by $A(\Omega)$ extends the subdivision induced by A, since a polyhedral complex is given by its maximal cells. Furthermore, if A induces a triangulation, so does $A(\Omega)$.

We say that the signed system $(A(\Omega), \Xi)$ *extends* the signed system (A, Σ).

Lemma 4.39. *For the matrix $A \in \mathbb{T}_{\min}^{n\times d}$, let $(A(\Omega), \Xi)$ be an extension of the signed system (A, Σ). For any $x \in \mathbb{TA}^d$, the generalized covector graph $G_A(x)$ is infeasible, if the generalized covector graph $G_{A(\Omega)}(x)$ is infeasible.*

Proof. Within the proof, we denote $A(\Omega)$ by $(\widetilde{a_{ji}}) = \widetilde{A}$. Fix an arbitrary $x \in \mathbb{TA}^d$. If the generalized covector graph $G_{\widetilde{A}}(x)$ is infeasible, there is a $j_0 \in [n]$, which is only incident with negative edges in $G_{\widetilde{A}}(x)$. Let I be the set of coordinate nodes adjacent to j_0. Since the entries of $\widetilde{A}$ are finite, $G_{\widetilde{A}}(x)$ is a proper covector graph on the support of x. Hence, using the definition of the covector graph, we see that x fulfills the inequalities

$$\widetilde{a_{j_0 i}} + x_i < \widetilde{a_{j_0 \ell}} + x_\ell \qquad \text{for all } i \in I \text{ and } \ell \in \operatorname{supp}(x) \setminus I \ .$$

Each entry $\widetilde{a_{j_0 i}}$ with $i \in I$ equals $a_{j_0 i} \neq \infty$ because (j_0, i) is negative. With $\widetilde{a_{j_0 \ell}} \leq a_{j_0 \ell}$ for $\ell \in \operatorname{supp}(x) \setminus I$, we obtain

$$a_{j_0 i} + x_i < a_{j_0 \ell} + x_\ell \qquad \text{for all } \ell \in \operatorname{supp}(x) \setminus \{i\} \ .$$

This implies that $G_A(x)$ is infeasible. $\square$

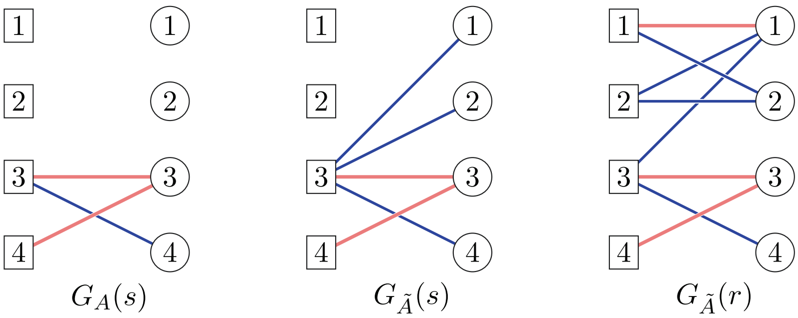

Figure 4.14: Three covector graphs for Example 4.40.

Example 4.40. Consider the signed systems (A, Σ) and $(\widetilde{A}, \Xi)$ with

$$A = \begin{pmatrix} 0 & 0 & \infty & \infty \\ 1 & 1 & \infty & \infty \\ \infty & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \Sigma = \begin{pmatrix} - & + & \bullet & \bullet \\ + & - & \bullet & \bullet \\ \bullet & + & - & - \\ + & - & + & + \end{pmatrix}$$

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & \Omega_1 & \Omega_2 \\ 1 & 1 & \Omega_3 & \Omega_4 \\ \Omega_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \Xi = \begin{pmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & - \\ + & - & + & + \end{pmatrix} .$$

They yield the Cramer solutions $s = \mathcal{C}_A([3],[4]) = (\infty,\infty,1,1)$ and $r = \mathcal{C}_{\widetilde{A}}([3],[4]) = (\Omega_1+1,\Omega_1+1,1,1)$. The corresponding covector graphs are left and right in Figure 4.14. The relation between the left and middle covector illustrates Lemma 4.39.

4.5.3 Splitting Apex Nodes

To apply the algorithms that will be presented in Section 4.6 and 4.7 to an STM (A,Σ), we require that each row of Σ contains at most one negative entry. We call this property *trimmed*.

In the realizable case, this can be obtained very easily. Through the conversion

$$c_0 \le \bigoplus_{\ell\in[m]} c_\ell \quad \Leftrightarrow \quad \left(c_0 \le c_\ell \quad \forall \ell \in [m]\right) \ , \tag{4.10}$$

for arbitrary $c_0, c_1, \ldots, c_m \in \mathbb{T}_{\min}$ each tropical inequality system is equivalent to a system for which the minimum on the bigger side of the new inequalities contains only one term. Here, the number of inequalities is increased by a factor which is at most the number of coordinates, see Figure 4.15.

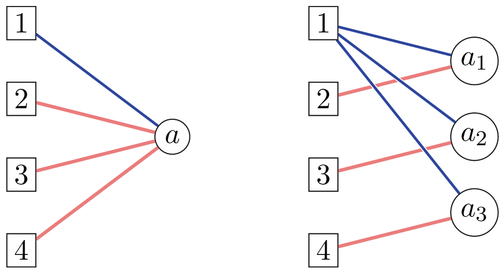

Figure 4.15: An apex node whose corresponding row in the sign matrix has three negative entries is replaced by three apex nodes.

This splitting of apex nodes was similarly used in [MSS04, §7.4].

For the non-realizable case, we use a more complicated polyhedral construction, which uses local changes. In two steps, we obtain a bigger STM which mimics a splitting of the inequalities in its covector graphs. A similar technique was used in [Hor16, §7.2]. We know how to extend a non-full STM, by Subsection 4.5.2, and can assume that the STM is full.

Let $k > 1$ entries of the nth row of Σ be $-$.

Define the projection $\pi\colon \mathbb{R}^{n-1+k} \times \mathbb{R}^d \to \mathbb{R}^n \times \mathbb{R}^d$ as

$$(y_1,\ldots,y_{n-1},y_n,\ldots,y_{n+k-1},z_1,\ldots,z_d) \mapsto \left(y_1,\ldots,y_{n-1},\sum_{\ell=0}^{k-1} y_{n+\ell},z_1,\ldots,z_d\right) \ .$$

This defines a surjective mapping from $\Delta_{n-1+k-1} \times \Delta_{d-1}$ onto $\Delta_{n-1} \times \Delta_{d-1}$ and furthermore, a surjective mapping from the subgraphs of $K_{d,n+k-1}$ to the subgraphs of $K_{d,n}$.

Lemma 4.41. *The preimage under π of a simplex in $\Delta_{n-1} \times \Delta_{d-1}$, given by the bipartite graph G, is $G \cup \{(i, n+\ell) \mid (i,n) \in G, \ell \in [k-1]\}$.*

Proof. Let H be any spanning subgraph of $K_{d,n+k-1}$. This defines a subpolytope of $\Delta_{n-1+k-1} \times \Delta_{d-1}$. A convex combination of its vertices is given by $\sum_{(i,j)\in H} \lambda_{i,j}(e_j, e_i)$ with $\sum_{(i,j)\in H} \lambda_{ij} = 1$. With the linearity of π, the projection of this point is

$$\sum_{(i,j)\in H, j\le n-1} \lambda_{i,j}\pi((e_j, e_i)) + \sum_{(i,j)\in H, j\ge n} \lambda_{i,j}\pi((e_j, e_i))$$

which evaluates to

$$\sum_{(i,j)\in H, j\le n-1} \lambda_{i,j}(e_j, e_i) + \sum_{(i,j)\in H, j\ge n} \lambda_{i,j}(e_n, e_i) \ .$$

Such a point lies in $\operatorname{conv}\{(e_j, e_i) \mid (i,j) \in G\}$ if and only if, for $\lambda_{ij} \neq 0$,

$$(i,j) \in H \Leftrightarrow \begin{cases} (i,j) \in G & \text{for } j \le n-1 \\ (i,n) \in G & \text{for } j > n-1 \ . \end{cases}$$

With the linearity of π, the claim follows. □

Fix an arbitrary $\varepsilon > 0$ and let $i_1, \dots, i_k$ be the indices where the nth row of Σ is '$-$'. We define the matrix $(m_{ji}) = M \in \mathbb{R}^{(n+k-1)\times d}$ by

$$m_{ji} = \begin{cases} \varepsilon & \text{for } j \ge n, i = i_j \\ 0 & \text{else} \ . \end{cases}$$

We refine the subdivision of $\Delta_{n-1+k-1} \times \Delta_{d-1}$, which we just constructed, with this matrix M to obtain a subdivision $\widehat{\mathcal{S}}$.

Additionally, we replace the nth row of Σ with k copies of this row, where we replace all the $-$ entries in every row j for $j > n-1$ by $+$ except for $(j, i_{j-(n-1)})$, where we keep the $-$.

Finally, the following is similar to Lemma 4.34 and justifies the construction. Let $(\mathcal{S}, \Sigma)$ be the original and $(\widehat{\mathcal{S}}, \widehat{\Sigma})$ the modified STM.

Proposition 4.42. *Let G be a maximal covector graph of $\mathcal{S}$ and $G_1, \dots, G_m$ the maximal covector graphs of $\widehat{\mathcal{S}}$ which is mapped to G by π. Then G is infeasible if and only if G_ℓ is infeasible for every $\ell \in [k]$.*

Proof. Let $\widehat{G}$ be the covector graph from Lemma 4.41 which is obtained by adding k copies of the apex node n. We define the matrix $M|_{\widehat{G}}$ by replacing every entry m_{ji} of M by ∞ for which (i,j) is not an edge of $\widehat{G}$.

By construction, $G_1, \dots, G_m$ are exactly the maximal covector graphs with respect to $M|_{\widehat{G}}$.

Since feasibility is a property which can be checked independently for all apex nodes, it suffices to consider the apex node n in G resp. $n, \dots, n+k-1$ in $G_1, \dots, G_m$.

Hence, the rows $n, \dots, n+k-1$ of $M|_{\widehat{G}}$ are, up to reordering of columns, of the form

$$\left(\begin{array}{cccc|ccc} 0 & \varepsilon & \cdots & \varepsilon & 0 & \cdots & 0 \\ \varepsilon & \ddots & \varepsilon & \vdots & 0 & \cdots & 0 \\ \vdots & \varepsilon & \ddots & \varepsilon & 0 & \cdots & 0 \\ \varepsilon & \cdots & \varepsilon & 0 & 0 & \cdots & 0 \end{array}\right)$$

where each 0 entry in the left part of the matrix is assigned a $-$ in $\widehat{\Sigma}$.

If G is infeasible, the right part of the matrix does not contain any columns and the corresponding inequality system is infeasible.

Otherwise, 0 is a feasible point. Therefore, at least one of the covectors $G_1, \dots, G_m$ is feasible. □

In this way, we can construct a signed tropical matroid $(\widehat{\mathcal{S}}, \widehat{\Sigma})$ such that the number of apex nodes is bounded by $n \cdot d$ and every row of $\widehat{\Sigma}$ contains exactly one negative entry.

In the realizable case, this translates to the following.

Corollary 4.43. *Let $I \subseteq \mathbb{N}$ be a finite index set, $b_0, b_i \in \mathbb{T}_{\min}$ for $i \in I$ and $\varepsilon > 0$ an arbitrary positive number.*

Then $b_0 \leq \bigoplus_{i \in I} b_i$ if and only if $b_0 \oplus \bigoplus_{i \in I \setminus \{\ell\}} (b_i + \varepsilon) \leq b_\ell$ for all $\ell \in I$.

Example 4.44. The left picture of Figure 4.16 visualizes the inequality $x_1 \leq x_2 \oplus x_3$ where again the infeasible region is marked. The middle one depicts the replacement by the two inequalities $x_1 \oplus (\varepsilon \odot x_2) \leq x_3$ and $x_1 \oplus (\varepsilon \odot x_3) \leq x_2$ as in Corollary 4.43. Finally, the right one illustrates the conversion from Equation 4.10. The resulting inequalities are $x_1 \leq x_2$ and $x_1 \leq x_3$.

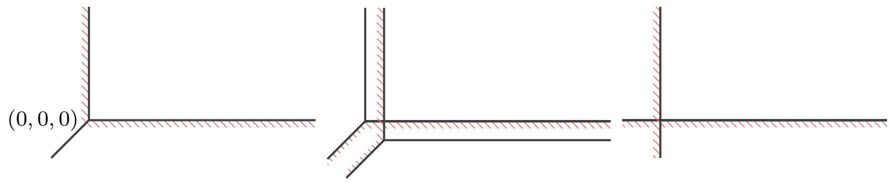

Figure 4.16: Starting from the left depiction, the middle one illustrates the construction of Corollary 4.43 and the right one illustrates Equation 4.10 applied to the left configuration, see Example 4.44.

4.6 Abstract Tropical Linear Programming

4.6.1 A Generalized Feasibility Problem

The tropical linear feasibility problem has connections to several other problems as we saw in Section 4.3. Therefore, algorithms for scheduling with AND-OR-networks [MSS04], mean payoff games [EM79, ZP96, GKK88] and classical linear programming

[ABGJ15, ABGJ14a, Ben14] are applicable to this problem. Furthermore, beside the algorithms for tropical inequality systems [BA09, But10], one can also use algorithms for tropical equality systems [Gri13, BZ06] which are equivalent via the reformulation $a \leq b \Leftrightarrow a = \min(a, b)$.

Our approach is motivated by the connection with the simplex method. Inspired by classical oriented matroid programming, cf. [Bla77, Fuk82, Tod85, Ter85], we will now describe an algorithm for solving the feasibility problem for an STM as an abstraction of the feasibility problem for signed systems.

Recall that a signed system (A, Σ), with coefficient matrix $A \in \mathbb{T}_{\min}^{n \times d}$, is feasible if and only if there is a point $x \in \mathbb{TA}^d$ which fulfills the corresponding homogeneous tropical inequality system. Otherwise, we call it infeasible.

With Lemma 4.25, this translates to the following for systems with finite coefficients.

Corollary 4.45. *A signed system (A, Σ), with finite coefficients $A \in \mathbb{R}^{n \times d}$, is infeasible if and only if every covector graph in every contraction is infeasible.*

This motivates the definition of the feasibility of a full STM as generalization of the feasibility of a tropical linear inequality system. A full STM $(\mathcal{T}, \Sigma)$ is *feasible* if there is a contraction which contains a feasible covector graph, otherwise we call it *infeasible*.

We do not give the definition of feasibility for a general non-full STM, as a more axiomatic approach for collections of generalized covectors would be necessary. Our suggestion is the following: An STM is feasible if there is an extension that is feasible. For this, it would be nice to show that this is indeed the case if and only if all extensions are feasible.

4.6.2 Description of the Algorithm

We introduce an algorithm which either finds a feasible or a totally infeasible covector graph in an STM, which is full, generic and trimmed (see Definition 4.21 and Subsection 4.5.3). By Lemma 4.26, a totally infeasible covector is a certificate that such an STM does not contain a feasible covector.

Like the variant of the simplex method presented in Subsection 4.3.3, the algorithm constructs a sequence of subsets (a basis) of apex nodes (which correspond to inequalities). In each step, we consider a covector which is defined by this sequence and check if it is feasible. If it is not feasible yet, there is an apex node which is only incident with negative edges (corresponding to a violated inequality). This determines which apex (variable) will enter the basis. For classical oriented matroid programming, this is described in, e.g., [Bla77, Theorem 4.5] .

Now, our approach diverges. While in the simplex method, one has to compute which variable leaves the basis, we deduce from Lemma 4.47 with the properties of a *basic covector* which apex leaves the basis. This can already be seen in Figure 4.17. To arrive at this insight, we will prove in Subsection 4.6.3 that moving along abstract tropical lines yields a basic covector if we start from one.

Furthermore, the termination of the simplex method is guaranteed by the increase of a linear functional. As we are working in a setting without weights such an argument is not at hand. However, again the special structure, in particular the preservation of the *distinguished direction*, of the basic covectors yields a purely combinatorial tool to measure the progress of the algorithm.

The powerful definition of a basic covector comes with the additional difficulty to find one. We will solve this in Subsection 4.6.4 by an inductive construction via contractions of an STM.

Throughout this section we assume that $(\mathcal{T}, \Sigma)$ is a signed tropical (n, d)-matroid, which is full, generic and trimmed. In particular, we are in the situation of Proposition 4.22. With the operations from Section 4.5, we can modify a general STM to an STM with this particular structure and the same feasibility status. This follows from Lemma 4.34, Proposition 4.37 and Proposition 4.42.

To emphasize that covector graphs take the role of vectors in the classical simplex method we denote them by y.

A *basic covector (graph)* y with *distinguished direction* δ and *support* $(D \cup \{\delta\}) \subseteq [d]$ with $D \subseteq [d] \setminus \{\delta\}$ is a covector graph on $[d] \sqcup [n]$ such that

1. it is a spanning tree on $(D \cup \{\delta\}) \sqcup N$,
2. each coordinate node in $[d] \setminus (D \cup \{\delta\})$ is isolated,
3. there is a $|D|$-set of apex nodes $N \subseteq [n]$, called *basis*, so that each node in N has degree 2 in y,
4. δ is not adjacent to an apex node in N via a negative edge,
5. each apex node in N is incident with a positive and a negative edge,
6. no two negative edges, each of which is incident with some node in N, are adjacent.

The apex nodes in the *basis* are called *basic apices*, the others *non-basic apices*. If Σ has a '$-$' at position $i \in [d]$ in row $j \in [n]$, we say that the apex node j has *shape* i resp. it is i*-shaped*.

Later on, we will construct a sequence of basic covectors. If there are apex nodes $p \neq q \in [n]$ so that N and $N \setminus \{p\} \cup \{q\}$ are bases, we say that p is the *leaving apex* and q is the *entering apex*.

Example 4.46. The graphs at the bottom of Figure 4.17 are the covector graphs of the points P_1, P_2 and P_3 in the top part. They are all basic covectors. The distinguished direction is $\delta = 1$. The corresponding bases are $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$. The apices 2 and 4 are 2-shaped, the apices 1 and 3 are 3-shaped.

We start with the nice structural property of basic covectors which connect the sign structure with the matching structure.

Lemma 4.47. *The negative edges which are incident with a basic apex form a perfect matching on $D \sqcup N$ in y. Furthermore, the edges in a path emerging from δ to another coordinate node are alternatingly positive and negative.*

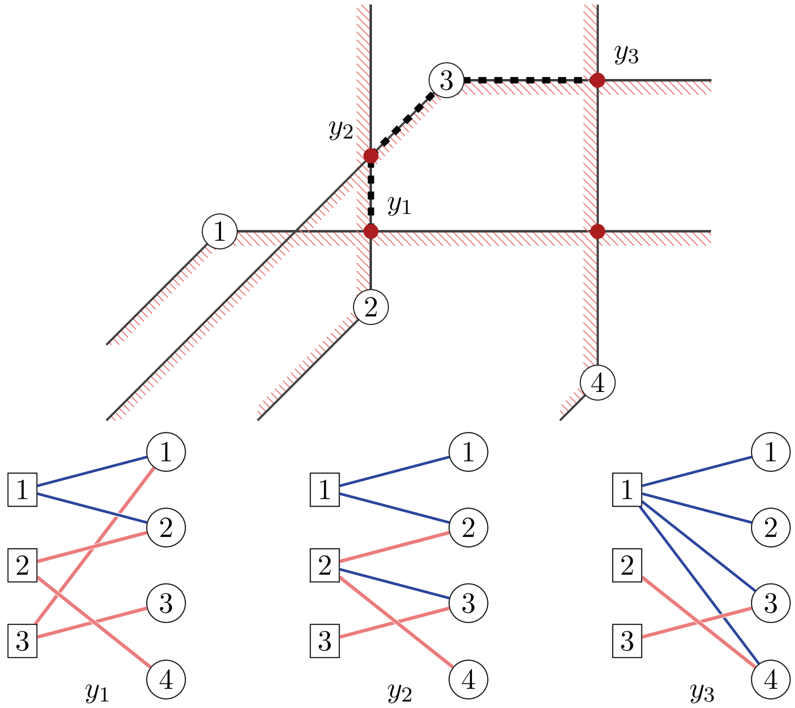

Figure 4.17: A path (dashed) along points with basic covectors (the four red points). The infeasible region is marked. In each step, a negative edge is removed from the covector graph. The bases are $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$.

Proof. Consider the induced subgraph $\tilde{y}$ of y on $(D \cup \{\delta\}) \sqcup N$. Each apex node is incident with a negative edge. By (5) and (6) in the definition, no two negative edges are incident, and by (4), δ is not incident with a negative edge. Hence, the negative edges define an injective function from N to D. Because of $|N| = |D|$, this function is also bijective. This yields the required matching.

Since each node in N has degree 2 and the nodes in $[d] \setminus (D \cup \{\delta\})$ are isolated, $\tilde{y}$ is a tree. Fix an arbitrary $i \in D$ and let $\rho = (e^0, e^1, \ldots, e^k)$ be the edge sequence from δ to i in $\tilde{y}$. Since e^0 is positive and incident with the same apex node as e^1 we conclude that e^1 is negative. Therefore, e^2 has to be positive again as it is incident with the same coordinate node as e^1. Iterating this argument, we obtain that the edges in ρ are alternatingly positive and negative. □

The former lemma tells us that there is exactly one i-shaped apex node for each $i \in D$ in the basis N. From Proposition 4.28, we know that there is at most one basic covector defined by $(D \cup \{\delta\})$ and N. If the Cramer covector $\mathcal{C}(N, D \cup \{\delta\})$ fulfills the conditions 4, 5 and 6, it is the basic covector with these parameters and we denote it by $\mathcal{B}(N, D, \delta)$.

Corollary 4.48. *The Cramer covector $\mathcal{C}(N, (D \cup \{\delta\}))$ is the basic covector $\mathcal{B}(N, D, \delta)$ if and only if the negative edges, which are incident with the basic apices, form a perfect matching on $D \sqcup N$.*

4.6.3 Pivoting between Basic Covectors

The crucial piece for our feasibility algorithm is a method to find a new basic covector which is "in the right direction" and "similar to the old one". In particular, the new basic covector should have the same distinguished direction. We present two variants for this in Algorithm 2 and Algorithm 3. The second one will evolve as an iteration over the first one. We need the first one for technical reasons in the proofs. The idea is the following.

If we remove a negative edge e which is incident to a basic apex p in a basic covector y with basis N then we obtain the covector graph $y - e$ having two trees as connected components and p leaves the basis. In this context, $-$ denotes set difference of the edge sets. We know by Proposition 4.22 that there is exactly one other tree w containing this graph. Hence, there is an edge f such that $w = y - e + f$ where $+$ denotes union.

Now, three cases can occur. If w is again a basic covector graph with distinguished direction δ, we are done. Otherwise, either an apex node in N has degree 3 or another apex node has degree 2. We continue the iteration by removing an edge. This edge is chosen such that no node becomes isolated and all nodes in $N \setminus \{p\}$ have degree ≥ 2 as well as one negative incident edge. This ensures that δ remains the distinguished direction and yields the case distinction of Algorithm 2. A closer inspection reveals that we do not need to iterate over all these covectors to find another basic covector but can construct it directly which results in Algorithm 3. For the proof of this, we assigned the variable *completed* in Line 11 of Algorithm 2. The latter algorithm is merely a technical tool to show that the other algorithms building on it behave correctly.

Remark 4.49. The iteration in Algorithm 2 moves along an abstract version of a "tropical line". A tropical line is a sequence of ordinary lines as explained in [DS04, Proposition 3]. A more refined version for this is given in [ABGJ15, §4]. Note that their description in terms of the "tangent digraph" is essentially the same as in terms of covector graphs in the realizable case. However, our approach also works in the non-realizable case.

We build our arguments for the correctness of the algorithms on properties of the paths in basic covectors. Let the *length of a path* in a graph be the number of nodes contained in the path. Define the *δ-distance of an edge e* in the covector graph y as the minimum of the two lengths of the paths from a fixed coordinate node δ to the nodes which are incident with e. Note that the path between two nodes in a tree is unique. We call the edge e *even* in y if the distance to the coordinate node δ is even, otherwise *odd*. We call this property the *δ-parity of an edge* in y.

Finding the next basic covector

Let y^0 be the input covector, r the input basic apex and p the leaving basic apex of shape i. We consider the sequence $y^1, y^2, \ldots$ of covectors which arise in Algorithm 2 in Line 6. Such a sequence is depicted in Figure 4.18. Then we can write $y^1 = y^0 - e^0 + f^1, y^2 = y^1 - e^1 + f^2, \ldots$ for appropriate edges e^ℓ and f^ℓ with $\ell \in \mathbb{N}$. Furthermore, let q^ℓ be the apex node, which is incident with f^ℓ in y^ℓ.

Example 4.50. Figure 4.18 depicts a possible sequence of covectors arising in Algorithm 2 Line 6. The first and the last covector are basic with basis $\{2, 3, 4\}$ resp. $\{2, 3, 5\}$. The

Algorithm 2 Finding the next basic covector; see also Algorithm 3

Input: Basic covector graph $y = \mathcal{B}(N, D, \delta)$ and a non-basic apex r that is adjacent to D via a negative edge in y

Output: Basic covector graph with support $D \cup \delta$ and distinguished direction δ

procedure NEXTBASICCOVECTOR(y,r)
    $i \leftarrow$coordinate node adjacent to r
    $p \leftarrow$basic apex adjacent to i via a negative edge $\triangleright$ *the i-shaped basic apex of the basis N. It leaves the basis.*
    $e \leftarrow$edge connecting i and p
    do
        $w \leftarrow$ unique covector $\neq y$ in $\mathcal{T}|_{D \cup \{\delta\}}$ containing $y - e$ $\triangleright$ *see Prop. 4.22*
        $f \leftarrow w - (y - e)$
        $q \leftarrow$ the apex node incident with f
        if q is adjacent to i via a negative edge **then**
            $\triangleright$ *w is the basic covector $\mathcal{B}(N \setminus p \cup q, D, \delta)$.*
            completed$\leftarrow (q = r)$
        else if q has degree 3 in w **then**
            $e \leftarrow$ the positive edge incident with q in $y - e = w - f$.
        else $\triangleright$ *In this case, q is incident with two edges.*
            $e \leftarrow$ the edge incident with q in $y - e = w - f$.
        end if
        $y \leftarrow w$
    while y is no basic covector
    return y
end procedure

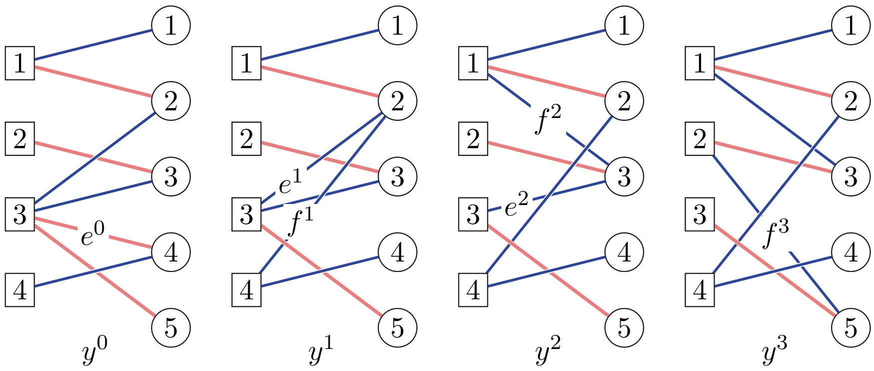

Figure 4.18: A possible sequence of covector graphs starting with an infeasible and ending with a feasible basic covector. Negative edges are light red, coordinate nodes left, apex nodes right, $\delta = 4$. The intermediate covectors are not basic.

distinguished direction is $\delta = 4$.

In the realizable case, the two apices 2 and 3 would define a tropical line which eventually has to hit the halfspace defined by the apex node 5.

Lemma 4.51. *The covector graph $y^\ell - e^\ell$ has two connected components for all $\ell \geq 0$. Each node in $N \setminus \{p\}$ has degree 2 and is incident with a positive and a negative edge. All other apex nodes have degree 1. The negative edges, which are incident with a node in $N \setminus \{p\}$, are pairwise not adjacent.*

Proof. By construction, y^ℓ is always a tree, hence $y^\ell - e^\ell$ has two connected components. Line 13 ensures the properties of the nodes in $N \setminus \{p\}$. Line 15 guarantees that the other apex nodes have degree 1. The last claim follows as the negative edges, which are incident with a node in $N \setminus \{p\}$, are the same as in y^0. □

Since we started the iteration with a basic covector, we obtain a nice invariant which is fulfilled by the edges which are removed and added.

Lemma 4.52. *Let y^ℓ and $y^{\ell+1} = y^\ell - e^\ell + f^{\ell+1}$ be two consecutive covector graphs for $\ell \geq 0$. Then e^ℓ is even in y^ℓ and $f^{\ell+1}$ is odd in $y^{\ell+1}$.*

Proof. We proceed by induction. The first covector graph y^0 in the iteration is a basic covector.

From Lemma 4.47, we know that the paths from δ to another coordinate node are alternatingly positive and negative. We conclude that all the negative edges which are incident with a basic apex are even. Hence, line 4 in Algorithm 2 yields that e^0 is even as it is negative.

Now fix an $\ell \geq 1$ and consider the union $Y^\ell := y^{\ell-1} + f^\ell = y^\ell + e^{\ell-1}$ of $y^{\ell-1}$ and y^ℓ. There is a unique fundamental cycle in Y^ℓ which contains f^ℓ and $e^{\ell-1}$. An example for this is depicted in Figure 4.19. Consider the path ρ in Y^ℓ that contains $e^{\ell-1}$ and goes from δ to the first node incident with f^ℓ. By the induction hypothesis, $e^{\ell-1}$ is even in

$y^{\ell-1}$. By the comparability condition in Proposition 4.22, the fundamental cycle must not be alternating between edges of $y^{\ell-1}$ and y^{ℓ}. Therefore, with the evenness of $e^{\ell-1}$, the number of nodes in ρ must be even as well. Since the number of edges forming a cycle in a bipartite graph is even, this implies that the other path from δ to the first node incident with f^{ℓ} in Y^{ℓ} contains an odd number of nodes. This is exactly the path defining the δ-distance of f^{ℓ} in y^{ℓ}, hence, this δ-distance is odd.

To show that f^{ℓ} and e^{ℓ} have different parity in y^{ℓ} we consider the two cases in Algorithm 2 lines 13 and 15. The first case occurs if q^{ℓ} is a basic apex. Consider the path from δ to q^{ℓ}. By Lemma 4.51, the apex nodes along this path are only nodes in $N \setminus \{p\}$ and analogously to Lemma 4.47, we get that the path is alternatingly positive and negative. In particular, the path to the positive edge incident with q^{ℓ} with the higher δ-distance contains the other positive edge. Therefore, these two edges have different parity.

The second case occurs if q^{ℓ} is an apex node in $[n] \setminus (N \setminus \{p\})$ which has degree 2 in y^{ℓ} but is not of shape i. In this case, f^{ℓ} and e^{ℓ} are again incident with the same apex node q^{ℓ}. There is a unique path from δ to q^{ℓ}. Since it has to contain one of the two edges the claim follows. □

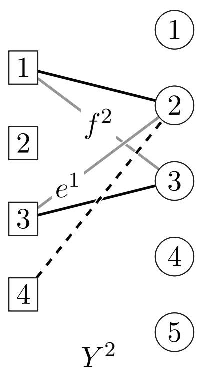

Figure 4.19: The fundamental cycle for y^1 and y^2 in Figure 4.18. The two graphs coincide in the black edges and differ in the green edges. The dashed edge connects the cycle with $\delta = 4$.

Now, we have the tools to prove a first lemma which guarantees termination.

Lemma 4.53. *For $\ell \geq 1$, let $C^{\ell-1}$ be the set of nodes in the connected component of the distinguished direction δ in $y^{\ell-1} - e^{\ell-1}$. Then $q^{\ell} \notin C^{\ell-1}, q^{\ell} \in C^{\ell}$ and $C_1 \subsetneq C_2 \subsetneq \dots$.*

Proof. Fix an arbitrary $\ell \geq 1$ indexing an element of the sequence (q^{ℓ}).

Not both endpoints of f^{ℓ} can be contained in $C^{\ell-1}$ as f^{ℓ} connects the two components of $y^{\ell-1} - e^{\ell-1}$. The path from δ to the endpoint of f^{ℓ} in y^{ℓ} has to be odd, by Lemma 4.52. Since such a path has to alternate between coordinate and apex nodes, this endpoint has to be a coordinate node. Hence, q^{ℓ} is not contained in $C^{\ell-1}$.

By the choice of e^{ℓ} in Line 13 or Line 15 of Algorithm 2, e^{ℓ} is incident with q^{ℓ}. Since e^{ℓ} is contained in $y^{\ell-1} - e^{\ell-1}$, the endpoint of e^{ℓ} different from q^{ℓ} must not lie in $C^{\ell-1}$, otherwise q^ℓ would lie in $C^{\ell-1}$. Subsuming, no endpoint of e^ℓ lies in $C^{\ell-1}$. Therefore, q^ℓ and the nodes in $C^{\ell-1}$ cannot be disconnected from δ in $y^\ell - e^\ell$. Hence, $q^\ell \in C^\ell$ and $C^{\ell-1} \subsetneq C^\ell$. $\square$

Example 4.54. The connected components of δ in the covector graphs in Figure 4.18 are $\{4, \overline{4}\}$, $\{1, 4, \overline{1}, \overline{2}, \overline{4}\}$, $\{1, 2, 4, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$, where the numbers with the line on top denote apex nodes.

Theorem 4.55. *Algorithm 2 does not cycle and yields a new basic covector with distinguished direction δ and support $(D \cup \delta)$ after less than n iterations.*

Proof. Note that the condition in Line 9 is fulfilled if q equals r. By Lemma 4.53, the set C^ℓ is increased by at least one apex node. Since there are only n apex nodes and the set fulfilling the condition in Line 9 is not empty, the algorithm terminates after less than n iterations.

Furthermore, the condition that $w \in \mathcal{T}|_{D \cup \{\delta\}}$ ensures that each coordinate node in $[d] \setminus (D \cup \{\delta\})$ is isolated. The condition in Line 9 together with Lemma 4.51 yields that the resulting covector graph is indeed a basic covector with distinguished direction δ. $\square$

If r does not enter the basis to form the new basic covector in Algorithm 2, it is still a non-basic apex, which is incident with a negative edge. Therefore, the following block yields the basic covector $y = \mathcal{B}(N \setminus p \cup r, D, \delta)$ where p is the leaving basic variable which has the same shape as r.

completed← FALSE
while not completed **do**
  NEXTBASICCOVECTOR(y,r) ▷ *see Algorithm 2*
end while ▷ *If r does not become a basic apex it can be used again.*

This implies that $\mathcal{C}(N \setminus p \cup r, D \cup \{\delta\})$ is indeed a basic covector.

Algorithm 3 Simplified variant of Algorithm 2 for finding the next basic covector

Input: Basic covector graph $y = \mathcal{B}(N, D, \delta)$ and a non-basic apex r that is adjacent to D via a negative edge in y
Output: The basic covector graph $\mathcal{B}(N \setminus p \cup r, D, \delta)$ where p is of the same shape as r
procedure NEXTBASICCOVECTOR(y,r)
  i ←coordinate node adjacent to r
  p ←i-shaped basic apex of the basis N
  return $\mathcal{C}(N \setminus p \cup r, D \cup \{\delta\})$
end procedure

The former observations imply the following.

Corollary 4.56. *Algorithm 3 is correct and has the same result as an iterative application of Algorithm 2.*

Example 4.57. Observe that y^0 is the basic covector $\mathcal{B}(\{2,3,4\},\{1,2,3\},4)$ and y^3 is the basic covector $\mathcal{B}(\{2,3,5\},\{1,2,3\},4)$ in Figure 4.18. That illustrates Corollary 4.56 as the apex nodes 4 and 5 are both 3-shaped and 5 is a non-basic apex node incident with a negative edge in y^0.

Finding an extreme basic covector

Eventually, we want to determine a feasible or totally infeasible basic covector. A feasible covector cannot have an apex node of degree one which is incident with a negative edge. Therefore, we want to construct a new basic covector if there is such an edge. We know from the former section how this can be achieved. Iterating this approach yields Algorithm 5. To check if we reached a feasible or totally infeasible basic covector we need the subroutine CheckFeasible from Algorithm 4. It is just the algorithmic manifestation of Definition 4.23.

Remark 4.58. We are left with some freedom of choice for the entering apex at each basic covector. We do not specify a rule to choose the apex, the algorithms work for any choice. For an implementation we suggest to use the smallest index, like in Bland's rule for the simplex method.

Algorithm 4 Checking feasibility of a basic covector

Input: Basic covector graph $y = \mathcal{B}(N, D, \delta)$
Output: A classification of y based on the signs of the edges

1: **procedure** CheckFeasible(y,δ)
2:     **if** there is a non-basic apex node only incident with a negative edge **then**
3:         **if** there is a negative edge incident with δ **then**
4:             **return** TOTALLY–INFEASIBLE
5:         **else**
6:             **return** INFEASIBLE
7:         **end if**
8:     **else**
9:         **return** FEASIBLE
10:     **end if**
11: **end procedure**

Lemma 4.59. *Algorithm 4 correctly determines if $y = \mathcal{B}(N, D, \delta)$ is feasible, infeasible or totally infeasible in the sense of Definition 4.23.*

Proof. If the condition in Line 2 is fulfilled, the covector y is surely infeasible. Since, in a basic covector graph, all the coordinate nodes in D are incident to a basic apex via a negative edge, the condition in Line 3 implies that y is totally infeasible. The claim follows as feasible is the opposite of infeasible. □

Algorithm 5 successively constructs basic covector graphs with Algorithm 3 until the result is feasible or totally infeasible.

Algorithm 5 Iterating over basic covectors

Input: Basic covector graph $y = \mathcal{B}(N, D, \delta)$

Output: A basic covector with support $(D \cup \delta)$ and distinguished direction δ which is either totally infeasible or feasible

1: **procedure** FindExtremeCovector(y)
2:     **while** (CheckFeasible(y, δ) = INFEASIBLE) **do**
3:         $r \leftarrow$non-basic apex in y which is incident to D via a negative edge $\triangleright$ *such an r exists if y is infeasible, see Algorithm 4 Line 2 and 3*
4:         $p \leftarrow$basic apex of y of the same shape as r
5:         $y \leftarrow \mathcal{C}(N \setminus p \cup r, D \cup \{\delta\})$
6:     **end while**
7:     **return** y
8: **end procedure**

At first, it is not clear that this terminates. We consider a run of this algorithm starting with the arbitrary basic covector y^0. Let y^k be a basic covector which is assigned in Line 5 of Algorithm 5 during this run. By Corollary 4.56, there is a sequence of covectors $y^0, y^1, \ldots, y^k$ (most of them not basic) which would occur as intermediate results by using Algorithm 2 instead of Algorithm 3.

Albeit the following lemma just applies to the realizable case, we state it here to provide more intuition for the general argument in Proposition 4.61. When the covectors are defined by a matrix A, the termination can be shown by bounding the increase of the coordinates of the occuring points. This follows with Lemma 4.33 from the next lemma. Later, this result is needed to deduce the complexity of our algorithm in the realizable case in Section 4.7.

Lemma 4.60. *Let $x^\ell \in \mathbb{T}^d_{\min}$ such that y^ℓ is the covector graph of x^ℓ, which can be constructed from A by Lemma 4.30. For each $\ell \in [k]$, we get the inequalities*

$$x_i^{\ell-1} - x_\delta^{\ell-1} \le x_i^\ell - x_\delta^\ell \qquad \text{for all } i \in (D \cup \{\delta\}) \ .$$

Proof. Lemma 4.30 allows us to express $x_i^{\ell-1} - x_\delta^{\ell-1}$ resp. $x_i^\ell - x_\delta^\ell$ as a sum along the path from δ to i in $y^{\ell-1}$ resp. y^ℓ, with the weights given by A.

For each i in the connected component $C^{\ell-1}$ of δ in $y^{\ell-1} - e^{\ell-1}$, there is exactly one path from δ to i and it is the same in $y^{\ell-1}$ and y^ℓ. Therefore, we obtain $x_i^{\ell-1} - x_\delta^{\ell-1} \le x_i^\ell - x_\delta^\ell$.

Now, let i be a node in $[d] \setminus C^{\ell-1}$. Then the path from δ to i in $y^{\ell-1}$ contains $e^{\ell-1}$ and the one in y^ℓ contains f^ℓ. Denote the paths by $\rho^{\ell-1}$ and ρ^ℓ. Their symmetric sum is a subgraph of $y^{\ell-1} + f^\ell$ and is a union of cycles. Since $y^{\ell-1}$ is a tree, $y^{\ell-1} + f^\ell$ contains only the elementary cycle formed by f^ℓ. It decomposes into two matchings μ_0 and μ_1 where one of them, without loss of generality μ_0, contains both the edges $e^{\ell-1}$ and f^ℓ by the comparability condition in Proposition 4.22.

In the formula for Lemma 4.30, odd edges get a positive sign and even edges a negative sign. Furthermore, we see that $(x_i^\ell - x_\delta^\ell) - (x_i^{\ell-1} - x_\delta^{\ell-1})$ is given by the difference of the sums over the two matchings μ_0 and μ_1. By Lemma 4.52, f^ℓ is odd in y^ℓ. This implies

$$(x_i^\ell - x_\delta^\ell) - (x_i^{\ell-1} - x_\delta^{\ell-1}) = \sum_{(j,i)\in\mu_0} a_{ji} - \sum_{(j,i)\in\mu_1} a_{ji} \ .$$

Finally, Proposition 4.4 yields that the difference $\sum_{(j,i)\in\mu_0} a_{ji} - \sum_{(j,i)\in\mu_1} a_{ji}$ is positive, since μ_1 is contained in the covector graph y^ℓ and hence minimal. $\square$

Now, we tackle the less intuitive general case. Let $\mathcal{E}$ be the graph on $(D \cup \{\delta\}) \sqcup [n]$ whose set of edges are exactly those which are contained in all the graphs $y^0, \ldots, y^k$. Denote by $\mathcal{E}(\delta)$ the connected component in $\mathcal{E}$ containing δ and by $I(\delta)$ the subset of the coordinate nodes in $\mathcal{E}(\delta)$.

Proposition 4.61. *There is an apex node $j \in [n]$ and an $h \in [k]$ such that j has degree 2 in y^0 and degree 1 in y^ℓ for $\ell \geq h$. In particular, $y^k \neq y^0$.*

Proof. Since y^0 is connected there is an apex node j in y^0 which is connected to $I(\delta)$ and to $(D \cup \{\delta\}) \setminus I(\delta)$. The covector y^0 is basic and j has degree 2. Therefore, j is a basic apex.

If both edges incident with j are contained in $\mathcal{E}$ this would contradict the definition of $I(\delta)$. Therefore, there is an h so that the edge e^h, which is removed in step h, is incident with j. Since the edges of $\mathcal{E}$ are contained in all the graphs $y^0, \ldots, y^k$, the edge e^h has the same δ-distance in y^h as in y^0. With Lemma 4.47 and 4.52, the edge e^h is even and negative in y^h. Furthermore, the positive edge incident with j is incident with $I(\delta)$.

For $\ell \geq h$, no edge in $\mathcal{E}(\delta)$ is removed. Assume there would be an $\ell_0 \geq h$ so that f^{ℓ_0} is incident with j. Then f^{ℓ_0} would be even in y^{ℓ_0}. However, this contradicts Lemma 4.52. Subsuming, j has degree 1 in y^ℓ for $\ell \geq h$. $\square$

Remark 4.62. Geometrically, for the realizable case the set $\mathcal{E}(\delta)$ defines a lower dimensional tropical hyperplane, which contains all the points $y_1, \ldots, y_{k+1}$. It is given by the intersection of the boundaries of the tropical halfspaces which correspond to the apex nodes which are internal nodes of $\mathcal{E}(\delta)$.

For the non-realizable case, we only give the following rough upper bound. It is just the number of $|D|$-tuples analogously to the number of possible bases for the classical simplex method. We will give a better upper bound for the realizable case in Theorem 4.79.

Theorem 4.63. *Algorithm 5 terminates after less than $\binom{n}{|D|}$ iterations.*

Proof. By Proposition 4.61, any two basic covectors arising in Line 5 are distinct. Furthermore, the assignment of y as Cramer covector in that line yields an injective function from the $|D|$-subsets of $[n]$ to the basic covectors. This implies the claim. $\square$

Remark 4.64. In Algorithm 5, we could continue the iteration until only δ is incident with non-basic apices via negative edges. For other basic covectors, one still can apply Algorithm 3 to construct a new basic covector.

4.6.4 Finding a Basic Covector and Even More

Until now, we assumed a basic covector to be given. Indeed, one easily finds a basic covector for each $\delta \in [d]$, namely the Cramer covector $\mathcal{C}(\emptyset, \{\delta\})$. Algorithm 5 allows us to determine a feasible or totally infeasible covector. This covector lives in $\mathcal{T}|_{(D \cup \{\delta\}}$. If it is feasible then we are finished as we are only looking for a feasible covector in a contraction. However, a totally infeasible covector in $\mathcal{T}|_{(D \cup \{\delta\}}$ is not enough to guarantee the infeasibility of $\mathcal{T}$. On the other hand, we demonstrate how one can construct a new basic covector in a contraction with a bigger support from a totally infeasible basic covector $y = \mathcal{B}(N, D, \delta)$.

By Definition 4.23 resp. Algorithm 4, there is a non-basic apex j in y which is incident to δ via a negative edge. Therefore, y contains a perfect matching μ on $(D \cup \{\delta\}) \sqcup (N \cup \{j\})$ which consists of negative edges. Consider an additional element $\delta' \in [n] \setminus (D \cup \{\delta\})$. By Proposition 4.29, the covector $y' = \mathcal{C}((N \cup \{j\}), (D \cup \{\delta\} \cup \{\delta'\}))$ also contains μ. With Corollary 4.48, we conclude that y' is the basic covector $\mathcal{B}((N \cup \{j\}), (D \cup \{\delta\}), \delta')$. Note that this argument works for any covector y which contains a matching of negative edges on $(D \cup \{\delta\}) \sqcup (N \cup \{j\})$.

Algorithm 6 Finding a feasible or totally infeasible covector graph

Input: A full generic trimmed STM $(\mathcal{T}, \Sigma)$
Output: A totally infeasible basic covector or a feasible covector in a contraction of $\mathcal{T}$

$\delta \leftarrow$ an element of $[d]$
$D \leftarrow \emptyset$, $N \leftarrow \emptyset$
$y \leftarrow \mathcal{C}(\emptyset, \{\delta\})$
while TRUE **do**
  check $\leftarrow$ CheckFeasible(y, δ) ▷ *see Algorithm 4*
  if *check* = INFEASIBLE **then**
    $y \leftarrow$ FindExtremeCovector(y) ▷ *see Algorithm 5*
    check $\leftarrow$ CheckFeasible(y, δ)
  end if ▷ *at this point y is guaranteed to be feasible or totally infeasible*
  if *check* = FEASIBLE **then**
    return "feasible",y
  end if ▷ *at this point y is guaranteed to be totally infeasible*
  if $D \cup \{\delta\} = [d]$ **then**
    return "infeasible",y
  else
    $j \leftarrow$non-basic apex incident with δ via a negative edge ▷ *exists by Algorithm 4 Line 3*
    $D \leftarrow D \cup \{\delta\}$
    $\delta \leftarrow$ node in $[d] \setminus D$.
    $N \leftarrow N \cup \{j\}$
    $y \leftarrow \mathcal{C}(N, D \cup \{\delta\})$
  end if
end while

Theorem 4.65. *Algorithm 6 correctly determines a totally infeasible basic covector in $\mathcal{T}$ or a feasible covector in a contraction of $\mathcal{T}$ in at most $d-1$ iterations of Algorithm 5.*

Proof. From the discussion above the theorem, we know that the covector in Line 20 is indeed a basic covector. By Theorem 4.63, y is a feasible or totally infeasible basic covector after Line 9, and Lemma 4.59 shows that CheckFeasible correctly determines the feasibility status of a basic covector. In each iteration of the while-loop in Line 4, the algorithm either terminates or D is increased by one element.

Since D is a subset of $[d]$ with at most $d-1$ elements, the claim follows. □

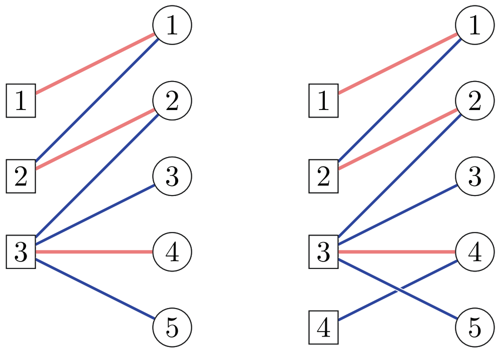

Figure 4.20: Constructing a basic covector with bigger support from a totally infeasible basic covector

Remark 4.66. The only passages in the algorithm where the data of the STM is needed are the assignments of the Cramer covectors. In the realizable case, the input for Algorithm 6 is supposed to be given as a signed system (A, Σ). By Remark 4.31, we obtain them as covector graph for the Cramer solutions.

In the non-realizable case, we assume to have an oracle which returns the Cramer covectors. Recall their guaranteed existence by Proposition 4.28. The requirements on this oracle should be further investigated in the context of *matching ensembles* [OY13].

Corollary 4.67. *Algorithm 6 needs at most $\sum_{k=1}^{d} \binom{n}{k}$ calls to the oracle that encodes $(\mathcal{T}, \Sigma)$ and returns Cramer covectors.*

Furthermore, the algorithm yields a partial generalization of [GP15, Lemma 11]. It is a theorem of alternatives for the feasibility of an STM. It covers a slightly different aspect than the "Tropical Farkas Lemma" [DS04, Proposition 9].

Theorem 4.68 (Tropical Farkas Lemma for STM)**.** *A full generic STM contains*

- *either a feasible covector in a contraction,*
- *or a totally infeasible covector,*

but not both.

Proof. By Theorem 4.65, Algorithm 6 returns a feasible or a totally infeasible covector. If the result is totally infeasible, Lemma 4.26 implies that the STM does not contain a feasible covector. This implies the claim. □

We demonstrate the course of the algorithms on two non-regular triangulations of $\Delta_5 \times \Delta_2$ and $\Delta_3 \times \Delta_3$ from [Hor12, DLRS10] which are listed in Table 4.1. The rows contain the covectors corresponding to the maximal simplices. The jth entry of a tuple contains the coordinate nodes which are adjacent to the apex node j. This is the compact form to write a covector, which was also used in, e.g., [DS04, AD09].

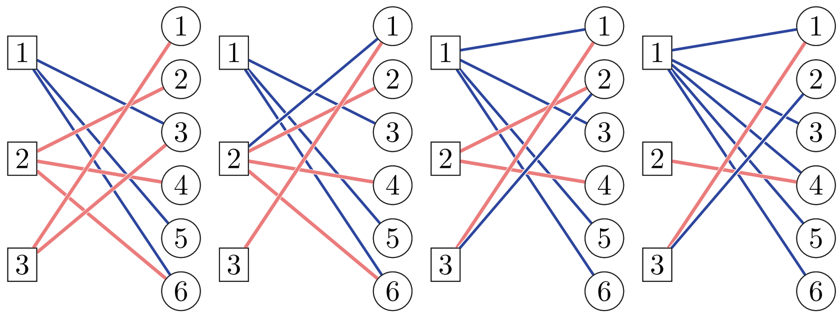

Figure 4.21: A sequence of basic covector graphs produced by a run of Algorithm 5, see Example 4.69. The first one is infeasible, the last one is feasible.

Example 4.69. Figure 4.21 shows a sequence of basic covector graphs from the STM given by the non-regular triangulation on the left of Table 4.1 and the sign matrix

$$\Sigma = \begin{pmatrix} + & + & - \\ + & - & + \\ + & + & - \\ + & - & + \\ + & + & - \\ + & - & + \end{pmatrix} .$$

If we start Algorithm 6 with $\delta = 2$ then a possible sequence is given by the following table.

δ	Cramer covector	label	possible entering apex
2	$\mathcal{C}(\emptyset, \{2\}) = (2, 2, 2, 2, 2, 2)$	y^1	$2, 4, 6$
3	$\mathcal{C}(\{6\}, \{2, 3\}) = (3, 3, 3, 3, 3, 123)$	y^2	$1, 3, 5$
1	$\mathcal{C}(\{3, 6\}, \{1, 2, 3\}) = (3, 3, 13, 2, 1, 12)$	y^3	$1, 4$
	$\mathcal{C}(\{1, 6\}, \{1, 2, 3\}) = (23, 2, 1, 2, 1, 12)$	y^4	$2, 4$
	$\mathcal{C}(\{1, 2\}, \{1, 2, 3\}) = (13, 23, 1, 2, 1, 1)$	y^5	4
	$\mathcal{C}(\{1, 4\}, \{1, 2, 3\}) = (13, 3, 1, 12, 1, 1)$	y^6	

The last four covectors are depicted in Figure 4.21.

The non-regular subdivision is visualized in Figure 4.22 as a mixed subdivision via the Cayley trick. The black lines form "tropical pseudohyperplanes" in the sense of [AD09, §5] and [Hor16, Theorem 4.2] which are dual to the mixed subdivision. The red points mark the cells which correspond to the basic covector graphs shown in Figure 4.21.

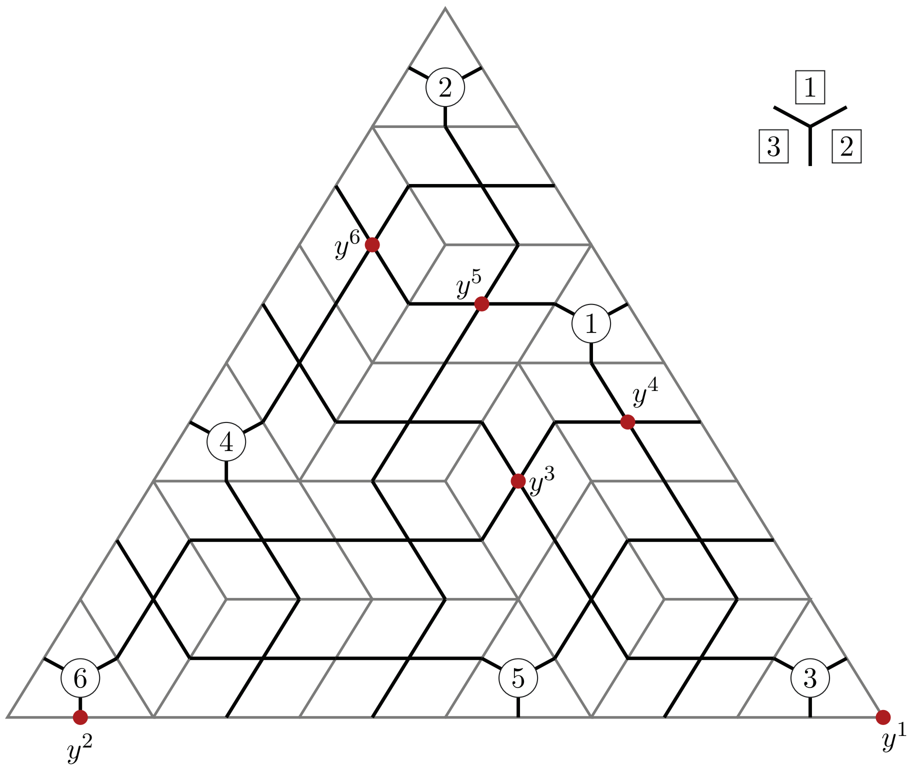

Figure 4.22: The non-regular subdivision from Example 4.69 represented as mixed subdivision of $6 \cdot \Delta_2$ which is possible through the Cayley trick. The black lines are tropical pseudohyperplanes in the sense of [Hor16, Theorem 4.2]. The red intersection points correspond to basic covectors. This figure is basically the same as [HJJS09, Figure 3].

Example 4.70. Furthermore, we demonstrate a run of Algorithm 6 on the STM given by the non-regular triangulation $\mathcal{T}$ on the right of Table 4.1 and the sign matrix

$$\Sigma = \begin{pmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{pmatrix} .$$

We start the algorithm with $\delta = 1$. The maximal covectors in the contractions are found by removing the nodes in $[d] \setminus (D \cup \{\delta\})$ and taking only those resulting graphs without isolated apex nodes.

The only covector in $\mathcal{T}|_{\{1\}}$ is $(1,1,1,1)$. It is a totally infeasible basic covector and, with the new $\delta = 2$, we construct the basic covector $\mathcal{C}(\{1\},\{1,2\})$. The list of maximal covectors in the contraction $\mathcal{T}|_{(\{1\}\cup\{2\})}$ is

$$(1,12,1,1),(12,2,2,2),(1,2,12,1),(1,2,2,12)\ .$$

So, the next basic covector is $(12,2,2,2)$. It is already totally infeasible and no call to FindExtreme is necessary. With the new $\delta = 4$, we get $\mathcal{C}(\{1,2\},\{1,2,4\})$, which yields the covector $(14,24,4,4)$.

Finally, the algorithm results in the totally infeasible basic covector $\mathcal{C}(\{1,2,4\},[4])$. The just constructed sequence of basic covector graphs is depicted in Figure 4.23.

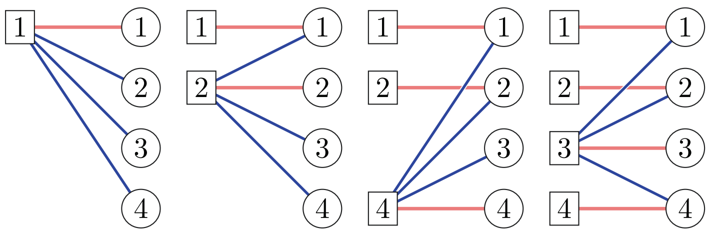

Figure 4.23: A sequence of basic covector graphs produced by a run of Algorithm 6, see Example 4.70.

(1,	123,	1,	1,	1,	1)
(1,	23,	1,	12,	1,	1)
(123,	2,	1,	2,	1,	1)
(23,	2,	1,	2,	1,	12)
(23,	2,	1,	2,	12,	2)
(13,	23,	1,	2,	1,	1)
(13,	3,	1,	12,	1,	1)
(23,	2,	13,	2,	2,	2)
(2,	2,	123,	2,	2,	2)
(3,	2,	13,	2,	12,	2)
(3,	2,	13,	2,	1,	12)
(3,	23,	13,	2,	1,	1)
(3,	3,	13,	12,	1,	1)
(3,	3,	3,	123,	1,	1)
(3,	3,	3,	23,	1,	12)
(3,	3,	3,	23,	13,	2)
(3,	23,	3,	2,	1,	12)
(3,	23,	3,	2,	13,	2)
(3,	2,	3,	2,	123,	2)
(3,	3,	3,	3,	3,	123)
(3,	3,	3,	3,	13,	12)

(1234,	2,	3,	4)
(1,	1234,	3,	4)
(1,	2,	1234,	4)
(1,	2,	3,	1234)
(1,	12,	13,	14)
(12,	2,	23,	24)
(13,	23,	3,	34)
(14,	24,	34,	4)
(123,	2,	3,	24)
(13,	2,	3,	234)
(134,	23,	3,	4)
(14,	234,	3,	4)
(1,	123,	3,	34)
(1,	12,	3,	134)
(1,	124,	13,	4)
(1,	24,	134,	4)
(1,	2,	123,	14)
(1,	2,	23,	124)
(12,	2,	234,	4)
(124,	2,	34,	4)

Table 4.1: Non-regular triangulations of $\Delta_5 \times \Delta_2$ and $\Delta_3 \times \Delta_3$ from [Hor12, DLRS10]. The rows contain the covectors of the maximal simplices. The jth entry of a tuple contains the coordinate nodes which are adjacent to the apex node j.

4.7 Feasibility of Signed Systems

We developed an algorithm to examine if a signed tropical matroid contains a feasible covector. The version given in the last section requires some additional assumptions which had to be fulfilled through the constructions from Section 4.5. We show that they are not necessary under the assumption of realizability. In this case, we also derive a stronger upper bound on the runtime.

Furthermore, we describe how one can find the maximal support of a feasible point. We finish be demonstrating how this relates to mean payoff games again.

In this section we assume that (A, Σ) is a trimmed signed system. We can always transform a general signed system to a trimmed one with Equation 4.10. Note that this is not a restriction on the corresponding inequality system but merely a requirement on the representation.

4.7.1 Solving General Signed Systems

We explain how the algorithms of the former section can be made applicable to general signed systems.

The part in Algorithm 6, where the data of the STM is invoked, is the computation of a Cramer covector. For a general non-full STM, the Cramer covectors can be quite degenerated as one can see in Figure 4.14. However, Proposition 4.75 will ensure that it carries all the necessary information.

Furthermore, the role of a "totally infeasible" covector is not so clear as Lemma 4.26 shows the infeasibility implication only under the condition, that the STM is generic and full. However, we will see that this termination criterion can be replaced by a similar condition.

Again, we start with an element $\delta \in [d]$, $D = \emptyset$ and $N = \emptyset$. As long as there is an apex node in $[n] \setminus N$ of degree 1 in y incident to D via a negative edge this apex enters the basis N and the apex of the same shape is removed from N. Note that in the non-generic case there can be non-basic apex nodes of degree ≥ 2. However, since we assume that the STM is trimmed they cannot be incident with more than one negative edge. After this iteration two cases can occur. If the result is already feasible, we terminate and return this feasible point. Otherwise, there is still an apex node of degree 1 incident with a negative edge. By construction, it cannot be adjacent to D and hence it is adjacent to δ. If the Cramer covector is already defined on the whole of $[d]$ this yields a point which certifies infeasibility. If this is not the case, we can add δ to D and obtain a covector graph which is defined on a bigger set of coordinates. Due to infinite entries of A, its coordinates in $D \cup \{\delta\}$ can be infinite, though.

Remark 4.71. For the realizable case, it is interesting to know the complexity of the computation of the Cramer covectors. The Cramer solution can be computed in $\mathcal{O}(d^3)$ by [AGG14, Remark 8.2]. One derives the covector by evaluating the minimum in each row which needs $\mathcal{O}(dn)$ steps. Note that not all the edges of the covector graph are

Algorithm 7 Determine the feasibility of a signed system

Input: A signed system (A, Σ) so that each row of Σ contains at most one $-$ entry.

Output: A feasible point or a point which guarantees the infeasibility of the signed system.

procedure FINDWITNESS((A, Σ))
    $\delta \leftarrow$ an element of $[d]$
    $D \leftarrow \emptyset$, $N \leftarrow \emptyset$
    $y \leftarrow \mathcal{C}_A(N, (D \cup \{\delta\}))$
    while TRUE **do**
        while there is a non-basic apex node of degree 1 in y incident to D via a negative edge **do**
            $r \leftarrow$the apex fulfilling the while-condition
            $p \leftarrow$basic apex of y of the same shape as r $\triangleright$ *p is an element of N.*
            $N \leftarrow N \setminus \{p\} \cup \{r\}$
            $y \leftarrow \mathcal{C}_A(N, (D \cup \{\delta\}))$
        end while $\triangleright$ *at this point, δ is the only coordinate node which can be incident with an apex node of degree 1 via a negative edge*
        if δ is incident with an apex node of degree 1 via a negative edge **then**
            if $|D| = d - 1$ **then**
                return "infeasible", $A[N|(D \cup \{\delta\})]$
            else
                $j \leftarrow$non-basic apex of degree 1 incident with δ via a negative edge
                $N \leftarrow N \cup \{j\}$
                $D \leftarrow D \cup \{\delta\}$
                $\delta \leftarrow$ node in $[d] \setminus D$.
                $y \leftarrow \mathcal{C}_A(N, (D \cup \{\delta\}))$
            end if
        else
            return "feasible", $A[N|(D \cup \{\delta\})]$
        end if
    end while
end procedure

needed and therefore, this computation could be reduced. Subsuming, a Cramer covector $\mathcal{C}_A(N, (D \cup \{\delta\}))$ can be computed in $\mathcal{O}(d^3 + dn)$.

To deduce the correctness of Algorithm 7, we relate the sequence of points in the iteration in the non-generic non-full situation with a run of Algorithm 6. To simplify the connection between the termination criterion for the general case and for a generic full STM, we chose a more canonical extreme covector, see Remark 4.64. This leads to the while-loop starting in Line 6.

4.7.2 Correctness and Implications of the Algorithm

To show the correctness of Algorithm 7, we reduce it to the correctness for full generic signed systems by exploiting the techniques established in Section 4.5. For this, fix an arbitrary trimmed signed system (A, Σ) and subsets $J \subseteq [n]$ and $I \subseteq [d]$ with $|J| = |I| - 1$.

Let $(A(\Omega), \Xi)$ be an extension of (A, Σ) in the sense of Subsection 4.5.2.

Lemma 4.72. *Each apex node of degree 1 in $\mathcal{C}_A(J, I)$ also has degree 1 in $\mathcal{C}_{A(\Omega)}(J, I)$ and is incident with the same coordinate node.*

Proof. Let (i, j) be an edge in $\mathcal{C}_A(N, D \cup \{\delta\})$ so that j has degree 1. For all $\ell \in I$, the choice of Ω in Equation 4.9 implies that $\operatorname{tdet}(A(\Omega)_{J,(I \setminus \{\ell\})})$ either equals $\operatorname{tdet}(A_{J,(I \setminus \{\ell\})})$ or it contains an Ω summand and $\operatorname{tdet}(A_{J,(I \setminus \{\ell\})}) = \infty$. The definition of a generalized covector graph yields $a_{ji} < \infty$ and $\operatorname{tdet}(A_{J,(I \setminus \{i\})}) < \infty$. Hence, $a_{ji} + \operatorname{tdet}(A(\Omega)_{J,(I \setminus \{i\})})$ is the minimum in row j and $(i, j) \in \mathcal{C}_{A(\Omega)}(N, D \cup \{\delta\})$. □

Example 4.73. Lemma 4.72 is illustrated in Figure 4.14. The covectors on the left and on the right both contain the edge $(3, 4)$.

Let $(\widehat{A(\Omega)}, \Xi)$ by a refinement of the signed system $(A(\Omega), \Xi)$ in the sense of Subsection 4.5.1.

Lemma 4.74. *The covector graph $\mathcal{C}_{A(\Omega)}(J, I)$ contains the covector graph $\mathcal{C}_{\widehat{A(\Omega)}}(J, I)$. Furthermore, each apex node of degree 1 in $\mathcal{C}_{A(\Omega)}(J, I)$ also has degree 1 in $\mathcal{C}_{\widehat{A(\Omega)}}(J, I)$ and is incident with the same coordinate node.*

Proof. The containment follows from Lemma 4.36. The fact that $\mathcal{C}_{\widehat{A(\Omega)}}(J, I)$ is a spanning tree implies the second claim. □

Combining these two lemmata yields the desired relation between the covector in the original and the modified signed system.

Proposition 4.75. *Each apex node of degree 1 in $\mathcal{C}_A(J, I)$ also has degree 1 in $\mathcal{C}_{\widehat{A(\Omega)}}(J, I)$ and is incident with the same coordinate node.*

Proof. By Lemma 4.72, the edge is also an edge of $\mathcal{C}_{A(\Omega)}(J, I)$. Furthermore, by Lemma 4.74, it is an edge of $\mathcal{C}_{\widehat{A(\Omega)}}(J, I)$. □

Now, we gathered the necessary tools to prove termination and correctness.

Theorem 4.76. *Algorithm 7 computes a covector graph, which certifies the feasibility or infeasibility of the signed system (A, Σ).*

Proof. Fix a $\delta \in [d]$ and a subset $D \subseteq [d] \setminus \{\delta\}$. Assume that $N^1 \subseteq [n]$ is a subset of the apex nodes for which $\mathcal{C}_{\widehat{A(\Omega)}}(N^1, D \cup \{\delta\})$ is a basic covector (which is the case for $D = N = \emptyset$). Let $k \in \mathbb{N}$ so that $N^1, N^2, \ldots, N^k$ is the sequence of the set N in Line 9 for the first k iterations of the while-loop starting in Line 6, beginning with N^1. Then for all $\ell \in [k-1]$ there are $r^\ell, p^\ell \in [n]$ so that $N^{\ell+1} = N^\ell \setminus \{p^\ell\} \cup \{r^\ell\}$.

By the iteration condition of the while-loop in Line 6, the apex node r^ℓ is not in N^ℓ, it is of degree 1, and it is incident with a negative edge (i^ℓ, r^ℓ) in $\mathcal{C}_A(N^\ell, D \cup \{\delta\})$. Proposition 4.75 implies that r^ℓ also has degree 1 and is incident with (i^ℓ, r^ℓ) in $\mathcal{C}_{\widehat{A(\Omega)}}(N^{\ell+1}, D \cup \{\delta\})$. Now, Corollary 4.56 implies that $\mathcal{C}_{\widehat{A(\Omega)}}(N^{\ell+1}, D \cup \{\delta\})$ is a basic covector if so is $\mathcal{C}_{\widehat{A(\Omega)}}(N^\ell, D \cup \{\delta\})$, since p^ℓ is chosen just to match the shape of r^ℓ, independent of the covector graph.

Hence by induction, $\mathcal{C}_{\widehat{A(\Omega)}}(N^\ell, D \cup \{\delta\})$ is a basic covector for all $\ell \in [k]$. By Theorem 4.63 and Remark 4.64, there is an $h \in \mathbb{N}$ so that in $\mathcal{C}_{\widehat{A(\Omega)}}(N^h, D \cup \{\delta\})$ no non-basic apex node is incident with D via a negative edge. Proposition 4.75 yields that no non-basic apex node of degree 1 is of degree 1 in $\mathcal{C}_A(N^h, D \cup \{\delta\})$. Therefore, this covector graph is either feasible, which means that we are finished, or δ is incident with an apex node j of degree 1 via a negative edge. In the latter case, again with Proposition 4.75, this also holds in $\mathcal{C}_{\widehat{A(\Omega)}}(N^h, D \cup \{\delta\})$.

If $D \cup \{\delta\} = [d]$, then $\mathcal{C}_{\widehat{A(\Omega)}}(N^h, D \cup \{\delta\})$ is totally infeasible. By Lemma 4.26, all covector graphs in all contractions for $(\widehat{A(\Omega)}, \Xi)$ are infeasible. Combining Lemma 4.34 and Lemma 4.39 implies the infeasibility of the signed system (A, Σ).

Otherwise, for any $\delta' \in [d] \setminus (D \cup \{\delta\})$, the Cramer covector $\mathcal{C}_{\widehat{A(\Omega)}}(N^h \cup \{j\}, D \cup \{\delta\} \cup \{\delta'\})$ is the basic covector $\mathcal{B}_{\widehat{A(\Omega)}}(N^h \cup \{j\}, D \cup \{\delta\}, \delta')$ and we can continue the iteration of the while-loop in Line 5. The termination is guaranteed as D grows in each iteration of this while-loop. □

Corollary 4.77. *The following statements are equivalent:*

a) (A, Σ) *is feasible.*

b) $(A(\Omega), \Xi)$ *is feasible.*

c) $(\widehat{A}, \Sigma)$ *is feasible.*

d) $(\widehat{A(\Omega)}, \Xi)$ *is feasible.*

Proof. The equivalence of all statements follows from the equivalence of the feasibility of (A, Σ) and $(\widehat{A(\Omega)}, \Xi)$. Now, if (A, Σ) is feasible, there is a point $x \in \mathbb{TA}^d$ so that $G_A(x)$ is feasible. By Lemma 4.39, $G_{A(\Omega)}(x)$ is feasible as well. Lemma 4.34 implies that $(\widehat{A(\Omega)}, \Xi)$ contains a feasible covector. If (A, Σ) is infeasible, Algorithm 7 implicitly computes a totally infeasible covector graph in $(\widehat{A(\Omega)}, \Xi)$. That ensures the infeasibility of $(\widehat{A(\Omega)}, \Xi)$. $\square$

Example 4.78. Consider the signed system (A, Σ) with

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \infty & 0 & 0 & \infty \\ \infty & 4 & 2 & \infty \\ 1 & -5 & \infty & 0 \\ 4 & 0 & -7 & 3 \\ 0 & \infty & -9 & \infty \\ 0 & \infty & \infty & 3 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} + & - & + & + \\ \bullet & + & - & \bullet \\ \bullet & - & + & \bullet \\ - & + & \bullet & + \\ + & + & + & - \\ + & \bullet & - & \bullet \\ + & \bullet & \bullet & - \end{pmatrix} .$$

Note that the last two rows are obtained by splitting the inequality $x_1 \leq (-9) \odot x_3 \oplus 3 \odot x_4$ into $x_1 \leq (-9) \odot x_3$ and $x_1 \leq 3 \odot x_4$.

We want to execute Algorithm 7 for (A, Σ) and start with $\delta = 2$. The iterations are shown in the table. We choose $j = 1$ as first entering apex.

δ	Cramer solution	violated inequalities
2	$A[\emptyset \vert \{2\}] = (\infty, 0, \infty, \infty)$	$j = 1, 3$
1	$A[\{1\} \vert \{1,2\}] = (1, 0, \infty, \infty)$	$r = 3$
	$A[\{3\} \vert \{1,2\}] = (4, \infty, \infty, \infty)$	$j = 4$
3	$A[\{3,4\} \vert \{1,2,3\}] = (-3, 3, 5, \infty)$	$j = 6$
4	$A[\{3,4,6\} \vert \{1,2,3,4\}] = (-5, 2, 4, -4)$	

The final result $(-5, 2, 4, -4)$ is a feasible point for the signed system.

4.7.3 Refined Analysis of the Runtime

For the abstract setting in Section 4.6, we gave only a rough upper bound on the number of iterations. For the realizable case, we obtain a better bound with Lemma 4.60. We show that Algorithm 7 is pseudopolynomial and only depends on the combinatorial structure of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

Let $\widetilde{A}$ be any matrix which induces the same triangulation as $\widehat{A(\Omega)}$. Recall the sequence $N^1, N^2, \ldots, N^h$ from the proof of Theorem 4.76. Then $\mathcal{C}_{\widetilde{A}}(N^1, D \cup \{\delta\}), \ldots, \mathcal{C}_{\widetilde{A}}(N^h, D \cup \{\delta\})$ is a sequence of basic covector graphs. With Corollary 4.56, we can apply Lemma 4.60 to the associated points $\widetilde{A}[N^1 | D \cup \{\delta\}], \ldots, \widetilde{A}[N^h | D \cup \{\delta\}]$. Let $z^1, \ldots, z^h$ be the representatives of this sequence modulo $\mathbb{R} \cdot \mathbf{1}$ with $z^\ell_\delta = 0$. In this way, for all $\ell \in [h-1]$ this yields the inequalities

$$z^\ell_i \leq z^{\ell+1}_i \qquad \text{for all } i \in (D \cup \{\delta\}) \ ,$$

where at least one inequality is strict for each ℓ.

If $\widetilde{A}$ is an integer matrix, then the points z^ℓ have only integer entries. Hence, for all $\ell \in [h-1]$, the difference $z^{\ell+1} - z^\ell$ is a non-negative integer vector with at least one non-zero entry. We deduce $\sum_{i\in(D\cup\{\delta\})}(z_i^h - z_i^1) \geq h$.

Furthermore, defining $\omega = \max\left\{|\widetilde{a_{ij}}| \mid (i,j) \in [d]\times[n]\right\}$, Lemma 4.33 yields the inequality

$$\left|\sum_{i\in(D\cup\{\delta\})}(z_i^h - z_i^1)\right| = \left|\sum_{i\in(D\cup\{\delta\})}(z_i^h - z_\delta^h + z_\delta^h - z_i^1)\right| \leq \sum_{i\in(D\cup\{\delta\})}|z_i^h - z_\delta^h| + \sum_{i\in(D\cup\{\delta\})}|z_\delta^h - z_i^1| \leq 2\cdot d\cdot 2\cdot\omega = 4d\omega \enspace .$$

We conclude the following.

Theorem 4.79. *The maximal number of iterations h of the while loop in Line 6 of Algorithm 7 fulfills $h \leq 4d\omega$.*

Note that a similar idea is used to give bounds on the runtime in [Ben14, §5.2] by using [FT87, Theorem 3.3].

To examine the parameter ω further, recall that each matrix in $\mathbb{R}^{n\times d}$ defines a height function for a regular subdivision of $\Delta_{n-1}\times\Delta_{d-1}$.

The set of all matrices which induce the same regular subdivision defines an open polyhedral cone. The collection of these cones is a complete fan, the *secondary fan*. For an introduction to secondary fans see [DLRS10, §5].

Since the secondary fan is the normal fan of the secondary polytope, see [GKZ94, §7] or [DLRS10, §5], which is a rational polytope for $\Delta_{n-1}\times\Delta_{d-1}$, every cone contains a rational and, hence, an integer vector.

Inspired by Theorem 4.79, we leave it as future work to give bounds on the minimal integer vectors in the cones of the secondary fan of $\Delta_{n-1}\times\Delta_{d-1}$. This might reveal either a good upper bound on the runtime of Algorithm 7 or special classes of instances which are particularly hard. Furthermore, it is interesting to consider the cones in the secondary fan which contain the weight functions describing parity games, see Subsection 4.3.2. We will take this up in the Conclusion 5.

4.7.4 Maximal support

Since the tropical sum of two feasible vectors is feasible again, the union of the supports of the feasible points is the support of a feasible point, see also [AGG12, Theorem 3.2]. We call this the *feasible support*.

Algorithm 7 determines a feasible point of a signed system or certifies that there is none. However, a resulting feasible point does not need to have the full feasible support. We show how one can use Algorithm 7 to determine the feasible support. The interest to determine this is motivated by the interpretation of the feasible points as vectors of feasible starting times or winning positions in a mean payoff game presented in Section 4.3.

We need some technical observations to achieve this.

Lemma 4.80. *Let (A, Σ) be a signed system for which the ith column of Σ only contains '$+$' entries. Then for any point $(z_1, \dots, z_d) \in \mathbb{T}^d_{\min}$ there is a number $\xi \in \mathbb{R}$ for which $(z_1, \dots, z_{i-1}, \xi, z_{i+1}, \dots, z_d)$ is feasible.*

Proof. We can assume, without loss of generality, that $i = 1$. Now, let $k_j \in [d]$ be the index in row $j \in [n]$ for which $\sigma_{jk_j} = -$. For $\xi \leq \min\left\{ z_{k_j} + a_{jk_j} - a_{j1} \,\middle|\, j \in [n] \right\}$ we obtain

$$(a_{j1} \odot \xi) \oplus \bigoplus_{\ell \in [d], \ell \neq k_j, 1} a_{j\ell} \odot z_\ell \leq \xi + a_{j1} \leq z_{k_j} + a_{jk_j} \qquad \text{for all } j \in [n] \ .$$

Hence, $(\xi, z_2, \dots, z_d)$ is feasible. □

Observation 4.81. If w and z are feasible solutions with $\operatorname{supp}(w) \cap \operatorname{supp}(z) = \{k\}$ for some $k \in [d]$, then the point $v = (-w_k) \odot w \oplus (-z_k) \odot z$ has the same pairwise coordinate differences as w and z on its support. By this we mean that $v_i - v_\ell = w_i - w_\ell$ for $i, \ell \in \operatorname{supp}(w)$ and $v_i - v_\ell = z_i - z_\ell$ for $i, \ell \in \operatorname{supp}(z)$.

Observation 4.82. The inequality $x_2 \oplus x_1 \leq (x_1 \odot a)$ is tautological for $a \geq 0$ and equivalent to $x_2 \leq x_1 \odot a$ for $a < 0$. Furthermore, $x_1 \oplus (x_1 \odot a)$ equals x_1 for $a \geq 0$ and $x_1 \odot a$ for $a < 0$.

To determine the feasible support, we run Algorithm 7 several times with a successively reduced input. As long as the algorithm terminates with a feasible point z we modify the system and restart with the reduced system.

If the support of z consists only of one element i, we omit all the inequalities which contain the variable x_i. By Lemma 4.80, these inequalities are fulfilled for every point for which the ith component is sufficiently small.

Now, assume that the support of z consists of the indices $i_1, \dots, i_k$ with $k \geq 2$. We replace x_{i_ℓ} in each inequality of the signed system by means of the equation

$$x_{i_\ell} = x_{i_k} + z_{i_\ell} - z_{i_k} \ . \tag{4.11}$$

With Observation 4.82, we can restore the property that each variable occurs on at most one side of each inequality. Furthermore, the reduced system has a feasible solution if and only if the original system has one since we can construct a solution, which fulfills all the Equations 4.11, by Observation 4.81.

As soon as we reach a totally infeasible point in a reduced system we can deduce that the complement of the current coordinate nodes forms the feasible support of the original system.

Example 4.83. Algorithm 7 behaves pairwise differently on the examples depicted in Figure 4.24 concerning the determination of the feasible support.

For the top left one, it finds a feasible point with support $\{3\}$ but needs a second run to find the certificate that this is already the feasible support.

For the top right one, it finds a feasible point whose support has 2 elements and needs a second run to determine the feasible support $\{1, 2, 3\}$.

For the bottom left one, it needs only one run to determine that the support is just the empty set.

For the bottom right one, starting with $\delta = 1$ and continuing with $\delta = 2$ or $\delta = 3$ yields feasible points with different supports. In the former case, we arrive at a basic point with support $\{1, 2, 3\}$. For the latter, the resulting basic point only has support $\{1, 3\}$.

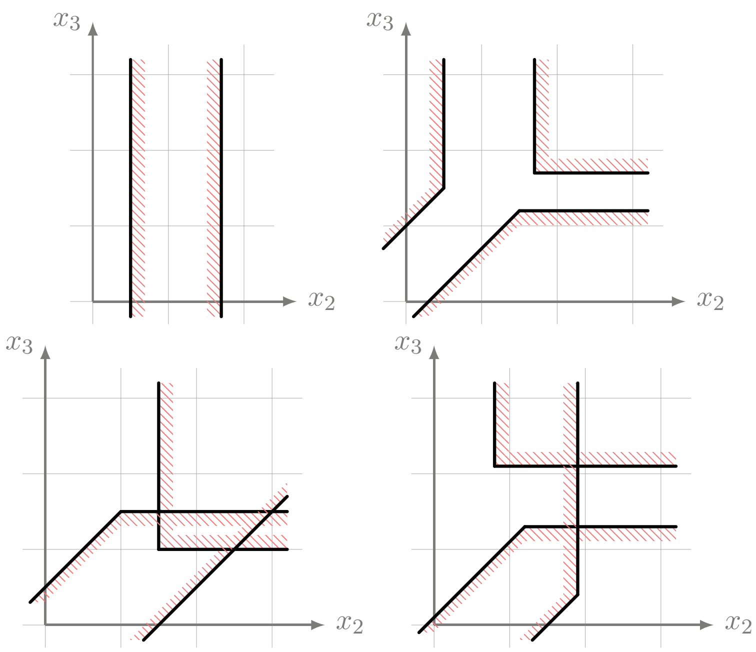

Figure 4.24: The bars indicate the *infeasible* regions. The supports of the feasible sets defined by the tropical halfspaces are different.

Moreover, we can use the former considerations to find a point, whose support is the feasible support, and a point which certifies that the feasible support cannot be bigger.

Definition 4.84. A covector graph G in $(\mathcal{S}(A), \Sigma)$ is *sufficiently infeasible* and *negatively covers* $D \subseteq [d]$ if there is a subset $N \subseteq [n]$ with $|N| = |D|$, for which $D = \bigcup_{j \in N} \operatorname{supp}(a_{j \cdot})$ and the induced subgraph of G on $D \sqcup N$ is a perfect matching consisting of negative edges.

The sufficiently infeasible covector graphs correspond to the generalized cycles with negative weight in [MSS04]. We show how one can construct a sufficiently infeasible covector graph for a signed system.

Theorem 4.85. *If F is the feasible support of the signed system (A, Σ) then there is a sufficiently infeasible covector graph G which negatively covers $([d] \setminus F)$.*

Proof. We discussed how an iterative application of Algorithm 7 can be used to determine the feasible support of a signed system. For $F = [d]$ there is nothing to show. Otherwise, let (R, Υ) be last reduced system in the sequence of successively reduced signed systems;

by construction, it is infeasible. Furthermore, let $(\widehat{R(\Omega)}, \Xi)$ be a refinement of an extension of (R, Υ).

Since (R, Υ) is infeasible, there is a totally infeasible covector graph H for $(\widehat{R(\Omega)}, \Xi)$. By the genericity of this system, there is a point x whose covector graph is contained in H and for which each basic apex of H is only incident with the negative edge.

We embed x into $\mathbb{T}^d_{\min}$ by setting the coordinates in F to ∞. Then by the construction of the extension and the refinement, x has the same covector graph with respect to A. Its covector graph G is sufficiently infeasible.

□

Example 4.86. Consider the following four matrices:

$$A = \begin{pmatrix} 0 & 0 & \infty & \infty \\ 0 & 2 & 11 & \infty \\ \infty & \infty & 0 & 0 \\ \infty & \infty & 2 & 0 \end{pmatrix} \quad \text{and} \quad \Sigma_1 = \begin{pmatrix} - & + & \bullet & \bullet \\ + & - & + & \bullet \\ \bullet & \bullet & - & + \\ \bullet & \bullet & + & - \end{pmatrix} ,$$

$$A(\Omega) = \begin{pmatrix} 0 & 0 & \Omega_1 & \Omega_2 \\ 0 & 2 & 11 & \Omega_3 \\ \Omega_4 & \Omega_5 & 0 & 0 \\ \Omega_6 & \Omega_7 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \Sigma_2 = \begin{pmatrix} + & - & \bullet & \bullet \\ - & + & \bullet & \bullet \\ \bullet & \bullet & - & + \\ \bullet & \bullet & + & - \end{pmatrix} .$$

At first, we examine the signed system (A, Σ_1). Starting with $\delta = 1$ we obtain:

δ	Cramer solution	violated inequalities
1	$A[\emptyset \mid \{1\}] = (0, \infty, \infty, \infty)$	$j = 1$
2	$A[\{1\} \mid \{1,2\}] = (0, 0, \infty, \infty)$	

The point $(0, 0, \infty, \infty)$ is feasible and the algorithm stops. We reduce the system by replacing x_1 with x_2 and, by using Observation 4.82, arrive at the system

$$(A', \Sigma_1') = \left(\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} - & + \\ + & - \end{pmatrix} \right) .$$

The Cramer solution $C_{A'}(\{3\}, \{3,4\}) = (0, 0)$ certifies the infeasibility of this reduced system. The point $x = (\infty, \infty, 0, 1)$ has a sufficiently infeasible covector graph.

As a second example, we consider the signed system (A, Σ_2). We construct Ξ by replacing the $\bullet$ entries in Σ_2 by $+$. Then $(A(\Omega), \Xi)$ is a generic extension of (A, Σ_2). The Cramer solution $\mathcal{C}_{A(\Omega)}(\{1,3,4\}, [4]) = (0, 0, \Omega_4, \Omega_4 + 2)$ has a totally infeasible covector graph. From this, we can obtain the point $x = (1, 0, \Omega_4, \Omega_4 + 1)$ which has a sufficiently infeasible covector graph. This point also yields a sufficiently infeasible covector graph for the signed system (A, Σ_2).

We conclude by interpreting a sufficiently infeasible covector graph in terms of mean payoff games. Recall the connection from Theorem 4.13. Extending the notions from

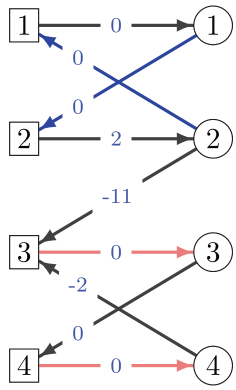

Figure 4.25: Graph for the mean payoff game corresponding to (A, Σ_1) from Example 4.86.

Subsection 4.3.2, we say that a coordinate node or an apex node is *winning* for the player on the coordinate nodes if there is a *winning strategy* meaning that the value of the game is negative when we start from such a position and this strategy is used on the coordinate nodes.

Let H be a sufficiently infeasible covector graph for the signed system (A, Σ) which negatively covers $D \subseteq [d]$.

Theorem 4.87. *The coordinate nodes in D and the apex nodes, whose support is contained in D, are winning positions for the strategy formed by the perfect matching μ consisting of negative edges contained in H.*

Proof. Let N be the set of the apex nodes, whose support is contained in D. Then the player on the apex nodes is forced to go back to D on N. Furthermore, the arcs formed from μ only go to N by the properties of H. Since H is a covector graph, Proposition 4.4 implies with the construction of the mean payoff graph in Equation 4.6 that all cycles reachable from N and from D through μ are negative. □

With Theorem 4.13, we deduce an extension of Lemma 4.26 for the realizable case.

Corollary 4.88. *If $(\mathcal{S}(A), \Sigma)$ contains a sufficiently infeasible covector graph G which negatively covers D, then $\operatorname{supp}(z) \subseteq [d] \setminus D$ for every feasible point z of (A, Σ).*

Proof. Theorem 4.87 implies that the player on the coordinate nodes has a winning strategy which secures a negative value. Therefore, there cannot be a feasible point z with $\operatorname{supp}(z) \cap D \neq \emptyset$ since this would imply a non-losing strategy for the player on the apex nodes with starting positions $\operatorname{supp}(z)$ by Theorem 4.13. □

Example 4.89. Figure 4.25 shows winning strategies in the mean payoff game corresponding to the signed system (A, Σ_1) from Example 4.86. The blue arcs form a non-loosing strategy for the player on the circle nodes. They are the positive edges in the covector graph of the feasible point $(0, 0, \infty, \infty)$. The purple arcs form a winning strategy for the player on the square nodes. They are the edges in the sufficiently infeasible covector graph of the point $(\infty, \infty, 0, 1)$.

5 Conclusion

We started by extending the theory of tropical convexity for infinite coordinates through methods from polyhedral geometry. This lead to the study of a tropical analogue of an oriented matroid, namely a signed tropical matroid. As a helpful tool to examine the relation between classical and tropical configurations we introduced the field of Puiseux fractions. Eventually, we could deduce a rather general algorithm for the tropical feasibility problem which has several nice properties and is related to the classical simplex method. The results of this thesis can be extended in several directions.

Our starting point was a thorough investigation of covector graphs and covector decompositions with infinite coordinates. As an extension, we introduced the concept of a signed tropical matroid as generalization of tropical linear inequality systems. We achieved this by deriving them from not necessarily regular subdivisions of a subpolytope of $\Delta_{n-1} \times \Delta_{d-1}$. It would be nice to have a different description which relies on purely combinatorial axioms similar to the axioms of a tropical oriented matroid [AD09]. A possible description could go through matching ensembles [OY13]. This might enable us to give a proper definition of feasibility for a not-necessarily full signed tropical matroid, see Subsection 4.6.1. It should fulfill that an extension is feasible if and only if all extensions are feasible.

We introduced Puiseux fractions as a suitable field for computations. It can be used to compute dual tropical convex hull in the generic case. We think that a proper description of the complexity of the arithmetic operations for Puiseux fractions is in place. With that, one should answer the question if there is a way to solve tropical linear programs through a linear program over Puiseux fractions in polynomial time.

Our feasibility Algorithm 6 for a signed tropical matroid differs in some aspects from the simplex method. With the insights from our kind of pivoting, we propose to study the pivoting in the tropicalized simplex method to determine if it can also be proven to solve abstract tropical linear programming. On the other hand, we suggest to formulate Algorithm 7 as a variant of the simplex method.

We came up with an upper bound for the number of iterations in terms of integer vectors in the secondary fan of $\Delta_{n-1} \times \Delta_{d-1}$. Computational experiments with `polymake` [GJ00] have shown that the algorithm needs only linearly many steps in d and n for all sample instances. We conjecture that the maximal entry of a minimal integer vector in each full-dimensional cone of the secondary fan of $\Delta_{n-1} \times \Delta_{d-1}$ is bounded by a polynomial in d and n.

Parity games can be considered as mean payoff games whose coefficient matrix has a particular form, see Subsection 4.3.2. The cones of the secondary fan of $\Delta_{n-1} \times \Delta_{d-1}$ which contain these coefficient matrices form a subfan. A closer examination of this subfan should reveal if it is proper and if we can give upper bounds on the minimal integer vectors in each cone. This provides a better understanding of how special parity games actually are compared with mean payoff games. Indeed, recall that they were used to demonstrate the worst-case complexity of several algorithms for mean payoff games and linear programming [Fri11, Han12]. However, they were also already shown to be quasi-polynomial [CJK+].

Additionally, we want to point out the close relation between strongly polynomial, weakly polynomial and pseudopolynomial behavior of algorithms which was demonstrated in the tropical setting in [Ben14] and in Subsection 4.7.3. Therefore we believe that it is a promising approach for settling the complexity of mean payoff games and tackling the question of the strong polynomiality of classical linear programming.

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