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POSITIVE REAL AND BOUNDED REAL BALANCING FOR MODEL REDUCTION OF DESCRIPTOR SYSTEMS

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Abstract. We present an extension of the positive real and bounded real balanced truncation model reduction methods to large-scale descriptor systems. These methods are based on balancing the solutions of the projected Lur'e matrix equations. Important properties of these methods are that, respectively, passivity and contractivity are preserved in the reduced-order models and that there exist approximation error bounds. We also discuss the numerical solution of the projected Lur'e equations. Numerical examples are given.

Key words. descriptor systems, passivity, contractivity, positive real, bounded real, model reduction, balanced truncation, Lur'e equations

AMS subject classifications. 15A22, 15A24, 34A09, 93C05

1. Introduction. Consider a linear time-invariant continuous-time descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control input and $y(t) \in \mathbb{R}^p$ is an output. The number of state variables n is called the *order* of system (1.1). The matrix E may be singular, but we will assume that a matrix pencil $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. Descriptor systems with singular E arise in a variety of applications including design of micro-electro-mechanical systems (MEMS) and circuit simulation, e.g., [4, 16, 22].

As physical models get more complex and different coupling effects have to be taken into account, the development of efficient modelling and simulation tools for very large systems is highly required. In this context, model order reduction is of crucial importance, especially if simulation of large-scale systems has to be done in a short time or it has to be repeated for different input signals. A general idea of model reduction is to approximate the large-scale descriptor system (1.1) by a reduced-order model

$$\begin{aligned} \widetilde{E}\,\dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t), \end{aligned} \tag{1.2}$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{p,\ell}$, $\widetilde{D} \in \mathbb{R}^{p,m}$ and $\ell \ll n$. It is required that the approximate model (1.2) preserves essential properties of (1.1) like stability, passivity and contractivity and that the approximation error is small.

Moment matching approximation based on Krylov subspace methods, e.g., [3,14], is commonly used model reduction approach in circuit simulation and MEMS design. Until recently, it was the only model reduction technique available for descriptor systems. Although the Krylov subspace model reduction methods are efficient for

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very large sparse problems, stability and passivity are not necessarily preserved in the reduced-order model, so that usually a post-processing is needed to guarantee these properties. Recently, passivity-preserving model reduction methods based on Krylov subspaces have been developed for standard state space systems with $E = I$, see [2,40], and also for structured descriptor systems describing linear RLC circuits [15,17,25,31]. Despite the successful application of these methods in practice, they provide a good local approximation only and, so far, there exist no global error bounds.

Another model order reduction approach widely used in control design is balanced truncation. In order to capture specific system properties, different balancing techniques have been developed for standard state space systems, see [21] for comprehensive review. An important property of balancing-related model reduction is the existence of computable error bounds that allow an adaptive choice of the order of the approximate model. The classical balanced truncation method has been extended to descriptor systems in [42]. This method is based on balancing the proper and improper controllability and observability Gramians of system (1.1) that satisfy projected generalized Lyapunov equations. Note that Lyapunov-based balanced truncation, in general, neither preserves passivity nor contractivity in the reduced-order model. To guarantee these properties one can employ the positive real and bounded real balanced truncation methods [23, 30, 32] that relay on solutions of two Lur'e equations. An extension of these methods to descriptor systems has been considered in [34]. The methods proposed there are based on computing a Weierstrass-like form of the pencil $\lambda E - A$. However, the computation of this form is very expensive for large-scale problems and ignores the sparsity and structure of matrix coefficients.

In this paper, we present a generalization of the positive real and bounded real balanced truncation methods for descriptor systems with singular E that avoids the explicit computation of the Weierstrass canonical form. We introduce projected generalized Lur'e matrix equations that can be used to define positive real and bounded real Gramians for descriptor systems. A different type of generalized Lur'e equations has been used in [45, 50] to characterize the positive real and bounded real properties of descriptor systems. However, the application of such equations is limited to index one problems, whereas the existence results for the projected Lur'e equations can be stated independently of the index of the pencil $\lambda E - A$.

Throughout the paper $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ denote the spaces of $n \times m$ real and complex matrices. The open left and right half-planes are denoted by $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively, and $i = \sqrt{-1}$. The negative and positive real half-axes are denoted by $\mathbb{R}_- = (-\infty, 0\,]$ and $\mathbb{R}_+ = [\,0, \infty)$, respectively. The matrices A^T and A^* denote, respectively, the transpose and the conjugate transpose of $A \in \mathbb{C}^{n,m}$, and $A^{-T} = (A^{-1})^T$. An identity matrix of order n is denoted by I_n or simply by I. The zero $n \times m$ matrix is denoted by $0_{n,m}$ or simply by 0. We denote by $\operatorname{rank}(A)$ the rank, by $\operatorname{im}(A)$ the image, by $\ker(A)$ the kernel and by $\operatorname{Sp}(A)$ the spectrum of a matrix A. Further, for Hermitian matrices $P, Q \in \mathbb{C}^{n,n}$, we write $P > Q$ ($P \geq Q$) if $P - Q$ is positive (semi)definite. The Euclidean vector norm is denoted by $\|\cdot\|$. For some interval $\mathbb{I} \subseteq \mathbb{R}$, we use $\mathbb{L}_2(\mathbb{I}, \mathbb{R}^m)$ to denote the Hilbert space of square integrable $\mathbb{R}^m$-valued functions. Let $\mathbb{H}_\infty$ be a space of all functions that are analytic and bounded in $\mathbb{C}_+$. The $\mathbb{H}_\infty$-*norm* of $\boldsymbol{G} \in \mathbb{H}_\infty$ is defined by

$$\|\boldsymbol{G}\|_{\mathbb{H}_\infty} = \sup_{s \in \mathbb{C}_+} \|\boldsymbol{G}(s)\|_2 = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} \sup_{\omega \in \mathbb{R}} \|\boldsymbol{G}(\sigma + i\omega)\|_2,$$

where $\|\cdot\|_2$ denotes the spectral matrix norm.

The paper is organized as follows. Section 2 contains some background material for descriptor systems. Sections 3 and 4 deal with balanced truncation model reduction methods based on bounded real and positive real balancing, respectively. We introduce projected Lur'e equations that play an essential role in these model reduction methods. In Section 5, we study the numerical solution of the projected Lur'e equations via a method based on deflating subspaces of a structured matrix pencil. Under some additional conditions the projected Lur'e equations can be rewritten as the projected algebraic Riccati equations. We also briefly discuss solving these equations using Newton's method. Furthermore, the computation of low-rank approximations of the solutions of the projected Lur'e and Riccati equations is considered. Finally, in Section 6, we present some numerical examples.

2. Preliminaries. In this section, we give basic definitions and some properties of matrix pencils and descriptor systems that will be used in the following.

Any regular matrix pencil $\lambda E - A$ with $E, A \in \mathbb{R}^{n,n}$ can be reduced to the Weierstrass canonical form

$$E = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & E_\infty \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} A_f & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T_r, \tag{2.1}$$

where T_l and T_r are the left and right nonsingular transformation matrices, $A_f \in \mathbb{R}^{n_f,n_f}$ and nilpotent $E_\infty \in \mathbb{R}^{n_\infty,n_\infty}$ with index of nilpotency ν, see [19]. The eigenvalues of A_f are the finite eigenvalues of $\lambda E - A$, and E_∞ corresponds to an eigenvalue at infinity. The number ν is called the *index* of $\lambda E - A$.

Subspaces $\mathcal{W}, \mathcal{T} \subset \mathbb{R}^n$ are called *left* and *right deflating subspaces* of the pencil $\lambda E - A$ if $\dim(\mathcal{W}) = \dim(\mathcal{T})$ and $\mathcal{W} = E\mathcal{T} + A\mathcal{T}$. The deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues in the open left (right) half-plane are called *stable* (*antistable*). The matrices

$$P_r = T_r^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_r, \qquad P_l = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1} \tag{2.2}$$

are the *spectral projectors* onto the right and left deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues. These subspaces have the dimension n_f. The complementary projectors to P_r and P_l are given by $Q_r = I - P_r$ and $Q_l = I - P_l$, respectively. They are the spectral projectors onto the left and right deflating subspaces of $\lambda E - A$ corresponding to the eigenvalue at infinity. These subspaces have the dimension $n_\infty = n - n_f$. A method for the computation of Q_r and Q_l avoiding the reduction to the Weierstrass canonical form is presented in [28]. For some structured problems arising in computational fluid dynamics, constrained multibody systems and circuit simulation, such projectors can be computed in explicit form, see [13,36,39,43].

Consider a descriptor system (1.1) and assume that $\lambda E - A$ is regular. The *index* of (1.1) is identified with the index of $\lambda E - A$. The *transfer function* of (1.1) is given by

$$\boldsymbol{G}(s) = C(sE - A)^{-1}B + D. \tag{2.3}$$

This rational matrix-valued function describes the input-output relation of (1.1) in the frequency domain. The transfer function $\boldsymbol{G}$ is called *proper* if $\lim\limits_{s\to\infty} \boldsymbol{G}(s) < \infty$, and *improper*, otherwise. If $\lim\limits_{s\to\infty} \boldsymbol{G}(s) = 0$, then $\boldsymbol{G}$ is called *strictly proper*. We identify properness or improperness of the descriptor system (1.1) with the respective property of its transfer function $\boldsymbol{G}$.

Using the Weierstrass canonical form (2.1), the transfer function $\boldsymbol{G}$ of system (1.1) can be additively decomposed into the *strictly proper part* and the *polynomial part*, that is, $\boldsymbol{G}(s) = \boldsymbol{G}_{sp}(s) + \boldsymbol{P}(s)$ with strictly proper $\boldsymbol{G}_{sp}$ and

$$\boldsymbol{P}(s) = \sum_{k=0}^{\nu-1} M_k s^k,$$

where

$$M_k = CT_r^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -E_\infty^k \end{bmatrix} T_l^{-1} B + \delta_{0,k} D \tag{2.4}$$

and $\delta_{0,k}$ denotes the Kronecker symbol.

For any rational function $\boldsymbol{G}$, one can find matrices E, A, B, C and D such that (2.3) holds [12]. A descriptor system (1.1) with these matrices is called a *realization* of $\boldsymbol{G}$ and denoted by $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$. The dimension of the matrices E and A defines the *order of the realization.* A realization $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ is called

- *R-minimal* if it is *R-controllable* and *R-observable*, i.e., for all $\lambda \in \mathbb{C}$ holds

$$\operatorname{rank}[\,\lambda E - A,\ B\,] = \operatorname{rank}[\,\lambda E^T - A^T,\ C^T\,] = n; \tag{2.5}$$

- *S-minimal* if it is R-minimal and also *controllable and observable at infinity*, i.e., for matrices S_∞ and T_∞ with $\operatorname{im}(S_\infty) = \ker(E)$ and $\ker(T_\infty) = \operatorname{im}(E)$, respectively, holds

$$\operatorname{rank}[\,E\,,\ AS_\infty\,,\ B\,] = \operatorname{rank}[\,E^T\,,\ A^T T_\infty^T\,,\ C^T\,] = n. \tag{2.6}$$

The descriptor system (1.1) is *stable* if all the finite eigenvalues of the pencil $\lambda E - A$ lie in the closed left half-plane and the eigenvalues on the imaginary axis are semisimple, i.e., they have the same algebraic and geometric multiplicity. System (1.1) is *asymptotically stable* if all the finite eigenvalues of the pencil $\lambda E - A$ lie in the open left half-plane.

For a descriptor system (1.1) and an interval $\mathbb{I} \subset \mathbb{R}$, we say that the input $u \in \mathbb{L}_2(\mathbb{I}, \mathbb{R}^m)$ is *consistent*, if (1.1) has a continuous solution $x : \mathbb{I} \to \mathbb{R}^n$. For $\mathbb{I}$ with $\min \mathbb{I} = t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, we say that $u \in \mathbb{L}_2(\mathbb{I}, \mathbb{R}^m)$ is *consistent with* $x(t_0) = x_0$, if (1.1) possesses a continuous solution $x : \mathbb{I} \to \mathbb{R}^n$ with $x(t_0) = x_0$. Note that if $\mathbb{I}$ is bounded from below and ν is the index of (1.1), then the ν-times differentiability of u is sufficient for consistency. In order to guarantee consistency with a prescribed initial value, the Weierstrass canonical form (2.1) can be used to formulate further conditions on u, see [12].

2.1. Contractivity and bounded realness. An important class of dynamical systems are the contractive systems. Contractivity means that the $\mathbb{L}_2$-norm of the output does not exceed the $\mathbb{L}_2$-norm of the input. This type of systems is used, for example, in the parametrization of all stabilizing controllers such that the closed-loop system satisfies a $\mathbb{L}_2$-gain constraint [20].

DEFINITION 2.1. *A descriptor system* (1.1) *is called* contractive *if*

$$\int_0^t \|u(\tau)\|^2 - \|y(\tau)\|^2 \, d\tau \geq 0 \tag{2.7}$$

for all $t \in \mathbb{R}_+$ *and all* $u \in \mathbb{L}_2([0,t], \mathbb{R}^m)$ *consistent with* $x(0) = 0$.

The quantity on the left-hand side of (2.7) expresses the difference between the input and output energies. Contractivity of the descriptor system (1.1) is closely related to the bounded realness of its transfer function $\boldsymbol{G}$.

DEFINITION 2.2. *A transfer function* $\boldsymbol{G}$ *is called* bounded real *if*

(BR1) $\boldsymbol{G}$ *is analytic in* $\mathbb{C}_+$,
(BR2) $\boldsymbol{G}(\overline{s}) = \overline{\boldsymbol{G}(s)}$ *for all* $s \in \mathbb{C}$,
(BR3) $\boldsymbol{G}(s)^*\boldsymbol{G}(s) \le I$ *for all* $s \in \mathbb{C}_+$.

Note that the bounded real transfer function $\boldsymbol{G}$ is necessarily proper. For the descriptor system (1.1) with real matrix coefficients E, A, B, C and D, condition (BR2) is always fulfilled. Moreover, if system (1.1) is asymptotically stable, then condition (BR1) is satisfied, and condition (BR3) is equivalent to the bound $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} \le 1$.

PROPOSITION 2.3. [1] *A descriptor system* (1.1) *is contractive if and only if its transfer function* $\boldsymbol{G}$ *is bounded real.*

2.2. Passivity and positive realness. Passivity is another crucial property of dynamical systems especially in circuit simulation and system design. Generally speaking, passivity means that system does not produce energy via the input-output channel. Mathematically, this property is defined as follows.

DEFINITION 2.4. *A descriptor system* (1.1) *is called* passive *if* $m = p$ *and*

$$\int_0^t u(\tau)^T y(\tau)\, d\tau \ge 0 \tag{2.8}$$

for all $t \in \mathbb{R}_+$ *and all* $u \in \mathbb{L}_2([0,t],\mathbb{R}^m)$ *consistent with* $x(0) = 0$.

Note that the quantity on the left-hand side of (2.8) stands for the energy that can be extracted from the system. Passivity of the descriptor system (1.1) is closely related to the positive realness of its transfer function $\boldsymbol{G}$.

DEFINITION 2.5. *A square transfer function* $\boldsymbol{G}$ *is called* positive real *if*

(PR1) $\boldsymbol{G}$ *is analytic in* $\mathbb{C}_+$,
(PR2) $\boldsymbol{G}(\overline{s}) = \overline{\boldsymbol{G}(s)}$ *for all* $s \in \mathbb{C}$,
(PR3) $\boldsymbol{G}(s) + \boldsymbol{G}(s)^* \ge 0$ *for all* $s \in \mathbb{C}_+$.

If system (1.1) is (asymptotically) stable, then condition (PR1) holds, and (PR3) is equivalent to the condition that $\boldsymbol{G}(i\omega) + \boldsymbol{G}(i\omega)^* \ge 0$ for all $\omega \in \mathbb{R}$ whenever $i\omega$ is not a pole of $\boldsymbol{G}$. Moreover, one can show that if the descriptor system (1.1) is R-minimal and passive, then it is stable [1]. In the following, we collect some further properties of passive systems and positive real transfer functions.

PROPOSITION 2.6. [1] *A descriptor system* (1.1) *is passive if and only if its transfer function* $\boldsymbol{G}$ *is positive real.*

PROPOSITION 2.7. [1] *A transfer function* $\boldsymbol{G}$ *is positive real if and only if the following conditions hold:*

1. *the proper part* $\boldsymbol{G}_p(s) = \boldsymbol{G}_{sp}(s) + M_0$ *of* $\boldsymbol{G}$ *is positive real,*
2. M_1 *is symmetric and positive semidefinite,*
3. $M_k = 0$ *for* $k > 1$.

A *Moebius transformation* of a square transfer function $\boldsymbol{G}$ with $\det(I+\boldsymbol{G}(s)) \not\equiv 0$ is defined as

$$\mathcal{M}(\boldsymbol{G})(s) = (I - \boldsymbol{G}(s))(I + \boldsymbol{G}(s))^{-1}. \tag{2.9}$$

It is a self-inverse bijection, i.e., $\mathcal{M}(\mathcal{M}(\boldsymbol{G})) = \boldsymbol{G}$. For a square transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ with the property that $I + D$ is invertible, a realization of the Moebius-transformed system $\boldsymbol{H}(s) = \mathcal{M}(\boldsymbol{G})(s)$ is given by $\boldsymbol{H} = [\,\mathcal{E},\, \mathcal{A},\, \mathcal{B},\, \mathcal{C},\, \mathcal{D}\,]$ with

$$\begin{aligned} &\mathcal{E} = E, \quad \mathcal{A} = A - B(I+D)^{-1}C, \quad \mathcal{B} = -\sqrt{2}B(I+D)^{-1}, \\ &\mathcal{C} = \sqrt{2}(I+D)^{-1}C, \qquad \mathcal{D} = (I-D)(I+D)^{-1}. \end{aligned} \tag{2.10}$$

Observe that the property $\det(I + \boldsymbol{G}(s)) \not\equiv 0$ is equivalent to the regularity of the pencil $\lambda E - A + B(I+D)^{-1}C$.

The following theorem gives a connection between the positive real and bounded real transfer functions related via the Moebius transformation.

THEOREM 2.8.

1. *Let $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ be positive real with invertible $I + D$. Then the Moebius-transformed transfer function $\boldsymbol{H}(s) = \mathcal{M}(\boldsymbol{G})(s)$ is bounded real.*
2. *Let a square bounded real transfer function $\boldsymbol{H}(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ with $\det(I + \boldsymbol{H}(s)) \not\equiv 0$ and invertible $I + \mathcal{D}$ be given. Then the Moebius-transformed transfer function $\boldsymbol{G}(s) = \mathcal{M}(\boldsymbol{H})(s)$ is positive real.*

Proof. The results can be proved analogously to the standard state space case, see [1, 30]. □

Note that in the first part of Theorem 2.8 we do not have the additional requirement that $\det(I + \boldsymbol{G}(s)) \not\equiv 0$, since it follows automatically from the positive realness of $\boldsymbol{G}$.

3. Bounded real balanced truncation. A bounded real balanced truncation model reduction method for contractive standard state space systems has been considered in [21, 30, 32]. A generalization of this method to the descriptor system (1.1) with nonsingular E is trivial. In the case of singular E, it has been proposed in [34] first to transform the pencil $\lambda E - A$ into the Weierstrass canonical form (2.1) and then apply the classical bounded real balanced truncation method to the standard state space system $[\,I,\, A_f,\, B_f,\, C_f,\, D - B_\infty C_\infty\,]$. Here, the matrices

$$B = T_l \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}, \qquad C = [\,C_f,\; C_\infty\,]\,T_r \tag{3.1}$$

are partitioned in accordance with the block structure of E and A in (2.1). However, the computation of the Weierstrass canonical form may be ill-conditioned problem and it is very expensive for large-scale problems. In this section, we present an extension of the bounded real balanced truncation method to descriptor systems with singular E that avoids the explicit computation of the Weierstrass canonical form. This form will be used for theoretical purposes only.

3.1. Bounded real lemma for descriptor systems. Bounded realness of standard state space system can be characterized by the bounded real lemma [1] that gives a connection between bounded realness and solvability of so-called *bounded real Lur'e equations*. Such equations are also known as Kalman-Yakubovich-Popov equations [24]. A generalization of this lemma to descriptor systems has been considered in [35, 45, 50]. However, the solvability of generalized Lur'e equations presented there is restricted to systems with the transfer function $\boldsymbol{G}$ satisfying $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} < 1$. It is furthermore required that the index of the descriptor system is at most one. We present here a generalized bounded real lemma for contractive descriptor systems whose transfer function $\boldsymbol{G}$ may also satisfy $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} = 1$. Moreover, by making use of the spectral projectors P_r and P_l introduced in the previous section, we may consider descriptor systems of arbitrary index.

THEOREM 3.1 (Generalized bounded real lemma). *Consider a descriptor system* (1.1) *with a transfer function* $\boldsymbol{G}$. *Let* P_r *and* P_l *be the spectral projectors onto the right and left deflating subspaces of* $\lambda E - A$ *corresponding to the finite eigenvalues and let* M_k *be as in* (2.4).

1. *If system* (1.1) *is R-minimal and* $\boldsymbol{G}$ *is bounded real, then the projected Lur'e equations*
$$\begin{aligned} AXE^T + EXA^T + P_lBB^TP_l^T &= -K_cK_c^T, \quad X = P_rXP_r^T \geq 0,\\ EXC^T + P_lBM_0^T &= -K_cJ_c^T, \quad I - M_0M_0^T = J_cJ_c^T \end{aligned} \tag{3.2}$$
are solvable for $K_c \in \mathbb{R}^{n,p}$, $J_c \in \mathbb{R}^{p,p}$ *and* $X \in \mathbb{R}^{n,n}$.
2. *If* $M_k = 0$ *for* $k \geq 1$ *and the projected Lur'e equations* (3.2) *are solvable, then* $\boldsymbol{G}$ *is bounded real.*

Proof. Substituting (2.1), (2.2) and (3.1) in the projected Lur'e equations (3.2), we find that the solution matrices have the form
$$X = T_r^{-1}\begin{bmatrix} X_f & 0\\ 0 & 0\end{bmatrix}T_r^{-T}, \qquad K_c = T_l\begin{bmatrix} K_{c,f}\\ 0\end{bmatrix}, \tag{3.3}$$
where $X_f \in \mathbb{R}^{n_f,n_f}$ and $K_{c,f} \in \mathbb{R}^{n_f,p}$ satisfy the Lur'e equations
$$\begin{aligned} A_fX_f + X_fA_f^T + B_fB_f^T &= -K_{c,f}K_{c,f}^T,\\ X_fC_f^T + B_fM_0^T &= -K_{c,f}J_c^T. \end{aligned} \tag{3.4}$$
If (1.1) is R-minimal and $\boldsymbol{G}$ is bounded real, then the standard state space system $[\,I,\, A_f,\, B_f,\, C_f,\, M_0\,]$ is minimal and $\boldsymbol{G}(s) = C_f(sI - A_f)^{-1}B_f + M_0$. On the other hand, if $M_k = 0$ for $k \geq 1$ and the projected Lur'e equations (3.2) have a solution, then $\boldsymbol{G}(s) = C_f(sI - A_f)^{-1}B_f + M_0$ and (3.4) is solvable. Thus, the result immediately follows from the bounded real lemma for standard state space systems [1]. ◻

Observe that the bounded realness of $\boldsymbol{G}$ can also be characterized via the solvability of the dual projected Lur'e equations
$$\begin{aligned} A^TYE + E^TYA + P_r^TC^TCP_r &= -K_o^TK_o, \quad Y = P_l^TYP_l \geq 0,\\ E^TYB + P_r^TC^TM_0 &= -K_o^TJ_o, \quad I - M_0^TM_0 = J_o^TJ_o \end{aligned} \tag{3.5}$$
for some $K_o \in \mathbb{R}^{m,n}$, $J_o \in \mathbb{R}^{m,m}$ and $Y \in \mathbb{R}^{n,n}$.

Next we introduce a generalization of the inverse of a matrix.

DEFINITION 3.2. *Let a matrix* $M \in \mathbb{R}^{m,n}$ *and projectors* $P_1 \in \mathbb{R}^{m,m}$, $P_2 \in \mathbb{R}^{n,n}$ *be given such that* $\mathrm{im}(P_1) = \mathrm{im}(M)$ *and* $\ker(P_2) = \ker(M)$. *Then a matrix* $M^- \in \mathbb{R}^{n,m}$ *is called a* reflexive generalized inverse of M with respect to P_1 and P_2 *if it satisfies the matrix equations*
$$MM^- = P_1, \qquad M^-M = P_2, \qquad M^-MM^- = M^-.$$
Note that for prescribed P_1 and P_2, the reflexive generalized inverse exists and it is unique [37]. In the sequel, we will denote by $(M)_l^-$ the reflexive generalized inverse with respect to the projectors P_l and P_l^T and by $(M)_r^-$ the reflexive generalized inverse with respect to the projectors P_r^T and P_r. Furthermore, for given $M \in \mathbb{R}^{m,n}$ of full column rank and a projector $P \in \mathbb{R}^{m,m}$ such that $\mathrm{im}(P) = \mathrm{im}(M)$, we will denote by $(M)_P^-$ a reflexive generalized inverse of M with respect to P and I_n.

The following theorem gives some relationships for the solutions of the bounded real projected Lur'e equations (3.2) and (3.5).

THEOREM 3.3. *Consider a descriptor system* (1.1) *that is R-minimal and has a bounded real transfer function* $\boldsymbol{G}$.

1. *If the matrix X is a solution of (3.2), then $Y = (EXE^T)_l^-$ satisfies (3.5).*
2. *If the matrix Y is a solution of (3.5), then $X = (E^TYE)_r^-$ satisfies (3.2).*
3. *The projected Lur'e equations (3.2) and (3.5) have two extremal solutions that satisfy*

$$0 \le X_{\min} \le X \le X_{\max}, \qquad 0 \le Y_{\min} \le Y \le Y_{\max} \tag{3.6}$$

for all solutions X and Y of (3.2) and (3.5), respectively.

4. *The extremal solutions $X_{\min}$, $X_{\max}$ of (3.2) and $Y_{\min}$, $Y_{\max}$ of (3.5) satisfy the relations*

$$\begin{array}{ll} X_{\min} = (E^TY_{\max}E)_r^-, & X_{\max} = (E^TY_{\min}E)_r^-, \\ Y_{\min} = (EX_{\max}E^T)_l^-, & Y_{\max} = (EX_{\min}E^T)_l^-, \\ EX_{\min}E^TY_{\max} = EX_{\max}E^TY_{\min} = P_l, & \\ X_{\min}E^TY_{\max}E = X_{\max}E^TY_{\min}E = P_r. & \end{array} \tag{3.7}$$

5. *The eigenvalues of the matrix $X_{\min}E^TY_{\min}E$ are real, non-negative and they do not exceed one, i.e., $0 \le \lambda_j(X_{\min}E^TY_{\min}E) \le 1$.*

Proof. Let $\lambda E - A$ be in Weierstrass canonical form (2.1) and let the matrices B and C be as in (3.1). Then the solutions of (3.2) have the form (3.3), whereas the solutions of (3.5) are given by

$$Y = T_l^{-T}\begin{bmatrix} Y_f & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad K_o = [\, K_{o,f},\ 0\,]\, T_r,$$

where $Y_f \in \mathbb{R}^{n_f,n_f}$ and $K_{o,f} \in \mathbb{R}^{m,n_f}$ satisfy the Lur'e equations

$$\begin{aligned} A_f^TY_f + Y_fA_f + C_f^TC_f &= -K_{o,f}^TK_{o,f}, \\ Y_fB_f + C_f^TM_0 &= -K_{o,f}^TJ_o. \end{aligned} \tag{3.8}$$

1. and 2.: By the results for bounded real standard state space systems [30, 32], the matrix X_f solves (3.4) if and only if $Y_f = X_f^{-1}$ solves (3.8). Thus, if X and Y are the solution of (3.2) and (3.5), respectively, then

$$(EXE^T)_l^- = T_l^{-T}\begin{bmatrix} X_f^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad (E^TYE)_r^- = T_r^{-1}\begin{bmatrix} Y_f^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_r^{-T}$$

satisfy (3.5) and (3.2), respectively.

3. and 4.: Furthermore, the Lur'e equations (3.4) and (3.8) have the extremal solutions that satisfy

$$0 < X_{\min,f} \le X_f \le X_{\max,f}, \qquad 0 < Y_{\min,f} \le Y_f \le Y_{\max,f}$$

for all solutions X_f and Y_f of (3.4) and (3.8), respectively. We also obtain that $Y_{\max,f} = X_{\min,f}^{-1}$ and $Y_{\min,f} = X_{\max,f}^{-1}$. Thus, the matrices

$$\begin{aligned} X_{\min} &= T_r^{-1}\begin{bmatrix} X_{\min,f} & 0 \\ 0 & 0 \end{bmatrix} T_r^{-T}, \quad X_{\max} = T_r^{-1}\begin{bmatrix} X_{\max,f} & 0 \\ 0 & 0 \end{bmatrix} T_r^{-T}, \\ Y_{\min} &= T_l^{-T}\begin{bmatrix} Y_{\min,f} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \quad Y_{\max} = T_l^{-T}\begin{bmatrix} Y_{\max,f} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1} \end{aligned} \tag{3.9}$$

are the extremal solutions of the projected Lur'e equations (3.2) and (3.5). Clearly, they satisfy (3.6) and (3.7).

5. In order to show that the eigenvalues of $X_{\min}E^TY_{\min}E$ are real and located in the interval $[0,1]$, we make use of

$$X_{\min}E^TY_{\min}E = T_r^{-1}\begin{bmatrix} X_{\min,f}Y_{\min,f} & 0 \\ 0 & 0 \end{bmatrix} T_r.$$

Then the required result follows again from the known facts for standard state space systems, see [30, 32]. □

It is known that for $E = I$, the extremal solutions of the Lur'e equations have an interpretation in terms of energy that can be extracted from a given state [30,32,47]. We will now extend this result to descriptor systems.

THEOREM 3.4. *Consider a descriptor system* (1.1) *that is S-minimal and contractive. Let $Y_{\max}$ and $Y_{\min}$ be the extremal solutions of* (3.5)*. Then for all $x_0 \in \mathbb{R}^n$ the following identities hold:*

$$x_0^TE^TY_{\max}Ex_0 = \inf\left\{ \int_{-\infty}^{0} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m) \textit{ consistent and control-} \\ \textit{ling to some } x(0) \textit{ with } Ex(0) = Ex_0 \end{array} \right\}, \quad (3.10)$$

$$x_0^TE^TY_{\min}Ex_0 = \sup\left\{ -\int_{0}^{\infty} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_+,\mathbb{R}^m) \textit{ consistent with some} \\ x(0) \textit{ satisfying } Ex(0) = Ex_0 \end{array} \right\}. \quad (3.11)$$

Proof. Since the system is proper and both controllable and observable at infinity, it has index at most one [12]. Without loss of generality we may assume that the pencil $\lambda E - A$ is in Weierstrass canonical form (2.1). Then the extremal solutions of (3.5) have the form (3.9), where $Y_{\min,f}$ and $Y_{\max,f}$ are the extremal solutions of the Lur'e equations (3.8). Let $x_0 = [\,x_{0,f}^T,\, x_{0,\infty}^T\,]^T$ with $x_{0,f} \in \mathbb{R}^{n_f}$ and $x_{0,\infty} \in \mathbb{R}^{n_\infty}$.

We first show the equality (3.10). Since $\boldsymbol{G}_f = [\,I,\, A_f,\, B_f,\, C_f,\, M_0\,]$ is controllable, there exists a sequence of smooth (and, thus, consistent) controls $u_n \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m)$ with corresponding outputs y_n and states $x_{n,f}$ [48] such that $x_{n,f}(0) = x_{0,f}$ and

$$\lim_{n\to\infty}\int_{-\infty}^{0} \|u_n(\tau)\|^2 - \|y_n(\tau)\|^2 d\tau$$
$$= \inf\left\{ \int_{-\infty}^{0} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; u \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m) \text{ controlling to } x_f(0) = x_{0,f} \right\}.$$

This together with the fact that the set of consistent inputs is contained in $\mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m)$ leads to

$$\inf\left\{ \int_{-\infty}^{0} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; u \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m) \text{ controlling to } x_f(0) = x_{0,f} \right\}$$
$$= \inf\left\{ \int_{-\infty}^{0} \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m) \text{ consistent and controlling} \\ \text{to } x_f(0) = x_{0,f} \end{array} \right\}.$$

The left-hand side equals $x_{0,f}^TY_{\max,f}x_{0,f} = x_0^TE^TY_{\max}Ex_0$ by the result for standard systems. Furthermore, the expression on the right-hand side is equal to the right-hand side of (3.10).

We now show relation (3.11). Since the set of inputs $u \in \mathbb{L}_2(\mathbb{R}_+, \mathbb{R}^m)$ consistent with some $x(0)$ satisfying $Ex_0 = Ex(0)$ is dense in $\mathbb{L}_2(\mathbb{R}_+, \mathbb{R}^m)$, we have

$$\begin{aligned}
&\sup\left\{ -\int_0^\infty \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_+, \mathbb{R}^m) \text{ consistent with some } x(0) \\ \text{satisfying } Ex(0) = Ex_0 \end{array} \right\} \\
=&\sup\left\{ -\int_0^\infty \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \;\middle|\; u \in \mathbb{L}_2(\mathbb{R}_+, \mathbb{R}^m) \text{ with } x_f(0) = x_{0,f} \right\}.
\end{aligned}$$

The right-hand side equals $x_{0,f}^T Y_{\min,f} x_{0,f} = x_0^T E^T Y_{\min} E x_0$ by the result for standard systems. Furthermore, the expression on the left-hand side is equal to the right-hand side of (3.11). ∎

Assuming that the amount of energy provided by the environment to the descriptor system (1.1) is the difference between input and output energy, the quantity (3.10) can be interpreted as the *required supply*, i.e., the minimum amount of energy that must be provided to control system (1.1) to the state $x(0)$ satisfying $Ex(0) = Ex_0$ over any possible trajectory. Further, (3.11) is the *available storage energy* or the maximum amount of energy that can be extracted from the system over any possible trajectory of the state from an initial state.

The difference to the corresponding results for standard state space systems is that we have the additional consistency conditions for the input due to the algebraic equations contained in the descriptor system (1.1). One can see that the consistency condition in (3.11) becomes trivial in the case $E = I$.

3.2. Bounded real balancing. Subsequently, we define the notions of Gramians and bounded real balanced realizations for descriptor systems.

DEFINITION 3.5. *Let $\boldsymbol{G} = [\,E,\, A,\, B,\, C,\, D\,]$ be R-minimal and contractive. The minimal solutions of the projected Lur'e equations* (3.2) *and* (3.5)*, denoted by X_{br} and Y_{br}, respectively, are called the* bounded real controllability *and* observability Gramians. *The square roots of non-zero eigenvalues of the matrix $X_{br}E^TY_{br}E$, denoted by ξ_j, are called* bounded real characteristic values *of $\boldsymbol{G}$.*

We will assume that the characteristic values are ordered decreasingly. It follows from Theorem 3.3 that the R-minimal descriptor system (1.1) has n_f non-zero characteristic values that do not exceed one. Note that ξ_j are input-output invariant, i.e., they are preserved under a system equivalence transformation.

DEFINITION 3.6. *A realization $\boldsymbol{G} = [\,E,\, A,\, B,\, C,\, D\,]$ is called* bounded real balanced *if $X_{br} = Y_{br} = \operatorname{diag}(\Sigma, 0)$ with $\Sigma = \operatorname{diag}(\xi_1, \ldots, \xi_{n_f})$.*

For any R-minimal, contractive realization $\boldsymbol{G} = [\,E,\, A,\, B,\, C,\, D\,]$, there always exist transformation matrices W_b and T_b such that the transformed realization $[W_bET_b, W_bAT_b, W_bB, CT_b, D]$ is bounded real balanced. Such a balancing transformation can be constructed analogously to the Lyapunov-based balancing [42]. Note that the matrix pencil of a bounded real balanced realization is automatically in Weierstrass-like form

$$\lambda W_bET_b - W_bAT_b = \begin{bmatrix} \lambda I - A_1 & 0 \\ 0 & \lambda E_2 - A_2 \end{bmatrix} \tag{3.12}$$

for some $A_1 \in \mathbb{R}^{n_f, n_f}$ and $E_2, A_2 \in \mathbb{R}^{n_\infty, n_\infty}$, where A_2 is nonsingular and $A_2^{-1}E_2$ is nilpotent.

Now we consider model reduction of contractive descriptor systems. Due to the variational representation of the bounded real Gramians in Theorem 3.4, it is evident to remove those states $x_0 \in \mathbb{R}^n$ for which $x_0^T E^T Y_{\max} E x_0$ is large and at the same time $x_0^T E^T Y_{\min} E x_0$ is small. If the descriptor system (1.1) is bounded real balanced, i.e.,

$$Y_{\min} = Y_{br} = \mathrm{diag}(\Sigma, 0), \qquad Y_{\max} = (E X_{br} E^T)_l^- = \mathrm{diag}(\Sigma^{-1}, 0),$$

then the states corresponding to the small bounded real characteristic values are good candidates for the reduction. This leads to the method of bounded real balanced truncation. The following theorem introduces this method and gives an error bound.

THEOREM 3.7. *Let a bounded real balanced system $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ be given with the bounded real characteristic values $\xi_1 \geq \ldots \geq \xi_{\ell_f} > \xi_{\ell_f+1} \geq \ldots \geq \xi_{n_f}$. Then a reduced-order system $\widetilde{\boldsymbol{G}} = [\,Z^T E Z,\ Z^T A Z,\ Z^T B,\ CZ,\ D\,]$ with*

$$Z = \begin{bmatrix} I_{\ell_f} & 0 \\ 0 & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \tag{3.13}$$

has the bounded real transfer function, and we have the error bound

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \leq 2(\xi_{\ell_f+1} + \ldots + \xi_{n_f}). \tag{3.14}$$

Proof. Since $\boldsymbol{G}$ is bounded real balanced, it has a standard state space realization $[\,I_{n_f},\ A_1,\ B_1,\ C_1,\ M_0\,]$. The transfer function of the reduced-order model is then given by $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}_1(sI - \widetilde{A}_1)^{-1}\widetilde{B}_1 + M_0$, where $\widetilde{A}_1 = \widetilde{Z}^T A_1 \widetilde{Z}$, $\widetilde{B}_1 = \widetilde{Z}^T B_1$, $\widetilde{C}_1 = C_1 \widetilde{Z}$ with $\widetilde{Z} = [\,I_{\ell_f},\ 0_{\ell_f, n_f - \ell_f}\,]^T$. Then the required results follow from the properties of classical bounded real balanced truncation [32]. ∎

The state-space dimension of the approximate system $\widetilde{\boldsymbol{G}}$ can be reduced further if we truncate the states that are uncontrollable and unobservable at infinity. The input-output relation of the system remains thereby unchanged and the error bound (3.14) still holds. Such states can be determined using the *improper controllability* and *observability Gramians* X_{imp} and Y_{imp} of the descriptor system (1.1), see [42]. These Gramians are defined as unique symmetric, positive semidefinite solutions of the projected discrete-time Lyapunov equations

$$A X_{imp} A^T - E X_{imp} E^T = Q_l B B^T Q_l^T, \quad X_{imp} = Q_r X_{imp} Q_r^T, \tag{3.15}$$

$$A^T Y_{imp} A - E^T Y_{imp} E \;= Q_r^T C^T C Q_r, \quad Y_{imp} = Q_l^T Y_{imp} Q_l. \tag{3.16}$$

The matrix $X_{imp} A^T Y_{imp} A$ has non-negative eigenvalues, and the square roots of its largest n_∞ eigenvalues define the *improper Hankel singular values* θ_j of system (1.1). States that are uncontrollable and unobservable at infinity correspond to the zero improper Hankel singular values.

Analogous to [42], we can now formulate an algorithm for computing the contractive reduced-order models via the bounded real balanced truncation method.

ALGORITHM 3.8. Bounded real balanced truncation for descriptor systems.
Given a contractive system $\boldsymbol{G} = [\,E,\ A,\ B,\ C,\ D\,]$ and the projectors P_l, P_r, compute a reduced-order contractive system $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D}\,]$.

1. Compute the Cholesky factors $\hat{R}$ and $\hat{L}$ of the solutions $X_{imp} = \hat{R}\hat{R}^T$ and $Y_{imp} = \hat{L}\hat{L}^T$ of the projected Lyapunov equations (3.15) and (3.16), respectively.
2. Compute the singular value decomposition $\hat{L}^T A \hat{R} = U_3 \Theta V_3^T$, where U_3 and V_3 have orthonormal columns, and $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_{\ell_\infty})$ is nonsingular.

3. Compute the matrix $M_0 = D - C\hat{R}V_3\Theta^{-1}U_3^T\hat{L}^TB$.
4. Compute the Cholesky factors R and L of the bounded real Gramians $X_{br} = RR^T$ and $Y_{br} = LL^T$ that are the minimal solutions of the bounded real projected Lur'e equations (3.2) and (3.5), respectively.
5. Compute the singular value decomposition

$$L^TER = [\,U_1,\,U_2\,]\begin{bmatrix}\Sigma_1 & \\ & \Sigma_2\end{bmatrix}[\,V_1,\,V_2\,]^T,$$

where $[\,U_1, U_2\,]$, $[\,V_1, V_2\,]$ have orthonormal columns, $\Sigma_1 = \operatorname{diag}(\xi_1, \ldots, \xi_{\ell_f})$ and $\Sigma_2 = \operatorname{diag}(\xi_{\ell_f+1}, \ldots, \xi_{n_f})$.
6. Compute the reduced system $[\,\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\,] = [\,WET,\,WAT,\,WB,\,CT,\,D\,]$ with $W = [\,LU_1\Sigma_1^{-1/2},\ \hat{L}U_3\Theta^{-1/2}\,]^T$ and $T = [\,RV_1\Sigma_1^{-1/2},\ \hat{R}V_3\Theta^{-1/2}\,]$.

The numerical solution of the projected Lur'e equations (3.2) and (3.5) will be considered in more detail in Section 5. For solving the projected Lyapunov equations (3.15) and (3.16) we can use the generalized Smith method [43]. This method provides the Cholesky factors $\hat{R} \in \mathbb{R}^{n,\nu m}$ and $\hat{L} \in \mathbb{R}^{n,\nu p}$ of the improper Gramians $X_{imp} = \hat{R}\hat{R}^T$ and $Y_{imp} = \hat{L}\hat{L}^T$ without computing these Gramians explicitly.

Remark 3.9. The reduced-order system computed by Algorithm 3.8 preserves the matrix D. A further elimination of the non-dynamic modes leads to another system $[\,W_fET_f,\,W_fAT_f,\,W_fB,\,CT_f,\,M_0\,]$ with $W_f = \Sigma_1^{-1/2}U_1^TL^T$ and $T_f = RV_1\Sigma_1^{-1/2}$, which has the same transfer function and even lower order. In the next section we will see that not always it makes sense to eliminate non-dynamic modes in bounded real balanced truncation.

4. Positive real balanced truncation. Passivity-preserving model reduction of standard state space systems via positive real balanced truncation was considered in [11, 21, 23, 30, 34, 46]. In this method the truncation of states is performed on the basis of solutions of so-called *positive real Lur'e equations*. As in the previous section, here we present a projector-based generalization of the positive real balanced truncation method to descriptor systems.

4.1. Positive real lemma for descriptor systems. First of all we establish an equivalence between positive realness and the solvability of certain projected Lur'e equations.

THEOREM 4.1 (Generalized positive real lemma). *Consider a descriptor system (1.1) with a square transfer function $\boldsymbol{G}$. Let P_r and P_l be the spectral projectors onto the right and left deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues and let $M_k \in \mathbb{R}^{m,m}$ be as in (2.4).*

1. *If system (1.1) is R-minimal and $\boldsymbol{G}$ is positive real, then the projected Lur'e equations*

$$\begin{aligned} AXE^T + EXA^T &= -K_cK_c^T, & X = P_rXP_r^T \ge 0,\\ EXC^T - P_lB &= -K_cJ_c^T, & M_0 + M_0^T = J_cJ_c^T \end{aligned} \tag{4.1}$$

are solvable for $K_c \in \mathbb{R}^{n,m}$, $J_c \in \mathbb{R}^{m,m}$ and $X \in \mathbb{R}^{n,n}$.
2. *If $M_k = 0$ for $k > 1$, $M_1 = M_1^T \ge 0$ and the projected Lur'e equations (4.1) have a solution, then $\boldsymbol{G}$ is positive real.*

Proof. The desired result can be proved analogously to Theorem 3.1 using the Weierstrass canonical form (2.1), Proposition 2.7 and the positive real lemma for standard state space systems [1]. ∎

Note that the solvability of (4.1) is equivalent to the solvability of the dual projected Lur'e equations

$$\begin{aligned} A^TYE + E^TYA &= -K_o^TK_o, \quad Y = P_l^TYP_l \geq 0, \\ E^TYB - P_r^TC^T &= -K_o^TJ_o, \quad M_0 + M_0^T = J_o^TJ_o \end{aligned} \tag{4.2}$$

for $K_o \in \mathbb{R}^{m,n}$, $J_o \in \mathbb{R}^{m,m}$ and $Y \in \mathbb{R}^{n,n}$. A counterpart of Theorem 3.3 can also be established for the positive real projected Lur'e equations (4.1) and (4.2).

Remark 4.2. In [18], positive realness of descriptor systems is characterized via feasibility of certain linear matrix inequalities. The existence of a solution of these inequalities was shown to be sufficient for positive realness. However, for necessity, the additional condition $D + D^T \geq M_0 + M_0^T$ further has to be valid.

4.2. Positive real balancing. We now define a positive real balanced realization and related notions for descriptor systems.

DEFINITION 4.3. *Let $\boldsymbol{G} = [\,E,\, A,\, B,\, C,\, D\,]$ be R-minimal and passive. The minimal solutions of the projected Lur'e equations* (4.1) *and* (4.2)*, denoted by X_{br} and Y_{br}, respectively, are called the* positive real controllability *and* observability Gramians. *The square roots of non-zero eigenvalues of the matrix $X_{br}E^TY_{br}E$, denoted by π_j, are called* positive real characteristic values *of $\boldsymbol{G}$. The system $\boldsymbol{G}$ is called* positive real balanced, *if $X_{pr} = Y_{pr} = \operatorname{diag}(\Pi, 0)$ with $\Pi = \operatorname{diag}(\pi_1, \ldots, \pi_{n_f})$.*

In a straightforward way, we can now introduce the positive real balanced truncation method for descriptor systems.

ALGORITHM 4.4. Positive real balanced truncation for descriptor systems.

Given a passive system $\boldsymbol{G} = [\,E,\, A,\, B,\, C,\, D\,]$ and the projectors P_l, P_r, compute a reduced-order passive system $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D}\,]$.

1. Compute the Cholesky factors $\hat{R}$ and $\hat{L}$ of the solutions $X_{imp} = \hat{R}\hat{R}^T$ and $Y_{imp} = \hat{L}\hat{L}^T$ of the projected Lyapunov equations (3.15) and (3.16), respectively.
2. Compute the singular value decomposition $\hat{L}^TA\hat{R} = U_3\Theta V_3^T$, where U_3 and V_3 have orthonormal columns, and $\Theta = \operatorname{diag}(\theta_1, \ldots, \theta_{\ell_\infty})$ is nonsingular.
3. Compute the matrix $M_0 = D - C\hat{R}V_3\Theta^{-1}U_3^T\hat{L}^TB$.
4. Compute the Cholesky factors R and L of the positive real Gramians $X_{pr} = RR^T$ and $Y_{pr} = LL^T$ that are the minimal solutions of the positive real projected Lur'e equations (4.1) and (4.2), respectively.
5. Compute the singular value decomposition
$$L^TER = [\,U_1,\, U_2\,]\begin{bmatrix} \Pi_1 & \\ & \Pi_2 \end{bmatrix}[\,V_1,\, V_2\,]^T,$$
where $[\,U_1, U_2\,]$, $[\,V_1, V_2\,]$ have orthonormal columns, $\Pi_1 = \operatorname{diag}(\pi_1, \ldots, \pi_{\ell_f})$ and $\Pi_2 = \operatorname{diag}(\pi_{\ell_f+1}, \ldots, \pi_{n_f})$.
6. Compute the reduced system $[\,\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\,] = [\,WET, WAT, WB, CT, D\,]$ with $W = [\,LU_1\Pi_1^{-1/2},\, \hat{L}U_3\Theta^{-1/2}\,]^T$ and $T = [\,RV_1\Pi_1^{-1/2},\, \hat{R}V_3\Theta^{-1/2}\,]$.

Similarly to the standard state space case [23], one can show that the reduced-order model computed by this algorithm is passive. Moreover, its transfer function $\widetilde{\boldsymbol{G}}$ has the same polynomial part as the transfer function $\boldsymbol{G}$ of the original system, and we have the following error bounds

$$\|\widetilde{\boldsymbol{G}}-\boldsymbol{G}\|_{\mathbb{H}_\infty} \le \left\|M_0+M_0^T\right\|_2 \sum_{j=\ell_f+1}^{n_f} \frac{2\pi_j}{(1-\pi_j)^2}\left(1+\sum_{k=1}^{j-1}\frac{2\pi_k}{1-\pi_k}\right)^2, \tag{4.3}$$

$$\|\widetilde{\boldsymbol{G}}-\boldsymbol{G}\|_{\mathbb{H}_\infty} \le 2\left\|(M_0+M_0^T)^{-1}\right\|_2 \|\boldsymbol{G}+M_0^T\|_{\mathbb{H}_\infty}\|\widetilde{\boldsymbol{G}}+M_0^T\|_{\mathbb{H}_\infty}\sum_{j=\ell_f+1}^{n_f}\pi_j \tag{4.4}$$

that can be proved as in [11] and [21], respectively.

Another possible approach for passivity-preserving model reduction is based on the Moebius transformation of the transfer function $\boldsymbol{G}$ as presented in (2.9). If $I+D$ is nonsingular, then a state space realization $[\,\mathcal{E},\,\mathcal{A},\,\mathcal{B},\,\mathcal{C},\,\mathcal{D}\,]$ of $\boldsymbol{H}(s)=\mathcal{M}(\boldsymbol{G})(s)$ is given in (2.10). In this case we can compute a reduced-order contractive system $\widetilde{\boldsymbol{H}}=[\,W\mathcal{E}T,\,W\mathcal{A}T,\,W\mathcal{B},\,\mathcal{C}T,\,\mathcal{D}\,]$ using Algorithm 3.8. Finally, a back transformation $\widetilde{\boldsymbol{G}}(s)=\mathcal{M}(\widetilde{\boldsymbol{H}})(s)$ leads to a reduced-order passive model that has the realization

$$\widetilde{\boldsymbol{G}}=[\,\widetilde{E},\,\widetilde{A},\,\widetilde{B},\,\widetilde{C},\,\widetilde{D}\,]=[\,WET,\,WAT,\,WB,\,CT,\,D\,].$$

Indeed, taking into account (2.10) and using the relation

$$(I+\mathcal{D})^{-1}=(I+(I-D)(I+D)^{-1})^{-1}=(I+D)/2,$$

we have $\widetilde{E}=W\mathcal{E}T=WET$ and

$$\begin{aligned}
\widetilde{A} &= W\mathcal{A}T-W\mathcal{B}(I+\mathcal{D})^{-1}\mathcal{C}T\\
&= W\big(A-B(I+D)^{-1}C+B(I+D)^{-1}C\big)T=WAT,\\
\widetilde{B} &= -\sqrt{2}\,W\mathcal{B}(I+\mathcal{D})^{-1}=WB,\\
\widetilde{C} &= \;\;\sqrt{2}\,(I+\mathcal{D})^{-1}\mathcal{C}T=CT,\\
\widetilde{D} &= (I-\mathcal{D})(I+\mathcal{D})^{-1}=D.
\end{aligned}$$

Remark 4.5. It should be noted that if we eliminate the non-dynamic modes of $\widetilde{\boldsymbol{H}}$ *as described in Remark 3.9, then* $I+\widetilde{D}$ *with the feedthrough matrix* $\widetilde{D}$ *of the resulting reduced-order system may not be invertible. In this case the Moebius transformation of* $\widetilde{\boldsymbol{H}}$ *may not exist.*

Summarizing, we obtain the following algorithm.

Algorithm 4.6. Passivity-preserving model reduction method via bounded real balanced truncation.

Given a passive system $\boldsymbol{G}=[\,E,\,A,\,B,\,C,\,D\,]$ with nonsingular $I+D$,
compute a reduced-order passive system $\widetilde{\boldsymbol{G}}=[\,\widetilde{E},\,\widetilde{A},\,\widetilde{B},\,\widetilde{C},\,\widetilde{D}\,]$.

1. Compute the projection matrices W and T using Algorithm 3.8 applied to the Moebius-transformed system $\boldsymbol{H}=\mathcal{M}(\boldsymbol{G})$ with a realization as in (2.10).
2. Compute the reduced system $[\,\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}\,]=[\,WET,WAT,WB,CT,D\,]$.

The following theorem provides an error bound for the reduced-order system computed by Algorithm 4.6.

Theorem 4.7. *Consider a passive descriptor system* (1.1) *with a transfer function* $\boldsymbol{G}$. *Let* $\widetilde{\boldsymbol{G}}$ *be a transfer function of a reduced-order model* (1.2) *computed by Algorithm* 4.6. *If*

$$\|I+\boldsymbol{G}\|_{\mathbb{H}_\infty}(\xi_{\xi_f+1}+\ldots+\xi_{n_f})<1, \tag{4.5}$$

where ξ_j are the bounded real characteristic values of the Moebius-transformed system $\boldsymbol{H} = \mathcal{M}(\boldsymbol{G})$, then

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le \frac{\|I + \boldsymbol{G}\|^2_{\mathbb{H}_\infty}(\xi_{\ell_f+1} + \ldots + \xi_{n_f})}{1 - \|I + \boldsymbol{G}\|_{\mathbb{H}_\infty}(\xi_{\ell_f+1} + \ldots + \xi_{n_f})}. \tag{4.6}$$

Proof. We have

$$\begin{aligned} \widetilde{\boldsymbol{G}} - \boldsymbol{G} &= (I - \widetilde{\boldsymbol{H}})(I + \widetilde{\boldsymbol{H}})^{-1} - (I - \boldsymbol{H})(I + \boldsymbol{H})^{-1} \\ &= 2\,(I + \widetilde{\boldsymbol{H}})^{-1}(\boldsymbol{H} - \widetilde{\boldsymbol{H}})(I + \boldsymbol{H})^{-1}. \end{aligned}$$

Then

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le \frac{2\|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathbb{H}_\infty}\|(I + \boldsymbol{H})^{-1}\|^2_{\mathbb{H}_\infty}}{1 - \|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathbb{H}_\infty}\|(I + \boldsymbol{H})^{-1}\|_{\mathbb{H}_\infty}}$$

provided that

$$\|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathbb{H}_\infty}\|(I + \boldsymbol{H})^{-1}\|_{\mathbb{H}_\infty} < 1. \tag{4.7}$$

It follows from Theorem 3.7 that $\|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathbb{H}_\infty} \le 2(\xi_{\ell_f+1} + \ldots + \xi_{n_f})$. Furthermore,

$$(I + \boldsymbol{H})^{-1} = \big(I + (I - \boldsymbol{G})(I + \boldsymbol{G})^{-1}\big)^{-1} = (I + \boldsymbol{G})/2.$$

Thus, the assumption (4.5) guarantees (4.7) and yields bound (4.6). □

Note that Algorithm 4.6, in general, does not preserve the polynomial part of the transfer function $\boldsymbol{G}$ and, therefore, it delivers a different result compared to those provided by Algorithm 4.4. However, in the case of proper $\boldsymbol{G}$ we can show that the reduced-order models of the same order have the equal transfer functions.

THEOREM 4.8. *Consider a descriptor system* (1.1) *that has a proper and positive real transfer function $\boldsymbol{G}$. Let $\widetilde{\boldsymbol{G}}_1$ and $\widetilde{\boldsymbol{G}}_2$ be the transfer function of the reduced models of the same order obtained by Algorithms 4.4 and 4.6, respectively. Then $\widetilde{\boldsymbol{G}}_1(s) \equiv \widetilde{\boldsymbol{G}}_2(s)$.*

Proof. Without loss of generality we may assume that (1.1) is R-minimal. Applying Algorithm 4.4 to $\boldsymbol{G}$, we obtain a reduced-order model $\widetilde{\boldsymbol{G}}_1$. By the results in [30], we have that a reduced-order model obtained by bounded real balanced truncation of the Moebius-transformed system $\boldsymbol{H} = \mathcal{M}(\boldsymbol{G})$ satisfies $\widetilde{\boldsymbol{H}} = \mathcal{M}(\widetilde{\boldsymbol{G}}_1)$. Thus, the reduced model computed by Algorithm 4.6 has the transfer function $\widetilde{\boldsymbol{G}}_2 = \mathcal{M}(\widetilde{\boldsymbol{H}}) = \widetilde{\boldsymbol{G}}_1$. □

From Theorem 3.4, we know that the bounded real Gramians have an interpretation in terms of energy inflow and outflow of the contractive descriptor system. We will show that an analogous result holds for the passive descriptor systems if one defines the amount of energy provided by the environment to the system as a scalar product of input and output and one takes the extremal solutions of the bounded real Lur'e equations of the Moebius-transformed system.

THEOREM 4.9. *Consider a descriptor system* (1.1) *that is S-minimal and has a positive real transfer function $\boldsymbol{G}$. Let $\mathcal{Y}_{\min}$ and $\mathcal{Y}_{\max}$ be the extremal solutions of the bounded real projected Lur'e equations for the Moebius-transformed system $\boldsymbol{H} = \mathcal{M}(\boldsymbol{G})$. Then the following relations*

$$\begin{aligned} & x_0^T E^T \mathcal{Y}_{\max} E x_0 \\ &= \inf\left\{ 2\int_{-\infty}^0 u(\tau)^T y(\tau)d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_-, \mathbb{R}^m) \textit{ consistent and controlling} \\ \textit{to some } x(0) \textit{ with } Ex(0) = Ex_0 \end{array} \right\}, \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&x_0^T E^T \mathcal{Y}_{\min} E x_0 \\
&= \sup\left\{ -2\int_0^\infty u(\tau)^T y(\tau)d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_+,\mathbb{R}^m) \textit{ and } u \textit{ is consistent with} \\ \textit{some } x(0) \textit{ satisfying } Ex(0)=Ex_0 \end{array} \right\}
\end{aligned} \tag{4.9}
$$

hold for all $x_0 \in \mathbb{R}^n$*.*

Proof. Since $\boldsymbol{G}$ is S-minimal, this also holds true for the Moebius-transformed system $\boldsymbol{H} = \mathcal{M}(\boldsymbol{G})$. Let $\hat{u}(t)$, $\hat{x}(t)$ and $\hat{y}(t)$ be respectively, the input, state and output of $\boldsymbol{H} = \mathcal{M}(\boldsymbol{G})$. Then

$$
u(t) = -\frac{1}{\sqrt{2}}\hat{u}(t) - \frac{1}{\sqrt{2}}\hat{y}(t), \quad x(t) = \hat{x}(t), \quad y(t) = -\frac{1}{\sqrt{2}}\hat{u}(t) + \frac{1}{\sqrt{2}}\hat{y}(t)
$$

are the input, state and output of system (1.1). We have $2u(t)^T y(t) = \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2$ for all $t \in \mathbb{R}$. Using Theorem 3.4, we obtain that

$$
\begin{aligned}
&\inf\left\{ 2\int_{-\infty}^0 u(\tau)^T y(\tau)d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m) \text{ consistent and controlling to} \\ \text{some } x(0) \text{ with } Ex(0)=Ex_0 \end{array} \right\} \\
&= \inf\left\{ \int_{-\infty}^0 \|\hat{u}(\tau)\|^2 - \|\hat{y}(\tau)\|^2 d\tau \;\middle|\; \begin{array}{l} \hat{u} \in \mathbb{L}_2(\mathbb{R}_-,\mathbb{R}^m) \text{ consistent and control-} \\ \text{ling to some } \hat{x}(0) \text{ with } E\hat{x}(0)=Ex_0 \end{array} \right\} \\
&= x_0^T E^T \mathcal{Y}_{\max} E x_0
\end{aligned}
$$

and, further,

$$
\begin{aligned}
&\sup\left\{ -2\int_0^\infty u(\tau)^T y(\tau)d\tau \;\middle|\; \begin{array}{l} u \in \mathbb{L}_2(\mathbb{R}_+,\mathbb{R}^m) \text{ consistent with some } x(0) \\ \text{satisfying } Ex(0)=Ex_0 \end{array} \right\} \\
&= \sup\left\{ -\int_0^\infty \left(\|\hat{u}(\tau)\|^2 - \|\hat{y}(\tau)\|^2\right) d\tau \;\middle|\; \begin{array}{l} \hat{u} \in \mathbb{L}_2(\mathbb{R}_+,\mathbb{R}^m) \text{ consistent with some } \hat{x}(0) \\ \text{satisfying } E\hat{x}(0)=Ex_0 \end{array} \right\} \\
&= x_0^T E^T \mathcal{Y}_{\min} E x_0.
\end{aligned}
$$

□

5. Numerical solution of projected Lur'e equations. In this section, we discuss the numerical solution of the projected Lur'e equations (3.2), (3.5), (4.1) and (4.2). We will treat both the positive real and bounded real cases simultaneously by considering the projected Lur'e equations

$$
\begin{aligned}
A^T Y E + E^T Y A + P_r^T Q^T Q P_r &= -K_o^T K_o, \qquad Y = P_l^T Y P_l \geq 0, \\
E^T Y B + P_r^T H^T &= -K_o^T J,
\end{aligned} \tag{5.1}
$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $H \in \mathbb{R}^{m,n}$, $Q \in \mathbb{R}^{p,n}$ and $J \in \mathbb{R}^{m,m}$ are given matrices and $Y \in \mathbb{R}^{n,n}$, $K_o \in \mathbb{R}^{m,n}$ are unknown matrices. We have the bounded real projected Lur'e equations (3.2) if $Q = C$, $H = M_0^T C$ and $J^T J = I - M_0^T M_0$. Furthermore, for $Q = 0$, $H = -C$ and $J^T J = M_0 + M_0^T$, we obtain the positive real projected Lur'e equations (4.1).

5.1. Deflating subspaces and projected Lur'e equations. Similarly to the standard state space case [24, 26], the extremal solutions $Y_{\min}$ and $Y_{\max}$ of (5.1) can be determined from deflating subspaces of an extended Hamiltonian pencil

$$
\lambda\mathcal{M} - \mathcal{N} = \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & P_l B \\ P_r^T Q^T Q P_r & -A^T & P_r^T H^T \\ -H P_r & B^T P_l^T & J^T J \end{bmatrix}. \tag{5.2}
$$

Note that if $\lambda \in \mathbb{C}$ is an eigenvalue of $\lambda\mathcal{M} - \mathcal{N}$, then $-\overline{\lambda}$ is also an eigenvalue of this pencil. Since $\mathcal{M}$ and $\mathcal{N}$ are both real, the finite eigenvalues of $\lambda\mathcal{M} - \mathcal{N}$ occur in quadruples $(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda})$. This property is known as Hamiltonian symmetry of the pencil [29]. Analogously to the standard case [24, Section 5.1], it can be shown that if J is nonsingular and the pencil $\lambda\mathcal{M} - \mathcal{N}$ has no finite eigenvalues on the imaginary axis, then $\lambda\mathcal{M} - \mathcal{N}$ has the stable (antistable) right and left deflating subspaces of dimension n_f. We now establish the connection between the deflating subspaces of $\lambda\mathcal{M} - \mathcal{N}$ and the solution of the projected Lur'e equations (5.1). Before, we need some preliminary results.

LEMMA 5.1. *Consider the pencil $\lambda\mathcal{M} - \mathcal{N}$ given in* (5.2). *Let the columns of a matrix*

$$\mathcal{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \tag{5.3}$$

with Z_1, $Z_2 \in \mathbb{R}^{n,k}$ and $Z_3 \in \mathbb{R}^{m,k}$ span a right deflating subspace of $\lambda\mathcal{M} - \mathcal{N}$, i.e., $\mathcal{M}\mathcal{Z}\Lambda = \mathcal{N}\mathcal{Z}$ for some $\Lambda \in \mathbb{R}^{k,k}$. If $\mathrm{Sp}(\Lambda) \cap \mathrm{Sp}(-\Lambda) = \emptyset$, *then*

$$Z_2^T E Z_1 = Z_1^T E^T Z_2, \tag{5.4}$$

$$Z_2^T A Z_1 + Z_1^T A^T Z_2 = Z_3^T J^T J Z_3. \tag{5.5}$$

Proof. It follows from the equation $\mathcal{M}\mathcal{Z}\Lambda = \mathcal{N}\mathcal{Z}$ that

$$E Z_1 \Lambda = A Z_1 + P_l B Z_3, \tag{5.6}$$

$$E^T Z_2 \Lambda = P_r^T Q^T Q P_r Z_1 - A^T Z_2 + P_r^T H^T Z_3, \tag{5.7}$$

$$0 = -H P_r Z_1 + B^T P_l^T Z_2 + J^T J Z_3. \tag{5.8}$$

Premultiplying (5.6) by Z_2^T and postmultiplying the transposed equation (5.7) by Z_1, we obtain

$$Z_2^T E Z_1 \Lambda = Z_2^T A Z_1 + Z_2^T P_l B Z_3, \tag{5.9}$$

$$\Lambda^T Z_2^T E Z_1 = -Z_2^T A Z_1 + Z_1^T P_r^T Q^T Q P_r Z_1 + Z_3^T H P_r Z_1. \tag{5.10}$$

Adding these equations and using (5.8), we get the Lyapunov equation

$$\begin{aligned} \Lambda^T (Z_2^T E Z_1) + (Z_2^T E Z_1)\Lambda = {} & Z_1^T P_r^T H^T Z_3 + Z_3^T H P_r Z_1 \\ & + Z_1^T P_r^T Q^T Q P_r Z_1 - Z_3^T J^T J Z_3 \end{aligned}$$

with a symmetric right-hand side. Since $\mathrm{Sp}(\Lambda) \cap \mathrm{Sp}(-\Lambda) = \emptyset$, such an equation has a unique symmetric solution $Z_2^T E Z_1$, see [19]. Thus, equation (5.4) holds. Furthermore, subtracting the transposed equation (5.10) from (5.9) and using (5.4), (5.8), we obtain relation (5.5). □

THEOREM 5.2. *Let the columns of a matrix $\mathcal{Z}$ as in* (5.3) *span a stable right deflating subspace of the pencil $\lambda\mathcal{M} - \mathcal{N}$ that has the dimension $k = n_f$. If Z_1 is of full column rank, then the minimal solution of the projected Lur'e equations* (5.1) *is given by $Y_{\min} = -Z_2(EZ_1)^-_{P_l}$.*

Proof. If the columns of $\mathcal{Z}$ form a basis of a stable right deflating subspace of $\lambda\mathcal{M} - \mathcal{N}$, then there exists a matrix Λ such that $\mathcal{M}\mathcal{Z}\Lambda = \mathcal{N}\mathcal{Z}$ and $\mathrm{Sp}(\Lambda) \subset \mathbb{C}_-$.

Using the Weierstrass canonical form (2.1) and partitioning the matrices

$$\begin{aligned} T_r Z_1 &= \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix}, \qquad T_l^T Z_2 = \begin{bmatrix} Z_{12} \\ Z_{22} \end{bmatrix}, \qquad T_l^{-1} B = \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}, \\ QT_r^{-1} &= [\, Q_f,\, Q_\infty \,], \qquad HT_r^{-1} = [\, H_f,\, H_\infty \,] \end{aligned} \tag{5.11}$$

such that $Z_{11}, Z_{12} \in \mathbb{R}^{n_f,n_f}$, $B_f \in \mathbb{R}^{n_f,m}$, $Q_f \in \mathbb{R}^{p,n_f}$, and $H_f \in \mathbb{R}^{m,n_f}$, we obtain from $\mathcal{M}\mathcal{Z}\Lambda = \mathcal{N}\mathcal{Z}$ the equations

$$Z_{11}\Lambda = A_f Z_{11} + B_f Z_3, \tag{5.12}$$
$$E_\infty Z_{21}\Lambda = Z_{21}, \tag{5.13}$$
$$Z_{12}\Lambda = Q_f^T Q_f Z_{11} - A_f^T Z_{12} + H_f^T Z_3, \tag{5.14}$$
$$E_\infty^T Z_{22}\Lambda = -Z_{22}, \tag{5.15}$$
$$0 = -H_f Z_{11} + B_f^T Z_{12} + J^T J Z_3. \tag{5.16}$$

Since Λ is nonsingular and E_∞ is nilpotent, it follows from (5.13) and (5.15) that $Z_{21} = 0$ and $Z_{22} = 0$. Therefore, Z_{11} is nonsingular. Multiplying equations (5.12), (5.14) and (5.16) from the right by Z_{11}^{-1}, we find that the matrices $Y_f = -Z_{12}Z_{11}^{-1}$ and $K_f = -Z_3 Z_{11}^{-1}$ satisfy the equations

$$\begin{aligned} &A_f - B_f K_f = Z_{11}\Lambda Z_{11}^{-1}, \\ &A_f^T Y_f + Q_f^T Q_f - H_f^T K_f = -Y_f (Z_{11}\Lambda Z_{11}^{-1}), \\ &Y_f B_f + H_f^T + K_f^T J^T J = 0. \end{aligned}$$

Furthermore, Lemma 5.1 implies that Y_f is symmetric. Taking into account that

$$(Z_1)^-_{P_r} = [\, Z_{11}^{-1},\ 0 \,] T_r, \qquad (EZ_1)^-_{P_l} = [\, Z_{11}^{-1},\ 0 \,] T_l^{-1},$$

we have that the pair

$$Y = T_l^{-T} \begin{bmatrix} Y_f & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1} = -Z_2 (EZ_1)^-_{P_l}, \quad K = [\, K_f,\ 0 \,] T_r = -Z_3 (Z_1)^-_{P_r} \tag{5.17}$$

is the solution of the system

$$\begin{aligned} A^T Y E + E^T Y A + P_r^T Q^T Q P_r &= -K^T J^T J K, \quad Y = P_l^T Y P_l \geq 0, \\ E^T Y B + P_r^T H^T &= -K^T J^T J. \end{aligned} \tag{5.18}$$

Then Y and $K_o = JK$ satisfy the projected Lur'e equations (5.1).

To show that Y is the minimal solution of (5.1), it is sufficient to prove that the matrix $\Delta Y = \hat{Y} - Y$ is positive semidefinite for any other solution $\hat{Y}$ of system (5.1). Substituting $\hat{Y} = Y + \Delta Y$ and $\hat{K} = K + \Delta K$ in (5.18) we obtain that

$$(A - P_l BK)^T \Delta Y E + E^T \Delta Y (A - P_l BK) = -P_r^T G P_r, \quad \Delta Y = P_l^T \Delta Y P_l, \tag{5.19}$$

where $G = E^T \Delta Y B B^T \Delta Y E$. Using (2.1), (2.2), (3.1) and (5.17), we have

$$A - P_l BK = T_l \begin{bmatrix} A_f - B_f K_f & 0 \\ 0 & I \end{bmatrix} T_r.$$

This implies that all the finite eigenvalues of the pencil $\lambda E - (A - P_l BK)$ have negative real part. In this case the projected Lyapunov equation (5.19) with symmetric, positive semidefinite G has a unique symmetric, positive semidefinite solution ΔY, see [41]. Thus, $Y \leq \hat{Y}$ for any solution $\hat{Y}$ of system (5.1), and, hence, $Y_{\min} = -Z_2 (EZ_1)^-_{P_l}$. □

The following lemma gives sufficient conditions for both matrices Z_1 and Z_2 in (5.3) to be of full column rank.

LEMMA 5.3. *Consider the pencil $\lambda\mathcal{M} - \mathcal{N}$ given in* (5.2)*. Let the matrix $\mathcal{Z}$ in* (5.3) *have full column rank and satisfies $\mathcal{M}\mathcal{Z}\Lambda = \mathcal{N}\mathcal{Z}$ for some nonsingular $\Lambda \in \mathbb{R}^{k,k}$. If the matrix J is nonsingular and if the descriptor system* (1.1) *is R-minimal, then Z_1 and Z_2 both have full column rank.*

Proof. Assume that Z_1 is rank deficient, i.e., there exists $v \neq 0$ such that $Z_1 v = 0$. Then it follows from equation (5.5) that also $Z_3 v = 0$. Since $\mathcal{Z}$ has full rank, we have $\mathcal{Z}v = [\,0,\, (Z_2 v)^T,\, 0\,]^T \neq 0$. In the proof of Theorem 5.2 it has been shown that Z_1 and Z_2 have the form

$$Z_1 = T_r^{-1}\begin{bmatrix} Z_{11} \\ 0 \end{bmatrix}, \qquad Z_2 = T_l^{-T}\begin{bmatrix} Z_{12} \\ 0 \end{bmatrix},$$

where Z_{11} and Z_{12} satisfy equations (5.12), (5.14) and (5.16). Multiplying these equations from the left by v, we obtain that

$$\begin{aligned} Z_{11}\Lambda v &= A_f Z_{11} v + B_f Z_3 v = 0, \\ Z_{12}\Lambda v &= Q_f^T Q_f Z_{11} v - A_f^T Z_{12} v + H_1^T Z_3 v = -A_f^T Z_{12} v, \\ 0 &= -H_1 Z_{11} v + B_f^T Z_{12} v + J^T J Z_3 v = B_f^T Z_{12} v. \end{aligned}$$

Let V be a matrix whose columns form a basis of $\ker(Z_1)$. Then $\Lambda V = VS$ for some square S and $A_f^T Z_{12} V = -Z_{12}\Lambda V = -Z_{12} V S$, $B_f^T Z_{12} V = 0$. If $\hat{v}$ is an eigenvector of S corresponding to an eigenvalue λ, i.e., $S\hat{v} = \lambda\hat{v}$, then for $z = Z_{12}V\hat{v} \neq 0$ we have $A_f^T z = -\lambda z$ and $B_f^T z = 0$. Hence, $\operatorname{rank}[\,\lambda E - A,\, B\,] < n$ that contradicts to the assumption that system (1.1) is R-controllable.

Analogously, we can show that the assumption of rank deficiency of Z_2 will lead to a contradiction to R-observability of (1.1). Thus, the proof is completed. □

The maximal solution $Y_{\max}$ of (5.1) can be determined from an antistable right deflating subspace of the pencil $\lambda\mathcal{M} - \mathcal{N}$ in a similar way. If this subspace has the dimension n_f and is spanned by columns of a matrix $\hat{\mathcal{Z}} = [\hat{Z}_1^T,\, \hat{Z}_2^T,\, \hat{Z}_3^T\,]^T$, where $\hat{Z}_1, \hat{Z}_2 \in \mathbb{R}^{n,n_f}$, $\hat{Z}_3 \in \mathbb{R}^{m,n_f}$ and $\hat{Z}_2$ has full column rank, then the minimal solution of the projected Lur'e equations dual to (5.1) is given by

$$X_{\min} = (E^T Y_{\max} E)_r^- = -\hat{Z}_1 (E^T \hat{Z}_2)_{P_r^T}^-.$$

5.2. Minimal solutions in factored form and their low-rank approximations. Next we show that $Y_{\min}$ and $X_{\min}$ can be computed in factored form directly from the matrices Z_1, Z_2 and $\hat{Z}_1$, $\hat{Z}_2$ defined above. Since $Y_{\min} = -Z_2(EZ_1)_{P_l}^-$ is symmetric and positive semidefinite, the symmetric matrix $-Z_1^T E^T Z_2$ is also positive semidefinite. Consider a decomposition

$$-Z_1^T E^T Z_2 = U_Y \Sigma_Y U_Y^T,$$

where the orthonormal columns of U_Y are the eigenvectors of $-Z_1^T E^T Z_2$ and Σ_Y is diagonal with positive diagonal elements that are non-zero eigenvalues of $-Z_1^T E^T Z_2$. Taking into account that $Z_2 = P_l^T Z_2$ and $EZ_1(EZ_1)_{P_l}^- = P_l$, we have

$$\begin{aligned} Y_{\min} &= -P_l^T Z_2 (EZ_1)_{P_l}^- = -((EZ_1)_{P_l}^-)^T (Z_1^T E^T Z_2)(EZ_1)_{P_l}^- \\ &= ((EZ_1)_{P_l}^-)^T U_Y \Sigma_Y U_Y^T (EZ_1)_{P_l}^- = \big(\Sigma_Y^{1/2} U_Y^T (EZ_1)_{P_l}^-\big)^T \big(\Sigma_Y^{1/2} U_Y^T (EZ_1)_{P_l}^-\big). \end{aligned}$$

Furthermore, using $\Sigma_Y^{1/2} = \Sigma_Y^{-1/2} U_Y^T U_Y \Sigma_Y$ we obtain

$$\begin{aligned}\Sigma_Y^{1/2} U_Y^T (EZ_1)_{P_l}^- &= \Sigma_Y^{-1/2} U_Y^T (U_Y \Sigma_Y U_Y^T)(EZ_1)_{P_l}^- \\ &= -\Sigma_Y^{-1/2} U_Y^T Z_2^T (EZ_1)(EZ_1)_{P_l}^- = -\Sigma_Y^{-1/2} U_Y^T Z_2^T.\end{aligned}$$

Thus, $Y_{\min}$ can be factored as

$$Y_{\min} = \big(\Sigma_Y^{-1/2} U_Y^T Z_2^T\big)^T \big(\Sigma_Y^{-1/2} U_Y^T Z_2^T\big) = LL^T.$$

Analogously, from a decomposition $-\hat{Z}_1^T E^T \hat{Z}_2 = U_X \Sigma_X U_X^T$, where $U_X^T U_X = I$ and Σ_X is diagonal and nonsingular, we obtain that

$$X_{\min} = \big(\Sigma_X^{-1/2} U_X^T \hat{Z}_1^T\big)^T \big(\Sigma_X^{-1/2} U_X^T \hat{Z}_1^T\big) = RR^T.$$

The established relationship between deflating subspaces of $\lambda\mathcal{M} - \mathcal{N}$ and the solution of the projected Lur'e equations turns out to be useful for the construction of numerical algorithms for computing the Gramians. For small and medium problems, the deflating subspaces of $\lambda\mathcal{M}-\mathcal{N}$ corresponding to the finite eigenvalues with positive and negative real part can be determined from the generalized Schur form computed using structured methods [6, 8]. This costs $\mathcal{O}(n^3)$ flops and requires storage of dense matrices $\mathcal{Z}$ and $\hat{\mathcal{Z}}$ of size $(2n+m)\times n_f$ even if $\mathcal{M}$ and $\mathcal{N}$ are sparse. Therefore, for large-scale sparse problems, computing the generalized Schur form would be impractical and inefficient. For this reason, we would prefer to approximate the solution of the projected Lur'e equations by low-rank matrices as it has been done for solutions of Lyapunov and Riccati equations [7, 27, 33, 43]. Similarly to the standard state space case [7], a low-rank approximate solution $\widetilde{Y}$ of the projected Lur'e equations (5.1) can be constructed in factored form from small dimensional deflating subspaces of $\lambda\mathcal{M} - \mathcal{N}$ spanned by the columns of $\mathcal{Z} \in \mathbb{R}^{n,k}$ in (5.3) with $k < n_f$ as

$$\widetilde{Y} = \big(\Sigma_Y^{-1/2} U_Y^T Z_2^T\big)^T \big(\Sigma_Y^{-1/2} U_Y^T Z_2^T\big).$$

Such subspaces can be computed using Krylov subspace methods like Lanzcos and Arnoldi processes. Since the matrices $\mathcal{M}$ and $\mathcal{N}$ have a very special block structure, it seems reasonable to use structure-preserving Krylov subspace methods in order to achieve better numerical accuracy. The development of such methods for the pencil $\lambda\mathcal{M} - \mathcal{N}$ with singular $\mathcal{M}$ remains for future work.

5.3. Newton's method for the projected Riccati equation. If the matrix J is nonsingular, then the projected Lur'e equations (5.1) are equivalent to the projected generalized algebraic Riccati equation

$$\begin{aligned}&A^T Y E + E^T Y A + P_r^T Q^T Q P_r + (B^T Y E + H P_r)^T (J^T J)^{-1} (B^T Y E + H P_r) = 0,\\ &Y = P_l^T Y P_l.\end{aligned} \tag{5.20}$$

If $Y_{\min}$ is a minimal solution of (5.1), then it is a *stabilizing* solution of (5.20) in the sense that all the finite eigenvalues of the pencil

$$\lambda E - A - P_l B (J^T J)^{-1} (H P_r + B^T Y_{\min} E)$$

have negative real part. Conversely, if Y is a stabilizing solution of the projected Riccati equation (5.20), then Y and $K_o = -J^{-T}(HP_r + B^T Y E)$ satisfy the projected

Lur'e equations (5.1) and $Y_{\min} = Y$. Thus, the Gramians of system (1.1) can also be computed by solving the projected Riccati equation (5.20) and its dual. Similarly to the standard state space case [5, 44, 49], these equations can be solved via Newton's method.

Introducing the matrices $\hat{A} = A + P_l B(J^T J)^{-1} H P_r$, $\hat{B} = BJ^{-1}$, $\hat{H} = J^{-T} H$ and $\hat{Q} = [Q^T,\ H^T J^{-1}]^T$, we rewrite the projected Riccati equation (5.20) as

$$\hat{A}^T Y E + E^T Y \hat{A} + P_r^T \hat{Q}^T \hat{Q} P_r + E^T Y \hat{B}\hat{B}^T Y E = 0, \quad Y = P_l^T Y P_l. \tag{5.21}$$

Newton's method for this equation is given as follows.

ALGORITHM 5.4. *Newton's method for the projected Riccati equation.*

Given E, $\hat{A} \in \mathbb{R}^{n,n}$, $\hat{B} \in \mathbb{R}^{n,m}$, $\hat{H} \in \mathbb{R}^{p,n}$, $\hat{Q} \in \mathbb{R}^{m,n}$, the projectors P_r and P_l, an initial guess Y_0 such that $Y_0 = P_l^T Y_0 P_l$ and all the finite eigenvalues of the pencil $\lambda E - \hat{A} - P_l \hat{B}\hat{B}^T Y_0 E$ lie in $\mathbb{C}_-$, compute an approximate solution of the projected Riccati equation (5.21).

FOR $j = 1, 2, \ldots$

1. Compute $K_j = \hat{B}^T Y_{j-1} E$ and $A_j = \hat{A} + P_l \hat{B} K_j$.
2. Solve the projected Lyapunov equation

$$A_j^T Y_j E + E^T Y_j A_j = -P_r^T (\hat{Q}^T \hat{Q} - K_j^T K_j) P_r, \qquad Y_j = P_l^T Y_j P_l.$$

END FOR

Using the Weierstrass canonical form (2.1) and (2.2), one can show similarly to the standard state space case [5, 44] that the finite eigenvalues of all $\lambda E - A_j$ have negative real part and $\lim\limits_{j\to\infty} Y_j = Y$ quadratically. To speed up the possibly slow convergence of the Newton iteration, we can use an exact line search method that can be derived for the projected Riccati equation (5.21) analogously to the standard case [5].

If the eigenvalues of Y decay to zero very rapidly, then Y can be well approximated by a matrix of low rank. Such a low-rank approximation can be computed in factored form $Y \approx \widetilde{L}\widetilde{L}^T$ with $\widetilde{L} \in \mathbb{R}^{n,k}$, $k \ll n$. To determine the low-rank factor $\widetilde{L}$ we can use the same approach as in [9]. Starting with $Y_{1,0} = Y_0$ and $Y_{2,0} = 0$, in each Newton iteration we compute $K_j = \hat{B}^T (Y_{1,j-1} - Y_{2,j-1}) E$ and $A_j = \hat{A} + P_l \hat{B} K_j$ and then solve two projected Lyapunov equations

$$A_j^T Y_{1,j} E + E^T Y_{1,j} A_j = -P_r^T \hat{Q}^T \hat{Q} P_r, \qquad Y_{1,j} = P_l^T Y_{1,j} P_l, \tag{5.22}$$

$$A_j^T Y_{2,j} E + E^T Y_{2,j} A_j = -P_r^T K_j^T K_j P_r, \qquad Y_{2,j} = P_l^T Y_{2,j} P_l, \tag{5.23}$$

for the low-rank factors $L_{1,j}$ and $L_{2,j}$ such that $Y_{1,j} \approx L_{1,j} L_{1,j}^T$ and $Y_{2,j} \approx L_{2,j} L_{2,j}^T$, respectively. Once the convergence is observed, an approximate solution $Y \approx \widetilde{L}\widetilde{L}^T$ of the projected Riccati equation (5.21) can be computed in factored form by solving either the projected Lyapunov equation

$$A^T Y E + E^T Y A = -P_r^T \hat{C}^T \hat{C} P_r, \qquad Y = P_l^T Y P_l \tag{5.24}$$

with $\hat{C} = [\,Q^T,\ \hat{H}^T + E^T (Y_{1,j_{\max}} - Y_{2,j_{\max}}) \hat{B}\,]^T$ or

$$\hat{A}_0^T Y E + E^T Y \hat{A}_0 = -P_r^T \hat{Q}^T \hat{Q} P_r, \qquad Y = P_l^T Y P_l, \tag{5.25}$$

where $\hat{A}_0 = \hat{A} + 1/2 P_l \hat{B}\hat{B}^T (Y_{1,j_{\max}} - Y_{2,j_{\max}}) E$. For computing the low-rank factors of the solutions of the projected Lyapunov equations (5.22)–(5.25) we can use the generalized alternating direction implicit method [43].

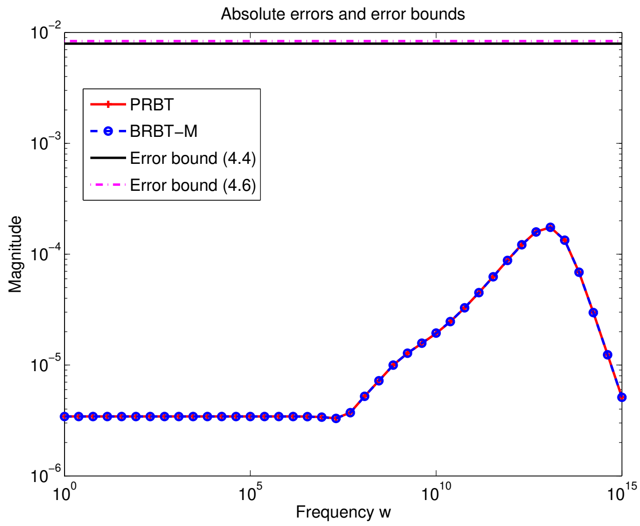

FIG. 6.1. *RC circuit: the absolute error for the reduced-order system computed by the positive real balanced truncation method.*

A slightly different way to determine the low-rank factors of the solution of the Riccati equation with $E = I$ via Newton's method has been considered in [44, 49]. This approach can also be extended to the projected Riccati equation (5.21), see [10] for details.

6. Numerical examples. In this section, we present some numerical examples to demonstrate the feasibility of the described model reduction methods for large scale descriptor systems. The computations were done on IBM RS 6000 44P Model 270 with machine precision $\varepsilon = 2.22 \times 10^{16}$ using MATLAB 7.0.4. The Gramians were computed by solving the projected Riccati equations via Newton's method.

Example 6.1. The first example is a three-port RC circuit provided by NEC Laboratories Europe, IT Research Division, NEC Europe Ltd. We have the passive descriptor system of order $n = 2007$ with $m = 3$ inputs and outputs. We approximate the descriptor system by two models of order $\ell = 41$ computed by the positive real balanced truncation (PRBT) method and the bounded real balanced truncation (BRBT-M) method applied to the Moebius-transformed system. The positive real Gramians of $\boldsymbol{G}$ and the bounded real Gramians of the Moebius-transformed system $\boldsymbol{H} = \mathcal{M}(\boldsymbol{G})$ have been approximated by the low-rank matrices $X_{\min} \approx \widetilde{R}\widetilde{R}^T$ and $Y_{\min} \approx \widetilde{L}\widetilde{L}^T$ with $\widetilde{R}, \widetilde{L} \in \mathbb{R}^{n,235}$. The frequency responses of the full-order and the reduced-order models are not presented, since they were impossible to distinguish. In Figure 6.1 we display the absolute errors $\|\widetilde{\boldsymbol{G}}(i\omega) - \boldsymbol{G}(i\omega)\|_2$ for a frequency range $\omega \in [1,\, 10^{15}]$ for both systems and also the error bounds (4.4) and (4.6). Note that due to the properness of the system, the PRBT and BRBT-M methods are equivalent and provide the similar results.

Example 6.2. Consider the 2D instationary Stokes equation that describes the flow of an incompressible fluid in a domain, see [38, Section 3.7.1] for details. The spatial discretization of this equation by the finite volume method on a uniform staggered grid leads to the descriptor system (1.1) which is contractive. In our experiments, the state space dimension is $n = 29799$ and the number of inputs and outputs is $m = p = 5$.

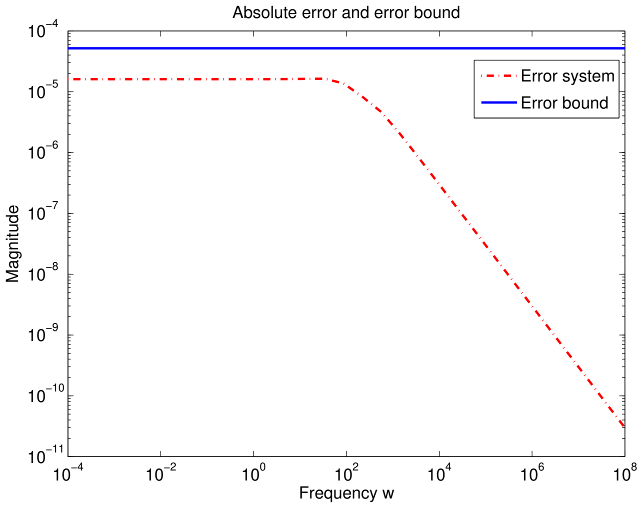

Fig. 6.2. *Semidiscretized Stokes equation: the absolute error and error bound for the reduced-order system computed by the BRBT method.*

This system has been approximated by a contractive reduced system of order $\ell = 15$ computed by the bounded real balanced truncation (BRBT) method. The bounded real Gramians have been approximated by the low-rank matrices $X_{br} \approx \widetilde{R}\widetilde{R}^T$ and $Y_{br} \approx \widetilde{L}\widetilde{L}^T$ with $\widetilde{R} \in \mathbb{R}^{n,217}$ and $\widetilde{R} \in \mathbb{R}^{n,103}$. Figure 6.2 shows the absolute error $\|\widetilde{\boldsymbol{G}}(i\omega) - \boldsymbol{G}(i\omega)\|_2$ for $\omega \in [10^{-4}, 10^8\,]$ as well as the error bound computed as the twice the sum of the truncated bounded real characteristic values.

7. Conclusion. In this paper we have considered model order reduction of descriptor systems using the positive real and bounded real balanced truncation methods. An advantage of these methods over the Krylov subspace model reduction technique is that computable error bounds are available. We have introduced the projected Lur'e and Riccati equations that can be used to define the positive real and bounded real Gramians for descriptor systems. We have also discussed the computation of these Gramians using the method based on deflating subspaces of the associated extended Hamiltonian pencil and Newton's method. The numerical examples demonstrate that the presented model reduction methods can be applied for large scale descriptor systems.

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