Small-Time Large Deviations for Sample Paths of Infinite-Dimensional Symmetric Dirichlet Processes

vorgelegt von
Mag. rer. nat. Stephan Sturm
aus Wien

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Introduction

The objective of the present thesis is to study the small-time asymptotic behavior of diffusion processes on general state spaces. In particular we want to determine the convergence speed of the weak law of large numbers, so we study the asymptotics on an exponential scale, as developed starting with the pioneering work of Cramér [C] (but already Khinchin [Kh] used in the early 1920s the notion of 'large deviations' for the asymptotics of very improbable events). Moreover, we are interested in the behavior of the whole path of the diffusion. So we consider a diffusion process (X_t) on the state space $\mathcal{X}$, and define the family of diffusions (X_t^ε), $\varepsilon > 0$, $X_t^\varepsilon = X_{\varepsilon t}$. For this family of sample paths we want to prove large deviation principle

$$- \inf_{\omega \in A^\circ} I(\omega) \leq \liminf_{\varepsilon \to 0} \varepsilon \log P_\mu [X_\cdot^\varepsilon \in A] \leq \limsup_{\varepsilon \to 0} \varepsilon \log P_\mu [X_\cdot^\varepsilon \in A] \leq - \inf_{\omega \in \overline{A}} I(\omega)$$

for any measurable set A in the space of sample paths $\Omega := ([0,1]; \mathcal{X})$, with some rate function I which will be determined.
First results in this direction were achieved by Varadhan [Va67b], who proved the small-time asymptotics on an exponential scale for elliptic, finite-dimensional diffusion processes. Generalizations to the hypoelliptic case can by found in Azencott [Az81]. Schilder's Theorem for Brownian motion can be understood as predecessor. It states that for a Brownian motion (W_t) it holds that

$$- \inf_{\omega \in A^\circ} I(\omega) \leq \liminf_{\varepsilon \to 0} \varepsilon \log P [W_\cdot^\varepsilon \in A] \leq \limsup_{\varepsilon \to 0} \varepsilon \log P [W_\cdot^\varepsilon \in A] \leq - \inf_{\omega \in \overline{A}} I(\omega)$$

with rate function

$$I(\omega) = \begin{cases} \frac{1}{2} \int_0^1 |\omega'(t)|^2 \, dt & \omega \in H; \\ \infty & \text{otherwise,} \end{cases}$$

where H is the Cameron Martin space of absolutely continuous functions with square integrable derivatives. By the scaling $\sqrt{\varepsilon} W_\cdot \sim W_{\varepsilon \cdot}$ it is equivalent to a statement of large deviations for an $\sqrt{\varepsilon}$-spatial scaling of Brownian motion (cf. [DZ], Chapter 5.2).
Our aim is now to extend this kind of results, assuming merely that $\mathcal{X}$ is a Polish space. In particular we do not assume that it is finite-dimensional or locally compact. Some results for specific processes in this setting were already proved, e.g for the Ornstein-Uhlenbeck process on abstract Wiener spaces (Fang and Zhang, [FZ99]), the Brownian motion on loop groups (Fang and Zhang, [FZ01]), super-Brownian motion (Schied, [Sch96]), diffusions on configuration spaces (Zhang, [Z01]; Röckner and Zhang, [RZ]; compare also Röckner and Schied, [RSchi]), diffusions on Hilbert spaces (Zhang, [Z00])

and Fleming-Viot processes (Xiang and Zhang, [XZ]). All these papers have in common that their techniques of proof, in particular for the central property of exponential tightness, are very specific and adjusted to these concrete settings. Our goal is to develop a general approach which does not depend on the specific setting.

The only restriction we pose from the beginning is that we restrict ourselves to diffusion processes generated by symmetric Dirichlet forms. Dirichlet forms, introduced by Beurling and Deny [BD] in a pure analytical/potential theoretic context, are for us of interest due to their connections to stochastic processes. The relation between strong Markov processes and Dirichlet forms was first pointed out by Fukushima [Fu], a detailed analysis on this association was given by Albeverio, Ma and Röckner ([AM91], [AM92] and [MR]).

A cornerstone in our treatment are the available results on Varadhan's principle on Dirichlet spaces. This principle, the logarithmic (integrated) heat-kernel asymptotics in small time, goes back to Varadhan [Va67a], who proved it for finite-dimensional elliptic operators. Important generalizations are, e.g., due to Léandre (for the hypoelliptic case, [L87a] and [L87b]) and Norris (on Lipschitz Riemannian Manifolds, [No]); sharp bounds are due to Carlen, Kusuoka and Stroock [CKS] and Davies [D]. The result for Dirichlet spaces over a probability space was proved by Hino and Ramírez [HR], based on earlier, independent work of both authors ([H] and [Ra]) and preceeding results for abstract Wiener spaces (Aida and Kawabi, [AK]) and path groups (Aida and Zhang, [AZ]). A generalization to general σ-additive measure spaces was given by Ariyoshi and Hino [AH].

The generality of our approach clarifies also the geometry implied by Dirichlet forms and their associated diffusions. Stroock [St] claimed in his review of the books of Fukushima, Oshima and Takeda [FOT] and Ma and Röckner [MR] for the Bulletin of the American Mathematical Society that

> *Sometimes the value of a theory is that it allows progress even in the absence of understanding, and, in many of its applications, that is precisely what the theory of Dirichlet forms does allow.*

While ironically exaggerated, Stroock's claim is not completely false. One essential point in understanding Dirichlet forms, is to understand the geometry which they generate. In the case of local Dirichlet forms over locally compact spaces, this understanding was achieved by Sturm ([S94], [S95a], [S95b] and [S96]). We hope that by introducing a concept of pointwise distance also in the case of non-locally compact state spaces, we can lay the ground for a better understanding of the geometry of Dirichlet forms also in this broader setting.

The first part of the thesis is devoted to the proof of the small-time large deviation principle for the sample paths of a diffusion on a Polish space with invariant probability measure. After a description of the setting the setting (Section 1), we explore in Section 2 the geometry associated to the Dirichlet form. We introduce a convenient notion of pointwise distance and give convenient sufficient conditions under which it is compatible with the notion of a setwise distance introduced by Hino and Ramírez [HR]. This enables us to define the notions of length and energy of a path on the state space and give also an alternative description of energy. Section 3 discusses the associated Markov process. We derive with Lemma 6 a crucial result for further work, based on techniques using the forward-backward martingale decomposition by Lyons and Zheng [LZhe]. The main result is achieved in Section 4: First

we discuss the important property of exponential tightness and give with Assumption (D) a useful criterion under which exponential tightness is satisfied: the distances from sets of a compact exhaustion of the state space have to increase towards infinity outside of increasing sets of the exhaustion. A central argument in the proof is the exponentially fast Jakubowski criterion and the exponential Hölder estimate by Schied [Sch95]. Based on this fact we derive then the upper bound of a large deviation principle for the whole sample paths. The last part of this section provides an integral representation of the rate function in terms of metric derivatives of absolutely continuous curves in the sense of Ambrosio, Gigli and Savaré [AGS05]. Section 5 treats the degenerate case where the set-distance is not even square integrable and discusses under which conditions the large deviation principle can be achieved nevertheless.
The second part of this work is devoted to concrete examples. First we take up the case of the Ornstein-Uhlenbeck process on abstract Wiener spaces, already treated by Fang and Zhang [FZ99]. Using earlier results by Hirsch [Hir] and a Rademacher theorem by Enchev and Stroock [ES] (compare also the preceding results by Kusuoka, [Kus82a] and [Kus82b]), the proof of our assumptions reduces to topological arguments. This applies also to the case of linear unbounded drifts, generated by a strongly elliptic operator. The second example is the Wasserstein diffusion on the space of probability measures on the unit interval, as introduced by von Renesse and Sturm [vRS]. We discuss first the construction of this process - relying on the isometry of the space of probability measures to the space of quantile functions - motivated by questions of optimal transportation. Then we prove the pathwise small time asymptotics for both processes, the diffusion on the space of quantile functions as well as for the Wasserstein process on the space of probability measures itself. Unifying this separate results, we show the relation between Fréchet derivatives on the space of quantile functions and tangent velocity fields along paths on the space of probability measures. Besides some concrete examples, we discuss also the question of the existence of a flow of probability measures along sample paths.
Four appendices conclude the present work. The first one gives some results on backward martingales and recalls the forward-backward martingale decomposition by Lyons and Zheng. The second one discusses contraction principles for large deviations and the Dawson-Gärtner theorem for large deviations of projective limits, the third one a lemma for the forthcoming proof of the lower bound. And the last one gives some results on the geometry in the case of Dirichlet forms over σ-finite measure spaces. Following Ariyoshi and Hino [AH], the notion of intrinsic metric has to be generalized in this case. Interestingly, in the case of locally compact state spaces this generalization of the intrinsic metric does not change the notion of energy of a sample path. To make the reading of the present work more convenient, we also provide a list of symbols.

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Contents

Part I

General Results

Chapter 1

Setting

First we give the setting in which we will work in the first part of the present text. Only in Section D we will relax these assumptions to explore the chance of generalizations.
We consider a probability space $(\mathcal{X}, \mathcal{B}, \mu)$ where $\mathcal{X}$ is a Polish space and $\mathcal{B}$ is the induced Borel-σ-algebra. Let L be an unbounded, self-adjoint and negative defined operator on $L^2(\mathcal{X}, \mu)$ with domain $D(L)$ (i.e. $\langle Lf, g\rangle_{L^2(\mu)} = \langle f, Lg\rangle_{L^2(\mu)}$ and $\langle Lf, f\rangle_{L^2(\mu)} \leq 0$ for all f, $g \in D(L)$) associated with a strongly continuous semigroup of operators (T_t), $T_t = e^{tL}$ for $t > 0$, which is sub-Markovian, i.e. $0 \leq f \leq 1$ implies $0 \leq T_t f \leq 1$ for each $t > 0$ and $f \in L^2(\mathcal{X}, \mu)$. Further let $(\mathcal{E}, D(\mathcal{E}))$ be the corresponding symmetric Dirichlet form, defined as the closure of

$$\mathcal{E}(f, g) := \langle -Lf, g\rangle_{L^2(\mu)}, \qquad f, g \in D(L).$$

We use the shorthand $\mathcal{E}(f)$ to abbreviate $\mathcal{E}(f, f)$. The Dirichlet space $D(\mathcal{E})$ is endowed with the inner product

$$\langle f, g\rangle_{\mathcal{E}} := \langle f, g\rangle_{L^2(\mu)} + \mathcal{E}(f, g),$$

and thus a Hilbert space. The associated norm $\|f\|_{\mathcal{E}} := \langle f, f\rangle_{\mathcal{E}}^{\frac{1}{2}}$ is called the $\mathcal{E}$-norm.
We suppose that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, admits an opérateur carré du champ Γ, is conservative and satisfies the local property. To make these assumptions concrete:

- *Carré du champ*: We say that a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ admits a carré du champ operator, if there exists a subspace $D(\Gamma) \subseteq D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu)$, dense in $D(\mathcal{E})$, such that for all $f \in D(\Gamma)$ there exists a function $\gamma \in L^1(\mathcal{X}, \mu)$, such that for all μ-a.e. bounded $h \in D(\mathcal{E})$

$$2\mathcal{E}(fh, f) - \mathcal{E}(h, f^2) = \int_{\mathcal{X}} h\gamma \, d\mu.$$

  In this case, there exists a unique continuous bilinear form $\Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(\mathcal{X}, \mu)$, such that for all μ-a.e. bounded f, g, $h \in D(\mathcal{E})$

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = \int_{\mathcal{X}} h\Gamma(f, g) \, d\mu. \tag{1.1}$$

For $\Gamma(f,f)$ we use the shorthand $\Gamma(f)$. In particular, if for all $f \in D(\Gamma)$ it is the case that $f^2 \in D(L)$, then our notion of the carré du champ operator boils down to the classical case where

$$\Gamma(f,g) = L(fg) - fLg - gLf$$

for all f, $g \in D(\Gamma)$ (compare [BH], Chapter I.4).

- *Quasi-regularity*: We define the capacity $\mathrm{Cap}_1(\cdot)$ for $U \subseteq \mathcal{X}$, U open, as

$$\mathrm{Cap}_1(U) := \inf\{\|w\|_{\mathcal{E}} \,:\, w \in D(\mathcal{E}),\, w \geq 1 \ \mu\text{-a.e. on } U\},$$

and for general $A \subseteq \mathcal{X}$

$$\mathrm{Cap}_1(A) := \inf\{\mathrm{Cap}_1(U) \,:\, A \subseteq U \subseteq \mathcal{X},\, U \text{ open}\}.$$

Note that Cap_1 is a Choquet capacity and $\mu(A) \leq \mathrm{Cap}_1(A)$ for all $A \in \mathcal{X}$. We say that a property holds $\mathcal{E}$-quasi-everywhere, if it holds on $\mathcal{X} \setminus N$ for some exceptional set N of capacity zero. A function $f \in D(\mathcal{E})$ is said to be $\mathcal{E}$-quasi-continuous, if for all $\varepsilon > 0$ there exists a set $G \subseteq \mathcal{X}$ with $\mathrm{Cap}_1(\mathcal{X} \setminus G) < \varepsilon$, such that the restriction of f to G is continuous. And a Dirichlet form is called quasi-regular, if it satisfies the following three conditions.

  i) There exists an increasing sequence $(E_k)_{k\in\mathbb{N}}$ of compact sets (called compact $\mathcal{E}$-nest), such that $\mathrm{Cap}_1(\mathcal{X} \setminus E_k)$ tends to zero for $k \to \infty$.
  ii) There exists an $\|\cdot\|_{\mathcal{E}}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous μ-versions.
  iii) There exists a countable family $\mathcal{U} = (u_n)$, $u_n \in D(\mathcal{E})$, and a set a set N of zero capacity, such that the family $(\tilde{u}_n)$ of $\mathcal{E}$-quasi-continuous μ-versions of the elements of $\mathcal{U}$ separates points on $\mathcal{X} \setminus N$.

In addition, we will impose a further assumption:

Assumption (BC). *The family $\mathcal{U}$ can be chosen such that its elements have μ-a.e. bounded carré du champ.*

- *Conservativeness*: We say that a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is conservative, if $1 \in D(\mathcal{E})$ with $\mathcal{E}(1) = 0$.

- *Local property*: We say that a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ satisfies the local property, if for f, $g \in D(\mathcal{E})$ with $(f + a)g = 0$ for all $a \in \mathbb{R}$, it follows that $\mathcal{E}(f,g) = 0$. Note that this is the easiest way to define locality in our setting, for a more general discussion compare [BH], Chapter I.5.

We note that the class of quasi-regular Dirichlet forms encompasses in particular the regular Dirichlet forms on a locally compact space $\mathcal{X}$, i.e. Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathcal{X},\mu)$ where $C_0(\mathcal{X}) \cap D(\mathcal{E})$ is dense both in $D(\mathcal{E})$ with respect to $\|\cdot\|_{\mathcal{E}}$ and in $C_0(\mathcal{X})$ with respect to the supremum norm $\|\cdot\|_\infty$

(cf. [MR], Section IV.4.a)).
To every local, quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ there exists a diffusion process (X_t) on $\mathcal{X}$ with transition kernels (p_t), $p_t f(x) = E_x[f(X_t)]$ for every positive measurable function f, which is properly associated to $(\mathcal{E}, D(\mathcal{E}))$: For every bounded measurable function $f \in L^2(\mathcal{X}, \mu)$ and $t > 0$ it holds that $p_t f$ is a μ-version of $T_t f$ and $p_t f$ is $\mathcal{E}$-quasi-continuous, moreover $p_t f$ is μ-almost surely unique. Conversely, we can find to every diffusion process on $\mathcal{X}$ a properly associated Dirichlet form which is local and quasi-regular ([MR], Theorems IV.3.5, IV.5.1, IV.6.4 and V.1.11). The conservativeness of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is equivalent to the conservativeness of the associated diffusion process in the sense that it has infinite lifetime.
Defining the first hitting time of the measurable set $B \subseteq \mathcal{X}$ by

$$\tau_B := \inf\{t > 0 \,:\, X_t \in B\},$$

it holds for any open set U that

$$\mathrm{Cap}_1(U) = E_\mu[e^{-\tau_U}] = \int_{\mathcal{X}} E_x[e^{-\tau_U}]\mu(dx).$$

In particular every $\mathcal{E}$-exceptional set N (i.e. a set of zero capacity) is an (X_t)-exceptional set in the sense that $P_\mu[\tau_N < \infty] = 0$ and vice versa ([MR], Theorem IV.5.29).

Chapter 2

Geometry

Our first goal is to develop a concept of intrinsic geometry associated to a given Dirichlet form. For this the notion of a pointwise distance will be crucial. It will enable us to define length and energy of a path in the state space $\mathcal{X}$.

2.1 Metric

Hino and Ramírez prove in [HR] (see also [Ra], [H]) a general Varadhan large deviation principle for local, conservative Dirichlet forms,

$$\lim_{t\to 0} t \log P_t(A,B) = -\frac{d(A,B)^2}{2}, \qquad A, B \in \mathcal{B}.$$

Here the integrated heat kernel P_t is given by

$$P_t(A,B) := \int_A T_t \mathbb{1}_B \, d\mu$$

and the metric functional d as

$$d(A,B) := \sup_{u\in\mathcal{G}} \big(\operatorname*{ess\,inf}_{x\in B} u(x) - \operatorname*{ess\,sup}_{y\in A} u(y) \big) \tag{2.1}$$

with

$$\mathcal{G} := \big\{ u \in D(\mathcal{E}) \cap L^\infty(\mathcal{X},\mu) \,:\, 2\mathcal{E}(uh,u) - \mathcal{E}(h,u^2) \le \|h\|_{L^1(\mu)}, \text{ for all } h \in D(\mathcal{E}) \cap L^\infty(\mathcal{X},\mu) \big\}.$$

Note that there is a set distance $d_A(\cdot)$ associated to the metric functional d, that is given in the following way (cf. [HR], Theorem 1.2. and Section 2.3.): Define for a measurable set A

$$V_A^M := \{ u \in \mathcal{G} \,:\, u|_A = 0,\, 0 \le u \le M \ \mu\text{-a.e.} \}, \tag{2.2}$$

and further

$$d_A(x) := \lim_{M\to\infty} \operatorname*{ess\,sup}_{u\in V_A^M} u(x). \tag{2.3}$$

Then $d_A \wedge M \in D(\mathcal{E})$ and

$$d(A,B) = \operatorname*{ess\,inf}_{x\in B} d_A(x).$$

This definition of the metric functional is elegant and applies to a quite general setting, since it requires only, that the Dirichlet form is local and conservative. But in our situation we have a carré du champ operator, so it is convenient to show that the metric functional d coincides with a more intuitive one. We will show that we get the metric functional also if we maximize over the functions $u \in D(\mathcal{E})$ with carré du champ bounded by 1 instead of the functions $u \in \mathcal{G}$. Moreover, we need a pointwise concept of distance since we want to introduce the notions of length and energy of a path $\omega : [a,b] \to \mathcal{X}$.

Lemma 1. *For u, $v \in D(\mathcal{E})$ it holds for the carré du champ operator Γ that*

$$|\Gamma(u,|v|)| \le |\Gamma(u,v)| \quad \mu\text{-}a.e.$$

Proof: Following [BH], Corollary I.6.1.3., the locality of the Dirichlet form implies for every continuously differentiable and Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$

$$\Gamma(u,\varphi(v)) = \varphi'(v)\Gamma(u,v) \quad \mu\text{-a.e.}, \qquad u,v \in D(\mathcal{E}).$$

In particular we have for φ with $|\varphi'| \le 1$

$$|\Gamma(u,\varphi(v))| \le |\varphi'(v)\Gamma(u,v)| \le |\Gamma(u,v)| \quad \mu\text{-a.e.}$$

Hence we can find a sequence φ_n of continuously differentiable and Lipschitz continuous even functions, such that $\varphi_n(0) = 0$, $|\varphi_n'(x)| \le 1$ and $\varphi_n(x) \to |x|$ for $n \to \infty$. Thus $|\varphi_n(u) - \varphi_n(v)| \le \big||u| - |v|\big|$ which implies by [MR], Theorem I.4.12,

$$\mathcal{E}(\varphi_n(u), \varphi_n(v)) = \mathcal{E}(\varphi_n(|u|), \varphi_n(|v|)) \le \mathcal{E}(|u|,|v|).$$

Moreover,

$$\langle \varphi_n(u), \varphi_n(v)\rangle_{\mathcal{E}} \le \langle |u|, |v|\rangle_{\mathcal{E}}\,,$$

and we have

$$\||u| - \varphi_n(u)\|_{\mathcal{E}}^2 \le 2\big(\||u|\|_{\mathcal{E}}^2 - \langle |u|, \varphi_n(u)\rangle_{\mathcal{E}}\big). \tag{2.4}$$

But since [MR], Lemma I.2.12 implies

$$\langle w, \varphi_n(u)\rangle_{\mathcal{E}} \to \langle w, |u|\rangle_{\mathcal{E}}$$

for every $w \in D(\mathcal{E})$ and $n \to \infty$, we have $\varphi_n(u) \to |u|$ with respect to $\|\cdot\|_{\mathcal{E}}$ by (2.4) and hence

$$|\Gamma(u,|v|)| \le |\Gamma(u,v)| \quad \mu\text{-a.e.}$$

(compare [RSchm92], Lemma 3.2). ∎

Lemma 2. *The set*
$$\mathcal{G}' := \{u \in D(\mathcal{E}) \,:\, \Gamma(u) \le 1\,\mu\text{-a.e.}\}$$
is directed upwards, i.e. with u and v, also $u \vee v$ is in $\mathcal{G}'$. Moreover also $u \wedge v$ is in $\mathcal{G}'$.

Proof: On the one hand the Dirichlet space is closed under extrema ([FOT], Theorem 1.4.2), so $u \vee v \in D(\mathcal{E})$. On the other hand, following Röckner and Schmuland ([RSchm92], Lemma 3.2), we get by Lemma 1
$$\begin{aligned}
\Gamma(u) \vee \Gamma(v) &= \frac{1}{2}\Big(\Gamma(u) + \Gamma(v) + |\Gamma(u) - \Gamma(v)|\Big)\\
&= \frac{1}{4}\Big(\Gamma(u+v) + 2|\Gamma(u) - \Gamma(v)| + \Gamma(u-v)\Big)\\
&= \frac{1}{4}\Big(\Gamma(u+v) + 2|\Gamma(u+v, u-v)| + \Gamma(u-v)\Big)\\
&\ge \frac{1}{4}\Big(\Gamma(u+v) + 2|\Gamma(u+v, |u-v|)| + \Gamma(|u-v|)\Big)\\
&\ge \frac{1}{4}\Big(\Gamma(u+v) + 2\Gamma\big(u+v, |u-v|\big) + \Gamma\big(|u-v|\big)\Big)\\
&= \frac{1}{4}\Gamma\big(u+v+|u-v|\big) = \Gamma(u \vee v),
\end{aligned}$$
and hence also $|\Gamma(u \vee v)| \le 1$ for u, $v \in \mathcal{G}'$. The proof for the minimum is analogous, $\Gamma(u) \vee \Gamma(v) \ge \Gamma(u \wedge v)$ and $|\Gamma(u \wedge v)| \le 1$ for u, $v \in \mathcal{G}'$. ■

Lemma 3. *The metric functional d coincides with the metric functional d' given by*
$$d'(A, B) := \sup_{u \in \mathcal{G}'}\big(\operatorname*{ess\,inf}_{x \in B} u(x) - \operatorname*{ess\,sup}_{y \in A} u(y)\big).$$

Proof: Note that it is sufficient to prove
$$\mathcal{G} = \{u \in D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu) \,:\, \Gamma(u) \le 1\,\mu\text{-a.e.}\}, \tag{2.5}$$
since we can approximate all elements of $\mathcal{G}'$ by bounded ones. Indeed, for $u \in \mathcal{G}'$ we can define its cut-off $u_k \in \mathcal{G}' \cap L^\infty(\mathcal{X}, \mu)$ via
$$u_k := (-k) \vee u \wedge k, \qquad k \in \mathbb{N}.$$
Note that $\Gamma(u) \le 1$ μ-almost everywhere implies by Lemma 2 $\Gamma(u_k) \le 1$ μ-almost everywhere. Proposition I.4.17 of [MR] entails that $u_k \to u$ with respect to $\|\cdot\|_{\mathcal{E}}$ for $k \to \infty$.
To prove (2.5), we suppose first that u is a function in the right-hand set. Then its carré du champ is bounded by 1 μ-almost everywhere and so it follows by (1.1) that
$$2\mathcal{E}(uh, u) - \mathcal{E}(h, u^2) = \int_{\mathcal{X}} h\Gamma(u)\, d\mu \le \|h\|_{L^1(\mu)}.$$

Conversely, if for $u \in \mathcal{G}$ there would exist a set of positive measure where $\Gamma(u) > 1$, we could find $\varepsilon > 0$ such that $\mu(E) > 0$ for the set $E := \{x \in \mathcal{X} : \Gamma(u)(x) \geq 1 + \varepsilon\}$. But since the Dirichlet space $D(\mathcal{E})$ is dense in $L^2(\mathcal{X}, \mu)$, it is also dense in $L^1(\mathcal{X}, \mu)$ and we can find for every $\delta \in]0, 1[$ some $h' \in D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu)$ such that $\|h' - \mathbb{1}_E\|_{L^1(\mu)} < \delta$. Without loss of generality we can assume $h' \geq 0$. Indeed, otherwise we can take $h' \vee 0$ instead. Moreover, $h'' := h' \vee \delta \in D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu)$ satisfies

$$\|h'' - \mathbb{1}_E\|_{L^1(\mu)} \leq \int_E |h' \vee \delta - \mathbb{1}_E| \, d\mu + \int_{E^c} |h' + \delta| \, d\mu < \delta + 2\delta = 3\delta.$$

This implies

$$\mu(E) - 3\delta < \|h''\|_{L^1(\mu)} < \mu(E) + 3\delta,$$

since by the triangle inequality, we get on the one hand

$$\mu(E) - 3\delta = \|\mathbb{1}_E\|_{L^1(\mu)} - 3\delta \leq \|\mathbb{1}_E - h''\|_{L^1(\mu)} + \|h''\|_{L^1(\mu)} - 3\delta < \|h''\|_{L^1(\mu)}$$

and on the other hand

$$\|h''\|_{L^1(\mu)} \leq \|\mathbb{1}_E - h''\|_{L^1(\mu)} + \|\mathbb{1}_E\|_{L^1(\mu)} < \mu(E) + 3\delta.$$

Thus we get

$$2\mathcal{E}(uh'', u) - \mathcal{E}(h'', u^2) = \int_{\mathcal{X}} h'' \Gamma(u) \, d\mu \geq (1 + \varepsilon) \int_E h'' \, d\mu > (1 + \varepsilon)\big(\mu(E) - 3\delta\big).$$

Choosing now δ such that

$$0 < \delta \leq \frac{\varepsilon \mu(E)}{3(2 + \varepsilon)},$$

we get by

$$2\mathcal{E}(uh'', u) - \mathcal{E}(h'', u^2) > \mu(E) + 3\delta > \|h''\|_{L^1(\mu)}$$

a contradiction to the assumption $u \in \mathcal{G}$. ■

In the next step, we show that we can write the essential extrema in 2.1 in terms of a single, two-dimensional essential infimum. Therefore we recall that the essential extrema are defined as

$$\operatorname*{ess\,sup}_{x \in \mathcal{X}}{}^{\mu} u(x) := \inf\{c : c \geq u(x) \ \mu\text{-a.e.}\},$$
$$\operatorname*{ess\,inf}_{x \in \mathcal{X}}{}^{\mu} u(x) := \sup\{c : c \leq u(x) \ \mu\text{-a.e.}\}.$$

Lemma 4. *For u, $v \in L^0(\mathcal{X}, \mu)$,*

$$\operatorname*{ess\,inf}_{x \in \mathcal{X}}{}^{\mu} u(x) - \operatorname*{ess\,sup}_{y \in \mathcal{X}}{}^{\mu} v(y) = \operatorname*{ess\,inf}_{(x,y) \in \mathcal{X}^2}{}^{\mu \otimes \mu} \big(u(x) - v(y)\big).$$

Proof: Note that $v(y) \leq \operatorname{ess\,sup}^{\mu}_{z \in \mathcal{X}} v(z)$ μ-almost everywhere and hence

$$\begin{aligned} \operatorname*{ess\,inf}_{(x,y) \in \mathcal{X}^2}{}^{\mu \otimes \mu} \big(u(x) - v(y)\big) &\geq \operatorname*{ess\,inf}_{(x,y) \in \mathcal{X}^2}{}^{\mu \otimes \mu} \big(u(x) - \operatorname{ess\,sup}^{\mu}_{z \in \mathcal{X}} v(z)\big) \\ &= \operatorname*{ess\,inf}_{x \in \mathcal{X}}{}^{\mu} u(x) - \operatorname{ess\,sup}^{\mu}_{y \in \mathcal{X}} v(y). \end{aligned}$$

To prove the other direction, define for $\varepsilon > 0$ the set

$$C := \big\{(x,y) \,:\, u(x) \leq \operatorname*{ess\,inf}_{x\in\mathcal{X}}{}^{\mu} u(x) + \frac{\varepsilon}{2};\, v(y) \geq \operatorname*{ess\,sup}_{y\in\mathcal{X}}{}^{\mu} v(y) - \frac{\varepsilon}{2}\big\}$$

and note that for ε small enough $\mu(C) > 0$ by the definition of the essential extrema. It follows that

$$\begin{aligned}\big\{c : u(x) - v(y) \geq c \;\mu\otimes\mu\text{-a.e.}\big\} &\subseteq \big\{c : \mu\otimes\mu(\{u(x) - v(y) < c\} \cap C) = 0\big\} \\ &\subseteq \big\{c : \operatorname*{ess\,inf}_{x\in\mathcal{X}}{}^{\mu} u(x) + \frac{\varepsilon}{2} - \big(\operatorname*{ess\,sup}_{y\in\mathcal{X}}{}^{\mu} v(y) - \frac{\varepsilon}{2}\big) \geq c \;\mu\otimes\mu\text{-a.e.}\big\}.\end{aligned}$$

Since the function in the right hand set is constant, taking the supremum on both sides and sending ε to zero yields the result. ■

Since we can restrict ourselves to functions $u \geq 0$, we choose the functions $u\mathbb{1}_A$, $u\mathbb{1}_B \in L^0(\mathcal{X},\mu)$ to get

$$\begin{aligned} d'(A,B) &= \sup_{u\in\mathcal{G}'}\big(\operatorname*{ess\,inf}_{x\in B}{}^{\mu} u(x) - \operatorname*{ess\,sup}_{y\in A}{}^{\mu} u(y)\big) \\ &= \sup_{u\in\mathcal{G}'}\big(\operatorname*{ess\,inf}_{x\in\mathcal{X}}{}^{\mu} (u\mathbb{1}_B)(x) - \operatorname*{ess\,sup}_{y\in\mathcal{X}}{}^{\mu} (u\mathbb{1}_A)(y)\big) \\ &= \sup_{u\in\mathcal{G}'}\operatorname*{ess\,inf}_{(x,y)\in\mathcal{X}^2}{}^{\mu\otimes\mu}\big((u\mathbb{1}_B)(x) - (u\mathbb{1}_A)(y)\big) \\ &= \sup_{u\in\mathcal{G}'}\operatorname*{ess\,inf}_{(x,y)\in B\times A}{}^{\mu\otimes\mu}\big(u(x) - u(y)\big). \end{aligned}$$

In the following we drop the superscripts μ and $\mu\otimes\mu$ if it is clear to which measure the essential extremum refers. Now we want to go further and construct the pointwise intrinsic distance ρ. Here, the quasi-regularity plays an essential role since it implies that every function $u \in D(\mathcal{E})$ has an $\mathcal{E}$-quasi-continuous μ-version $\tilde{u}$ ([MR], Proposition IV.3.3), i.e., a μ-version which is continuous on every set of an increasing sequence (E_k^u) of closed sets E_k^u with $\mathrm{Cap}_1(\mathcal{X}\setminus E_k^u) \to 0$. However, note that we have by definition of quasi-regularity even an increasing sequence of compact sets E_k with this property. So we can for every countable family of $\mathcal{E}$-quasi-continuous functions assume that they have μ-versions continuous on every of these compact sets (since otherwise we could create a sequence of smaller compact sets with vanishing capacity - cf. [MR], Proposition III.3.3.). In particular the point-separating family $(\tilde{u}_n)$ is continuous on every E_k. From now on we fix a compact $\mathcal{E}$-nest (E_k) and let $N = \bigcap_{k=1}^{\infty} E_k^c$.

Definition 1. *The pointwise distance ρ on $\mathcal{X}$ is given by*

$$\rho(x,y) := \begin{cases} \sup_{u\in\mathcal{G}'}\big(\tilde{u}(x) - \tilde{u}(y)\big) & x,y \in \mathcal{X}\setminus N; \\ \infty & \textit{otherwise.}\end{cases}$$

The distance of some point x from a set A is hence given by

$$\rho_A(x) := \inf_{y\in A}\rho(x,y).$$

Next we will show that ρ defines an extended pseudometric, i.e. a pseudometric which admits also ∞ as value.

Proposition 1. *$\rho(\cdot,\cdot)$ defines an extended pseudometric on $\mathcal{X}$ and it holds that $d_A \leq \rho_A$ for any set $A \in \mathcal{B}$*

Proof: Symmetry and triangle inequality are obvious. The dominance over d_A follows from

$$\begin{aligned}\rho_A(x) &= \inf_{y\in A}\sup_{u\in\mathcal{G}'}\big(\tilde{u}(x)-\tilde{u}(y)\big) \geq \inf_{y\in A}\lim_{M\to\infty}\operatorname*{ess\,sup}_{u\in V_A^M}\big(\tilde{u}(x)-\tilde{u}(y)\big)\\ &\geq \inf_{y\in A}\lim_{M\to\infty}\operatorname*{ess\,sup}_{u\in V_A^M} u(x) = \lim_{M\to\infty}\operatorname*{ess\,sup}_{u\in V_A^M} u(x) = d_A(x)\end{aligned}$$

for every $x \in \mathcal{X}$ and $A \subseteq \mathcal{X}\setminus N$. If $A \nsubseteq \mathcal{X}\setminus N$, it is enough to take the supremum over $A\setminus N$, since for $y \in N$, $\rho(x,y)$ is trivially infinite for every $x \in \mathcal{X}$. ■

Remark 1. *Note that the mapping $(x,y)\mapsto \tilde{u}(x)-\tilde{u}(y)$ is continuous on $N^c \times N^c$, so $\rho(\cdot,\cdot)$ is as supremum lower semi-continuous on $N^c\times N^c$. However, it is not clear that it is also lower semi-continuous on $\mathcal{X}\times\mathcal{X}$, e.g. if $\rho(\cdot,\cdot)$ is bounded on $N^c\times N^c$, but N is closed with respect to the original topology.*

Length L and the *energy* E of a path $\omega \in \Omega := C([a,b];\mathcal{X})$ with respect to ρ are given by

$$\begin{aligned}\mathsf{L}_{a,b}(\omega) &:= \sup_{\Delta}\sum_{t_{i-1},t_i\in\Delta}\rho(\omega_{t_{i-1}},\omega_{t_i});\\ \mathsf{E}_{a,b}(\omega) &:= \sup_{\Delta}\sum_{t_{i-1},t_i\in\Delta}\frac{\rho^2(\omega_{t_{i-1}},\omega_{t_i})}{2(t_i-t_{i-1})},\end{aligned}$$

where the supremum is taken over all partitions $\Delta = \{t_0,\dots,t_n\}$, $a = t_0 < t_1 < \cdots < t_n = b$, $n \in \mathbb{N}$.

2.2 Energy

Lemma 5. *The energy functional is additive, i.e. for any c, d, $e \in [a,b]$, $c<d<e$, it holds for any path $\omega\in\Omega$ that*

$$\mathsf{E}_{c,d}(\omega)+\mathsf{E}_{d,e}(\omega)=\mathsf{E}_{c,e}(\omega)$$

Proof: The set of all partitions of $[c,e]$ is of course larger then the set of all partitions which contain the point d, so “$\leq$“ holds. To prove the other direction, we claim that adding an additional partitioning point, the energy can only increase. It remains to show that

$$\frac{\rho^2(\omega_{t_1},\omega_{t_2})}{(t_2-t_1)}+\frac{\rho^2(\omega_{t_2},\omega_{t_3})}{(t_3-t_2)} \geq \frac{\rho^2(\omega_{t_1},\omega_{t_3})}{(t_3-t_1)}.$$

Choosing $\lambda\in(0,1)$ such that $t_2 = \lambda t_1 + (1-\lambda)t_3$, we get that the above inequality is equivalent to

$$\frac{\rho^2(\omega_{t_1},\omega_{t_2})}{1-\lambda}+\frac{\rho^2(\omega_{t_2},\omega_{t_3})}{\lambda} \geq \rho^2(\omega_{t_1},\omega_{t_3})$$

which is clearly true by the triangle inequality and since

$$\left(\sqrt{\frac{\lambda}{1-\lambda}}\rho(\omega_{t_1},\omega_{t_2}) - \sqrt{\frac{1-\lambda}{\lambda}}\rho(\omega_{t_2},\omega_{t_3})\right)^2 \geq 0.$$

■

This result can be used to give an alternative characterization of the energy of a path ω. Therefore we define for a path $\omega : [a,b] \to \mathcal{X}$ and $u \in D(\mathcal{E})$ the functional

$$J_{a,b}(u,\omega) := \tilde{u}(\omega_b) - \tilde{u}(\omega_a) - \frac{1}{2}\int_a^b \Gamma(u)(\omega_r)\,dr$$

and

$$\mathcal{J}_\Delta^{\vec{u}}(\omega) := \sum_{i=1}^n J_{t_{i-1},t_i}(u_i,\omega)$$

for fixed $n \in \mathbb{N}$, $\vec{u} = (u_1,\dots u_n) \in D(\mathcal{E})^n$ and a partition $\Delta = \{t_0,\dots,t_n\}$, $a = t_0 < t_1 < \dots < t_n = b$, $n \in \mathbb{N}$.

Theorem 1. *Given a path $\omega : [a,b] \to \mathcal{X}$,*

$$\mathsf{E}_{a,b}(\omega) = \sup_{\Delta;\vec{u}} \mathcal{J}_\Delta^{\vec{u}}(\omega). \tag{2.6}$$

Proof: We can assume that ω stays in E_k for some set E_k of our compact $\mathcal{E}$-nest (since otherwise both sides of the equality would trivially be infinite). Note that the set $\mathcal{H} := \{u \in D(\mathcal{E}) : \Gamma(u) \text{ is bounded}\}$ is point-separating by Assumption (BC) and it is a sub-algebra of $D(\mathcal{E})$ by

$$\Gamma(u+v) = \Gamma(u) + \Gamma(v) + 2\Gamma(u,v) \leq 2\big(\Gamma(u) + \Gamma(v)\big). \tag{2.7}$$

So it is a point-separating sub-algebra which contains the constants, whence it is dense in $C(E_k)$ with respect to the uniform topology by the Stone-Weierstraß theorem. But since every $u \in D(\mathcal{E})$ has a μ-version which is continuous on E_k (and in fact we take the supremum only over this modifications), it is enough to consider the supremum over $\vec{u} \in \mathcal{H}^n$.
So for any path $\omega : [a,b] \to E_k$ and c, $d \in [a,b]$, $c < d$, we get on the one hand by maximizing over λ

$$\begin{aligned}
\sup_{u\in\mathcal{H}} J_{c,d}(u,\omega) &= \sup_{u\in\mathcal{G}',\lambda\in\mathbb{R}} \Big(\lambda\tilde{u}(\omega_d) - \lambda\tilde{u}(\omega_c) - \frac{1}{2}\int_c^d \Gamma(\lambda u)(\omega_t)\,dt\Big) \\
&= \sup_{u\in\mathcal{G}'} \frac{(\tilde{u}(\omega_d)-\tilde{u}(\omega_c))^2}{2\int_c^d \Gamma(u)(\omega_t)\,dt} \qquad (2.8)\\
&\geq \sup_{u\in\mathcal{G}'} \frac{(\tilde{u}(\omega_d)-\tilde{u}(\omega_c))^2}{2(d-c)} = \frac{\rho^2(\omega_c,\omega_d)}{2(d-c)}. \qquad (2.9)
\end{aligned}$$

Summing up and taking the supremum over all partitions yields "≤" in (2.6).
On the other hand we can choose for some $u \in \mathcal{H}$, $\gamma_i := \sup_{s_{i-1}\leq r\leq s_i}\Gamma(u)(\omega_r) < \infty$ and every given $\varepsilon > 0$ a partition $a \leq c = s_0 < \dots < s_m = d \leq b$, such that

$$\sum_{i=1}^m (s_i - s_{i-1})\gamma_i \leq \varepsilon + \int_c^d \Gamma(u)(\omega_t)\,dt. \tag{2.10}$$

With Hölder's inequality we get by (2.10) and (2.9)

$$\begin{aligned}\tilde{u}(\omega_d) - \tilde{u}(\omega_c) &= \sum_{i=1}^{n} \big(\tilde{u}(\omega_{s_i}) - \tilde{u}(\omega_{s_{i-1}})\big) = \sum_{i=1}^{n} \frac{\tilde{u}(\omega_{s_i}) - \tilde{u}(\omega_{s_{i-1}})}{\sqrt{(s_i - s_{i-1})\gamma_i}} \sqrt{(s_i - s_{i-1})\gamma_i} \\ &\leq \left(\sum_{i=1}^{n} \frac{(\tilde{u}(\omega_{s_i}) - \tilde{u}(\omega_{s_{i-1}}))^2}{(s_i - s_{i-1})\gamma_i} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} (s_i - s_{i-1})\gamma_i \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{n} \frac{\left(\frac{\tilde{u}}{\sqrt{\gamma_i}}(\omega_{s_i}) - \frac{\tilde{u}}{\sqrt{\gamma_i}}(\omega_{s_{i-1}}) \right)^2}{s_i - s_{i-1}} \right)^{\frac{1}{2}} \left(\int_a^b \Gamma(u)(\omega_t)\, dt + \varepsilon \right)^{\frac{1}{2}} \\ &\leq \left(2\, \mathsf{E}_{c,d}(\omega)\right)^{\frac{1}{2}} \left(\int_c^d \Gamma(u)(\omega_t)\, dt + \varepsilon \right)^{\frac{1}{2}}\end{aligned}$$

since $\Gamma(u/\sqrt{\gamma_i})(\omega_{s_i}) = \Gamma(u)(\omega_{s_i})/\gamma_i \leq 1$. Taking the infimum over all $\varepsilon > 0$ and, after rearranging the inequality, the supremum over $u \in \mathcal{G}'$, we get by (2.8)

$$\mathsf{E}_{c,d}(\omega) \geq \sup_{u \in \mathcal{H}} J_{c,d}(u, \omega).$$

The additivity of the energy (Lemma 5) yields the result. ■

2.3 More on the Distances ρ and d

To prove the small-time large deviation principle, we need more than the dominance of ρ_A over d_A (as proven in Proposition 1). Actually we need (besides lower semicontinuity) that the distances ρ_{E_k} from the sets of the compact $\mathcal{E}$-nest are in $\mathcal{G}'$. We can prove this only for locally compact state spaces. Therefore we will need an additional assumption for the infinite dimensional setting. But it is not clear whether it holds in full generality.

2.3.1 I. The Locally Compact Case

We assume that the state space $\mathcal{X}$ is locally compact and the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is regular, i.e. $D(\mathcal{E}) \cap C_0(\mathcal{X})$ is dense in $D(\mathcal{E})$ with respect to $\|\cdot\|_{\mathcal{E}}$ and in $C_0(\mathcal{X})$ with respect to $\|\cdot\|_\infty$. The geometry of this setting was studied by a series of papers by Sturm ([S94], [S95a], [S95b] and [S96]; actually his setting is more general, since he considers local Dirichlet forms on Hausdorff spaces) under the following additional assumption:

Assumption (A). *The topology induced by ρ is equivalent to the original topology on $\mathcal{X}$.*

This implies in particular that ρ is not degenerate and that the open ρ-balls $B_r^\rho(x) := \{y \in \mathcal{X} : \rho(x, y) < r\}$ are relatively compact - at least if $\mathcal{X}$ is connected, since $\mathcal{X}$ is as Polish space complete

([S95b], Theorem 2). Moreover, the pointwise distance ρ_x is for every $x \in \mathcal{X}$ continuous and lies in the Dirichlet space $D(\mathcal{E})$ ([S94], Lemma 1'). But even more, also the setwise distances ρ_A satisfy these properties for every measurable subset $A \subseteq \mathcal{X}$ and $\Gamma(\rho_A) \leq 1$ μ-almost everywhere ([S95a], Lemma 1.9 and the following remark). Moreover, he proves, that for compact sets A, $B \subseteq \mathcal{X}$ it holds even that $\rho(A, B) = d(A, B)$ ([S95a], Lemma 1.10).
But in fact, using the results by Hino and Ramírez [HR], we can prove even more:

Proposition 2. *Let $A \subseteq \mathcal{X}$ a subset with $\mu(A) > 0$. Then $\rho_A = d_A$ μ-almost everywhere.*

We will give the proof below, together with that of the infinite-dimensional version.

2.3.2 II. The Infinite-Dimensional Case

For the prove of the upper bound of the large deviation principle the following assumption will be sufficient.

Assumption (B). *For all sets E_k of the compact $\mathcal{E}$-nest it holds that $\rho_{E_k} \in \mathcal{G}'$.*

To have a result on the equality we will need a little bit more, actually

Assumption (B*). *For all sets A, closed and of positive measure, it holds that $\rho_A \in \mathcal{G}'$.*

We note that all closed balls (with respect to the original topology) around points in the support of the measure are of positive measure. Moreover, we can easily see that this implies also Assumption (B) since we can choose the compact $\mathcal{E}$-nest in such a way, that all E_k are of positive measure: Since $\mathrm{Cap}_1(E_k^c) \to 0$ for $k \to \infty$ there exists a k_0 such that $\mu(E_k) > 0$ for all $k \geq k_0$. So starting the sequence only at k_0 yields such an $\mathcal{E}$-nest.

Proposition 3. *Let $A \subseteq \mathcal{X}$ a closed set of positive measure. Then $\rho_A = d_A$ μ-almost everywhere.*

Proof: By Hino and Ramírez (cf. [HR], Theorem 1.2), the set distance d_A defined in (2.3) is μ-almost everywhere 0 on A, the truncated functional $d_A \wedge M$ lies in $\mathcal{G}$ (and so in $D(\mathcal{E})$) for every $M \geq 0$ and it is the μ-almost everywhere largest function which satisfies this two conditions. But for the functional ρ_A it holds obviously that $\rho_A = 0$ on A. But since (by the consequences of Assumption (A) resp. Assumption (B*)) $\rho_A \in D(\mathcal{E})$ with $\Gamma(\rho_A) \leq 1$ μ-almost everywhere, it follows that for every $M \geq 0$ also $\rho_A \wedge M \in D(\mathcal{E})$ with

$$\Gamma(\rho_A \wedge M) \leq \Gamma(\rho_A) \leq 1 \quad \mu\text{-a.e.}$$

by Lemma 2. Hence $d_A \geq \rho_A$ μ-almost everywhere and equality follows by Proposition 1. ∎
A partial weakening of Assumption (B) will be discussed in Section 5.

Chapter 3

The Associated Markov Process

In this section we want to look on the probabilistic counterpart of our Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Since it is quasi-regular, there exists an associated Markov process (X_t) (cf. [MR], Section IV.3) which is by locality in fact a diffusion ([MR], Section V.1). We are interested in small time asymptotics, so we define for $\varepsilon > 0$ the process X^ε by $X_t^\varepsilon := X_{\varepsilon t}$. Further we define for $n \in \mathbb{N}$, $\vec{u} \in D(L)^n$ and a partition $\Delta = \{t_0, \ldots, t_n\}$ the process M_t^ε by

$$M_t^{\varepsilon,i} := \Big(\tilde{u}_i(X_{t\wedge t_i}^\varepsilon) - \tilde{u}_i(X_{t_{i-1}}^\varepsilon) - \varepsilon \int_{t_{i-1}}^{t\wedge t_i} Lu_i(X_s^\varepsilon)\, ds\Big) \mathbb{1}_{\{t_{i-1} \leq t\}}$$

$$M_t^\varepsilon := \sum_{i=1}^n M_t^{\varepsilon,i}.$$

We denote the exit times when X_t^ε leaves E_k by

$$\tau_k := \inf\{t \,:\, X_t^\varepsilon \notin E_k\}.$$

We know that the $(M_t^\varepsilon)^{\tau_k}$ are real-valued, continuous $L^2(P_\mu)$-martingales and since (E_k) is a compact $\mathcal{E}$-nest, $\tau_k \to \infty$ P_μ-almost surely for $k \to \infty$ (cf. [MR], Theorem IV.5.29). So M_t^ε is a continuous local martingale with quadratic variation

$$\langle M^\varepsilon \rangle_t = \sum_{i=1}^n \varepsilon \mathbb{1}_{\{t_{i-1} \leq t\}} \int_{t_{i-1}}^{t\wedge t_i} \Gamma(u_i)(X_s^\varepsilon)\, ds. \tag{3.1}$$

We shift our attention to the Doléans-Dade exponential $\exp\big(M_t^\varepsilon - \frac{1}{2}\langle M^\varepsilon \rangle_t\big)$; it is well known that a sufficient condition that this continuous local martingale is a true martingale, is given by Novikov's criterion

$$E_\mu\left[\exp\big(\frac{1}{2}\langle M^\varepsilon \rangle_{t_n}\big)\right] < \infty.$$

This criterion holds true by (3.1) for those $\vec{u}$ where the $\Gamma(u_i)$ are μ-a.e. bounded - moreover, then even

$$E_\mu\left[\exp\big(\lambda \langle M^\varepsilon \rangle_{t_n}\big)\right] < \infty$$

holds for every $\lambda \in \mathbb{R}$.
In the following we will use the forward-backward-martingale decomposition of Lyons and Zheng [LZhe]: Given a diffusion X_t, then for every continuous $f \in D(L)$ there exist a continuous forward martingale

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds, \qquad t \in [0,T],$$

and a continuous backward martingale

$$\hat{M}_t^f := f(X_{T-t}) - f(X_T) - \int_{T-t}^T Lf(X_s)\,ds, \qquad t \in [0,T],$$

both square integrable and starting in 0, such that

$$f(X_t) - f(X_0) = \frac{1}{2}(M_t^f + \hat{M}_{T-t}^f - \hat{M}_T^f).$$

This decomposition extends to $f \in D(\mathcal{E})$; for details we refer to Appendix A.2. To apply this to our situation, we introduce for $u_i \in D(L)$ and $T = t_i$ the backward martingale associated to $M_t^{\varepsilon,i}$:

$$\hat{M}_t^{\varepsilon,i} = \Big(\tilde{u}_i(X^\varepsilon_{(t_i-t+t_{i-1})\wedge t_i}) - \tilde{u}_i(X^\varepsilon_{t_i}) - \varepsilon\int_{(t_i-t+t_{i-1})\wedge t_i}^{t_i} Lu_i(X_s^\varepsilon)\,ds\Big)\mathbb{1}_{\{t_{i-1}\le t\le t_i\}}.$$

The Lyons-Zheng decomposition is then given by

$$\Big(\tilde{u}_i(X^\varepsilon_{t\wedge t_i}) - \tilde{u}_i(X^\varepsilon_{t_{i-1}})\Big)\,\mathbb{1}_{\{t_{i-1}\le t\}} = \frac{1}{2}\left(M_t^{\varepsilon,i} + \hat{M}^{\varepsilon,i}_{t_i-t+t_{i-1}} - \hat{M}^{\varepsilon,i}_{t_i}\right)\mathbb{1}_{\{t_{i-1}\le t\le t_i\}},$$

and we get by summing up

$$\sum_{i=1}^n \Big(\tilde{u}_i(X^\varepsilon_{t_i}) - \tilde{u}_i(X^\varepsilon_{t_{i-1}})\Big) = \sum_{i=1}^n \frac{1}{2}\left(M^{\varepsilon,i}_{t_i} + \hat{M}^{\varepsilon,i}_{t_{i-1}} - \hat{M}^{\varepsilon,i}_{t_i}\right) = \frac{1}{2}\Big(M^\varepsilon_{t_n} - \hat{M}^\varepsilon_{t_n}\Big)$$

where we write $\hat{M}_t^\varepsilon$ for $\sum_{i=1}^n \hat{M}_t^{\varepsilon,i}$. As shown in the Appendix A.2 this generalizes to $\vec{u} \in D(\mathcal{E})^n$. We denote by $\langle\hat{M}^\varepsilon\rangle_t$ the quadratic variation of the backward martingale $\hat{M}^\varepsilon$ at time t and note that $\big\langle\hat{M}^\varepsilon\big\rangle_{t_n} = \big\langle M^\varepsilon\big\rangle_{t_n}$.

Lemma 6. *For every $\varepsilon > 0$ and $\vec{u} \in D(\mathcal{E})^n$ with $\Gamma(u_i)$ μ-a.e. bounded it holds that*

$$E_\mu\left[\exp\left(\frac{\mathcal{J}^{\vec{u}}_\Delta(X^\varepsilon)}{\varepsilon}\right)\right] \le 1. \tag{3.2}$$

Proof: By

$$\mathcal{J}^{\vec{u}}_\Delta(X^\varepsilon) = \frac{1}{2}\Big(M^\varepsilon_{t_n} - \frac{1}{2\varepsilon}\big\langle M^\varepsilon\big\rangle_{t_n}\Big) + \frac{1}{2}\Big(-\hat{M}^\varepsilon_{t_n} - \frac{1}{2\varepsilon}\big\langle\hat{M}^\varepsilon\big\rangle_{t_n}\Big)$$

it follows that

$$\exp\left(\frac{\mathcal{J}^{\vec{u}}_\Delta(X^\varepsilon)}{\varepsilon}\right) = \exp\Big(\frac{1}{2}\big(\frac{1}{\varepsilon}M^\varepsilon_{t_n} - \frac{1}{2\varepsilon^2}\big\langle M^\varepsilon\big\rangle_{t_n}\big)\Big)\exp\Big(\frac{1}{2}\big(-\frac{1}{\varepsilon}\hat{M}^\varepsilon_{t_n} - \frac{1}{2\varepsilon^2}\big\langle\hat{M}^\varepsilon\big\rangle_{t_n}\big)\Big)$$

since the quadratic variation is of course non-negative. So the Cauchy-Schwarz inequality implies with Lemma 18

$$\begin{aligned} E_\mu\left[\exp\left(\frac{\mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon)}{\varepsilon}\right)\right] &\leq E_\mu\left[\exp\left(\frac{1}{\varepsilon}M^\varepsilon_{t_n} - \frac{1}{2\varepsilon^2}\langle M^\varepsilon\rangle_{t_n}\right)\right]^{\frac{1}{2}} E_\mu\left[\exp\left(-\frac{1}{\varepsilon}\hat{M}^\varepsilon_{t_n} - \frac{1}{2\varepsilon^2}\langle \hat{M}^\varepsilon\rangle_{t_n}\right)\right]^{\frac{1}{2}} \\ &\leq 1. \end{aligned}$$

■

Chapter 4

Large Deviations

4.1 Exponential Tightness

The aim of this section is to find a sufficient and tractable condition which implies the exponential tightness of (X^ε) with respect to the compact-open topology on the path space $\Omega := C([0,1];\mathcal{X})$. This means that for every $N > 0$ there exists a compact set $K_N \in \Omega$ with

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[X^\varepsilon_\cdot \notin K_N] \leq -N.$$

In this generality this will not be possible, unless there exists a compact set with μ-full measure. Instead we will show exponential tightness for families (X^ε) of diffusions starting out of the compact $\mathcal{E}$-nest. This means to show that for every set E_k of the compact $\mathcal{E}$-nest and every $N > 0$, there exists a compact set $K_N \in \Omega$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[X^\varepsilon_\cdot \notin K_N \,|\, X_0 \in E_k] \leq -N.$$

Subsequently, we will work under the following assumption (only noticing that in the case that $\mathcal{X}$ is compact it has not to be imposed).

Assumption (D). *We require that for every $k \in \mathbb{N}$ and $N > 0$ there exists some $l \in \mathbb{N}$, such that $\rho_{E_k} \geq N$ on the complement of one of the sets E_l in the compact $\mathcal{E}$-nest (E_k).*

We will first show that this Assumption is equivalent to the assumption of this property for pointwise distances and that each of them implies lower semi-continuity of $\rho(\cdot,\cdot)$ on $\mathcal{X}\times\mathcal{X}$.

Lemma 7. *Suppose $\rho(\cdot,\cdot)$ is lower semi-continuous on $\mathcal{X}\times\mathcal{X}$ with respect to the original topology, then for any compact set K the set distance $\rho_K(\cdot)$ is lower semi-continuous.*

Proof: Since K is compact, the infimum in the definition of the set distance is attained. Thus there exists to every $x \in \mathcal{X}$ a point $y_x \in K$ such that $\rho_K(x) = \rho(y_x, x)$. Fix now a point $x \in \mathcal{X}$ and take a sequence (x_n), $x_n \to x$ for $n \to \infty$. Since K is compact, there exists a subsequence (x_{n_k}) and some $y' \in K$ such that $y_{x_{n_k}} \to y'$ for $k \to \infty$. Now it follows by the lower semi-continuity of $\rho(\cdot,\cdot)$ that

$$\liminf_{k\to\infty} \rho_K(x_{n_k}) = \liminf_{k\to\infty} \rho(y_{x_{n_k}}, x_{n_k}) \geq \rho(y', x) \geq \rho_K(x),$$

hence $\rho_K(\cdot)$ is lower semi-continuous. ■

Proposition 4. *Assumption (D) is equivalent to the fact that for every $x \in \mathcal{X}$ and $N > 0$ there exists some $l \in \mathbb{N}$, such that $\rho_x \geq N$ on the complement of one of the sets E_l in the compact $\mathcal{E}$-nest (E_k). Moreover, it implies that $\rho(\cdot,\cdot)$ is lower semi-continuous on $\mathcal{X} \times \mathcal{X}$ with respect to the original topology.*

Proof: We show fist that Assumption (D) implies the growth condition for ρ_x: For $x \in N$ the assumption holds trivially, for $x \in N^c$ there exists a compact set E_k of the compact $\mathcal{E}$-nest such that $x \in E_k$. So it follows by Assumption (D) that there exists to every $N > 0$ some $l \in \mathbb{N}$ that

$$\rho_x \geq \rho_{E_k} \geq N \qquad \text{on } E_l^c.$$

Next we show that the condition on the groth of ρ_x implies the lower semi-continuity of ρ. This property is clear on $N^c \times N^c$ by Remark 1 and trivial if one of the points is in the interior of N. So it remains to prove the lower semi-continuity in the case that $x \in (N \cap \partial N)$ and $y \in \mathcal{X}$. We note that to every neighborhood U of x, also $U_k := U \cap E_k^c$ is a neighborhood. Thus we can extract of any sequence (x_n) with $x_n \to x$ a subsequence (x_{n_k}) with $x_{n_k} \in U_k \subseteq E_k^c$. So we have by the assumption on the growth of ρ_x for every $N > 0$ some $l \in \mathbb{N}$ that

$$\rho(x_{n_k}, y) = \rho_y(x_{n_k}) \geq N \qquad \text{for all } k \geq l,$$

whence $\liminf_{k\to\infty} \rho(x_{n_k}, y) = \infty$, proving the lower semi-continuity.
Finally the growth assumption on ρ_x together with the derived lower semi-continuity imply Assumption (D) by Lemma 7. ■

Lemma 8. *Let Assumption (D) hold, then we can find for every $N > 0$ and every E_k of the compact $\mathcal{E}$-nest a compact set $A_N \in \mathcal{X}$ with*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[\exists t \,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k] \leq -N.$$

Proof: Note first that the set $\{\exists t \,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k\}$ is measurable, since every set $\{\exists t \,:\, X_t \notin A\}$ is Borel-measurable given that A is open or closed (cf. [J], Proposition 1.6.(v) and [Sch95], Remark 3). We define now the stopping times

$$\tau_l := \inf\{t \,:\, X_t^\varepsilon \notin E_l\}$$

and the corresponding stopped processes $Y_t^{\varepsilon,l} := X_{t\wedge\tau_l}^\varepsilon$. Let n_l the smallest number such that $\rho_{E_k} \geq n_l$ on E_l^c for the sets $E_l \supseteq E_k$ of the compact $\mathcal{E}$-nest. The sequence (n_l) converges to infinity for $l \to \infty$

by Assumption (D). Note now that

$$\begin{aligned}
P_\mu[\exists t\,:\,X^\varepsilon_t\notin E_l, X_0\in E_k] &\le P_\mu[\exists t\,:\,\rho_{E_k}(X^\varepsilon_t)\ge n_l, X_0\in E_k]\\
&= P_\mu\Big[\max_{t\in[0,1]}\rho_{E_k}(X^\varepsilon_t)\ge n_l, X_0\in E_k\Big]\\
&= P_\mu\Big[\rho_{E_k}(Y^{\varepsilon,l}_1)\ge n_l, Y^{\varepsilon,l}_0\in E_k\Big]\\
&= P_\mu\Big[e^{\frac{1}{\varepsilon}\rho_{E_k}(Y^{\varepsilon,l}_1)}\ge e^{\frac{n_l}{\varepsilon}}, Y^{\varepsilon,l}_0\in E_k\Big]\\
&\le e^{-\frac{n_l}{\varepsilon}}E_\mu\Big[e^{\frac{1}{\varepsilon}\rho_{E_k}(Y^{\varepsilon,l}_1)}, Y^{\varepsilon,l}_0\in E_k\Big]\\
&\le e^{-\frac{n_l}{\varepsilon}}E_\mu\Big[e^{\frac{1}{2\varepsilon}\int_0^1\Gamma(\rho_{E_k})(Y^{\varepsilon,l}_r)\,dr}e^{\frac{1}{\varepsilon}J^{\rho_{E_k}}_{0,1}(Y^{\varepsilon,l}_t)}, Y^{\varepsilon,l}_0\in E_k\Big]\\
&\le e^{\frac{1-2n_l}{2\varepsilon}}E_\mu\Big[e^{\frac{1}{\varepsilon}J^{\rho_{E_k}}_{0,1}(Y^{\varepsilon,l}_t)}, Y^{\varepsilon,l}_0\in E_k\Big]\le e^{\frac{1-2n_l}{2\varepsilon}}
\end{aligned}$$

by Chebyshev's inequality, Lemma 7, since $\Gamma(\rho_{E_k})\le 1$ μ-almost everywhere by Proposition 2 / Assumption (B), and an application of Lemma 6. So for every $N>0$ we get a compact set A_N with

$$\limsup_{\varepsilon\to 0}\varepsilon\log P_\mu[\exists t\,:\,X^\varepsilon_t\notin A_N, X_0\in E_k]\le -N$$

directly as the appropriate E_l. ■

We denote by

$$|\tilde u(X^\varepsilon_\cdot)|_\alpha := \sup_{\substack{s,t\in[0,1]\\ s\ne t}}\frac{|\tilde u(X^\varepsilon_t)-\tilde u(X^\varepsilon_s)|}{|t-s|^\alpha}$$

the Hölder-norm of the real-valued process $u(X^\varepsilon_t)$ on $[0,1]$ with exponent α.

Lemma 9. *For every $u\in\mathcal{H}$ (i.e. $u\in D(\mathcal{E})$ with bounded carré du champ), $\alpha\in]0,1/2[$ and $N>0$, there exists an $R>0$, such that*

$$\limsup_{\varepsilon\to 0}\varepsilon\log P_\mu[|\tilde u(X^\varepsilon_\cdot)|_\alpha\ge R]\le -N.$$

Proof: Fix s, $t\in[0,1]$, $s\ne t$, and set $G(u):=\operatorname{ess\,sup}_{x\in\mathcal{X}}\Gamma(u)(x)$. We define $u':=\tilde u/\sqrt{|t-s|}$, then $\Gamma(u')\le G(u)/|t-s|$ and Lemma 6 implies for every $\varepsilon>0$

$$E_\mu\left[\exp\left(\frac{\tilde u(X^\varepsilon_t)-\tilde u(X^\varepsilon_s)}{\varepsilon\sqrt{|t-s|}}\right)\right]\le e^{\frac{G(u)}{2\varepsilon}}E_\mu\left[\exp\left(\frac{J^{u'}_{s,t}(X^\varepsilon)}{\varepsilon}\right)\right]\le e^{\frac{G(u)}{2\varepsilon}}.$$

By $\exp|x|\le\exp(x)+\exp(-x)$ it follows that

$$E_\mu\left[\exp\left(\frac{|\tilde u(X^\varepsilon_t)-\tilde u(X^\varepsilon_s)|}{\varepsilon\sqrt{|t-s|}}\right)\right]\le 2e^{\frac{G(u)}{2\varepsilon}}\le\kappa^{\frac{1}{\varepsilon}}$$

for $\varepsilon\in]0,1]$ and a certain constant κ. Now we can use Corollary 7.1. of [Sch97a] and get that for all $0<\alpha<1/2$ there exists a constant $C>0$ (depending only on α), such that

$$P_\mu\left[|u(X^\varepsilon_\cdot)|_\alpha\ge R\right]\le(1+\kappa)^{\frac{1}{\varepsilon}}e^{-\frac{R}{\varepsilon C}}$$

for all $R \geq 0$. So we can set $R := C(N + \log(1+\kappa))$ and choose K as in the Lemma above to get the result. ■

Theorem 2. *Under Assumption (D) the family (X^ε) is exponentially tight in Ω with respect to every of the probability measures $P_\mu[\,\cdot\,|\,X_0 \in E_k]$ and the compact open topology.*

Proof: To prove exponential tightness, we will use the exponentially fast Jakubowski criterion ([Sch95], Theorem 1) which states that the family (X^ε) is exponentially tight in the Skorohod space $D([0,1];\mathcal{X})$, exactly if on the one hand (X^ε) satisfies an exponential compact containment condition and on the other hand there exists an additive family of continuous, real valued functions, separating points on $\mathcal{X}$, such that for every function f of this family $f(X^\varepsilon)$ is exponentially tight in the Skorohod space $D([0,1];\mathbb{R})$.

In our case it is easy to adopt the above lemmata under the conditions given there to get the existence of A_N and R such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[\exists t\,:\, X_t^\varepsilon \notin A_N \,|\, X_0 \in E_k] \leq -N;$$
$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[|\tilde{u}(X_\cdot^\varepsilon)|_\alpha \geq R \,|\, X_0 \in E_k] \leq -N.$$

This follows, since $0 < \mu(E_k) \leq 1$, simply from

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[\exists t\,:\, X_t^\varepsilon \notin A_N \,|\, X_0 \in E_k] \\
=&\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[\exists t\,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k]/\mu(E_k) \\
=&\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[\exists t\,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k] \leq -N
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[|\tilde{u}(X_\cdot^\varepsilon)|_\alpha \geq R \,|\, X_0 \in E_k] \\
=&\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[|\tilde{u}(X_\cdot^\varepsilon)|_\alpha \geq R, X_0 \in E_k]/\mu(E_k) \\
\leq&\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[|\tilde{u}(X_\cdot^\varepsilon)|_\alpha \geq R] \leq -N.
\end{aligned}$$

Moreover, even if $\mathcal{U}$ itself has not to be additive, the $\mathbb{Q}$-vector space $\tilde{\mathcal{U}}$ over $\mathcal{U}$ is an additive family of quasi-continuous functionals on $\mathcal{X}$. Indeed, by (2.7) we have $\Gamma(u+v) \leq 2\big(\Gamma(u) + \Gamma(v)\big)$ for u, $v \in D(\mathcal{E})$. Lemma 9 implies the exponential tightness in $D([0,1];\mathbb{R})$ by Theorem 3 of [Sch95] since

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[\tilde{u}(X_0) > R \,|\, X_0 \in E_k] = -\infty,$$

as the function $\tilde{u}$ is continuous on the compact set E_k and hence bounded. (Note that in Theorem 1 of [Sch95] continuity for the members of the point-separating family is required, but quasi-continuity is sufficient, since in the proof continuity is used only on compacts. Compare also [Sch99], 7.Appendix, in particular Proposition 4.). The exponential tightness in the Skorohod space is of course inherited to the subspace Ω endowed with the compact-open topology since the Skorohod and the compact-open topology coincide on Ω (cf. [J], Proposition 1.6.). ■

4.2 Upper Bound

The proof of the upper bound relies on the characterization of the energy in Theorem 1 an the estimate for the $\mathcal{J}$-functional, Lemma 6. Using these results we can adapt a general approach (cf. [DZ], Theorem 4.5.3) to our setting. This yields the upper bound for compact sets, but having already established exponential tightness it holds all closed sets in the path space.

Proposition 5. *Given $\vec{u} \in \mathcal{H}^n$ and a partition Δ for some $n \in \mathbb{N}$. Then for any path ω and $\delta > 0$ there exists a neighborhood V_ω of ω such that*

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_\mu[X^\varepsilon \in V_\omega] \leq \begin{cases} -\mathcal{J}_\Delta^{\vec{u}}(\omega) + \delta & \omega_0 \in \operatorname{supp} \mu; \\ -\infty & \textit{else.} \end{cases}$$

Proof: 1. If $\omega_0 \notin \operatorname{supp} \mu$, then there exists a neighborhood U_0, $\omega_0 \in U_0$, such that $\mu(U_0) = 0$. Hence we can choose V_ω as the set of all paths ζ such that $\zeta_0 \in U_0$. Thus $P_\mu[X^\varepsilon \in V_\omega] = 0$ and the statement holds true.
2. If $\omega_0 \in \operatorname{supp} \mu$, we define

$$V_\omega := \{\zeta \,:\, \mathcal{J}_\Delta^{\vec{u}}(\zeta) > \mathcal{J}_\Delta^{\vec{u}}(\omega) - \delta\}$$

which is open in the compact-open topology by the definition of the functional $\mathcal{J}_\Delta^{\vec{u}}(\cdot)$. Thus we can conclude by Lemma 6

$$\begin{aligned}
-\mathcal{J}_\Delta^{\vec{u}}(\omega) &\geq \limsup_{\varepsilon \to 0} \varepsilon \log E_\mu\Big[\exp\Big(-\frac{1}{\varepsilon}\mathcal{J}_\Delta^{\vec{u}}(\omega)\Big)\Big] + \limsup_{\varepsilon \to 0} \varepsilon \log E_\mu\Big[\exp\Big(-\frac{1}{\varepsilon}\mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon)\Big)\Big] \\
&= \limsup_{\varepsilon \to 0} \varepsilon \log E_\mu\Big[\exp\Big(-\frac{1}{\varepsilon}\big(\mathcal{J}_\Delta^{\vec{u}}(\omega) - \mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon)\big)\Big)\Big] \\
&\geq \limsup_{\varepsilon \to 0} \varepsilon \log E_\mu\Big[\exp\Big(-\frac{1}{\varepsilon}\big(\mathcal{J}_\Delta^{\vec{u}}(\omega) - \mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon)\big)\Big);\, X^\varepsilon \in V_\omega\Big] \\
&\geq \limsup_{\varepsilon \to 0} \varepsilon \log E_\mu\Big[\exp\Big(-\frac{\delta}{\varepsilon}\Big);\, X^\varepsilon \in V_\omega\Big] \\
&= -\delta + \limsup_{\varepsilon \to 0} \varepsilon \log P_\mu[X^\varepsilon \in V_\omega].
\end{aligned}$$

■

Proposition 6. *For any compact set K in the path space Ω it holds that*

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_\mu[X^\varepsilon \in K] \leq \begin{cases} -\inf_{\omega \in K} \mathsf{E}_{0,1}(\omega) & \exists \bar{\omega} \in K \textit{ with } \bar{\omega}_0 \in \operatorname{supp} \mu; \\ -\infty & \textit{else.} \end{cases}$$

Proof: Note first that for every $\omega \in K$ and $\delta > 0$ we can find by (the proof of) Theorem 1 some $n \in \mathbb{N}$ that there exists a partition Δ and some $\vec{u} \in \mathcal{H}^n$ such that $\mathcal{J}_\Delta^{\vec{u}}(\omega) \geq I^\delta(\omega)$ for the δ-rate function by

$$I^\delta(\omega) := \min\Big(\mathsf{E}_{0,1}(\omega) - \delta, \frac{1}{\delta}\Big).$$

Moreover, we can find for now for this path $\omega \in K$ an open cylinder set V_ω, such that Lemma 5 holds. So (V_ω) constitutes an open cover of K and since K is compact there exists a finite subcover $(V_i)_{i=1,\dots,r}$, $V_i := \{\zeta \,:\, \mathcal{J}^{\vec{u}}_\Delta(\zeta) > \mathcal{J}^{\vec{u}}_\Delta(\omega^i) - \delta\}$. Thus we get by the above lemma

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0} \varepsilon \log P_\mu[X^\varepsilon \in K] &\leq \limsup_{\varepsilon\downarrow 0} \varepsilon \log P_\mu\big[X^\varepsilon \in \bigcup_{i=1}^r V_i\big] \\ &\leq \limsup_{\varepsilon\downarrow 0} \varepsilon \log\left(r \max_{1\leq j\leq r} P_\mu[X^\varepsilon \in V_j]\right) \\ &= \limsup_{\varepsilon\downarrow 0} \varepsilon \log r + \limsup_{\varepsilon\downarrow 0} \varepsilon \log \max_{1\leq j\leq r} P_\mu[X^\varepsilon \in V_j] \\ &\leq \begin{cases} \max_{1\leq j\leq r} -\mathcal{J}^{\vec{u}}_\Delta(\omega^j) + \delta & \exists\, \bar{\omega} \in K \text{ with } \bar{\omega}_0 \in \operatorname{supp}\mu; \\ -\infty & \omega_0 \notin \operatorname{supp}\mu \quad \forall\, \omega \in K. \end{cases}\end{aligned}$$

By the definition of the δ-rate function we have also

$$\max_{1\leq j\leq r} -\mathcal{J}^{\vec{u}}_\Delta(\omega^j) + \delta \leq \max_{1\leq j\leq r} -I^\delta(\omega^j) \leq \sup_{\omega\in K} -I^\delta(\omega) = -\inf_{\omega\in K} I^\delta(\omega),$$

whence we can conclude that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P_\mu[X^\varepsilon \in K] \leq \begin{cases} -\inf_{\omega\in K} I^\delta(\omega) & \exists\, \bar{\omega} \in K \text{ with } \bar{\omega}_0 \in \operatorname{supp}\mu; \\ -\infty & \omega_0 \notin \operatorname{supp}\mu \quad \forall\, \omega \in K. \end{cases}$$

Sending now $\delta \downarrow 0$ yields the result. ■

Since we have established in Theorem 2 exponential tightness for the case that we start out of the compact $\mathcal{E}$-nest, it is immediately clear (cf. [DZ], Lemma 1.2.18 (a)) that the general upper bound holds true.

Theorem 3. *For every closed thet $F \subseteq \Omega$ it holds that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[X^\varepsilon \in F \,|\, X_0 \in E_k] \leq -\inf_{\omega\in F} I_{E_k}(\omega)$$

with rate function

$$I_{E_k}(\omega) = \begin{cases} \mathsf{E}_{0,1}(\omega) & \omega_0 \in \operatorname{supp}\mu \cap E_k; \\ \infty & \textit{else.} \end{cases}$$

If we assume now for a moment that also the lower bound would hold in general (which it actually does in many cases), we can derive the following corollary.

Corollary 1. *For x, $y \in \operatorname{supp}\mu$ joined by a path of finite energy, we can describe the intrinsic metric ρ by*

$$\rho^2(x,y) = 2 \inf\{\mathsf{E}_{0,1}(\omega) \,:\, \pi_{\{0,1\}}(\omega) = (x,y)\}.$$

Furthermore

$$\rho(x,y) = \inf\{\mathsf{L}_{0,1}(\omega) \,:\, \pi_{\{0,1\}}(\omega) = (x,y)\}.$$

Proof: Note first that since x and y are joined by a path of finite energy, there exists a compact set E_k of the compact $\mathcal{E}$-nest such that $x \in E_k$. If we look at the projection $\pi_{\{0,1\}}$, the contraction principle (cf. [DZ], Theorem 4.2.1, resp. Appendix B, Proposition 16) entails a large deviation principle on $\mathcal{X} \times \mathcal{X}$ with respect to $P_\mu[\cdot \,|\, X_0 \in E_k]$ with good rate function

$$\tilde{I}(x,y) = \begin{cases} \inf_{\{\omega:\pi_{\{0,1\}}(\omega)=(x,y)\}} \mathsf{E}_{0,1}(\omega), & x \in \operatorname{supp}\mu \cap E_k \\ \infty & \text{else.} \end{cases}$$

But by Propositions 6 there holds also a large deviations principle upper bound with good rate function

$$I_{\{0,1\}}(x,y) = \begin{cases} \frac{\rho^2(x,y)}{2}, & x \in \operatorname{supp}\mu \cap E_k \\ \infty & \text{else.} \end{cases}$$

More precisely: Proposition 6 states the upper bound only for compact sets, but the exponential tightness implies that it holds for all closed sets. The respective lower bound was assumed to hold true as well.
By the definition of the energy it is clear that $\tilde{I} \geq I$. To prove the converse, we will mimic the proof of the uniqueness of the rate function of a large deviation principle (cf. [DZ], Lemma 4.1.4). Suppose now that there would exist a $(x_0, y_0) \in \mathcal{X}^2$ with $\tilde{I}(x_0,y_0) > I(x_0,y_0)$. So we can find for $\delta > 0$ small enough a neighborhood B of (x_0, y_0) such that

$$\inf_{(x,y)\in\overline{B}} \tilde{I}(x,y) \geq \big(\tilde{I}(x_0,y_0) - \delta\big) \wedge \frac{1}{\delta}.$$

On the other hand side the respective large deviation principles imply

$$\begin{aligned} -\inf_{(x,y)\in\overline{B}} \tilde{I}(x,y) &\geq \limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[(X_0, X_1^\varepsilon) \in B \,|\, X_0 \in E_k] \\ &\geq \liminf_{\varepsilon\to 0} \varepsilon \log P_\mu[(X_0, X_1^\varepsilon) \in B \,|\, X_0 \in E_k] \geq -\inf_{(x,y)\in B} I(x,y). \end{aligned}$$

So we get

$$I(x_0,y_0) \geq \inf_{(x,y)\in B} I(x,y) \geq \inf_{(x,y)\in\overline{B}} \tilde{I}(x,y) \geq \big(\tilde{I}(x_0,y_0) - \delta\big) \wedge \frac{1}{\delta}$$

for delta arbitrarily small and hence a contradiction to the Assumption $\tilde{I}(x_0,y_0) > I(x_0,y_0)$.
To prove the second statement, we remark that the Cauchy-Schwarz inequality implies

$$\begin{aligned} \left(\sum_{i=1}^n \rho(\omega_{t_{i-1}}, \omega_{t_i})\right)^2 &= \left(\sum_{i=1}^n \frac{\rho(\omega_{t_{i-1}}, \omega_{t_i})}{\sqrt{t_i - t_{i-1}}} \sqrt{t_i - t_{i-1}}\right)^2 \\ &\leq \left(\sum_{i=1}^n \frac{\rho^2(\omega_{t_{i-1}}, \omega_{t_i})}{t_i - t_{i-1}}\right)\left(\sum_{i=1}^n (t_i - t_{i-1})\right) = 2\sum_{i=1}^n \frac{\rho^2(\omega_{t_{i-1}}, \omega_{t_i})}{2(t_i - t_{i-1})}, \end{aligned}$$

whence $\mathsf{L}_{0,1}(\omega)^2 \leq 2\mathsf{E}_{0,1}(\omega)$. But $\rho(x,y) \leq \mathsf{L}_{0,1}(\omega)$ holds trivially, so also the second result follows.

■

4.3 An Integral Representation of the Rate Function

In this chapter, our aim is to give an integral representation of the rate function. For this purpose we recall the notion of absolutely continuous curves in a metric space (cf. [AT], [AGS04], [AGS05]).

Definition 2. *Given an complete metric space (E, d_E), we define the space $AC^p(E)$ of p-integrable absolutely continuous curves on E as the set of all curves (i.e. continuous paths) $\omega \in C([0,1];E)$, such there exists a non-negative $M \in L^p([0,1],\lambda)$, $p \in [1,\infty]$, such that*

$$d_E(\omega_r, \omega_t) \le \int_r^t M(s)\,ds, \qquad \text{for all } 0 \le r < t \le 1. \tag{4.1}$$

We say that a curve is absolutely continuous, if it belongs to $AC^1(E)$.

Given an p-integrable absolutely continuous curve ω, we can define its metric derivative $|\omega'|$ by

$$|\omega'|(t) := \lim_{r \to t} \frac{d_E(\omega_r, \omega_t)}{|r-t|}$$

which exists (Lebesgue-)almost everywhere, lies in $L^p([0,1],\lambda)$ and $|\omega'|$ is the (almost everywhere) smallest p-integrable function for which (4.1) holds ([AGS04], Theorem 2.2, [AGS05], Theorem 1.1.2). A direct application of this result is not possible, since our intrinsic metric functional ρ defines only an extended pseudo-metric on $\mathcal{X}$. But a close inspection of the proof shows, that in reality, it is only required that d_E restricted to any absolutely continuous curve has to be a true metric. To show that this holds true in our setting, we prove the following lemma.

Lemma 10. *Given a continuous path $\omega : [0,1] \to \mathcal{X}$, that satisfies $\rho(\omega_r, \omega_t) < \infty$ for every $0 \le r < t \le 1$. Then the restriction of ρ to the range R_ω of ω, $R_\omega := \omega([0,1])$, is a true metric.*

Proof: By the assumption it is clear that ρ is finite on R_ω. But this implies that ω lies completely in one of the sets E_k of the compact $\mathcal{E}$-nest (since otherwise the distance from at least one point would be infinite - cf. proof of Theorem 1). But on E_k we have by Assumption (BC) a point separating family with $\mathcal{E}$-quasi continuous μ-versions which separates points and has bounded carré du champ. By simple scaling, the family can be chosen to have carré du champ bounded by 1, and so $\rho(x,y) > 0$ for $x \neq y$ by definition of ρ. ■

This enables us to point out the relation between finite energy and absolute continuity.

Proposition 7. *Given a curve ω on $\mathcal{X}$, then the following statements are equivalent:*

(i) $\mathsf{E}_{0,1}(\omega)$ *is finite;*

(ii) ω *belongs to* $AC^2(\mathcal{X})$.

If one (hence both) of these conditions holds, the energy functional is given by

$$\mathsf{E}_{0,1}(\omega) = \frac{1}{4} \int_0^1 |\omega'|^2(t)\,dt.$$

Proof: To prove that (i) implies (ii), we note first that $\mathsf{E}_{0,1}(\omega) < \infty$ implies $d(\omega_r, \omega_t) < \infty$ for all $0 \le r < t \le 1$. Thus ρ is a true metric on R_ω by the above lemma. We want now prove first that every path of finite energy is absolutely continuous. To do so, we define for every open interval I in $[0,1]$ the set-function m by

$$m(I) := \sup_{\Delta} \sum_{t_{i-1}, t_i \in \Delta \cap I} \rho(\omega_{t_{i-1}}, \omega_{t_i}).$$

We denote by $\mathcal{I}$ the algebra generated by the open intervals and define for $A \in \mathcal{B}([0,1])$

$$m^*(A) := \inf \left\{ \sum_{k=1}^{\infty} m(I_k) \;:\; A \subseteq \bigcup_{k=1}^{\infty} I_k,\, I_k \in \mathcal{I} \right\}.$$

Moreover, since

$$m^*([0,1]) = m([0,1]) = \mathsf{L}_{0,1}(\omega) \le \sqrt{2E_{0,1}(\omega)} < \infty$$

by Corollary 1, m^* defines an outer measure on $([0,1], \mathcal{B}([0,1])$ which is in fact a true measure by [Bo], Section 1.5.

In a next step, we want to show that m is absolutely continuous with respect to the Lebesgue measure λ. We take some $I \in \mathcal{I}$ and assume without respect of generality that it is the disjoint union of finitely many intervals $I_k =]r_k, s_k[$. So it follows that

$$\begin{aligned}
m\left(\bigcup_{k=1}^{n} I_k\right) &\le \sum_{k=1}^{n} m(I_k) = \sum_{k=1}^{n} \sup_{\Delta} \sum_{t_{i-1}, t_i \in \Delta \cap I_k} \rho(\omega_{t_{i-1}}, \omega_{t_i}) \\
&= \sum_{k=1}^{n} \sup_{\Delta} \sum_{t_{i-1}, t_i \in \Delta \cap I_k} \sqrt{t_i - t_{i-1}} \frac{\rho(\omega_{t_{i-1}}, \omega_{t_i})}{\sqrt{t_i - t_{i-1}}} \\
&\le \sum_{k=1}^{n} \sqrt{\left(\sup_{\Delta} \sum_{t_{i-1}, t_i \in \Delta \cap I_k} |t_i - t_{i-1}| \right) \left(\sup_{\Delta} \sum_{t_{i-1}, t_i \in \Delta \cap I_k} \frac{\rho^2(\omega_{t_{i-1}}, \omega_{t_i})}{|t_i - t_{i-1}|} \right)} \\
&\le \sum_{k=1}^{n} \sqrt{\lambda(I_k)} \sqrt{2\mathsf{E}_{r_k, s_k}(\omega)},
\end{aligned}$$

whence the finiteness of the energy implies that $m \ll \lambda$. Thus there exists a Radon-Nikodym derivative $M = dm/d\lambda$ and we have

$$d(\omega_r, \omega_t) \le \int_r^t M(s)\, ds.$$

Thus it remains only to show that $M \in L^2([0,1], \lambda)$. Defining (using techniques analogous to [Sch96], Lemma 26) the functions $F(s,t) := \mathsf{E}_{s,t}(\omega)$ and $f(t) := F(0,t)$, f is a non-negative, non-decreasing and bounded function. Moreover, $F(r,t) = F(r,s) + F(s,t)$ for $0 \le r < s < t \le 1$ by Lemma 5 and f is almost everywhere differentiable with almost everywhere non-negative derivative $f'(t)$ which satisfies $\int_0^1 f'(t)\, dt \le f(1) = F(0,1)$. So it holds that

$$\begin{aligned}
\frac{1}{4} \int_0^1 |\omega'|^2(t)\, dt &= \frac{1}{4} \int_0^1 \left(\lim_{h \to 0} \frac{\rho(\omega_{s+h}, \omega_s)}{h} \right)^2 dt \le \int_0^1 \lim_{h \to 0} \frac{1}{h} F(s, s+h)\, ds \\
&= \int_0^1 f'(s)\, ds \le F(0,1) = \mathsf{E}_{0,1}(\omega) < \infty. \qquad (4.2)
\end{aligned}$$

To show that (ii) implies (i) we fix some $\omega \in AC^2(\mathcal{X})$. It follows by Jensen's inequality that

$$\begin{aligned} \mathsf{E}_{0,1}(\omega) &= \sup_{\Delta} \sum_{t_{i-1},t_i\in\Delta} \frac{\rho^2(\omega_{t_{i-1}},\omega_{t_i})}{4(t_i-t_{i-1})} \leq \sup_{\Delta} \sum_{t_{i-1},t_i\in\Delta} \left(\frac{1}{2(t_i-t_{i-1})}\int_{t_{i-1}}^{t_i} |\omega'|(s)\,ds\right)^2 \cdot (t_i-t_{i-1}) \\ &\leq \sup_{\Delta} \sum_{t_{i-1},t_i\in\Delta} \frac{1}{4(t_i-t_{i-1})}\int_{t_{i-1}}^{t_i} |\omega'|^2(s)\,ds \cdot (t_i-t_{i-1}) = \frac{1}{4}\int_0^1 |\omega'|^2(s)\,ds < \infty \end{aligned} \tag{4.3}$$

Combining (4.3) and (4.2) yields the integral representation of the energy functional. ■

This integral representation of the rate function is elegant, but note, that in the case where $\mathcal{X}$ is e.g. a manifold, we are rather interested in an integral representation via paths in the tangent bundle. Such a representation has to be done case by case, cf Chapter 7.

Remark 2. *We note only the case that $\mathcal{X}$ is a reflexive Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ (with respect to the norm generating the intrinsic metric!): The fact that $\omega \in AC^2(\mathcal{X})$ is equivalent to almost everywhere Fréchet differentiability of ω with $\dot{\omega} \in L^2([0,1];\mathcal{X})$ and*

$$\omega_t - \omega_r = \int_r^t \dot{\omega}_s\,ds, \qquad 0 \leq r < t \leq 1.$$

Here $\dot{\omega}$ denotes the Fréchet derivative and the integral is understood as Bochner integral (cf. [AGS05], Remark 1.1.3). So we get as integral representation of the finite energy of a path

$$\mathsf{E}_{0,1}(\omega) = \frac{1}{4}\int_0^1 \|\dot{\omega}_s\|_{\mathcal{X}}^2\,ds.$$

Chapter 5

Degenerate Cases

In this section we consider the degenerate cases, ruled out by Assumptions (A) and (B). We start with a little example, a degenerate Ornstein-Uhlenbeck process on $\mathbb{R}^2$.

Example 1. *We consider* $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \nu)$ *with*

$$d\nu = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}\, dx_1 dx_2,$$

the two-dimensional Gaussian measure. We define on $C_b^2(\mathbb{R}^2, \mathbb{R})$ *the operator* D *by* $Df := \frac{\partial f}{\partial x_1}$ *and compute its adjoint operator* D^* *by*

$$\begin{aligned}
\langle Df, g\rangle_{L^2(\nu)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{x_1}(x_1,x_2) g(x_1,x_2) e^{-\frac{x_1^2+x_2^2}{2}}\, dx_1 dx_2 \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)\frac{\partial}{\partial x_1}\Big(g(x_1,x_2)e^{-\frac{x_1^2+x_2^2}{2}}\Big)\, dx_1 dx_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)\big(-g_{x_1}(x_1,x_2) + x_1 g(x_1,x_2)\big)e^{-\frac{x_1^2+x_2^2}{2}}\, dx_1 dx_2 \\
&= \langle f, D^* g\rangle_{L^2(\nu)},
\end{aligned}$$

where D^* *is given by* $D^*f = -Df + x_1 f$. *So we can define a degenerate Ornstein-Uhlenbeck operator* L *as* $L := -D^*D$. *By construction this is a self-adjoint operator on* $L^2(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \nu)$. *The form* $\mathcal{E}(f,g) = \langle -Lf, g\rangle_{L^2(\nu)} = \langle Df, Dg\rangle_{L^2(\nu)}$, f, $g \in D(L)$, *is closable, since* L *is negative definite and self-adjoint (cf. [MR], Proposition I.3.3). Thus we can define the Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *as its closure. Obviously this Dirichlet form is local and conservative and has as carré du champ operator* $\Gamma(f) = 2(Df)^2$. *But it is also quasi-regular: We define the compact* $\mathcal{E}$*-nest simply as* $E_k := [-k,k]^2$. *By the conservativeness of the Dirichlet form we have*

$$\mathrm{Cap}_1(\mathbb{R}^2) = \|\mathbb{1}_{\mathbb{R}^2}\|_{\mathcal{E}} = 1.$$

But equally we have for $k \to \infty$ *the convergence*

$$\mathrm{Cap}_1(E_k) \geq \|\mathbb{1}_{E_k}\|_{L^2(\nu)} \to 1,$$

so $\mathcal{E}$ is quasi-regular. The intrinsic metric generated by this Dirichlet form is

$$\rho(x,y) = \begin{cases} \frac{|x_1-y_1|}{\sqrt{2}} & \text{if } x_2 = y_2; \\ \infty & \text{else.} \end{cases}$$

So every set A for which it holds that $\rho_A \in L^2(\mathbb{R}^2, \nu)$ has to have the property that its projection to the x_2-axis covers this axis up to a null set. Hence there exists no compact set K such that $\rho_K \in D(\mathcal{E})$.

This example indicates also what we can yet expect as result when we drop the Assumptions (A) and (B): If we start in a compact set, then we will get exponential tightness, since the diffusion process generated by L evolves only along lines parallel to the x_1-axis. This intuitive consideration we can make rigorous, so back to the general setting.

Definition 3. *To a given closed set F of positive measure we define the set of F-accessible points as*

$$\mathrm{acc}(F) := \{x \in \mathcal{X} \,:\, P_\mu[X_0 \in F, X_1 \in B_r(x)] > 0 \text{ for every } r > 0\},$$

where $B_r(x)$ denotes the open ball of radius r around x with respect to the original topology. Measurable subsets $C \subseteq \mathrm{acc}(F)^c$ we will call F-inaccessible.

We note that this definition has several different formulations, we note here

$$\begin{aligned} \mathrm{acc}(F) &= \{x \in \mathcal{X} \,:\, \exists t > 0,\, P_\mu[X_0 \in F, X_t \in B_r(x)] > 0 \text{ for every } r > 0\} \\ &= \{x \in \mathcal{X} \,:\, P_\mu[X_0 \in F, X_t \in B_r(x)] > 0 \text{ for every } t > 0 \text{ and } r > 0\} \\ &= \{x \in \mathcal{X} \,:\, d_F(B_r(x)) < \infty \text{ for every } r > 0\}. \end{aligned}$$

This follows from [AH], Proposition 5.1 resp. [HR], Lemma 2.16. Note that $\mathrm{acc}(F) \subseteq \operatorname{supp}\mu$ since

$$P_\mu[X_0 \in F, X_1 \in B_r(x)] = P_\mu[X_0 \in B_r(x), X_1 \in F] \leq \mu(B_r(x))$$

which vanishes for some small enough r if x is not in the support of the measure μ.

Proposition 8. *The Assumptions (A) and (B) imply that for every closed set F of positive measure it holds that $\mathrm{acc}(F) = \operatorname{supp}\mu$.*

Proof: Fix $x \in \operatorname{supp}\mu$ and some $r > 0$. Then $\mu(B_r(x)) > 0$, so there exists some $z \in B_r(x)$ with $\rho_F(z) < \infty$ (otherwise this would be a contradiction to $\rho_F \in L^2(\mathcal{X}, \mu)$). This implies by the definition of the setwise distance and Assumption (B) (resp. Assumption (A)) $d_F(B_r(x)) = \rho_F(B_r(x)) < \infty$, so $\mathrm{acc}^F(X) \supseteq \operatorname{supp}\mu$. But since $\mathrm{acc}^F(X) \subseteq \operatorname{supp}\mu$ holds in general, this yields the result. ■

That is also exactly what fails in Example 1. There we have on the one hand $\operatorname{supp}\mu = \mathbb{R}^2$ but on the other only

$$\mathrm{acc}(F) = \mathbb{R} \times \pi_2(F),$$

where π_2 is the projection on the x_2-axis, $\pi_2(x_1, x_2) = x_2$. So $\mathrm{acc}(F) \neq \operatorname{supp}\mu$ for every compact set F.

In the next lemma we collect a few additional facts of accessible sets:

Proposition 9. *For any closed set F of positive measure the following properties of* $\mathrm{acc}(F)$ *hold:*

$$\begin{aligned}
&\mathrm{acc}(F) \text{ is closed;}\\
&P_\mu[X_t \notin \mathrm{acc}(F); X_0 \in F] = 0 \quad \text{for every } t > 0;\\
&P_\mu[\exists t \in \mathbb{R}_{\geq 0} \,:\, X_t \notin \mathrm{acc}(F), X_0 \in F] = 0;\\
&\tau_{\mathrm{acc}(F)^c} = \infty \ P_\mu[\cdot\,|X_0 \in F]\text{-a.s.;}\\
&d_F(\mathrm{acc}(F)^c) = \rho_F(\mathrm{acc}(F)^c) = \infty.
\end{aligned}$$

Proof: We start by proving that the set $\mathrm{acc}(F)$ is closed: Given a sequence (x_n), $x_n \in \mathrm{acc}(F)$, which converges to some $x \in \mathcal{X}$. Then for every $r > 0$ there exists a x_n such that $x_n \in B_{r/2}(x)$. But since $x_n \in \mathrm{acc}(F)$ and $B_r(x) \supseteq B_{r/2}(x_n)$, we have clearly $\mu(B_r(x)) \geq \mu(B_{r/2}(x_n)) > 0$ and so $x \in \mathrm{acc}(F)$.
By the definition of accessible sets, we can find for every $x \notin \mathrm{acc}(F)$ a radius $r_x > 0$ such that

$$P_\mu[X_0 \in F; X_t \in B_{r_x}(x)] = 0$$

for the open balls $B_{r_x}(x)$. So the collection of these balls constitutes an open cover of $\mathrm{acc}(F)^c$. Since $\mathcal{X}$ is as Polish space Lindelöf (cf. [E], Corollary 4.1.16), we can extract a countable subcover $B_{r_i}(x_i)$, $i \in \mathbb{N}$. And since the countable union of null sets is again a null set, we can conclude by

$$P_\mu[X_t \notin \mathrm{acc}(F); X_0 \in F] \leq P_\mu\Big[\bigcup_{i=0}^{\infty}\big\{X_t \in B_{r_i}(x_i); X_0 \in F\big\}\Big] = 0. \tag{5.1}$$

Since $\mathrm{acc}(F)^c$ is open, the continuity of the paths of X_t imply

$$\begin{aligned}
P_\mu[\exists t \in \mathbb{R}_{\geq 0} \,:\, X_t \notin \mathrm{acc}(F), X_0 \in F] &= P_\mu[\exists t \in \mathbb{Q}_{\geq 0} \,:\, X_t \notin \mathrm{acc}(F), X_0 \in F]\\
&= P_\mu\Big[\bigcup_{t \in \mathbb{Q}_{\geq 0}} \{X_t \in \mathrm{acc}(F)^c, X_0 \in F\}\Big] = 0
\end{aligned}$$

by (5.1) as countable union of null sets. That the first hitting time of $\mathrm{acc}(F)^c$ is $P_\mu[\cdot\,|X_0 \in F]$-a.s. infinite is a direct consequence.
The last statement follows directly from Varadhan's principle and Proposition 1,

$$-\infty = \lim_{t\to 0} t \log P_\mu[X_t \notin \mathrm{acc}(F); X_0 \in F] = -\frac{d_F^2\big(\mathrm{acc}(F)^c\big)}{2} \geq -\frac{\rho_F^2\big(\mathrm{acc}(F)^c\big)}{2}.$$

■

Note in particular that every $\mathcal{X}$-inaccessible set C is an (X_t)- and $\mathcal{E}$-exceptional set (of zero capacity and with $P_\mu[\tau_C < \infty] = 0$ for the first hitting time τ_C). So we can understand the notion of F-inaccessible sets as a generalization of exceptional sets.
The moral is hence that for the large deviation principle we do not have to care about inaccessible sets. To make this rigorous, we have to find a weaker alternative to Assumption (B) which allows also to treat degenerate cases. Since Assumption (B) was needed to prove the exponential compact containment condition (Lemma 8), we have to prove these results then under the new Assumption.

Assumption (B'). *For all sets E_k of the compact $\mathcal{E}$-nest and all $M > 0$ it holds that $\rho_{E_k} \wedge M \in \mathcal{G}'$.*

Note that this implies immediately that ρ_{E_k} itself is lower semi-continuous. It remains only to prove the exponential compact containment condition. To do so, we need the following Lemma.

Lemma 11. *Let Assumption (B') hold, then we can find for every $N > 0$ and every E_k of the compact $\mathcal{E}$-nest a compact set $A_N \in \mathcal{X}$ with*

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_\mu[\exists t \,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k] \leq -N.$$

Proof: Note first that the set $\{\exists t \,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k\}$ is measurable, since very set $\{\exists t \,:\, X_t \notin A\}$ is Borel-measurable given that A is open or closed (cf. [J], Proposition 1.6.(v) and [Sch95], Remark 3). We define now the stopping times

$$\tau_l := \inf\{t \,:\, X_t^\varepsilon \notin E_l\}$$

and the corresponding stopped processes $Y_t^{\varepsilon,l} := X_{t\wedge\tau_l}^\varepsilon$. Let n_l the smallest number such that $\rho_{E_k} \geq n_l$ on E_l^c for the sets $E_l \supseteq E_k$ of the compact $\mathcal{E}$-nest. The sequence (n_l) converges to infinity for $l \to \infty$ by Assumption (D).
So for every given n_l we can set $\tilde{\rho}_{E_k} := \rho_{E_k} \wedge (n_l + 1)$. Then $\tilde{\rho}_{E_k} \in \mathcal{G}'$ by Assumption (B') and we can conclude

$$\begin{aligned}
P_\mu[\exists t \,:\, X_t^\varepsilon \notin E_l, X_0 \in E_k] &\leq P_\mu[\exists t \,:\, \rho_{E_k}(X_t^\varepsilon) \geq n_l, X_0 \in E_k] \\
&= P_\mu[\exists t \,:\, \tilde{\rho}_{E_k}(X_t^\varepsilon) \geq n_l, X_0 \in E_k] \\
&= P_\mu\Big[\max_{t\in[0,1]} \tilde{\rho}_{E_k}(X_t^\varepsilon) \geq n_l, X_0 \in E_k\Big] \\
&= P_\mu\Big[\tilde{\rho}_{E_k}(Y_1^{\varepsilon,l}) \geq n_l, Y_0^{\varepsilon,l} \in E_k\Big] \\
&= P_\mu\Big[e^{\frac{1}{\varepsilon}\tilde{\rho}_{E_k}(Y_1^{\varepsilon,l})} \geq e^{\frac{n_l}{\varepsilon}}, Y_0^{\varepsilon,l} \in E_k\Big] \\
&\leq e^{-\frac{n_l}{\varepsilon}} E_\mu\Big[e^{\frac{1}{\varepsilon}\tilde{\rho}_{E_k}(Y_1^{\varepsilon,l})}, Y_0^{\varepsilon,l} \in E_k\Big] \\
&\leq e^{-\frac{n_l}{\varepsilon}} E_\mu\Big[e^{\frac{1}{2\varepsilon}\int_0^1 \Gamma(\tilde{\rho}_{E_k})(Y_r^{\varepsilon,l})\,dr} e^{\frac{1}{\varepsilon}J_{0,1}^{\tilde{\rho}_{E_k}}(Y_t^{\varepsilon,l})}, Y_0^{\varepsilon,l} \in E_k\Big] \\
&\leq e^{\frac{1-2n_l}{2\varepsilon}} E_\mu\Big[e^{\frac{1}{\varepsilon}J_{0,1}^{\tilde{\rho}_{E_k}}(Y_t^{\varepsilon,l})}, Y_0^{\varepsilon,l} \in E_k\Big] \leq e^{\frac{1-2n_l}{2\varepsilon}}
\end{aligned}$$

by Chebyshev's inequality and an application of Lemma 6. So for every $N > 0$ we get a compact set A_N with

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_\mu[\exists t \,:\, X_t^\varepsilon \notin A_N, X_0 \in E_k] \leq -N$$

directly as the appropriate E_l. ■

Part II

Examples

Chapter 6

Ornstein-Uhlenbeck Process on Abstract Wiener Space

In this section we apply our general results to derive the pathwise small-time large deviation principle for the Ornstein-Uhlenbeck process on an abstract Wiener space. We then generalize this result to the case of Ornstein-Uhlenbeck processes with linear unbounded drift, given by a strongly elliptic operator.

6.1 Abstract Wiener Spaces and Malliavin Calculus

Let $(H, \langle\cdot,\cdot\rangle_H)$ a real separable Hilbert space endowed with a mean zero Gaussian cylinder set measure $\tilde{\mu}$. We take a measurable norm $\|\cdot\|$ to get a Banach space $(E, \|\cdot\|)$ as closure of H with respect to $\|\cdot\|$. Note that this implies that $\|\cdot\|$ is weaker then $\|\cdot\|_H$, i.e., there exists $c_1 > 0$ such that $\|x\| \leq c_1\|x\|_H$ for all $x \in H$. (cf. [Ku], Lemma I.4.2). $\tilde{\mu}$ induces a mean zero Gaussian cylinder measure μ on the on the σ-ring $\mathcal{R}$ generated by the cylinder subsets of E. Then E, endowed with the σ-algebra $\mathcal{F}$ generated by $\mathcal{R}$ (which is in fact the Borel-σ-algebra on E) and the measure μ, is a measure space. The embedding $\iota : H \to E$ is dense and continuous, the triple (E, H, μ) is called *abstract Wiener space* in the sense of Gross ([G]). Note that $\iota' : E' \to H'$ as embedding of the dual space is also dense and we can present the whole structure in the following diagram:

$$E' \overset{\iota'}{\hookrightarrow} H' \leftrightarrow H \overset{\iota}{\hookrightarrow} E.$$

In particular, given a separable Banach space $(E, \|\cdot\|)$ endowed with a mean-zero Gaussian measure μ, we can find an Hilbert space H (the *reproducing kernel Hilbert space*), such that (E, H, μ) is an abstract Wiener space (cf. [Ku], Theorem I.4.4). A stochastic process W indexed by the Hilbert space H is called an isonormal Gaussian process, if the random variables $W(h_1), \dots, W(h_n)$ are jointly mean zero Gaussian for every finite choice $h_1, \dots, h_n \in H$ and the covariance is given by $E[W(g)W(h)] = \langle g, h\rangle_H$.

(cf. [Nua], [CT], [Le]).

Example 2. *As concrete example we consider the classical Wiener space* $E = (C_0([0,T];\mathbb{R}^d), \|\cdot\|_\infty)$ *as probability space* $(E, \mathcal{B}, \gamma)$ *with Borel-σ-algebra* $\mathcal{B}$ *(induced by the norm* $\|\cdot\|_\infty$*) and Wiener measure* γ*. The Hilbert space* H *is in this case the Cameron-Martin space*

$$H := \left\{ h \in C_0^1([0,T];\mathbb{R}^d) \,:\, h(t) = \int_0^t h'(s)\,ds,\ h'(s) \in L^2([0,T];\mathbb{R}^d) \right\}$$

endowed with the inner product $\langle f, g\rangle_H := \int_0^T f'(s)g'(s)\,ds$*. If* (X_t) *denotes the coordinate process* $X_t(x) = x(t)$*,* $x \in E$*, and* $(\mathcal{F}_t)$ *the filtration generated by* (X_t)*, then* $X(h) := \int_0^T h'(s)\,dX_s$ *is an isonormal Gaussian process.*

Define now the set of smooth functions

$$\mathcal{S} := \left\{ f\big(W(h_1), \dots, W(h_n)\big) \,:\, f \in C_b^\infty(\mathbb{R}^n;\mathbb{R}),\, h_i \in H,\, n \in \mathbb{N} \right\}.$$

Their directional derivatives are given for $g \in H$ and $F \in \mathcal{S}$ by

$$\lim_{\varepsilon \to 0} \frac{F\big(W(h_1) + \varepsilon\langle g, h_1\rangle_H, \dots, W(h_n) + \varepsilon\langle g, h_n\rangle_H\big) - f\big(W(h_1), \dots, W(h_n)\big)}{\varepsilon} = \langle DF, g\rangle_H.$$

The gradient DF is called the *Malliavin derivative*, H plays here the role of the tangent space of E. For F, $G \in \mathcal{S}$ we define the bilinear form

$$\langle F, G\rangle_{\mathbb{D}^{1,2}} := E[FG] + E[\langle DF, DG\rangle_H],$$

the completion of $\mathcal{S}$ with respect to the associated norm $\|\cdot\|_{\mathbb{D}^{1,2}}$ is the Hilbert space $\mathbb{D}^{1,2}$. The integration by parts formula reads here for $F \in \mathbb{D}^{1,2}$, $G \in \mathcal{S}$ and $h \in H$

$$E[G\langle DF, h\rangle_H] = E[FGW(h)] - E[F\langle DG, h\rangle_H].$$

Note that the Malliavin derivative D is a mapping $L^2(E) \supseteq \mathbb{D}^{1,2} \to L^2(E;H)$. The adjoint operator $\delta \,:\, L^2(E;H) \supseteq D(\delta) \to L^2(E)$, defined via

$$E[\langle DF, u\rangle_H] = E[F\delta(u)], \qquad \text{for all } F \in \mathbb{D}^{1,2},\ u \in D(\delta),$$

is called the *Skorohod integral*.

The operator $L := -\delta \circ D$ is the Ornstein-Uhlenbeck operator on the abstract Wiener space, it is for $F = f(W(h_1), \dots, W(h_n)) \in \mathcal{S}$ concretely given by

$$LF = (-\delta \circ D)F = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial y_i \partial y_j}\big(W(h_1), \dots, W(h_n)\big)\langle h_i, h_j\rangle - \sum_{i=1}^n \frac{\partial f}{\partial y_i}\big(W(h_1), \dots, W(h_n)\big)W(h_i).$$

It holds that

$$\mathcal{E}(F, G) := E[(-LF)G] = E[\langle DF, DG\rangle_H], \qquad F, G \in \mathcal{S},$$

so this defines a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ with domain $D(\mathcal{E}) = \mathbb{D}^{1,2}$ and inner product $\langle F, G\rangle_{\mathcal{E}} = \langle F, G\rangle_{\mathbb{D}^{1,2}}$ (cf. [Nua], Chapter 1). It is quasi-regular by [MR], IV.4.b), and obviously conservative and local, the associated carré du champ operator is given by

$$\Gamma(F) = 2\|DF\|_H^2.$$

The diffusion process (X_t) associated to the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is the Ornstein-Uhlenbeck process on the abstract Wiener space (E, H, μ). It is the weak solution of the stochastic differential equation

$$dX_t = dW_t - \frac{1}{2}X_t\,dt,$$

where (W_t) is a Brownian motion on E, starting in $0 \in E$, with covariance $\langle\cdot,\cdot\rangle_H$ (cf. [FZ99]). The corresponding intrinsic metric on E is given by the Cameron-Martin distance ρ^H

$$\rho^H(f,g) = \begin{cases} \frac{\|f-g\|_H}{\sqrt{2}} & \text{if } (f-g) \in H; \\ \infty & \text{otherwise.} \end{cases}$$

6.2 Small-Time Asymptotics for the Ornstein-Uhlenbeck Process on Abstract Wiener Spaces

Now we want to prove the small-time large deviation principle for the Ornstein-Uhlenbeck process on an abstract Wiener space. Therefore we need some topological preliminaries.

Lemma 12. *Given a compact $\mathcal{E}$-nest (E_k), then $(\tilde{E}_k)$ defined by*

$$\tilde{E}_k := \{x \in E \,:\, \|x - E_k\|_H \le k\}$$

is also a compact $\mathcal{E}$-nest.

Proof: Note first that the $\|\cdot\|_H$-closed unit ball is $\|\cdot\|$-compact (cf. [Le], Chapter and [Kue], Lemma 2.1). Moreover, it is clear that

$$\tilde{E}_k = E_k + \overline{B}_k^H = \{x + y \,:\, x \in E_k,\, y \in \overline{B}_k^H\},$$

where $\overline{B}_k^H$ denotes the closed $\|\cdot\|_H$-ball of radius k around the origin. Thus so $\tilde{E}_k$ is $\|\cdot\|$-compact as sum of two $\|\cdot\|$-compact sets (cf. [E], Theorem 3.2.3). However, since $\tilde{E}_k \supseteq E_k$, also $\mathrm{Cap}_1(E \setminus \tilde{E}_k)$ tends obviously to zero, so $(\tilde{E}_k)$ is a compact $\mathcal{E}$-nest. ■

We define the space of absolutely continuous paths on E with square integrable derivatives in H by

$$H(E) := \left\{ h \in C([0,1], E) \,:\, \exists \dot{h} \in L^1([0,1]; H),\, h_t = h_0 + \int_0^t \dot{h}_r\,dr,\, \int_0^1 \|\dot{h}_r\|_H^2\,dr < \infty \right\}.$$

The integral $\int_0^t \dot{h}_r\,dr$ is an H-valued Bochner integral and $\dot{h}$ can be understood as Fréchet derivative.

Proposition 10. *The Ornstein-Uhlenbeck process* (X_t) *on the abstract Wiener space satisfies a small-time large deviation principle upper bound*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[X_t^\varepsilon \in B \,|\, X_0 \in \tilde{E}_k] \leq - \inf_{h\in\overline{B}} I^H_{\tilde{E}_k}(h)$$

for any set $B \subseteq C([0,1];E)$ *with rate function*

$$I^H_{\tilde{E}_k}(h) = \begin{cases} \frac{1}{4}\int_0^1 \|\dot{h}_t\|_H^2\, dt & h\in H(E),\, h_0 \in \operatorname{supp}\mu \cap \tilde{E}_k; \\ \infty & \textit{otherwise.} \end{cases}$$

Proof: Proving the proposition means essentially to show that the Assumptions (BC), (B) and (D) are fulfilled.
For any σ-compact positive measure set $K \subseteq E$ the distance functional ρ_K^H is μ-almost everywhere H-Lipschitz continuous in the sense of Enchev and Stroock ([ES], cf. also Kusuoka, [Kus82a] and [Kus82b]). This means that there exists a $C > 0$ and a μ-version $\tilde{\rho}_K^H$ of ρ_K^H such that

$$\left|\tilde{\rho}_K^H(x+h) - \tilde{\rho}_K^H(x)\right| \leq C\|h\|_H, \qquad \text{for all } x\in E \text{ and } h \in H.$$

This implies that $\rho_K^H \in D(\mathcal{E})$ with carré du champ bounded by 1 (cf. [Hir], Proposition 4.4, Proposition 4.5 and Theorem 4.2, compare also [ES] and [FLP], Proposition 4). Assumption (BC) is satisfied, since we can easily construct a point-separating family: Let $\mathcal{Y}$ a countable, dense subset of $E \setminus N$ (such a set exists by construction of the $\mathcal{E}$-nest (E_k), cf. [MR], Proposition IV.4.2) and take x, $y \in \mathcal{Y}$ with $x \neq y$. Then we can separate x and y by open $\|\cdot\|$-balls, i.e. there exists an $r \in \mathbb{Q}$, $r > 0$ such that $B_r^E(x) \cap B_r^E(y) = \emptyset$. Thus, by the separation version of the Hahn-Banach theorem (cf. [W], Theorem III.2.4) we can found a linear functional $l' \in E'$ such that

$$l'(z) < l'(w) \qquad \forall z \in B_r^E(x),\, w \in B_r^E(y).$$

However, these linear functionals are of course $\|\cdot\|_E$-Lipschitz and thus also $\|\cdot\|_H$-Lipschitz, so they are in the Dirichlet space with bounded carré du champ (cf. [Hir], Theorem 4.2). Hence this countable family of Lipschitz functionals separating balls of rational radius around points in the countable dense subset $\mathcal{Y}$ is our point-separating family. (An alternative family is given directly by $(u_{k,x})_{k\in\mathbb{N}, x\in\mathcal{Y}}$, $u_{k,x} := \rho^H_{\overline{B}^E_{1/k}(x)}(\cdot) \wedge M$ for some $M > 0$. The rational balls are closed sets of positive measure, so the set distances are (by Assumption(B) proved below) in the Dirichlet space with bounded carré du champ.)
For Assumption (D) we construct to (E_k) the sequence $(\tilde{E}_k)$ of the above lemma. Then it holds by construction of the compact $\mathcal{E}$-nest $(\tilde{E}_k)$, that for every $y \in \tilde{E}_l^c$, $l \geq k$,

$$\rho_{\tilde{E}_k}(y) = \inf_{x\in\tilde{E}_k} \frac{\|x-y\|_H}{\sqrt{2}} \geq \frac{l-k}{\sqrt{2}}$$

and so Assumption (D) holds true.
Let now A be a closed set of positive measure. Then $\rho_A = \rho_{A\setminus N}$ since N is a null set where the distance is infinite. But since $A_k := A \cap E_k$ is a compact set, it follows that $A \setminus N = \bigcup_{k=1}^\infty A_k$ is

σ-compact and hence Assumption (B) holds true, since $\rho_{A\setminus N} \in D(\mathcal{E})$.
So the large deviations principle holds with rate function

$$I_H(h) = \mathsf{E}_{0,1}(h) = \sup_{\Delta} \sum_{t_{i-1},t_i\in\Delta} \frac{\|h_{t_i} - h_{t_{i-1}}\|_H^2}{2(t_i - t_{i-1})}.$$

Moreover, Proposition 7 provides the integral representation of the rate function. ■

6.3 Ornstein-Uhlenbeck Process with Unbounded Linear Drift

Now we want to generalize this process. We take therefore a self-adjoint, strongly elliptic (i.e. $A \geq c_2 \operatorname{Id}_{H_0}$ for some $c_2 > 0$) operator A on H with domain $H_0 := D(A) \subseteq H$ and we define $L_A := -\delta \circ A \circ D$. The space H_0 with inner product $\langle \cdot, \cdot \rangle_{H_0} = \langle A^{\frac{1}{2}} \cdot, A^{\frac{1}{2}} \cdot \rangle_H$ is another Hilbert space contained in H; the embedding $H_0 \hookrightarrow H$ is dense and continuous. So we can define the Dirichlet form $(\mathcal{E}_A, D(\mathcal{E}_A))$ as closure of

$$\mathcal{E}_A(F, G) := E[-(L_A F)G] = E[\langle A^{\frac{1}{2}} DF, A^{\frac{1}{2}} Dg \rangle_H] = E[\langle DF, DG \rangle_{H_0}], \qquad F, G \in \mathcal{S}.$$

This Dirichlet form is also quasi-regular ([AR], Section 7.I) and [R92], Section 6.(a)), local and conservative, it admits a carré du champ operator given by

$$\Gamma(F) = 2\|DF\|_{H_0}^2.$$

The associated diffusion (X_t) we call generalized Ornstein Uhlenbeck process. The intrinsic metric is given by

$$\rho^{H_0}(f, g) = \begin{cases} \frac{\|f-g\|_{H_0}}{\sqrt{2}} & \text{if } (f - g) \in H_0; \\ \infty & \text{otherwise.} \end{cases}$$

The generalized Ornstein-Uhlenbeck process can be understood informally as weak solution of the SDE

$$dX_t = dW_t - \frac{1}{2} A(X_t)\, dt, \tag{6.1}$$

therefore we can look at A as a drift. But to be more precise, let's define X_h for $h \in H$ and a sequence (l_n), $l_n \in E'$, $l_n \to h$ in $\|\cdot\|_H$, as $L^2(E, \mu)$-limit $X_h := \lim_{n\to\infty} {}_{E'}\langle l_n, \cdot \rangle_E$. Let now $\beta : E \to E$ an $\mathcal{F}$-$\mathcal{F}$-measurable mapping that satisfies

$${}_{E'}\langle k, \beta \rangle_E = X_{Ak} \quad \mu\text{-a.e. for all } k \in E' \cap H_0$$

and $\int_E \|\beta\|\, d\mu < \infty$. (Note that such a mapping exists at least in the case where the embedding $H \hookrightarrow E$ is Hilbert-Schmidt and E is chosen as particular Hilbert space, cf. [R92], Proposition 4.15 and Section 6.(a). The proof relies on the Gross-Minlos-Sazonov theorem, for generalizations see [Y], Section 3) Then (X_t) is the solution of the SDE

$$dX_t = dW_t - \frac{1}{2} \beta(X_t)\, dt, \tag{6.2}$$

where (W_t) is a Brownian motion on E, starting in $0 \in E$, with covariance $\langle \cdot, \cdot \rangle_H$. The reason why we called the equation (6.1) an "informal expression" of (6.2) is the following: In the case that $A(E') \subseteq E'$ and $A|_{E'} : E' \to E'$ is $\| \cdot \|_{E'}$-continuous, it holds that $\beta(z) = A'(z)$ for μ-a.e. $z \in E$ for the operator A' defined as

$$_{E'}\langle k, A'(z) \rangle_E = {}_{E'}\langle Ak, z \rangle_E \quad \text{for all } z \in E,\ k \in E'.$$

(cf. [R], Section 6.(a); for a different approach to the SDE via "second quantization" we refer to [AR], Section 7.I) and [FZ99]).
In the following we will show that the generalized Ornstein-Uhlenbeck process obeys a large deviations principle.

Lemma 13. *The closed $\| \cdot \|_{H_0}$-unit ball*

$$\overline{B}_1^{H_0} := \{x \in E : \|x\|_{H_0} \leq 1\}$$

is weakly-$\| \cdot \|_{H_0}$-compact and $\| \cdot \|$-compact.

Proof: Weak compactness holds for every unit ball of a reflexive Banach space (cf. [W], Satz VIII.3.18). To prove $\| \cdot \|$-compactness, we note that strong ellipticity implies directly

$$\|x\|_{H_0}^2 = \|A^{\frac{1}{2}} x\|_H^2 \geq c_2 \|x\|_H^2.$$

So $\overline{B}_1^{H_0} \subseteq \overline{B}_{1/c_2}^{H}$ which is a $\| \cdot \|$-compact set. So $\overline{B}_1^{H_0}$ is relatively $\| \cdot \|$-compact.
But $\overline{B}_1^{H_0}$ is also $\| \cdot \|$-closed: Fix a sequence (x_n), $x_n \in \overline{B}_1^{H_0}$ converging to some $x \in E$ in $\| \cdot \|$. Since $\overline{B}_1^{H_0}$ is weak-$\| \cdot \|_{H_0}$-compact, so there exists a subsequence (x_{n_j}) which converges weakly to some $z \in \overline{B}_1^{H_0}$. E' separates points in E, so $x = z \in \overline{B}_1^{H_0}$, whence the unit ball is $\| \cdot \|$-closed and so $\| \cdot \|$-compact. ■

Lemma 14. $(\mathcal{E}_A, D(\mathcal{E}_A))$ *satisfies Assumption (D), Assumption (B') and Assumption (BC).*

Proof: Since Lemma 13 states the compactness properties required in the proof of Lemma 12, we can copy directly this proof to get to a given compact $\mathcal{E}$-nest (E_k) a compact $\mathcal{E}$-nest $(\tilde{E}_k)$ given by

$$\tilde{E}_k := \{x \in E : \|x - E_k\|_{H_0} \leq k\}.$$

So (analogously to the proof of Proposition 10) Assumption (D) holds.
To prove Assumption (B'), we note that for every $M > 0$ it holds that

$$\left| \rho_K^{H_0}(x+h) \wedge M - \rho_K^{H_0}(x) \wedge M \right| \leq \rho^{H_0}(x, x+h) = \|h\|_{H_0}. \tag{6.3}$$

Indeed, suppose first that K is compact and without loss of generality $\rho_K^{H_0}(x) \geq \rho_K^{H_0}(y)$. Since K is compact, it holds that $\rho_K^{H_0}(y) = \rho^{H_0}(y, k)$ for some $k \in K$. Then

$$\rho^{H_0}(x, y) \wedge M \geq \rho^{H_0}(x, k) \wedge M - \rho^{H_0}(y, k) \wedge M \geq \rho_K^{H_0}(x) \wedge M - \rho_K^{H_0}(y) \wedge M$$

and (6.3) holds. If K is merely σ-compact, we can write it as increasing union of compacts K_n, so $\rho_{K_n}^{H_0}(x)$ is a decreasing sequence and the inequality (6.3) remains true. Thus we can conclude that $\rho_K^{H_0} \in D(\mathcal{E}_A)$ with carré du champ bounded by 1.
Again we can fix a countable dense subset $\mathcal{Y}$ of $E \setminus N$ by construction of the compact $\mathcal{E}$-nest (E_k) (cf. [RSchm95], Proposition 3.1). To prove now Assumption (BC) we can repeat the proof of Proposition 10, since the point-separating linear functionals are also H_0-Lipschitz by the ellipticity of the operator A. ■

Remark 3. *If A is additionally of trace class, then $(\mathcal{E}_A, D(\mathcal{E}_A))$ satisfies also Assumption (B): Since A is of trace class, it holds for $z \in H$ for any orthonormal basis $(e_k)_k$ of H that*

$$\begin{aligned}\|z\|_{H_0}^2 = \|A^{\frac{1}{2}}z\|_H^2 &= \sum_{k=1}^{\infty}\langle A^{\frac{1}{2}}z, e_k\rangle_H^2 = \sum_{k=1}^{\infty}\langle z, A^{\frac{1}{2}}e_k\rangle_H^2 \leq \|z\|_H^2 \sum_{k=1}^{\infty}\|A^{\frac{1}{2}}e_k\|_H^2 \\ &= \|z\|_H^2 \sum_{k=1}^{\infty}\langle Ae_k, e_k\rangle_H = \|z\|_H^2 \cdot \mathrm{tr}\,(A). \end{aligned} \tag{6.4}$$

by Parseval's identity and Cauchy's inequality. So we get for every σ-compact set $K \subseteq E$, $\mu(K) > 0$,

$$\rho_K^{H_0}(x) = \inf_{y \in K} \frac{\|x-y\|_{H_0}^2}{\sqrt{2}} \leq \inf_{y \in K} \frac{\mathrm{tr}\,(A) \cdot \|x-y\|_H^2}{\sqrt{2}} = \mathrm{tr}\,(A) \cdot \rho_K^H(x),$$

whence $\rho_K^{H_0} \in L^2(E,\mu)$. But by (6.3) with (6.4) it follows even that $\rho_K^{H_0} \in D(\mathcal{E}_A)$, so Assumption (B) is fulfilled.

So we can collect all these results and define

$$H^0(E) := \left\{ h \in C([0,1],E) \,:\, \exists \dot{h} \in L^1([0,1];H_0),\, h_t = h_0 + \int_0^t \dot{h}_r\,dr,\, \int_0^1 \|\dot{h}_r\|_{H_0}^2\,dr < \infty \right\}.$$

The integral representation of the rate function holds by Proposition 7, so we get the following proposition.

Proposition 11. *The generalized Ornstein-Uhlenbeck process (X_t) on the abstract Wiener space satisfies a small-time large deviation principle upper bound*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu[X_t^\varepsilon \in B \,|\, X_0 \in \tilde{E}_k] \leq -\inf_{h\in\overline{B}} I_{\tilde{E}_k}^{H_0}(h)$$

for any set $B \subseteq C([0,1];E)$ with rate function

$$I_{\tilde{E}_k}^{H_0}(h) = \begin{cases} \frac{1}{4}\int_0^1 \|\dot{h}_t\|_{H_0}^2\,dt & h \in H^0(E),\, h_0 \in \mathrm{supp}\,\mu \cap \tilde{E}_k; \\ \infty & \textit{otherwise}. \end{cases}$$

Note that this result generalizes the work of Fang and Zhang ([FZ99], Proposition 2.5), who proved the pathwise large deviations for a fixed starting point by a direct approach via the associated semigroup in the case that the embedding $H_0 \hookrightarrow E$ is trace class.

Chapter 7

Wasserstein Diffusion on $\mathcal{P}([0,1])$

The Wasserstein diffusion $(\mu_t)_{t\geq 0}$ on the space $\mathcal{P}([0,1])$ of probability measures over the unit interval $[0,1]$ equipped with the L^2-Wasserstein distance was introduced by von Renesse and Sturm in their paper [vRS] whose presentation we follow closely. The L^2-Wasserstein distance is given by

$$d_W(\mu,\nu) := \inf_{\gamma\in\Gamma^{\mu,\nu}} \left(\iint_{[0,1]^2} |x-y|^2 \gamma(dx,dy) \right)^{\frac{1}{2}}$$

where γ ranges over all probability measures on $[0,1]^2$ with marginals μ and ν,

$$\Gamma^{\mu,\nu} := \left\{ \gamma \in \mathcal{P}([0,1]^2) \,:\, \gamma(A\times[0,1]) = \mu(A),\, \gamma([0,1]\times B) = \nu(B),\, A,B\subseteq[0,1] \right\}.$$

The Wasserstein diffusion on the space $(\mathcal{P}([0,1]), d_W)$ is constructed via a Dirichlet form defined in terms of an entropic measure on $\mathcal{P}([0,1])$ and a gradient on a convenient tangent space in such a way, that the associated intrinsic metric is the L^2-Wasserstein distance.
The construction draws heavily on the isometry between $\mathcal{P}([0,1])$ and the non-decreasing, right-continuous functions on $[0,1]$: Define

$$\mathcal{G}_0 := \{g : [0,1]\to[0,1] \,:\, g \text{ is right-continuous, non-decreasing with } g(1)=1\},$$

then there exists a bijection

$$\chi \,:\, (\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)}) \to (\mathcal{P}([0,1]), d_W)$$

given by the push-forward of the Lebesgue measure

$$\chi(g)(\cdot) = (g_*\lambda)(\cdot) = \lambda(g^{-1}(\cdot)).$$

The inverse of χ is given by the quantile function

$$\begin{aligned} \chi^{-1} \,:\, (\mathcal{P}([0,1]), d_W) &\to (\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)}) \\ \mu &\mapsto g_\mu := \inf\{s\in[0,1] \,:\, \mu([0,s]) > t\} \end{aligned}$$

(with $\inf \emptyset = 1$). Moreover, this bijection is in fact an isometry,

$$d_W(\mu,\nu) = \|g_\mu - g_\nu\|_{L^2(\lambda)}.$$

Indeed, this is nothing else than the Fréchet-Hoeffding theorem for optimal transport of probability measures on the real line with quadratic cost functional. For the background we refer to Villani's book [Vi]. The concrete statement there is Theorem 2.18 (cf. also [AGS05], Theorem 6.0.2). The L^2-topology on $\mathcal{G}_0$ and the image of the Wasserstein topology on $\mathcal{P}([0,1])$ under the map $\chi^{-1} : g \mapsto g_\mu$ coincide and $\mathcal{G}_0$ is compact with respect to this topology ([vRS], Section 2.1).

7.1 Entropic Measure

Using this isometry, we can construct the desired objects on $\mathcal{G}_0$ and pushing them forward to $\mathcal{P}([0,1])$. To construct the entropic measure on $\mathcal{G}_0$ we fix some $\beta > 0$. Then for $N \in \mathbb{N}$, a partition $\Delta_N := \{t_0, \ldots, t_{N+1}\}$, $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = 1$ and the simplex

$$\Sigma_N := \{(x_1, \ldots, x_N) \in [0,1]^N : 0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1\},$$

we define the marginals of the measure Q_0^β by requiring

$$\begin{aligned}&\int_{\mathcal{G}_0} u(g_{t_1}, \ldots, g_{t_N})\, dQ_0^\beta(g)\\ =&\frac{\Gamma(\beta)}{\prod_{i=1}^{N+1}\Gamma(\beta(t_i - t_{i-1}))}\int_{\Sigma_N} u(x_1,\ldots,x_N)\prod_{i=1}^{N+1}(x_i - x_{i-1})^{\beta(t_i - t_{i-1})}\frac{dx_1\cdots dx_N}{\prod_{i=1}^{N+1}(x_i - x_{i-1})}\end{aligned} \tag{7.1}$$

for every bounded and measurable $u : [0,1]^N \to \mathbb{R}$. Here Γ stands for the Gamma function. We define $\mathcal{G}^i$ as the set of mappings $\Delta_i \to \Sigma_i$. The above family of marginals is consistent, so Kolmogorov's extension theorem implies the existence of a unique measure $\tilde{Q}_0^\beta$ on the projective limit $\varprojlim \mathcal{G}^i$. The canonical projections $p_i : \mathcal{G}_0 \to \mathcal{G}^i$ are consistent with the canonical projections $\pi_i^j : \mathcal{G}^j \to \mathcal{G}^i$, $j \geq i$. So $\mathcal{G}_0$ is homeomorphic to a subset of $\prod_{i\in\mathbb{N}} \mathcal{G}^i$ and we can embed $\mathcal{G}_0$ continuously into $\varprojlim \mathcal{G}^i$. The restriction of the measure $\tilde{Q}_0^\beta$ to $\mathcal{G}_0$ defines the measure Q_0^β on $\mathcal{G}_0$ (compare Appendix 2).
Note that the measure Q_0^β is nothing else then a Dirichlet distribution. To show this, we define the measure ν on the unit interval by

$$\nu(]a,b]) := \int_a^b \beta\, ds, \qquad 0 \leq a < b \leq 1.$$

Applying the substitution $y_i := x_i - x_{i-1}$, $i = 1, \ldots, N$, to (7.1), we get

$$\begin{aligned}&\int_{\mathcal{G}_0} u(g_{t_1}, \ldots, g_{t_N})\, dQ_0^\beta(g)\\ =&\frac{\Gamma(\nu(]0,1]))}{\prod_{i=1}^{N+1}\Gamma(\nu(]t_{i-1},t_i]))}\int_{[0,1]^N} u(y_1,\ldots,y_N)\prod_{i=1}^{N+1} y_i^{\nu(]t_{i-1},t_i])-1}\delta_{1-\sum_{k=1}^{N-1} y_k}(dy_N)dy_{N-1}\cdots dy_1,\end{aligned}$$

which is the Dirichlet distribution on the unit interval with parameter ν (cf. [Sch02], (6)).
The *entropic measure* P_0^β on $\mathcal{P}([0,1])$ we get by pushing Q_0^β forward, hence by requiring that

$$\int_{\mathcal{P}([0,1])} u(\mu)\, dP_0^\beta(\mu) = \int_{\mathcal{G}_0} u(g_*\lambda)\, dQ_0^\beta(g)$$

for all bounded and measurable $u : \mathcal{P}([0,1]) \to \mathbb{R}$ ([vRS], Section 3.3). We note that the measures Q_0^β have full support on $\mathcal{G}_0$, that Q_0^β-almost surely the function $s \mapsto g(s)$ is strictly increasing, but it increases only by jumps (by heights adding up to 1, the jump locations are dense in $[0,1]$), and that for every fixed $s_0 \in [0,1]$, the function $s \mapsto g(s)$ is Q_0^β-almost surely continuous in s_0. The asymptotics of the family $(Q_0^\beta)_{\beta\in]0,\infty[}$ in β are the following: For $\beta \to \infty$ the measures Q_0^β converge weakly to the Dirac measure δ_e on the identity map e of $[0,1]$; for $\beta \to 0$ they converge weakly to the uniform distribution on the set $\{\mathbb{1}_{[s,1]} : s \in]0,1]\}$.
In terms of P_0^β, this means that $\mu \in \mathcal{P}([0,1])$ is P_0^β-almost surely singular continuous (i.e., it has no atoms, is not absolutely continuous and behaves 'Cantor-like') and the measures P_0^β converge weakly to δ_λ for $\beta \to \infty$ and to the uniform distribution on the set $\{(1-s)\delta_{\{0\}} + s\delta_{\{1\}} : s \in [0,1]\}$ for $\beta \to 0$ ([vRS], Section 3.5).
For both measures, Q_0^β and P_0^β, a Girsanov style change of variable formula holds. Let $h \in \mathcal{G}_0$ a C^2-isomorphism, hence an increasing homeomorphism $h : [0,1] \to [0,1]$, such that h and h^{-1} are both bounded in $C^2([0,1])$. Then under the translation $\tau_h : \mathcal{G}_0 \to \mathcal{G}_0$ given by $g \mapsto h \circ g$ for every $g \in \mathcal{G}_0$, the measure Q_0^β is quasi-invariant, so is P_0^β under $\tilde{\tau}_h : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ given by $\mu \mapsto h_*\mu$. More explicitly, the densities (bounded and bounded away from 0) are given by

$$\begin{aligned}\frac{dQ_0^\beta(h\circ g)}{dQ_0^\beta(g)} =& \frac{d(\tau_{h^{-1}})_* Q_0^\beta(g)}{dQ_0^\beta(g)} \\ =& e^{\beta\int_0^1 \log h'(g(s))\,ds} \frac{1}{\sqrt{h'(0)h'(1)}} \prod_{a\in J_g} \frac{\sqrt{h'(g(a-)h'(g(a+))}}{\frac{h(g(a+))-h(g(a-))}{g(a+)-g(a-)}}\end{aligned}$$

where $J_g \subseteq]0,1[$ denotes the set of the jump locations of g on $]0,1[$; Analogously

$$\frac{dP_0^\beta(h_*\mu)}{dP_0^\beta(\mu)} = e^{\beta\int_0^1 \log h'(s)\,\mu(ds)} \frac{1}{\sqrt{h'(0)h'(1)}} \prod_{I\in \text{gaps}\,(\mu)} \frac{\sqrt{h'(I_-)h'(I_+)}}{|h(I)|/|I|}$$

where gaps (μ) denotes the set of intervals $I =]I_-, I_+[\subseteq [0,1]$ of maximal length with $\mu(I) = 0$ and $|I|$ denotes the length of such an interval ([vRS], Section 4.3).

7.2 Directional Derivatives and Integration by Parts on $\mathcal{G}_0$

For each $\varphi \in C^\infty([0,1];\mathbb{R})$ with $\varphi(0) = \varphi(1) = 0$, we define the flow generated by φ as the map

$$e_\varphi : \mathbb{R} \times [0,1] \to [0,1],$$

that assigns to each $x \in [0,1]$ the function

$$\begin{aligned} e_\varphi(\cdot, x) \,:\, \mathbb{R} \to & [0,1]; \\ t \mapsto & e_\varphi(t,x). \end{aligned}$$

$e_\varphi(t,x)$ denotes here the unique solution of the ordinary differential equation

$$\begin{aligned} \frac{dx_t}{dt} =& \varphi(x_t), \\ x_0 =& x. \end{aligned}$$

The property $e_\varphi(t,x) = e_{t\varphi}(1,x)$ allows us the use of the shorthand $e_{t\varphi}(x) := e_\varphi(t,x)$. For every $\varphi \in C^\infty([0,1];\mathbb{R})$, the family $(e_{t\varphi})_{t\in\mathbb{R}}$ is a group of order-preserving C^∞-diffeomorphisms of $[0,1]$. Note that $e_{t\varphi}(0) = 0$ and $e_{t\varphi}(1) = 1$ for all $t \in \mathbb{R}$ and $\frac{\partial}{\partial t} e_{t\varphi}(x)|_{t=0} = \varphi(x)$.
Given a function $u \in \mathcal{G}_0$, we can define for every φ the directional derivative along φ by

$$D_\varphi u(g) := \lim_{t\to\infty} \frac{u(e_{t\varphi} \circ g) - u(g)}{t}$$

in the case that this limit exists.
This is in particular the case for the following family of functions $u : \mathcal{G}_0 \to \mathbb{R}$, for which the derivative $D_\varphi u(g)$ exists for every point $g \in \mathcal{G}_0$ in every direction φ: Define $\mathfrak{Z}^k(\mathcal{G}_0)$ by

$$\mathfrak{Z}^k(\mathcal{G}_0) := \\ \Big\{ u : u(g) = U\Big(\int_0^1 \alpha_1(g(s))ds, \ldots, \int_0^1 \alpha_m(g(s))\,ds\Big), \alpha_i \in C^k([0,1],\mathbb{R}), U \in C^k(\mathbb{R}^m,\mathbb{R}), m \in \mathbb{N}\Big\},$$

then we get for $u \in \mathfrak{Z}^1(\mathcal{G}_0)$

$$D_\varphi u(g) = \sum_{i=1}^m \frac{\partial}{\partial y_i} U\Big(\int_0^1 \alpha_1(g(s))\,ds, \ldots, \int_0^1 \alpha_m(g(s))\,ds\Big) \cdot \int_0^1 \alpha_i'(g(s))\varphi(g(s))\,ds.$$

This operator is closeable in $L^2(\mathcal{G}_0, Q_0^\beta)$ and we will denote the closure by $(D_\varphi, D(D_\varphi))$.
Denoting for $\varphi \in C^\infty([0,1];\mathbb{R})$ by D_φ^* the operator adjoint to D_φ in $L^2(\mathcal{G}_0, Q_0^\beta)$, we get for u in the family $\mathfrak{Z}^1(\mathcal{G}_0)$ the integration by parts formula

$$D_\varphi^* u = -D_\varphi u - V_\varphi^\beta \cdot u.$$

The drift operator $V_\varphi^\beta : \mathcal{G}_0 \to \mathbb{R}$ is given at the point $g \in \mathcal{G}_0$ by

$$\sum_{a\in J_g} \left(\frac{\varphi'(g(a+)) + \varphi'(g(a-))}{2} - \frac{\varphi(g(a+)) - \varphi(g(a-))}{g(a+) - g(a-)} \right) + \beta \int_{]0,1[} \varphi'(g(x))\,dx - \frac{\varphi'(0) + \varphi'(1)}{2}$$

(cf. [vRS], Section 5.4).

7.3 Gradient and Dirichlet Form on $\mathcal{G}_0$

To define a gradient for functionals on $\mathcal{G}_0$, we have to specify the tangent space in the points $g \in \mathcal{G}_0$. In accordance to our definition of the directional derivatives we choose

$$\|\varphi\|_{T_g} := \left(\int_0^1 \varphi(g(s))^2\, ds\right)^{\frac{1}{2}}$$

to get

$$T_g\mathcal{G}_0 = L^2([0,1], g_*\lambda).$$

So we can define the gradient $Du(g) \in T_g\mathcal{G}_0$ by

$$D_\varphi u(g) = \langle Du(g), \varphi\rangle_{T_g} \qquad \text{for all } \varphi \in T_g.$$

It exists if and only if

$$\sup_{\varphi\in T_g} \frac{D_\varphi u(g)}{\|\varphi \circ g\|_{L^2(\lambda)}} < \infty.$$

This definition of a tangent space is isometrically isomorphic to another one: If we understand $\mathcal{G}_0$ as subset of $L^2([0,1], \lambda)$, we can choose

$$\mathbb{T}_g\mathcal{G}_0 := L^2([0,1], \lambda)$$

and define the the directional derivative $\mathbb{D}_f$ for $u : \mathcal{G}_0 \to \mathbb{R}$ in direction $f \in \mathbb{T}_g$ as Gâteaux derivative

$$\mathbb{D}_f u(g) := \lim_{t\to 0} \frac{u(g + tf) - u(g)}{t}$$

which is in fact a Fréchet derivative and so $\mathbb{D}u$ by

$$\mathbb{D}_f u(g) = \langle \mathbb{D}u(g), f\rangle_{\mathbb{T}_g}, \qquad \text{for all } f \in \mathbb{T}_g.$$

It exists if and only if

$$\sup_{f\in\mathbb{T}_g} \frac{\mathbb{D}_f u(g)}{\|f\|_{L^2(\lambda)}} < \infty.$$

In particular we get

$$D_\varphi u(g) = \mathbb{D}_{\varphi\circ g} u(g) \qquad \text{and} \qquad \|Du(g)\|_{T_g} = \|\mathbb{D}u(g)\|_{\mathbb{T}_g}$$

(cf. [vRS], Section 7.1). For $u \in \mathfrak{Z}^1(\mathcal{G}_0)$ we can understand the gradient $\mathbb{D}u$ as mapping

$$\begin{aligned} \mathbb{D}u \,:\, \mathcal{G}_0 \times [0,1] &\to \mathbb{R}; \\ (g,t) &\mapsto \mathbb{D}u(g)(t). \end{aligned} \tag{7.2}$$

So $(\mathbb{D}, \mathfrak{Z}^1(\mathcal{G}_0))$ is an operator

$$\mathbb{D} \,:\, \mathfrak{Z}^1(\mathcal{G}_0) \to L^2(\mathcal{G}_0 \times [0,1], Q_0^\beta \otimes \lambda)$$

and as such it is closable in $L^2(\mathcal{G}_0, Q_0^\beta)$ with closure $(\mathbb{D}, D(\mathbb{D}))$.
For u, $v \in \mathfrak{Z}^1(\mathcal{G}_0)$ we can hence define the so called *Wasserstein Dirichlet integral*

$$\mathbb{E}(u,v) := \int_{\mathcal{G}_0} \langle \mathbb{D}u(g), \mathbb{D}v(g)\rangle_{L^2(\lambda)}\, dQ_0^\beta(g).$$

$(\mathbb{E}, \mathfrak{Z}^1(\mathcal{G}_0))$ is closeable, too, and its closure $(\mathbb{E}, D(\mathbb{E}))$ defines a regular, recurrent and local Dirichlet form. Its domain coincides with $D(\mathbb{D})$, since both are the closure of $\mathfrak{Z}^1(\mathcal{G}_0)$. We get the representation

$$\mathbb{E}(u,v) = \int_{\mathcal{G}_0\times[0,1]} \mathbb{D}u \cdot \mathbb{D}v\, d(Q_0^\beta \otimes \lambda),$$

understanding the gradient as mapping $\mathcal{G}_0 \times [0,1] \to \mathbb{R}$ as in (7.2). Furthermore $(\mathbb{E}, D(\mathbb{E}))$ admits a carré du champ operator $\Gamma(u,v) \in L^1(\mathcal{G}_0, Q_0^\beta)$ with domain $D(\Gamma) = D(\mathbb{E}) \cap L^\infty(\mathcal{G}_0, Q_0^\beta)$ which is given by

$$\Gamma(u,v)(g) = 2\,\langle \mathbb{D}u(g), \mathbb{D}v(g)\rangle_{L^2(\lambda)}$$

(cf. [vRS], Section 7.2). We note two important classes of functions which are contained in the domain of the Dirichlet form. On the one hand, for every $f \in \mathcal{G}_0$ the function

$$\begin{aligned} u_f^0 \,:\, \mathcal{G}_0 \to& \mathbb{R}; \\ g \mapsto& \langle f, g\rangle_{L^2(\lambda)} \end{aligned}$$

belongs to $D(\mathbb{E})$. And on the other hand also the function

$$\begin{aligned} u_f^1 \,:\, \mathcal{G}_0 \to& \mathbb{R}; \\ g \mapsto& \frac{\|f-g\|_{L^2(\lambda)}}{\sqrt{2}}, \end{aligned} \tag{7.3}$$

belongs to $D(\mathbb{E})$ with the additional property $\Gamma(u_f^1) \le 1$ Q_0^β-almost surely. Moreover, we have for functions in the domain of the Dirichlet form the following Rademacher property: Every $1/\sqrt{2}$-Lipschitz function u on $\mathcal{G}_0$ is contained in $D(\mathbb{E})$ with $\Gamma(u) \le 1$ Q_0^β-almost surely. Vice versa, every continuous function $u \in D(\mathbb{E})$ with $\Gamma(u) \le 1$ Q_0^β-almost surely is $1/\sqrt{2}$-Lipschitz. The intrinsic metric ρ on $\mathcal{G}_0$ generated by the carré du champ Γ is the L^2-metric:

$$\rho(f,g) = \frac{\|f-g\|_{L^2(\lambda)}}{\sqrt{2}} \qquad \text{for all } f, g \in \mathcal{G}_0$$

(cf. [vRS], Section 7.3, compare also [RSchi]). The Dirichlet form has a generator $\frac{1}{2}L$ and an associated Markov process, the diffusion $(g_t)_{t\ge 0}$ on $\mathcal{G}_0$ starting in some $g_0 \in \mathcal{G}_0$.

7.4 Wasserstein Dirichlet Form and Wasserstein Diffusion

Now we can push forward the Dirichlet form from $\mathcal{G}_0$ to $\mathcal{P}([0,1])$ by means of the isometry χ. Therefore we define the set $\mathfrak{Z}^k(\mathcal{P}([0,1]))$ by

$$\mathfrak{Z}^k(\mathcal{P}([0,1])) :=$$
$$\Big\{u : u(\mu) = U\Big(\int_0^1 \alpha_1\, d\mu, \dots, \int_0^1 \alpha_m\, d\mu\Big), \alpha_i \in C^k([0,1],\mathbb{R}), U \in C^k(\mathbb{R}^m,\mathbb{R}), m \in \mathbb{N}\Big\}.$$

For any $\mu \in \mathcal{P}([0,1])$ we can identify the tangent space $T_\mu\mathcal{P}([0,1])$ with $L^2([0,1],\mu)$. So for $u \in \mathfrak{Z}^1(\mathcal{P}([0,1]))$ the gradient $Du(\mu)$ is given by

$$Du(\mu) = \sum_{i=1}^m \frac{\partial}{\partial y_i} U\Big(\int_0^1 \alpha_1\, d\mu, \dots, \int_0^1 \alpha_m d\mu\Big) \cdot \alpha_i'(\cdot)$$

with norm

$$\|Du(\mu)\|_{T_\mu} = \left(\int_0^1 \left|\sum_{i=1}^m \frac{\partial}{\partial y_i} U\Big(\int_0^1 \alpha_1\, d\mu, \dots, \int_0^1 \alpha_m d\mu\Big) \cdot \alpha_i'\right|^2 d\mu\right)^{\frac{1}{2}}.$$

The associated flow by means of $\varphi \in T_\mu$ is given by the push-forward $\varphi_*\mu$.
Pushing the Dirichlet form $(\mathbb{E}, D(\mathbb{E}))$ on $\mathcal{G}_0$ forward yields the *Wasserstein Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathcal{P}([0,1]), P_0^\beta)$ which is regular, recurrent and local and on $\mathfrak{Z}^1(\mathcal{P}([0,1]))$ given by

$$\mathcal{E}(u,v) = \int_{\mathcal{P}([0,1])} \langle Du(\mu), Dv(\mu)\rangle_{L^2(\mu)}\, dP_0^\beta(\mu), \qquad u,v \in \mathfrak{Z}^1(\mathcal{P}([0,1])).$$

The associated carré du champ operator is defined on $D(\mathcal{E}) \cap L^\infty(\mathcal{P}([0,1]), P_0^\beta)$ and has on $\mathfrak{Z}^1(\mathcal{P}([0,1]))$ the representation

$$\Gamma(u,v)(\mu) = 2\,\langle Du(\mu), Dv(\mu)\rangle^2_{L^2(\mu)}, \qquad u,v \in \mathfrak{Z}^1(\mathcal{P}([0,1])).$$

The Rademacher property is here given in the following sense: Every $1/\sqrt{2}$-Lipschitz function u on $\mathcal{P}([0,1])$ (the Lipschitz property is understood with respect to the L^2-Wasserstein distance d_W) is contained in $D(\mathcal{E})$ with $\Gamma(u) \le 1$ P_0^β-almost surely. Conversely, every continuous function $u \in D(\mathcal{E})$ with $\Gamma(u) \le 1$ P_0^β-almost surely is $1/\sqrt{2}$-Lipschitz. The intrinsic metric on $\mathcal{P}([0,1])$ generated by Γ is the L^2-Wasserstein metric:

$$\rho(\mu,\nu) = \frac{d_W(\mu,\nu)}{\sqrt{2}} \qquad \text{for all } \mu,\nu \in \mathcal{P}([0,1]).$$

Moreover, we can represent the generator $\frac{1}{2}L$ of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathfrak{Z}_0^2(\mathcal{P}([0,1]))$ defined by

$$\mathfrak{Z}_0^2(\mathcal{P}([0,1])) := \Big\{u \in \mathfrak{Z}^2(\mathcal{P}([0,1])) \,:\, \alpha_i'(0) = \alpha_i'(1) = 0 \text{ for all } i \in \{1,\dots,m\},\, m \in \mathbb{N}\Big\}$$

as sum $L = L_1 + L_2 + \beta L_3$ with

$$\begin{aligned}
L_1 u(\mu) &= \sum_{i=1}^m \sum_{i=1}^m \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} U\Big(\int_0^1 \alpha_1\, d\mu, \dots, \int_0^1 \alpha_m d\mu\Big) \cdot \int_0^1 \alpha_i'\alpha_j'\, d\mu \\
L_2 u(\mu) &= \sum_{i=1}^m \frac{\partial}{\partial y_i} U\Big(\int_0^1 \alpha_1\, d\mu, \dots, \int_0^1 \alpha_m d\mu\Big) \cdot \\
&\quad \cdot \left(\sum_{I \in \mathrm{gaps}\,\mu} \left(\frac{\alpha_i''(I_-) + \alpha_i''(I_+)}{2} - \frac{\alpha_i'(I_-) - \alpha_i'(I_+)}{|I|}\right) - \frac{\alpha_i''(0) + \alpha_i''(1)}{2}\right) \\
L_3 u(\mu) &= \sum_{i=1}^m \frac{\partial}{\partial y_i} U\Big(\int_0^1 \alpha_1\, d\mu, \dots, \int_0^1 \alpha_m d\mu\Big) \cdot \int_0^1 \alpha_i''\, d\mu.
\end{aligned}$$

Here L_1 describes a diffusion on $\mathcal{P}([0,1])$ in all directions, L_2 is the drift part due to the change of variable formula and L^3 is the generator of the deterministic Neumann heat flow $(H_t)_{t\geq 0}$ on $L^2(\mathcal{P}([0,1]), P_0^\beta)$ given by

$$H_t u(\mu) = u(h_t\mu).$$

h_t is the heat kernel on $[0,1]$ with reflecting boundary conditions and

$$h_t\mu(\cdot) = \int_0^1 h_t(\cdot, y)\mu(dy).$$

Note that for $\beta \to \infty$ we have the convergence $1/\beta L \to L_3$.
For P_0^β-almost every $\mu_0 \in \mathcal{P}([0,1])$ we can associated a Markov process $(\mu_t)_{t\geq 0}$ starting in μ_0 to the generator $\frac{1}{2}L$. This process $(\mu_t)_{t\geq 0}$ we will call the *Wasserstein diffusion* on $(\mathcal{P}([0,1]), P_0^\beta)$. Note that it is related to the diffusion process on $\mathcal{G}_0$ by

$$\mu_t = (g_t)_*\lambda,$$

where $(g_t)_{t\geq 0}$ is the Markov process associated to the Dirichlet form $(\mathbb{E}, D(\mathbb{E}))$ on $L^2(\mathcal{G}_0, Q_0^\beta)$ with starting point $g_0 := \chi^{-1}(\mu_0)$ (cf. [vRS], Section 7.5).

7.5 Small Time Large Deviations on $\mathcal{G}_0$

Now we want to apply our general results on pathwise small-time large deviations to the case of the Wasserstein diffusion. We start with $\mathcal{G}_0$ and define the space $H(\mathcal{G}_0)$ as subspace of the sample path space $C([0,1], \mathcal{G}_0)$ by

$$H(\mathcal{G}_0) :=$$
$$\left\{ h \in C([0,1], \mathcal{G}_0) \,:\, \exists \dot{h} \in L^1([0,1]; L^2([0,1];\mathbb{R})),\, h_t = h_0 + \int_0^t \dot{h}_r\, dr,\, \int_0^1 \|\dot{h}_r\|^2_{L^2(\lambda)}\, dr < \infty \right\}.$$

Note that the integral $\int_0^t \dot{h}_r\, dr$ is an $L^2([0,1];\mathbb{R})$-valued Bochner integral; $H(\mathcal{G}_0)$ is the space of absolutely continuous paths on $(\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)})$ with square integrable Fréchet derivative (cf. Remark 2). Furthermore we denote by $H_{g_0}(\mathcal{G}_0)$ the subspace of all sample paths starting g_0.

Proposition 12. *The diffusion process $(g_t)_{t\geq 0}$ on $(\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)})$ associated to the Dirichlet form $(\mathbb{E}, D(\mathbb{E}))$ and starting in g_0 obeys a large deviation upper bound with rate function*

$$\tilde{\mathsf{E}}_{0,1}(h) = \tilde{I}_{g_0}(h) := \begin{cases} \frac{1}{4}\int_0^1 \|\dot{h}_r\|^2_{L^2(\lambda)}\, dr & \text{for } h \in H_{g_0}(\mathcal{G}_0); \\ \infty & \text{otherwise.} \end{cases}$$

Proof: To prove this result means essentially to check if our conditions for the large deviation principle hold. The Dirichlet form is quasi-regular (since regular), local, conservative (since recurrent, cf.

[FOT], Theorem 1.6.3) and admits a carré du champ operator. So Assumption (A) is fulfilled. A point-separating family with bounded square field operator is given by $(u^1_f)_{f\in\mathcal{Y}}$,

$$u^1_f(g) = \|f-g\|_{L^2(\lambda)}/\sqrt{2} = \rho(f,g) = \rho_f(g)$$

for a countable dense subset $\mathcal{Y}$ of $\mathcal{G}_0$ (see (7.3) above). In particular $\rho_f \in D(\mathbb{E})$ for every $f \in \mathcal{G}_0$; the exponential compact containment condition holds true (even without Assumption (D)) since $\mathcal{G}_0$ itself is compact (cf. Section 5). The integral representation holds true by Proposition 7. ■

But our general results imply in fact more, we can randomize the initial condition to get the following Corollary.

Corollary 2. *The diffusion $(g_t)_{t\geq 0}$ on $\mathcal{G}_0$ satisfies the small time large deviation upper bound*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_{Q_0^\beta}[g^\varepsilon_t \in A] \leq -\inf_{h\in\overline{A}} \tilde{I}(h)$$

for any set $A \subseteq C([0,1];\mathcal{G}_0)$ with rate function

$$\tilde{I}(h) := \begin{cases} \frac{1}{4}\int_0^1 \|\dot{h}_r\|^2_{L^2(\lambda)}\,dr & h \in H(\mathcal{G}_0),\ h_0 \in \operatorname{supp} Q_0^\beta; \\ \infty & \textit{otherwise.} \end{cases}$$

7.6 Small Time Large Deviations on Wasserstein Space

Now we want to derive the corresponding small time large deviation principle on the Wasserstein space. The isometry between $\mathcal{G}_0$ and $\mathcal{P}([0,1])$ implies that such a principle holds with rate function

$$I(\omega) := \mathsf{E}_{0,1}(\omega) = \sup_{\Delta} \sum_{t_{i-1},t_i\in\Delta} \frac{d^2_W(\omega_{t_{i-1}},\omega_{t_i})}{4(t_i-t_{i-1})},$$

ω a sample path on $\Omega := C([0,1];\mathcal{P}([0,1]))$. In particular $\rho_\mu \in D(\mathcal{E})$ for every $\mu \in \mathcal{P}([0,1])$ and $(\rho_\mu)_\mu$ separates points with $\Gamma(\rho_\mu) \leq 1$. But instead of directly carrying over the integral representation on $\mathcal{G}_0$, we prefer to give first a representation directly in terms of the Wasserstein geometry.

To introduce the notion of a tangent velocity field, we look at the so called continuity equation: We say that a curve ω on $\mathcal{P}([0,1])$ solves the continuity equation

$$\frac{\partial}{\partial t}\omega_t + \operatorname{div}(v_t\omega_t) = 0,$$

for a Borel vector field v, $v_t \in L^2([0,1],\omega_t)$, $v_t(x) : [0,1]\times]0,1[\to [0,1]$, $(x,t)\mapsto v_t(x)$, if it solves the equation in the sense of distributions. This means that for every smooth test function Φ on $[0,1]\times]0,1[$ with compact support, the equation

$$\int_0^1\int_0^1 \Big(\frac{\partial}{\partial t}\Phi(x,t) + \langle v_t(x), \nabla_x\Phi(x,t)\rangle_{L^2(\omega_t)}\Big)\,\omega_t(dx)\,dt = 0$$

is satisfied. If ω is absolutely continuous, then there exists a vector field v with $\|v_t\|_{L^2(\omega_t)} \leq |\omega'|(t)$ almost everywhere such that ω solves the equation with respect to v. Conversely, if ω satisfies the continuity equation for some Borel vector field v with $\|v_t\|_{L^2(\omega_t)} \in L^2([0,1],\lambda)$, then ω is absolutely continuous and $|\omega'|(t) \leq \|v_t\|_{L^2(\omega_t)}$ almost everywhere. (cf. [AGS04], Theorem 3.6 and [AGS05], Theorem 8.3.1). To a given curve ω there exists indeed no unique vector field v such that ω solves the continuity equation with respect to v. If ω is a solution with respect to v, then it is also a solution with respect to $v+w$ for every w which satisfies $\operatorname{div}(w_t\omega_t) = 0$. But there exists a minimal solution, that is unique up to (Lebesgue-)null sets and for which

$$\|v_t\|_{L^2(\omega_t)} = |\omega'|(t) \qquad \lambda\text{-almost surely} \tag{7.4}$$

holds (cf. [AGS04], Section 3.2 and [AGS05], p.167). This vector field v is called the tangent velocity field associated to ω. It can be understood as a sample path on the tangent bundle of $\mathcal{P}([0,1])$, $t \mapsto v_t$ defines a mapping $[0,1] \to T\mathcal{P}([0,1])$.

Proposition 13. *Given a curve ω on $\mathcal{P}([0,1])$, the following statements are equivalent:*

(i) $\mathsf{E}_{0,1}(\omega)$ is finite;

(ii) ω belongs to $AC^2(\mathcal{P}([0,1]))$;

(iii) there exists a tangent velocity field v associated to ω such that $\|v_t\|_{L^2(\omega_t)} \in L^2([0,1],\lambda)$.

If one (hence all) of these conditions holds, then the energy functional is given by

$$\mathsf{E}_{0,1}(\omega) = \frac{1}{4}\int_0^1 |\omega'|^2(t)\,dt = \frac{1}{4}\int_0^1 \|v_t\|^2_{L^2(\omega_t)}\,dt.$$

Proof: The equivalence of (i) and (ii) is true by Proposition 7. That implication (iii) entails (ii) is clearly true: Since v is the tangent velocity field such that ω solves the continuity equation, ω is absolutely continuous. But since $\|v_t\|_{L^2(\omega_t)} \in L^2([0,1],\lambda)$, it follows that $|\omega'|(t) \in L^2([0,1],\lambda)$ and so

$$\rho(\omega_r,\omega_t) \leq \int_r^t |\omega'|^2(s)\,ds, \qquad 0 \leq r < t \leq 1.$$

Conversely, the existence of a tangent velocity field implies that ω is absolutely continuous. Hence the metric derivative exists, and since $\|v_t\|_{L^2(\omega_t)} \in L^2([0,1],\lambda)$, it is square integrable by (7.4), proving the equivalence of (ii) entails (iii). The integral representation follows also directly by Remark 2 and (7.4). ■

From this proposition, we can directly derive the small time asymptotics of the Wasserstein diffusion.

Theorem 4. *The Wasserstein diffusion $(\mu_t)_{t\geq 0}$ satisfies a small time large deviation upper bound*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_{P_0^\beta}[\mu_t^\varepsilon \in A] \leq -\inf_{\omega\in\overline{A}} I(\omega)$$

for any set $A \subseteq \Omega = C([0,1];\mathcal{P}([0,1]))$ with rate function

$$I(\omega) = \begin{cases} \frac{1}{4}\int_0^1 \|v_t\|^2_{L^2(\omega_t)}\,dt & \omega \in AC^2(\mathcal{P}([0,1])),\ \omega_0 \in \operatorname{supp} P_0^\beta; \\ \infty & \textit{otherwise}, \end{cases}$$

where v is the tangent velocity field associated to ω.

Proof: This is straightforward from Proposition 13 and our general result Theorem 3. ∎
A remarkable result gives the reformulation of Corollary 1 in terms of the Wasserstein geometry, it recovers actually the celebrated Benamou-Brenier formula (cf. [AGS04], Theorem 3.9, [AGS05], p.168 and [Vi], Theorem 8.1):

Corollary 3. *Suppose that also the lower bound of the large deviation principle holds. Let μ_0, $\mu_1 \in \mathcal{P}([0,1))$ two measures connected by a curve in $AC^2(\mathcal{P}([0,1]))$. We define by $\pi_{\{0,1\}} : \Omega \to \mathcal{P}([0,1])^2$ the projection $\omega \mapsto (\omega_0, \omega_1)$ to get*

$$d_W^2(\mu_0, \mu_1) = \inf_{\{\omega : \pi_{\{0,1\}}(\omega) = (\mu_0, \mu_1)\}} \frac{1}{2} \int_0^1 \|v_t\|^2_{L^2(\omega_t)} \, dt.$$

7.7 More on the Relation between $\mathcal{G}_0$ and the Wasserstein Space

Till now, we used extensively the isometry of $(\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)})$ and $(\mathcal{P}([0,1]), d_W)$ to construct the Wasserstein diffusion, but developed the large deviation principles in both cases independently. Now we want to relate this two approaches and point out the relation between the vector fields v and the derivatives $\dot{g}$.

Proposition 14. *Suppose that a curve $\omega \in AC^2(\mathcal{P}([0,1]))$ solves the continuity equation with respect to the tangent velocity field v. Then (g_t), $g_t = \chi^{-1}(\omega_t)$, lies in $H(\mathcal{G}_0)$ with derivative $\dot{g}_t = v_t \circ g_t$. Conversely, if $g \in H(\mathcal{G}_0)$ with derivative $\dot{g}$, then the curve ω on $\mathcal{P}([0,1])$, $\omega_t = \chi(g_t)$, is in $AC^2(\mathcal{P}([0,1]))$ and solves the continuity equation with respect to the tangent velocity field v, $v_t = \dot{g}_t \circ g_t^{-1}$.*

Proof: We note first the equivalence of the integrability conditions,

$$\|v_t\|_{L^2(\omega_t)} = \|v_t \circ g_t\|_{L^2(\lambda)} = \|\dot{g}_t\|_{L^2(\lambda)}.$$

Assume now that $g \in H(\mathcal{G}_0)$ with derivative $\dot{g}$. Then ω, $\omega_t = \chi(g_t) = (g_t)_*\lambda$, solves the continuity equation with respect to the vector field v, $v_t = \dot{g}_t \circ g_t^{-1}$, since for any test function Φ

$$\begin{aligned}
&\int_0^1 \int_0^1 \Big(\Phi_t(x,t) + \langle v_t(x), \Phi_x(x,t) \rangle_{L^2(\omega_t)} \Big)\, \omega_t(dx)\, dt \\
=&\int_0^1 \langle 1, \Phi_t(\cdot,t)\rangle_{L^2(\omega_t)} + \langle v_t, \Phi_x(\cdot,t)\rangle_{L^2(\omega_t)}\, dt \\
=&\int_0^1 \langle 1, \Phi_t(g_t,t)\rangle_{L^2(\lambda)} + \langle v_t \circ g_t, \Phi_x(g_t,t)\rangle_{L^2(\lambda)}\, dt \\
=&\int_0^1 \langle 1, \Phi_t(g_t,t)\rangle_{L^2(\lambda)} + \langle \dot{g}_t, \Phi_x(g_t,t)\rangle_{L^2(\lambda)}\, dt \\
=&\int_0^1 \langle 1, \Phi_t(g_t,t) + \dot{g}_t \Phi_x(g_t,t)\rangle_{L^2(\lambda)}\, dt \\
=&\int_0^1 \Big\langle 1, \frac{\partial}{\partial t}\big(\Phi(g_t,t)\big)\Big\rangle_{L^2(\lambda)}\, dt \\
=&\langle 1, \Phi(g_1,1) - \Phi(g_0,0)\rangle_{L^2(\lambda)} = 0.
\end{aligned}$$

But v is also the minimal vector field which solves the continuity equation, since

$$\|v_t\|_{L^2(\omega_t)} = \|\dot{g}_t\|_{L^2(\lambda)} = \lim_{h\to 0} \left\| \frac{g_{t+h} - g_t}{h} \right\|_{L^2(\lambda)} = \lim_{h\to 0} \frac{d_W(\omega_{t+h}, \omega_t)}{h} = |\omega|'(t),$$

so it is the tangent velocity field to ω.
To prove the other direction, assume that $\omega \in AC^2(\mathcal{P}([0,1]))$ solves the continuity equation with respect to the tangent velocity field v. The above considerations imply

$$\int_0^1 \langle v_t \circ g_t, \Phi_x(g_t,t)\rangle_{L^2(\lambda)} + \langle 1, \Phi_x(g_t,t)\rangle_{L^2(\lambda)}\, dt = \int_0^1 \Big\langle 1, \frac{\partial}{\partial t}\Phi(g_t,t)\Big\rangle_{L^2(\lambda)}\, dt$$

for every smooth test function on $[0,1]\times]0,1[$ with compact support. So $v_t \circ g_t$ is a weak derivative of g_t. Moreover, it is clear that we have by the isometry between $(\mathcal{P}([0,1]), d_W)$ and $(\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)})$ the identity $\mathsf{E}_{0,1}(\omega) = \mathsf{E}_{0,1}(g)$ for the energy functionals defined by the respective metrics. Thus, since $\omega \in AC^2(\mathcal{P}([0,1]))$, it has by Proposition 7 finite energy; so has g, which is thus in $AC^2(\mathcal{G}_0)$ by Proposition 7. It follows that g_t is Fréchet differentiable, and since the Fréchet derivative has to be the same as the weak derivative, we have $\dot{g}_t = v_t \circ g_t$. ■
This machinery allows us to calculate in an easy way the tangent velocity field v, avoiding the messy task to find a vector field for which ω solves the continuity equation.

7.8 Some Examples

We start with an easy example:

Example 3. *We consider the curve $\omega_t = \delta_t$ on the Wasserstein space. Direct calculations yield that the distribution function is given by $g_t^{-1}(x) = \mathbb{1}_{[t,1]}(x)$ and the quantile function by $g_t = t\mathbb{1}_{[0,1[}(x) + (1-t)\mathbb{1}_{\{1\}}(x)$ with derivative $\dot{g}_t(x) = \mathbb{1}_{[0,1[}(x) - \mathbb{1}_{\{1\}}(x)$. So the tangent velocity field is given by*

$$v_t(x) = (\dot{g}_t \circ g_t^{-1})(x) = \mathbb{1}_{[0,t[}(x) - \mathbb{1}_{[t,1]}(x).$$

In this case, divergence free vector fields are given by $w_t = f \circ g_t^{-1}$ for every $f : [0,1] \to \mathbb{R}$, that satisfies $\int_0^1 f(x)\,dx = 0$: For any test function Φ it holds that

$$\int_0^1 \langle w_t, \Phi_x(\cdot,t)\rangle_{L^2(\omega_t)}\,dt = \int_0^1 \langle f, \Phi_x(g_t,t)\rangle_{L^2(\lambda)}\,dt = \int_0^1 f\,d\lambda \int_0^1 \Phi_x(t,t)\,dt = 0,$$

so any vector field of the form $v+w$ satisfies the continuity equation.

The following example is intended as warning that regularity properties in the space variable (as Lipschitz continuity) are not preserved under the transformation mapping g_t^{-1}.

Example 4. *We consider the path $g_t(x) = x^{2+t}$. The derivative with respect to time t is $\dot{g}_t(x) = x^{2+t}\ln x$ which is for every $t \in [0,1]$ globally Lipschitz in x with some Lipschitz constant uniformly bounded in t. The inverse of g_t is $g_t^{-1}(x) = x^{\frac{1}{2+t}}$, so the tangent velocity field is given by*

$$v_t(x) = \frac{x \ln x}{2+t}$$

which is for every $t \in [0,1]$ clearly not globally Lipschitz in x (and in particular also not locally Lipschitz in 0, with exponentially exploding Lipschitz constants approaching this value): Thus v does not induce a flow in the sense of Ambrosio and Gigli (cf. next section) along the curve ω_t (given by their distribution functions $x^{\frac{1}{2+t}}$).

7.9 Flows along Regular Curves in the Wasserstein Space

One question remains yet open, namely if the velocity field v induces a flow of the curve ω on $\mathcal{P}([0,1])$. Note that till now, we had only the notion of a flow with respect to some $\varphi \in C^\infty([0,1];\mathbb{R})$ which was pushed forward from $\mathcal{G}_0$. But now we need a definition of a flow that is associated to the tangent velocity field v, so v_t belongs to a different tangent space for every t. So we will understand now under a flow along ω a mapping $\mathbf{T}(s,t,x) : [0,1]^2 \times [0,1] \to [0,1]$, that is absolutely continuous in t, Lipschitz in x, and satisfies

$$\begin{cases} \mathbf{T}(s,s,x) &= x & \text{for all } x, s \in [0,1]; \\ \frac{d}{d\tau}\big|_{\tau=t}\mathbf{T}(s,\tau,x) &= v_t(\mathbf{T}(s,t,x)) & \text{for all } x,\ s, \text{ and almost every } t \in [0,1]; \\ \mathbf{T}(s,t,\mathbf{T}(r,s,x)) &= \mathbf{T}(r,t,x) & \text{for all } x, r, s, t \in [0,1]; \\ \mathbf{T}(s,t,\cdot)_*\omega_s &= \omega_t(\cdot) & \text{for all } s, t \in [0,1]. \end{cases}$$

In general, we cannot associate to every absolutely continuous curve ω a flow $\mathbf{T}$ which is Lipschitz in the space variable (cf. Example 4); so to describe the results, we introduce the notion of regular curves ([AG], Definition 5.1).

Definition 4. *Let ω an absolutely continuous curve on $\mathcal{P}([0,1])$ with associated tangent velocity field v, $v_t \in L^2([0,1],\omega_t)$. We call ω regular, if*

$$\int_0^1 \mathrm{Lip}(v_t)\,dt < \infty.$$

If a curve ω is regular, then there exists a flow $\mathbf{T}$ induced by the tangent velocity field v of ω, that pushes the probability measure ω_0 forward along the curve ω (cf. [AG], Section 5). Regularity is in general not fulfilled, as pointed out in Example 4. As shown there, we have also no direct relationship between the Lipschitz properties (in x) of $\dot{g}$ and v. But we can at least formulate a sufficient condition for regularity:

Proposition 15. *Suppose that $\dot{g}_t$ and g_t^{-1} are Lipschitz in x for almost every t and that it holds for the respective Lipschitz constants*

$$\int_0^1 \mathrm{Lip}\,(\dot{g}_t) \cdot \mathrm{Lip}\,(g_t^{-1})\,dt < \infty.$$

Then the absolutely continuous curve ω, that solves the continuity equation for v, $v_t = \dot{g}_t \circ g_t^{-1}$, is regular.

Proof: The proof is straight forward,

$$\begin{aligned}|v_t(x_1) - v_t(x_2)| =& \left|\left(\dot{g}_t \circ g_t^{-1}\right)(x_1) - \left(\dot{g}_t \circ g_t^{-1}\right)(x_2)\right| \\ \leq& \mathrm{Lip}\,(\dot{g}_t) \cdot |g_t^{-1}(x_1) - g_t^{-1}(x_2)| \leq \mathrm{Lip}\,(\dot{g}_t) \cdot \mathrm{Lip}\,(g_t^{-1}) \cdot |x_1 - x_2|,\end{aligned}$$

which implies the integrability condition. ∎

Moreover, Ambrosio and Gigli [AG] pointed out, that the set of regular curves on $\mathcal{P}([0,1])$ is dense in the set of absolutely continuous curves. More precisely: Given an absolutely continuous curve ω with tangent velocity field v, we can construct a sequence of regular curves ω^n with associated tangent velocity fields v^n, such that for $n \to \infty$ both $\sup_{t\in[0,1]} d_W(\omega_t^n, \omega_t) \to 0$ and $\|v_t^n\|_{L^2(\omega_t^n)} \to \|v_t\|_{L^2(\omega_t)}$. For details on this rather tricky approximation result we refer to [AG], Section 6.

Part III

Appendices

Appendix A

Backward Martingales and the Lyons-Zheng Decomposition

A.1 Backward Martingales

In the following we consider a probability space $(\Omega, \mathcal{F}, P)$ and a symmetric diffusion process $(X_t)_{t\in[0,T]}$ on some Polish space $\mathcal{X}$ as introduced in Chapter 1. We define the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ by $\mathcal{F}_t = \sigma(X_s,\, 0 \le s \le t)$ and the *backward filtration* $(\hat{\mathcal{F}}_t)_{t\in[0,T]}$ by $\hat{\mathcal{F}}_t := \sigma(X_{T-s},\, 0 \le s \le t)$.

Definition 5. *A martingale with respect to the filtration $(\hat{\mathcal{F}}_t)$ we will call a* backward martingale*, while a "classical" martingale (hence with respect to the original filtration $(\mathcal{F}_t)$) we will call* forward martingale*. More precisely: $(\hat{M}_t)$ is a backward martingale, if it is an integrable, $(\hat{\mathcal{F}}_t)$-adapted stochastic process such that*

$$E[\hat{M}_t \,|\, \hat{\mathcal{F}}_s] = \hat{M}_s \qquad \text{for all } 0 \le s \le t \le T$$

holds.

Note that this definition of a backward martingale is in accordance with that of Lyons and Zhang [LZha], in particular it is a special case of the two-parameter backward martingale of Kunita (cf. [Kun], pp. 111f.) with the second parameter fixed at T. Next we will assemble some results on the Itô-calculus for backward martingales.

Definition 6. *For a given backward martingale $(\hat{M}_t)$ and some continuous function f, we define the* stochastic backward integral *with respect to $(\hat{M}_t)$ as L^2-limes along a refining sequence (ζ_n) of partitions of $[0,T]$ with mesh tending to zero,*

$$\int_0^t f(\hat{M}_s)\, d\hat{M}_s := \lim_{n\to\infty} \sum_{\substack{\tau_i \in \zeta_n \\ \tau_i \le t}} f(\hat{M}_{\tau_i})(\hat{M}_{\tau_{i+1}} - \hat{M}_{\tau_i}).$$

Of course this definition generalizes straightforward to integrals with respect to a local backward martingale by stopping along an increasing sequence of stopping times diverging to infinity almost surely. Remark that by definition the stochastic integral is anticipating with respect to the filtration $(\mathcal{F})$, but it is non-anticipating for our backward filtration $(\hat{\mathcal{F}}_t)$. This is also in accordance with the definition of Kunita [Kun] who defines the backward integral as

$$\int_r^T f(\hat{M}_s)\, d\hat{M}_s := \lim_{n\to\infty} \sum_{\substack{t_i\in\zeta_n\\ t_i\geq r}} f(\hat{M}_{t_{i+1}})(\hat{M}_{t_{i+1}} - \hat{M}_{t_i})$$

for the refining sequences $\tilde{\zeta}_n = \{0 = t_0 < t_1 < \cdots < t_{n-1} = t_n = T\}$ (although requiring only convergence in probability!). This on the one hand from

$$\int_0^t f(\hat{M}_s)\, d\hat{M}_s = -\int_{T-t}^T f(\hat{M}_s)\, d\hat{M}_s$$

and on the other, setting for the partition $\tilde{\zeta}_n$ $t_i := T - \tau_{n-i}$

$$\begin{aligned}\lim_{n\to\infty} \sum_{\substack{\tau_i\in\zeta_n\\ \tau_{i+1}\leq t}} f(\hat{M}_{\tau_i})(\hat{M}_{\tau_{i+1}} - \hat{M}_{\tau_i}) &= \lim_{n\to\infty} \sum_{\substack{t_i\in\zeta_n\\ t_{n-i}\geq T-t}} f(\hat{M}_{T-t_{n-i-1}})(\hat{M}_{T-t_{n-i-1}} - \hat{M}_{T-t_{n-i}})\\ &= -\lim_{n\to\infty} \sum_{\substack{t_i\in\zeta_n\\ t_{n-i}\geq T-t}} f(\hat{M}_{T-t_{n-i-1}})(\hat{M}_{T-t_{n-i}} - \hat{M}_{T-t_{n-i-1}}).\end{aligned}$$

Since we are thus in the framework of classical Itô-calculus, we can use straightforward the classical proofs of the following statements

Lemma 15. *For every continuous function f the stochastic backward integral is a local martingale,*

$$E\left[\int_0^t f(\hat{M}_r)\, d\hat{M}_r \,\middle|\, \hat{\mathcal{F}}_s\right] = \int_0^s f(\hat{M}_r)\, d\hat{M}_r \qquad 0\leq s\leq t\leq T.$$

Lemma 16. *The backward Itô formula reads for $g\in C^2(\mathbb{R})$ and a $(\hat{\mathcal{F}}_t)$-semimartingale $\hat{S} = \hat{M} + \hat{A}$*

$$g(\hat{S}_t) = g(\hat{S}_0) + \int_0^t g'(\hat{S}_s)\, d\hat{S}_s + \frac{1}{2}\int_0^t g''(\hat{S}_s)\ d\langle\hat{M}\rangle_s.$$

Lemma 17. *The* backward stochastic exponential $\exp\left(\hat{M}_t - \frac{1}{2}\langle\hat{M}\rangle_t\right)$ *is a local backward martingale.*

Lemma 18. *For the backward stochastic exponential we have*

$$E\big[\exp\big(\hat{M}_t - \frac{1}{2}\langle\hat{M}\rangle_t\big)\big] \leq 1.$$

Lemma 19. *Suppose that*

$$E\big[\exp\big(\hat{M}_t - \frac{1}{2}\langle\hat{M}\rangle_t\big)\big] < \infty.$$

Then $\exp\left(\hat{M}_t - \frac{1}{2}\langle\hat{M}\rangle_t\right)$ *is a uniformly integrable backward martingale, in particular*

$$E\big[\exp\big(\hat{M}_t - \frac{1}{2}\langle\hat{M}\rangle_t\big)\big] = 1.$$

A.2 Lyons-Zheng Decomposition

In the following we want to present the Lyons-Zheng (cf. [LZhe]) forward-backward martingale decomposition, relying on the approach given by Lyons and Zhang ([LZha]). We will remain in the setting laid out in Chapter 1.

Theorem 5. *For every continuous $f \in D(\mathcal{E})$ there exist a continuous forward P_μ-martingale M_t^f and a continuous backward P_μ-martingale $\hat{M}_t^f$, both square integrable with respect to P_μ and with $M_0^f = \hat{M}_0^f = 0$, such that*

$$f(X_t) - f(X_0) = \frac{1}{2}(M_t^f + \hat{M}_{T-t}^f - \hat{M}_T^f). \tag{A.1}$$

Proof: In a first step we restrict ourselves to the case $f \in D(L)$. Then it is clear that

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds, \qquad t \in [0,T],$$

is a square-integrable (forward) P_μ martingale. But since the operator L is self-adjoint, it follows that X_{T-t} has under P_μ the same law as X_t and hence

$$\begin{aligned}\hat{M}_t^f :=& f(X_{T-t}) - f(X_T) - \int_0^t Lf(X_s)\,ds \\ =& f(X_{T-t}) - f(X_T) - \int_0^t Lf(X_{T-s})\,ds \\ =& f(X_{T-t}) - f(X_T) - \int_{T-t}^T Lf(X_s)\,ds, \qquad t \in [0,T],\end{aligned}$$

is a square-integrable backward P_μ-martingale and $\langle \hat{M}^f \rangle_T = \langle M^f \rangle_T$. Putting these equations together, this yields

$$M_t^f + \hat{M}_{T-t}^f - \hat{M}_T^f = 2(f(X_t) - f(X_0))$$

and hence (A.1).

The extension to the general case $f \in D(\mathcal{E})$ relies on the isometry

$$\begin{aligned}E_\mu\big[\langle \hat{M}^f \rangle_T\big] =& E_\mu\big[\langle M^f \rangle_T\big] = E_\mu\Big[\int_0^T \Gamma(f)(X_s)\,ds\Big] = \int_0^T E_\mu[\Gamma(f)(X_s)]\,ds \\ =& \int_0^T \int_{\mathcal{X}} \Gamma(f)\,d\mu\,ds = 2T\mathcal{E}(f).\end{aligned} \tag{A.2}$$

For $f \in D(\mathcal{E})$ we can choose a sequence $f_n \in D(L)$ in such a way that

$$\mathcal{E}(f - f_n) + \int_{\mathcal{X}} |f - f_n|^2\,d\mu \to 0.$$

Since the f_n have $\mathcal{E}$-quasi-continuous μ-versions $\tilde{f}_n$, there exist by [MR], Proposition III.3.5 a subsequence (f_{n_k}) such that $(\tilde{f}_{n_k})$ converges $\mathcal{E}$-quasi uniformly to $\tilde{f}$, an $\mathcal{E}$-quasi-continuous μ-versions of f.

Since we have by the quasi-regularity of the Dirichlet form on the other hand for the diffusion process X_t and τ_N its first hitting time of the $\mathcal{E}$-exceptional set N, we have $P_x[\tau_N < \infty] = 0$ for $\mathcal{E}$-quasi every $x \in \mathcal{X}$ (cf. [MR], Proposition IV.5.30), we can conclude that for the subsequence (f_{n_k}) we have for $\mathcal{E}$-quasi-every $x \in \mathcal{X}$.

$$P_x[f_{n_k}(X_t) \text{ converges uniformly in } t \text{ on every compact interval of } [0,\infty[\,] = 1.$$

Now we define $M_t^f = \lim_{n\to\infty} M_t^{f_n}$ and, analogously, $\hat{M}_t^f = \lim_{n\to\infty} \hat{M}_t^{f_n}$ as $L^2(\Omega, P_\mu)$-limits via the isometry (A.2). Since, of course, also $f_n(X_t)$ converges almost surely to $f(X_t)$ for each $t \in [0,T]$, the decomposition extends to all continuous functions in the domain of the Dirichlet form. ■

Remark 4. *We note that the result in [LZha] is even more precise: For the above constructed decomposition it holds that A given by*

$$A_t := M_t^f - \hat{M}_{T-t}^f + \hat{M}_T^f$$

is a continuous additive functional of zero energy, i.e.

$$e(A) := \lim_{t\to 0} \frac{1}{2t} E_\mu[A_t^2] = 0.$$

Moreover, requiring that A is a continuous additive functional of zero energy, the decomposition is unique.

Appendix B

Projective Limits and their Large Deviations

B.1 Contraction Principles

For convenience we note here some results on the transformation of large deviation principles under continuous mappings, proofs can be found in [DZ], Section 4.2. We start with the contraction principle.

Proposition 16. *Suppose we have given two topological Hausdorff spaces $\mathcal{X}$, $\mathcal{Y}$ and a continuous function $f : \mathcal{X} \to \mathcal{Y}$. Suppose further that the family of probability measures $(\mu_\varepsilon)_{\varepsilon \geq 0}$ satisfies a large deviation principle on $\mathcal{X}$ with good rate function I. Then the family of probability measures $(\mu_\varepsilon \circ f^{-1})_{\varepsilon \geq 0}$ on $\mathcal{Y}$ satisfies a large deviation principle with good rate function*

$$\tilde{I}(y) = \inf_{x \in f^{-1}(\{y\})} I(x), \qquad y \in \mathcal{Y}.$$

In the case that the function is a bijection, we can derive also the so called inverse contraction principle.

Proposition 17. *Let $\mathcal{X}$, $\mathcal{Y}$ two topological Hausdorff spaces and $g : \mathcal{Y} \to \mathcal{X}$ a continuous bijection. If $(\nu_\varepsilon)_{\varepsilon \geq 0}$ is a family of exponentially tight probability measures on $\mathcal{Y}$ such that $(\nu_\varepsilon \circ g^{-1})_{\varepsilon \geq 0}$ satisfies a large deviation principle on $\mathcal{X}$ with rate function I, then the family $(\nu_\varepsilon)_{\varepsilon \geq 0}$ satisfies a large deviation principle on $\mathcal{Y}$ with good rate function*

$$\tilde{I}(y) = I(g(y)), \qquad y \in \mathcal{Y}.$$

An easy corollary of this theorem is the following possibility to strengthen a large deviation principle to a finer topology.

Corollary 4. *Suppose we have given two Hausdorff topologies σ, τ on the topological Hausdorff space $\mathcal{X}$, τ finer then σ. If the family of probability measures $(\mu_\varepsilon)_{\varepsilon \geq 0}$ on $\mathcal{X}$ is exponentially tight with respect* *to τ and obeys a large deviation principle with respect to σ, then it satisfies also a large deviation principle on $\mathcal{X}$ with respect to τ.*

B.2 Projective Limits

Let $(J, \leq)$ a partially ordered, right filtering (i.e. for i, $j \in J$ there exists $k \in J$ with $i \leq k$ and $j \leq k$) index set such that for every $i \in J$ we have a topological space $\mathcal{X}_i$ given in such a way that for i, $j \in J$, $i \leq j$ a continuous mapping $\pi_i^j : \mathcal{X}_j \to \mathcal{X}_i$ is defined. Furthermore we require the following consistency: Given i, j, $k \in J$ with $i \leq j \leq k$, we have $\pi_i^j \circ \pi_j^k = \pi_i^k$ and for every $i \in J$ the mapping π_i^i is given by the identity map on $\mathcal{X}_i$. The family $(\mathcal{X}_i, \pi_i^j)_{i,\, j\in J}$ is called the *inverse system of the spaces* $\mathcal{X}_i$ *under the bonding maps* π_i^j. The subspace of $\prod_{i\in J} \mathcal{X}_i$ consisting of those elements (called *threads*) $\{x_i\}_{i\in J} \in \prod_{i\in J} \mathcal{X}_i$, that satisfy $\pi_i^j(x_j) = x_i$ for any i, $j \in J$, $i \leq j$, is called the *limit of the inverse system* $(\mathcal{X}_i, \pi_i^j)_{i,\, j\in J}$ and is denoted by $\varprojlim(\mathcal{X}_i, \pi_i^j)_{i,\, j\in J}$ or shortly $\varprojlim \mathcal{X}_i$. The canonical topology on $\varprojlim \mathcal{X}_i$ is the topology induced by the product topology on $\prod_{i\in J} \mathcal{X}_i$.
Note that the limit of an inverse system of topological Hausdorff spaces $\mathcal{X}_i$ is a closed subset of the Cartesian product $\prod_{i\in J} \mathcal{X}_i$ and as such Hausdorff itself. In the case that π_i is the projection $\pi_i : \prod_{j\in J} \mathcal{X}_j \to \mathcal{X}_i$, we denote by a slight abuse of notation the continuous mapping $\pi_i|_{\varprojlim \mathcal{X}_j} : \varprojlim \mathcal{X}_j \to \mathcal{X}_i$ also by π_i. For any i, $j \in J$, $i \leq j$, the projections π_i and π_j are consistent in the sense that $\pi_i = \pi_i^j \circ \pi_j$. The mappings π_i are called *projections of the limit of* $(\mathcal{X}_i, \pi_i^j)_{i,\, j\in J}$ *to* $\mathcal{X}_i$ and $\varprojlim \mathcal{X}_j$ the *projective limit*. A basis of the the projective limit is given by a family of open subsets $\pi_j^{-1}(U_j)$ where U_j is an open subset of $\mathcal{X}_j$ and j ranges over the cofinal subsets $J' \subseteq J$ (this means that there exists a $k \in J'$ such that $j \leq k$ for any $j \in J$).
Every closed subset F of $\varprojlim \mathcal{X}_i$ is itself a projective limit, indeed the system $(\overline{F}_j, \tilde{\pi}_i^j)$ with $F_j := \pi_j(F)$, $\tilde{\pi}_i^j := \pi_i^j|_{F_j}$ is an inverse system with projective limit $\varprojlim \overline{F}_j = F$. Moreover, a projective limit of non-empty, compact sets is non-empty. Given a topological property $\mathfrak{P}$ that is hereditary with respect to closed subsets and finitely multiplicative. Then a topological vector space $\mathcal{X}$ is homeomorphic to a projective limit of topological Hausdorff spaces with property $\mathfrak{P}$ exactly iff $\mathcal{X}$ is homeomorphic to a closed subspace of a Cartesian product of topological Hausdorff spaces with property $\mathfrak{P}$. For proofs of these statements we refer to Engelking ([E], pp. 98ff.).

B.3 Large Deviations for Projective Limits

Projective limits as tool to lift large deviation principles from projections to the whole space were introduced by Dawson and Gärtner ([DG], Section 3.3; see also [DZ], Section 4.6). Here the spaces $\mathcal{X}_i$ are topological Hausdorff spaces and we require that the projective limit $\mathcal{X} := \varprojlim \mathcal{X}_i$ is not empty. Suppose now that $(\mu_\varepsilon)_{\varepsilon\geq 0}$ is a family of probability measures on $\mathcal{X}$ and define $\mu_\varepsilon^i := \mu_\varepsilon \circ \pi_i^{-1}$, so μ_ε can be regarded as kind of projective limit of the probability measures μ_ε^i.

Theorem 6. *The family $(\mu_\varepsilon)_{\varepsilon\geq 0}$ satisfies a large deviation principle on $\mathcal{X}$ with with speed $1/\varepsilon$ and* *good rate function I exactly if for every $i \in J$, the family $(\mu_\varepsilon^i)_{\varepsilon \geq 0}$ satisfies a large deviation principle on $\mathcal{X}_i$ with speed $1/\varepsilon$ and good rate function I_i. Moreover, the rate functions are related by*

$$\begin{aligned} I(x) &= \sup_{i \in J} I_i\big(\pi_i(x)\big), \qquad && x \in \mathcal{X}; \\ I_i(z) &= \inf_{x \in \pi_i^{-1}(\{z\})} I(x), \qquad && z \in \mathcal{X}_i, i \in J. \end{aligned}$$

Proof: That the large deviation principle on the projective limit implies the large deviation principles on the spaces $\mathcal{X}_i$ with good rate function I_i is a direct consequence of the contraction principle Proposition 16.
Assume now that the large deviation principles on the spaces $\mathcal{X}_i$ hold with good rate functions I_i and the function I is given as above. First we want to show that I is a good rate function; therefore we define for $\alpha \in [0, \infty[$ the level sets

$$\Psi_{I_i}(\alpha) := \{x_i \in \mathcal{X}_i \,:\, I(x_i) \leq \alpha\}, \qquad \Psi_I(\alpha) := \{x \in \mathcal{X} \,:\, I(x) \leq \alpha\}.$$

Since the mappings π_i^j are continuous and consistent with the families $(\mu_\varepsilon^i)_{\varepsilon \geq 0}$ in the sense that $\mu_\varepsilon^i = \pi_i^j \circ \mu_\varepsilon^j$ for $i \leq j$, we can write the rate function I_i as

$$I_i(z) = \inf_{y_j \in (\pi_i^j)^{-1}(\{z\})} I(y_j), \qquad z \in \mathcal{X}_i, i \leq j,$$

again by contraction principle. So it follows for the level sets that $\Psi_{I_i}(\alpha) = \pi_i^j\big(\Psi_{I_j}(\alpha)\big)$ and hence $(\Psi_{I_i}(\alpha), \pi_i^j)_{i,j \in J}$ is a projective system with projective limit

$$\Psi_I(\alpha) = \varprojlim \Psi_{I_i}(\alpha) = \varprojlim \Big(\mathcal{X}_i \cap \prod_{i \in J} \Psi_{I_i}(\alpha)\Big).$$

But as product of compact subsets, the set $\prod_{i \in J} \Psi_{I_i}(\alpha)$ is compact by Tychonov's theorem and so is $\Psi_I(\alpha)$ as closed relative compact set in a topological Hausdorff space. So the rate function I is good.
Since the rate function I is good, it is sufficient to show that for any measurable set $A \subseteq \mathcal{X}$ and any $x \in A^\circ$

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) \geq -I(x)$$

to prove the lower bound. Note that a basis of the topology of $\mathcal{X}$ is given by the collection $\{\pi_j^{-1}(U_j) : U_j \subseteq \mathcal{X}_j,\, U_j \text{ open},\, j \in J',\, J' \subseteq J \text{ cofinal}\}$ (see above). So there exists some $j \in J$ and some $U_j \subseteq \mathcal{X}_j$ such that $\pi_j^{-1}(U_j) \subseteq A^\circ$ and we get by the lower bound on $\mathcal{X}_j$

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) \geq \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon^j(U_j) \geq -\inf_{y \in U_j} I_j(y) \geq -I_j\big(\pi_j(x)\big) \geq -I(x).$$

Finally, to prove the upper bound, we note that the upper bound holds trivially if $\inf_{x \in F} I(x) = 0$ for some non-empty, closed set $F \subseteq \mathcal{X}$. Otherwise we can find some $\alpha \in [0, \infty[$, $\alpha < \inf_{x \in F} I(x)$, such that $F \cap \Psi_I(\alpha) = \emptyset$. Since F is closed, we have for $F_i := \pi_i(F)$ the representation $F = \varprojlim \overline{F}_i$, and since moreover $\Psi_I(\alpha) = \varprojlim \Psi_{I_i}(\alpha)$ it follows that

$$F \cap \Psi_I(\alpha) = \varprojlim \big(\overline{F}_i \cap \Psi_{I_i}(\alpha)\big).$$

Our assumption that $F \cap \Psi_I(\alpha) = \emptyset$ implies the existence of some $i \in J$ with $\overline{F}_i \cap \Psi_{I_i}(\alpha) = \emptyset$, since the projective limit of non-empty compact sets would be non-empty (cf. [E], Theorem 3.2.13). So the upper bound on $\mathcal{X}_i$ implies

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(F) = \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon^i(\overline{F}_i) \leq -\alpha$$

which conversely implies the upper bound on $\mathcal{X}$ since I is a good rate function. ■

The following version of the large deviation principle for projective limits by Schied (cf. [Sch96], Lemma 19) is in particular useful for our setting since it is based on the notion of exponential tightness.

Corollary 5. *Suppose we have given a topological Hausdorff space E and a family of mappings $(p_i)_{i \in J}$, $p_i : E \to \mathcal{X}_i$ that is point-separating on E and consistent with π_i^j in the sense that $p_i = \pi_i^j \circ p_j$ for i, $j \in J$ and $i \leq j$. If (μ_ε) is an exponentially tight family of probability measures on E, such that for every $i \in J$, $(\mu_\varepsilon^i)_{\varepsilon \geq 0}$, $\mu_\varepsilon^i := \mu_\varepsilon \circ p_i$, satisfies a large deviation principle on $\mathcal{X}_i$ with speed $1/\varepsilon$ and rate function I_i, then $(\mu_\varepsilon)_{\varepsilon \geq 0}$ satisfies a large deviation principle on E with speed $1/\varepsilon$ and rate function*

$$I(x) = \sup_{i \in J} I_i\big(p_i(x)\big), \qquad x \in E.$$

Proof: Since the set $P(y) := \{x \in \prod_{i \in J} : x = \pi_i^{-1} \circ p_i(y),\, y \in E\}$ is well defined and not dependent on i by the consistency of p_i and π_i with π_i^j, the space E is homeomorphic to a subset of $\prod_{i \in J} \mathcal{X}_i$ and we can find a continuous embedding $\iota : E \to \iota E \subseteq \mathcal{X}$ of E into the projective limit $\mathcal{X} = \varprojlim \mathcal{X}_i$. Since the family of measures (μ_ε) is exponentially tight on E, it is also exponentially tight on $\mathcal{X}$ and so a fortiori the family (μ_ε^i) on $\mathcal{X}_i$. So the large deviation principle on $\mathcal{X}_i$ holds with rate function I_i. Moreover this rate function is good: Fix some $\alpha \in [0, \infty[$, so exponential tightness implies the existence of a compact set K_α with $\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(K_\alpha^c) < -\alpha$ and hence $\Psi_{I_i}(\alpha) \subseteq K_\alpha$ which yields the compactness of the level set. Theorem 6 implies also a large deviation principle on $\mathcal{X}$ with good rate function

$$\tilde{I}(\tilde{x}) = \sup_{i \in J} I_i\big(\pi_i(\tilde{x})\big), \qquad \tilde{x} \in \mathcal{X}.$$

Fix now some $y \in \mathcal{X} \setminus \iota E$, then $\tilde{I}(y) = \infty$ by the exponential tightness of (μ_ε). So for any measurable set $A \subseteq \mathcal{X}$ we have

$$\inf_{x \in A} \tilde{I}(x) = \inf_{x \in A \cap \iota E} \tilde{I}(x)$$

and so both bounds, upper and lower, hold for all measurable subsets of ιE. But since the level sets $\Psi_{\tilde{I}}(\alpha)$ are closed subsets of ιE, the rate function remains also lower semi-continuous when restricted to ιE. So the large deviation principle on ιE holds with rate function

$$\tilde{I}(\tilde{x}) = \sup_{i \in J} \tilde{I}_i\big(\pi_i(\tilde{x})\big), \qquad \tilde{x} \in \iota E,$$

with respect to the topology induced by the product topology. But Corollary 4 enables us to strengthen the topology to the topology induced by the original topology on E via ι and hence a large deviations principle holds on E with good rate function

$$I(x) = \sup_{i \in J} I_i\big(p_i(x)\big) = \sup_{i \in J} \tilde{I}_i\big(\pi_i(\iota x)\big) = \tilde{I}(\iota x), \qquad x \in E,$$

with respect to the original topology on E. ■

Appendix C

Preliminary results for the lower bound

Till now we have no complete prove for the lower bound of the large deviation principle, but we will recollect yet a lemma which will play a prominent role therein: an estimate in the spirit ofLemma 6, which gives a bound for the second moment of the Doléans-Dade exponentials.

Lemma 20. *For every $\vec{u} \in D(\mathcal{E})^n$ with $\Gamma(u_i)$ μ-a.e. bounded it holds that*

$$E_\mu \left[\exp\left(\frac{2}{\varepsilon} \mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon) \right) \right] \leq \exp\left(\frac{G(\vec{u})}{\varepsilon} \right). \tag{C.1}$$

with $G(\vec{u}) := \max_{1 \leq i \leq n} \operatorname{ess\,sup}_{x \in \mathcal{X}} 1/t_n \Gamma(u_i)$. Moreover, we have thus

$$\lim_{\varepsilon \to 0} \varepsilon \log E_\mu \left[\exp\left(\frac{2}{\varepsilon} \mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon) \right) \right] \leq G(\vec{u}). \tag{C.2}$$

Proof: We start by giving a moment estimate for the stochastic exponential Since the exponential satisfies the stochastic differential equation

$$\exp\left(\frac{1}{\varepsilon} M_{t_n}^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_{t_n} \right) = 1 + \int_0^{t_n} \exp\left(\frac{1}{\varepsilon} M_s^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_s \right) d\left(\frac{1}{\varepsilon} M_s^\varepsilon\right)$$

we can derive that

$$\begin{aligned}
E_\mu \left[\left(\exp\left(\frac{1}{\varepsilon} M_{t_n}^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_{t_n} \right) \right)^2 \right] &= E_\mu \left[\left\langle \exp\left(\frac{1}{\varepsilon} M^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle \right) \right\rangle_{t_n} \right] \\
&= E_\mu \left[\left\langle \int_0^{\cdot} \exp\left(\frac{1}{\varepsilon} M_s^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_s \right) d\left(\frac{1}{\varepsilon} M_s^\varepsilon\right) \right\rangle_{t_n} \right] \\
&= E_\mu \left[\int_0^{t_n} \left(\exp\left(\frac{1}{\varepsilon} M_s^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_s \right) \right)^2 d\left\langle \frac{1}{\varepsilon} M^\varepsilon \right\rangle_s \right] \\
&= \frac{1}{\varepsilon^2} E_\mu \left[\int_0^{t_n} \left(\exp\left(\frac{1}{\varepsilon} M_s^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_s \right) \right)^2 d \langle M^\varepsilon \rangle_s \right] \\
&\leq \frac{G(\vec{u})}{\varepsilon} \int_0^{t_n} E_\mu \left[\left(\exp\left(\frac{1}{\varepsilon} M_s^\varepsilon - \frac{1}{2\varepsilon^2} \langle M^\varepsilon \rangle_s \right) \right)^2 \right] ds
\end{aligned}$$

by (3.1). Applying now Gronwall's inequality leads to

$$E_\mu\left[\left(\exp\left(\frac{1}{\varepsilon}M_s^\varepsilon - \frac{1}{2\varepsilon^2}\langle M^\varepsilon\rangle_s\right)\right)^2\right] \leq \exp\left(\frac{G(\vec{u})}{\varepsilon}\right),$$

such that we can conclude

$$\lim_{\varepsilon\to 0}\varepsilon\log E_\mu\left[\left(\exp\left(\frac{1}{\varepsilon}M_s^\varepsilon - \frac{1}{2\varepsilon^2}\langle M^\varepsilon\rangle_s\right)\right)^2\right] \leq G(\vec{u}).$$

The same reasoning applies of course for the stochastic backward exponential

$$\exp\left(1/\varepsilon M_{t_n}^\varepsilon - 1/(2\varepsilon^2)\langle M^\varepsilon\rangle_{t_n}\right).$$

Thus we can conclude as in Lemma 6 by Cauchy's inequality,

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon\log E_\mu\left[\exp\left(\frac{2}{\varepsilon}\mathcal{J}_\Delta^{\vec{u}}(X^\varepsilon)\right)\right] &\leq \lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\log E_\mu\left[\left(\exp\left(\frac{1}{\varepsilon}M_{t_n}^\varepsilon - \frac{1}{2\varepsilon^2}\langle M^\varepsilon\rangle_{t_n}\right)\right)^2\right]\\ &\quad + \lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\log E_\mu\left[\left(\exp\left(-\frac{1}{\varepsilon}\hat{M}_{t_n}^\varepsilon - \frac{1}{2\varepsilon^2}\langle \hat{M}^\varepsilon\rangle_{t_n}\right)\right)^2\right]\\ &\leq G(\vec{u}).\end{aligned}$$

■

Appendix D

Some Remarks on the Geometry Generated by Diffusions on σ-finite Measure Spaces

In this appendix we will give some remarks on the intrinsic geometry generated by diffusions on σ-finite measure spaces. Thus μ is no longer a probability measure but merely a σ-finite measure. This implies of course, that the Dirichlet form will be no longer conservative. As classical example one can think of Brownian motion on $\mathbb{R}^d$. Hence we will now consider quasi-regular, local Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ defined on a σ-finite measure space $(\mathcal{X}, \mathcal{B}, \mu)$ which admit a carré du champ operator.
A Varadhan type large deviation principle for local Dirichlet forms was proved (generalizing the work of Hino and Ramírez, [HR]) by Ariyoshi and Hino ([AH], compare also [ERS]). They insist, that also in the case we drop only conservativeness, another concept of distance is necessary, since, roughly speaking, the set $\mathcal{G}$ of Section 2.1. is to small. A convenient example herefore is Brownian motion on the interval $[-1, 1]$ killed in 0 (cf. [AH], Example 2.9.(i)).
The work of Ariyoshi and Hino is built up on the concept of a (topology free) measure theoretic nest.

Definition 7. *An increasing sequence (E_k) of measurable sets is called a measure theoretic nest, if*

i) for every $k \in \mathbb{N}$ there exists a function $\chi_{E_k} \in D(\mathcal{E})$ such that $\chi_{E_k} \geq 1$ μ-almost everywhere on E_k;

ii) the union of the sets

$$D(\mathcal{E})_{E_k} := \left\{ f \in D(\mathcal{E}) \,:\, f = 0 \ \mu\text{-a.e. on } E_k^c \right\}$$

is dense in $D(\mathcal{E})$ with respect to $\| \cdot \|_{\mathcal{E}}$.

In particular we can choose the function χ_{E_k} in such a way that $0 \leq \chi_{E_k} \leq 1$, since $0 \vee \chi_{E_k} \wedge 1$ is also in $D(\mathcal{E})$ and satisfies $i)$. We define the domain of local Dirichlet forms $D_{loc}^{(E_k)}(\mathcal{E})$ with respect to a given measure-theoretic nest (E_k) as the set of all measurable functions $u \in L^0(\mathcal{X}, \mu)$, that there

exists a sequence $(u_k)_k$ such that $u_k \in D(\mathcal{E})$ and $u_k = u$ μ-almost everywhere on E_k. The set of all such functions that are bounded will be denoted by $D_{loc,b}^{(E_k)}(\mathcal{E})$.
For u, $h \in D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu)$ we define the functional

$$I_u(h) := 2\mathcal{E}(uh, u) - \mathcal{E}(h, u^2).$$

The locality of $(\mathcal{E}, D(\mathcal{E}))$ implies that $I_u(h) = 0$ whenever $(a+h)u = 0$ μ-almost everywhere for some $a \in \mathbb{R}$. So we can extend the definition of $I_u(h)$ to $u \in D(\mathcal{E})_{loc,b}^{(E_k)}$ and $h \in D(\mathcal{E})_{E_k} \cap L^\infty(\mathcal{X}, \mu)$ for some E_k by setting $I_u(h) = I_{u_k}(h)$ where $u_k \in D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu)$ and $u_k = u$ μ-almost everywhere on E_k. We can hence define the set

$$\breve{\mathcal{G}} := \Big\{ u \in D_{loc,b}^{(E_k)}(\mathcal{E}) \,:\, I_u(h) \le \|h\|_{L^1(\mu)}, \text{ for all } h \in \bigcup_{k=1}^\infty \Big(D_{E_k}(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu) \Big) \Big\}$$

and the metric functional $\breve{d}$ as

$$\breve{d}(A, B) := \sup_{u \in \breve{\mathcal{G}}} \big(\operatorname*{ess\,inf}_{x \in B} u(x) - \operatorname*{ess\,sup}_{y \in A} u(y) \big).$$

Note that the definition of $\breve{\mathcal{G}}$ (and hence $\breve{d}$) does not depend on the particular choice of the measure-theoretic nest (E_k) (cf. [AH], Proposition 3.9).
As in Chapter 2, we can associate a set distance $\breve{d}_A$ to the metric functional $\breve{d}$. We define analogously for $A \in \mathcal{B}$ and $M \ge 0$

$$\breve{V}_A^M := \big\{ u \in \breve{\mathcal{G}} \,:\, u\big|_A = 0,\, 0 \le u \le M \,\mu\text{-a.e.} \big\}$$

and

$$\breve{d}_A(x) := \lim_{M \to \infty} \operatorname*{ess\,sup}_{u \in \breve{V}_A^M} u(x).$$

Again $\breve{d}_A \wedge M$ is the almost everywhere maximal element of $\breve{V}_A^M$ and

$$\breve{d}(A, B) = \operatorname*{ess\,inf}_{x \in B} \breve{d}_A(x)$$

(cf. [AH], Proposition 3.11).
The main result of the article by Ariyoshi and Hino([AH], Theorem 2.7.) is that this metric is the correct generalization of d, so it holds for all sets A, $B \in \mathcal{X}$ with positive, but finite measure that

$$\lim_{t \to 0} t \log P_t(A, B) = -\frac{\breve{d}(A, B)^2}{2}.$$

In accordance with the definition of $\breve{\mathcal{G}}$ we will also relax Assumption (BC), requiring only

Assumption (BC'). *There exists a countable family $\breve{\mathcal{U}} = (u_n)$, $u_n \in L^0(\mathcal{X}, \mu)$, point-separating on $\mathcal{X} \setminus N$, such that $u_n = v_n^k$ μ-almost everywhere on E_k for functions $v_n^k \in D(\mathcal{E})$ with carré du champ uniformly bounded on E_k, i.e.*

$$\sup_{k \in \mathbb{N}} \operatorname*{ess\,sup}_{x \in E_k} \Gamma(v_n^k)(x) < \infty.$$

In the same spirit we can weaken Assumption (B'), where $\breve{\rho}$ is the generalized pointwise distance defined later on:

Assumption (B"). *For all sets A of positive measure and closed with respect to the original topology and all $M > 0$ it holds that $\breve{\rho}_A \wedge M \in \breve{\mathcal{G}}'$ and $\breve{\rho}_A \wedge M$ is lower semi-continuous.*

Moreover, we will require, of course, Assumption(D).

D.1 Generalized Pointwise Distance

The first task in generalizing the small time large deviation principle to this setting is to establish the connection to the pointwise distance. To do so, we need first the correct definitions of quasi-regularity (and hence capacity and compact $\mathcal{E}$-nest) in this case (cf. [MR], Chapter III and [AM91]). Let $(G_\alpha)_{\alpha>0}$ denote the resolvent of the semigroup (T_t). We define the set

$$\mathbb{H} = \{h \in D(\mathcal{E}) \,:\, h = G_1 f,\, f \in L^2(\mathcal{X}, \mu), 0 < f \leq 1\, \mu\text{-a.e.}\}.$$

The h-weighted capacity $\mathrm{Cap_h}(\cdot)$ is given for $h \in \mathbb{H}$, $U \subseteq \mathcal{X}$, U open, by

$$\mathrm{Cap_h}(U) := \inf\{\|w\|_{\mathcal{E}} \,:\, w \in D(\mathcal{E}),\, w \geq h\ \mu\text{-a.e. on } U\},$$

and for general $A \subseteq \mathcal{X}$ by

$$\mathrm{Cap_h}(A) := \inf\{\mathrm{Cap_h}(U) \,:\, A \subseteq U \subseteq \mathcal{X},\, U \text{ open}\}.$$

For h, $h' \in \mathbb{H}$, $h \leq h'$, it follows that $\mathrm{Cap_h} \leq \mathrm{Cap}_{h'}$ and in particular $\mathrm{Cap_h} \leq \mathrm{Cap_1}$ (but note that 1 has not to be in $\mathbb{H}$).
$\mathrm{Cap_h}$ is again a Choquet capacity and we will call an increasing sequence of closed sets (F_k) a closed $\mathcal{E}$-nest, if for some $h \in \mathbb{H}$ the capacities $\mathrm{Cap_h}(\mathcal{X} \setminus F_k)$ tend to zero for $k \to \infty$. Note that this implies the asymptotic vanishing of the capacities for every $h \in \mathbb{H}$. Moreover, by [MR], Theorem III.2.11, (see also [AM91], Proposition 1.5) this is equivalent to the fact that $\bigcup_{k=1}^{\infty} D_{E_k}(\mathcal{E})$ is $\|\cdot\|_{\mathcal{E}}$-dense in $D(\mathcal{E})$. With this definition of capacity we can define the notions compact $\mathcal{E}$-nest, $\mathcal{E}$-quasi-everywhere, $\mathcal{E}$-quasi-continuity and quasi-regularity as in Section 1. Note that for a compact $\mathcal{E}$-nest (E_k), also $\mu(\mathcal{X} \setminus \bigcup_{k=1}^{\infty} E_k) = 0$.
Now we have to join the different notions of nests, compact $\mathcal{E}$-nests and measure theoretic nests.

Lemma 21. *There exist measure theoretic nests which are also compact $\mathcal{E}$-nests. Moreover, every measure theoretic nest consisting of compacts is a compact $\mathcal{E}$-nest.*

Proof: Note that if (F_k) is a compact $\mathcal{E}$-nest, then it satisfies condition *ii*) of the definition of the measure theoretic nest. Fix now $f \in L^2(\mathcal{X}, \mu)$ and define $h = G_1 f$, then (E_k) defined by $E_k := \{h \geq 1/k\}$ is a measure theoretic nest by [AH], Lemma 3.1. In particular the sets E_k are closed. Take now an arbitrary compact $\mathcal{E}$-nest (F_k) and define $E'_k := E_k \cap F_k$. Then (E'_k) is also a compact $\mathcal{E}$-nest, since E'_k is as closed, relatively compact set compact itself and

$$\mathrm{Cap_h}({E'_k}^c) = \mathrm{Cap_h}(E_k^c \cup F_k^c) \leq \mathrm{Cap_h}(E_k^c) + \mathrm{Cap_h}(F_k^c)$$

by the subadditivity of the capacity. But for $k \to \infty$ we have $\mathrm{Cap_h}(E_k^c) \to 0$ by definition and $\mathrm{Cap_h}(F_k^c) \to 0$ by the above characterization of closed $\mathcal{E}$-nests. Moreover, $\chi_{E_k} \geq 1$ μ-almost everywhere on $E'_k \subseteq E_k$ trivially, so (E'_k) is also a measure theoretic nest. ■

So from now on, if we mention a compact $\mathcal{E}$-nest, we assume implicitly that it is also a measure theoretic nest. We define for a given compact $\mathcal{E}$-nest (E_k)

$$\breve{\mathcal{G}}' := \left\{ u \in D_{loc}^{(E_k)}(\mathcal{E}) \,:\, \forall k\, \exists u_k \in D(\mathcal{E}),\ \Gamma(u_k)|_{E_k} \leq 1\, \mu\text{-a.e.},\ u = u_k\, \mu\text{-a.e. on } E_k \right\}.$$

Lemma 22. *The metric functional $\breve{d}$ coincides with the metric functional $\breve{d}'$ given by*

$$\breve{d}'(A,B) := \sup_{u \in \breve{\mathcal{G}}'} \left(\operatorname*{ess\,inf}_{x\in B} u(x) - \operatorname*{ess\,sup}_{y \in A} u(y) \right).$$

Proof: As in the proof of Lemma 3, we can approximate elements of $\breve{\mathcal{G}}'$ by bounded ones, since with w_1 and w_2 also $w_1 \wedge w_2$ and $w_1 \vee w_2$ are elements of $\breve{\mathcal{G}}'$ (the proof of Lemma 2 remains true in our generalized setting): Given $v \in \breve{\mathcal{G}}'$, $v_k := (-k) \vee v \wedge k$, $k \in \mathbb{N}$, converges for $k \to \infty$ to v μ-almost everywhere and on every E_k also in $\|\cdot\|_{\mathcal{E}}$. So it is sufficient to prove

$$\breve{\mathcal{G}} = \left\{ u \in D_{loc,b}^{(E_k)}(\mathcal{E}) \,:\, \forall k\, \exists u_k \in D(\mathcal{E}),\ \Gamma(u_k)|_{E_k} \leq 1\, \mu\text{-a.e.},\ u = u_k\, \mu\text{-a.e. on } E_k \right\}. \tag{D.1}$$

Suppose now that u is a function in the right-hand set. Then for a given compact $\mathcal{E}$-nest, the carré du champ of the approximating functions u_k is bounded by 1 μ-almost everywhere on E_k and so it follows by the definition of the carré du champ operator in (1.1) that for every $h \in \bigcup_{k=1}^\infty \left(D_{E_k}(\mathcal{E}) \cap L^\infty(\mathcal{X},\mu) \right)$

$$I_{u_k}(h) = 2\mathcal{E}(u_k h, u_k) - \mathcal{E}(h, u_k^2) = \int_{\mathcal{X}} h \Gamma(u_k)\, d\mu \leq \|h\|_{L^1(\mu)}.$$

Choose now h_k in $D(\mathcal{E})_{E_k} \cap L^\infty(\mathcal{X},\mu)$ and $h_k \to h$ in $\|\cdot\|_{\mathcal{E}}$ such that (h_k) is uniformely bounded. (This is possible by taking an arbitrary approximating sequence (g_k) in $D(\mathcal{E})$ which converges with respect to $\|\cdot\|_{\mathcal{E}}$ and setting $h_k := 0 \vee g_k \wedge h$.) Then it holds true that

$$I_{u_k}(h_k) = I_u(h_k) \leq \|h_k\|_{L^1(\mu)}.$$

Sending k to infinity, Lemma 3.3.(iii) of [AH] implies $I_u(h) \leq \|h\|_{L^1(\mu)}$ and so $u \in \breve{G}$.

Conversely, suppose that for $u \in \breve{\mathcal{G}}$ there would exist for some $k \in \mathbb{N}$ a set of positive measure where $\Gamma(u_k)|_{E_k} > 1$ at least for one u_k of the approximating sequence (u_k), $u_k \in D(\mathcal{E})$, with respect to some compact $\mathcal{E}$-nest (E_k). Then we could find $\varepsilon > 0$ such that there exists a set $E \subseteq E_k$ of positive, but finite measure on which $\Gamma(u_k)(x) \geq 1+\varepsilon$. But since the Dirichlet space $D(\mathcal{E})$ is dense in $L^2(\mathcal{X},\mu)$, it is also dense in $L^1(\mathcal{X},\mu)$ and we can find for every $\delta \in]0,1[$ some $h' \in D(\mathcal{E}) \cap L^\infty(\mathcal{X},\mu)$ such that

$\|h' - \mathbb{1}_E\|_{L^1(\mu)} < \delta$. Without loss of generality we can assume $h' \geq 0$. Indeed, otherwise we can take $h' \vee 0$ instead. Moreover, $h'' := h' \vee \delta\chi^2_{E_k} \in D(\mathcal{E}) \cap L^\infty(\mathcal{X}, \mu)$ satisfies

$$\begin{aligned}\|h'' - \mathbb{1}_E\|_{L^1(\mu)} \leq &\int_E |h' \vee \delta - \mathbb{1}_E|\, d\mu + \int_{E^c} |h' \vee \delta\chi^2_{E_k}|\, d\mu < \delta + \int_{E^c} h'\, d\mu + \delta \int_{E^c} \chi^2_{E_k}\, d\mu \\ \leq &2\delta + \delta \int_{E^c} \chi^2_{E_k}\, d\mu \leq \delta\big(2 + \|\chi_{E_k}\|^2_{L^2(\mu)}\big).\end{aligned}$$

Since $\chi_{E_k} \in D(\mathcal{E})$, $\|\chi_{E_k}\|_{L^2(\mu)}$ is finite, so from here on we can copy the proof of Lemma 3 to get a contradiction to the assumption $u \in \breve{\mathcal{G}}$. ■

Note that (D.1) and the approximation result imply in particular that $\breve{G}'$ does not depend on the choice of the compact $\mathcal{E}$-nest. So we can introduce the pointwise distance $\breve{\rho}$ as

$$\breve{\rho}(x, y) := \begin{cases} \sup\limits_{u \in \breve{\mathcal{G}}'} \big(\tilde{u}(x) - \tilde{u}(y)\big) & x, y \in \mathcal{X} \setminus N, \\ \infty & \text{else.} \end{cases}$$

Here $\tilde{u}$ is of course the μ-version of u such that $\tilde{u}$ is continuous on every E_k. Indeed, such a μ-version exists: Since we can restrict ourselves to the case that u is nonnegative, it is directly given as $\sup_{k \in \mathbb{N}} \tilde{u}_k \mathbb{1}_{E_k}$. Moreover we note that by the μ-a.e. identity with functions in the Dirichlet space on every E_k, also the carré du champ operator is μ-a.e. well defined for $u \in D(\mathcal{E})^{(E_k)}_{loc}$. The proof of Lemma 4 remains true exactly as it is, to show Proposition 1, we substitute the generalizations $\breve{\mathcal{G}}'$, $\breve{\rho}_A$, $\breve{d}_A$ and $\breve{V}^M_A$ for $\mathcal{G}'$, ρ_A, d_A and V^M_A. So immediately we get the following Proposition:

Proposition 18. *$\breve{\rho}(\cdot, \cdot)$ defines an extended pseudometric on $\mathcal{X}$ and it holds that $\breve{d}_A \leq \breve{\rho}_A$ for an arbitrary set $A \in \mathcal{B}$.*

But by Assumption (B”) we can conclude exactly as in Section 2.3:

Proposition 19. *Let $A \subseteq \mathcal{X}$ a closed set of positive measure. Then $\breve{\rho}_A = \breve{d}_A$ μ-almost everywhere.*

The only difference in the proof is that we have to use now the corresponding result by Ariyoshi and Hino ([AH], Proposition 3.11) on the characterization of $\breve{d}_A$ instead that of Hino and Ramírez ([HR], Theorem 1.2). Clearly, the energy functional associated to $\breve{\rho}$ is given by

$$\breve{\mathsf{E}}_{a,b}(\omega) := \sup_{\Delta} \sum_{t_{i-1}, t_i \in \Delta} \frac{\breve{\rho}^2(\omega_{t_{i-1}}, \omega_{t_i})}{2(t_i - t_{i-1})}.$$

Lemma 5 (additivity of the energy functional) remains still true, to show Theorem 1 (characterization of the energy via $\mathcal{J}^{\vec{u}}_\Delta$) in this generalized setting, we have to adapt the proof slightly. We define for a path $\omega : [a, b] \to \mathcal{X}$ and $u \in D(\mathcal{E})^{(E_k)}_{loc}$ the functional

$$J_{a,b}(u, \omega) := \tilde{u}(\omega_b) - \tilde{u}(\omega_a) - \frac{1}{2} \int_a^b \Gamma(u)(\omega_r)$$

and

$$\mathcal{J}^{\vec{u}}_\Delta(\omega) := \sum_{i=1}^{n} J_{t_{i-1}, t_i}(u_i, \omega)$$

for fixed $n \in \mathbb{N}$, $\vec{u} = (u_1, \dots u_n) \in (D(\mathcal{E})^{(E_k)}_{loc})^n$ and a partition $\Delta = \{t_0, \dots, t_n\}$, $a = t_0 < t_1 < \cdots < t_n = b$, $n \in \mathbb{N}$.

Theorem 7. *Given a path $\omega : [a,b] \to \mathcal{X}$,*

$$\breve{\mathsf{E}}_{a,b}(\omega) = \sup_{\Delta;\vec{u}} \mathcal{J}^{\vec{u}}_{\Delta}(\omega). \tag{D.2}$$

Proof: We can assume that ω stays in E_k for some set E_k of our compact $\mathcal{E}$-nest (since otherwise both sides of the equality would trivially be infinite). Note that the set

$$\breve{\mathcal{H}} := \{u \in D(\mathcal{E})^{(E_k)}_{loc} \,:\, u = v_k \; \mu\text{-a.e. on } E_k,\, v_k \in D(\mathcal{E}),\, \sup_{k\in\mathbb{N}} \operatorname*{ess\,sup}_{x\in E_k} \Gamma(v_k)(x) < \infty\}$$

is point-separating by Assumption (BC') and it is a sub-algebra of $D(\mathcal{E})$ by

$$\Gamma(u+v) = \Gamma(u) + \Gamma(v) + 2\Gamma(u,v) \le 2\big(\Gamma(u) + \Gamma(v)\big). \tag{D.3}$$

So it is a point-separating sub-algebra which contains the constants, whence it is dense in $C(E_k)$ with respect to the uniform topology by the Stone-Weierstraß theorem. But since every $u \in D(\mathcal{E})^{(E_k)}_{loc}$ has a μ-version which is continuous on E_k (and in fact we take the supremum only over this modifications), it is enough to consider the supremum over $\vec{u} \in \breve{\mathcal{H}}^n$.
So for any path $\omega : [a,b] \to E_k$ and c, $d \in [a,b]$, $c < d$, we get on the one hand by maximizing over λ

$$\begin{aligned}
\sup_{u\in\breve{\mathcal{H}}} J_{c,d}(u,\omega) &= \sup_{u\in\breve{\mathcal{G}}',\lambda\in\mathbb{R}} \Big(\lambda\tilde{u}(\omega_d) - \lambda\tilde{u}(\omega_c) - \frac{1}{2}\int_c^d \Gamma(\lambda u)(\omega_t)\,dt\Big) \\
&= \sup_{u\in\breve{\mathcal{G}}'} \frac{(\tilde{u}(\omega_d) - \tilde{u}(\omega_c))^2}{2\int_c^d \Gamma(u)(\omega_t)\,dt} && \text{(D.4)} \\
&\ge \sup_{u\in\breve{\mathcal{G}}'} \frac{(\tilde{u}(\omega_d) - \tilde{u}(\omega_c))^2}{2(d-c)} = \frac{\breve{\rho}^2(\omega_c,\omega_d)}{2(d-c)}. && \text{(D.5)}
\end{aligned}$$

Summing up and taking the supremum over all partitions yields "$\le$" in (D.2).
On the other hand we can choose for some $u \in \breve{\mathcal{H}}$, $\gamma_i := \sup_{s_{i-1}\le r\le s_i} \Gamma(u)(\omega_r) < \infty$ and every given $\varepsilon > 0$ a partition $a \le c = s_0 < \cdots < s_m = d \le b$, such that

$$\sum_{i=1}^m (s_i - s_{i-1})\gamma_i \le \varepsilon + \int_c^d \Gamma(u)(\omega_t)\,dt. \tag{D.6}$$

With Hölder's inequality we get by (D.6) and (D.5)

$$\begin{aligned}
\tilde{u}(\omega_d) - \tilde{u}(\omega_c) &= \sum_{i=1}^{n} \big(\tilde{u}(\omega_{s_i}) - \tilde{u}(\omega_{s_{i-1}})\big) = \sum_{i=1}^{n} \frac{\tilde{u}(\omega_{s_i}) - \tilde{u}(\omega_{s_{i-1}})}{\sqrt{(s_i - s_{i-1})\gamma_i}} \sqrt{(s_i - s_{i-1})\gamma_i} \\
&\leq \left(\sum_{i=1}^{n} \frac{(\tilde{u}(\omega_{s_i}) - \tilde{u}(\omega_{s_{i-1}}))^2}{(s_i - s_{i-1})\gamma_i} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} (s_i - s_{i-1})\gamma_i \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n} \frac{\left(\frac{\tilde{u}}{\sqrt{\gamma_i}}(\omega_{s_i}) - \frac{\tilde{u}}{\sqrt{\gamma_i}}(\omega_{s_{i-1}}) \right)^2}{s_i - s_{i-1}} \right)^{\frac{1}{2}} \left(\int_a^b \Gamma(u)(\omega_t)\, dt + \varepsilon \right)^{\frac{1}{2}} \\
&\leq \left(2\, \breve{\mathsf{E}}_{c,d}(\omega) \right)^{\frac{1}{2}} \left(\int_c^d \Gamma(u)(\omega_t)\, dt + \varepsilon \right)^{\frac{1}{2}}
\end{aligned}$$

since $\Gamma(u/\sqrt{\gamma_i})(\omega_{s_i}) = \Gamma(u)(\omega_{s_i})/\gamma_i \leq 1$. Taking the infimum over all $\varepsilon > 0$ and, after rearranging the inequality, the supremum over $u \in \breve{\mathcal{G}}'$, we get by (D.4)

$$\breve{\mathsf{E}}_{c,d}(\omega) \geq \sup_{u \in \breve{\mathcal{H}}} J_{c,d}(u, \omega).$$

The additivity of the energy yields the result. ■

D.2 A Note on the Locally Compact Case

In the locally compact case (under Assumption (A)) we can even prove that the two notions of energy, $\breve{\mathsf{E}}$ and E coincide: By Assumptions (A) and (D) we can find to every compact set E_k of the compact $\mathcal{E}$-nest another compact set E_l such that

$$E_k \subseteq \mathring{E}_l \subseteq E_l.$$

Lemma 23. *Suppose that $x \in \mathring{E}_k$ for some set E_k of the compact $\mathcal{E}$-nest. Then we can find a finite family $v_i \in D(\mathcal{E})$ with bounded carré du champ such that*

$$\bigcap_{i=1}^{n} \big\{ y \,:\, \tilde{v}_i(y) - \tilde{v}_i(x) > \varepsilon \big\} \subseteq \mathring{E}_k.$$

Proof: Assumption (D) implies that we can find a set E_l of the compact $\mathcal{E}$-nest such that $E_k \subseteq \mathring{E}_l$. By Assumption(BC') there exists to every $y \in \partial E_k$ a function $u_y \in D(\mathcal{E})$, such that $\tilde{u}_y(x) = -1$ and $\tilde{u}_y(y) = 1$. We set now

$$V_y := \{z \,:\, \tilde{u}_y(z) > 0\}.$$

The family $(V_y)_y$ constitutes now an open cover of the compact set ∂E_k. So we can extract a finite subcover $(V_i)_{i=1,\ldots n}$ generated by functions $(\tilde{u}_i)_{i=1,\ldots n}$. Set now $W := \bigcup_{i=1}^{n} V_i$. Since $W \setminus E_k$ is open, we can define a function $\chi \in D(\mathcal{E})$, $0 \leq \chi \leq 1$, with $\chi|_{E_k} = 1$ and $\chi|_{E_k^c \cap W^c} = 0$ which has bounded carré du champ. Indeed, we choose a compact set E_m of the compact $\mathcal{E}$-nest (E_k) such that

$$E_k \subseteq \overset{\circ}{E}_l \subseteq E_l \subseteq \overset{\circ}{E}_m \subseteq E_m$$

and set

$$\tilde{W} := (E_m \setminus \overset{\circ}{E}_l) \cup (E_l \setminus W).$$

This is a closed set of positive, but finite measure. Moreover, $\rho(\tilde{W}, E_k) > 0$ by the compactness of E_k and since $\rho(\tilde{W}, x) > 0$ for every $x \in E_k$. So we can define the required $\chi \in D(\mathcal{E})$ directly by setting

$$\chi(x) := \begin{cases} \frac{\rho_{\tilde{W}}(x)}{\rho(\tilde{W}, E_k)} \wedge 1 & x \in E_m; \\ 0 & \text{else.} \end{cases}$$

Set now $v_i := u_i \cdot \chi$, then v_i lies in the Dirichlet space $D(\mathcal{E})$ and has bounded carré du champ, so the v_i satisfy the required condition. ■

Lemma 24. *Let $x \in E_k$ for some fixed E_k and $u \in \breve{\mathcal{G}}'$. Then there exists a neighborhood $U(x)$ and a function $w \in D(\mathcal{E})$ such that*

i) $\tilde{u}(y) - \tilde{u}(x) \leq \tilde{w}(y) - \tilde{w}(x)$ for all $y \in U(x)$;

ii) $\Gamma(w) \leq 1$ μ-almost everywhere.

Proof: By the introductory remark of this section there exists a set E_l of the compact $\mathcal{E}$-nest such that $x \in \overset{\circ}{E}_l$. The function u coincides on E_l with a function in the Dirichlet space with carré du champ bounded by 1, its continuous version we will denote by $\tilde{u}_l$. Moreover, by the previous lemma there exists functions v_i, $i = 1, \dots, k$ with bounded carré du champ of the point separating family $\breve{\mathcal{U}}$ and $\varepsilon > 0$ such that

$$V := \{y \,:\, \tilde{u}_l(y) - \tilde{u}_l(x) < \varepsilon\} \cap \bigcap_{i=1}^{k} \{y \,:\, \tilde{v}_i(y) - \tilde{v}_i(x) < \varepsilon\} \subseteq \overset{\circ}{E}_l$$

We define now a function $\bar{v} \in D(\mathcal{E})$ by

$$\bar{v}(y) := \bigvee_{i=1}^{k} \big((v_i(y) - v_i(x)\big) \vee \big((u_l(y) - u_l(x)\big).$$

Since $\bar{v}(x) = 0$, it follows that

$$\bar{v}(y) - \bar{v}(x) = \bigvee_{i=1}^{k} \big((v_i(y) - v_i(x)\big) \vee \big((u_l(y) - u_l(x)\big) \geq u_l(y) - u_l(x)$$

globally. Moreover, we have by Lemma 2

$$\Gamma(\bar{v}) \leq \max_{i=1,\dots n} G(v_i), \qquad G(v_i) = \operatorname{ess\,sup}_{x \in \mathcal{X}} \Gamma(v_i)(x),$$

and it follows that

$$\left\{y\,:\,\tilde{\bar{v}}(y) < \frac{2\varepsilon}{3}\right\} \subseteq V \subseteq \overset{\circ}{E}_l.$$

Defining now $v(y) := \bar{v}(y) \wedge 2\varepsilon/3$ and $U(x) := \{y\,:\,\tilde{v}(y) < \frac{\varepsilon}{3}\}$, we have $U(x) \subseteq V \subseteq \overset{\circ}{E}_l$. Again by Lemma 2, $\Gamma(v) = \Gamma(\bar{v})\mathbb{1}_{\overset{\circ}{E}_l} \leq \max_{i=1,\dots n} G(v_i)$ μ-almost everywhere, and furthermore

$$\tilde{v}(y) - \tilde{v}(x) = \tilde{\bar{v}}(y) - \tilde{\bar{v}}(x) \geq \tilde{u}_l(y) - \tilde{u}_l(x) = \tilde{u}(y) - \tilde{u}(x)$$

holds in $U(x)$. The same is true for $w := v/(\max_{i=1,\dots n} G(v_i) \vee 1)$ with the additional property that $\Gamma(w) \leq 1$ μ-almost everywhere. ■

This implies immediately the following lemma.

Lemma 25. *Let $x \in E_k$ for some fixed E_k, then there exists a neighborhood $U(x)$ where $\breve{\rho}(y,x) = \rho(y,x)$ for all $y \in U(x)$.*

Proof: Take an arbitrary function $u \in \breve{\mathcal{G}}'$. By the previous Lemma there exists a function $w \in \mathcal{G}'$ such that $\tilde{w}(\cdot) - \tilde{w}(x) \geq \tilde{u}(\cdot) - \tilde{u}(x)$ on a neighborhood $U(x)$. So $\rho(y,x) \geq \breve{\rho}(x,y)$ for $y \in U(x)$, but the other direction holds trivially, so the metric functionals coincide. ■

This corollary implies the equivalence of the two notions of energy.

Proposition 20. *For every path $\omega \in \Omega = C([a,b];\mathcal{X})$ it holds that $\mathsf{E}_{a,b}(\omega) = \breve{\mathsf{E}}_{a,b}(\omega)$*

Proof: Suppose first that $\{\omega_t\,:\,a \leq t \leq b\} \cap N \neq \emptyset$, then we have trivially $\mathsf{E}_{a,b}(\omega) = \infty = \breve{\mathsf{E}}_{a,b}(\omega)$. Otherwise $\{\omega_t\,:\,a \leq t \leq b\} \subseteq E_k$ for some set E_k of the compact $\mathcal{E}$-nest. But by the previous lemma we can find to every point $x \in \{\omega_t\,:\,a \leq t \leq b\}$ a neighborhood where the metric functionals coincide. But since $\{\omega_t\,:\,a \leq t \leq b\}$ is compact, we can extract a finite subcover $U(\omega_{t_i})$, $t_i \in [a,b]$, $i = 1,\dots,n$. Set $\Delta' = \{t_1,\cdots,t_n\}$. So for any partition Δ of $[a,b]$. $\Delta \cup \Delta'$ is a subpartition and

$$\sum_{t_{i-1},t_i \in \Delta\cup\Delta'} \frac{\breve{\rho}^2(\omega_{t_{i-1}},\omega_{t_i})}{2(t_i - t_{i-1})} = \sum_{t_{i-1},t_i \in \Delta\cup\Delta'} \frac{\rho^2(\omega_{t_{i-1}},\omega_{t_i})}{2(t_i - t_{i-1})}.$$

Clearly also the suprema coincide, too, so the Proposition holds true. ■

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Nomenclature

General Results

(E, d_E)	Complete metric space, p. 28
$(\mathcal{X}, \mathcal{B}, \mu)$	Probability space, p. 3
$\langle \cdot \rangle_t$	Quadratic variation process, p. 17
$\langle \cdot, \cdot \rangle_{\mathcal{E}}$	Standard inner product on the Dirichlet space $D(\mathcal{E})$, p. 3
$\langle \cdot, \cdot \rangle_{L^2(\mu)}$	Standard inner product on $L^2(\mathcal{X}, \mu)$, p. 3
$\lvert \cdot \rvert_\alpha$	Hölder norm with exponent α, p. 23
$\\| \cdot \\|_\infty$	Supremum norm on $L^\infty(\mathcal{X})$, p. 4
$\\| \cdot \\|_{\mathcal{E}}$	Standard norm on the Dirichlet space $D(\mathcal{E})$, p. 3
$\mathbb{1}_E$	Indicator function of the set E, p. 10
Γ	Carré du champ operator, p. 3
Δ	Partition of an interval, p. 12
Ω	Space of continuous paths on $\mathcal{X}$, p. 21
λ	Lebesgue measure, p. 28
μ	Probability measure, p. 3
$\rho(\cdot, \cdot)$	Pointwise intrinsic distance, p. 11
ρ_A	Intrinsic set distance from A, p. 11
τ_B	First hitting time of B, p. 5
τ_k	E_k-exit time of (X_t^ε), p. 17
$\dot{\omega}$	Fréchet derivative of ω, p. 30

$\lvert\omega'\rvert$	Metric derivative of ω, p. 28
$\mathrm{acc}(F)$	Set of F-accessible points of (X_t), p. 32
$AC^p(E)$	Space of p-integrable absolutely continuous curves on E, p. 28
$\mathcal{B}$	Borel-σ-Algebra on $\mathcal{X}$, p. 3
$B_r^\rho(x)$	Open ρ-ball around x with radius r, p. 14
$\mathrm{Cap}_1(\cdot)$	Capacity of a subset of $\mathcal{X}$, p. 4
$C_b^k(\mathbb{R}^2, \mathbb{R})$	Space of k-times continuously differentiable functions on $\mathbb{R}^2$ with bounded derivatives, p. 31
$C_0(\mathcal{X})$	Space of continuous functions on $\mathcal{X}$ with compact support, p. 4
$D(\mathcal{E})$	Domain of the symmetric Dirichlet form $\mathcal{E}(\cdot, \cdot)$, p. 3
$D(\Gamma)$	Domain of the carré du champ operator, p. 3
$D(L)$	Domain of the operator L, p. 3
$D([0, 1]; \mathbb{R})$	Skorohod space over the unit interval, p. 24
$d(\cdot, \cdot)$	Metric functional of Hino & Ramírez, p. 7
d_A	Set distance from A in the sense of Hino & Ramírez, p. 7
$d'(\cdot, \cdot)$	Alternative representation of the metric functional $d(\cdot, \cdot)$, p. 9
D^*	Adjoint of the Operator D, p. 31
$\mathcal{E}(\cdot, \cdot)$	Symmetric Dirichlet form on $L^2(\mathcal{X}, \mu)$, p. 3
$\mathsf{E}_{a,b}(\cdot)$	Energy of a path, p. 12
(E_k)	Compact $\mathcal{E}$-nest, p. 4
$E_\mu[\cdot]$	Expectation with respect to μ as initial distribution of (X_t), p. 5
$E_x[\cdot]$	Expectation with respect to $X_0 = x$, p. 5
$\mathcal{G}$	Index set of Hino & Ramírez, p. 7
$\mathcal{G}'$	Set of Dirichlet functions with carré du champ bounded by 1, p. 9
$\mathcal{H}$	Set of Dirichlet functions with bounded carré du champ, p. 13
$\tilde{I}$	Rate function implied by the contraction principle, p. 27
$J_{a,b}(u, \omega)$	J-functional, p. 13

$\mathcal{J}_{\Delta}^{\vec{u}}(\omega)$	Generalized J-functional, p. 13
L	unbounded, self-adjoint and negative definite operator on $L^2(\mathcal{X},\mu)$, p. 3
$\mathsf{L}_{a,b}(\cdot)$	Length of a path, p. 12
$L^\infty(\mathcal{X},\mu)$	Equivalence class of μ-a.e. bounded functions on $\mathcal{X}$, p. 3
$L^0(\mathcal{X},\mu)$	Set of functions on $\mathcal{X}$ measurable with respect to μ, p. 10
$L^p(\mathcal{X},\mu)$	Equivalence class of p-integrable functions on $\mathcal{X}$ with respect to μ, p. 3
M_t^ε	Forward martingale of the Lyons-Zheng decomposition, p. 17
$\hat{M}_t^\varepsilon$	Backward martingale of the Lyons-Zheng decomposition, p. 18
(p_t)	Semigroup of transition kernels for the Markov process (X_t), p. 5
$P_t(\cdot,\cdot)$	Transition probability, p. 7
R_ω	Range of the path ω, p. 28
(T_t)	Semigroup of operators associated with L, p. 3
$\mathcal{U}$	Countable family of functions with $\mathcal{E}$-quasi continuous, point-separating μ-versions, p. 4
$u\|_A$	Restriction of the function u to the set A, p. 8
u_k	Cut-off of the function u, p. 9
$\tilde{u}$	$\mathcal{E}$-quasi continuous version of u, p. 4
$\vec{u}$	Vector valued function with components in the Dirichlet space, p. 13
V_A^M	Set of functions in $\mathcal{G}$, μ-a.e. bounded and vanishing on A, p. 8
$\mathcal{X}$	Polish space, p. 3
(X_t^ε)	Family of ε-scaled diffusions, p. 17

Ornstein-Uhlenbeck Processes on Abstract Wiener Space

$(E,\|\cdot\|)$	Banach space, p. 37
(E,H,μ)	Abstract Wiener space, p. 37
$(\mathcal{E}_A,D(\mathcal{E}_A))$	Dirichlet space of the generalized Ornstein-Uhlenbeck process, p. 41
$(H,\langle\cdot,\cdot\rangle_H)$	Cameron-Martin space / Reproducing kernel Hilbert space, p. 37
$\langle\cdot,\cdot\rangle_{H_0}$	Standard inner product on H_0, p. 41

$\langle\cdot,\cdot\rangle_H$	Cameron-Martin inner product on H, p. 37
$\|\cdot\|$	Banach space norm on E, p. 37
$\|\cdot\|_H$	Cameron-Martin norm, p. 37
$\|\cdot\|_{\mathbb{D}^{1,2}}$	Standard norm on $\mathbb{D}^{1,2}$, p. 38
$\|\cdot\|_{H_0}$	Standard norm on H_0, p. 42
β	Drift of the Ornstein-Uhlenbeck SDE, p. 41
δ	Skorohod integral, p. 38
ι	Embedding of H into E, p. 37
ι'	Embedding of E' into H', p. 37
μ	Probability measure on $(E,\mathcal{F})$, p. 37
$\tilde{\mu}$	Gaussian cylinder set measure, p. 37
ρ^{H_0}	Intrinsic metric of the generalized OU process, p. 41
ρ^H	Cameron-Martin distance on abstract Wiener space, p. 39
A	Self-adjoint, strongly elliptic operator on H, p. 41
A'	Dual operator to A, p. 42
$\overline{B}_1^{H_0}$	Closed unit ball with respect to the $\|\cdot\|_{H_0}$-norm, p. 42
$D(A)$	Domain of the operator A ($\triangleq H_0$), p. 41
$D(\mathcal{E})$	Dirichlet space of the Ornstein-Uhlenbeck process, p. 39
$D(L)$	Domain of the Ornstein-Uhlenbeck operator L, p. 38
$D(L_A)$	Domain of the generalized Ornstein-Uhlenbeck operator L_A, p. 41
$\mathbb{D}^{1,2}$	Hilbert space of Malliavin calculus, p. 38
DF	Malliavin derivative of F, p. 38
$\mathcal{E}(\cdot,\cdot)$	Dirichlet form of the Ornstein-Uhlenbeck process, p. 39
$\mathcal{E}_A(\cdot,\cdot)$	Dirichlet form of the generalized Ornstein-Uhlenbeck process, p. 41
$(\tilde{E}_k)$	Enlarged compact $\mathcal{E}$-nest, p. 39
E'	Dual space of E, p. 37
$\mathcal{F}$	σ-algebra generated by $\mathcal{R}$, p. 37

H_0	Domain of the operator A ($\triangleq D(A)$), p. 41
$H^0(E)$	Space of absolutely continuous paths on E with square-integrable derivatives in H_0, p. 43
$H(E)$	Space of absolutely continuous paths on E with square-integrable derivatives, p. 39
H'	Dual space of H, p. 37
$I^H_{\tilde{E}_k}$	Rate function for the OU process on abstract Wiener space, p. 40
$I^{H_0}_{\tilde{E}_k}$	Rate function for the generalized OU process on abstract Wiener space, p. 43
L	Ornstein-Uhlenbeck operator on abstract Wiener space, p. 38
L_A	Generalized Ornstein-Uhlenbeck operator, p. 41
$\mathcal{R}$	σ-ring generated by the cylinder subsets of E, p. 37
$\mathcal{S}$	Set of smooth test functions, p. 38
$\mathrm{tr}\,(A)$	Trace of the operator A, p. 43
W	Isonormal Gaussian process, p. 37
(X_t)	(Generalized) Ornstein-Uhlenbeck process on abstract Wiener space, p. 39

Wasserstein Diffusion

$(\mathcal{G}_0, \\|\cdot\\|_{L^2(\lambda)})$	Space of distribution functions equipped with L^2-norm, p. 45
$(\mathcal{P}([0,1]), d_W)$	Wasserstein space, p. 45
$\langle\cdot,\cdot\rangle_{L^2(\omega_t)}$	Standard inner product on $L^2([0,1],\omega_t)$, p. 54
$\\|\cdot\\|_{L^2(\lambda)}$	L^2-norm with respect to the Lebesgue measure, p. 45
$\\|\cdot\\|_{L^2(\omega_t)}$	Standard norm on $L^2([0,1],\omega_t)$, p. 54
$\\|\cdot\\|_{T_g}$	Norm of the tangent space $T_g\mathcal{G}_0$, p. 49
$\\|\cdot\\|_{T_\mu}$	Norm on the tangent space $T_\mu\mathcal{P}([0,1])$, p. 51
$\Gamma^{\mu,\nu}$	Set of probability measures on $[0,1]$ with marginals μ and ν, p. 45
Δ_N	Partition of $[0,1]$, p. 46
Σ_N	N-dimensional simplex, p. 46
Ω	Path space over $\mathcal{P}([0,1])$, p. 53

β	Parameter of the entropic measure, p. 46
λ	Lebesgue measure on $[0,1]$, p. 45
(μ_t)	Wasserstein diffusion, p. 52
$\rho(\cdot,\cdot)$	Intrinsic metric of the Wasserstein diffusion on $\mathcal{P}([0,1])$, p. 51
τ_h	Translation on $\mathcal{G}_0$ by means of h, p. 47
$\tilde{\tau}_h$	Translation on $\mathcal{P}([0,1])$ by means of h, p. 47
χ	Isometry between $(\mathcal{G}_0, \|\cdot\|_{L^2(\lambda)})$ and $(\mathcal{P}([0,1]), d_W)$, p. 45
$AC^2(\mathcal{P}([0,1]))$	Space of square-integrable absolutely continuous curves on $\mathcal{P}([0,1])$, p. 54
$C^\infty([0,1];\mathbb{R})$	Space of smooth functions on the unit interval, p. 47
$C^k([0,1],\mathbb{R})$	k times continuously differentiable functions on $[0,1]$, p. 48
D^*_φ	Adjoint of the directional derivative D_φ, p. 48
D_φ	Directional derivative in direction $\varphi \in T_g$, p. 48
$D(D_\varphi)$	Domain of D_φ, p. 48
$D(\mathbb{D})$	Domain of $\mathbb{D}$, p. 50
$D(\mathcal{E})$	Domain of the Wasserstein Dirichlet form, p. 51
$D(\mathbb{E})$	Wasserstein Dirichlet space, p. 50
$\mathbb{D}_f$	Directional derivative in direction $f \in \mathbb{T}_g$, p. 49
$\mathbb{D}u(g)$	Gradient of u at g in the tangent space $\mathbb{T}_g\mathcal{G}_0$, p. 49
$Du(\mu)$	Gradient of u in $T_\mu\mathcal{P}([0,1])$, p. 51
$Du(g)$	Gradient of u at g in the tangent space $T_g\mathcal{G}_0$, p. 49
div	Divergence, p. 53
$d_W(\cdot,\cdot)$	L^2-Wasserstein distance on $\mathcal{P}([0,1])$, p. 45
e_φ	Flow generated by φ, p. 48
$\mathbb{E}(\cdot,\cdot)$	Wasserstein Dirichlet integral, p. 50
$\mathcal{E}(\cdot,\cdot)$	Wasserstein Dirichlet form, p. 51
$\mathsf{E}_{0,1}$	Intrinsic energy of the Wasserstein diffusion on $\mathcal{P}([0,1])$, p. 53
$\tilde{\mathsf{E}}_{0,1}$	Intrinsic energy of (g_t) on $\mathcal{G}_0$, p. 52

gaps (μ)	Set of maximal intervals with vanishing μ, p. 47
$g_*\lambda$	Push-forward of the Lebesgue measure by means of g, p. 46
$\mathcal{G}_0$	Set of probability distribution functions on $[0,1]$, p. 45
(g_t)	Diffusion process on $\mathcal{G}_0$, p. 50
(H_t)	Neumann heat flow, p. 52
$\dot{h}$	Fréchet derivative of h, p. 52
$H_{g_0}(\mathcal{G}_0)$	Subspace of $H(\mathcal{G}_0)$ of paths starting in h_0, p. 52
$H(\mathcal{G}_0)$	Space of absolutely continuous paths on $\mathcal{G}_0$ with square-integrable derivative, p. 52
h_t	Heat kernel on $[0,1]$, p. 52
$I(\omega)$	Rate function of the Wasserstein diffusion, p. 53
$\tilde{I}(h)$	Rate function of the diffusion (g_t), p. 53
J_g	Set of the jump locations of g, p. 47
L	Generator of the Wasserstein diffusion, p. 51
$L^2([0,1],\lambda)$	Equivalence class of square integrable functions on $[0,1]$ with respect to the Lebesgue measure, p. 49
$L^2([0,1],\mu)$	Equivalence class of square-integrable functions with respect to μ, p. 51
$L^2([0,1],\omega_t)$	Equivalence class of square integrable functions on $[0,1]$ with respect to the measure ω_t, p. 53
$L^2([0,1],g_*\lambda)$	Equivalence class of square integrable functions on $[0,1]$ with respect to the measure $g_*\lambda$, p. 49
$L^2(\mathcal{G}_0,Q_0^\beta)$	Equivalence class of square-integrable functions on $\mathcal{G}_0$ with respect to Q_0^β, p. 48
$L^2(\mathcal{P}([0,1]),P_0^\beta)$	Equivalence class of square integrable functions on $\mathcal{P}([0,1])$ with respect to P_0^β, p. 51
$\mathrm{Lip}(\cdot)$	Lipschitz constant, p. 58
P_0^β	Entropic measure on $\mathcal{P}([0,1])$, p. 47
$\mathcal{P}([0,1])$	Space of probability measures over the unit interval, p. 45
Q_0^β	Entropic measure on $\mathcal{G}_0$, p. 46
$T_\mu\mathcal{P}([0,1])$	Tangent space of $\mathcal{P}([0,1])$ at μ, p. 51

$T\mathcal{P}([0,1])$	Tangent bundle over $\mathcal{P}([0,1])$, p. 54
$\mathbb{T}_g\mathcal{G}_0$	Tangent space in g with respect to the linear structure of $\mathcal{G}_0$, p. 49
$\mathbf{T}(s,t,x)$	Flow along a curve on $\mathcal{P}([0,1])$, p. 57
$T_g\mathcal{G}_0$	Tangent space in g with respect to the group structure of $\mathcal{G}_0$, p. 49
V_φ^β	Drift operator for the directional derivative, p. 48
v	Tangent velocity field, p. 54
$\mathfrak{Z}^k(\mathcal{G}_0)$	Cylinder functions on $\mathcal{G}_0$, p. 48
$\mathfrak{Z}^k(\mathcal{P}([0,1]))$	Cylinder functions on $\mathcal{P}([0,1])$, p. 50

σ-finite Measures

$(\mathcal{X},\mathcal{B},\mu)$	σ-finite measure space, p. 71
μ	σ-finite measure, p. 71
$\breve{\rho}(\cdot,\cdot)$	Generalized pointwise intrinsic distance, p. 75
$\breve{\rho}_A$	Generalized intrinsic set distance from A, p. 75
χ_{E_k}	Smoothing indicator function of E_k, p. 71
$\mathrm{Cap_h}(\cdot)$	h-weighted capacity, p. 73
$D(\mathcal{E})_{E_k}$	Subspace of $D(\mathcal{E})$ of functions vanishing outside of E_k, p. 71
$\breve{d}(\cdot,\cdot)$	Metric functional of Ariyoshi & Hino, p. 72
$\breve{d}_A$	Set distance from A in the sense of Ariyoshi & Hino, p. 72
$D_{loc,b}^{(E_k)}(\mathcal{E})$	Subset of bounded functions in $D_{loc}^{(E_k)}(\mathcal{E})$, p. 72
$D_{loc}^{(E_k)}(\mathcal{E})$	Domain of local Dirichlet forms, p. 71
$(G_\alpha)_{\alpha>0}$	Resolvent of the semigroup T_t, p. 73
$\breve{\mathcal{G}}$	Index set of Ariyoshi & Hino, p. 72
$\breve{\mathcal{G}}'$	Set of local Dirichlet functions with carré du champ bounded by 1, p. 74
$\breve{\mathcal{H}}$	Set of local Dirichlet functions with uniformly bounded carré du champ, p. 76
$I.(\cdot)$	I-functional, p. 72
$\breve{\mathcal{U}}$	Generalized point-separating family, p. 72
$\breve{V}_A^M$	Set of functions in $\breve{\mathcal{G}}$, μ-a.e. bounded and vanishing on A, p. 72