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Abstract

We examine the centroaffine geometry of Tchebychev surfaces. We introduce regular and singular surfaces. By understanding the local integrability conditions, we will classify the centroaffine Tchebychev surfaces of constant curvature metric.
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1 Introduction

1.1 Centroaffine Geometry

Let M be a connected oriented manifold of dimension n. A hypersurface immersion $x\colon M^n \to \mathbb{R}^{n+1}$ is called *centroaffine*, if the position vector is nowhere tangent to the surface. For centroaffine hypersurfaces, the second directional derivative $\overline{\nabla}^2$ of the position vector x can be decomposed as follows:

$$\overline{\nabla}_U dx(V) = dx(\nabla_U V) + h(U,V)(-x), \quad U,V \in \Gamma(TM).$$

This is the Gauss structure equation. ∇ is called the *induced connection*, h the *centroaffine metric*. We assume h to be semi-Riemannian, so it defines raising and lowering $(^\flat)$ of indices as well as $\|\cdot\|$ on tensors. Its Levi-Civita connection and curvature tensor will be denoted by $\widehat{\nabla}$, $\widehat{\mathrm{R}}$, respectively. Let

∇^* denote the conormal connection. We define the *(1,2)-difference tensor* C by

$$C := \frac{1}{2}(\nabla - \nabla^*) = \nabla - \widehat{\nabla} = \widehat{\nabla} - \nabla^*.$$

The $(0,3)$-tensor $C^\flat$ is called the *cubic form* of the hypersurface. The *Tchebychev form* $T^\flat$ is obtained by contracting C:

$$nT^\flat(\partial_i) = \sum_{j=1}^{n} C_{ij}{}^j.$$

T is called centroaffine *Tchebychev vector field* and is linked to the equiaffine support function ρ at 0 by

$$T = \frac{n+2}{2n} \operatorname{grad} \log |\rho|.$$

Finally, we define the totally symmetric traceless cubic form $\tilde{C}$ introduced by U. Simon by

$$\begin{aligned}\tilde{C}(U,V,W) =& C(U,V,W) - \frac{n}{n+2}(T^\flat(W)h(U,V) + \\ & T^\flat(V)h(W,U) + T^\flat(U)h(V,W)).\end{aligned}$$

In centroaffine geometry we have the following integrability conditions ([5], 6.3.3.3):

$$C^\flat \text{ is totally symmetric,} \tag{1}$$

$$\begin{aligned}\widehat{\mathrm{R}}(U,V)W = & C(C(U,W),V) - C(C(V,W),U) + \\ & h(V,W)U - h(U,W)V,\end{aligned} \tag{2}$$

$$\widehat{\nabla} C^\flat \text{ is totally symmetric.} \tag{3}$$

1.2 Centroaffine Tchebychev surfaces

A centroaffine hypersurface is called *Tchebychev*, if the Tchebychev vector field T of x is conformal. As T is always closed [5, 4.4.8], this is equivalent to

$$\widehat{\nabla}_X T = \lambda X$$

for any $X \in \Gamma(TM)$. λ is called the (normalised) *centroaffine divergence* of T. For motivation of these definitions cf. [3]. Here are the known examples:

(a) Proper affine spheres ($T = 0$) with centroaffine origin in their center.
(b) Quadrics (characterised by $\tilde{C} = 0$) with origin chosen arbitrarily.

(c) Centroaffine canonical hypersurfaces ($\widehat{\nabla} C^\flat = 0$ and h is flat). For ind $h \leq 1$, there is a classification in [2].

(d) The family of centroaffine noncanonical ($\widehat{\nabla} C^\flat \neq 0$) flat Tchebychev surfaces:

$$x(t,s) = \Big(a(s)\, e^{c(t)}, b(s)\, e^{c(t)}, e^{d(s)}\Big),$$

where a, b are linearily independent and $c', d' \neq 0$.

For $n = 2$, Tchebychev surfaces with $\lambda = 0$ as well as those with vanishing centroaffine Gauss curvature $K = \widehat{K}$ are classified. In the first case, the complete list is given by (a), (c), (d) above ([3, Theorem 4.1]), whereas in the second the list is (c), (d) ([3, Theorem 4.2]).

2 Regular and Singular Tchebychev Surfaces

Let $x \colon M^2 \to \mathbb{R}^3$ be a Tchebychev surface with $\|T\|^2 \neq 0$ on M. In semi-Riemannian geometry, a conformal vector field which is either space- or timelike is called *inessential.*

2.1 Local Integrability Conditions

We apply [1, Lemma 2.7] which states that locally the existence of a closed conformal vector field is equivalent to a warped product structure of the manifold. In dimension $n = 2$ the fiber manifold is a curve. We assume centroaffine arc length parametrisation by s. The induced Gauss basis $\{\partial_t, \partial_s\}$ is unique up to sign. From now on we will abbreviate ∂_t by $'$. With the notation $f := \log|\rho|$ we get

$$h \;=\; \eta\, dt \otimes dt \;+\; \epsilon f'(t)^2\, ds \otimes ds$$

for $\epsilon = \pm 1$. The Tchebychev vector field of x is $T = \operatorname{grad} f = \eta f' \partial_t$. For the Tchebychev-Operator and $1 \leq i \leq n-1$ we get

$$\widehat{\nabla}_t T = \eta f'' \partial_t = f' \, \widehat{\nabla}_i \partial_t = \widehat{\nabla}_i T.$$

The general formula reduces to $\lambda = \eta f''$. Calculating $\widehat{\nabla}$ and K we get

$$\widehat{\nabla}_t \partial_t = 0, \qquad \widehat{\nabla}_t \partial_s = \widehat{\nabla}_s \partial_t = \frac{f''}{f'} \partial_s, \qquad \widehat{\nabla}_s \partial_s = -\eta \epsilon f' f'' \partial_t,$$

$$K = -\eta \frac{f'''}{f'}. \tag{4}$$

For complete determination of the centroaffine geometry we introduce the cubic form $C^\flat$ and collect integrability conditions.

2.1.1 Integrability condition (1)

This means $C^\flat$ has four essential components C_{111}, C_{112}, C_{122}, C_{222}. For tensor indices we use 1, t and 2, s synonymously. By definition of $T^\flat$ we have a "partial apolarity":

$$C_{11}{}^1 + C_{12}{}^2 = 2f', \qquad \text{or} \qquad \eta\, C_{111} + \epsilon\, (f')^{-2} C_{122} = 2f', \tag{5}$$

$$C_{21}{}^1 + C_{22}{}^2 = 0, \qquad \text{or} \qquad \eta\, C_{112} + \epsilon\, (f')^{-2} C_{222} = 0. \tag{6}$$

2.1.2 Integrability condition (2)

Writing (2) in coordinate form we observe that all symmetries of $\widehat{\mathrm{R}}$ are built into the formula. For $n = 2$, curvature tensors posess only one essential component $\widehat{\mathrm{R}}_{1212} = K \det h$. Therefore, (2) is equivalent to the following equation:

$$K \det h = \widehat{\mathrm{R}}_{1212} = C_{12}{}^1 C_{121} + C_{12}{}^2 C_{221} - C_{22}{}^1 C_{111} - C_{22}{}^2 C_{211} + h_{11} h_{22}. \tag{7}$$

Elimininating C_{122} and C_{222} from (7) using (5), (6) we obtain

$$\epsilon (C_{112})^2 + \eta f'^2 \Big(C_{111} - \frac{3}{2} \eta f' \Big)^2 = \frac{1}{2} f'^2 \Big(K + \frac{1}{2} \eta f'^2 - 1 \Big). \tag{8}$$

We introduce

$$r(t) := \frac{1}{\sqrt{2}} |K + \frac{1}{2} \eta f'^2 - 1|^{\frac{1}{2}}, \quad \nu := \operatorname{sgn}(K + \frac{1}{2} \eta f'^2 - 1). \tag{9}$$

In the following distinction, we do not have to discuss isolated zeros of r since they resemble a closed set without inner points.

Definition. We call a Tchebychev surface x with $||T||^2 \neq 0$ *regular* if $r > 0$, and *singular* if $r \equiv 0$.

2.1.3 Integrability condition (3)

Because of (1) the total symmetry of $\widehat{\nabla} C^\flat$ is equivalent to

$$(\widehat{\nabla}_t C^\flat)\ (\partial_s\,, \cdot\,, \cdot) \;=\; (\widehat{\nabla}_s C^\flat)\ (\partial_t\,, \cdot\,, \cdot).$$

Inserting (∂_t, ∂_t), (∂_t, ∂_s) and (∂_s, ∂_s) into the free slots yields

$$\partial_t C_{112} \;+\; 2\frac{f''}{f'} C_{112} \;=\; \partial_s C_{111}, \tag{10}$$

$$\partial_t C_{122} \;-\; \epsilon \eta f' f'' C_{111} \;=\; \partial_s C_{112}, \tag{11}$$

$$\partial_t C_{222} \;-\; 2\frac{f''}{f'} C_{222} \;-\; 2\epsilon \eta f' f'' C_{112} \;=\; \partial_s C_{122}. \tag{12}$$

By eliminating C_{122} and C_{222} through (5), (6) we realize the equivalence of (10) and (12). Only two PDE remain:

$$\begin{aligned} \partial_t C_{112} + 2\frac{f''}{f'}C_{112} &= \partial_s C_{111}, &(13)\\ 3\epsilon f' f''(2f' - \eta C_{111}) &= \partial_s C_{112} + \epsilon\eta f'^2 \partial_t C_{111}. &(14) \end{aligned}$$

Lemma 2.1. (i) *The components of* $\tilde{C}$ *read*

$$\tilde{C}_{111} = C_{111} - \frac{3}{2}\eta f', \qquad \tilde{C}_{112} = C_{112},$$
$$\tilde{C}_{122} = C_{122} - \frac{1}{2}\epsilon f'^3, \qquad \tilde{C}_{222} = C_{222},$$

(ii) $||\tilde{C}||^2 = 4\nu r^2$.

Proof. Straightforward calculations. □

Remark 2.2. x is an open part of a quadric if and only if $C_{111} = \frac{3}{2}\eta f'$, $C_{122} = \frac{1}{2}\epsilon f'^3$, $C_{112} = C_{222} = 0$. Obviously, the integrability conditions are fulfilled in this case. On the other hand, from (8) it is clear that any quadric is a singular Tchebychev surface.

2.2 Regular Tchebychev surfaces

For regular surfaces (8) takes the form

$$\epsilon\left(\frac{C_{112}}{f'r}\right)^2 + \eta\left(\frac{C_{111} - \frac{3}{2}\eta f'}{r}\right)^2 = \nu. \tag{15}$$

Exemplarily we consider definite surfaces ($\epsilon = \eta = \nu$). By the substitution

$$C_{112} = f' r \cos\varphi, \qquad C_{111} = r\sin\varphi + \frac{3}{2}\eta f' \tag{16}$$

for a function $\varphi = \varphi(t, s)$, (15) is fulfilled automatically. (13) and (14) modify as follows:

$$\begin{aligned} w\cos\varphi &= f'\sin\varphi\,\partial_t\varphi + \cos\varphi\,\partial_s\varphi, &(17)\\ -w\sin\varphi &= f'\cos\varphi\,\partial_t\varphi - \sin\varphi\,\partial_s\varphi &(18) \end{aligned}$$

for the function $w(t) := f'r'/r + 3f''$. Consider (17), (18) as a linear system in the partials. This system always has a unique solution:

$$\partial_t\varphi = 0, \quad \partial_s\varphi = w(t).$$

We get $\varphi = \varphi(s) = ws + c$ for some constant $c \in [0, 2\pi)$ and $w = \text{const}$. Considering c as a variable we obtain a 1-parameter family of Tchebychev surfaces ([3, Theorem 5.1]). For indefinite surfaces ($\epsilon\eta = -1$) the considerations are analogous. Depending on the sign $\epsilon\nu$ we substitute $\sinh\varphi$, $\cosh\varphi$ and choose $c \in \mathbb{R}$. Using (4) the condition $w = \text{const}$ leads to

$$f^{(4)} - 5\eta f'' K - 4f'^2 f'' + 6\eta f'' + wf'^2 + 2\eta w(K-1) = 0, \tag{19}$$

an explicit third order ODE for f'. For given initial values there is a unique solution.

Theorem 2.3. *Let $x\colon M^2 \to \mathbb{R}^3$ be a regular Tchebychev surface. Up to centroaffine equivalences and the freedom in t_0, x is uniquely determined by the set of data*

$$(\eta,\ \epsilon,\ t_0,\ f'(t_0),\ f''(t_0),\ f'''(t_0),\ c).$$

Remark 2.4. (i) The three initial conditions are not arbitrary in $\mathbb{R}^3$, since e.g. in the definite case $\nu(t_0) = \eta = \epsilon$ is a must.

(ii) The data sets $(\cdot, \cdot, \cdot, b \neq 0, 0, 0, \cdot)$ exactly resemble the (regular) flat Tchebychev surfaces (c), (d) from subsection 1.2, since $\lambda = 0$ if and only if $f'' = 0$ and since $f' = b = \text{const}$ is the unique solution of (19) for the given initial conditions.

(iii) There are no Tchebychev surfaces which induce a nontrivial homothetical Tchebychev vector field on M: On an open set $U \subseteq M$ with $||T||^2 \neq 0$ from $0 \neq \text{const} = \lambda = \eta f''$ we get $f'(t) = \eta\lambda t + \tilde{c}$. Because of $K = 0$ and (9) this surface would be regular. The linear f' leads to a contradiction in (19).

Theorem 2.5. *A regular Tchebychev surface has flat centroaffine metric if and only if $||\tilde{C}||^2 = \text{const}$:*

Proof. Due to 2.1.ii $||\tilde{C}||^2 = \text{const}$ is equivalent to $r' = 0$ or $w = 3f'' = \text{const}$, which in turn is equivalent to $K = 0$. □

2.3 Singular Tchebychev surfaces

For $r = 0$ we cannot avoid a consideration of the index of h. Singularity leads to the second order ODE

$$f''' = \frac{1}{2} f'^3 - \eta f' \tag{20}$$

for f'. For definite surfaces (8) implies that $C_{112} = 0$ and $C_{111} = \frac{3}{2}\eta f'$. With Remark 2.2 we get

Theorem 2.6. *Any definite singular Tchebychev surface is an open set of a quadric.*

For indefinite surfaces things are more complicated. From (8) we see

$$\xi\, C_{112} = f'(C_{111} - \frac{3}{2}\eta f')$$

for some $\xi \in \{-1, 0, 1\}$. As in the definite case we obtain quadrics for $\xi = 0$. If $\xi \neq 0$, then the PDE (13), (14) are equivalent to

$$-6\eta f' f'' + 3f'' C_{111} = -f' \partial_t C_{111} + \xi\, \partial_s C_{111}, \tag{21}$$

a quasilinear PDE for C_{111}. Considering C_{111} as a graph surface over $\mathbb{R}^2$ for uniqueness we have to desribe an "initial curve". Thus, there are "many" different indefinite singular Tchebychev surfaces which – in contrast with Theorem 2.3 – cannot be described by a finite number of parameters. These non-quadric singular surfaces make that [3, Theorem 5.1] mentioned above cannot be generalised to indefinite metrics.

2.4 Main Result

The following lemma treats the class of surfaces we skipped in the preceding section.

Lemma 2.7. *An arbitrary centroaffine indefinite Tchebychev surface with null vector field T must have flat centroaffine metric.*

Proof. We use asymptotic coordinates for the metric, i.e.

$$h = F(ds \otimes dt + dt \otimes ds).$$

Then the Levi-Civita connection of h reads

$$\widehat{\nabla}_t \partial_t = \frac{F_t}{F}\partial_t, \quad \widehat{\nabla}_t \partial_s = \widehat{\nabla}_s \partial_t = 0, \quad \widehat{\nabla}_s \partial_s = \frac{F_s}{F}\partial_s.$$

Without loss of generality we can write $T = \psi \partial_s$ for some $\psi \in C^\infty(M)$. For $\widehat{\nabla} T$ we get

$$\widehat{\nabla}_t T = \psi_t \partial_s, \quad \widehat{\nabla}_s T = (\psi_s + \psi\frac{F_s}{F})\partial_s.$$

The Tchebychev condition implies $\psi_t = 0$, $\psi_s + \psi F_s/F = 0$, which means $\psi = \psi(s) = c(t)/F(t,s)$ for some $c \in C^\infty(\mathbb{R})$. The integrability condition of the system of PDE is the multiplicative separability of F. Hence

$$K = -F^{-1}\partial_t\partial_s \log F = 0. \quad \square$$

Theorem 2.8. *Let $x\colon M^2 \to \mathbb{R}^3$ be a Tchebychev surface of constant curvature K. Then x is centroaffinely equivalent to either an affine sphere (see [4, Fig. 1]) or to one of the flat surfaces (c), (d) in subsection 1.2.*

Proof. (a) Suppose $\|T\|^2 = 0$ on an open submanifold U of M. Then either $T = 0$, which chararcterises affine spheres, or T is a null vector field. In the latter case $K = 0$ follows from Lemma 2.7.

(b) Otherwise, we apply the previous results to the Tchebychev surface $x\colon U \to \mathbb{R}^3$. If x is singular, then from $r = 0$ and $K = \text{const}$ it follows that $f' = \text{const} \neq 0$. Thus $K = 0$.

(c) If $x|_U$ is regular, then for $K = \text{const}$, $\kappa := |K|^{\frac{1}{2}}$, $\xi := -\eta \operatorname{sgn} K \in \{\pm 1\}$ and $\alpha, \beta \in \mathbb{R}$

$$f'(t) = \begin{cases} \alpha \cosh \kappa t + \beta \sinh \kappa t, & \text{if } \xi = +1 \text{ and} \\ \alpha \ \cos \kappa t + \beta \ \sin \kappa t, & \text{if } \xi = -1 \end{cases} \tag{22}$$

solves the ODE $f''' = -\eta K f'$. We assert that the system (19), (22) has nontrivial solutions for $K = 0$ only. Therefore, we insert (22) into (19):

$$6\eta(1-K)f'' - 4f'^2 f'' + w f'^2 + 2\eta w(K-1) = 0. \tag{23}$$

To treat $\xi = \pm 1$ at one time we write $\mathbf{co}(t) := \cosh \kappa t$, $\mathbf{si}(t) := \sinh \kappa t$ if $\xi = +1$ and $\mathbf{co}(t) := \cos \kappa t$, $\mathbf{si}(t) := \sin \kappa t$ if $\xi = -1$. Then $f' = \alpha\,\mathbf{co} + \beta\,\mathbf{si}$, $f'' = \alpha\kappa\xi\,\mathbf{si} + \beta\kappa\,\mathbf{co}$ and from (23) we get using $\mathbf{co}^2 - \xi\,\mathbf{si}^2 = 1$

$$\begin{aligned} 2\alpha\kappa\xi(3\eta(1-K) - 2(\alpha^2 + 2\xi\beta^2))\ \mathbf{si}\ &+ \\ 2\beta\kappa(3\eta(1-K) + 2(\alpha^2 + \xi\beta^2))\ \mathbf{co}\ &- \\ 4\alpha\kappa(\alpha^2 + 3\xi\beta^2)\,\mathbf{si}^3\ - 4\beta\kappa(\xi\beta^2 + 3\alpha^2)\,\mathbf{co}^3 &+ \\ w(\xi\alpha^2 + \beta^2)\,\mathbf{si}^2\ + 2w\alpha\beta\,\mathbf{si}\,\mathbf{co}\ + 2\eta w(K-1) + w\alpha^2 &= 0. \end{aligned} \tag{24}$$

It is easy to verify that $\{1, \mathbf{si}, \mathbf{co}, \mathbf{si}\,\mathbf{co}, \mathbf{si}^2, \mathbf{si}^3, \mathbf{co}^3\}$ is a linearily independent system of functions. Hence all coefficients in (24) vanish. Elementary considerations of the coefficients of $\mathbf{si}\,\mathbf{co}$, $\mathbf{si}^3$ und $\mathbf{co}^3$ lead to $\alpha = \beta = 0$, which contradicts the nondegeneracy of h. This proves the Theorem.

□

Remark 2.9. In [3, Theorem 4.4] there is a *global* classification of the *complete definite* Tchebychev surfaces of constant curvature $K \neq 0$: Ellipsoids and hyperboloids with center $0 \in \mathbb{R}^3$ (i.e. proper affine spheres) represent all possibilities.

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